CHAPTER 2 DIFFERENTIATION

2.1 TANGENTS AND DERIVATIVES AT A POINT

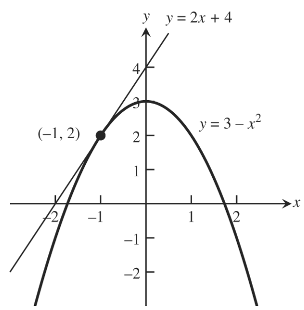
1. P_1: $m_1 = 1$, P_2: $m_2 = 5$

2. P_1: $m_1 = -2$, P_2: $m_2 = 0$

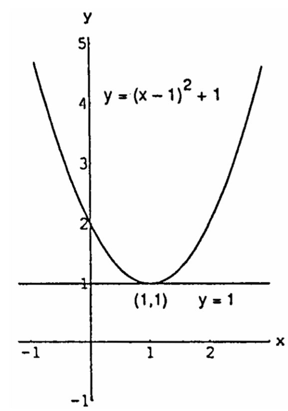
3. P_1: $m_1 = \frac{5}{2}$, P_2: $m_2 = -\frac{1}{2}$

4. P_1: $m_1 = 3$, P_2: $m_2 = -3$

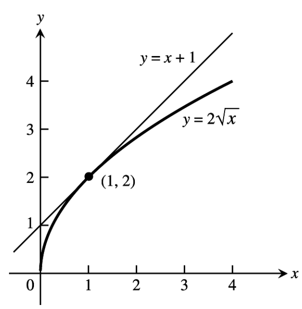
5. $m = \lim\limits_{h \to 0} \frac{[3-(-1+h)^2]-(3-(-1)^2)}{h}$
$= \lim\limits_{h \to 0} \frac{3-(1-2h+h^2)-2}{h} = \lim\limits_{h \to 0} \frac{h(2-h)}{h} = 2;$
at $(-1, 2)$: $y = 2 + 2(x - (-1)) \Rightarrow y = 2x + 4$,
tangent line



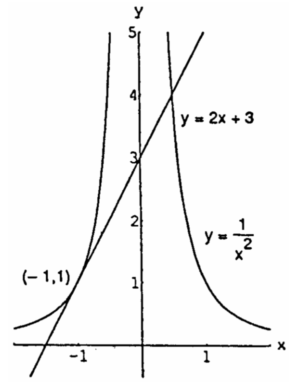
6. $m = \lim\limits_{h \to 0} \frac{[(1+h-1)^2+1]-[(1-1)^2+1]}{h} = \lim\limits_{h \to 0} \frac{h^2}{h}$
$= \lim\limits_{h \to 0} h = 0$; at $(1, 1)$: $y = 1 + 0(x - 1) \Rightarrow y = 1$,
tangent line



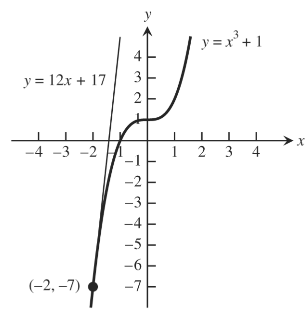
7. $m = \lim\limits_{h \to 0} \frac{2\sqrt{1+h}-2\sqrt{1}}{h} = \lim\limits_{h \to 0} \frac{2\sqrt{1+h}-2}{h} \cdot \frac{2\sqrt{1+h}+2}{2\sqrt{1+h}+2}$
$= \lim\limits_{h \to 0} \frac{4(1+h)-4}{2h\left(\sqrt{1+h}+1\right)} = \lim\limits_{h \to 0} \frac{2}{\sqrt{1+h}+1} = 1;$
at $(1, 2)$: $y = 2 + 1(x - 1) \Rightarrow y = x + 1$, tangent line



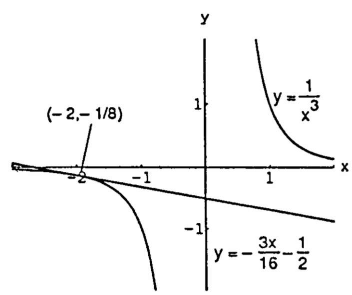
8. $m = \lim_{h \to 0} \frac{\frac{1}{(-1+h)^2} - \frac{1}{(-1)^2}}{h} = \lim_{h \to 0} \frac{1-(-1+h)^2}{h(-1+h)^2}$
$= \lim_{h \to 0} \frac{-(-2h+h^2)}{h(-1+h)^2} = \lim_{h \to 0} \frac{2-h}{(-1+h)^2} = 2;$
at $(-1, 1)$: $y = 1 + 2(x-(-1)) \Rightarrow y = 2x + 3$, tangent line



9. $m = \lim_{h \to 0} \frac{(-2+h)^3 + 1 - ((-2)^3+1)}{h} = \lim_{h \to 0} \frac{-8+12h-6h^2+h^3+8}{h}$
$= \lim_{h \to 0} (12 - 6h + h^2) = 12;$
at $(-2, -7)$: $y = -7 + 12(x-(-2)) \Rightarrow y = 12x + 17$, tangent line



10. $m = \lim_{h \to 0} \frac{\frac{1}{(-2+h)^3} - \frac{1}{(-2)^3}}{h} = \lim_{h \to 0} \frac{-8-(-2+h)^3}{-8h(-2+h)^3}$
$= \lim_{h \to 0} \frac{-(12h-6h^2+h^3)}{-8h(-2+h)^3} = \lim_{h \to 0} \frac{12-6h+h^2}{8(-2+h)^3}$
$= \frac{12}{8(-8)} = -\frac{3}{16};$
at $\left(-2, -\frac{1}{8}\right)$: $y = -\frac{1}{8} - \frac{3}{16}(x-(-2))$
$\Rightarrow y = -\frac{3}{16}x - \frac{1}{2}$, tangent line



11. $m = \lim_{h \to 0} \frac{[(2+h)^2+1]-5}{h} = \lim_{h \to 0} \frac{(5+4h+h^2)-5}{h} = \lim_{h \to 0} \frac{h(4+h)}{h} = 4;$
at $(2, 5)$: $y - 5 = 4(x - 2)$, tangent line

12. $m = \lim_{h \to 0} \frac{[(1+h)-2(1+h)^2]-(-1)}{h} = \lim_{h \to 0} \frac{(1+h-2-4h-2h^2)+1}{h} = \lim_{h \to 0} \frac{h(-3-2h)}{h} = -3;$
at $(1, -1)$: $y + 1 = -3(x - 1)$, tangent line

13. $m = \lim_{h \to 0} \frac{\frac{3+h}{(3+h)-2} - 3}{h} = \lim_{h \to 0} \frac{(3+h)-3(h+1)}{h(h+1)} = \lim_{h \to 0} \frac{-2h}{h(h+1)} = -2;$
at $(3, 3)$: $y - 3 = -2(x - 3)$, tangent line

14. $m = \lim_{h \to 0} \frac{\frac{8}{(2+h)^2} - 2}{h} = \lim_{h \to 0} \frac{8-2(2+h)^2}{h(2+h)^2} = \lim_{h \to 0} \frac{8-2(4+4h+h^2)}{h(2+h)^2} = \lim_{h \to 0} \frac{-2h(4+h)}{h(2+h)^2} = \frac{-8}{4} = -2;$
at $(2, 2)$: $y - 2 = -2(x - 2)$

15. $m = \lim_{h \to 0} \frac{((2+h)^3-(2+h))-6}{h} = \lim_{h \to 0} \frac{(6+11h+6h^2+h^3)-6}{h} = \lim_{h \to 0} \frac{h(11+6h+h^2)}{h} = 11;$
at $(2, 6)$: $y - 6 = 11(t - 2)$, tangent line

16. $m = \lim\limits_{h \to 0} \frac{[(1+h)^3 + 3(1+h)] - 4}{h} = \lim\limits_{h \to 0} \frac{(1 + 3h + 3h^2 + h^3 + 3 + 3h) - 4}{h} = \lim\limits_{h \to 0} \frac{h(6 + 3h + h^2)}{h} = 6;$
at $(1, 4)$: $y - 4 = 6(t - 1)$, tangent line

17. $m = \lim\limits_{h \to 0} \frac{\sqrt{4+h} - 2}{h} = \lim\limits_{h \to 0} \frac{\sqrt{4+h} - 2}{h} \cdot \frac{\sqrt{4+h}+2}{\sqrt{4+h}+2} = \lim\limits_{h \to 0} \frac{(4+h) - 4}{h\left(\sqrt{4+h}+2\right)} = \lim\limits_{h \to 0} \frac{h}{h\left(\sqrt{4+h}+2\right)} = \frac{1}{\sqrt{4}+2}$
$= \frac{1}{4}$; at $(4, 2)$: $y - 2 = \frac{1}{4}(x - 4)$, tangent line

18. $m = \lim\limits_{h \to 0} \frac{\sqrt{(8+h)+1} - 3}{h} = \lim\limits_{h \to 0} \frac{\sqrt{9+h} - 3}{h} \cdot \frac{\sqrt{9+h}+3}{\sqrt{9+h}+3} = \lim\limits_{h \to 0} \frac{(9+h) - 9}{h\left(\sqrt{9+h}+3\right)} = \lim\limits_{h \to 0} \frac{h}{h\left(\sqrt{9+h}+3\right)}$
$= \frac{1}{\sqrt{9}+3} = \frac{1}{6}$; at $(8, 3)$: $y - 3 = \frac{1}{6}(x - 8)$, tangent line

19. At $x = -1$, $y = 5 \Rightarrow m = \lim\limits_{h \to 0} \frac{5(-1+h)^2 - 5}{h} = \lim\limits_{h \to 0} \frac{5(1 - 2h + h^2) - 5}{h} = \lim\limits_{h \to 0} \frac{5h(-2+h)}{h} = -10$, slope

20. At $x = 2$, $y = -3 \Rightarrow m = \lim\limits_{h \to 0} \frac{[1 - (2+h)^2] - (-3)}{h} = \lim\limits_{h \to 0} \frac{(1 - 4 - 4h - h^2) + 3}{h} = \lim\limits_{h \to 0} \frac{-h(4+h)}{h} = -4$, slope

21. At $x = 3$, $y = \frac{1}{2} \Rightarrow m = \lim\limits_{h \to 0} \frac{\frac{1}{(3+h)-1} - \frac{1}{2}}{h} = \lim\limits_{h \to 0} \frac{2 - (2+h)}{2h(2+h)} = \lim\limits_{h \to 0} \frac{-h}{2h(2+h)} = -\frac{1}{4}$, slope

22. At $x = 0$, $y = -1 \Rightarrow m = \lim\limits_{h \to 0} \frac{\frac{h-1}{h+1} - (-1)}{h} = \lim\limits_{h \to 0} \frac{(h-1) + (h+1)}{h(h+1)} = \lim\limits_{h \to 0} \frac{2h}{h(h+1)} = 2$, slope

23. At a horizontal tangent the slope $m = 0 \Rightarrow 0 = m = \lim\limits_{h \to 0} \frac{[(x+h)^2 + 4(x+h) - 1] - (x^2 + 4x - 1)}{h}$
$= \lim\limits_{h \to 0} \frac{(x^2 + 2xh + h^2 + 4x + 4h - 1) - (x^2 + 4x - 1)}{h} = \lim\limits_{h \to 0} \frac{(2xh + h^2 + 4h)}{h} = \lim\limits_{h \to 0} (2x + h + 4) = 2x + 4;$
$2x + 4 = 0 \Rightarrow x = -2$. Then $f(-2) = 4 - 8 - 1 = -5 \Rightarrow (-2, -5)$ is the point on the graph where there is a horizontal tangent.

24. $0 = m = \lim\limits_{h \to 0} \frac{[(x+h)^3 - 3(x+h)] - (x^3 - 3x)}{h} = \lim\limits_{h \to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3 - 3x - 3h) - (x^3 - 3x)}{h}$
$= \lim\limits_{h \to 0} \frac{3x^2h + 3xh^2 + h^3 - 3h}{h} = \lim\limits_{h \to 0} (3x^2 + 3xh + h^2 - 3) = 3x^2 - 3$; $3x^2 - 3 = 0 \Rightarrow x = -1$ or $x = 1$. Then $f(-1) = 2$ and $f(1) = -2 \Rightarrow (-1, 2)$ and $(1, -2)$ are the points on the graph where a horizontal tangent exists.

25. $-1 = m = \lim\limits_{h \to 0} \frac{\frac{1}{(x+h)-1} - \frac{1}{x-1}}{h} = \lim\limits_{h \to 0} \frac{(x-1) - (x+h-1)}{h(x-1)(x+h-1)} = \lim\limits_{h \to 0} \frac{-h}{h(x-1)(x+h-1)} = -\frac{1}{(x-1)^2}$
$\Rightarrow (x-1)^2 = 1 \Rightarrow x^2 - 2x = 0 \Rightarrow x(x-2) = 0 \Rightarrow x = 0$ or $x = 2$. If $x = 0$, then $y = -1$ and $m = -1$ $\Rightarrow y = -1 - (x - 0) = -(x + 1)$. If $x = 2$, then $y = 1$ and $m = -1 \Rightarrow y = 1 - (x - 2) = -(x - 3)$.

26. $\frac{1}{4} = m = \lim\limits_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim\limits_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} = \lim\limits_{h \to 0} \frac{(x+h) - x}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$
$= \lim\limits_{h \to 0} \frac{h}{h\left(\sqrt{x+h} + \sqrt{x}\right)} = \frac{1}{2\sqrt{x}}$. Thus, $\frac{1}{4} = \frac{1}{2\sqrt{x}} \Rightarrow \sqrt{x} = 2 \Rightarrow x = 4 \Rightarrow y = 2$. The tangent line is $y = 2 + \frac{1}{4}(x - 4) = \frac{x}{4} + 1$.

27. $\lim\limits_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim\limits_{h \to 0} \frac{(100 - 4.9(2+h)^2) - (100 - 4.9(2)^2)}{h} = \lim\limits_{h \to 0} \frac{-4.9(4 + 4h + h^2) + 4.9(4)}{h}$
$= \lim\limits_{h \to 0} (-19.6 - 4.9h) = -19.6$. The minus sign indicates the object is falling downward at a speed of 19.6 m/sec.

28. $\lim\limits_{h \to 0} \frac{f(10+h) - f(10)}{h} = \lim\limits_{h \to 0} \frac{3(10+h)^2 - 3(10)^2}{h} = \lim\limits_{h \to 0} \frac{3(20h + h^2)}{h} = 60$ ft/sec.

29. $\lim\limits_{h \to 0} \frac{f(3+h)-f(3)}{h} = \lim\limits_{h \to 0} \frac{\pi(3+h)^2 - \pi(3)^2}{h} = \lim\limits_{h \to 0} \frac{\pi[9+6h+h^2-9]}{h} = \lim\limits_{h \to 0} \pi(6+h) = 6\pi$

30. $\lim\limits_{h \to 0} \frac{f(2+h)-f(2)}{h} = \lim\limits_{h \to 0} \frac{\frac{4\pi}{3}(2+h)^3 - \frac{4\pi}{3}(2)^3}{h} = \lim\limits_{h \to 0} \frac{\frac{4\pi}{3}[12h+6h^2+h^3]}{h} = \lim\limits_{h \to 0} \frac{4\pi}{3}[12+6h+h^2] = 16\pi$

31. $\lim\limits_{h \to 0} \frac{(m(x_0+h)+b)-(mx_0+b)}{h} = \lim\limits_{h \to 0} \frac{mx_0+mh+b-mx_0-b}{h} = \lim\limits_{h \to 0} \frac{mh}{h} = \lim\limits_{h \to 0} m = m$
$y - (mx_0 + b) = m(x - x_0) \to y - mx_0 - b = mx - mx_0 \to y = mx + b$

32. $\lim\limits_{h \to 0} \frac{\frac{1}{\sqrt{4+h}} - \frac{1}{2}}{h} = \lim\limits_{h \to 0} \frac{\frac{2-\sqrt{4+h}}{2\sqrt{4+h}}}{h} = \lim\limits_{h \to 0} \frac{2-\sqrt{4+h}}{2h\sqrt{4+h}} = \lim\limits_{h \to 0} \frac{\left(2-\sqrt{4+h}\right)\left(2+\sqrt{4+h}\right)}{2h\sqrt{4+h}\left(2+\sqrt{4+h}\right)} = \lim\limits_{h \to 0} \frac{4-(4+h)}{2h\sqrt{4+h}\left(2+\sqrt{4+h}\right)}$
$= \lim\limits_{h \to 0} \frac{-h}{2h\sqrt{4+h}\left(2+\sqrt{4+h}\right)} = \lim\limits_{h \to 0} \frac{-1}{2\sqrt{4+h}\left(2+\sqrt{4+h}\right)} = \frac{-1}{2\sqrt{4+0}(2+\sqrt{4+0})} = \frac{-1}{16} \to m = \frac{-1}{16}$

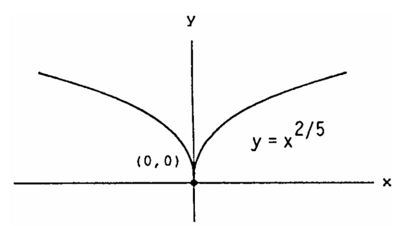
33. Slope at origin $= \lim\limits_{h \to 0} \frac{f(0+h)-f(0)}{h} = \lim\limits_{h \to 0} \frac{h^2 \sin\left(\frac{1}{h}\right)}{h} = \lim\limits_{h \to 0} h \sin\left(\frac{1}{h}\right) = 0 \Rightarrow$ yes, f(x) does have a tangent at the origin with slope 0.

34. $\lim\limits_{h \to 0} \frac{g(0+h)-g(0)}{h} = \lim\limits_{h \to 0} \frac{h \sin\left(\frac{1}{h}\right)}{h} = \lim\limits_{h \to 0} \sin \frac{1}{h}$. Since $\lim\limits_{h \to 0} \sin \frac{1}{h}$ does not exist, f(x) has no tangent at the origin.

35. $\lim\limits_{h \to 0^-} \frac{f(0+h)-f(0)}{h} = \lim\limits_{h \to 0^-} \frac{-1-0}{h} = \infty$, and $\lim\limits_{h \to 0^+} \frac{f(0+h)-f(0)}{h} = \lim\limits_{h \to 0^+} \frac{1-0}{h} = \infty$. Therefore, $\lim\limits_{h \to 0} \frac{f(0+h)-f(0)}{h} = \infty \Rightarrow$ yes, the graph of f has a vertical tangent at the origin.

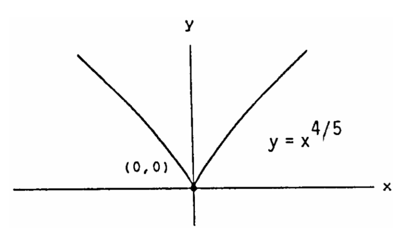
36. $\lim\limits_{h \to 0^-} \frac{U(0+h)-U(0)}{h} = \lim\limits_{h \to 0^-} \frac{0-1}{h} = \infty$, and $\lim\limits_{h \to 0^+} \frac{U(0+h)-U(0)}{h} = \lim\limits_{h \to 0^+} \frac{1-1}{h} = 0 \Rightarrow$ no, the graph of f does not have a vertical tangent at (0, 1) because the limit does not exist.

37. (a) The graph appears to have a cusp at $x = 0$.



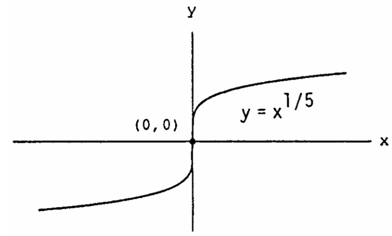
(b) $\lim\limits_{h \to 0^-} \frac{f(0+h)-f(0)}{h} = \lim\limits_{h \to 0^-} \frac{h^{2/5}-0}{h} = \lim\limits_{h \to 0^-} \frac{1}{h^{3/5}} = -\infty$ and $\lim\limits_{h \to 0^+} \frac{1}{h^{3/5}} = \infty \Rightarrow$ limit does not exist $\Rightarrow$ the graph of $y = x^{2/5}$ does not have a vertical tangent at $x = 0$.

38. (a) The graph appears to have a cusp at $x = 0$.



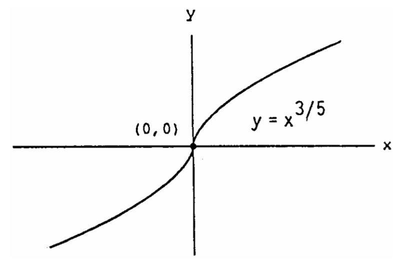
(b) $\lim\limits_{h \to 0^-} \frac{f(0+h)-f(0)}{h} = \lim\limits_{h \to 0^-} \frac{h^{4/5}-0}{h} = \lim\limits_{h \to 0^-} \frac{1}{h^{1/5}} = -\infty$ and $\lim\limits_{h \to 0^+} \frac{1}{h^{1/5}} = \infty \Rightarrow$ limit does not exist $\Rightarrow$ $y = x^{4/5}$ does not have a vertical tangent at $x = 0$.

39. (a) The graph appears to have a vertical tangent at $x = 0$.



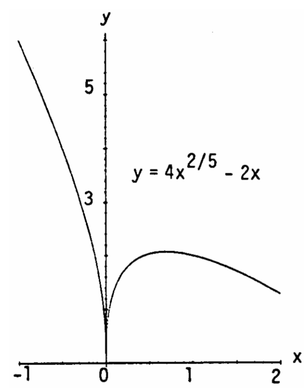
(b) $\lim\limits_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim\limits_{h \to 0} \frac{h^{1/5} - 0}{h} = \lim\limits_{h \to 0} \frac{1}{h^{4/5}} = \infty \Rightarrow y = x^{1/5}$ has a vertical tangent at $x = 0$.

40. (a) The graph appears to have a vertical tangent at $x = 0$.



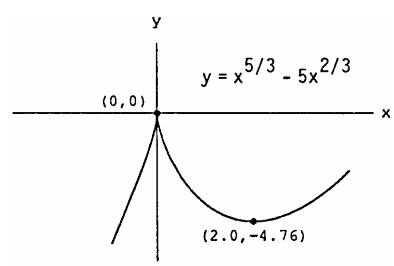
(b) $\lim\limits_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim\limits_{h \to 0} \frac{h^{3/5} - 0}{h} = \lim\limits_{h \to 0} \frac{1}{h^{2/5}} = \infty \Rightarrow$ the graph of $y = x^{3/5}$ has a vertical tangent at $x = 0$.

41. (a) The graph appears to have a cusp at $x = 0$.



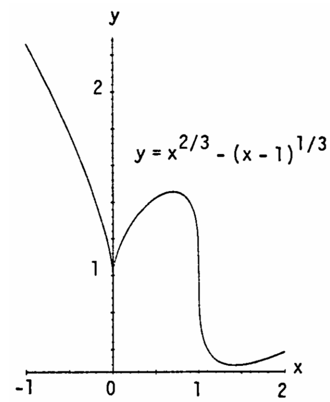
(b) $\lim\limits_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim\limits_{h \to 0^-} \frac{4h^{2/5} - 2h}{h} = \lim\limits_{h \to 0^-} \frac{4}{h^{3/5}} - 2 = -\infty$ and $\lim\limits_{h \to 0^+} \frac{4}{h^{3/5}} - 2 = \infty$
$\Rightarrow$ limit does not exist $\Rightarrow$ the graph of $y = 4x^{2/5} - 2x$ does not have a vertical tangent at $x = 0$.

42. (a) The graph appears to have a cusp at $x = 0$.



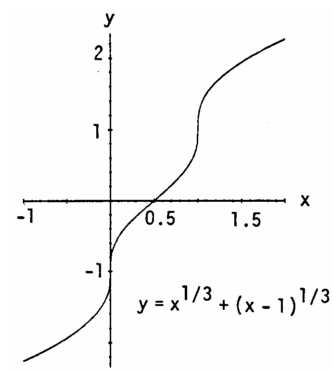
(b) $\lim\limits_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim\limits_{h \to 0} \frac{h^{5/3} - 5h^{2/3}}{h} = \lim\limits_{h \to 0} h^{2/3} - \frac{5}{h^{1/3}} = 0 - \lim\limits_{h \to 0} \frac{5}{h^{1/3}}$ does not exist $\Rightarrow$ the graph of $y = x^{5/3} - 5x^{2/3}$ does not have a vertical tangent at $x = 0$.

43. (a) The graph appears to have a vertical tangent at $x = 1$ and a cusp at $x = 0$.



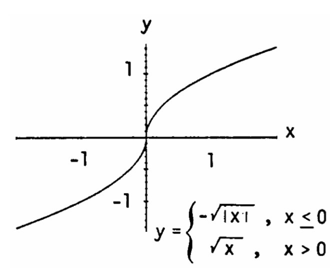
(b) $x = 1$: $\lim\limits_{h \to 0} \frac{(1+h)^{2/3} - (1+h-1)^{1/3} - 1}{h} = \lim\limits_{h \to 0} \frac{(1+h)^{2/3} - h^{1/3} - 1}{h} = -\infty$
$\Rightarrow\ y = x^{2/3} - (x-1)^{1/3}$ has a vertical tangent at $x = 1$;
$x = 0$: $\lim\limits_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim\limits_{h \to 0} \frac{h^{2/3} - (h-1)^{1/3} - (-1)^{1/3}}{h} = \lim\limits_{h \to 0} \left[\frac{1}{h^{1/3}} - \frac{(h-1)^{1/3}}{h} + \frac{1}{h}\right]$
does not exist $\Rightarrow\ y = x^{2/3} - (x-1)^{1/3}$ does not have a vertical tangent at $x = 0$.

44. (a) The graph appears to have vertical tangents at $x = 0$ and $x = 1$.



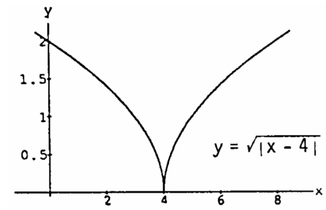
(b) $x = 0$: $\lim\limits_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim\limits_{h \to 0} \frac{h^{1/3} + (h-1)^{1/3} - (-1)^{1/3}}{h} = \infty\ \Rightarrow\ y = x^{1/3} + (x-1)^{1/3}$ has a vertical tangent at $x = 0$;
$x = 1$: $\lim\limits_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim\limits_{h \to 0} \frac{(1+h)^{1/3} + (1+h-1)^{1/3} - 1}{h} = \infty\ \Rightarrow\ y = x^{1/3} + (x-1)^{1/3}$ has a vertical tangent at $x = 1$.

45. (a) The graph appears to have a vertical tangent at $x = 0$.



