10 □ PARAMETRIC EQUATIONS AND POLAR COORDINATES

10.1 Curves Defined by Parametric Equations

1. $x = t^2 + t, \quad y = 3^{t+1}, \quad t = -2, -1, 0, 1, 2$

t	-2	-1	0	1	2
x	2	0	0	2	6
y	$\frac{1}{3}$	1	3	9	27

Therefore, the coordinates are $\left(2, \frac{1}{3}\right)$, $(0, 1)$, $(0, 3)$, $(2, 9)$, and $(6, 27)$.

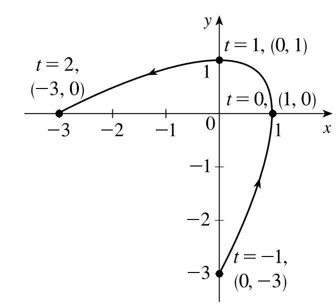
2. $x = \ln(t^2 + 1), \quad y = t/(t + 4), \quad t = -2, -1, 0, 1, 2$

t	-2	-1	0	1	2
x	$\ln 5$	$\ln 2$	0	$\ln 2$	$\ln 5$
y	-1	$-\frac{1}{3}$	0	$\frac{1}{5}$	$\frac{1}{3}$

Therefore, the coordinates are $(\ln 5, -1)$, $\left(\ln 2, -\frac{1}{3}\right)$, $(0, 0)$, $\left(\ln 2, \frac{1}{5}\right)$, and $\left(\ln 5, \frac{1}{3}\right)$.

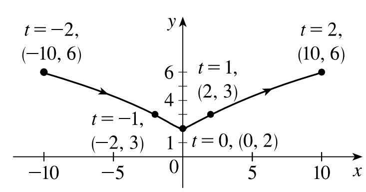
3. $x = 1 - t^2, \quad y = 2t - t^2, \quad -1 \leq t \leq 2$

t	-1	0	1	2
x	0	1	0	-3
y	-3	0	1	0



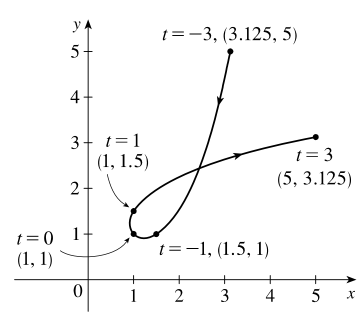
4. $x = t^3 + t, \quad y = t^2 + 2, \quad -2 \leq t \leq 2$

t	-2	-1	0	1	2
x	-10	-2	0	2	10
y	6	3	2	3	6



5. $x = 2^t - t, \quad y = 2^{-t} + t, \quad -3 \leq t \leq 3$

t	-3	-2	-1	0	1	2	3
x	3.125	2.25	1.5	1	1	2	5
y	5	2	1	1	1.5	2.25	3.125



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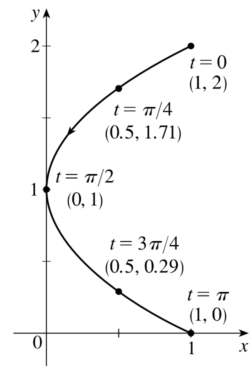
6. $x = \cos^2 t, \quad y = 1 + \cos t, \quad 0 \le t \le \pi$

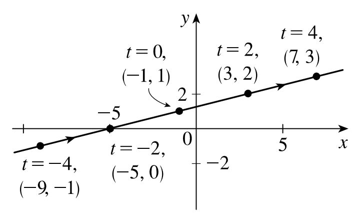
t	0	$\pi/4$	$\pi/2$	$3\pi/4$	π
x	1	0.5	0	0.5	1
y	2	1.707	1	0.293	0



7. $x = 2t - 1, \quad y = \frac{1}{2}t + 1$

(a)

t	-4	-2	0	2	4
x	-9	-5	-1	3	7
y	-1	0	1	2	3



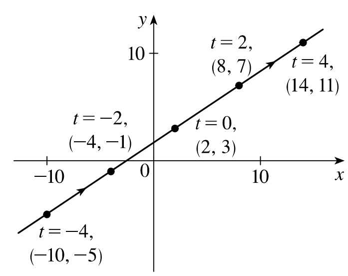
(b) $x = 2t - 1 \quad \Rightarrow \quad 2t = x + 1 \quad \Rightarrow \quad t = \frac{1}{2}x + \frac{1}{2}$, so

$y = \frac{1}{2}t + 1 = \frac{1}{2}\left(\frac{1}{2}x + \frac{1}{2}\right) + 1 = \frac{1}{4}x + \frac{1}{4} + 1 \quad \Rightarrow \quad y = \frac{1}{4}x + \frac{5}{4}$

8. $x = 3t + 2, \quad y = 2t + 3$

(a)

t	-4	-2	0	2	4
x	-10	-4	2	8	14
y	-5	-1	3	7	11



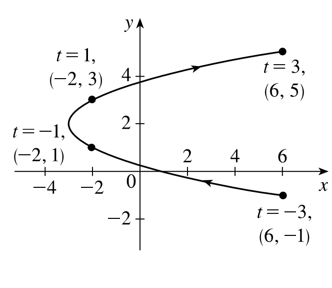
(b) $x = 3t + 2 \quad \Rightarrow \quad 3t = x - 2 \quad \Rightarrow \quad t = \frac{1}{3}x - \frac{2}{3}$, so

$y = 2t + 3 = 2\left(\frac{1}{3}x - \frac{2}{3}\right) + 3 = \frac{2}{3}x - \frac{4}{3} + 3 \quad \Rightarrow \quad y = \frac{2}{3}x + \frac{5}{3}$

9. $x = t^2 - 3, \quad y = t + 2, \quad -3 \le t \le 3$

(a)

t	-3	-1	1	3
x	6	-2	-2	6
y	-1	1	3	5



(b) $y = t + 2 \quad \Rightarrow \quad t = y - 2$, so

$x = t^2 - 3 = (y - 2)^2 - 3 = y^2 - 4y + 4 - 3 \quad \Rightarrow$

$x = y^2 - 4y + 1, -1 \le y \le 5$

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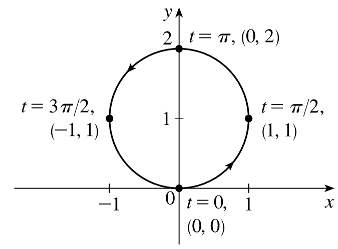
10. $x = \sin t, \quad y = 1 - \cos t, \quad 0 \le t \le 2\pi$

(a)

t	0	$\pi/2$	π	$3\pi/2$	2π
x	0	1	0	-1	0
y	0	1	2	1	0



(b) $x = \sin t, y = 1 - \cos t \quad [\text{or } y - 1 = -\cos t] \quad \Rightarrow$

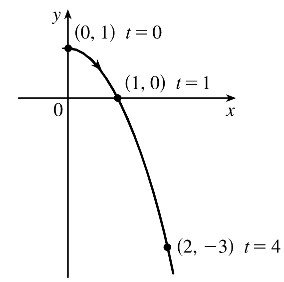
$x^2 + (y-1)^2 = (\sin t)^2 + (-\cos t)^2 \quad \Rightarrow \quad x^2 + (y-1)^2 = 1.$

As t varies from 0 to 2π, the circle with center $(0, 1)$ and radius 1 is traced out.

11. $x = \sqrt{t}, \; y = 1 - t$

(a)

t	0	1	2	3	4
x	0	1	1.414	1.732	2
y	1	0	-1	-2	-3



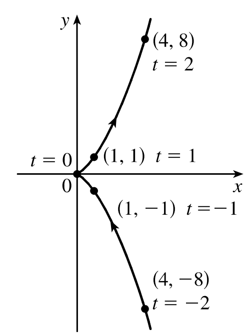
(b) $x = \sqrt{t} \quad \Rightarrow \quad t = x^2 \quad \Rightarrow \quad y = 1 - t = 1 - x^2$. Since $t \ge 0$, $x \ge 0$.

