81. $s = (1+4t)^{1/2} \Rightarrow v = \frac{ds}{dt} = \frac{1}{2}(1+4t)^{-1/2}(4) = 2(1+4t)^{-1/2} \Rightarrow v(6) = 2(1+4\cdot 6)^{-1/2} = \frac{2}{5}$ m/sec;
$v = 2(1+4t)^{-1/2} \Rightarrow a = \frac{dv}{dt} = -\frac{1}{2}\cdot 2(1+4t)^{-3/2}(4) = -4(1+4t)^{-3/2} \Rightarrow a(6) = -4(1+4\cdot 6)^{-3/2} = -\frac{4}{125}$ m/sec^2

82. We need to show $a = \frac{dv}{dt}$ is constant: $a = \frac{dv}{dt} = \frac{dv}{ds}\cdot\frac{ds}{dt}$ and $\frac{dv}{ds} = \frac{d}{ds}\left(k\sqrt{s}\right) = \frac{k}{2\sqrt{s}} \Rightarrow a = \frac{dv}{ds}\cdot\frac{ds}{dt} = \frac{dv}{ds}\cdot v$
$= \frac{k}{2\sqrt{s}}\cdot k\sqrt{s} = \frac{k^2}{2}$ which is a constant.

83. v proportional to $\frac{1}{\sqrt{s}} \Rightarrow v = \frac{k}{\sqrt{s}}$ for some constant k $\Rightarrow \frac{dv}{ds} = -\frac{k}{2s^{3/2}}$. Thus, $a = \frac{dv}{dt} = \frac{dv}{ds}\cdot\frac{ds}{dt} = \frac{dv}{ds}\cdot v$
$= -\frac{k}{2s^{3/2}}\cdot\frac{k}{\sqrt{s}} = -\frac{k^2}{2}\left(\frac{1}{s^2}\right) \Rightarrow$ acceleration is a constant times $\frac{1}{s^2}$ so a is inversely proportional to s^2.

84. Let $\frac{dx}{dt} = f(x)$. Then, $a = \frac{dv}{dt} = \frac{dv}{dx}\cdot\frac{dx}{dt} = \frac{dv}{dx}\cdot f(x) = \frac{d}{dx}\left(\frac{dx}{dt}\right)\cdot f(x) = \frac{d}{dx}(f(x))\cdot f(x) = f'(x)f(x)$, as required.

85. No. The chain rule says that when g is differentiable at 0 and f is differentiable at g(0), then $f\circ g$ is differentiable at 0. But the chain rule says nothing about what happens when g is not differentiable at 0 so there is no contradiction.

86. The graph of $y = (f\circ g)(x)$ has a horizontal tangent at $x = 1$ provided that $(f\circ g)'(1) = 0 \Rightarrow f'(g(1))g'(1) = 0$
$\Rightarrow$ either $f'(g(1)) = 0$ or $g'(1) = 0$ (or both) $\Rightarrow$ either the graph of f has a horizontal tangent at $u = g(1)$, or the graph of g has a horizontal tangent at $x = 1$ (or both).

87. From the power rule, with $y = x^{1/4}$, we get $\frac{dy}{dx} = \frac{1}{4}x^{-3/4}$. From the chain rule, $y = \sqrt{\sqrt{x}}$
$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{\sqrt{x}}}\cdot\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{\sqrt{x}}}\cdot\frac{1}{2\sqrt{x}} = \frac{1}{4}x^{-3/4}$, in agreement.

88. From the power rule, with $y = x^{3/4}$, we get $\frac{dy}{dx} = \frac{3}{4}x^{-1/4}$. From the chain rule, $y = \sqrt{x\sqrt{x}}$
$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x\sqrt{x}}}\cdot\frac{d}{dx}\left(x\sqrt{x}\right) \Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x\sqrt{x}}}\cdot\left(x\cdot\frac{1}{2\sqrt{x}} + \sqrt{x}\right) = \frac{1}{2\sqrt{x\sqrt{x}}}\cdot\left(\frac{3}{2}\sqrt{x}\right) = \frac{3\sqrt{x}}{4\sqrt{x\sqrt{x}}}$
$= \frac{3\sqrt{x}}{4\sqrt{x}\sqrt{\sqrt{x}}} = \frac{3}{4}x^{-1/4}$, in agreement.

2.8 IMPLICIT DIFFERENTIATION

1. $x^2y + xy^2 = 6$:
Step 1: $\left(x^2\frac{dy}{dx} + y\cdot 2x\right) + \left(x\cdot 2y\frac{dy}{dx} + y^2\cdot 1\right) = 0$
Step 2: $x^2\frac{dy}{dx} + 2xy\frac{dy}{dx} = -2xy - y^2$
Step 3: $\frac{dy}{dx}(x^2 + 2xy) = -2xy - y^2$
Step 4: $\frac{dy}{dx} = \frac{-2xy - y^2}{x^2 + 2xy}$

2. $x^3 + y^3 = 18xy \Rightarrow 3x^2 + 3y^2\frac{dy}{dx} = 18y + 18x\frac{dy}{dx} \Rightarrow (3y^2 - 18x)\frac{dy}{dx} = 18y - 3x^2 \Rightarrow \frac{dy}{dx} = \frac{6y - x^2}{y^2 - 6x}$

3. $2xy + y^2 = x + y$:
Step 1: $\left(2x\frac{dy}{dx} + 2y\right) + 2y\frac{dy}{dx} = 1 + \frac{dy}{dx}$
Step 2: $2x\frac{dy}{dx} + 2y\frac{dy}{dx} - \frac{dy}{dx} = 1 - 2y$
Step 3: $\frac{dy}{dx}(2x + 2y - 1) = 1 - 2y$
Step 4: $\frac{dy}{dx} = \frac{1 - 2y}{2x + 2y - 1}$

4. $x^3 - xy + y^3 = 1 \Rightarrow 3x^2 - y - x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0 \Rightarrow (3y^2 - x)\frac{dy}{dx} = y - 3x^2 \Rightarrow \frac{dy}{dx} = \frac{y-3x^2}{3y^2-x}$

5. $x^2(x-y)^2 = x^2 - y^2$:

Step 1: $x^2\left[2(x-y)\left(1-\frac{dy}{dx}\right)\right] + (x-y)^2(2x) = 2x - 2y\frac{dy}{dx}$

Step 2: $-2x^2(x-y)\frac{dy}{dx} + 2y\frac{dy}{dx} = 2x - 2x^2(x-y) - 2x(x-y)^2$

Step 3: $\frac{dy}{dx}\left[-2x^2(x-y) + 2y\right] = 2x\left[1 - x(x-y) - (x-y)^2\right]$

Step 4: $\frac{dy}{dx} = \frac{2x\left[1-x(x-y)-(x-y)^2\right]}{-2x^2(x-y)+2y} = \frac{x\left[1-x(x-y)-(x-y)^2\right]}{y-x^2(x-y)} = \frac{x\left(1-x^2+xy-x^2+2xy-y^2\right)}{x^2y-x^3+y}$

$= \frac{x-2x^3+3x^2y-xy^2}{x^2y-x^3+y}$

6. $(3xy+7)^2 = 6y \Rightarrow 2(3xy+7)\cdot\left(3x\frac{dy}{dx}+3y\right) = 6\frac{dy}{dx} \Rightarrow 2(3xy+7)(3x)\frac{dy}{dx} - 6\frac{dy}{dx} = -6y(3xy+7)$

$\Rightarrow \frac{dy}{dx}\left[6x(3xy+7)-6\right] = -6y(3xy+7) \Rightarrow \frac{dy}{dx} = -\frac{y(3xy+7)}{x(3xy+7)-1} = \frac{3xy^2+7y}{1-3x^2y-7x}$

7. $y^2 = \frac{x-1}{x+1} \Rightarrow 2y\frac{dy}{dx} = \frac{(x+1)-(x-1)}{(x+1)^2} = \frac{2}{(x+1)^2} \Rightarrow \frac{dy}{dx} = \frac{1}{y(x+1)^2}$

8. $x^2 = \frac{x-y}{x+y} \Rightarrow x^3 + x^2y = x - y \Rightarrow 3x^2 + 2xy + x^2y' = 1 - y' \Rightarrow (x^2+1)\,y' = 1 - 3x^2 - 2xy \Rightarrow y' = \frac{1-3x^2-2xy}{x^2+1}$

9. $x = \tan y \Rightarrow 1 = (\sec^2 y)\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y} = \cos^2 y$

10. $xy = \cot(xy) \Rightarrow x\frac{dy}{dx} + y = -\csc^2(xy)\left(x\frac{dy}{dx}+y\right) \Rightarrow x\frac{dy}{dx} + x\csc^2(xy)\frac{dy}{dx} = -y\csc^2(xy) - y$

$\Rightarrow \frac{dy}{dx}\left[x + x\csc^2(xy)\right] = -y\left[\csc^2(xy)+1\right] \Rightarrow \frac{dy}{dx} = \frac{-y\left[\csc^2(xy)+1\right]}{x\left[1+\csc^2(xy)\right]} = -\frac{y}{x}$

11. $e^{2x} = \sin(x+3y) \Rightarrow 2e^{2x} = (1+3y')\cos(x+3y) \Rightarrow 1 + 3y' = \frac{2e^{2x}}{\cos(x+3y)} \Rightarrow 3y' = \frac{2e^{2x}}{\cos(x+3y)} - 1$

$\Rightarrow y' = \frac{2e^{2x}-\cos(x+3y)}{3\cos(x+3y)}$

12. $x + \sin y = xy \Rightarrow 1 + (\cos y)\frac{dy}{dx} = y + x\frac{dy}{dx} \Rightarrow (\cos y - x)\frac{dy}{dx} = y - 1 \Rightarrow \frac{dy}{dx} = \frac{y-1}{\cos y - x}$

13. $y\sin\left(\frac{1}{y}\right) = 1 - xy \Rightarrow y\left[\cos\left(\frac{1}{y}\right)\cdot(-1)\frac{1}{y^2}\cdot\frac{dy}{dx}\right] + \sin\left(\frac{1}{y}\right)\cdot\frac{dy}{dx} = -x\frac{dy}{dx} - y \Rightarrow$

$\frac{dy}{dx}\left[-\frac{1}{y}\cos\left(\frac{1}{y}\right) + \sin\left(\frac{1}{y}\right) + x\right] = -y \Rightarrow \frac{dy}{dx} = \frac{-y}{-\frac{1}{y}\cos\left(\frac{1}{y}\right)+\sin\left(\frac{1}{y}\right)+x} = \frac{-y^2}{y\sin\left(\frac{1}{y}\right)-\cos\left(\frac{1}{y}\right)+xy}$