(b) $\lim\limits_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim\limits_{x \to 0^+} \frac{\sqrt{h} - 0}{h} = \lim\limits_{h \to 0} \frac{1}{\sqrt{h}} = \infty$;
$\lim\limits_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim\limits_{h \to 0^-} \frac{-\sqrt{|h|} - 0}{h} = \lim\limits_{h \to 0^-} \frac{-\sqrt{|h|}}{-|h|} = \lim\limits_{h \to 0^-} \frac{1}{\sqrt{|h|}} = \infty$
$\Rightarrow$ y has a vertical tangent at $x = 0$.

46. (a) The graph appears to have a cusp at $x = 4$.



(b) $\lim\limits_{h \to 0^+} \frac{f(4+h)-f(4)}{h} = \lim\limits_{h \to 0^+} \frac{\sqrt{|4-(4+h)|}-0}{h} = \lim\limits_{h \to 0^+} \frac{\sqrt{|h|}}{h} = \lim\limits_{h \to 0^+} \frac{1}{\sqrt{h}} = \infty$;

$\lim\limits_{h \to 0^-} \frac{f(4+h)-f(4)}{h} = \lim\limits_{h \to 0^-} \frac{\sqrt{|4-(4+h)|}}{h} = \lim\limits_{h \to 0^-} \frac{\sqrt{|h|}}{-|h|} = \lim\limits_{h \to 0^-} \frac{-1}{\sqrt{|h|}} = -\infty$

$\Rightarrow$ $y = \sqrt{4-x}$ does not have a vertical tangent at $x = 4$.

47-50. Example CAS commands:

<u>Maple</u>:

```
f := x -> x^3 + 2*x;x0 := 0;
plot( f(x), x=x0-1/2..x0+3, color=black,                # part (a)
        title="Section 2.1, #47(a)" );
q := unapply( (f(x0+h)-f(x0))/h, h );                   # part (b)
L := limit( q(h), h=0 );                                # part (c)
sec_lines := seq( f(x0)+q(h)*(x-x0), h=1..3 );          # part (d)
tan_line := f(x0) + L*(x-x0);
plot( [f(x),tan_line,sec_lines], x=x0-1/2..x0+3, color=black,
        linestyle=[1,2,5,6,7], title="Section 2.1, #47(d)",
        legend=["y=f(x)","Tangent line at x=0","Secant line (h=1)",
                "Secant line (h=2)","Secant line (h=3)"] );
```

<u>Mathematica</u>: (function and value for x0 may change)

```
Clear[f, m, x, h]
x0 = p;
f[x_]: = Cos[x] + 4Sin[2x]
Plot[f[x], {x, x0 − 1, x0 + 3}]
dq[h_]: = (f[x0+h] − f[x0])/h
m = Limit[dq[h], h → 0]
ytan: = f[x0] + m(x − x0)
y1: = f[x0] + dq[1](x − x0)
y2: = f[x0] + dq[2](x − x0)
y3: = f[x0] + dq[3](x − x0)
Plot[{f[x], ytan, y1, y2, y3}, {x, x0 − 1, x0 + 3}]
```

2.2 THE DERIVATIVE AS A FUNCTION

1. Step 1: $f(x) = 4 - x^2$ and $f(x+h) = 4 - (x+h)^2$

Step 2: $\frac{f(x+h)-f(x)}{h} = \frac{[4-(x+h)^2]-(4-x^2)}{h} = \frac{(4-x^2-2xh-h^2)-4+x^2}{h} = \frac{-2xh-h^2}{h} = \frac{h(-2x-h)}{h}$
$= -2x - h$

Step 3: $f'(x) = \lim\limits_{h \to 0} (-2x - h) = -2x$; $f'(-3) = 6$, $f'(0) = 0$, $f'(1) = -2$

2. $F(x) = (x-1)^2 + 1$ and $F(x+h) = (x+h-1)^2 + 1$ $\Rightarrow$ $F'(x) = \lim\limits_{h \to 0} \frac{[(x+h-1)^2+1]-[(x-1)^2+1]}{h}$
$= \lim\limits_{h \to 0} \frac{(x^2+2xh+h^2-2x-2h+1+1)-(x^2-2x+1+1)}{h} = \lim\limits_{h \to 0} \frac{2xh+h^2-2h}{h} = \lim\limits_{h \to 0} (2x+h-2)$
$= 2(x-1)$; $F'(-1) = -4$, $F'(0) = -2$, $F'(2) = 2$

3. Step 1: $g(t) = \frac{1}{t^2}$ and $g(t+h) = \frac{1}{(t+h)^2}$

Step 2: $\frac{g(t+h)-g(t)}{h} = \frac{\frac{1}{(t+h)^2} - \frac{1}{t^2}}{h} = \frac{\left(\frac{t^2-(t+h)^2}{(t+h)^2 \cdot t^2}\right)}{h} = \frac{t^2-(t^2+2th+h^2)}{(t+h)^2 \cdot t^2 \cdot h} = \frac{-2th-h^2}{(t+h)^2\, t^2 h}$

$= \frac{h(-2t-h)}{(t+h)^2\, t^2 h} = \frac{-2t-h}{(t+h)^2\, t^2}$

Step 3: $g'(t) = \lim\limits_{h \to 0} \frac{-2t-h}{(t+h)^2\, t^2} = \frac{-2t}{t^2 \cdot t^2} = \frac{-2}{t^3}$; $g'(-1) = 2$, $g'(2) = -\frac{1}{4}$, $g'\left(\sqrt{3}\right) = -\frac{2}{3\sqrt{3}}$

4. $k(z) = \frac{1-z}{2z}$ and $k(z+h) = \frac{1-(z+h)}{2(z+h)} \Rightarrow k'(z) = \lim\limits_{h \to 0} \frac{\left(\frac{1-(z+h)}{2(z+h)} - \frac{1-z}{2z}\right)}{h}$

$= \lim\limits_{h \to 0} \frac{(1-z-h)z-(1-z)(z+h)}{2(z+h)zh} = \lim\limits_{h \to 0} \frac{z-z^2-zh-z-h+z^2+zh}{2(z+h)zh} = \lim\limits_{h \to 0} \frac{-h}{2(z+h)zh} = \lim\limits_{h \to 0} \frac{-1}{2(z+h)z}$

$= \frac{-1}{2z^2}$; $k'(-1) = -\frac{1}{2}$, $k'(1) = -\frac{1}{2}$, $k'\left(\sqrt{2}\right) = -\frac{1}{4}$

5. Step 1: $p(\theta) = \sqrt{3\theta}$ and $p(\theta+h) = \sqrt{3(\theta+h)}$

Step 2: $\frac{p(\theta+h)-p(\theta)}{h} = \frac{\sqrt{3(\theta+h)}-\sqrt{3\theta}}{h} = \frac{\left(\sqrt{3\theta+3h}-\sqrt{3\theta}\right)}{h} \cdot \frac{\left(\sqrt{3\theta+3h}+\sqrt{3\theta}\right)}{\left(\sqrt{3\theta+3h}+\sqrt{3\theta}\right)} = \frac{(3\theta+3h)-3\theta}{h\left(\sqrt{3\theta+3h}+\sqrt{3\theta}\right)}$

$= \frac{3h}{h\left(\sqrt{3\theta+3h}+\sqrt{3\theta}\right)} = \frac{3}{\sqrt{3\theta+3h}+\sqrt{3\theta}}$

Step 3: $p'(\theta) = \lim\limits_{h \to 0} \frac{3}{\sqrt{3\theta+3h}+\sqrt{3\theta}} = \frac{3}{\sqrt{3\theta}+\sqrt{3\theta}} = \frac{3}{2\sqrt{3\theta}}$; $p'(1) = \frac{3}{2\sqrt{3}}$, $p'(3) = \frac{1}{2}$, $p'\left(\frac{2}{3}\right) = \frac{3}{2\sqrt{2}}$

6. $r(s) = \sqrt{2s+1}$ and $r(s+h) = \sqrt{2(s+h)+1} \Rightarrow r'(s) = \lim\limits_{h \to 0} \frac{\sqrt{2s+2h+1}-\sqrt{2s+1}}{h}$

$= \lim\limits_{h \to 0} \frac{\left(\sqrt{2s+h+1}-\sqrt{2s+1}\right)}{h} \cdot \frac{\left(\sqrt{2s+2h+1}+\sqrt{2s+1}\right)}{\left(\sqrt{2s+2h+1}+\sqrt{2s+1}\right)} = \lim\limits_{h \to 0} \frac{(2s+2h+1)-(2s+1)}{h\left(\sqrt{2s+2h+1}+\sqrt{2s+1}\right)}$

$= \lim\limits_{h \to 0} \frac{2h}{h\left(\sqrt{2s+2h+1}+\sqrt{2s+1}\right)} = \lim\limits_{h \to 0} \frac{2}{\sqrt{2s+2h+1}+\sqrt{2s+1}} = \frac{2}{\sqrt{2s+1}+\sqrt{2s+1}} = \frac{2}{2\sqrt{2s+1}}$

$= \frac{1}{\sqrt{2s+1}}$; $r'(0) = 1$, $r'(1) = \frac{1}{\sqrt{3}}$, $r'\left(\frac{1}{2}\right) = \frac{1}{\sqrt{2}}$

7. $y = f(x) = 2x^3$ and $f(x+h) = 2(x+h)^3 \Rightarrow \frac{dy}{dx} = \lim\limits_{h \to 0} \frac{2(x+h)^3-2x^3}{h} = \lim\limits_{h \to 0} \frac{2\left(x^3+3x^2h+3xh^2+h^3\right)-2x^3}{h}$

$= \lim\limits_{h \to 0} \frac{6x^2h+6xh^2+2h^3}{h} = \lim\limits_{h \to 0} \frac{h\left(6x^2+6xh+2h^2\right)}{h} = \lim\limits_{h \to 0} \left(6x^2+6xh+2h^2\right) = 6x^2$

8. $r = \frac{s^3}{2} + 1 \Rightarrow \frac{dr}{ds} = \lim\limits_{h \to 0} \frac{\left[\frac{(s+h)^3}{2}+1\right]-\left[\frac{s^3}{2}+1\right]}{h} = \frac{1}{2}\lim\limits_{h \to 0} \frac{\left[(s+h)^3+2\right]-\left[s^3+2\right]}{h}$

$= \frac{1}{2}\lim\limits_{h \to 0} \frac{s^3+3s^2h+3sh^2+h^3+2-s^3-2}{h} = \frac{1}{2}\lim\limits_{h \to 0} \frac{h\left[3s^2+3sh+h^2\right]}{h} = \frac{1}{2}\lim\limits_{h \to 0} \left(3s^2+3sh+h^2\right) = \frac{3}{2}s^2$

9. $s = r(t) = \frac{t}{2t+1}$ and $r(t+h) = \frac{t+h}{2(t+h)+1} \Rightarrow \frac{ds}{dt} = \lim\limits_{h \to 0} \frac{\left(\frac{t+h}{2(t+h)+1}\right)-\left(\frac{t}{2t+1}\right)}{h}$

$= \lim\limits_{h \to 0} \frac{\left(\frac{(t+h)(2t+1)-t(2t+2h+1)}{(2t+2h+1)(2t+1)}\right)}{h} = \lim\limits_{h \to 0} \frac{(t+h)(2t+1)-t(2t+2h+1)}{(2t+2h+1)(2t+1)h} = \lim\limits_{h \to 0} \frac{2t^2+t+2ht+h-2t^2-2ht-t}{(2t+2h+1)(2t+1)h}$

$= \lim\limits_{h \to 0} \frac{h}{(2t+2h+1)(2t+1)h} = \lim\limits_{h \to 0} \frac{1}{(2t+2h+1)(2t+1)} = \frac{1}{(2t+1)(2t+1)} = \frac{1}{(2t+1)^2}$

10. $\frac{dv}{dt} = \lim\limits_{h \to 0} \frac{\left[(t+h)-\frac{1}{t+h}\right]-\left(t-\frac{1}{t}\right)}{h} = \lim\limits_{h \to 0} \frac{h-\frac{1}{t+h}+\frac{1}{t}}{h} = \lim\limits_{h \to 0} \frac{\left(\frac{h(t+h)t-t+(t+h)}{(t+h)t}\right)}{h}$

$= \lim\limits_{h \to 0} \frac{ht^2+h^2t+h}{h(t+h)t} = \lim\limits_{h \to 0} \frac{t^2+ht+1}{(t+h)t} = \frac{t^2+1}{t^2} = 1 + \frac{1}{t^2}$

11. $p = f(q) = \frac{1}{\sqrt{q+1}}$ and $f(q+h) = \frac{1}{\sqrt{(q+h)+1}} \Rightarrow \frac{dp}{dq} = \lim_{h \to 0} \frac{\left(\frac{1}{\sqrt{(q+h)+1}}\right) - \left(\frac{1}{\sqrt{q+1}}\right)}{h}$

$= \lim_{h \to 0} \frac{\left(\frac{\sqrt{q+1} - \sqrt{q+h+1}}{\sqrt{q+h+1}\sqrt{q+1}}\right)}{h} = \lim_{h \to 0} \frac{\sqrt{q+1} - \sqrt{q+h+1}}{h\sqrt{q+h+1}\sqrt{q+1}} = \lim_{h \to 0} \frac{\left(\sqrt{q+1} - \sqrt{q+h+1}\right)}{h\sqrt{q+h+1}\sqrt{q+1}} \cdot \frac{\left(\sqrt{q+1} + \sqrt{q+h+1}\right)}{\left(\sqrt{q+1} + \sqrt{q+h+1}\right)}$

$= \lim_{h \to 0} \frac{(q+1) - (q+h+1)}{h\sqrt{q+h+1}\sqrt{q+1}\left(\sqrt{q+1} + \sqrt{q+h+1}\right)} = \lim_{h \to 0} \frac{-h}{h\sqrt{q+h+1}\sqrt{q+1}\left(\sqrt{q+1} + \sqrt{q+h+1}\right)}$

$= \lim_{h \to 0} \frac{-1}{\sqrt{q+h+1}\sqrt{q+1}\left(\sqrt{q+1} + \sqrt{q+h+1}\right)} = \frac{-1}{\sqrt{q+1}\sqrt{q+1}\left(\sqrt{q+1} + \sqrt{q+1}\right)} = \frac{-1}{2(q+1)\sqrt{q+1}}$

12. $\frac{dz}{dw} = \lim_{h \to 0} \frac{\left(\frac{1}{\sqrt{3(w+h)-2}} - \frac{1}{\sqrt{3w-2}}\right)}{h} = \lim_{h \to 0} \frac{\sqrt{3w-2} - \sqrt{3w+3h-2}}{h\sqrt{3w+3h-2}\sqrt{3w-2}}$

$= \lim_{h \to 0} \frac{\left(\sqrt{3w-2} - \sqrt{3w+3h-2}\right)}{h\sqrt{3w+3h-2}\sqrt{3w-2}} \cdot \frac{\left(\sqrt{3w-2} + \sqrt{3w+3h-2}\right)}{\left(\sqrt{3w-2} + \sqrt{3w+3h-2}\right)} = \lim_{h \to 0} \frac{(3w-2) - (3w+3h-2)}{h\sqrt{3w+3h-2}\sqrt{3w-2}\left(\sqrt{3w-2} + \sqrt{3w+3h-2}\right)}$

$= \lim_{h \to 0} \frac{-3}{\sqrt{3w+3h-2}\sqrt{3w-2}\left(\sqrt{3w-2} + \sqrt{3w+3h-2}\right)} = \frac{-3}{\sqrt{3w-2}\sqrt{3w-2}\left(\sqrt{3w-2} + \sqrt{3w-2}\right)} = \frac{-3}{2(3w-2)\sqrt{3w-2}}$

13. $f(x) = x + \frac{9}{x}$ and $f(x+h) = (x+h) + \frac{9}{(x+h)} \Rightarrow \frac{f(x+h) - f(x)}{h} = \frac{\left[(x+h) + \frac{9}{(x+h)}\right] - \left[x + \frac{9}{x}\right]}{h}$

$= \frac{x(x+h)^2 + 9x - x^2(x+h) - 9(x+h)}{x(x+h)h} = \frac{x^3 + 2x^2h + xh^2 + 9x - x^3 - x^2h - 9x - 9h}{x(x+h)h} = \frac{x^2h + xh^2 - 9h}{x(x+h)h}$

$= \frac{h(x^2 + xh - 9)}{x(x+h)h} = \frac{x^2 + xh - 9}{x(x+h)}$; $f'(x) = \lim_{h \to 0} \frac{x^2 + xh - 9}{x(x+h)} = \frac{x^2 - 9}{x^2} = 1 - \frac{9}{x^2}$; $m = f'(-3) = 0$

14. $k(x) = \frac{1}{2+x}$ and $k(x+h) = \frac{1}{2+(x+h)} \Rightarrow k'(x) = \lim_{h \to 0} \frac{k(x+h) - k(x)}{h} = \lim_{h \to 0} \frac{\left(\frac{1}{2+x+h} - \frac{1}{2+x}\right)}{h}$

$= \lim_{h \to 0} \frac{(2+x) - (2+x+h)}{h(2+x)(2+x+h)} = \lim_{h \to 0} \frac{-h}{h(2+x)(2+x+h)} = \lim_{h \to 0} \frac{-1}{(2+x)(2+x+h)} = \frac{-1}{(2+x)^2}$; $k'(2) = -\frac{1}{16}$

15. $\frac{ds}{dt} = \lim_{h \to 0} \frac{[(t+h)^3 - (t+h)^2] - (t^3 - t^2)}{h} = \lim_{h \to 0} \frac{(t^3 + 3t^2h + 3th^2 + h^3) - (t^2 + 2th + h^2) - t^3 + t^2}{h}$

$= \lim_{h \to 0} \frac{3t^2h + 3th^2 + h^3 - 2th - h^2}{h} = \lim_{h \to 0} \frac{h(3t^2 + 3th + h^2 - 2t - h)}{h} = \lim_{h \to 0} (3t^2 + 3th + h^2 - 2t - h)$

$= 3t^2 - 2t$; $m = \left.\frac{ds}{dt}\right|_{t=-1} = 5$

16. $\frac{dy}{dx} = \lim_{h \to 0} \frac{(x+h+1)^3 - (x+1)^3}{h} = \lim_{h \to 0} \frac{(x+1)^3 + 3(x+1)^2h + 3(x+1)h^2 + h^3 - (x+1)^3}{h}$

$= \lim_{h \to 0} [3(x+1)^2 + 3(x+1)h + h^2] = 3(x+1)^2$; $m = \left.\frac{dy}{dx}\right|_{x=-2} = 3$

17. $f(x) = \frac{8}{\sqrt{x-2}}$ and $f(x+h) = \frac{8}{\sqrt{(x+h)-2}} \Rightarrow \frac{f(x+h) - f(x)}{h} = \frac{\frac{8}{\sqrt{(x+h)-2}} - \frac{8}{\sqrt{x-2}}}{h}$

$= \frac{8\left(\sqrt{x-2} - \sqrt{x+h-2}\right)}{h\sqrt{x+h-2}\sqrt{x-2}} \cdot \frac{\left(\sqrt{x-2} + \sqrt{x+h-2}\right)}{\left(\sqrt{x-2} + \sqrt{x+h-2}\right)} = \frac{8[(x-2) - (x+h-2)]}{h\sqrt{x+h-2}\sqrt{x-2}\left(\sqrt{x-2} + \sqrt{x+h-2}\right)}$

$= \frac{-8h}{h\sqrt{x+h-2}\sqrt{x-2}\left(\sqrt{x-2} + \sqrt{x+h-2}\right)} \Rightarrow f'(x) = \lim_{h \to 0} \frac{-8}{\sqrt{x+h-2}\sqrt{x-2}\left(\sqrt{x-2} + \sqrt{x+h-2}\right)}$

$= \frac{-8}{\sqrt{x-2}\sqrt{x-2}\left(\sqrt{x-2} + \sqrt{x-2}\right)} = \frac{-4}{(x-2)\sqrt{x-2}}$; $m = f'(6) = \frac{-4}{4\sqrt{4}} = -\frac{1}{2} \Rightarrow$ the equation of the tangent line at $(6, 4)$ is $y - 4 = -\frac{1}{2}(x-6) \Rightarrow y = -\frac{1}{2}x + 3 + 4 \Rightarrow y = -\frac{1}{2}x + 7$.

18. $g'(z) = \lim_{h \to 0} \frac{\left(1 + \sqrt{4-(z+h)}\right) - \left(1 + \sqrt{4-z}\right)}{h} = \lim_{h \to 0} \frac{\left(\sqrt{4-z-h} - \sqrt{4-z}\right)}{h} \cdot \frac{\left(\sqrt{4-z-h} + \sqrt{4-z}\right)}{\left(\sqrt{4-z-h} + \sqrt{4-z}\right)}$

$= \lim_{h \to 0} \frac{(4-z-h) - (4-z)}{h\left(\sqrt{4-z-h} + \sqrt{4-z}\right)} = \lim_{h \to 0} \frac{-h}{h\left(\sqrt{4-z-h} + \sqrt{4-z}\right)} = \lim_{h \to 0} \frac{-1}{\left(\sqrt{4-z-h} + \sqrt{4-z}\right)} = \frac{-1}{2\sqrt{4-z}}$;

$m = g'(3) = \frac{-1}{2\sqrt{4-3}} = -\frac{1}{2} \Rightarrow$ the equation of the tangent line at $(3, 2)$ is $w - 2 = -\frac{1}{2}(z-3)$

$\Rightarrow w = -\frac{1}{2}z + \frac{3}{2} + 2 \Rightarrow w = -\frac{1}{2}z + \frac{7}{2}$.

19. $s = f(t) = 1 - 3t^2$ and $f(t+h) = 1 - 3(t+h)^2 = 1 - 3t^2 - 6th - 3h^2 \Rightarrow \frac{ds}{dt} = \lim\limits_{h \to 0} \frac{f(t+h) - f(t)}{h}$
$= \lim\limits_{h \to 0} \frac{(1 - 3t^2 - 6th - 3h^2) - (1 - 3t^2)}{h} = \lim\limits_{h \to 0} (-6t - 3h) = -6t \Rightarrow \left.\frac{ds}{dt}\right|_{t=-1} = 6$

20. $y = f(x) = 1 - \frac{1}{x}$ and $f(x+h) = 1 - \frac{1}{x+h} \Rightarrow \frac{dy}{dx} = \lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0} \frac{\left(1 - \frac{1}{x+h}\right) - \left(1 - \frac{1}{x}\right)}{h}$
$= \lim\limits_{h \to 0} \frac{\frac{1}{x} - \frac{1}{x+h}}{h} = \lim\limits_{h \to 0} \frac{h}{x(x+h)h} = \lim\limits_{h \to 0} \frac{1}{x(x+h)} = \frac{1}{x^2} \Rightarrow \left.\frac{dy}{dx}\right|_{x=\sqrt{3}} = \frac{1}{3}$

21. $r = f(\theta) = \frac{2}{\sqrt{4-\theta}}$ and $f(\theta + h) = \frac{2}{\sqrt{4-(\theta+h)}} \Rightarrow \frac{dr}{d\theta} = \lim\limits_{h \to 0} \frac{f(\theta+h) - f(\theta)}{h} = \lim\limits_{h \to 0} \frac{\frac{2}{\sqrt{4-\theta-h}} - \frac{2}{\sqrt{4-\theta}}}{h}$
$= \lim\limits_{h \to 0} \frac{2\sqrt{4-\theta} - 2\sqrt{4-\theta-h}}{h\sqrt{4-\theta}\sqrt{4-\theta-h}} = \lim\limits_{h \to 0} \frac{2\sqrt{4-\theta} - 2\sqrt{4-\theta-h}}{h\sqrt{4-\theta}\sqrt{4-\theta-h}} \cdot \frac{\left(2\sqrt{4-\theta} + 2\sqrt{4-\theta-h}\right)}{\left(2\sqrt{4-\theta} + 2\sqrt{4-\theta-h}\right)}$
$= \lim\limits_{h \to 0} \frac{4(4-\theta) - 4(4-\theta-h)}{2h\sqrt{4-\theta}\sqrt{4-\theta-h}\left(\sqrt{4-\theta} + \sqrt{4-\theta-h}\right)} = \lim\limits_{h \to 0} \frac{2}{\sqrt{4-\theta}\sqrt{4-\theta-h}\left(\sqrt{4-\theta} + \sqrt{4-\theta-h}\right)}$
$= \frac{2}{(4-\theta)\left(2\sqrt{4-\theta}\right)} = \frac{1}{(4-\theta)\sqrt{4-\theta}} \Rightarrow \left.\frac{dr}{d\theta}\right|_{\theta=0} = \frac{1}{8}$

22. $w = f(z) = z + \sqrt{z}$ and $f(z+h) = (z+h) + \sqrt{z+h} \Rightarrow \frac{dw}{dz} = \lim\limits_{h \to 0} \frac{f(z+h) - f(z)}{h}$
$= \lim\limits_{h \to 0} \frac{\left(z + h + \sqrt{z+h}\right) - \left(z + \sqrt{z}\right)}{h} = \lim\limits_{h \to 0} \frac{h + \sqrt{z+h} - \sqrt{z}}{h} = \lim\limits_{h \to 0} \left[1 + \frac{\sqrt{z+h} - \sqrt{z}}{h} \cdot \frac{\left(\sqrt{z+h} + \sqrt{z}\right)}{\left(\sqrt{z+h} + \sqrt{z}\right)}\right]$
$= 1 + \lim\limits_{h \to 0} \frac{(z+h) - z}{h\left(\sqrt{z+h} + \sqrt{z}\right)} = 1 + \lim\limits_{h \to 0} \frac{1}{\sqrt{z+h} + \sqrt{z}} = 1 + \frac{1}{2\sqrt{z}} \Rightarrow \left.\frac{dw}{dz}\right|_{z=4} = \frac{5}{4}$

23. $f'(x) = \lim\limits_{z \to x} \frac{f(z) - f(x)}{z - x} = \lim\limits_{z \to x} \frac{\frac{1}{z+2} - \frac{1}{x+2}}{z - x} = \lim\limits_{z \to x} \frac{(x+2) - (z+2)}{(z-x)(z+2)(x+2)} = \lim\limits_{z \to x} \frac{x - z}{(z-x)(z+2)(x+2)} = \lim\limits_{z \to x} \frac{-1}{(z+2)(x+2)} = \frac{-1}{(x+2)^2}$

24. $f'(x) = \lim\limits_{z \to x} \frac{f(z) - f(x)}{z - x} = \lim\limits_{z \to x} \frac{\frac{1}{(z-1)^2} - \frac{1}{(x-1)^2}}{z - x} = \lim\limits_{z \to x} \frac{(x-1)^2 - (z-1)^2}{(z-x)(z-1)^2(x-1)^2} = \lim\limits_{z \to x} \frac{[(x-1) - (z-1)][(x-1) + (z-1)]}{(z-x)(z-1)^2(x-1)^2}$
$= \lim\limits_{z \to x} \frac{(x-z)(x+z-2)}{(z-x)(z-1)^2(x-1)^2} = \lim\limits_{z \to x} \frac{-1(x+z-2)}{(z-1)^2(x-1)^2} = \frac{-1(2x-2)}{(x-1)^4} = \frac{-2(x-1)}{(x-1)^4} = \frac{-2}{(x-1)^3}$

25. $g'(x) = \lim\limits_{z \to x} \frac{g(z) - g(x)}{z - x} = \lim\limits_{z \to x} \frac{\frac{z}{z-1} - \frac{x}{x-1}}{z - x} = \lim\limits_{z \to x} \frac{z(x-1) - x(z-1)}{(z-x)(z-1)(x-1)} = \lim\limits_{z \to x} \frac{-z + x}{(z-x)(z-1)(x-1)} = \lim\limits_{z \to x} \frac{-1}{(z-1)(x-1)} = \frac{-1}{(x-1)^2}$

26. $g'(x) = \lim\limits_{z \to x} \frac{g(z) - g(x)}{z - x} = \lim\limits_{z \to x} \frac{(1 + \sqrt{z}) - (1 + \sqrt{x})}{z - x} = \lim\limits_{z \to x} \frac{\sqrt{z} - \sqrt{x}}{z - x} \cdot \frac{\sqrt{z} + \sqrt{x}}{\sqrt{z} + \sqrt{x}} = \lim\limits_{z \to x} \frac{z - x}{(z-x)\left(\sqrt{z} + \sqrt{x}\right)} = \lim\limits_{z \to x} \frac{1}{\sqrt{z} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$

27. Note that as x increases, the slope of the tangent line to the curve is first negative, then zero (when $x = 0$), then positive $\Rightarrow$ the slope is always increasing which matches (b).