So the curve is the right half of the parabola $y = 1 - x^2$.

12. $x = t^2, \quad y = t^3$

(a)

t	-2	-1	0	1	2
x	4	1	0	1	4
y	-8	-1	0	1	8

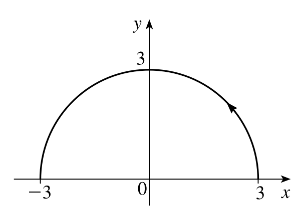


(b) $y = t^3 \quad \Rightarrow \quad t = \sqrt[3]{y} \quad \Rightarrow \quad x = t^2 = \left(\sqrt[3]{y}\right)^2 = y^{2/3}. \quad t \in \mathbb{R}, y \in \mathbb{R}, x \ge 0.$

13. (a) $x = 3\cos t, \quad y = 3\sin t, \quad 0 \le t \le \pi$

$x^2 + y^2 = 9\cos^2 t + 9\sin^2 t = 9(\cos^2 t + \sin^2 t) = 9$, which is the equation of a circle with radius 3. For $0 \le t \le \pi/2$, we have $3 \ge x \ge 0$ and $0 \le y \le 3$. For $\pi/2 < t \le \pi$, we have $0 > x \ge -3$ and $3 > y \ge 0$. Thus, the curve is the top half of the circle $x^2 + y^2 = 9$ traced counterclockwise.

(b)



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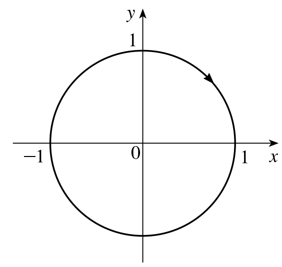
14. (a) $x = \sin 4\theta, \quad y = \cos 4\theta, \quad 0 \le \theta \le \pi/2$

$x^2 + y^2 = \sin^2 4\theta + \cos^2 4\theta = 1$, which is the equation of a circle with radius 1. When $\theta = 0$, we have $x = 0$ and $y = 1$. For $0 \le \theta \le \pi/4$, we have $x \ge 0$. For $\pi/4 < \theta \le \pi/2$, we have $x \le 0$. Thus, the curve is the circle $x^2 + y^2 = 1$ traced clockwise starting at $(0, 1)$.

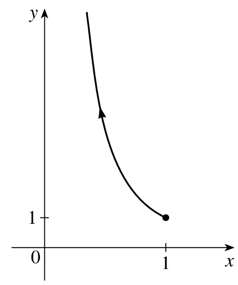
(b)



15. (a) $x = \cos\theta, \quad y = \sec^2\theta, \quad 0 \le \theta < \pi/2.$

$y = \sec^2\theta = \dfrac{1}{\cos^2\theta} = \dfrac{1}{x^2}$. For $0 \le \theta < \pi/2$, we have $1 \ge x > 0$ and $1 \le y$.

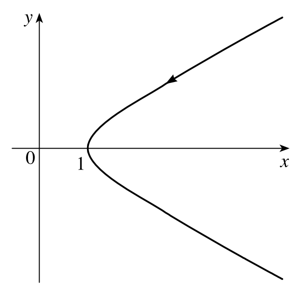
(b)



16. (a) $x = \csc t, y = \cot t, 0 < t < \pi$

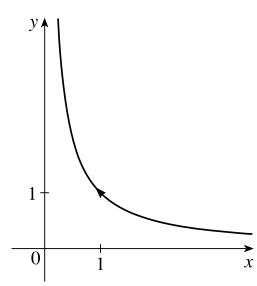
$y^2 - x^2 = \cot^2 t - \csc^2 t = 1$. For $0 < t < \pi$, we have $x > 1$. Thus, the curve is the right branch of the hyperbola $y^2 - x^2 = 1$.

(b)



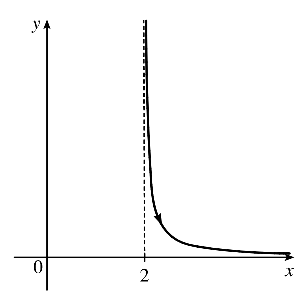
17. (a) $y = e^t = 1/e^{-t} = 1/x$ for $x > 0$ since $x = e^{-t}$. Thus, the curve is the portion of the hyperbola $y = 1/x$ with $x > 0$.

(b)



18. (a) $x = t + 2 \quad \Rightarrow \quad t = x - 2$. $y = 1/t = 1/(x - 2)$. For $t > 0$, we have $x > 2$ and $y > 0$. Thus, the curve is the portion of the hyperbola $y = 1/(x - 2)$ with $x > 2$.

(b)



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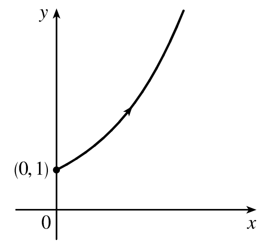
19. (a) $x = \ln t$, $y = \sqrt{t}$, $t \geq 1$.

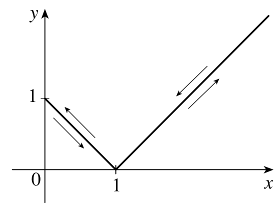
$x = \ln t \;\Rightarrow\; t = e^x \;\Rightarrow\; y = \sqrt{t} = e^{x/2}$, $x \geq 0$.

(b)



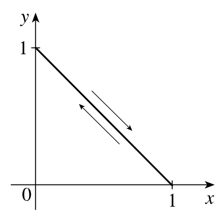
20. (a) $x = |t|$, $y = \left|1 - |t|\right| = |1 - x|$. For all t, we have $x \geq 0$ and $y \geq 0$. Thus, the curve is the portion of the absolute value function $y = |1 - x|$ with $x \geq 0$.

(b)



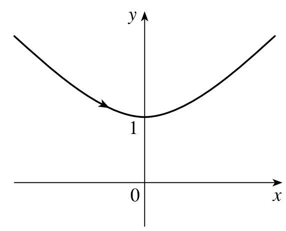
21. (a) $x = \sin^2 t$, $y = \cos^2 t$. $x + y = \sin^2 t + \cos^2 t = 1$. For all t, we have $0 \leq x \leq 1$ and $0 \leq y \leq 1$. Thus, the curve is the portion of the line $x + y = 1$ or $y = -x + 1$ in the first quadrant.

(b)



22. (a) $x = \sinh t$, $y = \cosh t \;\Rightarrow\; y^2 - x^2 = \cosh^2 t - \sinh^2 t = 1$. Since $y = \cosh t \geq 1$, we have the upper branch of the hyperbola $y^2 - x^2 = 1$.

(b)



23. The parametric equations $x = 5\cos t$ and $y = -5\sin t$ both have period 2π. When $t = 0$, we have $x = 5$ and $y = 0$. When $t = \pi/2$, we have $x = 0$ and $y = -5$. This is one-fourth of a circle. Thus, the object completes one revolution in $4 \cdot \frac{\pi}{2} = 2\pi$ seconds following a clockwise path.

24. The parametric equations $x = 3\sin\left(\frac{\pi}{4}t\right)$ and $y = 3\cos\left(\frac{\pi}{4}t\right)$ both have period $\dfrac{2\pi}{\pi/4} = 8$. When $t = 0$, we have $x = 0$ and $y = 3$. When $t = 2$, we have $x = 3$ and $y = 0$. This is one-fourth of a circle. Thus, the object completes one revolution in $4 \cdot 2 = 8$ seconds following a clockwise path.

25. $x = 5 + 2\cos \pi t$, $y = 3 + 2\sin \pi t \;\Rightarrow\; \cos \pi t = \dfrac{x-5}{2}$, $\sin \pi t = \dfrac{y-3}{2}$. $\cos^2(\pi t) + \sin^2(\pi t) = 1 \;\Rightarrow\; \left(\dfrac{x-5}{2}\right)^2 + \left(\dfrac{y-3}{2}\right)^2 = 1$. The motion of the particle takes place on a circle centered at $(5, 3)$ with a radius 2. As t goes from 1 to 2, the particle starts at the point $(3, 3)$ and moves counterclockwise along the circle $\left(\dfrac{x-5}{2}\right)^2 + \left(\dfrac{y-3}{2}\right)^2 = 1$ to $(7, 3)$ [one-half of a circle].