14. $e^{x^2y} = 2x + 2y \Rightarrow e^{x^2y}(x^2y' + 2xy) = 2 + 2y' \Rightarrow x^2e^{x^2y}y' + 2xye^{x^2y} = 2 + 2y' \Rightarrow x^2e^{x^2y}y' - 2y' = 2 - 2xye^{x^2y}$

$\Rightarrow y' = \frac{2-2xye^{x^2y}}{x^2e^{x^2y}-2}$

15. $\theta^{1/2} + r^{1/2} = 1 \Rightarrow \frac{1}{2}\theta^{-1/2} + \frac{1}{2}r^{-1/2}\cdot\frac{dr}{d\theta} = 0 \Rightarrow \frac{dr}{d\theta}\left[\frac{1}{2\sqrt{r}}\right] = \frac{-1}{2\sqrt{\theta}} \Rightarrow \frac{dr}{d\theta} = -\frac{2\sqrt{r}}{2\sqrt{\theta}} = -\frac{\sqrt{r}}{\sqrt{\theta}}$

16. $r - 2\sqrt{\theta} = \frac{3}{2}\theta^{2/3} + \frac{4}{3}\theta^{3/4} \Rightarrow \frac{dr}{d\theta} - \theta^{-1/2} = \theta^{-1/3} + \theta^{-1/4} \Rightarrow \frac{dr}{d\theta} = \theta^{-1/2} + \theta^{-1/3} + \theta^{-1/4}$

17. $\sin(r\theta) = \frac{1}{2} \Rightarrow [\cos(r\theta)]\left(r + \theta\frac{dr}{d\theta}\right) = 0 \Rightarrow \frac{dr}{d\theta}[\theta\cos(r\theta)] = -r\cos(r\theta) \Rightarrow \frac{dr}{d\theta} = \frac{-r\cos(r\theta)}{\theta\cos(r\theta)} = -\frac{r}{\theta}$,

$\cos(r\theta) \neq 0$

18. $\cos r + \cot\theta = e^{r\theta} \Rightarrow -\sin r \cdot \frac{dr}{d\theta} - \csc^2\theta = e^{r\theta}\left(r + \theta\frac{dr}{d\theta}\right) \Rightarrow -\sin r \cdot \frac{dr}{d\theta} - \csc^2\theta = r\,e^{r\theta} + \theta\,e^{r\theta}\frac{dr}{d\theta}$
$\Rightarrow -\sin r\,\frac{dr}{d\theta} - \theta\,e^{r\theta}\,\frac{dr}{d\theta} = r\,e^{r\theta} + \csc^2\theta \Rightarrow \frac{dr}{d\theta} = -\frac{r\,e^{r\theta} + \csc^2\theta}{\theta\,e^{r\theta} + \sin r}$

19. $x^2 + y^2 = 1 \Rightarrow 2x + 2yy' = 0 \Rightarrow 2yy' = -2x \Rightarrow \frac{dy}{dx} = y' = -\frac{x}{y}$; now to find $\frac{d^2y}{dx^2}$, $\frac{d}{dx}(y') = \frac{d}{dx}\left(-\frac{x}{y}\right)$
$\Rightarrow y'' = \frac{y(-1) + xy'}{y^2} = \frac{-y + x\left(-\frac{x}{y}\right)}{y^2}$ since $y' = -\frac{x}{y} \Rightarrow \frac{d^2y}{dx^2} = y'' = \frac{-y^2 - x^2}{y^3} = \frac{-y^2 - (1 - y^2)}{y^3} = \frac{-1}{y^3}$

20. $x^{2/3} + y^{2/3} = 1 \Rightarrow \frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx}\left[\frac{2}{3}y^{-1/3}\right] = -\frac{2}{3}x^{-1/3} \Rightarrow y' = \frac{dy}{dx} = -\frac{x^{-1/3}}{y^{-1/3}} = -\left(\frac{y}{x}\right)^{1/3}$;
Differentiating again, $y'' = \frac{x^{1/3}\cdot\left(-\frac{1}{3}y^{-2/3}\right)y' + y^{1/3}\left(\frac{1}{3}x^{-2/3}\right)}{x^{2/3}} = \frac{x^{1/3}\cdot\left(-\frac{1}{3}y^{-2/3}\right)\left(-\frac{y^{1/3}}{x^{1/3}}\right) + y^{1/3}\left(\frac{1}{3}x^{-2/3}\right)}{x^{2/3}}$
$\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{3}x^{-2/3}y^{-1/3} + \frac{1}{3}y^{1/3}x^{-4/3} = \frac{y^{1/3}}{3x^{4/3}} + \frac{1}{3y^{1/3}x^{2/3}}$

21. $y^2 = e^{x^2} + 2x \Rightarrow 2yy' = 2x\,e^{x^2} + 2 \Rightarrow \frac{dy}{dx} = \frac{x\,e^{x^2} + 1}{y} \Rightarrow \frac{d^2y}{dx^2} = \frac{y\left(2x^2e^{x^2} + e^{x^2}\right) - \left(x\,e^{x^2} + 1\right)y'}{y^2}$
$= \frac{y\left(2x^2e^{x^2} + e^{x^2}\right) - \left(x\,e^{x^2} + 1\right)\cdot\frac{xe^{x^2} + 1}{y}}{y^2} = \frac{y^2\left(2x^2e^{x^2} + e^{x^2}\right) - \left(x^2e^{2x^2} + 2x\,e^{x^2} + 1\right)}{y^3}$
$= \frac{\left(2x^2y^2 + y^2 - 2x\right)e^{x^2} - x^2e^{2x^2} - 1}{y^3}$

22. $y^2 - 2x = 1 - 2y \Rightarrow 2y\cdot y' - 2 = -2y' \Rightarrow y'(2y + 2) = 2 \Rightarrow y' = \frac{1}{y+1} = (y+1)^{-1}$; then $y'' = -(y+1)^{-2}\cdot y'$
$= -(y+1)^{-2}(y+1)^{-1} \Rightarrow \frac{d^2y}{dx^2} = y'' = \frac{-1}{(y+1)^3}$

23. $2\sqrt{y} = x - y \Rightarrow y^{-1/2}y' = 1 - y' \Rightarrow y'\left(y^{-1/2} + 1\right) = 1 \Rightarrow \frac{dy}{dx} = y' = \frac{1}{y^{-1/2} + 1} = \frac{\sqrt{y}}{\sqrt{y} + 1}$; we can differentiate the equation $y'\left(y^{-1/2} + 1\right) = 1$ again to find y'': $y'\left(-\frac{1}{2}y^{-3/2}y'\right) + \left(y^{-1/2} + 1\right)y'' = 0$
$\Rightarrow \left(y^{-1/2} + 1\right)y'' = \frac{1}{2}[y']^2y^{-3/2} \Rightarrow \frac{d^2y}{dx^2} = y'' = \frac{\frac{1}{2}\left(\frac{1}{y^{-1/2} + 1}\right)^2 y^{-3/2}}{\left(y^{-1/2} + 1\right)} = \frac{1}{2y^{3/2}\left(y^{-1/2} + 1\right)^3} = \frac{1}{2\left(1 + \sqrt{y}\right)^3}$

24. $xy + y^2 = 1 \Rightarrow xy' + y + 2yy' = 0 \Rightarrow xy' + 2yy' = -y \Rightarrow y'(x + 2y) = -y \Rightarrow y' = \frac{-y}{(x+2y)}$; $\frac{d^2y}{dx^2} = y''$
$= \frac{-(x+2y)y' + y(1 + 2y')}{(x+2y)^2} = \frac{-(x+2y)\left[\frac{-y}{(x+2y)}\right] + y\left[1 + 2\left(\frac{-y}{(x+2y)}\right)\right]}{(x+2y)^2} = \frac{\frac{1}{(x+2y)}\left[y(x+2y) + y(x+2y) - 2y^2\right]}{(x+2y)^2}$
$= \frac{2y(x+2y) - 2y^2}{(x+2y)^3} = \frac{2y^2 + 2xy}{(x+2y)^3} = \frac{2y(x+y)}{(x+2y)^3}$

25. $x^3 + y^3 = 16 \Rightarrow 3x^2 + 3y^2y' = 0 \Rightarrow 3y^2y' = -3x^2 \Rightarrow y' = -\frac{x^2}{y^2}$; we differentiate $y^2y' = -x^2$ to find y'':
$y^2y'' + y'[2y\cdot y'] = -2x \Rightarrow y^2y'' = -2x - 2y[y']^2 \Rightarrow y'' = \frac{-2x - 2y\left(-\frac{x^2}{y^2}\right)^2}{y^2} = \frac{-2x - \frac{2x^4}{y^3}}{y^2}$
$= \frac{-2xy^3 - 2x^4}{y^5} \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{(2,2)} = \frac{-32 - 32}{32} = -2$

26. $xy + y^2 = 1 \Rightarrow xy' + y + 2yy' = 0 \Rightarrow y'(x + 2y) = -y \Rightarrow y' = \frac{-y}{(x+2y)} \Rightarrow y'' = \frac{(x+2y)(-y') - (-y)(1 + 2y')}{(x+2y)^2}$;
since $y'|_{(0,-1)} = -\frac{1}{2}$ we obtain $y''|_{(0,-1)} = \frac{(-2)\left(\frac{1}{2}\right) - (1)(0)}{4} = -\frac{1}{4}$

27. $y^2 + x^2 = y^4 - 2x$ at $(-2, 1)$ and $(-2, -1) \Rightarrow 2y\frac{dy}{dx} + 2x = 4y^3\frac{dy}{dx} - 2 \Rightarrow 2y\frac{dy}{dx} - 4y^3\frac{dy}{dx} = -2 - 2x$
$\Rightarrow \frac{dy}{dx}(2y - 4y^3) = -2 - 2x \Rightarrow \frac{dy}{dx} = \frac{x+1}{2y^3 - y} \Rightarrow \left.\frac{dy}{dx}\right|_{(-2,1)} = -1$ and $\left.\frac{dy}{dx}\right|_{(-2,-1)} = 1$

28. $(x^2+y^2)^2 = (x-y)^2$ at $(1,0)$ and $(1,-1)$ $\Rightarrow$ $2(x^2+y^2)\left(2x+2y\,\frac{dy}{dx}\right) = 2(x-y)\left(1-\frac{dy}{dx}\right)$
$\Rightarrow$ $\frac{dy}{dx}\left[2y(x^2+y^2)+(x-y)\right] = -2x(x^2+y^2)+(x-y)$ $\Rightarrow$ $\frac{dy}{dx} = \frac{-2x(x^2+y^2)+(x-y)}{2y(x^2+y^2)+(x-y)}$ $\Rightarrow$ $\left.\frac{dy}{dx}\right|_{(1,0)} = -1$
and $\left.\frac{dy}{dx}\right|_{(1,-1)} = 1$