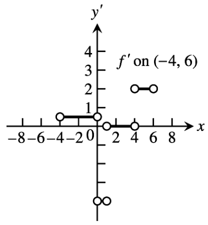
28. Note that the slope of the tangent line is never negative. For x negative, $f_2'(x)$ is positive but decreasing as x increases. When $x = 0$, the slope of the tangent line to x is 0. For $x > 0$, $f_2'(x)$ is positive and increasing. This graph matches (a).

29. $f_3(x)$ is an oscillating function like the cosine. Everywhere that the graph of f_3 has a horizontal tangent we expect f_3' to be zero, and (d) matches this condition.

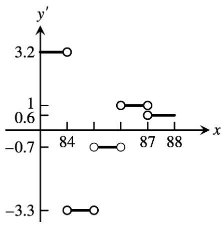
30. The graph matches with (c).

31. (a) f' is not defined at $x = 0, 1, 4$. At these points, the left-hand and right-hand derivatives do not agree. For example, $\lim\limits_{x \to 0^-} \frac{f(x)-f(0)}{x-0}$ = slope of line joining $(-4, 0)$ and $(0, 2) = \frac{1}{2}$ but $\lim\limits_{x \to 0^+} \frac{f(x)-f(0)}{x-0}$ = slope of line joining $(0, 2)$ and $(1, -2) = -4$. Since these values are not equal, $f'(0) = \lim\limits_{x \to 0} \frac{f(x)-f(0)}{x-0}$ does not exist.

(b)



32.



33. Left-hand derivative: For $h < 0$, $f(0+h) = f(h) = h^2$ (using $y = x^2$ curve) $\Rightarrow \lim\limits_{h \to 0^-} \frac{f(0+h)-f(0)}{h}$
$= \lim\limits_{h \to 0^-} \frac{h^2-0}{h} = \lim\limits_{h \to 0^-} h = 0$;
Right-hand derivative: For $h > 0$, $f(0+h) = f(h) = h$ (using $y = x$ curve) $\Rightarrow \lim\limits_{h \to 0^+} \frac{f(0+h)-f(0)}{h}$
$= \lim\limits_{h \to 0^+} \frac{h-0}{h} = \lim\limits_{h \to 0^+} 1 = 1$;
Then $\lim\limits_{h \to 0^-} \frac{f(0+h)-f(0)}{h} \neq \lim\limits_{h \to 0^+} \frac{f(0+h)-f(0)}{h} \Rightarrow$ the derivative $f'(0)$ does not exist.

34. Left-hand derivative: When $h < 0$, $1 + h < 1 \Rightarrow f(1+h) = 2 \Rightarrow \lim\limits_{h \to 0^-} \frac{f(1+h)-f(1)}{h} = \lim\limits_{h \to 0^-} \frac{2-2}{h} = \lim\limits_{h \to 0^-} 0 = 0$;
Right-hand derivative: When $h > 0$, $1 + h > 1 \Rightarrow f(1+h) = 2(1+h) = 2 + 2h \Rightarrow \lim\limits_{h \to 0^+} \frac{f(1+h)-f(1)}{h}$
$= \lim\limits_{h \to 0^+} \frac{(2+2h)-2}{h} = \lim\limits_{h \to 0^+} \frac{2h}{h} = \lim\limits_{h \to 0^+} 2 = 2$;
Then $\lim\limits_{h \to 0^-} \frac{f(1+h)-f(1)}{h} \neq \lim\limits_{h \to 0^+} \frac{f(1+h)-f(1)}{h} \Rightarrow$ the derivative $f'(1)$ does not exist.

35. Left-hand derivative: When $h < 0$, $1 + h < 1 \Rightarrow f(1+h) = \sqrt{1+h} \Rightarrow \lim\limits_{h \to 0^-} \frac{f(1+h)-f(1)}{h}$
$= \lim\limits_{h \to 0^-} \frac{\sqrt{1+h}-1}{h} = \lim\limits_{h \to 0^-} \frac{\left(\sqrt{1+h}-1\right)}{h} \cdot \frac{\left(\sqrt{1+h}+1\right)}{\left(\sqrt{1+h}+1\right)} = \lim\limits_{h \to 0^-} \frac{(1+h)-1}{h\left(\sqrt{1+h}+1\right)} = \lim\limits_{h \to 0^-} \frac{1}{\sqrt{1+h}+1} = \frac{1}{2}$;
Right-hand derivative: When $h > 0$, $1 + h > 1 \Rightarrow f(1+h) = 2(1+h) - 1 = 2h + 1 \Rightarrow \lim\limits_{h \to 0^+} \frac{f(1+h)-f(1)}{h}$
$= \lim\limits_{h \to 0^+} \frac{(2h+1)-1}{h} = \lim\limits_{h \to 0^+} 2 = 2$;
Then $\lim\limits_{h \to 0^-} \frac{f(1+h)-f(1)}{h} \neq \lim\limits_{h \to 0^+} \frac{f(1+h)-f(1)}{h} \Rightarrow$ the derivative $f'(1)$ does not exist.

36. Left-hand derivative: $\lim\limits_{h \to 0^-} \frac{f(1+h)-f(1)}{h} = \lim\limits_{h \to 0^-} \frac{(1+h)-1}{h} = \lim\limits_{h \to 0^-} 1 = 1$;
Right-hand derivative: $\lim\limits_{h \to 0^+} \frac{f(1+h)-f(1)}{h} = \lim\limits_{h \to 0^+} \frac{\left(\frac{1}{1+h}-1\right)}{h} = \lim\limits_{h \to 0^+} \frac{\left(\frac{1-(1+h)}{1+h}\right)}{h}$
$= \lim\limits_{h \to 0^+} \frac{-h}{h(1+h)} = \lim\limits_{h \to 0^+} \frac{-1}{1+h} = -1$;
Then $\lim\limits_{h \to 0^-} \frac{f(1+h)-f(1)}{h} \neq \lim\limits_{h \to 0^+} \frac{f(1+h)-f(1)}{h} \Rightarrow$ the derivative $f'(1)$ does not exist.

37. (a) The function is differentiable on its domain $-3 \le x \le 2$ (it is smooth)
    (b) none
    (c) none

38. (a) The function is differentiable on its domain $-2 \le x \le 3$ (it is smooth)
    (b) none
    (c) none

39. (a) The function is differentiable on $-3 \le x < 0$ and $0 < x \le 3$
    (b) none
    (c) The function is neither continuous nor differentiable at $x = 0$ since $\lim\limits_{x \to 0^-} f(x) \ne \lim\limits_{x \to 0^+} f(x)$

40. (a) f is differentiable on $-2 \le x < -1$, $-1 < x < 0$, $0 < x < 2$, and $2 < x \le 3$
    (b) f is continuous but not differentiable at $x = -1$: $\lim\limits_{x \to -1} f(x) = 0$ exists but there is a corner at $x = -1$ since $\lim\limits_{h \to 0^-} \frac{f(-1+h) - f(-1)}{h} = -3$ and $\lim\limits_{h \to 0^+} \frac{f(-1+h) - f(-1)}{h} = 3 \Rightarrow f'(-1)$ does not exist
    (c) f is neither continuous nor differentiable at $x = 0$ and $x = 2$:
    at $x = 0$, $\lim\limits_{x \to 0^-} f(x) = 3$ but $\lim\limits_{x \to 0^+} f(x) = 0 \Rightarrow \lim\limits_{x \to 0} f(x)$ does not exist;
    at $x = 2$, $\lim\limits_{x \to 2} f(x)$ exists but $\lim\limits_{x \to 2} f(x) \ne f(2)$

41. (a) f is differentiable on $-1 \le x < 0$ and $0 < x \le 2$
    (b) f is continuous but not differentiable at $x = 0$: $\lim\limits_{x \to 0} f(x) = 0$ exists but there is a cusp at $x = 0$, so $f'(0) = \lim\limits_{h \to 0} \frac{f(0+h) - f(0)}{h}$ does not exist
    (c) none

42. (a) f is differentiable on $-3 \le x < -2$, $-2 < x < 2$, and $2 < x \le 3$
    (b) f is continuous but not differentiable at $x = -2$ and $x = 2$: there are corners at those points
    (c) none

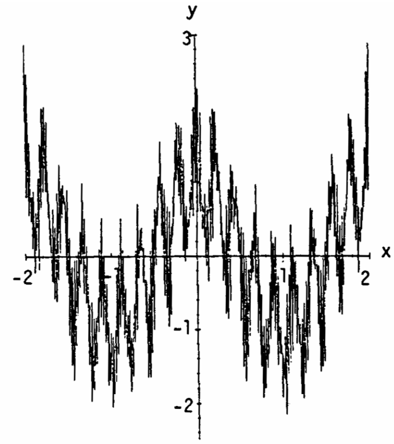
43. $y' = \lim\limits_{h \to 0} \frac{(2(x+h)^2 - 13(x+h) + 5) - (2x^2 - 13x + 5)}{h} = \lim\limits_{h \to 0} \frac{2x^2 + 4xh + 2h^2 - 13x - 13h + 5 - 2x^2 + 13x - 5}{h}$
$= \lim\limits_{h \to 0} \frac{4xh + 2h^2 - 13h}{h} = \lim\limits_{h \to 0} (4x + 2h - 13) = 4x - 13$, slope at x. The slope is -1 when $4x - 13 = -1$
$\Rightarrow 4x = 12 \Rightarrow x = 3 \Rightarrow y = 2 \cdot 3^2 - 13 \cdot 3 + 5 = -16$. Thus the tangent line is $y + 16 = (-1)(x - 3)$
$\Rightarrow y = -x - 13$ and the point of tangency is $(3, -16)$.

44. For the curve $y = \sqrt{x}$, we have $y' = \lim\limits_{h \to 0} \frac{\left(\sqrt{x+h} - \sqrt{x}\right)}{h} \cdot \frac{\left(\sqrt{x+h} + \sqrt{x}\right)}{\left(\sqrt{x+h} + \sqrt{x}\right)} = \lim\limits_{h \to 0} \frac{(x+h) - x}{\left(\sqrt{x+h} + \sqrt{x}\right) h}$
$= \lim\limits_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$. Suppose $\left(a, \sqrt{a}\right)$ is the point of tangency of such a line and $(-1, 0)$ is the point on the line where it crosses the x-axis. Then the slope of the line is $\frac{\sqrt{a} - 0}{a - (-1)} = \frac{\sqrt{a}}{a+1}$ which must also equal $\frac{1}{2\sqrt{a}}$; using the derivative formula at $x = a \Rightarrow \frac{\sqrt{a}}{a+1} = \frac{1}{2\sqrt{a}} \Rightarrow 2a = a + 1 \Rightarrow a = 1$. Thus such a line does exist: its point of tangency is $(1, 1)$, its slope is $\frac{1}{2\sqrt{a}} = \frac{1}{2}$; and an equation of the line is $y - 1 = \frac{1}{2}(x - 1) \Rightarrow y = \frac{1}{2}x + \frac{1}{2}$.

45. (a) Suppose $|f(x)| \le x^2$ for $-1 \le x \le 1$. Then $|f(0)| \le 0^2 \Rightarrow f(0) = 0$. Then $f'(0) = \lim\limits_{h \to 0} \frac{f(0+h) - f(0)}{h}$
$= \lim\limits_{h \to 0} \frac{f(h) - 0}{h} = \lim\limits_{h \to 0} \frac{f(h)}{h}$. For $|h| \le 1$, $-h^2 \le f(h) \le h^2 \Rightarrow -h \le \frac{f(h)}{h} \le h \Rightarrow f'(0) = \lim\limits_{h \to 0} \frac{f(h)}{h} = 0$
by the Sandwich Theorem for limits.

(b) Note that for $x \neq 0$, $|f(x)| = \left|x^2 \sin \frac{1}{x}\right| = |x^2|\,|\sin x| \leq |x^2| \cdot 1 = x^2$ (since $-1 \leq \sin x \leq 1$). By part (a), f is differentiable at $x = 0$ and $f'(0) = 0$.

46. Weierstrass's nowhere differentiable continuous function.



$$g(x) = \cos(\pi x) + \left(\frac{2}{3}\right)^1 \cos(9\pi x) + \left(\frac{2}{3}\right)^2 \cos(9^2\pi x) + \left(\frac{2}{3}\right)^3 \cos(9^3\pi x) + \cdots + \left(\frac{2}{3}\right)^7 \cos(9^7\pi x)$$

47-52. Example CAS commands:

Maple:

```
f := x -> x^3 + x^2 - x;
x0 := 1;
plot( f(x), x=x0-5..x0+2, color=black,
        title="Section 2.2, #47(a)" );
q := unapply( (f(x+h)-f(x))/h, (x,h) );                    # (b)
L := limit( q(x,h), h=0 );                                 # (c)
m := eval( L, x=x0 );
tan_line := f(x0) + m*(x-x0);
plot( [f(x),tan_line], x=x0-2..x0+3, color=black,
       linestyle=[1,7], title="Section 2.2 #47(d)",
       legend=["y=f(x)","Tangent line at x=1"] );
Xvals := sort( [ x0+2^(-k) $ k=0..5, x0-2^(-k) $ k=0..5 ] ):    # (e)
Yvals := map( f, Xvals ):
evalf[4](< convert(Xvals,Matrix) , convert(Yvals,Matrix) >);
plot( L, x=x0-5..x0+3, color=black, title="Section 2.2 #47(f)" );
```

Mathematica: (functions and x0 may vary):

```
<<Miscellaneous`RealOnly`
Clear[f, m, x, y, h]
x0= π /4;
f[x_]:=x^2 Cos[x]
Plot[f[x], {x, x0 - 3, x0 + 3}]
q[x_, h_]:=(f[x + h] - f[x])/h
m[x_]:=Limit[q[x, h], h → 0]
ytan:=f[x0] + m[x0] (x - x0)
Plot[{f[x], ytan},{x, x0 - 3, x0 + 3}]
m[x0 - 1]//N
```

m[x0 + 1]//N
Plot[{f[x], m[x]},{x, x0 − 3, x0 + 3}]

2.3 DIFFERENTIATION RULES

1. $y = -x^2 + 3 \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(-x^2) + \frac{d}{dx}(3) = -2x + 0 = -2x \Rightarrow \frac{d^2y}{dx^2} = -2$

2. $y = x^2 + x + 8 \Rightarrow \frac{dy}{dx} = 2x + 1 + 0 = 2x + 1 \Rightarrow \frac{d^2y}{dx^2} = 2$

3. $s = 5t^3 - 3t^5 \Rightarrow \frac{ds}{dt} = \frac{d}{dt}(5t^3) - \frac{d}{dt}(3t^5) = 15t^2 - 15t^4 \Rightarrow \frac{d^2s}{dt^2} = \frac{d}{dt}(15t^2) - \frac{d}{dt}(15t^4) = 30t - 60t^3$

4. $w = 3z^7 - 7z^3 + 21z^2 \Rightarrow \frac{dw}{dz} = 21z^6 - 21z^2 + 42z \Rightarrow \frac{d^2w}{dz^2} = 126z^5 - 42z + 42$

5. $y = \frac{4}{3}x^3 - x \Rightarrow \frac{dy}{dx} = 4x^2 - 1 \Rightarrow \frac{d^2y}{dx^2} = 8x$

6. $y = \frac{x^3}{3} + \frac{x^2}{2} + \frac{x}{4} \Rightarrow \frac{dy}{dx} = x^2 + x + \frac{1}{4} \Rightarrow \frac{d^2y}{dx^2} = 2x + 1 + 0 = 2x + 1$

7. $w = 3z^{-2} - z^{-1} \Rightarrow \frac{dw}{dz} = -6z^{-3} + z^{-2} = \frac{-6}{z^3} + \frac{1}{z^2} \Rightarrow \frac{d^2w}{dz^2} = 18z^{-4} - 2z^{-3} = \frac{18}{z^4} - \frac{2}{z^3}$

8. $s = -2t^{-1} + 4t^{-2} \Rightarrow \frac{ds}{dt} = 2t^{-2} - 8t^{-3} = \frac{2}{t^2} - \frac{8}{t^3} \Rightarrow \frac{d^2s}{dt^2} = -4t^{-3} + 24t^{-4} = \frac{-4}{t^3} + \frac{24}{t^4}$

9. $y = 6x^2 - 10x - 5x^{-2} \Rightarrow \frac{dy}{dx} = 12x - 10 + 10x^{-3} = 12x - 10 + \frac{10}{x^3} \Rightarrow \frac{d^2y}{dx^2} = 12 - 0 - 30x^{-4} = 12 - \frac{30}{x^4}$

10. $y = 4 - 2x - x^{-3} \Rightarrow \frac{dy}{dx} = -2 + 3x^{-4} = -2 + \frac{3}{x^4} \Rightarrow \frac{d^2y}{dx^2} = 0 - 12x^{-5} = \frac{-12}{x^5}$

11. $r = \frac{1}{3}s^{-2} - \frac{5}{2}s^{-1} \Rightarrow \frac{dr}{ds} = -\frac{2}{3}s^{-3} + \frac{5}{2}s^{-2} = \frac{-2}{3s^3} + \frac{5}{2s^2} \Rightarrow \frac{d^2r}{ds^2} = 2s^{-4} - 5s^{-3} = \frac{2}{s^4} - \frac{5}{s^3}$

12. $r = 12\theta^{-1} - 4\theta^{-3} + \theta^{-4} \Rightarrow \frac{dr}{d\theta} = -12\theta^{-2} + 12\theta^{-4} - 4\theta^{-5} = \frac{-12}{\theta^2} + \frac{12}{\theta^4} - \frac{4}{\theta^5} \Rightarrow \frac{d^2r}{d\theta^2} = 24\theta^{-3} - 48\theta^{-5} + 20\theta^{-6}$
$= \frac{24}{\theta^3} - \frac{48}{\theta^5} + \frac{20}{\theta^6}$

13. (a) $y = (3 - x^2)(x^3 - x + 1) \Rightarrow y' = (3 - x^2) \cdot \frac{d}{dx}(x^3 - x + 1) + (x^3 - x + 1) \cdot \frac{d}{dx}(3 - x^2)$
$= (3 - x^2)(3x^2 - 1) + (x^3 - x + 1)(-2x) = -5x^4 + 12x^2 - 2x - 3$
(b) $y = -x^5 + 4x^3 - x^2 - 3x + 3 \Rightarrow y' = -5x^4 + 12x^2 - 2x - 3$

14. (a) $y = (x - 1)(x^2 + x + 1) \Rightarrow y' = (x - 1)(2x + 1) + (x^2 + x + 1)(1) = 3x^2$
(b) $y = (x - 1)(x^2 + x + 1) = x^3 - 1 \Rightarrow y' = 3x^2$

15. (a) $y = (x^2 + 1)\left(x + 5 + \frac{1}{x}\right) \Rightarrow y' = (x^2 + 1) \cdot \frac{d}{dx}\left(x + 5 + \frac{1}{x}\right) + \left(x + 5 + \frac{1}{x}\right) \cdot \frac{d}{dx}(x^2 + 1)$
$= (x^2 + 1)(1 - x^{-2}) + (x + 5 + x^{-1})(2x) = (x^2 - 1 + 1 - x^{-2}) + (2x^2 + 10x + 2) = 3x^2 + 10x + 2 - \frac{1}{x^2}$
(b) $y = x^3 + 5x^2 + 2x + 5 + \frac{1}{x} \Rightarrow y' = 3x^2 + 10x + 2 - \frac{1}{x^2}$

16. $y = \left(x + \frac{1}{x}\right)\left(x - \frac{1}{x} + 1\right)$
(a) $y' = (x + x^{-1}) \cdot (1 + x^{-2}) + (x - x^{-1} + 1)(1 - x^{-2}) = 2x + 1 - \frac{1}{x^2} + \frac{2}{x^3}$
(b) $y = x^2 + x + \frac{1}{x} - \frac{1}{x^2} \Rightarrow y' = 2x + 1 - \frac{1}{x^2} + \frac{2}{x^3}$

17. $y = \frac{2x+5}{3x-2}$; use the quotient rule: $u = 2x + 5$ and $v = 3x - 2 \Rightarrow u' = 2$ and $v' = 3 \Rightarrow y' = \frac{vu' - uv'}{v^2}$
$= \frac{(3x-2)(2)-(2x+5)(3)}{(3x-2)^2} = \frac{6x-4-6x-15}{(3x-2)^2} = \frac{-19}{(3x-2)^2}$

18. $z = \frac{2x+1}{x^2-1} \Rightarrow \frac{dz}{dx} = \frac{(x^2-1)(2)-(2x+1)(2x)}{(x^2-1)^2} = \frac{2x^2-2-4x^2-2x}{(x^2-1)^2} = \frac{-2x^2-2x-2}{(x^2-1)^2} = \frac{-2(x^2+x+1)}{(x^2-1)^2}$

19. $g(x) = \frac{x^2-4}{x+0.5}$; use the quotient rule: $u = x^2 - 4$ and $v = x + 0.5 \Rightarrow u' = 2x$ and $v' = 1 \Rightarrow g'(x) = \frac{vu' - uv'}{v^2}$
$= \frac{(x+0.5)(2x)-(x^2-4)(1)}{(x+0.5)^2} = \frac{2x^2+x-x^2+4}{(x+0.5)^2} = \frac{x^2+x+4}{(x+0.5)^2}$

20. $f(t) = \frac{t^2-1}{t^2+t-2} = \frac{(t-1)(t+1)}{(t+2)(t-1)} = \frac{t+1}{t+2}, t \neq 1 \Rightarrow f'(t) = \frac{(t+2)(1)-(t+1)(1)}{(t+2)^2} = \frac{t+2-t-1}{(t+2)^2} = \frac{1}{(t+2)^2}$

21. $v = (1-t)(1+t^2)^{-1} = \frac{1-t}{1+t^2} \Rightarrow \frac{dv}{dt} = \frac{(1+t^2)(-1)-(1-t)(2t)}{(1+t^2)^2} = \frac{-1-t^2-2t+2t^2}{(1+t^2)^2} = \frac{t^2-2t-1}{(1+t^2)^2}$

22. $w = \frac{x+5}{2x-7} \Rightarrow w' = \frac{(2x-7)(1)-(x+5)(2)}{(2x-7)^2} = \frac{2x-7-2x-10}{(2x-7)^2} = \frac{-17}{(2x-7)^2}$

23. $f(s) = \frac{\sqrt{s}-1}{\sqrt{s}+1} \Rightarrow f'(s) = \frac{(\sqrt{s}+1)\left(\frac{1}{2\sqrt{s}}\right)-(\sqrt{s}-1)\left(\frac{1}{2\sqrt{s}}\right)}{(\sqrt{s}+1)^2} = \frac{(\sqrt{s}+1)-(\sqrt{s}-1)}{2\sqrt{s}(\sqrt{s}+1)^2} = \frac{1}{\sqrt{s}(\sqrt{s}+1)^2}$
NOTE: $\frac{d}{ds}(\sqrt{s}) = \frac{1}{2\sqrt{s}}$ from Example 2 in Section 2.1

24. $u = \frac{5x+1}{2\sqrt{x}} \Rightarrow \frac{du}{dx} = \frac{(2\sqrt{x})(5)-(5x+1)\left(\frac{1}{\sqrt{x}}\right)}{4x} = \frac{5x-1}{4x^{3/2}}$

25. $v = \frac{1+x-4\sqrt{x}}{x} \Rightarrow v' = \frac{x\left(1-\frac{2}{\sqrt{x}}\right)-(1+x-4\sqrt{x})}{x^2} = \frac{2\sqrt{x}-1}{x^2}$

26. $r = 2\left(\frac{1}{\sqrt{\theta}} + \sqrt{\theta}\right) \Rightarrow r' = 2\left(\frac{\sqrt{\theta}(0)-1\left(\frac{1}{2\sqrt{\theta}}\right)}{\theta} + \frac{1}{2\sqrt{\theta}}\right) = -\frac{1}{\theta^{3/2}} + \frac{1}{\theta^{1/2}}$

27. $y = \frac{1}{(x^2-1)(x^2+x+1)}$; use the quotient rule: $u = 1$ and $v = (x^2-1)(x^2+x+1) \Rightarrow u' = 0$ and
$v' = (x^2-1)(2x+1) + (x^2+x+1)(2x) = 2x^3 + x^2 - 2x - 1 + 2x^3 + 2x^2 + 2x = 4x^3 + 3x^2 - 1$
$\Rightarrow \frac{dy}{dx} = \frac{vu'-uv'}{v^2} = \frac{0-1(4x^3+3x^2-1)}{(x^2-1)^2(x^2+x+1)^2} = \frac{-4x^3-3x^2+1}{(x^2-1)^2(x^2+x+1)^2}$

28. $y = \frac{(x+1)(x+2)}{(x-1)(x-2)} = \frac{x^2+3x+2}{x^2-3x+2} \Rightarrow y' = \frac{(x^2-3x+2)(2x+3)-(x^2+3x+2)(2x-3)}{(x-1)^2(x-2)^2} = \frac{-6x^2+12}{(x-1)^2(x-2)^2} = \frac{-6(x^2-2)}{(x-1)^2(x-2)^2}$