26. $x = 2 + \sin t$, $y = 1 + 3\cos t \;\Rightarrow\; \sin t = x - 2$, $\cos t = \dfrac{y-1}{3}$. $\sin^2 t + \cos^2 t = 1 \;\Rightarrow\; (x-2)^2 + \left(\dfrac{y-1}{3}\right)^2 = 1$. The motion of the particle takes place on an ellipse centered at $(2, 1)$. As t goes from $\pi/2$ to 2π, the particle starts at the point $(3, 1)$ and moves counterclockwise three-fourths of the way around the ellipse to $(2, 4)$.

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27. $x = 5\sin t,\ y = 2\cos t \quad \Rightarrow \quad \sin t = \dfrac{x}{5},\ \cos t = \dfrac{y}{2}.\quad \sin^2 t + \cos^2 t = 1 \quad \Rightarrow \quad \left(\dfrac{x}{5}\right)^2 + \left(\dfrac{y}{2}\right)^2 = 1$. The motion of the particle takes place on an ellipse centered at $(0, 0)$. As t goes from $-\pi$ to 5π, the particle starts at the point $(0, -2)$ and moves clockwise around the ellipse 3 times.

28. $y = \cos^2 t = 1 - \sin^2 t = 1 - x^2$. The motion of the particle takes place on the parabola $y = 1 - x^2$. As t goes from -2π to $-\pi$, the particle starts at the point $(0, 1)$, moves to $(1, 0)$, and goes back to $(0, 1)$. As t goes from $-\pi$ to 0, the particle moves to $(-1, 0)$ and goes back to $(0, 1)$. The particle repeats this motion as t goes from 0 to 2π.

29. We must have $1 \le x \le 4$ and $2 \le y \le 3$. So the graph of the curve must be contained in the rectangle $[1, 4]$ by $[2, 3]$.

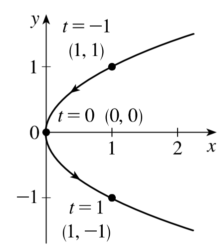
30. (a) From the first graph, we have $1 \le x \le 2$. From the second graph, we have $-1 \le y \le 1$. The only choice that satisfies either of those conditions is III.

(b) From the first graph, the values of x cycle through the values from -2 to 2 four times. From the second graph, the values of y cycle through the values from -2 to 2 six times. Choice I satisfies these conditions.

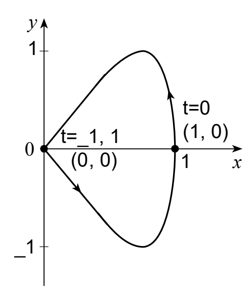
(c) From the first graph, the values of x cycle through the values from -2 to 2 three times. From the second graph, we have $0 \le y \le 2$. Choice IV satisfies these conditions.

(d) From the first graph, the values of x cycle through the values from -2 to 2 two times. From the second graph, the values of y do the same thing. Choice II satisfies these conditions.

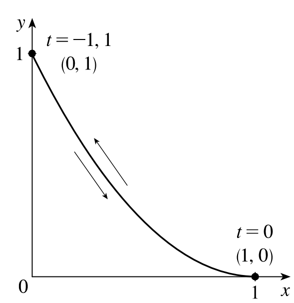
31. When $t = -1$, $(x, y) = (1, 1)$. As t increases to 0, x and y both decrease to 0. As t increases from 0 to 1, x increases from 0 to 1 and y decreases from 0 to -1. As t increases beyond 1, x continues to increase and y continues to decrease. For $t < -1$, x and y are both positive and decreasing. We could achieve greater accuracy by estimating x- and y-values for selected values of t from the given graphs and plotting the corresponding points.



32. When $t = -1$, $(x, y) = (0, 0)$. As t increases to 0, x increases from 0 to 1, while y first decreases to -1 and then increases to 0. As t increases from 0 to 1, x decreases from 1 to 0, while y first increases to 1 and then decreases to 0. We could achieve greater accuracy by estimating x- and y-values for selected values of t from the given graphs and plotting the corresponding points.



33. When $t = -1$, $(x, y) = (0, 1)$. As t increases to 0, x increases from 0 to 1 and y decreases from 1 to 0. As t increases from 0 to 1, the curve is retraced in the opposite direction with x decreasing from 1 to 0 and y increasing from 0 to 1. We could achieve greater accuracy by estimating x- and y-values for selected values of t from the given graphs and plotting the corresponding points.



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34. (a) $x = t^4 - t + 1 = (t^4 + 1) - t > 0$ [think of the graphs of $y = t^4 + 1$ and $y = t$] and $y = t^2 \geq 0$, so these equations are matched with graph V.

(b) $y = \sqrt{t} \geq 0$. $x = t^2 - 2t = t(t-2)$ is negative for $0 < t < 2$, so these equations are matched with graph I.

(c) $x = t^3 - 2t = t(t^2 - 2) = t\left(t + \sqrt{2}\right)\left(t - \sqrt{2}\right)$, $y = t^2 - t = t(t-1)$. The equation $x = 0$ has three solutions and the equation $y = 0$ has two solutions. Thus, the curve has three y-intercepts and two x-intercepts, which matches graph II.

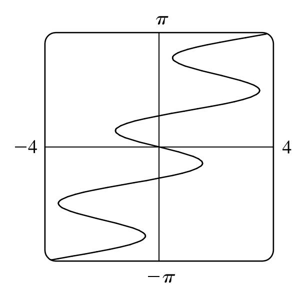
Alternate method: $x = t^3 - 2t$, $y = t^2 - t = \left(t^2 - t + \frac{1}{4}\right) - \frac{1}{4} = \left(t - \frac{1}{2}\right)^2 - \frac{1}{4}$ so $y \geq -\frac{1}{4}$ on this curve, whereas x is unbounded. These equations are matched with graph II.

(d) $x = \cos 5t$ has period $2\pi/5$ and $y = \sin 2t$ has period π, so x will take on the values -1 to 1, and then 1 to -1, before y takes on the values -1 to 1. Note that when $t = 0$, $(x, y) = (1, 0)$. These equations are matched with graph VI.

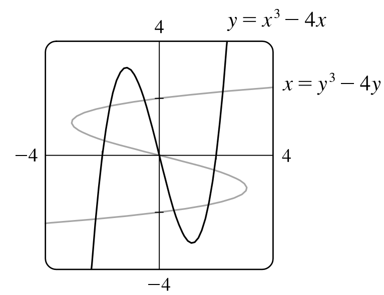
(e) $x = t + \sin 4t$, $y = t^2 + \cos 3t$. As t becomes large, t and t^2 become the dominant terms in the expressions for x and y, so the graph will look like the graph of $y = x^2$, but with oscillations. These equations are matched with graph IV.

(f) $x = t + \sin 2t$, $y = t + \sin 3t$. As t becomes large, t becomes the dominant term in the expressions for both x and y, so the graph will look like the graph of $y = x$, but with oscillations. These equations are matched with graph III.

35. Use $y = t$ and $x = t - 2\sin \pi t$ with a t-interval of $[-\pi, \pi]$.



36. Use $x_1 = t$, $y_1 = t^3 - 4t$ and $x_2 = t^3 - 4t$, $y_2 = t$ with a t-interval of $[-3, 3]$. There are 9 points of intersection; $(0, 0)$ is fairly obvious. The point in quadrant I is approximately $(2.2, 2.2)$, and by symmetry, the point in quadrant III is approximately $(-2.2, -2.2)$. The other six points are approximately $(\mp 1.9, \pm 0.5)$, $(\mp 1.7, \pm 1.7)$, and $(\mp 0.5, \pm 1.9)$.