29. $x^2+xy-y^2=1 \Rightarrow 2x+y+xy'-2yy'=0 \Rightarrow (x-2y)y' = -2x-y \Rightarrow y' = \frac{2x+y}{2y-x}$;
(a) the slope of the tangent line $m = y'|_{(2,3)} = \frac{7}{4} \Rightarrow$ the tangent line is $y-3 = \frac{7}{4}(x-2) \Rightarrow y = \frac{7}{4}x - \frac{1}{2}$
(b) the normal line is $y-3 = -\frac{4}{7}(x-2) \Rightarrow y = -\frac{4}{7}x + \frac{29}{7}$

30. $x^2+y^2=25 \Rightarrow 2x+2yy'=0 \Rightarrow y' = -\frac{x}{y}$;
(a) the slope of the tangent line $m = y'|_{(3,-4)} = -\left.\frac{x}{y}\right|_{(3,-4)} = \frac{3}{4} \Rightarrow$ the tangent line is $y+4 = \frac{3}{4}(x-3) \Rightarrow y = \frac{3}{4}x - \frac{25}{4}$
(b) the normal line is $y+4 = -\frac{4}{3}(x-3) \Rightarrow y = -\frac{4}{3}x$

31. $x^2y^2=9 \Rightarrow 2xy^2+2x^2yy'=0 \Rightarrow x^2yy' = -xy^2 \Rightarrow y' = -\frac{y}{x}$;
(a) the slope of the tangent line $m = y'|_{(-1,3)} = -\left.\frac{y}{x}\right|_{(-1,3)} = 3 \Rightarrow$ the tangent line is $y-3 = 3(x+1) \Rightarrow y = 3x+6$
(b) the normal line is $y-3 = -\frac{1}{3}(x+1) \Rightarrow y = -\frac{1}{3}x + \frac{8}{3}$

32. $y^2-2x-4y-1=0 \Rightarrow 2yy'-2-4y'=0 \Rightarrow 2(y-2)y'=2 \Rightarrow y' = \frac{1}{y-2}$;
(a) the slope of the tangent line $m = y'|_{(-2,1)} = -1 \Rightarrow$ the tangent line is $y-1 = -1(x+2) \Rightarrow y = -x-1$
(b) the normal line is $y-1 = 1(x+2) \Rightarrow y = x+3$

33. $6x^2+3xy+2y^2+17y-6=0 \Rightarrow 12x+3y+3xy'+4yy'+17y'=0 \Rightarrow y'(3x+4y+17) = -12x-3y$
$\Rightarrow y' = \frac{-12x-3y}{3x+4y+17}$;
(a) the slope of the tangent line $m = y'|_{(-1,0)} = \left.\frac{-12x-3y}{3x+4y+17}\right|_{(-1,0)} = \frac{6}{7} \Rightarrow$ the tangent line is $y-0 = \frac{6}{7}(x+1)$
$\Rightarrow y = \frac{6}{7}x + \frac{6}{7}$
(b) the normal line is $y-0 = -\frac{7}{6}(x+1) \Rightarrow y = -\frac{7}{6}x - \frac{7}{6}$

34. $x^2-\sqrt{3}xy+2y^2=5 \Rightarrow 2x-\sqrt{3}xy'-\sqrt{3}y+4yy'=0 \Rightarrow y'\left(4y-\sqrt{3}x\right) = \sqrt{3}y-2x \Rightarrow y' = \frac{\sqrt{3}y-2x}{4y-\sqrt{3}x}$;
(a) the slope of the tangent line $m = y'|_{(\sqrt{3},2)} = \left.\frac{\sqrt{3}y-2x}{4y-\sqrt{3}x}\right|_{(\sqrt{3},2)} = 0 \Rightarrow$ the tangent line is $y = 2$
(b) the normal line is $x = \sqrt{3}$

35. $2xy+\pi\sin y = 2\pi \Rightarrow 2xy'+2y+\pi(\cos y)y'=0 \Rightarrow y'(2x+\pi\cos y) = -2y \Rightarrow y' = \frac{-2y}{2x+\pi\cos y}$;
(a) the slope of the tangent line $m = y'|_{(1,\frac{\pi}{2})} = \left.\frac{-2y}{2x+\pi\cos y}\right|_{(1,\frac{\pi}{2})} = -\frac{\pi}{2} \Rightarrow$ the tangent line is
$y-\frac{\pi}{2} = -\frac{\pi}{2}(x-1) \Rightarrow y = -\frac{\pi}{2}x+\pi$
(b) the normal line is $y-\frac{\pi}{2} = \frac{2}{\pi}(x-1) \Rightarrow y = \frac{2}{\pi}x - \frac{2}{\pi} + \frac{\pi}{2}$

36. $x\sin 2y = y\cos 2x \Rightarrow x(\cos 2y)2y' + \sin 2y = -2y\sin 2x + y'\cos 2x \Rightarrow y'(2x\cos 2y - \cos 2x)$
$= -\sin 2y - 2y\sin 2x \Rightarrow y' = \frac{\sin 2y + 2y\sin 2x}{\cos 2x - 2x\cos 2y}$;
(a) the slope of the tangent line $m = y'|_{(\frac{\pi}{4},\frac{\pi}{2})} = \left.\frac{\sin 2y+2y\sin 2x}{\cos 2x-2x\cos 2y}\right|_{(\frac{\pi}{4},\frac{\pi}{2})} = \frac{\pi}{\frac{\pi}{2}} = 2 \Rightarrow$ the tangent line is
$y-\frac{\pi}{2} = 2\left(x-\frac{\pi}{4}\right) \Rightarrow y = 2x$
(b) the normal line is $y-\frac{\pi}{2} = -\frac{1}{2}\left(x-\frac{\pi}{4}\right) \Rightarrow y = -\frac{1}{2}x + \frac{5\pi}{8}$

37. $y = 2\sin(\pi x - y) \Rightarrow y' = 2[\cos(\pi x - y)] \cdot (\pi - y') \Rightarrow y'[1 + 2\cos(\pi x - y)] = 2\pi\cos(\pi x - y)$
$\Rightarrow y' = \frac{2\pi\cos(\pi x - y)}{1 + 2\cos(\pi x - y)}$;

(a) the slope of the tangent line $m = y'|_{(1,0)} = \left.\frac{2\pi\cos(\pi x - y)}{1 + 2\cos(\pi x - y)}\right|_{(1,0)} = 2\pi \Rightarrow$ the tangent line is
$y - 0 = 2\pi(x - 1) \Rightarrow y = 2\pi x - 2\pi$

(b) the normal line is $y - 0 = -\frac{1}{2\pi}(x - 1) \Rightarrow y = -\frac{x}{2\pi} + \frac{1}{2\pi}$

38. $x^2\cos^2 y - \sin y = 0 \Rightarrow x^2(2\cos y)(-\sin y)y' + 2x\cos^2 y - y'\cos y = 0 \Rightarrow y'[-2x^2\cos y\sin y - \cos y]$
$= -2x\cos^2 y \Rightarrow y' = \frac{2x\cos^2 y}{2x^2\cos y\sin y + \cos y}$;

(a) the slope of the tangent line $m = y'|_{(0,\pi)} = \left.\frac{2x\cos^2 y}{2x^2\cos y\sin y + \cos y}\right|_{(0,\pi)} = 0 \Rightarrow$ the tangent line is $y = \pi$

(b) the normal line is $x = 0$

39. Solving $x^2 + xy + y^2 = 7$ and $y = 0 \Rightarrow x^2 = 7 \Rightarrow x = \pm\sqrt{7} \Rightarrow \left(-\sqrt{7}, 0\right)$ and $\left(\sqrt{7}, 0\right)$ are the points where the curve crosses the x-axis. Now $x^2 + xy + y^2 = 7 \Rightarrow 2x + y + xy' + 2yy' = 0 \Rightarrow (x + 2y)y' = -2x - y$
$\Rightarrow y' = -\frac{2x+y}{x+2y} \Rightarrow m = -\frac{2x+y}{x+2y} \Rightarrow$ the slope at $\left(-\sqrt{7}, 0\right)$ is $m = -\frac{-2\sqrt{7}}{-\sqrt{7}} = -2$ and the slope at $\left(\sqrt{7}, 0\right)$ is
$m = -\frac{2\sqrt{7}}{\sqrt{7}} = -2$. Since the slope is -2 in each case, the corresponding tangents must be parallel.

40. Let p and q be integers with $q > 0$ and suppose that $y = \sqrt[q]{x^p} = x^{p/q}$. Then $y^q = x^p$. Since p and q are integers and assuming y is a differentiable function of x, $\frac{d}{dx}(y^q) = \frac{d}{dx}(x^p) \Rightarrow qy^{q-1}\frac{dy}{dx} = px^{p-1} \Rightarrow \frac{dy}{dx} = \frac{px^{p-1}}{qy^{q-1}} = \frac{p}{q} \cdot \frac{x^{p-1}}{y^{q-1}}$
$= \frac{p}{q} \cdot \frac{x^{p-1}}{(x^{p/q})^{q-1}} = \frac{p}{q} \cdot \frac{x^{p-1}}{x^{p-p/q}} = \frac{p}{q} \cdot x^{p-1-(p-p/q)} = \frac{p}{q} \cdot x^{(p/q)-1}$

41. $y^4 = y^2 - x^2 \Rightarrow 4y^3y' = 2yy' - 2x \Rightarrow 2(2y^3 - y)y' = -2x \Rightarrow y' = \frac{x}{y - 2y^3}$; the slope of the tangent line at
$\left(\frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{2}\right)$ is $\left.\frac{x}{y - 2y^3}\right|_{\left(\frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{2}\right)} = \frac{\frac{\sqrt{3}}{4}}{\frac{\sqrt{3}}{2} - \frac{6\sqrt{3}}{8}} = \frac{\frac{1}{4}}{\frac{1}{2} - \frac{3}{4}} = \frac{1}{2 - 3} = -1$; the slope of the tangent line at $\left(\frac{\sqrt{3}}{4}, \frac{1}{2}\right)$
is $\left.\frac{x}{y - 2y^3}\right|_{\left(\frac{\sqrt{3}}{4}, \frac{1}{2}\right)} = \frac{\frac{\sqrt{3}}{4}}{\frac{1}{2} - \frac{2}{8}} = \frac{2\sqrt{3}}{4 - 2} = \sqrt{3}$

42. $x^3 + y^3 - 9xy = 0 \Rightarrow 3x^2 + 3y^2y' - 9xy' - 9y = 0 \Rightarrow y'(3y^2 - 9x) = 9y - 3x^2 \Rightarrow y' = \frac{9y - 3x^2}{3y^2 - 9x} = \frac{3y - x^2}{y^2 - 3x}$

(a) $y'|_{(4,2)} = \frac{5}{4}$ and $y'|_{(2,4)} = \frac{4}{5}$;

(b) $y' = 0 \Rightarrow \frac{3y - x^2}{y^2 - 3x} = 0 \Rightarrow 3y - x^2 = 0 \Rightarrow y = \frac{x^2}{3} \Rightarrow x^3 + \left(\frac{x^2}{3}\right)^3 - 9x\left(\frac{x^2}{3}\right) = 0 \Rightarrow x^6 - 54x^3 = 0$
$\Rightarrow x^3(x^3 - 54) = 0 \Rightarrow x = 0$ or $x = \sqrt[3]{54} = 3\sqrt[3]{2} \Rightarrow$ there is a horizontal tangent at $x = 3\sqrt[3]{2}$. To find the corresponding y-value, we will use part (c).