29. $y = \frac{1}{2}x^4 - \frac{3}{2}x^2 - x \Rightarrow y' = 2x^3 - 3x - 1 \Rightarrow y'' = 6x^2 - 3 \Rightarrow y''' = 12x \Rightarrow y^{(4)} = 12 \Rightarrow y^{(n)} = 0$ for all $n \geq 5$

30. $y = \frac{1}{120}x^5 \Rightarrow y' = \frac{1}{24}x^4 \Rightarrow y'' = \frac{1}{6}x^3 \Rightarrow y''' = \frac{1}{2}x^2 \Rightarrow y^{(4)} = x \Rightarrow y^{(5)} = 1 \Rightarrow y^{(n)} = 0$ for all $n \geq 6$

31. $y = \frac{x^3+7}{x} = x^2 + 7x^{-1} \Rightarrow \frac{dy}{dx} = 2x - 7x^{-2} = 2x - \frac{7}{x^2} \Rightarrow \frac{d^2y}{dx^2} = 2 + 14x^{-3} = 2 + \frac{14}{x^3}$

32. $s = \frac{t^2+5t-1}{t^2} = 1 + \frac{5}{t} - \frac{1}{t^2} = 1 + 5t^{-1} - t^{-2} \Rightarrow \frac{ds}{dt} = 0 - 5t^{-2} + 2t^{-3} = -5t^{-2} + 2t^{-3} = \frac{-5}{t^2} + \frac{2}{t^3}$
$\Rightarrow \frac{d^2s}{dt^2} = 10t^{-3} - 6t^{-4} = \frac{10}{t^3} - \frac{6}{t^4}$

33. $r = \frac{(\theta-1)(\theta^2+\theta+1)}{\theta^3} = \frac{\theta^3-1}{\theta^3} = 1 - \frac{1}{\theta^3} = 1 - \theta^{-3} \Rightarrow \frac{dr}{d\theta} = 0 + 3\theta^{-4} = 3\theta^{-4} = \frac{3}{\theta^4} \Rightarrow \frac{d^2r}{d\theta^2} = -12\theta^{-5} = \frac{-12}{\theta^5}$

34. $u = \frac{(x^2+x)(x^2-x+1)}{x^4} = \frac{x(x+1)(x^2-x+1)}{x^4} = \frac{x(x^3+1)}{x^4} = \frac{x^4+x}{x^4} = 1 + \frac{x}{x^4} = 1 + x^{-3}$
$\Rightarrow \frac{du}{dx} = 0 - 3x^{-4} = -3x^{-4} = \frac{-3}{x^4} \Rightarrow \frac{d^2u}{dx^2} = 12x^{-5} = \frac{12}{x^5}$

35. $w = \left(\frac{1+3z}{3z}\right)(3-z) = \left(\frac{1}{3}z^{-1} + 1\right)(3-z) = z^{-1} - \frac{1}{3} + 3 - z = z^{-1} + \frac{8}{3} - z \Rightarrow \frac{dw}{dz} = -z^{-2} + 0 - 1 = -z^{-2} - 1$
$= \frac{-1}{z^2} - 1 \Rightarrow \frac{d^2w}{dz^2} = 2z^{-3} - 0 = 2z^{-3} = \frac{2}{z^3}$

36. $w = (z+1)(z-1)(z^2+1) = (z^2-1)(z^2+1) = z^4 - 1 \Rightarrow \frac{dw}{dz} = 4z^3 - 0 = 4z^3 \Rightarrow \frac{d^2w}{dz^2} = 12z^2$

37. $p = \left(\frac{q^2+3}{12q}\right)\left(\frac{q^4-1}{q^3}\right) = \frac{q^6 - q^2 + 3q^4 - 3}{12q^4} = \frac{1}{12}q^2 - \frac{1}{12}q^{-2} + \frac{1}{4} - \frac{1}{4}q^{-4} \Rightarrow \frac{dp}{dq} = \frac{1}{6}q + \frac{1}{6}q^{-3} + q^{-5} = \frac{1}{6}q + \frac{1}{6q^3} + \frac{1}{q^5}$
$\Rightarrow \frac{d^2p}{dq^2} = \frac{1}{6} - \frac{1}{2}q^{-4} - 5q^{-6} = \frac{1}{6} - \frac{1}{2q^4} - \frac{5}{q^6}$

38. $p = \frac{q^2+3}{(q-1)^3 + (q+1)^3} = \frac{q^2+3}{(q^3 - 3q^2 + 3q - 1) + (q^3 + 3q^2 + 3q + 1)} = \frac{q^2+3}{2q^3+6q} = \frac{q^2+3}{2q(q^2+3)} = \frac{1}{2q} = \frac{1}{2}q^{-1}$
$\Rightarrow \frac{dp}{dq} = -\frac{1}{2}q^{-2} = -\frac{1}{2q^2} \Rightarrow \frac{d^2p}{dq^2} = q^{-3} = \frac{1}{q^3}$

39. $u(0) = 5,\ u'(0) = -3,\ v(0) = -1,\ v'(0) = 2$
(a) $\frac{d}{dx}(uv) = uv' + vu' \Rightarrow \frac{d}{dx}(uv)\big|_{x=0} = u(0)v'(0) + v(0)u'(0) = 5 \cdot 2 + (-1)(-3) = 13$
(b) $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2} \Rightarrow \frac{d}{dx}\left(\frac{u}{v}\right)\big|_{x=0} = \frac{v(0)u'(0) - u(0)v'(0)}{(v(0))^2} = \frac{(-1)(-3) - (5)(2)}{(-1)^2} = -7$
(c) $\frac{d}{dx}\left(\frac{v}{u}\right) = \frac{uv' - vu'}{u^2} \Rightarrow \frac{d}{dx}\left(\frac{v}{u}\right)\big|_{x=0} = \frac{u(0)v'(0) - v(0)u'(0)}{(u(0))^2} = \frac{(5)(2) - (-1)(-3)}{(5)^2} = \frac{7}{25}$
(d) $\frac{d}{dx}(7v - 2u) = 7v' - 2u' \Rightarrow \frac{d}{dx}(7v - 2u)\big|_{x=0} = 7v'(0) - 2u'(0) = 7 \cdot 2 - 2(-3) = 20$

40. $u(1) = 2,\ u'(1) = 0,\ v(1) = 5,\ v'(1) = -1$
(a) $\frac{d}{dx}(uv)\big|_{x=1} = u(1)v'(1) + v(1)u'(1) = 2 \cdot (-1) + 5 \cdot 0 = -2$
(b) $\frac{d}{dx}\left(\frac{u}{v}\right)\big|_{x=1} = \frac{v(1)u'(1) - u(1)v'(1)}{(v(1))^2} = \frac{5 \cdot 0 - 2 \cdot (-1)}{(5)^2} = \frac{2}{25}$
(c) $\frac{d}{dx}\left(\frac{v}{u}\right)\big|_{x=1} = \frac{u(1)v'(1) - v(1)u'(1)}{(u(1))^2} = \frac{2 \cdot (-1) - 5 \cdot 0}{(2)^2} = -\frac{1}{2}$
(d) $\frac{d}{dx}(7v - 2u)\big|_{x=1} = 7v'(1) - 2u'(1) = 7 \cdot (-1) - 2 \cdot 0 = -7$

41. $y = x^3 - 4x + 1$. Note that $(2, 1)$ is on the curve: $1 = 2^3 - 4(2) + 1$
(a) Slope of the tangent at (x, y) is $y' = 3x^2 - 4 \Rightarrow$ slope of the tangent at $(2, 1)$ is $y'(2) = 3(2)^2 - 4 = 8$. Thus the slope of the line perpendicular to the tangent at $(2, 1)$ is $-\frac{1}{8} \Rightarrow$ the equation of the line perpendicular to the tangent line at $(2, 1)$ is $y - 1 = -\frac{1}{8}(x - 2)$ or $y = -\frac{x}{8} + \frac{5}{4}$.
(b) The slope of the curve at x is $m = 3x^2 - 4$ and the smallest value for m is -4 when $x = 0$ and $y = 1$.
(c) We want the slope of the curve to be 8 $\Rightarrow y' = 8 \Rightarrow 3x^2 - 4 = 8 \Rightarrow 3x^2 = 12 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$. When $x = 2$, $y = 1$ and the tangent line has equation $y - 1 = 8(x - 2)$ or $y = 8x - 15$; when $x = -2$, $y = (-2)^3 - 4(-2) + 1 = 1$, and the tangent line has equation $y - 1 = 8(x + 2)$ or $y = 8x + 17$.

42. (a) $y = x^3 - 3x - 2 \Rightarrow y' = 3x^2 - 3$. For the tangent to be horizontal, we need $m = y' = 0 \Rightarrow 0 = 3x^2 - 3$ $\Rightarrow 3x^2 = 3 \Rightarrow x = \pm 1$. When $x = -1$, $y = 0 \Rightarrow$ the tangent line has equation $y = 0$. The line perpendicular to this line at $(-1, 0)$ is $x = -1$. When $x = 1$, $y = -4 \Rightarrow$ the tangent line has equation $y = -4$. The line perpendicular to this line at $(1, -4)$ is $x = 1$.
(b) The smallest value of y' is -3, and this occurs when $x = 0$ and $y = -2$. The tangent to the curve at $(0, -2)$ has slope $-3 \Rightarrow$ the line perpendicular to the tangent at $(0, -2)$ has slope $\frac{1}{3} \Rightarrow y + 2 = \frac{1}{3}(x - 0)$ or $y = \frac{1}{3}x - 2$ is an equation of the perpendicular line.

43. $y = \frac{4x}{x^2+1} \Rightarrow \frac{dy}{dx} = \frac{(x^2+1)(4)-(4x)(2x)}{(x^2+1)^2} = \frac{4x^2+4-8x^2}{(x^2+1)^2} = \frac{4(-x^2+1)}{(x^2+1)^2}$. When $x = 0$, $y = 0$ and $y' = \frac{4(0+1)}{1}$ $= 4$, so the tangent to the curve at $(0,0)$ is the line $y = 4x$. When $x = 1$, $y = 2 \Rightarrow y' = 0$, so the tangent to the curve at $(1,2)$ is the line $y = 2$.

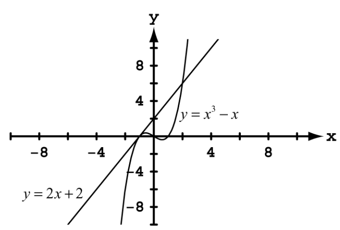
44. $y = \frac{8}{x^2+4} \Rightarrow y' = \frac{(x^2+4)(0)-8(2x)}{(x^2+4)^2} = \frac{-16x}{(x^2+4)^2}$. When $x = 2$, $y = 1$ and $y' = \frac{-16(2)}{(2^2+4)^2} = -\frac{1}{2}$, so the tangent line to the curve at $(2,1)$ has the equation $y - 1 = -\frac{1}{2}(x-2)$, or $y = -\frac{x}{2} + 2$.

45. $y = ax^2 + bx + c$ passes through $(0,0) \Rightarrow 0 = a(0) + b(0) + c \Rightarrow c = 0$; $y = ax^2 + bx$ passes through $(1,2)$ $\Rightarrow 2 = a + b$; $y' = 2ax + b$ and since the curve is tangent to $y = x$ at the origin, its slope is 1 at $x = 0 \Rightarrow y' = 1$ when $x = 0 \Rightarrow 1 = 2a(0) + b \Rightarrow b = 1$. Then $a + b = 2 \Rightarrow a = 1$. In summary $a = b = 1$ and $c = 0$ so the curve is $y = x^2 + x$.

46. $y = cx - x^2$ passes through $(1,0) \Rightarrow 0 = c(1) - 1 \Rightarrow c = 1 \Rightarrow$ the curve is $y = x - x^2$. For this curve, $y' = 1 - 2x$ and $x = 1 \Rightarrow y' = -1$. Since $y = x - x^2$ and $y = x^2 + ax + b$ have common tangents at $x = 0$, $y = x^2 + ax + b$ must also have slope -1 at $x = 1$. Thus $y' = 2x + a \Rightarrow -1 = 2 \cdot 1 + a \Rightarrow a = -3 \Rightarrow y = x^2 - 3x + b$. Since this last curve passes through $(1,0)$, we have $0 = 1 - 3 + b \Rightarrow b = 2$. In summary, $a = -3$, $b = 2$ and $c = 1$ so the curves are $y = x^2 - 3x + 2$ and $y = x - x^2$.

47. (a) $y = x^3 - x \Rightarrow y' = 3x^2 - 1$. When $x = -1$, $y = 0$ and $y' = 2 \Rightarrow$ the tangent line to the curve at $(-1,0)$ is $y = 2(x+1)$ or $y = 2x + 2$.

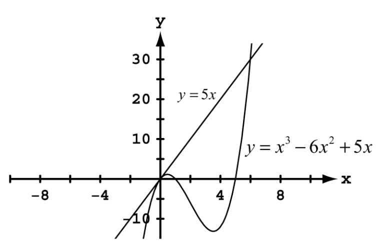
(b)



(c) $\left.\begin{array}{l} y = x^3 - x \\ y = 2x + 2 \end{array}\right\} \Rightarrow x^3 - x = 2x + 2 \Rightarrow x^3 - 3x - 2 = (x-2)(x+1)^2 = 0 \Rightarrow x = 2$ or $x = -1$. Since $y = 2(2) + 2 = 6$; the other intersection point is $(2,6)$

48. (a) $y = x^3 - 6x^2 + 5x \Rightarrow y' = 3x^2 - 12x + 5$. When $x = 0$, $y = 0$ and $y' = 5 \Rightarrow$ the tangent line to the curve at $(0,0)$ is $y = 5x$.

(b)



(c) $\left.\begin{array}{l} y = x^3 - 6x^2 + 5x \\ y = 5x \end{array}\right\} \Rightarrow x^3 - 6x^2 + 5x = 5x \Rightarrow x^3 - 6x^2 = 0 \Rightarrow x^2(x-6) = 0 \Rightarrow x = 0$ or $x = 6$. Since $y = 5(6) = 30$, the other intersection point is $(6,30)$.

49. $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0 \Rightarrow P'(x) = na_nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + 2a_2x + a_1$

50. $R = M^2\left(\frac{C}{2} - \frac{M}{3}\right) = \frac{C}{2}M^2 - \frac{1}{3}M^3$, where C is a constant $\Rightarrow \frac{dR}{dM} = CM - M^2$

51. Let c be a constant $\Rightarrow \frac{dc}{dx} = 0 \Rightarrow \frac{d}{dx}(u \cdot c) = u \cdot \frac{dc}{dx} + c \cdot \frac{du}{dx} = u \cdot 0 + c\,\frac{du}{dx} = c\,\frac{du}{dx}$. Thus when one of the functions is a constant, the Product Rule is just the Constant Multiple Rule $\Rightarrow$ the Constant Multiple Rule is a special case of the Product Rule.

52. (a) We use the Quotient rule to derive the Reciprocal Rule (with u = 1): $\frac{d}{dx}\left(\frac{1}{v}\right) = \frac{v \cdot 0 - 1 \cdot \frac{dv}{dx}}{v^2} = \frac{-1 \cdot \frac{dv}{dx}}{v^2} = -\frac{1}{v^2} \cdot \frac{dv}{dx}$.
(b) Now, using the Reciprocal Rule and the Product Rule, we'll derive the Quotient Rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{d}{dx}\left(u \cdot \frac{1}{v}\right)$
$= u \cdot \frac{d}{dx}\left(\frac{1}{v}\right) + \frac{1}{v} \cdot \frac{du}{dx}$ (Product Rule) $= u \cdot \left(\frac{-1}{v^2}\right)\frac{dv}{dx} + \frac{1}{v}\,\frac{du}{dx}$ (Reciprocal Rule) $\Rightarrow \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{-u\frac{dv}{dx} + v\frac{du}{dx}}{v^2} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$, the Quotient Rule.

53. $P = \frac{nRT}{V-nb} - \frac{an^2}{V^2}$. We are holding T constant, and a, b, n, R are also constant so their derivatives are zero
$\Rightarrow \frac{dP}{dV} = \frac{(V-nb)\cdot 0 - (nRT)(1)}{(V-nb)^2} - \frac{V^2(0) - (an^2)(2V)}{(V^2)^2} = \frac{-nRT}{(V-nb)^2} + \frac{2an^2}{V^3}$

54. $\frac{d}{dx}(x^{-m}) = \frac{d}{dx}\left(\frac{1}{x^m}\right) = \frac{x^m \cdot 0 - 1(m \cdot x^{m-1})}{(x^m)^2} = \frac{-m \cdot x^{m-1}}{x^{2m}} = -m \cdot x^{m-1-2m} = -m \cdot x^{-m-1}$

2.4 THE DERIVATIVE AS A RATE OF CHANGE

1. $s = t^2 - 3t + 2,\ 0 \le t \le 2$
(a) displacement $= \Delta s = s(2) - s(0) = 0\text{m} - 2\text{m} = -2$ m, $v_{av} = \frac{\Delta s}{\Delta t} = \frac{-2}{2} = -1$ m/sec
(b) $v = \frac{ds}{dt} = 2t - 3 \Rightarrow |v(0)| = |-3| = 3$ m/sec and $|v(2)| = 1$ m/sec;
$a = \frac{d^2s}{dt^2} = 2 \Rightarrow a(0) = 2$ m/sec^2 and $a(2) = 2$ m/sec^2
(c) $v = 0 \Rightarrow 2t - 3 = 0 \Rightarrow t = \frac{3}{2}$. v is negative in the interval $0 < t < \frac{3}{2}$ and v is positive when $\frac{3}{2} < t < 2 \Rightarrow$ the body changes direction at $t = \frac{3}{2}$.

2. $s = 6t - t^2,\ 0 \le t \le 6$
(a) displacement $= \Delta s = s(6) - s(0) = 0$ m, $v_{av} = \frac{\Delta s}{\Delta t} = \frac{0}{6} = 0$ m/sec
(b) $v = \frac{ds}{dt} = 6 - 2t \Rightarrow |v(0)| = |6| = 6$ m/sec and $|v(6)| = |-6| = 6$ m/sec;
$a = \frac{d^2s}{dt^2} = -2 \Rightarrow a(0) = -2$ m/sec^2 and $a(6) = -2$ m/sec^2
(c) $v = 0 \Rightarrow 6 - 2t = 0 \Rightarrow t = 3$. v is positive in the interval $0 < t < 3$ and v is negative when $3 < t < 6 \Rightarrow$ the body changes direction at t = 3.

3. $s = -t^3 + 3t^2 - 3t,\ 0 \le t \le 3$
(a) displacement $= \Delta s = s(3) - s(0) = -9$ m, $v_{av} = \frac{\Delta s}{\Delta t} = \frac{-9}{3} = -3$ m/sec
(b) $v = \frac{ds}{dt} = -3t^2 + 6t - 3 \Rightarrow |v(0)| = |-3| = 3$ m/sec and $|v(3)| = |-12| = 12$ m/sec; $a = \frac{d^2s}{dt^2} = -6t + 6$
$\Rightarrow a(0) = 6$ m/sec^2 and $a(3) = -12$ m/sec^2
(c) $v = 0 \Rightarrow -3t^2 + 6t - 3 = 0 \Rightarrow t^2 - 2t + 1 = 0 \Rightarrow (t-1)^2 = 0 \Rightarrow t = 1$. For all other values of t in the interval the velocity v is negative (the graph of $v = -3t^2 + 6t - 3$ is a parabola with vertex at t = 1 which opens downward $\Rightarrow$ the body never changes direction).

4. $s = \frac{t^4}{4} - t^3 + t^2,\ 0 \le t \le 3$
(a) $\Delta s = s(3) - s(0) = \frac{9}{4}$ m, $v_{av} = \frac{\Delta s}{\Delta t} = \frac{\frac{9}{4}}{3} = \frac{3}{4}$ m/sec
(b) $v = t^3 - 3t^2 + 2t \Rightarrow |v(0)| = 0$ m/sec and $|v(3)| = 6$ m/sec; $a = 3t^2 - 6t + 2 \Rightarrow a(0) = 2$ m/sec^2 and $a(3) = 11$ m/sec^2

(c) $v = 0 \Rightarrow t^3 - 3t^2 + 2t = 0 \Rightarrow t(t-2)(t-1) = 0 \Rightarrow t = 0, 1, 2 \Rightarrow v = t(t-2)(t-1)$ is positive in the interval for $0 < t < 1$ and v is negative for $1 < t < 2$ and v is positive for $2 < t < 3 \Rightarrow$ the body changes direction at $t = 1$ and at $t = 2$.

5. $s = \frac{25}{t^2} - \frac{5}{t}, 1 \le t \le 5$

(a) $\Delta s = s(5) - s(1) = -20$ m, $v_{av} = \frac{-20}{4} = -5$ m/sec

(b) $v = \frac{-50}{t^3} + \frac{5}{t^2} \Rightarrow |v(1)| = 45$ m/sec and $|v(5)| = \frac{1}{5}$ m/sec; $a = \frac{150}{t^4} - \frac{10}{t^3} \Rightarrow a(1) = 140$ m/sec^2 and $a(5) = \frac{4}{25}$ m/sec^2

(c) $v = 0 \Rightarrow \frac{-50+5t}{t^3} = 0 \Rightarrow -50 + 5t = 0 \Rightarrow t = 10 \Rightarrow$ the body does not change direction in the interval

6. $s = \frac{25}{t+5}, -4 \le t \le 0$

(a) $\Delta s = s(0) - s(-4) = -20$ m, $v_{av} = -\frac{20}{4} = -5$ m/sec

(b) $v = \frac{-25}{(t+5)^2} \Rightarrow |v(-4)| = 25$ m/sec and $|v(0)| = 1$ m/sec; $a = \frac{50}{(t+5)^3} \Rightarrow a(-4) = 50$ m/sec^2 and $a(0) = \frac{2}{5}$ m/sec^2

(c) $v = 0 \Rightarrow \frac{-25}{(t+5)^2} = 0 \Rightarrow$ v is never 0 $\Rightarrow$ the body never changes direction

7. $s = t^3 - 6t^2 + 9t$ and let the positive direction be to the right on the s-axis.

(a) $v = 3t^2 - 12t + 9$ so that $v = 0 \Rightarrow t^2 - 4t + 3 = (t-3)(t-1) = 0 \Rightarrow t = 1$ or 3; $a = 6t - 12 \Rightarrow a(1) = -6$ m/sec^2 and $a(3) = 6$ m/sec^2. Thus the body is motionless but being accelerated left when $t = 1$, and motionless but being accelerated right when $t = 3$.

(b) $a = 0 \Rightarrow 6t - 12 = 0 \Rightarrow t = 2$ with speed $|v(2)| = |12 - 24 + 9| = 3$ m/sec

(c) The body moves to the right or forward on $0 \le t < 1$, and to the left or backward on $1 < t < 2$. The positions are $s(0) = 0$, $s(1) = 4$ and $s(2) = 2 \Rightarrow$ total distance $= |s(1) - s(0)| + |s(2) - s(1)| = |4| + |-2| = 6$ m.

8. $v = t^2 - 4t + 3 \Rightarrow a = 2t - 4$

(a) $v = 0 \Rightarrow t^2 - 4t + 3 = 0 \Rightarrow t = 1$ or $3 \Rightarrow a(1) = -2$ m/sec^2 and $a(3) = 2$ m/sec^2

(b) $v > 0 \Rightarrow (t-3)(t-1) > 0 \Rightarrow 0 \le t < 1$ or $t > 3$ and the body is moving forward; $v < 0 \Rightarrow (t-3)(t-1) < 0 \Rightarrow 1 < t < 3$ and the body is moving backward

(c) velocity increasing $\Rightarrow a > 0 \Rightarrow 2t - 4 > 0 \Rightarrow t > 2$; velocity decreasing $\Rightarrow a < 0 \Rightarrow 2t - 4 < 0 \Rightarrow 0 \le t < 2$

9. $s_m = 1.86t^2 \Rightarrow v_m = 3.72t$ and solving $3.72t = 27.8 \Rightarrow t \approx 7.5$ sec on Mars; $s_j = 11.44t^2 \Rightarrow v_j = 22.88t$ and solving $22.88t = 27.8 \Rightarrow t \approx 1.2$ sec on Jupiter.

10. (a) $v(t) = s'(t) = 24 - 1.6t$ m/sec, and $a(t) = v'(t) = s''(t) = -1.6$ m/sec^2

(b) Solve $v(t) = 0 \Rightarrow 24 - 1.6t = 0 \Rightarrow t = 15$ sec

(c) $s(15) = 24(15) - .8(15)^2 = 180$ m

(d) Solve $s(t) = 90 \Rightarrow 24t - .8t^2 = 90 \Rightarrow t = \frac{30 \pm 15\sqrt{2}}{2} \approx 4.39$ sec going up and 25.6 sec going down

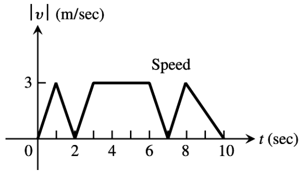
(e) Twice the time it took to reach its highest point or 30 sec

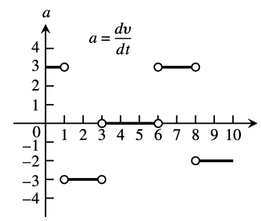
11. (a) $s = 179 - 16t^2 \Rightarrow v = -32t \Rightarrow$ speed $= |v| = 32t$ ft/sec and $a = -32$ ft/sec^2

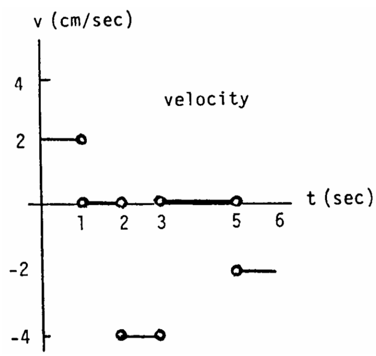
(b) $s = 0 \Rightarrow 179 - 16t^2 = 0 \Rightarrow t = \sqrt{\frac{179}{16}} \approx 3.3$ sec

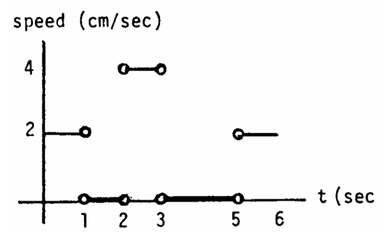
(c) When $t = \sqrt{\frac{179}{16}}$, $v = -32\sqrt{\frac{179}{16}} = -8\sqrt{179} \approx -107.0$ ft/sec

12. Solving $s_m = 832t - 2.6t^2 = 0 \Rightarrow t(832 - 2.6t) = 0 \Rightarrow t = 0$ or $320 \Rightarrow 320$ sec on the moon; solving $s_e = 832t - 16t^2 = 0 \Rightarrow t(832 - 16t) = 0 \Rightarrow t = 0$ or $52 \Rightarrow 52$ sec on the earth. Also, $v_m = 832 - 5.2t = 0 \Rightarrow t = 160$ and $s_m(160) = 66{,}560$ ft, the height it reaches above the moon's surface; $v_e = 832 - 32t = 0 \Rightarrow t = 26$ and $s_e(26) = 10{,}816$ ft, the height it reaches above the earth's surface.