37. (a) $x = x_1 + (x_2 - x_1)t$, $y = y_1 + (y_2 - y_1)t$, $0 \leq t \leq 1$. Clearly the curve passes through $P_1(x_1, y_1)$ when $t = 0$ and through $P_2(x_2, y_2)$ when $t = 1$. For $0 < t < 1$, x is strictly between x_1 and x_2 and y is strictly between y_1 and y_2. For every value of t, x and y satisfy the relation $y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)$, which is the equation of the line through $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$.

Finally, any point (x, y) on that line satisfies $\dfrac{y - y_1}{y_2 - y_1} = \dfrac{x - x_1}{x_2 - x_1}$; if we call that common value t, then the given parametric equations yield the point (x, y); and any (x, y) on the line between $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ yields a value of t in $[0, 1]$. So the given parametric equations exactly specify the line segment from $P_1(x_1, y_1)$ to $P_2(x_2, y_2)$.

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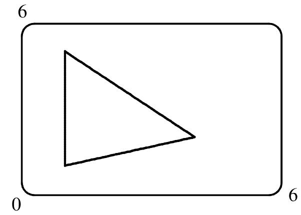
(b) $x = -2 + [3 - (-2)]t = -2 + 5t$ and $y = 7 + (-1 - 7)t = 7 - 8t$ for $0 \le t \le 1$.

38. For the side of the triangle from A to B, use $(x_1, y_1) = (1, 1)$ and $(x_2, y_2) = (4, 2)$. Hence, the equations are

$$x = x_1 + (x_2 - x_1)\,t = 1 + (4 - 1)\,t = 1 + 3t,$$
$$y = y_1 + (y_2 - y_1)\,t = 1 + (2 - 1)\,t = 1 + t.$$



Graphing $x = 1 + 3t$ and $y = 1 + t$ with $0 \le t \le 1$ gives us the side of the triangle from A to B. Similarly, for the side BC we use $x = 4 - 3t$ and $y = 2 + 3t$, and for the side AC we use $x = 1$ and $y = 1 + 4t$.

39. The result in Example 4 indicates the parametric equations have the form $x = h + r \sin bt$ and $y = k + r \cos bt$ where (h, k) is the center of the circle with radius r and $b = 2\pi/\text{period}$. (The use of positive sine in the x-equation and positive cosine in the y-equation results in a clockwise motion.) With $h = 0$, $k = 0$ and $b = 2\pi/4\pi = 1/2$, we have $x = 5\sin\left(\frac{1}{2}t\right)$, $y = 5\cos\left(\frac{1}{2}t\right)$.

40. As in Example 4, we use parametric equations of the form $x = h + r\cos bt$ and $y = k + r \sin bt$ where $(h, k) = (1, 3)$ is the center of the circle with radius $r = 1$ and $b = 2\pi/\text{period} = 2\pi/3$. (The use of positive cosine in the x-equation and positive sine in the y-equation results in a counterclockwise motion.) Thus, $x = 1 + \cos\left(\frac{2\pi}{3}t\right)$, $y = 3 + \sin\left(\frac{2\pi}{3}t\right)$.

41. The circle $x^2 + (y - 1)^2 = 4$ has center $(0, 1)$ and radius 2, so by Example 4 it can be represented by $x = 2\cos t$, $y = 1 + 2\sin t$, $0 \le t \le 2\pi$. This representation gives us the circle with a counterclockwise orientation starting at $(2, 1)$.

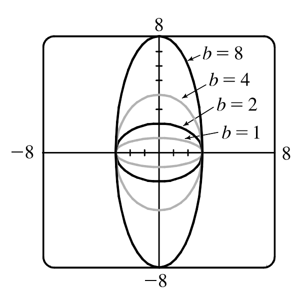
(a) To get a clockwise orientation, we could change the equations to $x = 2\cos t$, $y = 1 - 2\sin t$, $0 \le t \le 2\pi$.

(b) To get three times around in the counterclockwise direction, we use the original equations $x = 2\cos t$, $y = 1 + 2\sin t$ with the domain expanded to $0 \le t \le 6\pi$.

(c) To start at $(0, 3)$ using the original equations, we must have $x_1 = 0$; that is, $2\cos t = 0$. Hence, $t = \frac{\pi}{2}$. So we use $x = 2\cos t$, $y = 1 + 2\sin t$, $\frac{\pi}{2} \le t \le \frac{3\pi}{2}$.

Alternatively, if we want t to start at 0, we could change the equations of the curve. For example, we could use $x = -2\sin t$, $y = 1 + 2\cos t$, $0 \le t \le \pi$.

42. (a) Let $x^2/a^2 = \sin^2 t$ and $y^2/b^2 = \cos^2 t$ to obtain $x = a\sin t$ and $y = b\cos t$ with $0 \le t \le 2\pi$ as possible parametric equations for the ellipse $x^2/a^2 + y^2/b^2 = 1$.



(b) The equations are $x = 3\sin t$ and $y = b\cos t$ for $b \in \{1, 2, 4, 8\}$.

(c) As b increases, the ellipse stretches vertically.

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43. *Big circle:* It's centered at $(2, 2)$ with a radius of 2, so by Example 4, parametric equations are

$$x = 2 + 2\cos t, \qquad y = 2 + 2\sin t, \qquad 0 \le t \le 2\pi$$

Small circles: They are centered at $(1, 3)$ and $(3, 3)$ with a radius of 0.1. By Example 4, parametric equations are

$$\textit{(left)} \qquad x = 1 + 0.1\cos t, \qquad y = 3 + 0.1\sin t, \qquad 0 \le t \le 2\pi$$

and

$$\textit{(right)} \qquad x = 3 + 0.1\cos t, \qquad y = 3 + 0.1\sin t, \qquad 0 \le t \le 2\pi$$

Semicircle: It's the lower half of a circle centered at $(2, 2)$ with radius 1. By Example 4, parametric equations are

$$x = 2 + 1\cos t, \qquad y = 2 + 1\sin t, \qquad \pi \le t \le 2\pi$$

To get all four graphs on the same screen with a typical graphing calculator, we need to change the last t-interval to $[0, 2\pi]$ in order to match the others. We can do this by changing t to $0.5t$. This change gives us the upper half. There are several ways to get the lower half—one is to change the "+" to a "−" in the y-assignment, giving us

$$x = 2 + 1\cos(0.5t), \qquad y = 2 - 1\sin(0.5t), \qquad 0 \le t \le 2\pi$$

44. If you are using a calculator or computer that can overlay graphs (using multiple t-intervals), the following is appropriate.

Left side: $x = 1$ and y goes from 1.5 to 4, so use

$$x = 1, \qquad y = t, \qquad 1.5 \le t \le 4$$

Right side: $x = 10$ and y goes from 1.5 to 4, so use

$$x = 10, \qquad y = t, \qquad 1.5 \le t \le 4$$

Bottom: x goes from 1 to 10 and $y = 1.5$, so use

$$x = t, \qquad y = 1.5, \qquad 1 \le t \le 10$$

Handle: It starts at $(10, 4)$ and ends at $(13, 7)$, so use

$$x = 10 + t, \qquad y = 4 + t, \qquad 0 \le t \le 3$$

Left wheel: It's centered at $(3, 1)$, has a radius of 1, and appears to go about 30° above the horizontal, so use

$$x = 3 + 1\cos t, \qquad y = 1 + 1\sin t, \qquad \tfrac{5\pi}{6} \le t \le \tfrac{13\pi}{6}$$

Right wheel: Similar to the left wheel with center $(8, 1)$, so use

$$x = 8 + 1\cos t, \qquad y = 1 + 1\sin t, \qquad \tfrac{5\pi}{6} \le t \le \tfrac{13\pi}{6}$$

If you are using a calculator or computer that cannot overlay graphs (using one t-interval), the following is appropriate. We'll start by picking the t-interval $[0, 2.5]$ since it easily matches the t-values for the two sides. We now need to find parametric equations for all graphs with $0 \le t \le 2.5$.