(c) $\frac{dx}{dy} = 0 \Rightarrow \frac{y^2 - 3x}{3y - x^2} = 0 \Rightarrow y^2 - 3x = 0 \Rightarrow y = \pm\sqrt{3x}$; $y = \sqrt{3x} \Rightarrow x^3 + \left(\sqrt{3x}\right)^3 - 9x\sqrt{3x} = 0$
$\Rightarrow x^3 - 6\sqrt{3}\,x^{3/2} = 0 \Rightarrow x^{3/2}\left(x^{3/2} - 6\sqrt{3}\right) = 0 \Rightarrow x^{3/2} = 0$ or $x^{3/2} = 6\sqrt{3} \Rightarrow x = 0$ or $x = \sqrt[3]{108} = 3\sqrt[3]{4}$.
Since the equation $x^3 + y^3 - 9xy = 0$ is symmetric in x and y, the graph is symmetric about the line $y = x$. That is, if (a, b) is a point on the folium, then so is (b, a). Moreover, if $y'|_{(a,b)} = m$, then $y'|_{(b,a)} = \frac{1}{m}$. Thus, if the folium has a horizontal tangent at (a, b), it has a vertical tangent at (b, a) so one might expect that with a horizontal tangent at $x = \sqrt[3]{54}$ and a vertical tangent at $x = 3\sqrt[3]{4}$, the points of tangency are $\left(\sqrt[3]{54}, 3\sqrt[3]{4}\right)$ and $\left(3\sqrt[3]{4}, \sqrt[3]{54}\right)$, respectively. One can check that these points do satisfy the equation $x^3 + y^3 - 9xy = 0$.

43. $x^2 + 2xy - 3y^2 = 0 \Rightarrow 2x + 2xy' + 2y - 6yy' = 0 \Rightarrow y'(2x - 6y) = -2x - 2y \Rightarrow y' = \frac{x+y}{3y-x} \Rightarrow$ the slope of the tangent line $m = y'|_{(1,1)} = \left.\frac{x+y}{3y-x}\right|_{(1,1)} = 1 \Rightarrow$ the equation of the normal line at $(1, 1)$ is $y - 1 = -1(x - 1) \Rightarrow y = -x + 2$. To find where the normal line intersects the curve we substitute into its equation: $x^2 + 2x(2 - x) - 3(2 - x)^2 = 0$ $\Rightarrow x^2 + 4x - 2x^2 - 3(4 - 4x + x^2) = 0 \Rightarrow -4x^2 + 16x - 12 = 0 \Rightarrow x^2 - 4x + 3 = 0 \Rightarrow (x - 3)(x - 1) = 0$ $\Rightarrow x = 3$ and $y = -x + 2 = -1$. Therefore, the normal to the curve at $(1, 1)$ intersects the curve at the point $(3, -1)$. Note that it also intersects the curve at $(1, 1)$.

44. $xy + 2x - y = 0 \Rightarrow x\frac{dy}{dx} + y + 2 - \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{y+2}{1-x}$; the slope of the line $2x + y = 0$ is -2. In order to be parallel, the normal lines must also have slope of -2. Since a normal is perpendicular to a tangent, the slope of the tangent is $\frac{1}{2}$. Therefore, $\frac{y+2}{1-x} = \frac{1}{2} \Rightarrow 2y + 4 = 1 - x \Rightarrow x = -3 - 2y$. Substituting in the original equation, $y(-3 - 2y) + 2(-3 - 2y) - y = 0 \Rightarrow y^2 + 4y + 3 = 0 \Rightarrow y = -3$ or $y = -1$. If $y = -3$, then $x = 3$ and $y + 3 = -2(x - 3) \Rightarrow y = -2x + 3$. If $y = -1$, then $x = -1$ and $y + 1 = -2(x + 1) \Rightarrow y = -2x - 3$.

45. $xy^3 + x^2y = 6 \Rightarrow x\left(3y^2\frac{dy}{dx}\right) + y^3 + x^2\frac{dy}{dx} + 2xy = 0 \Rightarrow \frac{dy}{dx}(3xy^2 + x^2) = -y^3 - 2xy \Rightarrow \frac{dy}{dx} = \frac{-y^3-2xy}{3xy^2+x^2}$ $= -\frac{y^3+2xy}{3xy^2+x^2}$; also, $xy^3 + x^2y = 6 \Rightarrow x(3y^2) + y^3\frac{dx}{dy} + x^2 + y\left(2x\frac{dx}{dy}\right) = 0 \Rightarrow \frac{dx}{dy}(y^3 + 2xy) = -3xy^2 - x^2$ $\Rightarrow \frac{dx}{dy} = -\frac{3xy^2+x^2}{y^3+2xy}$; thus $\frac{dx}{dy}$ appears to equal $\frac{1}{\frac{dy}{dx}}$. The two different treatments view the graphs as functions symmetric across the line $y = x$, so their slopes are reciprocals of one another at the corresponding points (a, b) and (b, a).

46. $x^3 + y^2 = \sin^2 y \Rightarrow 3x^2 + 2y\frac{dy}{dx} = (2\sin y)(\cos y)\frac{dy}{dx} \Rightarrow \frac{dy}{dx}(2y - 2\sin y\cos y) = -3x^2 \Rightarrow \frac{dy}{dx} = \frac{-3x^2}{2y-2\sin y\cos y}$ $= \frac{3x^2}{2\sin y\cos y-2y}$; also, $x^3 + y^2 = \sin^2 y \Rightarrow 3x^2\frac{dx}{dy} + 2y = 2\sin y\cos y \Rightarrow \frac{dx}{dy} = \frac{2\sin y\cos y-2y}{3x^2}$; thus $\frac{dx}{dy}$ appears to equal $\frac{1}{\frac{dy}{dx}}$. The two different treatments view the graphs as functions symmetric across the line $y = x$ so their slopes are reciprocals of one another at the corresponding points (a, b) and (b, a).

47-54. Example CAS commands:

Maple:

```
q1 := x^3-x*y+y^3 = 7;
pt := [x=2,y=1];
p1 := implicitplot( q1, x=-3..3, y=-3..3 ):
p1;
eval( q1, pt );
q2 := implicitdiff( q1, y, x );
m := eval( q2, pt );
tan_line := y = 1 + m*(x-2);
p2 := implicitplot( tan_line, x=-5..5, y=-5..5, color=green ):
p3 := pointplot( eval([x,y],pt), color=blue ):
display( [p1,p2,p3], ="Section 2.8 #47(c)" );
```

Mathematica: (functions and x0 may vary):

Note use of double equal sign (logic statement) in definition of eqn and tanline.

```
<<Graphics`ImplicitPlot`
Clear[x, y]
{x0, y0}={1, π/4};
eqn=x + Tan[y/x]==2;
ImplicitPlot[eqn,{ x, x0 − 3, x0 + 3},{y, y0 − 3, y0 + 3}]
eqn/.{x → x0, y → y0}
eqn/.{ y → y[x]}
```

```
D[%, x]
Solve[%, y'[x]]
slope=y'[x]/.First[%]
m=slope/.{x → x0, y[x] → y0}
tanline=y==y0 + m (x − x0)
ImplicitPlot[{eqn, tanline}, {x, x0 − 3, x0 + 3},{y, y0 − 3, y0 + 3}]
```

2.9 INVERSE FUNCTIONS AND THEIR DERIVATIVES

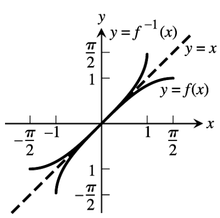
1. Yes one-to-one, the graph passes the horizontal test.

2. Not one-to-one, the graph fails the horizontal test.

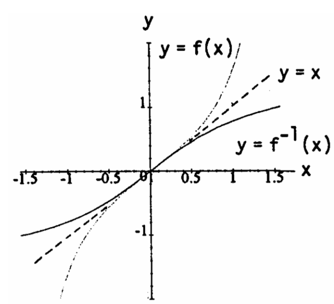
3. Not one-to-one since (for example) the horizontal line $y = 2$ intersects the graph twice.

4. Not one-to-one, the graph fails the horizontal test.

5. Domain: $-1 \le x \le 1$, Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$



6. Domain: $-\infty < x < \infty$, Range: $-\frac{\pi}{2} < y \le \frac{\pi}{2}$



7. Step 1: $y = x^2 + 1 \Rightarrow x^2 = y - 1 \Rightarrow x = \sqrt{y-1}$
   Step 2: $y = \sqrt{x-1} = f^{-1}(x)$

8. Step 1: $y = x^2 \Rightarrow x = -\sqrt{y}$, since $x \le 0$.
   Step 2: $y = -\sqrt{x} = f^{-1}(x)$

9. Step 1: $y = x^3 - 1 \Rightarrow x^3 = y + 1 \Rightarrow x = (y+1)^{1/3}$
   Step 2: $y = \sqrt[3]{x+1} = f^{-1}(x)$

10. Step 1: $y = x^2 - 2x + 1 \Rightarrow y = (x-1)^2 \Rightarrow \sqrt{y} = x - 1,$ since $x \ge 1 \Rightarrow x = 1 + \sqrt{y}$
    Step 2: $y = 1 + \sqrt{x} = f^{-1}(x)$

11. Step 1: $y = (x+1)^2 \Rightarrow \sqrt{y} = x + 1,$ since $x \ge -1 \Rightarrow x = \sqrt{y} - 1$
    Step 2: $y = \sqrt{x} - 1 = f^{-1}(x)$

12. Step 1: $y = x^{2/3} \Rightarrow x = y^{3/2}$
    Step 2: $y = x^{3/2} = f^{-1}(x)$

13. Step 1: $y = x^5 \Rightarrow x = y^{1/5}$
    Step 2: $y = \sqrt[5]{x} = f^{-1}(x);$

Domain and Range of f^{-1}: all reals;
$f(f^{-1}(x)) = (x^{1/5})^5 = x$ and $f^{-1}(f(x)) = (x^5)^{1/5} = x$