13. (a) at 2 and 7 seconds
(b) between 3 and 6 seconds: $3 \le t \le 6$
(c)


(d)


14. (a) P is moving to the left when $2 < t < 3$ or $5 < t < 6$; P is moving to the right when $0 < t < 1$; P is standing still when $1 < t < 2$ or $3 < t < 5$
(b)




15. (a) 190 ft/sec
(b) 2 sec
(c) at 8 sec, 0 ft/sec
(d) 10.8 sec, 90 ft/sec
(e) From $t = 8$ until $t = 10.8$ sec, a total of 2.8 sec
(f) Greatest acceleration happens 2 sec after launch
(g) From $t = 2$ to $t = 10.8$ sec; during this period, $a = \frac{v(10.8) - v(2)}{10.8 - 2} \approx -32$ ft/sec^2

16. (a) Forward: $0 \le t < 1$ and $5 < t < 7$; Backward: $1 < t < 5$; Speeds up: $1 < t < 2$ and $5 < t < 6$; Slows down: $0 \le t < 1$, $3 < t < 5$, and $6 < t < 7$
(b) Positive: $3 < t < 6$; negative: $0 \le t < 2$ and $6 < t < 7$; zero: $2 < t < 3$ and $7 < t < 9$
(c) $t = 0$ and $2 \le t \le 3$
(d) $7 \le t \le 9$

17. $s = 490t^2 \Rightarrow v = 980t \Rightarrow a = 980$
(a) Solving $160 = 490t^2 \Rightarrow t = \frac{4}{7}$ sec. The average velocity was $\frac{s(4/7) - s(0)}{4/7} = 280$ cm/sec.
(b) At the 160 cm mark the balls are falling at $v(4/7) = 560$ cm/sec. The acceleration at the 160 cm mark was 980 cm/sec^2.
(c) The light was flashing at a rate of $\frac{17}{4/7} = 29.75$ flashes per second.

18. $s = v_0t - 16t^2 \Rightarrow v = v_0 - 32t$; $v = 0 \Rightarrow t = \frac{v_0}{32}$; $1900 = v_0t - 16t^2$ so that $t = \frac{v_0}{32} \Rightarrow 1900 = \frac{v_0^2}{32} - \frac{v_0^2}{64}$
$\Rightarrow v_0 = \sqrt{(64)(1900)} = 80\sqrt{19}$ ft/sec and, finally, $\frac{80\sqrt{19}\text{ ft}}{\text{sec}} \cdot \frac{60\text{ sec}}{1\text{ min}} \cdot \frac{60\text{ min}}{1\text{ hr}} \cdot \frac{1\text{ mi}}{5280\text{ ft}} \approx 238$ mph.

19. (a) $c(100) = 11{,}000 \Rightarrow c_{av} = \frac{11{,}000}{100} = \$110$
(b) $c(x) = 2000 + 100x - .1x^2 \Rightarrow c'(x) = 100 - .2x$. Marginal cost $= c'(x) \Rightarrow$ the marginal cost of producing 100 machines is $c'(100) = \$80$
(c) The cost of producing the 101st machine is $c(101) - c(100) = 100 - \frac{201}{10} = \$79.90$

20. (a) $r(x) = 20000\left(1 - \frac{1}{x}\right) \Rightarrow r'(x) = \frac{20000}{x^2}$, which is marginal revenue.

(b) $r'(100) = \frac{20000}{100^2} = \$2$.

(c) $\lim_{x \to \infty} r'(x) = \lim_{x \to \infty} \frac{20000}{x^2} = 0$. The increase in revenue as the number of items increases without bound will approach zero.

21. $b(t) = 10^6 + 10^4 t - 10^3 t^2 \Rightarrow b'(t) = 10^4 - (2)(10^3 t) = 10^3(10 - 2t)$

(a) $b'(0) = 10^4$ bacteria/hr

(b) $b'(5) = 0$ bacteria/hr

(c) $b'(10) = -10^4$ bacteria/hr

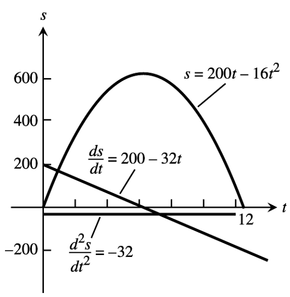
22. (a) $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dr} = 4\pi r^2 \Rightarrow \left.\frac{dV}{dr}\right|_{r=2} = 4\pi(2)^2 = 16\pi$ ft^3/ft

(b) When $r = 2$, $\frac{dV}{dr} = 16\pi$ so that when r changes by 1 unit, we expect V to change by approximately 16π. Therefore when r changes by 0.2 units V changes by approximately $(16\pi)(0.2) = 3.2\pi \approx 10.05$ ft^3. Note that $V(2.2) - V(2) \approx 11.09$ ft^3.

23. 200 km/hr $= 55\frac{5}{9}$ m/sec $= \frac{500}{9}$ m/sec, and $D = \frac{10}{9}t^2 \Rightarrow V = \frac{20}{9}t$. Thus $V = \frac{500}{9} \Rightarrow \frac{20}{9}t = \frac{500}{9} \Rightarrow t = 25$ sec. When $t = 25$, $D = \frac{10}{9}(25)^2 = \frac{6250}{9}$ m

24. $Q(t) = 200(30 - t)^2 = 200\left(900 - 60t + t^2\right) \Rightarrow Q'(t) = 200(-60 + 2t) \Rightarrow Q'(10) = -8{,}000$ gallons/min is the rate the water is running at the end of 10 min. Then $\frac{Q(10) - Q(0)}{10} = -10{,}000$ gallons/min is the average rate the water flows during the first 10 min. The negative signs indicate water is leaving the tank.

25.



(a) $v = 0$ when $t = 6.25$ sec

(b) $v > 0$ when $0 \le t < 6.25 \Rightarrow$ body moves up; $v < 0$ when $6.25 < t \le 12.5 \Rightarrow$ body moves down

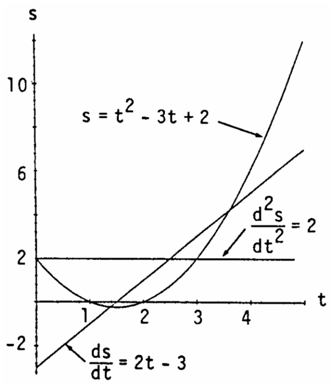
(c) body changes direction at $t = 6.25$ sec

(d) body speeds up on $(6.25, 12.5]$ and slows down on $[0, 6.25)$

(e) The body is moving fastest at the endpoints $t = 0$ and $t = 12.5$ when it is traveling 200 ft/sec. It's moving slowest at $t = 6.25$ when the speed is 0.

(f) When $t = 6.25$ the body is $s = 625$ m from the origin and farthest away.

26.



(a) $v = 0$ when $t = \frac{3}{2}$ sec

(b) $v < 0$ when $0 \le t < 1.5 \Rightarrow$ body moves down; $v > 0$ when $1.5 < t \le 5 \Rightarrow$ body moves up

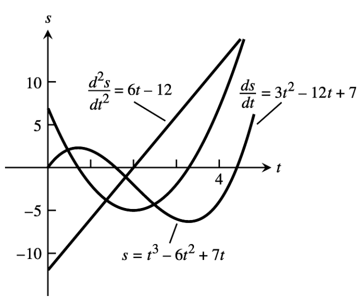
(c) body changes direction at $t = \frac{3}{2}$ sec

(d) body speeds up on $\left(\frac{3}{2}, 5\right]$ and slows down on $\left[0, \frac{3}{2}\right)$

(e) body is moving fastest at $t = 5$ when the speed $= |v(5)| = 7$ units/sec; it is moving slowest at $t = \frac{3}{2}$ when the speed is 0

(f) When $t = 5$ the body is $s = 12$ units from the origin and farthest away.

27.



(a) $v = 0$ when $t = \frac{6 \pm \sqrt{15}}{3}$ sec

(b) $v < 0$ when $\frac{6-\sqrt{15}}{3} < t < \frac{6+\sqrt{15}}{3} \Rightarrow$ body moves left; $v > 0$ when $0 \le t < \frac{6-\sqrt{15}}{3}$ or $\frac{6+\sqrt{15}}{3} < t \le 4$ $\Rightarrow$ body moves right

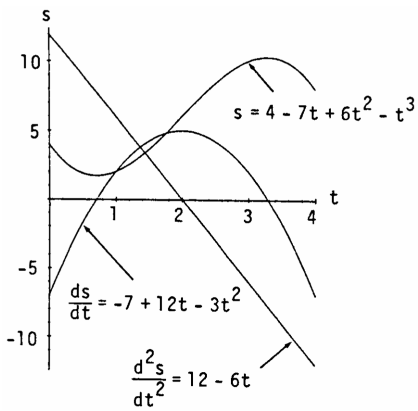
(c) body changes direction at $t = \frac{6 \pm \sqrt{15}}{3}$ sec

(d) body speeds up on $\left(\frac{6-\sqrt{15}}{3}, 2\right) \cup \left(\frac{6+\sqrt{15}}{3}, 4\right]$ and slows down on $\left[0, \frac{6-\sqrt{15}}{3}\right) \cup \left(2, \frac{6+\sqrt{15}}{3}\right)$.

(e) The body is moving fastest at $t = 0$ and $t = 4$ when it is moving 7 units/sec and slowest at $t = \frac{6 \pm \sqrt{15}}{3}$ sec

(f) When $t = \frac{6+\sqrt{15}}{3}$ the body is at position $s \approx -6.303$ units and farthest from the origin.

28.



(a) $v = 0$ when $t = \frac{6 \pm \sqrt{15}}{3}$

(b) $v < 0$ when $0 \le t < \frac{6-\sqrt{15}}{3}$ or $\frac{6+\sqrt{15}}{3} < t \le 4 \Rightarrow$ body is moving left; $v > 0$ when $\frac{6-\sqrt{15}}{3} < t < \frac{6+\sqrt{15}}{3} \Rightarrow$ body is moving right

(c) body changes direction at $t = \frac{6 \pm \sqrt{15}}{3}$ sec

(d) body speeds up on $\left(\frac{6-\sqrt{15}}{3}, 2\right) \cup \left(\frac{6+\sqrt{15}}{3}, 4\right]$ and slows down on $\left[0, \frac{6-\sqrt{15}}{3}\right) \cup \left(2, \frac{6+\sqrt{15}}{3}\right)$

(e) The body is moving fastest at 7 units/sec when $t = 0$ and $t = 4$; it is moving slowest and stationary at $t = \frac{6 \pm \sqrt{15}}{3}$

(f) When $t = \frac{6+\sqrt{15}}{3}$ the position is $s \approx 10.303$ units and the body is farthest from the origin.

2.5 DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

1. $y = -10x + 3\cos x \Rightarrow \frac{dy}{dx} = -10 + 3\frac{d}{dx}(\cos x) = -10 - 3\sin x$

2. $y = \frac{3}{x} + 5\sin x \Rightarrow \frac{dy}{dx} = \frac{-3}{x^2} + 5\frac{d}{dx}(\sin x) = \frac{-3}{x^2} + 5\cos x$

3. $y = \csc x - 4\sqrt{x} + 7 \Rightarrow \frac{dy}{dx} = -\csc x \cot x - \frac{4}{2\sqrt{x}} + 0 = -\csc x \cot x - \frac{2}{\sqrt{x}}$

4. $y = x^2 \cot x - \frac{1}{x^2} \Rightarrow \frac{dy}{dx} = x^2 \frac{d}{dx}(\cot x) + \cot x \cdot \frac{d}{dx}(x^2) + \frac{2}{x^3} = -x^2 \csc^2 x + (\cot x)(2x) + \frac{2}{x^3}$
$= -x^2 \csc^2 x + 2x \cot x + \frac{2}{x^3}$

5. $y = (\sec x + \tan x)(\sec x - \tan x) \Rightarrow \frac{dy}{dx} = (\sec x + \tan x)\frac{d}{dx}(\sec x - \tan x) + (\sec x - \tan x)\frac{d}{dx}(\sec x + \tan x)$
$= (\sec x + \tan x)(\sec x \tan x - \sec^2 x) + (\sec x - \tan x)(\sec x \tan x + \sec^2 x)$
$= (\sec^2 x \tan x + \sec x \tan^2 x - \sec^3 x - \sec^2 x \tan x) + (\sec^2 x \tan x - \sec x \tan^2 x + \sec^3 x - \tan x \sec^2 x) = 0.$
$\left(\text{Note also that } y = \sec^2 x - \tan^2 x = (\tan^2 x + 1) - \tan^2 x = 1 \Rightarrow \frac{dy}{dx} = 0.\right)$

6. $y = (\sin x + \cos x)\sec x \Rightarrow \frac{dy}{dx} = (\sin x + \cos x)\frac{d}{dx}(\sec x) + \sec x \frac{d}{dx}(\sin x + \cos x)$
$= (\sin x + \cos x)(\sec x \tan x) + (\sec x)(\cos x - \sin x) = \frac{(\sin x + \cos x)\sin x}{\cos^2 x} + \frac{\cos x - \sin x}{\cos x}$
$= \frac{\sin^2 x + \cos x \sin x + \cos^2 x - \cos x \sin x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$
$\left(\text{Note also that } y = \sin x \sec x + \cos x \sec x = \tan x + 1 \Rightarrow \frac{dy}{dx} = \sec^2 x.\right)$

7. $y = \frac{\cot x}{1 + \cot x} \Rightarrow \frac{dy}{dx} = \frac{(1 + \cot x)\frac{d}{dx}(\cot x) - (\cot x)\frac{d}{dx}(1 + \cot x)}{(1 + \cot x)^2} = \frac{(1 + \cot x)(-\csc^2 x) - (\cot x)(-\csc^2 x)}{(1 + \cot x)^2}$
$= \frac{-\csc^2 x - \csc^2 x \cot x + \csc^2 x \cot x}{(1 + \cot x)^2} = \frac{-\csc^2 x}{(1 + \cot x)^2}$

8. $y=\frac{\cos x}{1+\sin x} \Rightarrow \frac{dy}{dx}=\frac{(1+\sin x)\frac{d}{dx}(\cos x)-(\cos x)\frac{d}{dx}(1+\sin x)}{(1+\sin x)^2}=\frac{(1+\sin x)(-\sin x)-(\cos x)(\cos x)}{(1+\sin x)^2}$
$=\frac{-\sin x-\sin^2 x-\cos^2 x}{(1+\sin x)^2}=\frac{-\sin x-1}{(1+\sin x)^2}=\frac{-(1+\sin x)}{(1+\sin x)^2}=\frac{-1}{1+\sin x}$

9. $y=\frac{4}{\cos x}+\frac{1}{\tan x}=4\sec x+\cot x \Rightarrow \frac{dy}{dx}=4\sec x\tan x-\csc^2 x$

10. $y=\frac{\cos x}{x}+\frac{x}{\cos x} \Rightarrow \frac{dy}{dx}=\frac{x(-\sin x)-(\cos x)(1)}{x^2}+\frac{(\cos x)(1)-x(-\sin x)}{\cos^2 x}=\frac{-x\sin x-\cos x}{x^2}+\frac{\cos x+x\sin x}{\cos^2 x}$

11. $y=x^2\sin x+2x\cos x-2\sin x \Rightarrow \frac{dy}{dx}=\left(x^2\cos x+(\sin x)(2x)\right)+\left((2x)(-\sin x)+(\cos x)(2)\right)-2\cos x$
$=x^2\cos x+2x\sin x-2x\sin x+2\cos x-2\cos x=x^2\cos x$

12. $y=x^2\cos x-2x\sin x-2\cos x \Rightarrow \frac{dy}{dx}=\left(x^2(-\sin x)+(\cos x)(2x)\right)-\left(2x\cos x+(\sin x)(2)\right)-2(-\sin x)$
$=-x^2\sin x+2x\cos x-2x\cos x-2\sin x+2\sin x=-x^2\sin x$

13. $s=\tan t-e^{-t} \Rightarrow \frac{ds}{dt}=\sec^2 t+e^{-t}$

14. $s=t^2-\sec t+5e^t \Rightarrow \frac{ds}{dt}=2t-\sec t\tan t+5e^t$

15. $s=\frac{1+\csc t}{1-\csc t} \Rightarrow \frac{ds}{dt}=\frac{(1-\csc t)(-\csc t\cot t)-(1+\csc t)(\csc t\cot t)}{(1-\csc t)^2}=\frac{-\csc t\cot t+\csc^2 t\cot t-\csc t\cot t-\csc^2 t\cot t}{(1-\csc t)^2}=\frac{-2\csc t\cot t}{(1-\csc t)^2}$

16. $s=\frac{\sin t}{1-\cos t} \Rightarrow \frac{ds}{dt}=\frac{(1-\cos t)(\cos t)-(\sin t)(\sin t)}{(1-\cos t)^2}=\frac{\cos t-\cos^2 t-\sin^2 t}{(1-\cos t)^2}=\frac{\cos t-1}{(1-\cos t)^2}=-\frac{1}{1-\cos t}=\frac{1}{\cos t-1}$

17. $r=4-\theta^2\sin\theta \Rightarrow \frac{dr}{d\theta}=-\left(\theta^2\frac{d}{d\theta}(\sin\theta)+(\sin\theta)(2\theta)\right)=-\left(\theta^2\cos\theta+2\theta\sin\theta\right)=-\theta(\theta\cos\theta+2\sin\theta)$

18. $r=\theta\sin\theta+\cos\theta \Rightarrow \frac{dr}{d\theta}=(\theta\cos\theta+(\sin\theta)(1))-\sin\theta=\theta\cos\theta$

19. $r=\sec\theta\csc\theta \Rightarrow \frac{dr}{d\theta}=(\sec\theta)(-\csc\theta\cot\theta)+(\csc\theta)(\sec\theta\tan\theta)$
$=\left(\frac{-1}{\cos\theta}\right)\left(\frac{1}{\sin\theta}\right)\left(\frac{\cos\theta}{\sin\theta}\right)+\left(\frac{1}{\sin\theta}\right)\left(\frac{1}{\cos\theta}\right)\left(\frac{\sin\theta}{\cos\theta}\right)=\frac{-1}{\sin^2\theta}+\frac{1}{\cos^2\theta}=\sec^2\theta-\csc^2\theta$

20. $r=(1+\sec\theta)\sin\theta \Rightarrow \frac{dr}{d\theta}=(1+\sec\theta)\cos\theta+(\sin\theta)(\sec\theta\tan\theta)=(\cos\theta+1)+\tan^2\theta=\cos\theta+\sec^2\theta$

21. $p=5+\frac{1}{\cot q}=5+\tan q \Rightarrow \frac{dp}{dq}=\sec^2 q$

22. $p=(1+\csc q)\cos q \Rightarrow \frac{dp}{dq}=(1+\csc q)(-\sin q)+(\cos q)(-\csc q\cot q)=(-\sin q-1)-\cot^2 q=-\sin q-\csc^2 q$

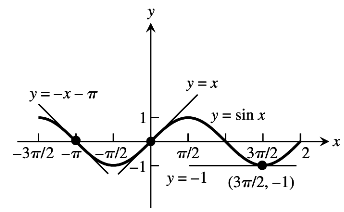
23. $p=\frac{\sin q+\cos q}{\cos q} \Rightarrow \frac{dp}{dq}=\frac{(\cos q)(\cos q-\sin q)-(\sin q+\cos q)(-\sin q)}{\cos^2 q}=\frac{\cos^2 q-\cos q\sin q+\sin^2 q+\cos q\sin q}{\cos^2 q}=\frac{1}{\cos^2 q}=\sec^2 q$

24. $p=\frac{\tan q}{1+\tan q} \Rightarrow \frac{dp}{dq}=\frac{(1+\tan q)(\sec^2 q)-(\tan q)(\sec^2 q)}{(1+\tan q)^2}=\frac{\sec^2 q+\tan q\sec^2 q-\tan q\sec^2 q}{(1+\tan q)^2}=\frac{\sec^2 q}{(1+\tan q)^2}$

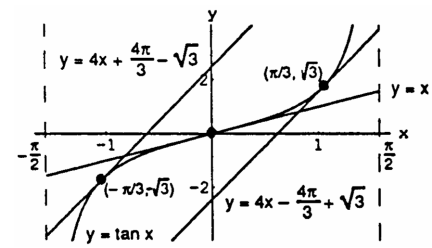
25. (a) $y=\csc x \Rightarrow y'=-\csc x\cot x \Rightarrow y''=-\left((\csc x)\left(-\csc^2 x\right)+(\cot x)(-\csc x\cot x)\right)=\csc^3 x+\csc x\cot^2 x$
$=(\csc x)\left(\csc^2 x+\cot^2 x\right)=(\csc x)\left(\csc^2 x+\csc^2 x-1\right)=2\csc^3 x-\csc x$
(b) $y=\sec x \Rightarrow y'=\sec x\tan x \Rightarrow y''=(\sec x)\left(\sec^2 x\right)+(\tan x)(\sec x\tan x)=\sec^3 x+\sec x\tan^2 x$
$=(\sec x)\left(\sec^2 x+\tan^2 x\right)=(\sec x)\left(\sec^2 x+\sec^2 x-1\right)=2\sec^3 x-\sec x$

26. (a) $y = -2 \sin x \Rightarrow y' = -2 \cos x \Rightarrow y'' = -2(-\sin x) = 2 \sin x \Rightarrow y''' = 2 \cos x \Rightarrow y^{(4)} = -2 \sin x$
(b) $y = 9 \cos x \Rightarrow y' = -9 \sin x \Rightarrow y'' = -9 \cos x \Rightarrow y''' = -9(-\sin x) = 9 \sin x \Rightarrow y^{(4)} = 9 \cos x$

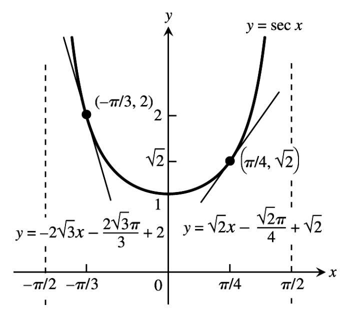
27. $y = \sin x \Rightarrow y' = \cos x \Rightarrow$ slope of tangent at $x = -\pi$ is $y'(-\pi) = \cos(-\pi) = -1$; slope of tangent at $x = 0$ is $y'(0) = \cos(0) = 1$; and slope of tangent at $x = \frac{3\pi}{2}$ is $y'\left(\frac{3\pi}{2}\right) = \cos \frac{3\pi}{2} = 0$. The tangent at $(-\pi, 0)$ is $y - 0 = -1(x + \pi)$, or $y = -x - \pi$; the tangent at $(0, 0)$ is $y - 0 = 1(x - 0)$, or $y = x$; and the tangent at $\left(\frac{3\pi}{2}, -1\right)$ is $y = -1$.



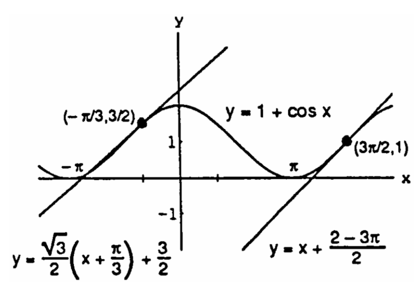
28. $y = \tan x \Rightarrow y' = \sec^2 x \Rightarrow$ slope of tangent at $x = -\frac{\pi}{3}$ is $\sec^2\left(-\frac{\pi}{3}\right) = 4$; slope of tangent at $x = 0$ is $\sec^2(0) = 1$; and slope of tangent at $x = \frac{\pi}{3}$ is $\sec^2\left(\frac{\pi}{3}\right) = 4$. The tangent at $\left(-\frac{\pi}{3}, \tan\left(-\frac{\pi}{3}\right)\right) = \left(-\frac{\pi}{3}, -\sqrt{3}\right)$ is $y + \sqrt{3} = 4\left(x + \frac{\pi}{3}\right)$; the tangent at $(0, 0)$ is $y = x$; and the tangent at $\left(\frac{\pi}{3}, \tan\left(\frac{\pi}{3}\right)\right) = \left(\frac{\pi}{3}, \sqrt{3}\right)$ is $y - \sqrt{3} = 4\left(x - \frac{\pi}{3}\right)$.



29. $y = \sec x \Rightarrow y' = \sec x \tan x \Rightarrow$ slope of tangent at $x = -\frac{\pi}{3}$ is $\sec\left(-\frac{\pi}{3}\right)\tan\left(-\frac{\pi}{3}\right) = -2\sqrt{3}$; slope of tangent at $x = \frac{\pi}{4}$ is $\sec\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{4}\right) = \sqrt{2}$. The tangent at the point $\left(-\frac{\pi}{3}, \sec\left(-\frac{\pi}{3}\right)\right) = \left(-\frac{\pi}{3}, 2\right)$ is $y - 2 = -2\sqrt{3}\left(x + \frac{\pi}{3}\right)$; the tangent at the point $\left(\frac{\pi}{4}, \sec\left(\frac{\pi}{4}\right)\right) = \left(\frac{\pi}{4}, \sqrt{2}\right)$ is $y - \sqrt{2} = \sqrt{2}\left(x - \frac{\pi}{4}\right)$.



30. $y = 1 + \cos x \Rightarrow y' = -\sin x \Rightarrow$ slope of tangent at $x = -\frac{\pi}{3}$ is $-\sin\left(-\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$; slope of tangent at $x = \frac{3\pi}{2}$ is $-\sin\left(\frac{3\pi}{2}\right) = 1$. The tangent at the point $\left(-\frac{\pi}{3}, 1 + \cos\left(-\frac{\pi}{3}\right)\right) = \left(-\frac{\pi}{3}, \frac{3}{2}\right)$ is $y - \frac{3}{2} = \frac{\sqrt{3}}{2}\left(x + \frac{\pi}{3}\right)$; the tangent at the point $\left(\frac{3\pi}{2}, 1 + \cos\left(\frac{3\pi}{2}\right)\right) = \left(\frac{3\pi}{2}, 1\right)$ is $y - 1 = x - \frac{3\pi}{2}$



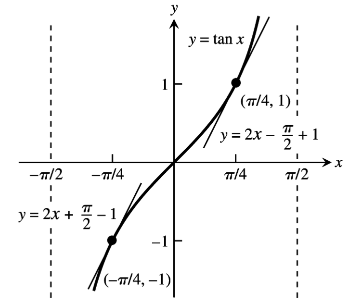
31. Yes, $y = x + \sin x \Rightarrow y' = 1 + \cos x$; horizontal tangent occurs where $1 + \cos x = 0 \Rightarrow \cos x = -1 \Rightarrow x = \pi$

32. No, $y = 2x + \sin x \Rightarrow y' = 2 + \cos x$; horizontal tangent occurs where $2 + \cos x = 0 \Rightarrow \cos x = -2$. But there are no x-values for which $\cos x = -2$.