Left side: $x = 1$ and y goes from 1.5 to 4, so use

$$x = 1, \qquad y = 1.5 + t, \qquad 0 \le t \le 2.5$$

[continued]

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Right side: $x = 10$ and y goes from 1.5 to 4, so use

$$x = 10, \qquad y = 1.5 + t, \qquad 0 \le t \le 2.5$$

Bottom: x goes from 1 to 10 and $y = 1.5$, so use

$$x = 1 + 3.6t, \qquad y = 1.5, \qquad 0 \le t \le 2.5$$

To get the x-assignment, think of creating a linear function such that when $t = 0$, $x = 1$ and when $t = 2.5$, $x = 10$. We can use the point-slope form of a line with $(t_1, x_1) = (0, 1)$ and $(t_2, x_2) = (2.5, 10)$.

$x - 1 = \dfrac{10 - 1}{2.5 - 0}(t - 0) \quad \Rightarrow \quad x = 1 + 3.6t.$

Handle: It starts at $(10, 4)$ and ends at $(13, 7)$, so use

$$x = 10 + 1.2t, \qquad y = 4 + 1.2t, \qquad 0 \le t \le 2.5$$

$(t_1, x_1) = (0, 10)$ and $(t_2, x_2) = (2.5, 13)$ gives us $x - 10 = \dfrac{13 - 10}{2.5 - 0}(t - 0) \quad \Rightarrow \quad x = 10 + 1.2t.$

$(t_1, y_1) = (0, 4)$ and $(t_2, y_2) = (2.5, 7)$ gives us $y - 4 = \dfrac{7 - 4}{2.5 - 0}(t - 0) \quad \Rightarrow \quad y = 4 + 1.2t.$

Left wheel: It's centered at $(3, 1)$, has a radius of 1, and appears to go about $30°$ above the horizontal, so use

$$x = 3 + 1\cos\left(\tfrac{8\pi}{15}t + \tfrac{5\pi}{6}\right), \qquad y = 1 + 1\sin\left(\tfrac{8\pi}{15}t + \tfrac{5\pi}{6}\right), \qquad 0 \le t \le 2.5$$

$(t_1, \theta_1) = \left(0, \frac{5\pi}{6}\right)$ and $(t_2, \theta_2) = \left(\frac{5}{2}, \frac{13\pi}{6}\right)$ gives us $\theta - \frac{5\pi}{6} = \dfrac{\frac{13\pi}{6} - \frac{5\pi}{6}}{\frac{5}{2} - 0}(t - 0) \quad \Rightarrow \quad \theta = \frac{5\pi}{6} + \frac{8\pi}{15}t.$

Right wheel: Similar to the left wheel with center $(8, 1)$, so use

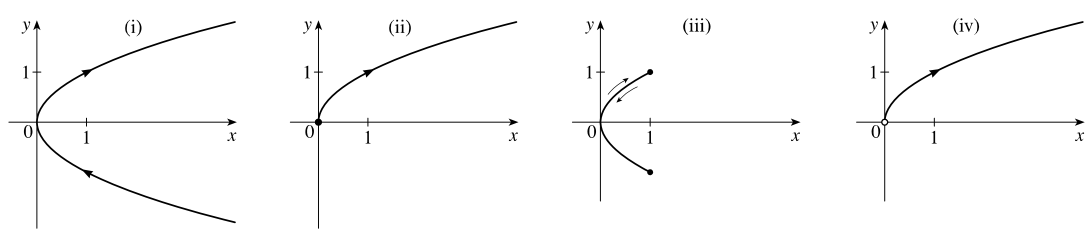
$$x = 8 + 1\cos\left(\tfrac{8\pi}{15}t + \tfrac{5\pi}{6}\right), \qquad y = 1 + 1\sin\left(\tfrac{8\pi}{15}t + \tfrac{5\pi}{6}\right), \qquad 0 \le t \le 2.5$$

45. (a) (i) $x = t^2, y = t \;\Rightarrow\; y^2 = t^2 = x$ (ii) $x = t, y = \sqrt{t} \;\Rightarrow\; y^2 = t = x$

(iii) $x = \cos^2 t, y = \cos t \;\Rightarrow\; y^2 = \cos^2 t = x$ (iv) $x = 3^{2t}, y = 3^t \;\Rightarrow\; y^2 = (3^t)^2 = 3^{2t} = x.$

Thus, the points on all four of the given parametric curves satisfy the Cartesian equation $y^2 = x$.

(b) The graph of $y^2 = x$ is a right-opening parabola with vertex at the origin. For curve (i), $x \ge 0$ and y is unbounded so the graph contains the entire parabola. For (ii), $y = \sqrt{t}$ requires that $t \ge 0$, so that both $x \ge 0$ and $y \ge 0$, which captures the upper half of the parabola, including the origin. For (iii), $-1 \le \cos t \le 1$ so the graph is the portion of the parabola contained in the intervals $0 \le x \le 1$ and $-1 \le y \le 1$. For (iv), $x > 0$ and $y > 0$, which captures the upper half of the parabola excluding the origin.



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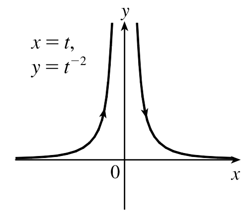
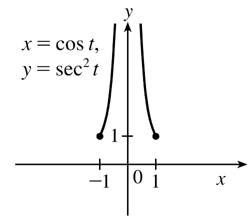
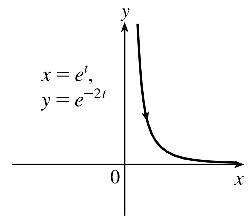
46. (a) $x = t$, so $y = t^{-2} = x^{-2}$. We get the entire curve $y = 1/x^2$ traversed in a left-to-right direction.



(b) $x = \cos t$, $y = \sec^2 t = \dfrac{1}{\cos^2 t} = \dfrac{1}{x^2}$. Since $\sec t \geq 1$, we only get the parts of the curve $y = 1/x^2$ with $y \geq 1$. We get the first quadrant portion of the curve when $x > 0$, that is, $\cos t > 0$, and we get the second quadrant portion of the curve when $x < 0$, that is, $\cos t < 0$.

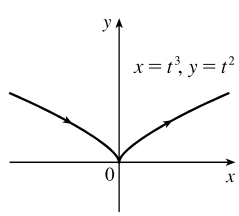


(c) $x = e^t$, $y = e^{-2t} = (e^t)^{-2} = x^{-2}$. Since e^t and e^{-2t} are both positive, we only get the first quadrant portion of the curve $y = 1/x^2$.



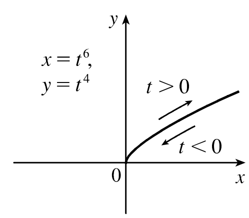
47. (a) $x = t^3 \quad \Rightarrow \quad t = x^{1/3}$, so $y = t^2 = x^{2/3}$.

We get the entire curve $y = x^{2/3}$ traversed in a left to right direction.



(b) $x = t^6 \quad \Rightarrow \quad t = x^{1/6}$, so $y = t^4 = x^{4/6} = x^{2/3}$.

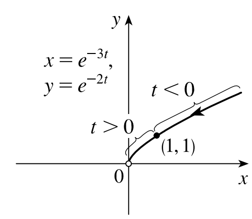
Since $x = t^6 \geq 0$, we only get the right half of the curve $y = x^{2/3}$.



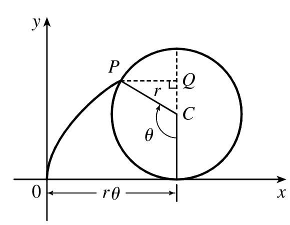
(c) $x = e^{-3t} = (e^{-t})^3 \quad [\text{so } e^{-t} = x^{1/3}]$,

$y = e^{-2t} = (e^{-t})^2 = (x^{1/3})^2 = x^{2/3}$.

If $t < 0$, then x and y are both larger than 1. If $t > 0$, then x and y are between 0 and 1. Since $x > 0$ and $y > 0$, the curve never quite reaches the origin.