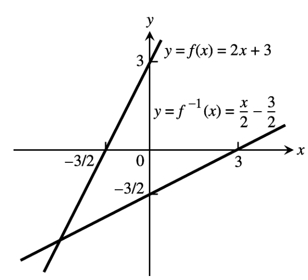
14. Step 1: $y = x^3 + 1 \Rightarrow x^3 = y - 1 \Rightarrow x = (y-1)^{1/3}$
Step 2: $y = \sqrt[3]{x-1} = f^{-1}(x)$;
Domain and Range of f^{-1}: all reals;
$f(f^{-1}(x)) = \left((x-1)^{1/3}\right)^3 + 1 = (x-1) + 1 = x$ and $f^{-1}(f(x)) = \left((x^3+1)-1\right)^{1/3} = (x^3)^{1/3} = x$

15. Step 1: $y = \frac{1}{x^2} \Rightarrow x^2 = \frac{1}{y} \Rightarrow x = \frac{1}{\sqrt{y}}$
Step 2: $y = \frac{1}{\sqrt{x}} = f^{-1}(x)$
Domain of f^{-1}: $x > 0$, Range of f^{-1}: $y > 0$;
$f(f^{-1}(x)) = \frac{1}{\left(\frac{1}{\sqrt{x}}\right)^2} = \frac{1}{\left(\frac{1}{x}\right)} = x$ and $f^{-1}(f(x)) = \frac{1}{\sqrt{\frac{1}{x^2}}} = \frac{1}{\left(\frac{1}{x}\right)} = x$ since $x > 0$

16. Step 1: $y = \frac{1}{x^3} \Rightarrow x^3 = \frac{1}{y} \Rightarrow x = \frac{1}{y^{1/3}}$
Step 2: $y = \frac{1}{x^{1/3}} = \sqrt[3]{\frac{1}{x}} = f^{-1}(x)$;
Domain of f^{-1}: $x \neq 0$, Range of f^{-1}: $y \neq 0$;
$f(f^{-1}(x)) = \frac{1}{(x^{-1/3})^3} = \frac{1}{x^{-1}} = x$ and $f^{-1}(f(x)) = \left(\frac{1}{x^3}\right)^{-1/3} = \left(\frac{1}{x}\right)^{-1} = x$

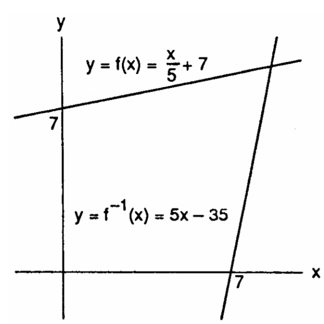
17. (a) $y = 2x + 3 \Rightarrow 2x = y - 3$
$\Rightarrow x = \frac{y}{2} - \frac{3}{2} \Rightarrow f^{-1}(x) = \frac{x}{2} - \frac{3}{2}$
(c) $\left.\frac{df}{dx}\right|_{x=-1} = 2$, $\left.\frac{df^{-1}}{dx}\right|_{x=1} = \frac{1}{2}$

(b)



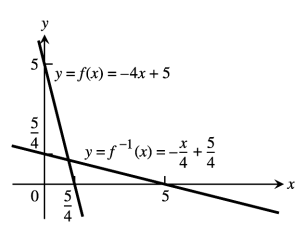
18. (a) $y = \frac{1}{5}x + 7 \Rightarrow \frac{1}{5}x = y - 7$
$\Rightarrow x = 5y - 35 \Rightarrow f^{-1}(x) = 5x - 35$
(c) $\left.\frac{df}{dx}\right|_{x=-1} = \frac{1}{5}$, $\left.\frac{df^{-1}}{dx}\right|_{x=34/5} = 5$

(b)

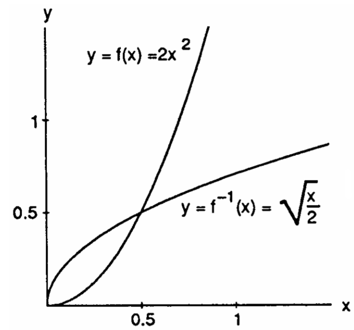


19. (a) $y = 5 - 4x \Rightarrow 4x = 5 - y$
$\Rightarrow x = \frac{5}{4} - \frac{y}{4} \Rightarrow f^{-1}(x) = \frac{5}{4} - \frac{x}{4}$
(c) $\left.\frac{df}{dx}\right|_{x=1/2} = -4$, $\left.\frac{df^{-1}}{dx}\right|_{x=3} = -\frac{1}{4}$

(b)

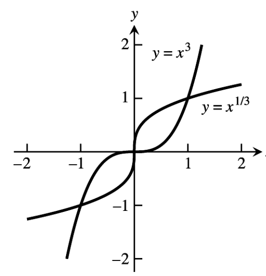


20. (a) $y = 2x^2 \Rightarrow x^2 = \frac{1}{2}\,y$
$\Rightarrow x = \frac{1}{\sqrt{2}}\sqrt{y} \Rightarrow f^{-1}(x) = \sqrt{\frac{x}{2}}$
(b)



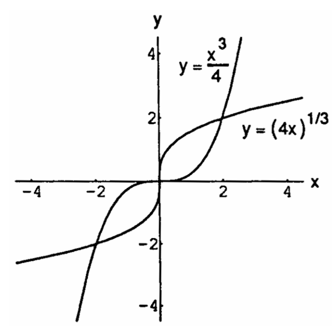
(c) $\left.\frac{df}{dx}\right|_{x=5} = \left.4x\right|_{x=5} = 20,$
$\left.\frac{df^{-1}}{dx}\right|_{x=50} = \left.\frac{1}{2\sqrt{2}}\,x^{-1/2}\right|_{x=50} = \frac{1}{20}$

21. (a) $f(g(x)) = \left(\sqrt[3]{x}\right)^3 = x,\ g(f(x)) = \sqrt[3]{x^3} = x$
(b)



(c) $f'(x) = 3x^2 \Rightarrow f'(1) = 3,\ f'(-1) = 3;$
$g'(x) = \frac{1}{3}\,x^{-2/3} \Rightarrow g'(1) = \frac{1}{3}\,,\ g'(-1) = \frac{1}{3}$
(d) The line $y = 0$ is tangent to $f(x) = x^3$ at $(0, 0)$; the line $x = 0$ is tangent to $g(x) = \sqrt[3]{x}$ at $(0, 0)$

22. (a) $h(k(x)) = \frac{1}{4}\left((4x)^{1/3}\right)^3 = x,$
$k(h(x)) = \left(4 \cdot \frac{x^3}{4}\right)^{1/3} = x$
(b)



(c) $h'(x) = \frac{3x^2}{4} \Rightarrow h'(2) = 3,\ h'(-2) = 3;$
$k'(x) = \frac{4}{3}\,(4x)^{-2/3} \Rightarrow k'(2) = \frac{1}{3},\ k'(-2) = \frac{1}{3}$
(d) The line $y = 0$ is tangent to $h(x) = \frac{x^3}{4}$ at $(0, 0)$; the line $x = 0$ is tangent to $k(x) = (4x)^{1/3}$ at $(0, 0)$

23. $\frac{df}{dx} = 3x^2 - 6x \Rightarrow \left.\frac{df^{-1}}{dx}\right|_{x=f(3)} = \left.\frac{1}{\frac{df}{dx}}\right|_{x=3} = \frac{1}{9}$

24. $\frac{df}{dx} = 2x - 4 \Rightarrow \left.\frac{df^{-1}}{dx}\right|_{x=f(5)} = \left.\frac{1}{\frac{df}{dx}}\right|_{x=5} = \frac{1}{6}$

25. (a) $y = mx \Rightarrow x = \frac{1}{m}\,y \Rightarrow f^{-1}(x) = \frac{1}{m}\,x$
(b) The graph of $y = f^{-1}(x)$ is a line through the origin with slope $\frac{1}{m}$.

26. $y = mx + b \Rightarrow x = \frac{y}{m} - \frac{b}{m} \Rightarrow f^{-1}(x) = \frac{1}{m}\,x - \frac{b}{m}$; the graph of $f^{-1}(x)$ is a line with slope $\frac{1}{m}$ and y-intercept $-\frac{b}{m}$.

27. Let $x_1 \neq x_2$ be two numbers in the domain of an increasing function f. Then, either $x_1 < x_2$ or $x_1 > x_2$ which implies $f(x_1) < f(x_2)$ or $f(x_1) > f(x_2)$, since f(x) is increasing. In either case, $f(x_1) \neq f(x_2)$ and f is one-to-one. Similar arguments hold if f is decreasing.

28. f(x) is increasing since $x_2 > x_1 \Rightarrow \frac{1}{3}\,x_2 + \frac{5}{6} > \frac{1}{3}\,x_1 + \frac{5}{6}$; $\frac{df}{dx} = \frac{1}{3} \Rightarrow \frac{df^{-1}}{dx} = \frac{1}{\left(\frac{1}{3}\right)} = 3$

29. f(x) is increasing since $x_2 > x_1 \Rightarrow 27x_2^3 > 27x_1^3$; $y = 27x^3 \Rightarrow x = \frac{1}{3}\,y^{1/3} \Rightarrow f^{-1}(x) = \frac{1}{3}\,x^{1/3}$;
$\frac{df}{dx} = 81x^2 \Rightarrow \frac{df^{-1}}{dx} = \left.\frac{1}{81x^2}\right|_{\frac{1}{3}x^{1/3}} = \frac{1}{9x^{2/3}} = \frac{1}{9}\,x^{-2/3}$

30. f(x) is decreasing since $x_2 > x_1 \Rightarrow 1 - 8x_2^3 < 1 - 8x_1^3$; $y = 1 - 8x^3 \Rightarrow x = \frac{1}{2}(1-y)^{1/3} \Rightarrow f^{-1}(x) = \frac{1}{2}(1-x)^{1/3}$;
$\frac{df}{dx} = -24x^2 \Rightarrow \frac{df^{-1}}{dx} = \frac{1}{-24x^2}\Big|_{\frac{1}{2}(1-x)^{1/3}} = \frac{-1}{6(1-x)^{2/3}} = -\frac{1}{6}(1-x)^{-2/3}$

31. f(x) is decreasing since $x_2 > x_1 \Rightarrow (1-x_2)^3 < (1-x_1)^3$; $y = (1-x)^3 \Rightarrow x = 1 - y^{1/3} \Rightarrow f^{-1}(x) = 1 - x^{1/3}$;
$\frac{df}{dx} = -3(1-x)^2 \Rightarrow \frac{df^{-1}}{dx} = \frac{1}{-3(1-x)^2}\Big|_{1-x^{1/3}} = \frac{-1}{3x^{2/3}} = -\frac{1}{3}x^{-2/3}$

32. f(x) is increasing since $x_2 > x_1 \Rightarrow x_2^{5/3} > x_1^{5/3}$; $y = x^{5/3} \Rightarrow x = y^{3/5} \Rightarrow f^{-1}(x) = x^{3/5}$;
$\frac{df}{dx} = \frac{5}{3}x^{2/3} \Rightarrow \frac{df^{-1}}{dx} = \frac{1}{\frac{5}{3}x^{2/3}}\Big|_{x^{3/5}} = \frac{3}{5x^{2/5}} = \frac{3}{5}x^{-2/5}$

33. The function g(x) is also one-to-one. The reasoning: f(x) is one-to-one means that if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$, so $-f(x_1) \neq -f(x_2)$ and therefore $g(x_1) \neq g(x_2)$. Therefore g(x) is one-to-one as well.