33. No, $y = x - \cot x \Rightarrow y' = 1 + \csc^2 x$; horizontal tangent occurs where $1 + \csc^2 x = 0 \Rightarrow \csc^2 x = -1$. But there are no x-values for which $\csc^2 x = -1$.

34. Yes, $y = x + 2 \cos x \Rightarrow y' = 1 - 2 \sin x$; horizontal tangent occurs where $1 - 2 \sin x = 0 \Rightarrow 1 = 2 \sin x \Rightarrow \frac{1}{2} = \sin x \Rightarrow x = \frac{\pi}{6}$ or $x = \frac{5\pi}{6}$

35. We want all points on the curve where the tangent line has slope 2. Thus, $y = \tan x \Rightarrow y' = \sec^2 x$ so that $y' = 2 \Rightarrow \sec^2 x = 2 \Rightarrow \sec x = \pm\sqrt{2}$ $\Rightarrow x = \pm\frac{\pi}{4}$. Then the tangent line at $\left(\frac{\pi}{4}, 1\right)$ has equation $y - 1 = 2\left(x - \frac{\pi}{4}\right)$; the tangent line at $\left(-\frac{\pi}{4}, -1\right)$ has equation $y + 1 = 2\left(x + \frac{\pi}{4}\right)$.



36. $y = 4 + \cot x - 2\csc x \Rightarrow y' = -\csc^2 x + 2\csc x \cot x = -\left(\frac{1}{\sin x}\right)\left(\frac{1 - 2\cos x}{\sin x}\right)$
(a) When $x = \frac{\pi}{2}$, then $y' = -1$; the tangent line is $y = -x + \frac{\pi}{2} + 2$.
(b) To find the location of the horizontal tangent set $y' = 0 \Rightarrow 1 - 2\cos x = 0 \Rightarrow x = \frac{\pi}{3}$ radians. When $x = \frac{\pi}{3}$, then $y = 4 - \sqrt{3}$ is the horizontal tangent.

37. $\lim\limits_{x \to 2} \sin\left(\frac{1}{x} - \frac{1}{2}\right) = \sin\left(\frac{1}{2} - \frac{1}{2}\right) = \sin 0 = 0$

38. $\lim\limits_{x \to -\frac{\pi}{6}} \sqrt{1 + \cos(\pi \csc x)} = \sqrt{1 + \cos\left(\pi \csc\left(-\frac{\pi}{6}\right)\right)} = \sqrt{1 + \cos(\pi \cdot (-2))} = \sqrt{2}$

39. $\lim\limits_{x \to 0} \sec\left[e^x + \pi \tan\left(\frac{\pi}{4 \sec x}\right) - 1\right] = \sec\left[1 + \pi \tan\left(\frac{\pi}{4 \sec 0}\right) - 1\right] = = \sec\left[\pi \tan\left(\frac{\pi}{4}\right)\right] = \sec \pi = -1$

40. $\lim\limits_{x \to 0} \sin\left(\frac{\pi + \tan x}{\tan x - 2\sec x}\right) = \sin\left(\frac{\pi + \tan 0}{\tan 0 - 2\sec 0}\right) = \sin\left(-\frac{\pi}{2}\right) = -1$

41. $\lim\limits_{t \to 0} \tan\left(1 - \frac{\sin t}{t}\right) = \tan\left(1 - \lim\limits_{t \to 0} \frac{\sin t}{t}\right) = \tan(1 - 1) = 0$

42. $\lim\limits_{\theta \to 0} \cos\left(\frac{\pi\theta}{\sin\theta}\right) = \cos\left(\pi \lim\limits_{\theta \to 0} \frac{\theta}{\sin\theta}\right) = \cos\left(\pi \cdot \frac{1}{\lim\limits_{\theta \to 0} \frac{\sin\theta}{\theta}}\right) = \cos\left(\pi \cdot \frac{1}{1}\right) = -1$

43. $s = 2 - 2\sin t \Rightarrow v = \frac{ds}{dt} = -2\cos t \Rightarrow a = \frac{dv}{dt} = 2\sin t \Rightarrow j = \frac{da}{dt} = 2\cos t$. Therefore, velocity $= v\left(\frac{\pi}{4}\right)$ $= -\sqrt{2}$ m/sec; speed $= \left|v\left(\frac{\pi}{4}\right)\right| = \sqrt{2}$ m/sec; acceleration $= a\left(\frac{\pi}{4}\right) = \sqrt{2}$ m/sec^2; jerk $= j\left(\frac{\pi}{4}\right) = \sqrt{2}$ m/sec^3.

44. $s = \sin t + \cos t \Rightarrow v = \frac{ds}{dt} = \cos t - \sin t \Rightarrow a = \frac{dv}{dt} = -\sin t - \cos t \Rightarrow j = \frac{da}{dt} = -\cos t + \sin t$. Therefore velocity $= v\left(\frac{\pi}{4}\right) = 0$ m/sec; speed $= \left|v\left(\frac{\pi}{4}\right)\right| = 0$ m/sec; acceleration $= a\left(\frac{\pi}{4}\right) = -\sqrt{2}$ m/sec^2; jerk $= j\left(\frac{\pi}{4}\right) = 0$ m/sec^3.

45. $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} \frac{\sin^2 3x}{x^2} = \lim\limits_{x \to 0} 9\left(\frac{\sin 3x}{3x}\right)\left(\frac{\sin 3x}{3x}\right) = 9$ so that f is continuous at $x = 0 \Rightarrow \lim\limits_{x \to 0} f(x) = f(0) \Rightarrow 9 = c$.

46. $\lim\limits_{x \to 0^-} g(x) = \lim\limits_{x \to 0^-} (x + b) = b$ and $\lim\limits_{x \to 0^+} g(x) = \lim\limits_{x \to 0^+} \cos x = 1$ so that g is continuous at $x = 0 \Rightarrow \lim\limits_{x \to 0^-} g(x)$ $= \lim\limits_{x \to 0^+} g(x) \Rightarrow b = 1$. Now g is not differentiable at $x = 0$: At $x = 0$, the left-hand derivative is $\frac{d}{dx}(x + b)\Big|_{x=0} = 1$, but the right-hand derivative is $\frac{d}{dx}(\cos x)\Big|_{x=0} = -\sin 0 = 0$. The left- and right-hand derivatives can never agree at $x = 0$, so g is not differentiable at $x = 0$ for any value of b (including $b = 1$).

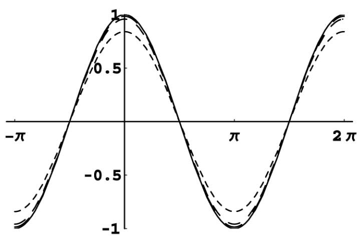
47. $\frac{d^{999}}{dx^{999}}(\cos x) = \sin x$ because $\frac{d^4}{dx^4}(\cos x) = \cos x \Rightarrow$ the derivative of cos x any number of times that is a multiple of 4 is cos x. Thus, dividing 999 by 4 gives $999 = 249 \cdot 4 + 3 \Rightarrow \frac{d^{999}}{dx^{999}}(\cos x) = \frac{d^3}{dx^3}\left[\frac{d^{249 \cdot 4}}{dx^{249 \cdot 4}}(\cos x)\right] = \frac{d^3}{dx^3}(\cos x) = \sin x$.

48. (a) $y = \sec x = \frac{1}{\cos x} \Rightarrow \frac{dy}{dx} = \frac{(\cos x)(0)-(1)(-\sin x)}{(\cos x)^2} = \frac{\sin x}{\cos^2 x} = \left(\frac{1}{\cos x}\right)\left(\frac{\sin x}{\cos x}\right) = \sec x \tan x \Rightarrow \frac{d}{dx}(\sec x) = \sec x \tan x$

(b) $y = \csc x = \frac{1}{\sin x} \Rightarrow \frac{dy}{dx} = \frac{(\sin x)(0)-(1)(\cos x)}{(\sin x)^2} = \frac{-\cos x}{\sin^2 x} = \left(\frac{-1}{\sin x}\right)\left(\frac{\cos x}{\sin x}\right) = -\csc x \cot x \Rightarrow \frac{d}{dx}(\csc x) = -\csc x \cot x$

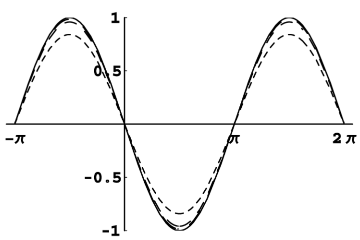
(c) $y = \cot x = \frac{\cos x}{\sin x} \Rightarrow \frac{dy}{dx} = \frac{(\sin x)(-\sin x)-(\cos x)(\cos x)}{(\sin x)^2} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \frac{-1}{\sin^2 x} = -\csc^2 x \Rightarrow \frac{d}{dx}(\cot x) = -\csc^2 x$

49. (a)



The dashed curves of $y = \frac{\sin(x+h)-\sin(x-h)}{2h}$ are closer to the black curve $y = \cos x$ than the corresponding dashed curves in Exercise 51 illustrating that the centered difference quotient is a better approximation of the derivative of this function.

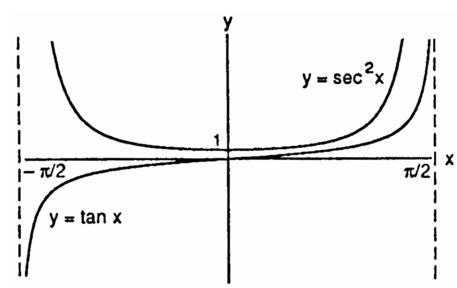
(b)



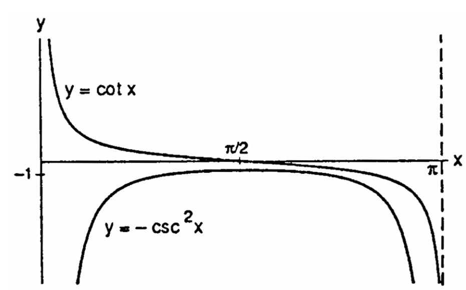
The dashed curves of $y = \frac{\cos(x+h)-\cos(x-h)}{2h}$ are closer to the black curve $y = -\sin x$ than the corresponding dashed curves in Exercise 52 illustrating that the centered difference quotient is a better approximation of the derivative of this function.

50. $\lim\limits_{h \to 0} \frac{|0+h|-|0-h|}{2h} = \lim\limits_{x \to 0} \frac{|h|-|h|}{2h} = \lim\limits_{h \to 0} 0 = 0 \Rightarrow$ the limits of the centered difference quotient exists even though the derivative of $f(x) = |x|$ does not exist at $x = 0$.

51. $y = \tan x \Rightarrow y' = \sec^2 x$, so the smallest value $y' = \sec^2 x$ takes on is $y' = 1$ when $x = 0$; y' has no maximum value since $\sec^2 x$ has no largest value on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$; y' is never negative since $\sec^2 x \geq 1$.

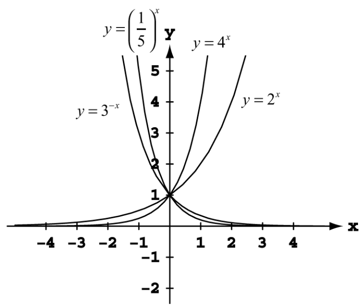


52. $y = \cot x \Rightarrow y' = -\csc^2 x$ so y' has no smallest value since $-\csc^2 x$ has no minimum value on $(0, \pi)$; the largest value of y' is -1, when $x = \frac{\pi}{2}$; the slope is never positive since the largest value $y' = -\csc^2 x$ takes on is -1.

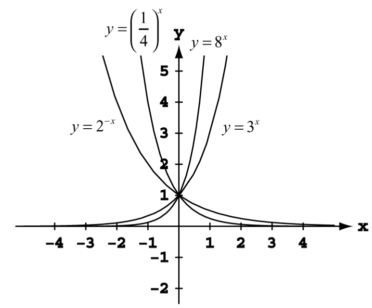


2.6 EXPONENTIAL FUNCTIONS

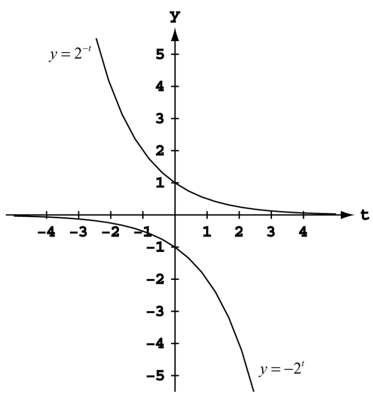
1.



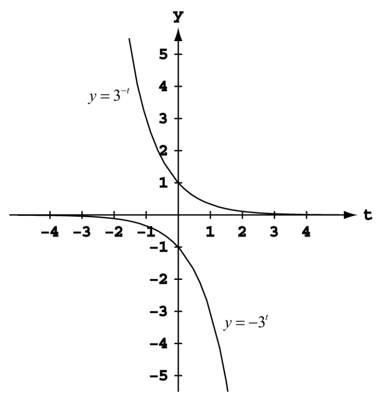
2.



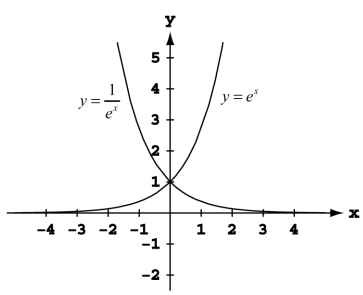
3.



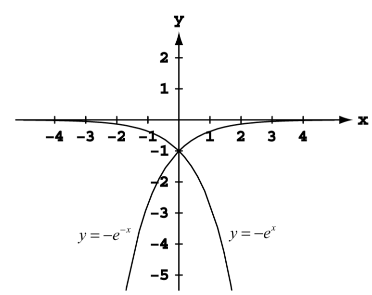
4.



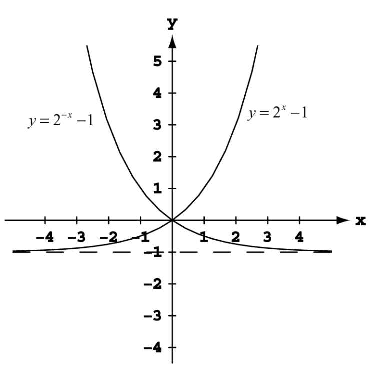
5.



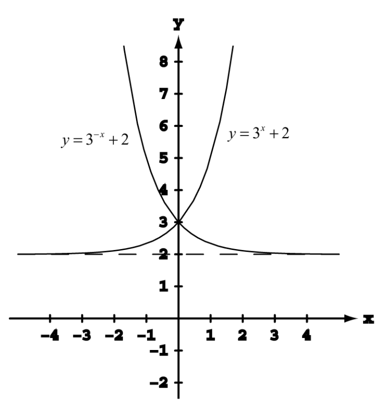
6.



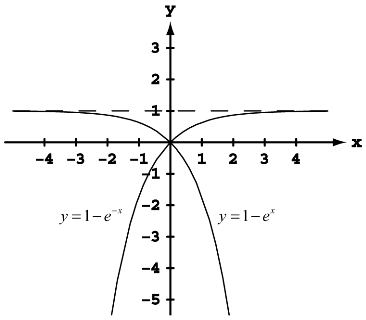
7.



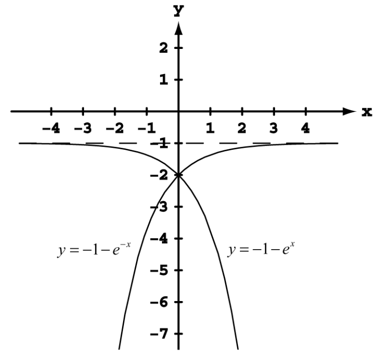
8.



9.



10.



11. $16^2 \cdot 16^{-1.75} = 16^{2+(-1.75)} = 16^{0.25} = 16^{1/4} = 2$

12. $9^{1/3} \cdot 9^{1/6} = 9^{\frac{1}{3}+\frac{1}{6}} = 9^{1/2} = 3$

13. $\frac{4^{4.2}}{4^{3.7}} = 4^{4.2-3.7} = 4^{0.5} = 4^{1/2} = 2$

14. $\frac{3^{5/3}}{3^{2/3}} = 3^{\frac{5}{3}-\frac{2}{3}} = 3^1 = 3$

15. $\left(25^{1/8}\right)^4 = 25^{4/8} = 25^{1/2} = 5$

16. $\left(13^{\sqrt{2}}\right)^{\sqrt{2}/2} = 13^{2/2} = 13$

17. $2^{\sqrt{3}} \cdot 7^{\sqrt{3}} = (2 \cdot 7)^{\sqrt{3}} = 14^{\sqrt{3}}$

18. $\left(\sqrt{3}\right)^{1/2}\left(\sqrt{12}\right)^{1/2} = \left(\sqrt{3} \cdot \sqrt{12}\right)^{1/2} = \left(\sqrt{36}\right)^{1/2} = 6^{1/2}$

19. $\left(\frac{2}{\sqrt{2}}\right)^4 = \frac{2^4}{\left(2^{1/2}\right)^4} = \frac{16}{2^2} = 4$

20. $\left(\frac{\sqrt{6}}{3}\right)^2 = \frac{\left(6^{1/2}\right)^2}{3^2} = \frac{6}{9} = \frac{2}{3}$

21. Domain: $(-\infty, \infty)$; y in range $\Rightarrow y = \frac{1}{2 + e^x}$. As x increases, e^x becomes infinitely large and y becomes a smaller and smaller positive real number. As x decreases, e^x becomes a smaller and smaller positive real number, $y < \frac{1}{2}$, and y gets arbitrarily close to $\frac{1}{2} \Rightarrow$ Range: $\left(0, \frac{1}{2}\right)$.

22. Domain: $(-\infty, \infty)$; y in range $\Rightarrow y = \cos(e^{-t})$. Since the values of e^{-t} are $(0, \infty)$ and $-1 \le \cos x \le 1 \Rightarrow$ Range: $[-1, 1]$.

23. Domain: $(-\infty, \infty)$; y in range $\Rightarrow y = \sqrt{1 + 3^{-t}}$. Since the values of 3^{-t} are $(0, \infty)$ $\Rightarrow$ Range: $(1, \infty)$.

24. If $e^{2x} = 1$, then $x = 0 \Rightarrow$ Domain: $(-\infty, 0) \cup (0, \infty)$; y in range $\Rightarrow y = \frac{3}{1 - e^{2x}}$. If $x > 0$, then $1 < e^{2x} < \infty$ $\Rightarrow -\infty < y < 0$. If $x < 0$, then $0 < e^{2x} < 1 \Rightarrow 3 < y < \infty \Rightarrow$ Range: $(-\infty, 0) \cup (3, \infty)$.

25. $y = x^3 e^x \Rightarrow y' = x^3 \cdot e^x + 3x^2 \cdot e^x = (x^3 + 3x^2)e^x$

26. $w = re^{-r} \Rightarrow w' = r \cdot e^{-r}(-1) + (1) \cdot e^{-r} = (1 - r)e^{-r}$

277. $y = x^{9/4} \Rightarrow y' = \frac{9}{4}x^{5/4}$

28. $y = x^{-3/5} + \pi^{3/2} \Rightarrow y' = -\frac{3}{5}x^{-8/5} + 0 \Rightarrow y' = -\frac{3}{5}x^{-8/5}$

29. $s = 2t^{3/2} + 3e^2 \Rightarrow s' = 3t^{1/2} + 0 \Rightarrow s' = 3t^{1/2}$

30. $w = \frac{1}{z^{1.4}} + \frac{\pi}{\sqrt{z}} = z^{-1.4} + \pi z^{-1/2} \Rightarrow w' = -1.4z^{-2.4} - \frac{\pi}{2} z^{-3/2} \Rightarrow w' = -\frac{1.4}{z^{2.4}} - \frac{\pi}{2z^{3/2}}$

31. $y = \sqrt[7]{x^2} - x^e = x^{2/7} - x^e \Rightarrow y' = \frac{2}{7} x^{-5/7} - e\, x^{e-1} \Rightarrow y' = \frac{2}{7x^{5/7}} - e\, x^{e-1}$

32. $y = \sqrt[3]{x^{9.6}} + 2e^{1.3} = x^{9.6/3} + 2e^{1.3} = x^{3.2} + 2e^{1.3} \Rightarrow y' = 3.2x^{2.2} + 0 \Rightarrow y' = 3.2x^{2.2}$

33. $r = \frac{e^s}{s} \Rightarrow r' = \frac{s \cdot e^s - e^s(1)}{s^2} \Rightarrow r' = \frac{e^s(s-1)}{s^2}$

34. $r = e^{\theta}\left(\frac{1}{\theta^2} + \theta^{-\pi/2}\right) = e^{\theta}\left(\theta^{-2} + \theta^{-\pi/2}\right) \Rightarrow r' = e^{\theta}\left(-2\theta^{-3} - \frac{\pi}{2}\,\theta^{-\pi/2-1}\right) + e^{\theta}\left(\theta^{-2} + \theta^{-\pi/2}\right)$
$\Rightarrow r' = e^{\theta}\left(-2\theta^{-3} - \frac{\pi}{2}\,\theta^{-\pi/2} + \theta^{-2} + \theta^{-\pi/2}\right) \Rightarrow r' = e^{\theta}\left(-\frac{2}{\theta^3} - \frac{\pi}{2\theta^{\pi/2+1}} + \frac{1}{\theta^2} + \frac{1}{\theta^{\pi/2}}\right)$
$\Rightarrow r' = e^{\theta}\left(\frac{\theta-2}{\theta^3} + \frac{2\theta-\pi}{2\theta^{\pi/2+1}}\right)$

35. $y = \frac{1}{2}x^4 - \frac{3}{2}x^2 - x \;\Rightarrow\; y' = 2x^3 - 3x - 1 \;\Rightarrow\; y'' = 6x^2 - 3 \;\Rightarrow\; y''' = 12x \;\Rightarrow\; y^{(4)} = 12 \;\Rightarrow\; y^{(n)} = 0$ for all $n \geq 5$

36. $y = \frac{1}{120}x^5 \;\Rightarrow\; y' = \frac{1}{24}x^4 \;\Rightarrow\; y'' = \frac{1}{6}x^3 \;\Rightarrow\; y''' = \frac{1}{2}x^2 \;\Rightarrow\; y^{(4)} = x \;\Rightarrow\; y^{(5)} = 1 \;\Rightarrow\; y^{(n)} = 0$ for all $n \geq 6$

37. $y = \frac{x^3+7}{x} = x^2 + 7x^{-1} \;\Rightarrow\; \frac{dy}{dx} = 2x - 7x^{-2} = 2x - \frac{7}{x^2} \Rightarrow \frac{d^2y}{dx^2} = 2 + 14x^{-3} = 2 + \frac{14}{x^3}$

38. $s = \frac{t^2+5t-1}{t^2} = 1 + \frac{5}{t} - \frac{1}{t^2} = 1 + 5t^{-1} - t^{-2} \;\Rightarrow\; \frac{ds}{dt} = 0 - 5t^{-2} + 2t^{-3} = -5t^{-2} + 2t^{-3} = \frac{-5}{t^2} + \frac{2}{t^3}$
$\Rightarrow \frac{d^2s}{dt^2} = 10t^{-3} - 6t^{-4} = \frac{10}{t^3} - \frac{6}{t^4}$

39. $r = \frac{(\theta-1)(\theta^2+\theta+1)}{\theta^3} = \frac{\theta^3-1}{\theta^3} = 1 - \frac{1}{\theta^3} = 1 - \theta^{-3} \;\Rightarrow\; \frac{dr}{d\theta} = 0 + 3\theta^{-4} = 3\theta^{-4} = \frac{3}{\theta^4} \Rightarrow \frac{d^2r}{d\theta^2} = -12\theta^{-5} = \frac{-12}{\theta^5}$

40. $u = \frac{(x^2+x)(x^2-x+1)}{x^4} = \frac{x(x+1)(x^2-x+1)}{x^4} = \frac{x(x^3+1)}{x^4} = \frac{x^4+x}{x^4} = 1 + \frac{x}{x^4} = 1 + x^{-3}$
$\Rightarrow \frac{du}{dx} = 0 - 3x^{-4} = -3x^{-4} = \frac{-3}{x^4} \Rightarrow \frac{d^2u}{dx^2} = 12x^{-5} = \frac{12}{x^5}$

41. $w = 3z^2e^z \Rightarrow w' = 3z^2 \cdot e^z + 6z \cdot e^z = (3z^2+6z)e^z$ and $w'' = (3z^2+6z)e^z + (6z+6)e^z = (3z^2+12z+6)e^z$

42. $w = e^z(z-1)(z^2+1) = e^z(z^3-z^2+z-1) \Rightarrow w' = e^z \cdot (z^3-z^2+z-1) + e^z \cdot (3z^2-2z+1) = e^z(z^3+2z^2-z)$
and $w'' = e^z \cdot (z^3+2z^2-z) + e^z \cdot (3z^2+4z-1) = e^z(z^3+5z^2+3z-1)$

43. $p = \left(\frac{q^2+3}{12q}\right)\left(\frac{q^4-1}{q^3}\right) = \frac{q^6-q^2+3q^4-3}{12q^4} = \frac{1}{12}q^2 - \frac{1}{12}q^{-2} + \frac{1}{4} - \frac{1}{4}q^{-4} \;\Rightarrow\; \frac{dp}{dq} = \frac{1}{6}q + \frac{1}{6}q^{-3} + q^{-5} = \frac{1}{6}q + \frac{1}{6q^3} + \frac{1}{q^5}$
$\Rightarrow \frac{d^2p}{dq^2} = \frac{1}{6} - \frac{1}{2}q^{-4} - 5q^{-6} = \frac{1}{6} - \frac{1}{2q^4} - \frac{5}{q^6}$

44. $p = \frac{q^2+3}{(q-1)^3+(q+1)^3} = \frac{q^2+3}{(q^3-3q^2+3q-1)+(q^3+3q^2+3q+1)} = \frac{q^2+3}{2q^3+6q} = \frac{q^2+3}{2q(q^2+3)} = \frac{1}{2q} = \frac{1}{2}q^{-1}$
$\Rightarrow \frac{dp}{dq} = -\frac{1}{2}q^{-2} = -\frac{1}{2q^2} \Rightarrow \frac{d^2p}{dq^2} = q^{-3} = \frac{1}{q^3}$

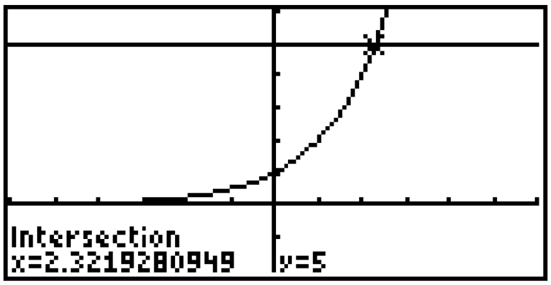
45. $\frac{dy}{dx} = e^{-x}(2x) + (x^2-3)e^{-x}(-1) = e^{-x}(2x-(x^2-3)) = e^{-x}(3+2x-x^2)$
$\Rightarrow \frac{d^2y}{dx^2} = e^{-x}(2-2x) + (3+2x-x^2)e^{-x}(-1) = e^{-x}(2-2x-3-2x+x^2) = e^{-x}(x^2-4x-1)$

46. $\frac{dy}{dx} = \frac{(x^2+1)e^{-x} - (1-e^{-x})(2x)}{(x^2+1)^2} = \frac{x^2e^{-x} + e^{-x} - 2x + 2xe^{-x}}{(x^2+1)^2} = \frac{e^{-x}(x^2+2x+1) - 2x}{(x^2+1)^2}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{(x^2+1)^2(e^{-x}(2x+2) - e^{-x}(x^2+2x+1) - 2) - (e^{-x}(x^2+2x+1) - 2x)2(x^2+1)(2x)}{\left[(x^2+1)^2\right]^2}$

$= \frac{(x^2+1)(e^{-x}(1-x^2) - 2) - (e^{-x}(4x^3+8x^2+4x) - 8x^2)}{(x^2+1)^3} = \frac{e^{-x}(-x^4 - 4x^3 - 8x^2 - 4x + 1) + 6x^2 - 2}{(x^2+1)^3}$

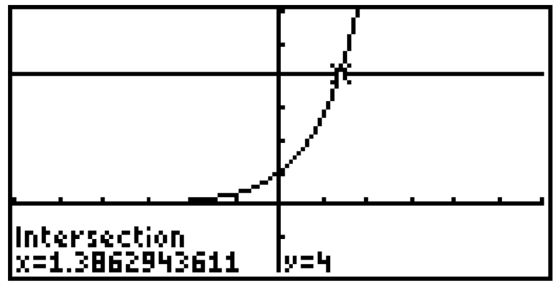
47.