48. The case $\frac{\pi}{2} < \theta < \pi$ is illustrated. C has coordinates $(r\theta, r)$ as in Example 7, and Q has coordinates $(r\theta, r + r\cos(\pi - \theta)) = (r\theta, r(1 - \cos\theta))$ [since $\cos(\pi - \alpha) = \cos\pi\cos\alpha + \sin\pi\sin\alpha = -\cos\alpha$], so P has coordinates $(r\theta - r\sin(\pi - \theta), r(1 - \cos\theta)) = (r(\theta - \sin\theta), r(1 - \cos\,\theta))$ [since $\sin(\pi - \alpha) = \sin\pi\cos\alpha - \cos\pi\sin\alpha = \sin\alpha$]. Again we have the parametric equations $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$.



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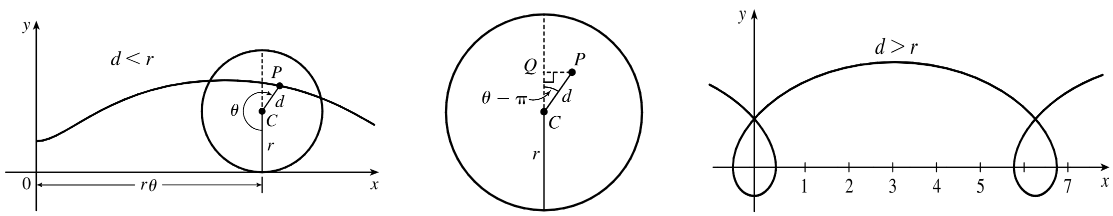
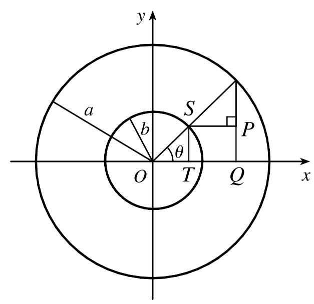
49. The first two diagrams depict the case $\pi < \theta < \frac{3\pi}{2}$, $d < r$. As in Example 7, C has coordinates $(r\theta, r)$. Now Q (in the second diagram) has coordinates $(r\theta, r + d\cos(\theta - \pi)) = (r\theta, r - d\cos\theta)$, so a typical point P of the trochoid has coordinates $(r\theta + d\sin(\theta - \pi), r - d\cos\theta)$. That is, P has coordinates (x, y), where $x = r\theta - d\sin\theta$ and $y = r - d\cos\theta$. When $d = r$, these equations agree with those of the cycloid.



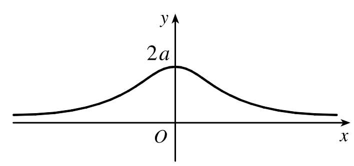
50. In polar coordinates, an equation for the circle is $r = 2a\sin\theta$. Thus, the coordinates of Q are $x = r\cos\theta = 2a\sin\theta\,\cos\theta$ and $y = r\sin\theta = 2a\sin^2\theta$. The coordinates of R are $x = 2a\cot\theta$ and $y = 2a$. Since P is the midpoint of QR, we use the midpoint formula to get $x = a(\sin\theta\,\cos\theta + \cot\theta)$ and $y = a(1 + \sin^2\theta)$.

51. It is apparent that $x = |OQ|$ and $y = |QP| = |ST|$. From the diagram, $x = |OQ| = a\cos\theta$ and $y = |ST| = b\sin\theta$. Thus, the parametric equations are $x = a\cos\theta$ and $y = b\sin\theta$. To eliminate θ we rearrange: $\sin\theta = y/b \;\Rightarrow\; \sin^2\theta = (y/b)^2$ and $\cos\theta = x/a \;\Rightarrow\; \cos^2\theta = (x/a)^2$. Adding the two equations: $\sin^2\theta + \cos^2\theta = 1 = x^2/a^2 + y^2/b^2$. Thus, we have an ellipse.



52. A has coordinates $(a\cos\theta, a\sin\theta)$. Since OA is perpendicular to AB, ΔOAB is a right triangle and B has coordinates $(a\sec\theta, 0)$. It follows that P has coordinates $(a\sec\theta, b\sin\,\theta)$. Thus, the parametric equations are $x = a\sec\theta$, $y = b\sin\theta$.

53. $C = (2a\cot\theta, 2a)$, so the x-coordinate of P is $x = 2a\cot\theta$. Let $B = (0, 2a)$. Then $\angle OAB$ is a right angle and $\angle OBA = \theta$, so $|OA| = 2a\sin\theta$ and $A = ((2a\sin\theta)\cos\theta, (2a\sin\theta)\sin\theta)$. Thus, the y-coordinate of P is $y = 2a\sin^2\theta$.



54. (a) Let θ be the angle of inclination of segment OP. Then $|OB| = \dfrac{2a}{\cos\theta}$.

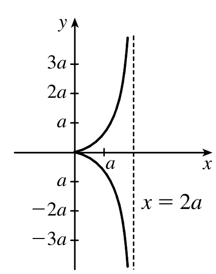
Let $C = (2a, 0)$. Then by use of right triangle OAC we see that $|OA| = 2a\cos\theta$.

Now

$$\begin{aligned}|OP| &= |AB| = |OB| - |OA| \\ &= 2a\left(\frac{1}{\cos\theta} - \cos\theta\right) = 2a\,\frac{1 - \cos^2\theta}{\cos\theta} = 2a\,\frac{\sin^2\theta}{\cos\theta} = 2a\sin\theta\,\tan\theta\end{aligned}$$

So P has coordinates $x = 2a\sin\theta\,\tan\theta \cdot \cos\theta = 2a\sin^2\theta$ and $y = 2a\sin\theta\,\tan\theta \cdot \sin\theta = 2a\sin^2\theta\,\tan\theta$.

(b)



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55. (a) *Red particle*: $x = t + 5$, $y = t^2 + 4t + 6$

Blue particle: $x = 2t + 1$, $y = 2t + 6$

Substituting $x = 1$ and $y = 6$ into the parametric equations for the red particle gives $1 = t + 5$ and $6 = t^2 + 4t + 6$, which are both satisfied when $t = -4$. Making the same substitution for the blue particle gives $1 = 2t + 1$ and $6 = 2t + 6$, which are both satisfied when $t = 0$. Repeating the process for $x = 6$ and $y = 11$, the red particle's equations become $6 = t + 5$ and $11 = t^2 + 4t + 6$, which are both satisfied when $t = 1$. Similarly, the blue particle's equations become $6 = 2t + 1$ and $11 = 2t + 6$, which are both satisfied when $t = 2.5$. Thus, $(1, 6)$ and $(6, 11)$ are both intersection points, but they are not collision points, since the particles reach each of these points at different times.

(b) *Blue particle*: $x = 2t + 1 \Rightarrow t = \frac{1}{2}(x - 1)$.

Substituting into the equation for y gives $y = 2t + 6 = 2\left[\frac{1}{2}(x - 1)\right] + 6 = x + 5$.

Green particle: $x = 2t + 4 \Rightarrow t = \frac{1}{2}(x - 4)$.

Substituting into the equation for y gives $y = 2t + 9 = 2\left[\frac{1}{2}(x - 4)\right] + 9 = x + 5$.

Thus, the green and blue particles both move along the line $y = x + 5$.

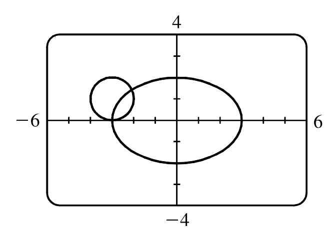
Now, the red and green particles will collide if there is a time t when both particles are at the same point. Equating the x parametric equations, we find $t + 5 = 2t + 4$, which is satisfied when $t = 1$, and gives $x = 1 + 5 = 6$. Substituting $t = 1$ into the red and green particles' y equations gives $y = (1)^2 + 4(1) + 6 = 11$ and $y = 2(1) + 9 = 11$, respectively. Thus, the red and green particles collide at the point $(6, 11)$ when $t = 1$.