34. The function h(x) is also one-to-one. The reasoning: f(x) is one-to-one means that if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$, so $\frac{1}{f(x_1)} \neq \frac{1}{f(x_2)}$, and therefore $h(x_1) \neq h(x_2)$.

35. The composite is one-to-one also. The reasoning: If $x_1 \neq x_2$ then $g(x_1) \neq g(x_2)$ because g is one-to-one. Since $g(x_1) \neq g(x_2)$, we also have $f(g(x_1)) \neq f(g(x_2))$ because f is one-to-one. Thus, $f \circ g$ is one-to-one because $x_1 \neq x_2 \Rightarrow f(g(x_1)) \neq f(g(x_2))$.

36. Yes, g must be one-to-one. If g were not one-to-one, there would exist numbers $x_1 \neq x_2$ in the domain of g with $g(x_1) = g(x_2)$. For these numbers we would also have $f(g(x_1)) = f(g(x_2))$, contradicting the assumption that $f \circ g$ is one-to-one.

37. $(g \circ f)(x) = x \Rightarrow g(f(x)) = x \Rightarrow g'(f(x))f'(x) = 1$

38. The function $f(x) = x^n$ is diferentiable everywhere, and $g(x) = x^{1/n}$ is the inverse of f, so by Theorem 4
$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{n\,(g(x))^{n-1}} = \frac{1}{n\,(x^{1/n})^{n-1}} = \frac{1}{n\,x^{(n-1)/n}} = \frac{1}{n\,x^{1-1/n}} = \frac{1}{n}x^{(1/n)-1}$ provided $x \neq 0$.

39-46. Example CAS commands:

<u>Maple</u>:

```
with( plots );#41
f := x -> sqrt(3*x-2);
domain := 2/3 .. 4;
x0 := 3;
Df := D(f);                        # (a)
plot( [f(x),Df(x)], x=domain, color=[red,blue], linestyle=[1,3], legend=["y=f(x)","y=f '(x)"],
    title="#41(a) (Section 2.9)" );
q1 := solve( y=f(x), x );          # (b)
g := unapply( q1, y );
m1 := Df(x0);                      # (c)
t1 := f(x0)+m1*(x-x0);
y=t1;
m2 := 1/Df(x0);                    # (d)
t2 := g(f(x0)) + m2*(x-f(x0));
y=t2;
domaing := map(f,domain);          # (e)
```

```
p1 := plot( [f(x),x], x=domain, color=[pink,green], linestyle=[1,9], thickness=[3,0] ):
p2 := plot( g(x), x=domaing, color=cyan, linestyle=3, thickness=4 ):
p3 := plot( t1, x=x0-1..x0+1, color=red, linestyle=4, thickness=0 ):
p4 := plot( t2, x=f(x0)-1..f(x0)+1, color=blue, linestyle=7, thickness=1 ):
p5 := plot( [ [x0,f(x0)], [f(x0),x0] ], color=green ):
display( [p1,p2,p3,p4,p5], scaling=constrained, title="#41(e) (Section 2.9)" );
```

<u>Mathematica:</u> (assigned function and values for a, b, and x0 may vary)

If a function requires the odd root of a negative number, begin by loading the RealOnly package that allows Mathematica to do this.

```
<<Miscellaneous `RealOnly`
Clear[x, y]
{a,b} = {−2, 1}; x0 = 1/2 ;
f[x_] = (3x + 2) / (2x − 11)
Plot[{f[x], f'[x]}, {x, a, b}]
solx = Solve[y == f[x], x]
g[y_] = x /. solx[[1]]
y0 = f[x0]
ftan[x_] = y0 + f'[x0] (x-x0)
gtan[y_] = x0 + 1/ f'[x0] (y − y0)
Plot[{f[x], ftan[x], g[x], gtan[x], Identity[x]},{x, a, b},
Epilog → Line[{{x0, y0},{y0, x0}}], PlotRange → {{a,b},{a,b}}, AspectRatio → Automatic]
```

47-48. Example CAS commands:

<u>Maple</u>:

```
with( plots );
eq := cos(y) = x^(1/5);
domain := 0 .. 1;
x0 := 1/2;
f := unapply( solve( eq, y ), x );   # (a)
Df := D(f);
plot( [f(x),Df(x)], x=domain, color=[red,blue], linestyle=[1,3], legend=["y=f(x)","y=f '(x)"],
      title="#48(a) (Section 2.9)" );
q1 := solve( eq, x );                # (b)
g := unapply( q1, y );
m1 := Df(x0);                        # (c)
t1 := f(x0)+m1*(x-x0);
y=t1;
m2 := 1/Df(x0);                      # (d)
t2 := g(f(x0)) + m2*(x-f(x0));
y=t2;
domaing := map(f,domain);            # (e)
p1 := plot( [f(x),x], x=domain, color=[pink,green], linestyle=[1,9], thickness=[3,0] ):
p2 := plot( g(x), x=domaing, color=cyan, linestyle=3, thickness=4 ):
p3 := plot( t1, x=x0-1..x0+1, color=red, linestyle=4, thickness=0 ):
p4 := plot( t2, x=f(x0)-1..f(x0)+1, color=blue, linestyle=7, thickness=1 ):
p5 := plot( [ [x0,f(x0)], [f(x0),x0] ], color=green ):
display( [p1,p2,p3,p4,p5], scaling=constrained, title="#48(e) (Section 2.9)" );
```

<u>Mathematica:</u> (assigned function and values for a, b, and x0 may vary)

For problems 47 and 48, the code is just slightly altered. At times, different "parts" of solutions need to be used, as in the definitions of f[x] and g[y]

```
Clear[x, y]
{a,b} = {0, 1}; x0 = 1/2 ;
eqn = Cos[y] == x^(1/5)
soly = Solve[eqn, y]
f[x_] = y /. soly[[2]]
Plot[{f[x], f'[x]}, {x, a, b}]
solx = Solve[eqn, x]
g[y_] = x /. solx[[1]]
y0 = f[x0]
ftan[x_] = y0 + f'[x0] (x − x0)
gtan[y_] = x0 + 1/ f'[x0] (y − y0)
Plot[{f[x], ftan[x], g[x], gtan[x], Identity[x]},{x, a, b},
Epilog → Line[{{x0, y0},{y0, x0}}], PlotRange → {{a, b}, {a, b}}, AspectRatio → Automatic]
```

2.10 LOGARITHMIC FUNCTIONS

1. (a) $\ln 0.75 = \ln \frac{3}{4} = \ln 3 - \ln 4 = \ln 3 - \ln 2^2 = \ln 3 - 2\ln 2$
   (b) $\ln \frac{4}{9} = \ln 4 - \ln 9 = \ln 2^2 - \ln 3^2 = 2\ln 2 - 2\ln 3$
   (c) $\ln \frac{1}{2} = \ln 1 - \ln 2 = -\ln 2$
   (d) $\ln \sqrt[3]{9} = \frac{1}{3}\ln 9 = \frac{1}{3}\ln 3^2 = \frac{2}{3}\ln 3$
   (e) $\ln 3\sqrt{2} = \ln 3 + \ln 2^{1/2} = \ln 3 + \frac{1}{2}\ln 2$
   (f) $\ln \sqrt{13.5} = \frac{1}{2}\ln 13.5 = \frac{1}{2}\ln \frac{27}{2} = \frac{1}{2}(\ln 3^3 - \ln 2) = \frac{1}{2}(3\ln 3 - \ln 2)$

2. (a) $\ln \frac{1}{125} = \ln 1 - 3\ln 5 = -3\ln 5$
   (b) $\ln 9.8 = \ln \frac{49}{5} = \ln 7^2 - \ln 5 = 2\ln 7 - \ln 5$
   (c) $\ln 7\sqrt{7} = \ln 7^{3/2} = \frac{3}{2}\ln 7$
   (d) $\ln 1225 = \ln 35^2 = 2\ln 35 = 2\ln 5 + 2\ln 7$
   (e) $\ln 0.056 = \ln \frac{7}{125} = \ln 7 - \ln 5^3 = \ln 7 - 3\ln 5$
   (f) $\frac{\ln 35 + \ln \frac{1}{7}}{\ln 25} = \frac{\ln 5 + \ln 7 - \ln 7}{2\ln 5} = \frac{1}{2}$

3. (a) $\ln \sin\theta - \ln\left(\frac{\sin\theta}{5}\right) = \ln\left(\frac{\sin\theta}{\left(\frac{\sin\theta}{5}\right)}\right) = \ln 5$
   (b) $\ln(3x^2 - 9x) + \ln\left(\frac{1}{3x}\right) = \ln\left(\frac{3x^2 - 9x}{3x}\right) = \ln(x-3)$
   (c) $\frac{1}{2}\ln(4t^4) - \ln 2 = \ln\sqrt{4t^4} - \ln 2 = \ln 2t^2 - \ln 2 = \ln\left(\frac{2t^2}{2}\right) = \ln(t^2)$

4. (a) $\ln \sec\theta + \ln\cos\theta = \ln[(\sec\theta)(\cos\theta)] = \ln 1 = 0$
   (b) $\ln(8x+4) - \ln 2^2 = \ln(8x+4) - \ln 4 = \ln\left(\frac{8x+4}{4}\right) = \ln(2x+1)$
   (c) $3\ln\sqrt[3]{t^2-1} - \ln(t+1) = 3\ln(t^2-1)^{1/3} - \ln(t+1) = 3\left(\frac{1}{3}\right)\ln(t^2-1) - \ln(t+1) = \ln\left(\frac{(t+1)(t-1)}{(t+1)}\right)$
   $= \ln(t-1)$

5. (a) $e^{\ln 7.2} = 7.2$
   (b) $e^{-\ln x^2} = \frac{1}{e^{\ln x^2}} = \frac{1}{x^2}$
   (c) $e^{\ln x - \ln y} = e^{\ln(x/y)} = \frac{x}{y}$