[-6, 6] by [-2, 6]

$x \approx 2.3219$

48.



[-6, 6] by [-2, 6]

$x \approx 1.3863$

49. Let t be the number of years. Solving $500{,}000(1.0375)^t = 1{,}000{,}000$ graphically, we find that $t \approx 18.828$. The population will reach 1 million in about 19 years.

50. (a) $A(t) = 6.6\left(\frac{1}{2}\right)^{t/14}$

(b) Solving $A(t) = 1$ graphically, we find that $t \approx 38$. There will be 1 gram remaining after about 38.1145 days.

51. Let A be the amount of the initial investment, and let t be the number of years. We wish to solve $A(1.0625)^t = 2A$, which is equivalent to $1.0625^t = 2$. Solving graphically, we find that $t \approx 11.433$. It will take about 11.433 years. (If the interest is credited at the end of each year, it will take 12 years.)

52. After t hours, the population is $P(t) = 2^{t/0.5}$, or equivalently, $P(t) = 2^{2t}$. After 24 hours, the population is $P(24) = 2^{48} \approx 2.815 \times 10^{14}$ bacteria.

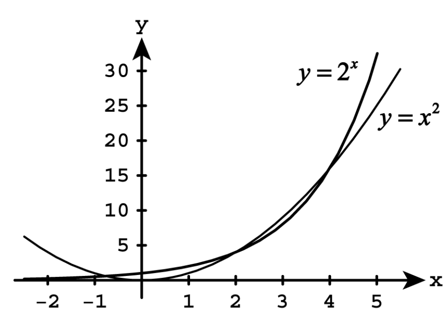
53. $-\frac{1}{e^x} \le \frac{\sin x}{e^x} \le \frac{1}{e^x} \Rightarrow \lim_{x \to \infty} e^{-x}\sin x = 0$ by the Sandwich Theorem.

54. $\lim_{x \to -\infty} (e^x)\left(\cos^2\left(\frac{1}{x}\right)\right) = 0 \cdot \cos^2 0 = 0 \cdot (1)^2 = 0$

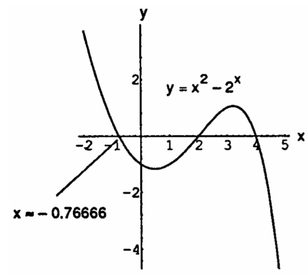
55. $\lim_{x \to -\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{x \to -\infty} \frac{e^x - \frac{1}{e^x}}{e^x + \frac{1}{e^x}} = \lim_{x \to -\infty} \frac{\frac{e^{2x}-1}{e^x}}{\frac{e^{2x}+1}{e^x}} = \lim_{x \to -\infty} \frac{e^{2x}-1}{e^{2x}+1} = \frac{0-1}{0+1} = -1$

56. $\lim_{x \to \infty} \frac{3x^2 + e^{-x}}{\sin(1/x) - 2x^2} = \lim_{x \to \infty} \frac{3 + \frac{1}{x^2e^x}}{\frac{\sin(1/x)}{x^2} - 2} = \frac{3+0}{0-2} = -\frac{3}{2}$

57. From the graph at the right, can see that the exponential function $y = 2^x$ grows more rapidly than $y = x^2$ if $x > 4$.



58. From zooming in on the graph at the right, we estimate the third root to be $x \approx -0.76666$



2.7 THE CHAIN RULE

1. $f(u) = 6u - 9 \Rightarrow f'(u) = 6 \Rightarrow f'(g(x)) = 6$; $g(x) = \frac{1}{2}x^4 \Rightarrow g'(x) = 2x^3$; therefore $\frac{dy}{dx} = f'(g(x))g'(x) = 6 \cdot 2x^3 = 12x^3$

2. $f(u) = 2u^3 \Rightarrow f'(u) = 6u^2 \Rightarrow f'(g(x)) = 6(8x-1)^2$; $g(x) = 8x - 1 \Rightarrow g'(x) = 8$; therefore $\frac{dy}{dx} = f'(g(x))g'(x)$ $= 6(8x-1)^2 \cdot 8 = 48(8x-1)^2$

3. $f(u) = \sin u \Rightarrow f'(u) = \cos u \Rightarrow f'(g(x)) = \cos(3x+1)$; $g(x) = 3x + 1 \Rightarrow g'(x) = 3$; therefore $\frac{dy}{dx} = f'(g(x))g'(x)$ $= (\cos(3x+1))(3) = 3\cos(3x+1)$

4. $f(u) = \cos u \Rightarrow f'(u) = -\sin u \Rightarrow f'(g(x)) = -\sin\left(\frac{-x}{3}\right)$; $g(x) = \frac{-x}{3} \Rightarrow g'(x) = -\frac{1}{3}$; therefore $\frac{dy}{dx} = f'(g(x))g'(x)$ $= -\sin\left(\frac{-x}{3}\right) \cdot \left(\frac{-1}{3}\right) = \frac{1}{3}\sin\left(\frac{-x}{3}\right)$

5. $f(u) = \cos u \Rightarrow f'(u) = -\sin u \Rightarrow f'(g(x)) = -\sin(\sin x)$; $g(x) = \sin x \Rightarrow g'(x) = \cos x$; therefore $\frac{dy}{dx} = f'(g(x))g'(x) = -(\sin(\sin x))\cos x$

6. $f(u) = \sin u \Rightarrow f'(u) = \cos u \Rightarrow f'(g(x)) = \cos(x - \cos x)$; $g(x) = x - \cos x \Rightarrow g'(x) = 1 + \sin x$; therefore $\frac{dy}{dx} = f'(g(x))g'(x) = (\cos(x - \cos x))(1 + \sin x)$

7. $f(u) = \tan u \Rightarrow f'(u) = \sec^2 u \Rightarrow f'(g(x)) = \sec^2(10x-5)$; $g(x) = 10x - 5 \Rightarrow g'(x) = 10$; therefore $\frac{dy}{dx} = f'(g(x))g'(x) = \left(\sec^2(10x-5)\right)(10) = 10\sec^2(10x-5)$

8. $f(u) = -\sec u \Rightarrow f'(u) = -\sec u \tan u \Rightarrow f'(g(x)) = -\sec(x^2+7x)\tan(x^2+7x)$; $g(x) = x^2 + 7x$ $\Rightarrow g'(x) = 2x + 7$; therefore $\frac{dy}{dx} = f'(g(x))g'(x) = -(2x+7)\sec(x^2+7x)\tan(x^2+7x)$

9. With $u = (2x+1)$, $y = u^5$: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 5u^4 \cdot 2 = 10(2x+1)^4$

10. With $u = (2x)$, $y = \sqrt[3]{u} = u^{1/3} \Rightarrow \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{3}u^{-2/3} \cdot 2 = \frac{2^{1/3}}{3x^{2/3}}$

11. With $u = \left(1 - \frac{x}{7}\right)$, $y = u^{-7}$: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -7u^{-8} \cdot \left(-\frac{1}{7}\right) = \left(1 - \frac{x}{7}\right)^{-8}$

12. With $u = \left(\frac{x}{2} - 1\right)$, $y = u^{-10}$: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -10u^{-11} \cdot \left(\frac{1}{2}\right) = -5\left(\frac{x}{2} - 1\right)^{-11}$

13. With $u = (1 - 6x)$, $y = u^{2/3} \Rightarrow \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{2}{3}u^{-1/3}(-6) = -4(1-6x)^{-1/3}$

14. With $u = \left(\frac{x}{5} + \frac{1}{5x}\right)$, $y = u^5$: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 5u^4 \cdot \left(\frac{1}{5} - \frac{1}{5x^2}\right) = \left(\frac{x}{5} + \frac{1}{5x}\right)^4\left(1 - \frac{1}{x^2}\right)$

15. With $u = \tan x$, $y = \sec u$: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (\sec u \tan u)(\sec^2 x) = (\sec(\tan x)\tan(\tan x))\sec^2 x$

16. With $u = \pi - \frac{1}{x}$, $y = \cot u$: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (-\csc^2 u)\left(\frac{1}{x^2}\right) = -\frac{1}{x^2}\csc^2\left(\pi - \frac{1}{x}\right)$

17. With $u = \sin x$, $y = u^3$: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 3u^2 \cos x = 3(\sin^2 x)(\cos x)$

18. With $u = \cos x$, $y = 5u^{-4}$: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (-20u^{-5})(-\sin x) = 20(\cos^{-5} x)(\sin x)$

19. With $u = -5x$, $y = e^u$: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = e^u(-5) = -5e^{-5x}$

20. With $u = \frac{2x}{3}$, $y = e^u$: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = e^u\left(\frac{2}{3}\right) = \frac{2}{3}e^{2x/3}$

21. With $u = 5-7x$, $y = e^u$: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = e^u(-7) = -7e^{5-7x}$

22. With $u = 4\sqrt{x}+x^2$, $y = e^u$: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = e^u\left(4 \cdot \frac{1}{2}x^{-1/2} + 2x\right) = \left(\frac{2}{\sqrt{x}} + 2x\right)e^{(4\sqrt{x}+x^2)}$

23. $p = \sqrt{3-t} = (3-t)^{1/2} \Rightarrow \frac{dp}{dt} = \frac{1}{2}(3-t)^{-1/2} \cdot \frac{d}{dt}(3-t) = -\frac{1}{2}(3-t)^{-1/2} = \frac{-1}{2\sqrt{3-t}}$

24. $y = x(x^2+1)^{1/2} \Rightarrow y' = x \cdot \frac{1}{2}(x^2+1)^{-1/2}(2x) + (x^2+1)^{1/2} \cdot 1 = (x^2+1)^{-1/2}(x^2+x^2+1) = \frac{2x^2+1}{\sqrt{x^2+1}}$

25. $s = \frac{4}{3\pi}\sin 3t + \frac{4}{5\pi}\cos 5t \Rightarrow \frac{ds}{dt} = \frac{4}{3\pi}\cos 3t \cdot \frac{d}{dt}(3t) + \frac{4}{5\pi}(-\sin 5t) \cdot \frac{d}{dt}(5t) = \frac{4}{\pi}\cos 3t - \frac{4}{\pi}\sin 5t = \frac{4}{\pi}(\cos 3t - \sin 5t)$

26. $y = \sin\left((2t+5)^{-2/3}\right) \Rightarrow \frac{dy}{dt} = \cos\left((2t+5)^{-2/3}\right) \cdot \left(-\frac{2}{3}\right)(2t+5)^{-5/3} \cdot 2 = -\frac{4}{3}(2t+5)^{-5/3}\cos\left((2t+5)^{-2/3}\right)$

27. $r = (\csc\theta + \cot\theta)^{-1} \Rightarrow \frac{dr}{d\theta} = -(\csc\theta + \cot\theta)^{-2}\frac{d}{d\theta}(\csc\theta + \cot\theta) = \frac{\csc\theta\cot\theta + \csc^2\theta}{(\csc\theta + \cot\theta)^2} = \frac{\csc\theta(\cot\theta + \csc\theta)}{(\csc\theta + \cot\theta)^2} = \frac{\csc\theta}{\csc\theta + \cot\theta}$

28. $r = -(\sec\theta + \tan\theta)^{-1} \Rightarrow \frac{dr}{d\theta} = (\sec\theta + \tan\theta)^{-2}\frac{d}{d\theta}(\sec\theta + \tan\theta) = \frac{\sec\theta\tan\theta + \sec^2\theta}{(\sec\theta + \tan\theta)^2} = \frac{\sec\theta(\tan\theta + \sec\theta)}{(\sec\theta + \tan\theta)^2} = \frac{\sec\theta}{\sec\theta + \tan\theta}$

29. $y = x^2\sin^4 x + x\cos^{-2} x \Rightarrow \frac{dy}{dx} = x^2\frac{d}{dx}(\sin^4 x) + \sin^4 x \cdot \frac{d}{dx}(x^2) + x\frac{d}{dx}(\cos^{-2} x) + \cos^{-2} x \cdot \frac{d}{dx}(x)$
$= x^2\left(4\sin^3 x \frac{d}{dx}(\sin x)\right) + 2x\sin^4 x + x\left(-2\cos^{-3} x \cdot \frac{d}{dx}(\cos x)\right) + \cos^{-2} x$
$= x^2(4\sin^3 x\cos x) + 2x\sin^4 x + x((-2\cos^{-3} x)(-\sin x)) + \cos^{-2} x$
$= 4x^2\sin^3 x\cos x + 2x\sin^4 x + 2x\sin x\cos^{-3} x + \cos^{-2} x$

30. $z = \cos\left((1-6t)^{2/3}\right) \Rightarrow \frac{dz}{dt} = -\sin\left((1-6t)^{2/3}\right) \cdot \frac{2}{3}(1-6t)^{-1/3}(-6) = 4(1-6t)^{-1/3}\sin\left((1-6t)^{2/3}\right)$

31. $y = \frac{1}{21}(3x-2)^7 + \left(4 - \frac{1}{2x^2}\right)^{-1} \Rightarrow \frac{dy}{dx} = \frac{7}{21}(3x-2)^6 \cdot \frac{d}{dx}(3x-2) + (-1)\left(4 - \frac{1}{2x^2}\right)^{-2} \cdot \frac{d}{dx}\left(4 - \frac{1}{2x^2}\right)$
$= \frac{7}{21}(3x-2)^6 \cdot 3 + (-1)\left(4 - \frac{1}{2x^2}\right)^{-2}\left(\frac{1}{x^3}\right) = (3x-2)^6 - \frac{1}{x^3\left(4 - \frac{1}{2x^2}\right)^2}$

32. $g(x) = 2\left(2x^{-1/2}+1\right)^{-1/3} \Rightarrow g'(x) = -\frac{2}{3}\left(2x^{-1/2}+1\right)^{-4/3} \cdot (-1)x^{-3/2} = \frac{2}{3}\left(2x^{-1/2}+1\right)^{-4/3}x^{-3/2}$

33. $y = (4x+3)^4(x+1)^{-3} \Rightarrow \frac{dy}{dx} = (4x+3)^4(-3)(x+1)^{-4} \cdot \frac{d}{dx}(x+1) + (x+1)^{-3}(4)(4x+3)^3 \cdot \frac{d}{dx}(4x+3)$
$= (4x+3)^4(-3)(x+1)^{-4}(1) + (x+1)^{-3}(4)(4x+3)^3(4) = -3(4x+3)^4(x+1)^{-4} + 16(4x+3)^3(x+1)^{-3}$
$= \frac{(4x+3)^3}{(x+1)^4}[-3(4x+3) + 16(x+1)] = \frac{(4x+3)^3(4x+7)}{(x+1)^4}$

34. $y = x\left(x^2+1\right)^{-1/2} \Rightarrow y' = x \cdot \left(-\frac{1}{2}\right)\left(x^2+1\right)^{-3/2}(2x) + \left(x^2+1\right)^{-1/2} \cdot 1 = \left(x^2+1\right)^{-3/2}\left(-x^2+x^2+1\right) = \frac{1}{\left(x^2+1\right)^{3/2}}$

35. $y = xe^{-x} + e^{3x} \Rightarrow y' = x \cdot e^{-x}(-1) + (1) \cdot e^{-x} + 3e^{3x} = (1-x)e^{-x} + 3e^{3x}$

36. $y = (1+2x)e^{-2x} \Rightarrow y' = (1+2x) \cdot e^{-2x}(-2) + (2) \cdot e^{-2x} = -4xe^{-2x}$

37. $y = \left(x^2-2x+2\right)e^{5x/2} \Rightarrow y' = \left(x^2-2x+2\right) \cdot e^{5x/2}\left(\frac{5}{2}\right) + (2x-2) \cdot e^{5x/2} = \left(\frac{5}{2}x^2 - 3x + 3\right)e^{5x/2}$

38. $y = \left(9x^2-6x+2\right)e^{x^3} \Rightarrow y' = \left(9x^2-6x+2\right) \cdot e^{x^3}\left(3x^2\right) + (18x-6) \cdot e^{x^3} = \left(27x^4 - 18x^3 + 6x^2 + 18x - 6\right)e^{x^3}$

39. $h(x) = x \tan\left(2\sqrt{x}\right) + 7 \Rightarrow h'(x) = x \frac{d}{dx}\left(\tan\left(2x^{1/2}\right)\right) + \tan\left(2x^{1/2}\right) \cdot \frac{d}{dx}(x) + 0$
$= x \sec^2\left(2x^{1/2}\right) \cdot \frac{d}{dx}\left(2x^{1/2}\right) + \tan\left(2x^{1/2}\right) = x \sec^2\left(2\sqrt{x}\right) \cdot \frac{1}{\sqrt{x}} + \tan\left(2\sqrt{x}\right) = \sqrt{x}\sec^2\left(2\sqrt{x}\right) + \tan\left(2\sqrt{x}\right)$

40. $k(x) = x^2 \sec\left(\frac{1}{x}\right) \Rightarrow k'(x) = x^2 \frac{d}{dx}\left(\sec\frac{1}{x}\right) + \sec\left(\frac{1}{x}\right) \cdot \frac{d}{dx}\left(x^2\right) = x^2 \sec\left(\frac{1}{x}\right)\tan\left(\frac{1}{x}\right) \cdot \frac{d}{dx}\left(\frac{1}{x}\right) + 2x \sec\left(\frac{1}{x}\right)$
$= x^2 \sec\left(\frac{1}{x}\right)\tan\left(\frac{1}{x}\right) \cdot \left(-\frac{1}{x^2}\right) + 2x\sec\left(\frac{1}{x}\right) = 2x\sec\left(\frac{1}{x}\right) - \sec\left(\frac{1}{x}\right)\tan\left(\frac{1}{x}\right)$

41. $f(\theta) = \left(\frac{\sin\theta}{1+\cos\theta}\right)^2 \Rightarrow f'(\theta) = 2\left(\frac{\sin\theta}{1+\cos\theta}\right) \cdot \frac{d}{d\theta}\left(\frac{\sin\theta}{1+\cos\theta}\right) = \frac{2\sin\theta}{1+\cos\theta} \cdot \frac{(1+\cos\theta)(\cos\theta)-(\sin\theta)(-\sin\theta)}{(1+\cos\theta)^2}$
$= \frac{(2\sin\theta)\left(\cos\theta+\cos^2\theta+\sin^2\theta\right)}{(1+\cos\theta)^3} = \frac{(2\sin\theta)(\cos\theta+1)}{(1+\cos\theta)^3} = \frac{2\sin\theta}{(1+\cos\theta)^2}$

42. $g(t) = \left(\frac{1+\cos t}{\sin t}\right)^{-1} \Rightarrow g'(t) = -\left(\frac{1+\cos t}{\sin t}\right)^{-2} \cdot \frac{d}{dt}\left(\frac{1+\cos t}{\sin t}\right) = -\frac{\sin^2 t}{(1+\cos t)^2} \cdot \frac{(\sin t)(-\sin t)-(1+\cos t)(\cos t)}{(\sin t)^2}$
$= \frac{-\left(-\sin^2 t - \cos t - \cos^2 t\right)}{(1+\cos t)^2} = \frac{1}{1+\cos t}$

43. $h(\theta) = \sqrt[3]{1+\cos(2\theta)} = (1+\cos 2\theta)^{1/3} \Rightarrow h'(\theta) = \frac{1}{3}(1+\cos 2\theta)^{-2/3} \cdot (-\sin 2\theta) \cdot 2 = -\frac{2}{3}(\sin 2\theta)(1+\cos 2\theta)^{-2/3}$

44. $r = \left(\sec\sqrt{\theta}\right)\tan\left(\frac{1}{\theta}\right) \Rightarrow \frac{dr}{d\theta} = \left(\sec\sqrt{\theta}\right)\left(\sec^2\frac{1}{\theta}\right)\left(-\frac{1}{\theta^2}\right) + \tan\left(\frac{1}{\theta}\right)\left(\sec\sqrt{\theta}\tan\sqrt{\theta}\right)\left(\frac{1}{2\sqrt{\theta}}\right)$
$= -\frac{1}{\theta^2}\sec\sqrt{\theta}\sec^2\left(\frac{1}{\theta}\right) + \frac{1}{2\sqrt{\theta}}\tan\left(\frac{1}{\theta}\right)\sec\sqrt{\theta}\tan\sqrt{\theta} = \left(\sec\sqrt{\theta}\right)\left[\frac{\tan\sqrt{\theta}\tan\left(\frac{1}{\theta}\right)}{2\sqrt{\theta}} - \frac{\sec^2\left(\frac{1}{\theta}\right)}{\theta^2}\right]$

45. $q = \sin\left(\frac{t}{\sqrt{t+1}}\right) \Rightarrow \frac{dq}{dt} = \cos\left(\frac{t}{\sqrt{t+1}}\right) \cdot \frac{d}{dt}\left(\frac{t}{\sqrt{t+1}}\right) = \cos\left(\frac{t}{\sqrt{t+1}}\right) \cdot \frac{\sqrt{t+1}\,(1) - t \cdot \frac{d}{dt}\left(\sqrt{t+1}\right)}{\left(\sqrt{t+1}\right)^2}$
$= \cos\left(\frac{t}{\sqrt{t+1}}\right) \cdot \frac{\sqrt{t+1} - \frac{t}{2\sqrt{t+1}}}{t+1} = \cos\left(\frac{t}{\sqrt{t+1}}\right)\left(\frac{2(t+1)-t}{2(t+1)^{3/2}}\right) = \left(\frac{t+2}{2(t+1)^{3/2}}\right)\cos\left(\frac{t}{\sqrt{t+1}}\right)$

46. $q = \cot\left(\frac{\sin t}{t}\right) \Rightarrow \frac{dq}{dt} = -\csc^2\left(\frac{\sin t}{t}\right) \cdot \frac{d}{dt}\left(\frac{\sin t}{t}\right) = \left(-\csc^2\left(\frac{\sin t}{t}\right)\right)\left(\frac{t\cos t - \sin t}{t^2}\right)$

47. $y = \cos\left(e^{-\theta^2}\right) \Rightarrow \frac{dy}{d\theta} = -\sin\left(e^{-\theta^2}\right)\frac{d}{d\theta}\left(e^{-\theta^2}\right) = \left(-\sin\left(e^{-\theta^2}\right)\right)\left(e^{-\theta^2}\right)\frac{d}{d\theta}\left(-\theta^2\right) = 2\theta e^{-\theta^2}\sin\left(e^{-\theta^2}\right)$

48. $y = \theta^3 e^{-2\theta}\cos 5\theta \Rightarrow \frac{dy}{d\theta} = \left(3\theta^2\right)\left(e^{-2\theta}\cos 5\theta\right) + \left(\theta^3\cos 5\theta\right)e^{-2\theta}\frac{d}{d\theta}(-2\theta) - 5(\sin 5\theta)\left(\theta^3 e^{-2\theta}\right)$
$= \theta^2 e^{-2\theta}(3\cos 5\theta - 2\theta\cos 5\theta - 5\theta\sin 5\theta)$

49. $y = \sin^2(\pi t - 2) \Rightarrow \frac{dy}{dt} = 2\sin(\pi t - 2) \cdot \frac{d}{dt}\sin(\pi t - 2) = 2\sin(\pi t - 2) \cdot \cos(\pi t - 2) \cdot \frac{d}{dt}(\pi t - 2)$
$= 2\pi\sin(\pi t - 2)\cos(\pi t - 2)$

50. $y = \sec^2\pi t \Rightarrow \frac{dy}{dt} = (2\sec\pi t) \cdot \frac{d}{dt}(\sec\pi t) = (2\sec\pi t)(\sec\pi t\tan\pi t) \cdot \frac{d}{dt}(\pi t) = 2\pi\sec^2\pi t\tan\pi t$

51. $y = (1+\cos 2t)^{-4} \Rightarrow \frac{dy}{dt} = -4(1+\cos 2t)^{-5} \cdot \frac{d}{dt}(1+\cos 2t) = -4(1+\cos 2t)^{-5}(-\sin 2t) \cdot \frac{d}{dt}(2t) = \frac{8\sin 2t}{(1+\cos 2t)^5}$