56. (a) $x = 3\sin t,\ y = 2\cos t,\ 0 \le t \le 2\pi$;

$x = -3 + \cos t,\ y = 1 + \sin t,\ 0 \le t \le 2\pi$

There are 2 points of intersection:

$(-3, 0)$ and approximately $(-2.1, 1.4)$.



(b) A collision point occurs when $x_1 = x_2$ and $y_1 = y_2$ for the same t. So solve the equations:

$$3\sin t = -3 + \cos t \quad \textbf{(1)}$$

$$2\cos t = 1 + \sin t \quad \textbf{(2)}$$

From **(2)**, $\sin t = 2\cos t - 1$. Substituting into **(1)**, we get $3(2\cos t - 1) = -3 + \cos t \Rightarrow 5\cos t = 0\ (\star) \Rightarrow \cos t = 0 \Rightarrow t = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. We check that $t = \frac{3\pi}{2}$ satisfies **(1)** and **(2)** but $t = \frac{\pi}{2}$ does not. So the only collision point occurs when $t = \frac{3\pi}{2}$, and this gives the point $(-3, 0)$. [We could check our work by graphing x_1 and x_2 together as functions of t and, on another plot, y_1 and y_2 as functions of t. If we do so, we see that the only value of t for which *both* pairs of graphs intersect is $t = \frac{3\pi}{2}$.]

(c) The circle is centered at $(3, 1)$ instead of $(-3, 1)$. There are still 2 intersection points: $(3, 0)$ and $(2.1, 1.4)$, but there are no collision points, since $(\star)$ in part (b) becomes $5\cos t = 6 \Rightarrow \cos t = \frac{6}{5} > 1$.

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57. (a) $x = 1 - t^2$, $y = t - t^3$. The curve intersects itself if there are two distinct times $t = a$ and $t = b$ (with $a < b$) such that $x(a) = x(b)$ and $y(a) = y(b)$. The equation $x(a) = x(b)$ gives $1 - a^2 = 1 - b^2$ so that $a^2 = b^2$. Since $a \neq b$ by assumption, we must have $a = -b$. Substituting into the equation for y gives $y(-b) = y(b) \Rightarrow$ $-b - (-b)^3 = b - b^3 \Rightarrow 2b^3 - 2b = 0 \Rightarrow 2b(b-1)(b+1) = 0 \Rightarrow b = -1, 0, 1$. Since $a < b$, the only valid solution is $b = 1$, which corresponds to $a = -1$ and results in the coordinates $x = 0$ and $y = 0$. Thus, the curve intersects itself at $(0, 0)$ when $t = -1$ and $t = 1$.

(b) $x = 2t - t^3$, $y = t - t^2$. Similar to part (a), we try to find the times $t = a$ and $t = b$ with $a < b$ such that $x(a) = x(b)$ and $y(a) = y(b)$. The equation $y(a) = y(b)$ gives $a - a^2 = b - b^2 \Rightarrow 0 = a^2 - a + (b - b^2)$. Using the quadratic formula to solve for a, we get

$$a = \frac{1 \pm \sqrt{1 - 4(b - b^2)}}{2} = \frac{1 \pm \sqrt{4b^2 - 4b + 1}}{2} = \frac{1 \pm \sqrt{(2b-1)^2}}{2} = \frac{1 \pm (2b-1)}{2} \Rightarrow a = b \text{ or } a = 1 - b.$$

Since $a < b$ by assumption, we reject the first solution and substitute $a = 1 - b$ into $x(a) = x(b) \Rightarrow x(1-b) = x(b) \Rightarrow$ $2(1-b) - (1-b)^3 = 2b - b^3$. Expanding and simplifying gives $2b^3 - 3b^2 - b + 1 = 0$. By graphing the equation, we see that $b = \frac{1}{2}$ is a zero, so $2b - 1$ is a factor, and by long division $b^2 - b - 1$ is another factor. Hence, the solutions are $b = \frac{1}{2}$ and $b = \frac{1}{2} \pm \frac{1}{2}\sqrt{5}$ (found using the quadratic formula). Since $a = 1 - b$ and we require $a < b$, the only valid solution is $b = \frac{1}{2} + \frac{1}{2}\sqrt{5}$, which corresponds to $a = \frac{1}{2} - \frac{1}{2}\sqrt{5}$ and results in the coordinates $x = 2\left(\frac{1}{2} - \frac{1}{2}\sqrt{5}\right) - \left(\frac{1}{2} - \frac{1}{2}\sqrt{5}\right)^3 = -1$ and $y = \frac{1}{2} - \frac{1}{2}\sqrt{5} - \left(\frac{1}{2} - \frac{1}{2}\sqrt{5}\right)^2 = -1$. Thus, the curve intersects itself at $(-1, -1)$ when $t = \frac{1}{2} - \frac{1}{2}\sqrt{5}$ and $t = \frac{1}{2} + \frac{1}{2}\sqrt{5}$.

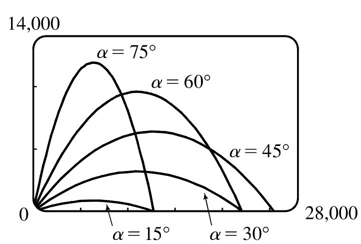
58. (a) If $\alpha = 30^\circ$ and $v_0 = 500$ m/s, then the equations become $x = (500\cos 30^\circ)t = 250\sqrt{3}t$ and $y = (500\sin 30^\circ)t - \frac{1}{2}(9.8)t^2 = 250t - 4.9t^2$. $y = 0$ when $t = 0$ (when the gun is fired) and again when $t = \frac{250}{4.9} \approx 51$ s. Then $x = \left(250\sqrt{3}\right)\left(\frac{250}{4.9}\right) \approx 22{,}092$ m, so the bullet hits the ground about 22 km from the gun.

The formula for y is quadratic in t. To find the maximum y-value, we will complete the square:

$$y = -4.9\left(t^2 - \tfrac{250}{4.9}t\right) = -4.9\left[t^2 - \tfrac{250}{4.9}t + \left(\tfrac{125}{4.9}\right)^2\right] + \tfrac{125^2}{4.9} = -4.9\left(t - \tfrac{125}{4.9}\right)^2 + \tfrac{125^2}{4.9} \leq \tfrac{125^2}{4.9}$$

with equality when $t = \frac{125}{4.9}$ s, so the maximum height attained is $\frac{125^2}{4.9} \approx 3189$ m.

(b)



As α ($0^\circ < \alpha < 90^\circ$) increases up to 45°, the projectile attains a greater height and a greater range. As α increases past 45°, the projectile attains a greater height, but its range decreases.

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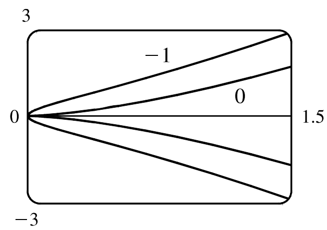
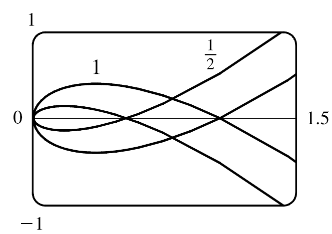
(c) $x = (v_0 \cos\alpha)t \;\Rightarrow\; t = \dfrac{x}{v_0 \cos\alpha}$.

$$y = (v_0 \sin\alpha)t - \tfrac{1}{2}gt^2 \;\Rightarrow\; y = (v_0 \sin\alpha)\frac{x}{v_0 \cos\alpha} - \frac{g}{2}\left(\frac{x}{v_0 \cos\alpha}\right)^2 = (\tan\alpha)x - \left(\frac{g}{2v_0^2 \cos^2\alpha}\right)x^2,$$

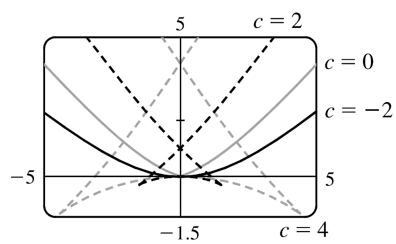
which is the equation of a parabola (quadratic in x).