6. (a) $e^{\ln(x^2+y^2)} = x^2 + y^2$
   (b) $e^{-\ln 0.3} = \frac{1}{e^{\ln 0.3}} = \frac{1}{0.3}$
   (c) $e^{\ln \pi x - \ln 2} = e^{\ln(\pi x/2)} = \frac{\pi x}{2}$

7. (a) $2\ln\sqrt{e} = 2\ln e^{1/2} = (2)\left(\frac{1}{2}\right)\ln e = 1$
   (b) $\ln(\ln e^e) = \ln(e\ln e) = \ln e = 1$
   (c) $\ln e^{(-x^2-y^2)} = (-x^2-y^2)\ln e = -x^2-y^2$

8. (a) $\ln\left(e^{\sec\theta}\right) = (\sec\theta)(\ln e) = \sec\theta$ (b) $\ln e^{(e^x)} = (e^x)(\ln e) = e^x$
(c) $\ln\left(e^{2\ln x}\right) = \ln\left(e^{\ln x^2}\right) = \ln x^2 = 2\ln x$

9. $\ln y = 2t + 4 \Rightarrow e^{\ln y} = e^{2t+4} \Rightarrow y = e^{2t+4}$

10. $\ln y = -t + 5 \Rightarrow e^{\ln y} = e^{-t+5} \Rightarrow y = e^{-t+5}$

11. $\ln(y - 40) = 5t \Rightarrow e^{\ln(y-40)} = e^{5t} \Rightarrow y - 40 = e^{5t} \Rightarrow y = e^{5t} + 40$

12. $\ln(1 - 2y) = t \Rightarrow e^{\ln(1-2y)} = e^t \Rightarrow 1 - 2y = e^t \Rightarrow -2y = e^t - 1 \Rightarrow y = -\left(\frac{e^t - 1}{2}\right)$

13. $\ln(y - 1) - \ln 2 = x + \ln x \Rightarrow \ln(y - 1) - \ln 2 - \ln x = x \Rightarrow \ln\left(\frac{y-1}{2x}\right) = x \Rightarrow e^{\ln\left(\frac{y-1}{2x}\right)} = e^x \Rightarrow \frac{y-1}{2x} = e^x$
$\Rightarrow y - 1 = 2xe^x \Rightarrow y = 2xe^x + 1$

14. $\ln(y^2 - 1) - \ln(y + 1) = \ln(\sin x) \Rightarrow \ln\left(\frac{y^2-1}{y+1}\right) = \ln(\sin x) \Rightarrow \ln(y - 1) = \ln(\sin x) \Rightarrow e^{\ln(y-1)} = e^{\ln(\sin x)}$
$\Rightarrow y - 1 = \sin x \Rightarrow y = \sin x + 1$

15. (a) $e^{2k} = 4 \Rightarrow \ln e^{2k} = \ln 4 \Rightarrow 2k\ln e = \ln 2^2 \Rightarrow 2k = 2\ln 2 \Rightarrow k = \ln 2$
(b) $100e^{10k} = 200 \Rightarrow e^{10k} = 2 \Rightarrow \ln e^{10k} = \ln 2 \Rightarrow 10k\ln e = \ln 2 \Rightarrow 10k = \ln 2 \Rightarrow k = \frac{\ln 2}{10}$
(c) $e^{k/1000} = a \Rightarrow \ln e^{k/1000} = \ln a \Rightarrow \frac{k}{1000}\ln e = \ln a \Rightarrow \frac{k}{1000} = \ln a \Rightarrow k = 1000\ln a$

16. (a) $e^{5k} = \frac{1}{4} \Rightarrow \ln e^{5k} = \ln 4^{-1} \Rightarrow 5k\ln e = -\ln 4 \Rightarrow 5k = -\ln 4 \Rightarrow k = -\frac{\ln 4}{5}$
(b) $80e^k = 1 \Rightarrow e^k = 80^{-1} \Rightarrow \ln e^k = \ln 80^{-1} \Rightarrow k\ln e = -\ln 80 \Rightarrow k = -\ln 80$
(c) $e^{(\ln 0.8)k} = 0.8 \Rightarrow \left(e^{\ln 0.8}\right)^k = 0.8 \Rightarrow (0.8)^k = 0.8 \Rightarrow k = 1$

17. (a) $e^{-0.3t} = 27 \Rightarrow \ln e^{-0.3t} = \ln 3^3 \Rightarrow (-0.3t)\ln e = 3\ln 3 \Rightarrow -0.3t = 3\ln 3 \Rightarrow t = -10\ln 3$
(b) $e^{kt} = \frac{1}{2} \Rightarrow \ln e^{kt} = \ln 2^{-1} = kt\ln e = -\ln 2 \Rightarrow t = -\frac{\ln 2}{k}$
(c) $e^{(\ln 0.2)t} = 0.4 \Rightarrow \left(e^{\ln 0.2}\right)^t = 0.4 \Rightarrow 0.2^t = 0.4 \Rightarrow \ln 0.2^t = \ln 0.4 \Rightarrow t\ln 0.2 = \ln 0.4 \Rightarrow t = \frac{\ln 0.4}{\ln 0.2}$

18. (a) $e^{-0.01t} = 1000 \Rightarrow \ln e^{-0.01t} = \ln 1000 \Rightarrow (-0.01t)\ln e = \ln 1000 \Rightarrow -0.01t = \ln 1000 \Rightarrow t = -100\ln 1000$
(b) $e^{kt} = \frac{1}{10} \Rightarrow \ln e^{kt} = \ln 10^{-1} = kt\ln e = -\ln 10 \Rightarrow kt = -\ln 10 \Rightarrow t = -\frac{\ln 10}{k}$
(c) $e^{(\ln 2)t} = \frac{1}{2} \Rightarrow \left(e^{\ln 2}\right)^t = 2^{-1} \Rightarrow 2^t = 2^{-1} \Rightarrow t = -1$

19. $e^{\sqrt{t}} = x^2 \Rightarrow \ln e^{\sqrt{t}} = \ln x^2 \Rightarrow \sqrt{t} = 2\ln x \Rightarrow t = 4(\ln x)^2$

20. $e^{x^2}e^{2x+1} = e^t \Rightarrow e^{x^2+2x+1} = e^t \Rightarrow \ln e^{x^2+2x+1} = \ln e^t \Rightarrow t = x^2 + 2x + 1$

21. (a) $5^{\log_5 7} = 7$ (b) $8^{\log_8 \sqrt{2}} = \sqrt{2}$ (c) $1.3^{\log_{1.3} 75} = 75$
(d) $\log_4 16 = \log_4 4^2 = 2\log_4 4 = 2 \cdot 1 = 2$ (e) $\log_3 \sqrt{3} = \log_3 3^{1/2} = \frac{1}{2}\log_3 3 = \frac{1}{2} \cdot 1 = \frac{1}{2} = 0.5$
(f) $\log_4\left(\frac{1}{4}\right) = \log_4 4^{-1} = -1\log_4 4 = -1 \cdot 1 = -1$

22. (a) $2^{\log_2 3} = 3$ (b) $10^{\log_{10}(1/2)} = \frac{1}{2}$ (c) $\pi^{\log_\pi 7} = 7$
(d) $\log_{11} 121 = \log_{11} 11^2 = 2\log_{11} 11 = 2 \cdot 1 = 2$
(e) $\log_{121} 11 = \log_{121} 121^{1/2} = \left(\frac{1}{2}\right)\log_{121} 121 = \left(\frac{1}{2}\right) \cdot 1 = \frac{1}{2}$
(f) $\log_3\left(\frac{1}{9}\right) = \log_3 3^{-2} = -2\log_3 3 = -2 \cdot 1 = -2$

23. (a) Let $z = \log_4 x \Rightarrow 4^z = x \Rightarrow 2^{2z} = x \Rightarrow (2^z)^2 = x \Rightarrow 2^z = \sqrt{x}$
(b) Let $z = \log_3 x \Rightarrow 3^z = x \Rightarrow (3^z)^2 = x^2 \Rightarrow 3^{2z} = x^2 \Rightarrow 9^z = x^2$
(c) $\log_2 \left(e^{(\ln 2)\sin x}\right) = \log_2 2^{\sin x} = \sin x$

24. (a) Let $z = \log_5 (3x^2) \Rightarrow 5^z = 3x^2 \Rightarrow 25^z = 9x^4$
(b) $\log_e (e^x) = x$
(c) $\log_4 \left(2^{e^x \sin x}\right) = \log_4 4^{(e^x \sin x)/2} = \frac{e^x \sin x}{2}$

25. (a) $\frac{\log_2 x}{\log_3 x} = \frac{\ln x}{\ln 2} \div \frac{\ln x}{\ln 3} = \frac{\ln x}{\ln 2} \cdot \frac{\ln 3}{\ln x} = \frac{\ln 3}{\ln 2}$
(b) $\frac{\log_2 x}{\log_8 x} = \frac{\ln x}{\ln 2} \div \frac{\ln x}{\ln 8} = \frac{\ln x}{\ln 2} \cdot \frac{\ln 8}{\ln x} = \frac{3 \ln 2}{\ln 2} = 3$
(c) $\frac{\log_x a}{\log_{x^2} a} = \frac{\ln a}{\ln x} \div \frac{\ln a}{\ln x^2} = \frac{\ln a}{\ln x} \cdot \frac{\ln x^2}{\ln a} = \frac{2 \ln x}{\ln x} = 2$

26. (a) $\frac{\log_9 x}{\log_3 x} = \frac{\ln x}{\ln 9} \div \frac{\ln x}{\ln 3} = \frac{\ln x}{2 \ln 3} \cdot \frac{\ln 3}{\ln x} = \frac{1}{2}$
(b) $\frac{\log_{\sqrt{10}} x}{\log_{\sqrt{2}} x} = \frac{\ln x}{\ln \sqrt{10}} \div \frac{\ln x}{\ln \sqrt{2}} = \frac{\ln x}{\left(\frac{1}{2}\right) \ln 10} \cdot \frac{\left(\frac{1}{2}\right) \ln 2}{\ln x} = \frac{\ln 2}{\ln 10}$
(c) $\frac{\log_a b}{\log_b a} = \frac{\ln b}{\ln a} \div \frac{\ln a}{\ln b} = \frac{\ln b}{\ln a} \cdot \frac{\ln b}{\ln a} = \left(\frac{\ln b}{\ln a}\right)^2$

27. $y = \ln 3x \Rightarrow y' = \left(\frac{1}{3x}\right)(3) = \frac{1}{x}$

28. $y = \ln kx \Rightarrow y' = \left(\frac{1}{kx}\right)(k) = \frac{1}{x}$

29. $y = \ln (t^2) \Rightarrow \frac{dy}{dt} = \left(\frac{1}{t^2}\right)(2t) = \frac{2}{t}$