52. $y = \left(1+\cot\left(\frac{t}{2}\right)\right)^{-2} \Rightarrow \frac{dy}{dt} = -2\left(1+\cot\left(\frac{t}{2}\right)\right)^{-3} \cdot \frac{d}{dt}\left(1+\cot\left(\frac{t}{2}\right)\right) = -2\left(1+\cot\left(\frac{t}{2}\right)\right)^{-3} \cdot \left(-\csc^2\left(\frac{t}{2}\right)\right) \cdot \frac{d}{dt}\left(\frac{t}{2}\right)$
$= \frac{\csc^2\left(\frac{t}{2}\right)}{\left(1+\cot\left(\frac{t}{2}\right)\right)^3}$

53. $y = e^{\cos^2(\pi t-1)} \Rightarrow \frac{dy}{dt} = e^{\cos^2(\pi t-1)} \cdot 2\cos(\pi t-1) \cdot (-\sin(\pi t-1)) \cdot \pi = -2\pi\sin(\pi t-1)\cos(\pi t-1)e^{\cos^2(\pi t-1)}$

54. $y = \left(e^{\sin(t/2)}\right)^3 \Rightarrow \frac{dy}{dt} = 3\left(e^{\sin(t/2)}\right)^2 \cdot e^{\sin(t/2)} \cdot \cos\left(\frac{t}{2}\right) \cdot \frac{1}{2} = \frac{3}{2}\cos\left(\frac{t}{2}\right)e^{\sin(t/2)}e^{2\sin(t/2)} = \frac{3}{2}\cos\left(\frac{t}{2}\right)e^{3\sin(t/2)}$

55. $y = \sin(\cos(2t-5)) \Rightarrow \frac{dy}{dt} = \cos(\cos(2t-5)) \cdot \frac{d}{dt}\cos(2t-5) = \cos(\cos(2t-5)) \cdot (-\sin(2t-5)) \cdot \frac{d}{dt}(2t-5)$
$= -2\cos(\cos(2t-5))(\sin(2t-5))$

56. $y = \cos\left(5\sin\left(\frac{t}{3}\right)\right) \Rightarrow \frac{dy}{dt} = -\sin\left(5\sin\left(\frac{t}{3}\right)\right) \cdot \frac{d}{dt}\left(5\sin\left(\frac{t}{3}\right)\right) = -\sin\left(5\sin\left(\frac{t}{3}\right)\right)\left(5\cos\left(\frac{t}{3}\right)\right) \cdot \frac{d}{dt}\left(\frac{t}{3}\right)$
$= -\frac{5}{3}\sin\left(5\sin\left(\frac{t}{3}\right)\right)\left(\cos\left(\frac{t}{3}\right)\right)$

57. $y = \left[1+\tan^4\left(\frac{t}{12}\right)\right]^3 \Rightarrow \frac{dy}{dt} = 3\left[1+\tan^4\left(\frac{t}{12}\right)\right]^2 \cdot \frac{d}{dt}\left[1+\tan^4\left(\frac{t}{12}\right)\right] = 3\left[1+\tan^4\left(\frac{t}{12}\right)\right]^2\left[4\tan^3\left(\frac{t}{12}\right) \cdot \frac{d}{dt}\tan\left(\frac{t}{12}\right)\right]$
$= 12\left[1+\tan^4\left(\frac{t}{12}\right)\right]^2\left[\tan^3\left(\frac{t}{12}\right)\sec^2\left(\frac{t}{12}\right) \cdot \frac{1}{12}\right] = \left[1+\tan^4\left(\frac{t}{12}\right)\right]^2\left[\tan^3\left(\frac{t}{12}\right)\sec^2\left(\frac{t}{12}\right)\right]$

58. $y = \frac{1}{6}\left[1+\cos^2(7t)\right]^3 \Rightarrow \frac{dy}{dt} = \frac{3}{6}\left[1+\cos^2(7t)\right]^2 \cdot 2\cos(7t)(-\sin(7t))(7) = -7\left[1+\cos^2(7t)\right]^2(\cos(7t)\sin(7t))$

59. $y = (1+\cos(t^2))^{1/2} \Rightarrow \frac{dy}{dt} = \frac{1}{2}(1+\cos(t^2))^{-1/2} \cdot \frac{d}{dt}(1+\cos(t^2)) = \frac{1}{2}(1+\cos(t^2))^{-1/2}\left(-\sin(t^2) \cdot \frac{d}{dt}(t^2)\right)$
$= -\frac{1}{2}(1+\cos(t^2))^{-1/2}(\sin(t^2)) \cdot 2t = -\frac{t\sin(t^2)}{\sqrt{1+\cos(t^2)}}$

60. $y = 4\sin\left(\sqrt{1+\sqrt{t}}\right) \Rightarrow \frac{dy}{dt} = 4\cos\left(\sqrt{1+\sqrt{t}}\right) \cdot \frac{d}{dt}\left(\sqrt{1+\sqrt{t}}\right) = 4\cos\left(\sqrt{1+\sqrt{t}}\right) \cdot \frac{1}{2\sqrt{1+\sqrt{t}}} \cdot \frac{d}{dt}\left(1+\sqrt{t}\right)$
$= \frac{2\cos\left(\sqrt{1+\sqrt{t}}\right)}{\sqrt{1+\sqrt{t}} \cdot 2\sqrt{t}} = \frac{\cos\left(\sqrt{1+\sqrt{t}}\right)}{\sqrt{t+t\sqrt{t}}}$

61. $y = \left(1+\frac{1}{x}\right)^3 \Rightarrow y' = 3\left(1+\frac{1}{x}\right)^2\left(-\frac{1}{x^2}\right) = -\frac{3}{x^2}\left(1+\frac{1}{x}\right)^2 \Rightarrow y'' = \left(-\frac{3}{x^2}\right) \cdot \frac{d}{dx}\left(1+\frac{1}{x}\right)^2 - \left(1+\frac{1}{x}\right)^2 \cdot \frac{d}{dx}\left(\frac{3}{x^2}\right)$
$= \left(-\frac{3}{x^2}\right)\left(2\left(1+\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)\right) + \left(\frac{6}{x^3}\right)\left(1+\frac{1}{x}\right)^2 = \frac{6}{x^4}\left(1+\frac{1}{x}\right) + \frac{6}{x^3}\left(1+\frac{1}{x}\right)^2 = \frac{6}{x^3}\left(1+\frac{1}{x}\right)\left(\frac{1}{x}+1+\frac{1}{x}\right)$
$= \frac{6}{x^3}\left(1+\frac{1}{x}\right)\left(1+\frac{2}{x}\right)$

62. $y = \left(1-\sqrt{x}\right)^{-1} \Rightarrow y' = -\left(1-\sqrt{x}\right)^{-2}\left(-\frac{1}{2}x^{-1/2}\right) = \frac{1}{2}\left(1-\sqrt{x}\right)^{-2}x^{-1/2}$
$\Rightarrow y'' = \frac{1}{2}\left[\left(1-\sqrt{x}\right)^{-2}\left(-\frac{1}{2}x^{-3/2}\right) + x^{-1/2}(-2)\left(1-\sqrt{x}\right)^{-3}\left(-\frac{1}{2}x^{-1/2}\right)\right]$
$= \frac{1}{2}\left[\frac{-1}{2}x^{-3/2}\left(1-\sqrt{x}\right)^{-2} + x^{-1}\left(1-\sqrt{x}\right)^{-3}\right] = \frac{1}{2}x^{-1}\left(1-\sqrt{x}\right)^{-3}\left[-\frac{1}{2}x^{-1/2}\left(1-\sqrt{x}\right)+1\right]$
$= \frac{1}{2x}\left(1-\sqrt{x}\right)^{-3}\left(-\frac{1}{2\sqrt{x}}+\frac{1}{2}+1\right) = \frac{1}{2x}\left(1-\sqrt{x}\right)^{-3}\left(\frac{3}{2}-\frac{1}{2\sqrt{x}}\right)$

63. $y = \frac{1}{9}\cot(3x-1) \Rightarrow y' = -\frac{1}{9}\csc^2(3x-1)(3) = -\frac{1}{3}\csc^2(3x-1) \Rightarrow y'' = \left(-\frac{2}{3}\right)\left(\csc(3x-1) \cdot \frac{d}{dx}\csc(3x-1)\right)$
$= -\frac{2}{3}\csc(3x-1)(-\csc(3x-1)\cot(3x-1) \cdot \frac{d}{dx}(3x-1)) = 2\csc^2(3x-1)\cot(3x-1)$

64. $y = 9\tan\left(\frac{x}{3}\right) \Rightarrow y' = 9\left(\sec^2\left(\frac{x}{3}\right)\right)\left(\frac{1}{3}\right) = 3\sec^2\left(\frac{x}{3}\right) \Rightarrow y'' = 3 \cdot 2\sec\left(\frac{x}{3}\right)\left(\sec\left(\frac{x}{3}\right)\tan\left(\frac{x}{3}\right)\right)\left(\frac{1}{3}\right) = 2\sec^2\left(\frac{x}{3}\right)\tan\left(\frac{x}{3}\right)$

65. $y = e^{x^2} + 5x \Rightarrow y' = 2xe^{x^2} + 5 \Rightarrow y'' = 2x \cdot e^{x^2}(2x) + 2e^{x^2} = (4x^2 + 2)e^{x^2}$

66. $y = \sin(x^2e^x) \Rightarrow y' = \cos(x^2e^x) \cdot (x^2e^x + 2xe^x) = (x^2 + 2x)e^x\cos(x^2e^x)$
$\Rightarrow$ (Use triple product rule: $D(fgh) = f'gh + fg'h + fgh'$)
$y'' = (2x+2)e^x\cos(x^2e^x) + (x^2+2x)e^x\cos(x^2e^x) + (x^2+2x)e^x(-\sin(x^2e^x) \cdot (x^2e^x + 2xe^x))$
$= (x^2 + 4x + 2)e^x\cos(x^2e^x) - xe^{2x}(x^3 + 4x^2 + 4x)\sin(x^2e^x)$

67. $g(x) = \sqrt{x} \Rightarrow g'(x) = \frac{1}{2\sqrt{x}} \Rightarrow g(1) = 1$ and $g'(1) = \frac{1}{2}$; $f(u) = u^5 + 1 \Rightarrow f'(u) = 5u^4 \Rightarrow f'(g(1)) = f'(1) = 5$; therefore, $(f \circ g)'(1) = f'(g(1)) \cdot g'(1) = 5 \cdot \frac{1}{2} = \frac{5}{2}$

68. $g(x) = (1-x)^{-1} \Rightarrow g'(x) = -(1-x)^{-2}(-1) = \frac{1}{(1-x)^2} \Rightarrow g(-1) = \frac{1}{2}$ and $g'(-1) = \frac{1}{4}$; $f(u) = 1 - \frac{1}{u}$
$\Rightarrow f'(u) = \frac{1}{u^2} \Rightarrow f'(g(-1)) = f'\left(\frac{1}{2}\right) = 4$; therefore, $(f \circ g)'(-1) = f'(g(-1))g'(-1) = 4 \cdot \frac{1}{4} = 1$

69. $g(x) = 5\sqrt{x} \Rightarrow g'(x) = \frac{5}{2\sqrt{x}} \Rightarrow g(1) = 5$ and $g'(1) = \frac{5}{2}$; $f(u) = \cot\left(\frac{\pi u}{10}\right) \Rightarrow f'(u) = -\csc^2\left(\frac{\pi u}{10}\right)\left(\frac{\pi}{10}\right)$
$= \frac{-\pi}{10}\csc^2\left(\frac{\pi u}{10}\right) \Rightarrow f'(g(1)) = f'(5) = -\frac{\pi}{10}\csc^2\left(\frac{\pi}{2}\right) = -\frac{\pi}{10}$; therefore, $(f \circ g)'(1) = f'(g(1))g'(1) = -\frac{\pi}{10} \cdot \frac{5}{2} = -\frac{\pi}{4}$

70. $g(x) = \pi x \Rightarrow g'(x) = \pi \Rightarrow g\left(\frac{1}{4}\right) = \frac{\pi}{4}$ and $g'\left(\frac{1}{4}\right) = \pi$; $f(u) = u + \sec^2 u \Rightarrow f'(u) = 1 + 2\sec u \cdot \sec u \tan u$
$= 1 + 2\sec^2 u \tan u \Rightarrow f'\left(g\left(\frac{1}{4}\right)\right) = f'\left(\frac{\pi}{4}\right) = 1 + 2\sec^2\frac{\pi}{4}\tan\frac{\pi}{4} = 5$; therefore, $(f \circ g)'\left(\frac{1}{4}\right) = f'\left(g\left(\frac{1}{4}\right)\right)g'\left(\frac{1}{4}\right) = 5\pi$

71. $g(x) = 10x^2 + x + 1 \Rightarrow g'(x) = 20x + 1 \Rightarrow g(0) = 1$ and $g'(0) = 1$; $f(u) = \frac{2u}{u^2+1} \Rightarrow f'(u) = \frac{(u^2+1)(2) - (2u)(2u)}{(u^2+1)^2}$
$= \frac{-2u^2+2}{(u^2+1)^2} \Rightarrow f'(g(0)) = f'(1) = 0$; therefore, $(f \circ g)'(0) = f'(g(0))g'(0) = 0 \cdot 1 = 0$

72. $g(x) = \frac{1}{x^2} - 1 \Rightarrow g'(x) = -\frac{2}{x^3} \Rightarrow g(-1) = 0$ and $g'(-1) = 2$; $f(u) = \left(\frac{u-1}{u+1}\right)^2 \Rightarrow f'(u) = 2\left(\frac{u-1}{u+1}\right)\frac{d}{du}\left(\frac{u-1}{u+1}\right)$
$= 2\left(\frac{u-1}{u+1}\right) \cdot \frac{(u+1)(1)-(u-1)(1)}{(u+1)^2} = \frac{2(u-1)(2)}{(u+1)^3} = \frac{4(u-1)}{(u+1)^3} \Rightarrow f'(g(-1)) = f'(0) = -4$; therefore,
$(f \circ g)'(-1) = f'(g(-1))g'(-1) = (-4)(2) = -8$

73. (a) $y = 2f(x) \Rightarrow \frac{dy}{dx} = 2f'(x) \Rightarrow \left.\frac{dy}{dx}\right|_{x=2} = 2f'(2) = 2\left(\frac{1}{3}\right) = \frac{2}{3}$
(b) $y = f(x) + g(x) \Rightarrow \frac{dy}{dx} = f'(x) + g'(x) \Rightarrow \left.\frac{dy}{dx}\right|_{x=3} = f'(3) + g'(3) = 2\pi + 5$
(c) $y = f(x) \cdot g(x) \Rightarrow \frac{dy}{dx} = f(x)g'(x) + g(x)f'(x) \Rightarrow \left.\frac{dy}{dx}\right|_{x=3} = f(3)g'(3) + g(3)f'(3) = 3 \cdot 5 + (-4)(2\pi) = 15 - 8\pi$
(d) $y = \frac{f(x)}{g(x)} \Rightarrow \frac{dy}{dx} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2} \Rightarrow \left.\frac{dy}{dx}\right|_{x=2} = \frac{g(2)f'(2) - f(2)g'(2)}{[g(2)]^2} = \frac{(2)\left(\frac{1}{3}\right) - (8)(-3)}{2^2} = \frac{37}{6}$
(e) $y = f(g(x)) \Rightarrow \frac{dy}{dx} = f'(g(x))g'(x) \Rightarrow \left.\frac{dy}{dx}\right|_{x=2} = f'(g(2))g'(2) = f'(2)(-3) = \frac{1}{3}(-3) = -1$
(f) $y = (f(x))^{1/2} \Rightarrow \frac{dy}{dx} = \frac{1}{2}(f(x))^{-1/2} \cdot f'(x) = \frac{f'(x)}{2\sqrt{f(x)}} \Rightarrow \left.\frac{dy}{dx}\right|_{x=2} = \frac{f'(2)}{2\sqrt{f(2)}} = \frac{\left(\frac{1}{3}\right)}{2\sqrt{8}} = \frac{1}{6\sqrt{8}} = \frac{1}{12\sqrt{2}} = \frac{\sqrt{2}}{24}$
(g) $y = (g(x))^{-2} \Rightarrow \frac{dy}{dx} = -2(g(x))^{-3} \cdot g'(x) \Rightarrow \left.\frac{dy}{dx}\right|_{x=3} = -2(g(3))^{-3}g'(3) = -2(-4)^{-3} \cdot 5 = \frac{5}{32}$
(h) $y = \left((f(x))^2 + (g(x))^2\right)^{1/2} \Rightarrow \frac{dy}{dx} = \frac{1}{2}\left((f(x))^2 + (g(x))^2\right)^{-1/2}(2f(x) \cdot f'(x) + 2g(x) \cdot g'(x))$
$\Rightarrow \left.\frac{dy}{dx}\right|_{x=2} = \frac{1}{2}\left((f(2))^2 + (g(2))^2\right)^{-1/2}(2f(2)f'(2) + 2g(2)g'(2)) = \frac{1}{2}\left(8^2 + 2^2\right)^{-1/2}\left(2 \cdot 8 \cdot \frac{1}{3} + 2 \cdot 2 \cdot (-3)\right) = -\frac{5}{3\sqrt{17}}$

74. (a) $y = 5f(x) - g(x) \Rightarrow \frac{dy}{dx} = 5f'(x) - g'(x) \Rightarrow \left.\frac{dy}{dx}\right|_{x=1} = 5f'(1) - g'(1) = 5\left(-\frac{1}{3}\right) - \left(\frac{-8}{3}\right) = 1$
(b) $y = f(x)(g(x))^3 \Rightarrow \frac{dy}{dx} = f(x)\left(3(g(x))^2g'(x)\right) + (g(x))^3f'(x) \Rightarrow \left.\frac{dy}{dx}\right|_{x=0} = 3f(0)(g(0))^2g'(0) + (g(0))^3f'(0)$
$= 3(1)(1)^2\left(\frac{1}{3}\right) + (1)^3(5) = 6$
(c) $y = \frac{f(x)}{g(x)+1} \Rightarrow \frac{dy}{dx} = \frac{(g(x)+1)f'(x) - f(x)g'(x)}{(g(x)+1)^2} \Rightarrow \left.\frac{dy}{dx}\right|_{x=1} = \frac{(g(1)+1)f'(1) - f(1)g'(1)}{(g(1)+1)^2} = \frac{(-4+1)\left(-\frac{1}{3}\right) - (3)\left(-\frac{8}{3}\right)}{(-4+1)^2} = 1$

(d) $y = f(g(x)) \Rightarrow \frac{dy}{dx} = f'(g(x))g'(x) \Rightarrow \left.\frac{dy}{dx}\right|_{x=0} = f'(g(0))g'(0) = f'(1)\left(\frac{1}{3}\right) = \left(-\frac{1}{3}\right)\left(\frac{1}{3}\right) = -\frac{1}{9}$

(e) $y = g(f(x)) \Rightarrow \frac{dy}{dx} = g'(f(x))f'(x) \Rightarrow \left.\frac{dy}{dx}\right|_{x=0} = g'(f(0))f'(0) = g'(1)(5) = \left(-\frac{8}{3}\right)(5) = -\frac{40}{3}$

(f) $y = \left(x^{11} + f(x)\right)^{-2} \Rightarrow \frac{dy}{dx} = -2\left(x^{11} + f(x)\right)^{-3}\left(11x^{10} + f'(x)\right) \Rightarrow \left.\frac{dy}{dx}\right|_{x=1} = -2(1 + f(1))^{-3}\left(11 + f'(1)\right)$
$= -2(1+3)^{-3}\left(11 - \frac{1}{3}\right) = \left(-\frac{2}{4^3}\right)\left(\frac{32}{3}\right) = -\frac{1}{3}$

(g) $y = f(x + g(x)) \Rightarrow \frac{dy}{dx} = f'(x + g(x))\left(1 + g'(x)\right) \Rightarrow \left.\frac{dy}{dx}\right|_{x=0} = f'(0 + g(0))\left(1 + g'(0)\right) = f'(1)\left(1 + \frac{1}{3}\right) = \left(-\frac{1}{3}\right)\left(\frac{4}{3}\right)$
$= -\frac{4}{9}$

75. $\frac{ds}{dt} = \frac{ds}{d\theta} \cdot \frac{d\theta}{dt}$: $s = \cos\theta \Rightarrow \frac{ds}{d\theta} = -\sin\theta \Rightarrow \left.\frac{ds}{d\theta}\right|_{\theta=\frac{3\pi}{2}} = -\sin\left(\frac{3\pi}{2}\right) = 1$ so that $\frac{ds}{dt} = \frac{ds}{d\theta} \cdot \frac{d\theta}{dt} = 1 \cdot 5 = 5$

76. $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$: $y = x^2 + 7x - 5 \Rightarrow \frac{dy}{dx} = 2x + 7 \Rightarrow \left.\frac{dy}{dx}\right|_{x=1} = 9$ so that $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = 9 \cdot \frac{1}{3} = 3$

77. With $y = x$, we should get $\frac{dy}{dx} = 1$ for both (a) and (b):

(a) $y = \frac{u}{5} + 7 \Rightarrow \frac{dy}{du} = \frac{1}{5}$; $u = 5x - 35 \Rightarrow \frac{du}{dx} = 5$; therefore, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{5} \cdot 5 = 1$, as expected

(b) $y = 1 + \frac{1}{u} \Rightarrow \frac{dy}{du} = -\frac{1}{u^2}$; $u = (x-1)^{-1} \Rightarrow \frac{du}{dx} = -(x-1)^{-2}(1) = \frac{-1}{(x-1)^2}$; therefore $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$
$= \frac{-1}{u^2} \cdot \frac{-1}{(x-1)^2} = \frac{-1}{\left((x-1)^{-1}\right)^2} \cdot \frac{-1}{(x-1)^2} = (x-1)^2 \cdot \frac{1}{(x-1)^2} = 1$, again as expected

78. With $y = x^{3/2}$, we should get $\frac{dy}{dx} = \frac{3}{2}x^{1/2}$ for both (a) and (b):

(a) $y = u^3 \Rightarrow \frac{dy}{du} = 3u^2$; $u = \sqrt{x} \Rightarrow \frac{du}{dx} = \frac{1}{2\sqrt{x}}$; therefore, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 3u^2 \cdot \frac{1}{2\sqrt{x}} = 3\left(\sqrt{x}\right)^2 \cdot \frac{1}{2\sqrt{x}} = \frac{3}{2}\sqrt{x}$, as expected.

(b) $y = \sqrt{u} \Rightarrow \frac{dy}{du} = \frac{1}{2\sqrt{u}}$; $u = x^3 \Rightarrow \frac{du}{dx} = 3x^2$; therefore, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{2\sqrt{u}} \cdot 3x^2 = \frac{1}{2\sqrt{x^3}} \cdot 3x^2 = \frac{3}{2}x^{1/2}$, again as expected.

79. $y = 2\tan\left(\frac{\pi x}{4}\right) \Rightarrow \frac{dy}{dx} = \left(2\sec^2\frac{\pi x}{4}\right)\left(\frac{\pi}{4}\right) = \frac{\pi}{2}\sec^2\frac{\pi x}{4}$

(a) $\left.\frac{dy}{dx}\right|_{x=1} = \frac{\pi}{2}\sec^2\left(\frac{\pi}{4}\right) = \pi \Rightarrow$ slope of tangent is 2; thus, $y(1) = 2\tan\left(\frac{\pi}{4}\right) = 2$ and $y'(1) = \pi \Rightarrow$ tangent line is given by $y - 2 = \pi(x-1) \Rightarrow y = \pi x + 2 - \pi$

(b) $y' = \frac{\pi}{2}\sec^2\left(\frac{\pi x}{4}\right)$ and the smallest value the secant function can have in $-2 < x < 2$ is $1 \Rightarrow$ the minimum value of y' is $\frac{\pi}{2}$ and that occurs when $\frac{\pi}{2} = \frac{\pi}{2}\sec^2\left(\frac{\pi x}{4}\right) \Rightarrow 1 = \sec^2\left(\frac{\pi x}{4}\right) \Rightarrow \pm 1 = \sec\left(\frac{\pi x}{4}\right) \Rightarrow x = 0$.

80. (a) $y = \sin 2x \Rightarrow y' = 2\cos 2x \Rightarrow y'(0) = 2\cos(0) = 2 \Rightarrow$ tangent to $y = \sin 2x$ at the origin is $y = 2x$; $y = -\sin\left(\frac{x}{2}\right) \Rightarrow y' = -\frac{1}{2}\cos\left(\frac{x}{2}\right) \Rightarrow y'(0) = -\frac{1}{2}\cos 0 = -\frac{1}{2} \Rightarrow$ tangent to $y = -\sin\left(\frac{x}{2}\right)$ at the origin is $y = -\frac{1}{2}x$. The tangents are perpendicular to each other at the origin since the product of their slopes is -1.

(b) $y = \sin(mx) \Rightarrow y' = m\cos(mx) \Rightarrow y'(0) = m\cos 0 = m$; $y = -\sin\left(\frac{x}{m}\right) \Rightarrow y' = -\frac{1}{m}\cos\left(\frac{x}{m}\right)$
$\Rightarrow y'(0) = -\frac{1}{m}\cos(0) = -\frac{1}{m}$. Since $m \cdot \left(-\frac{1}{m}\right) = -1$, the tangent lines are perpendicular at the origin.

(c) $y = \sin(mx) \Rightarrow y' = m\cos(mx)$. The largest value $\cos(mx)$ can attain is 1 at $x = 0 \Rightarrow$ the largest value y' can attain is $|m|$ because $|y'| = |m\cos(mx)| = |m|\,|\cos mx| \le |m| \cdot 1 = |m|$. Also, $y = -\sin\left(\frac{x}{m}\right)$
$\Rightarrow y' = -\frac{1}{m}\cos\left(\frac{x}{m}\right) \Rightarrow |y'| = \left|\frac{-1}{m}\cos\left(\frac{x}{m}\right)\right| \le \left|\frac{1}{m}\right|\left|\cos\left(\frac{x}{m}\right)\right| \le \frac{1}{|m|} \Rightarrow$ the largest value y' can attain is $\left|\frac{1}{m}\right|$.

(d) $y = \sin(mx) \Rightarrow y' = m\cos(mx) \Rightarrow y'(0) = m \Rightarrow$ slope of curve at the origin is m. Also, $\sin(mx)$ completes m periods on $[0, 2\pi]$. Therefore the slope of the curve $y = \sin(mx)$ at the origin is the same as the number of periods it completes on $[0, 2\pi]$. In particular, for large m, we can think of "compressing" the graph of $y = \sin x$ horizontally which gives more periods completed on $[0, 2\pi]$, but also increases the slope of the graph at the origin.