59. $x = t^2, y = t^3 - ct$. We use a graphing device to produce the graphs for various values of c with $-\pi \le t \le \pi$. Note that all the members of the family are symmetric about the x-axis. For $c < 0$, the graph does not cross itself, but for $c = 0$ it has a cusp at $(0, 0)$ and for $c > 0$ the graph crosses itself at $x = c$, so the loop grows larger as c increases.

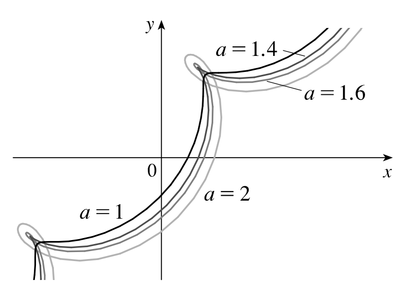




60. $x = 2ct - 4t^3, y = -ct^2 + 3t^4$. We use a graphing device to produce the graphs for various values of c with $-\pi \le t \le \pi$. Note that all the members of the family are symmetric about the y-axis. When $c < 0$, the graph resembles that of a polynomial of even degree, but when $c = 0$ there is a corner at the origin, and when $c > 0$, the graph crosses itself at the origin, and has two cusps below the x-axis. The size of the "swallowtail" increases as c increases.



61. $x = t + a\cos t, y = t + a\sin t, a > 0$. From the first figure, we see that curves roughly follow the line $y = x$, and they start having loops when a is between 1.4 and 1.6. The loops increase in size as a increases.



While not required, the following is a solution to determine the *exact* values for which the curve has a loop, that is, we seek the values of a for which there exist parameter values t and u such that $t < u$ and $(t + a\cos t, t + a\sin t) = (u + a\cos u, u + a\sin u)$.

[continued]

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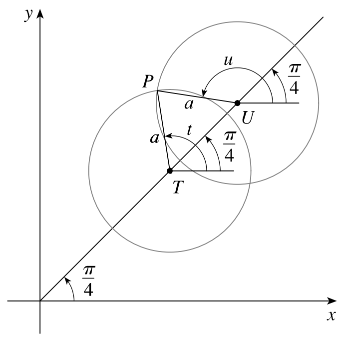
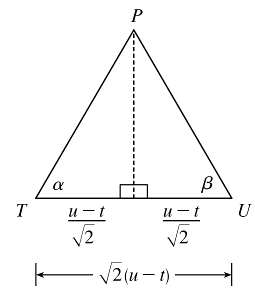


In the diagram at the left, T denotes the point (t,t), U the point (u,u), and P the point $(t+a\cos t, t+a\sin t) = (u+a\cos u, u+a\sin u)$. Since $\overline{PT} = \overline{PU} = a$, the triangle PTU is isosceles. Therefore its base angles, $\alpha = \angle PTU$ and $\beta = \angle PUT$ are equal. Since $\alpha = t - \frac{\pi}{4}$ and $\beta = 2\pi - \frac{3\pi}{4} - u = \frac{5\pi}{4} - u$, the relation $\alpha = \beta$ implies that $u + t = \frac{3\pi}{2}$ **(1)**.



Since $\overline{TU} = \text{distance}((t,t),(u,u)) = \sqrt{2(u-t)^2} = \sqrt{2}\,(u-t)$, we see that

$\cos\alpha = \dfrac{\frac{1}{2}\overline{TU}}{\overline{PT}} = \dfrac{(u-t)/\sqrt{2}}{a}$, so $u - t = \sqrt{2}\,a\cos\alpha$, that is,

$u - t = \sqrt{2}\,a\cos\!\left(t - \frac{\pi}{4}\right)$ **(2)**. Now $\cos\!\left(t-\frac{\pi}{4}\right) = \sin\!\left[\frac{\pi}{2} - \left(t - \frac{\pi}{4}\right)\right] = \sin\!\left(\frac{3\pi}{4} - t\right)$, so we can rewrite **(2)** as $u - t = \sqrt{2}\,a\sin\!\left(\frac{3\pi}{4} - t\right)$ **(2′)**. Subtracting **(2′)** from **(1)** and dividing by 2, we obtain $t = \frac{3\pi}{4} - \frac{\sqrt{2}}{2}a\sin\!\left(\frac{3\pi}{4} - t\right)$, or $\frac{3\pi}{4} - t = \frac{a}{\sqrt{2}}\sin\!\left(\frac{3\pi}{4} - t\right)$ **(3)**.

Since $a > 0$ and $t < u$, it follows from **(2′)** that $\sin\!\left(\frac{3\pi}{4} - t\right) > 0$. Thus from **(3)** we see that $t < \frac{3\pi}{4}$. [We have implicitly assumed that $0 < t < \pi$ by the way we drew our diagram, but we lost no generality by doing so since replacing t by $t + 2\pi$ merely increases x and y by 2π. The curve's basic shape repeats every time we change t by 2π.] Solving for a in

(3), we get $a = \dfrac{\sqrt{2}\left(\frac{3\pi}{4} - t\right)}{\sin\!\left(\frac{3\pi}{4} - t\right)}$. Write $z = \frac{3\pi}{4} - t$. Then $a = \dfrac{\sqrt{2}\,z}{\sin z}$, where $z > 0$. Now $\sin z < z$ for $z > 0$, so $a > \sqrt{2}$.

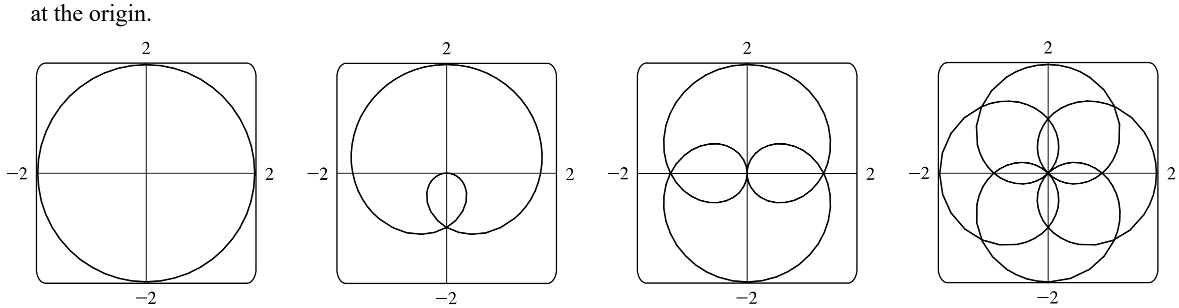
$\left[\text{As } z \to 0^+\text{, that is, as } t \to \left(\frac{3\pi}{4}\right)^-,\ a \to \sqrt{2}\right]$.

62. Consider the curves $x = \sin t + \sin nt$, $y = \cos t + \cos nt$, where n is a positive integer. For $n = 1$, we get a circle of radius 2 centered at the origin. For $n > 1$, we get a curve lying on or inside that circle that traces out $n - 1$ loops as t ranges from 0 to 2π.

Note:

$$\begin{aligned} x^2 + y^2 &= (\sin t + \sin nt)^2 + (\cos t + \cos nt)^2 \\ &= \sin^2 t + 2\sin t\,\sin nt + \sin^2 nt + \cos^2 t + 2\cos t\,\cos nt + \cos^2 nt \\ &= (\sin^2 t + \cos^2 t) + (\sin^2 nt + \cos^2 nt) + 2(\cos t\,\cos nt + \sin t\,\sin nt) \\ &= 1 + 1 + 2\cos(t - nt) = 2 + 2\cos((1-n)t) \le 4 = 2^2, \end{aligned}$$

with equality for $n = 1$. This shows that each curve lies on or inside the curve for $n = 1$, which is a circle of radius 2 centered at the origin.



$n = 1$ $\quad$ $n = 2$ $\quad$ $n = 3$ $\quad$ $n = 5$

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