30. $y = \ln \left(t^{3/2}\right) \Rightarrow \frac{dy}{dt} = \left(\frac{1}{t^{3/2}}\right)\left(\frac{3}{2} t^{1/2}\right) = \frac{3}{2t}$

31. $y = \ln \frac{3}{x} = \ln 3x^{-1} \Rightarrow \frac{dy}{dx} = \left(\frac{1}{3x^{-1}}\right)(-3x^{-2}) = -\frac{1}{x}$

32. $y = \ln \frac{10}{x} = \ln 10x^{-1} \Rightarrow \frac{dy}{dx} = \left(\frac{1}{10x^{-1}}\right)(-10x^{-2}) = -\frac{1}{x}$

33. $y = \ln (\theta + 1) \Rightarrow \frac{dy}{d\theta} = \left(\frac{1}{\theta + 1}\right)(1) = \frac{1}{\theta + 1}$

34. $y = \ln (2\theta + 2) \Rightarrow \frac{dy}{d\theta} = \left(\frac{1}{2\theta + 2}\right)(2) = \frac{1}{\theta + 1}$

35. $y = \ln x^3 \Rightarrow \frac{dy}{dx} = \left(\frac{1}{x^3}\right)(3x^2) = \frac{3}{x}$

36. $y = (\ln x)^3 \Rightarrow \frac{dy}{dx} = 3(\ln x)^2 \cdot \frac{d}{dx}(\ln x) = \frac{3(\ln x)^2}{x}$

37. $y = t(\ln t)^2 \Rightarrow \frac{dy}{dt} = (\ln t)^2 + 2t(\ln t) \cdot \frac{d}{dt}(\ln t) = (\ln t)^2 + \frac{2t \ln t}{t} = (\ln t)^2 + 2 \ln t$

38. $y = t\sqrt{\ln t} = t(\ln t)^{1/2} \Rightarrow \frac{dy}{dt} = (\ln t)^{1/2} + \frac{1}{2} t(\ln t)^{-1/2} \cdot \frac{d}{dt}(\ln t) = (\ln t)^{1/2} + \frac{t(\ln t)^{-1/2}}{2t} = (\ln t)^{1/2} + \frac{1}{2(\ln t)^{1/2}}$

39. $y = \frac{x^4}{4} \ln x - \frac{x^4}{16} \Rightarrow \frac{dy}{dx} = x^3 \ln x + \frac{x^4}{4} \cdot \frac{1}{x} - \frac{4x^3}{16} = x^3 \ln x$

40. $y = \frac{x^3}{3} \ln x - \frac{x^3}{9} \Rightarrow \frac{dy}{dx} = x^2 \ln x + \frac{x^3}{3} \cdot \frac{1}{x} - \frac{3x^2}{9} = x^2 \ln x$

41. $y = \frac{\ln t}{t} \Rightarrow \frac{dy}{dt} = \frac{t\left(\frac{1}{t}\right) - (\ln t)(1)}{t^2} = \frac{1 - \ln t}{t^2}$

42. $y = \frac{1 + \ln t}{t} \Rightarrow \frac{dy}{dt} = \frac{t\left(\frac{1}{t}\right) - (1 + \ln t)(1)}{t^2} = \frac{1 - 1 - \ln t}{t^2} = -\frac{\ln t}{t^2}$

43. $y = \frac{\ln x}{1 + \ln x} \Rightarrow y' = \frac{(1 + \ln x)\left(\frac{1}{x}\right) - (\ln x)\left(\frac{1}{x}\right)}{(1 + \ln x)^2} = \frac{\frac{1}{x} + \frac{\ln x}{x} - \frac{\ln x}{x}}{(1 + \ln x)^2} = \frac{1}{x(1 + \ln x)^2}$

44. $y = \frac{x \ln x}{1 + \ln x} \Rightarrow y' = \frac{(1+\ln x)\left(\ln x + x \cdot \frac{1}{x}\right) - (x \ln x)\left(\frac{1}{x}\right)}{(1+\ln x)^2} = \frac{(1 + \ln x)^2 - \ln x}{(1 + \ln x)^2} = 1 - \frac{\ln x}{(1 + \ln x)^2}$

45. $y = \ln(\ln x) \Rightarrow y' = \left(\frac{1}{\ln x}\right)\left(\frac{1}{x}\right) = \frac{1}{x \ln x}$

46. $y = \ln(\ln(\ln x)) \Rightarrow y' = \frac{1}{\ln(\ln x)} \cdot \frac{d}{dx}(\ln(\ln x)) = \frac{1}{\ln(\ln x)} \cdot \frac{1}{\ln x} \cdot \frac{d}{dx}(\ln x) = \frac{1}{x(\ln x)\ln(\ln x)}$

47. $y = \theta[\sin(\ln\theta) + \cos(\ln\theta)] \Rightarrow \frac{dy}{d\theta} = [\sin(\ln\theta) + \cos(\ln\theta)] + \theta\left[\cos(\ln\theta) \cdot \frac{1}{\theta} - \sin(\ln\theta) \cdot \frac{1}{\theta}\right]$
$= \sin(\ln\theta) + \cos(\ln\theta) + \cos(\ln\theta) - \sin(\ln\theta) = 2\cos(\ln\theta)$

48. $y = \ln(\sec\theta + \tan\theta) \Rightarrow \frac{dy}{d\theta} = \frac{\sec\theta\tan\theta + \sec^2\theta}{\sec\theta + \tan\theta} = \frac{\sec\theta(\tan\theta + \sec\theta)}{\tan\theta + \sec\theta} = \sec\theta$

49. $y = \ln\frac{1}{x\sqrt{x+1}} = -\ln x - \frac{1}{2}\ln(x+1) \Rightarrow y' = -\frac{1}{x} - \frac{1}{2}\left(\frac{1}{x+1}\right) = -\frac{2(x+1)+x}{2x(x+1)} = -\frac{3x+2}{2x(x+1)}$

50. $y = \frac{1}{2}\ln\frac{1+x}{1-x} = \frac{1}{2}[\ln(1+x) - \ln(1-x)] \Rightarrow y' = \frac{1}{2}\left[\frac{1}{1+x} - \left(\frac{1}{1-x}\right)(-1)\right] = \frac{1}{2}\left[\frac{1-x+1+x}{(1+x)(1-x)}\right] = \frac{1}{1-x^2}$

51. $y = \frac{1+\ln t}{1-\ln t} \Rightarrow \frac{dy}{dt} = \frac{(1-\ln t)\left(\frac{1}{t}\right) - (1+\ln t)\left(\frac{-1}{t}\right)}{(1-\ln t)^2} = \frac{\frac{1}{t} - \frac{\ln t}{t} + \frac{1}{t} + \frac{\ln t}{t}}{(1-\ln t)^2} = \frac{2}{t(1-\ln t)^2}$

52. $y = \sqrt{\ln\sqrt{t}} = \left(\ln t^{1/2}\right)^{1/2} \Rightarrow \frac{dy}{dt} = \frac{1}{2}\left(\ln t^{1/2}\right)^{-1/2} \cdot \frac{d}{dt}\left(\ln t^{1/2}\right) = \frac{1}{2}\left(\ln t^{1/2}\right)^{-1/2} \cdot \frac{1}{t^{1/2}} \cdot \frac{d}{dt}\left(t^{1/2}\right)$
$= \frac{1}{2}\left(\ln t^{1/2}\right)^{-1/2} \cdot \frac{1}{t^{1/2}} \cdot \frac{1}{2}t^{-1/2} = \frac{1}{4t\sqrt{\ln\sqrt{t}}}$

53. $y = \ln(\sec(\ln\theta)) \Rightarrow \frac{dy}{d\theta} = \frac{1}{\sec(\ln\theta)} \cdot \frac{d}{d\theta}(\sec(\ln\theta)) = \frac{\sec(\ln\theta)\tan(\ln\theta)}{\sec(\ln\theta)} \cdot \frac{d}{d\theta}(\ln\theta) = \frac{\tan(\ln\theta)}{\theta}$

54. $y = \ln\frac{\sqrt{\sin\theta\cos\theta}}{1+2\ln\theta} = \frac{1}{2}(\ln\sin\theta + \ln\cos\theta) - \ln(1+2\ln\theta) \Rightarrow \frac{dy}{d\theta} = \frac{1}{2}\left(\frac{\cos\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta}\right) - \frac{\frac{2}{\theta}}{1+2\ln\theta}$
$= \frac{1}{2}\left[\cot\theta - \tan\theta - \frac{4}{\theta(1+2\ln\theta)}\right]$

55. $y = \ln\left(\frac{(x^2+1)^5}{\sqrt{1-x}}\right) = 5\ln(x^2+1) - \frac{1}{2}\ln(1-x) \Rightarrow y' = \frac{5 \cdot 2x}{x^2+1} - \frac{1}{2}\left(\frac{1}{1-x}\right)(-1) = \frac{10x}{x^2+1} + \frac{1}{2(1-x)}$

56. $y = \ln\sqrt{\frac{(x+1)^5}{(x+2)^{20}}} = \frac{1}{2}[5\ln(x+1) - 20\ln(x+2)] \Rightarrow y' = \frac{1}{2}\left(\frac{5}{x+1} - \frac{20}{x+2}\right) = \frac{5}{2}\left[\frac{(x+2) - 4(x+1)}{(x+1)(x+2)}\right]$
$= -\frac{5}{2}\left[\frac{3x+2}{(x+1)(x+2)}\right]$

57. $y = \sqrt{x(x+1)} = (x(x+1))^{1/2} \Rightarrow \ln y = \frac{1}{2}\ln(x(x+1)) \Rightarrow 2\ln y = \ln(x) + \ln(x+1) \Rightarrow \frac{2y'}{y} = \frac{1}{x} + \frac{1}{x+1}$
$\Rightarrow y' = \left(\frac{1}{2}\right)\sqrt{x(x+1)}\left(\frac{1}{x} + \frac{1}{x+1}\right) = \frac{\sqrt{x(x+1)}(2x+1)}{2x(x+1)} = \frac{2x+1}{2\sqrt{x(x+1)}}$

58. $y = \sqrt{(x^2+1)(x-1)^2} \Rightarrow \ln y = \frac{1}{2}[\ln(x^2+1) + 2\ln(x-1)] \Rightarrow \frac{y'}{y} = \frac{1}{2}\left(\frac{2x}{x^2+1} + \frac{2}{x-1}\right)$
$\Rightarrow y' = \sqrt{(x^2+1)(x-1)^2}\left(\frac{x}{x^2+1} + \frac{1}{x-1}\right) = \sqrt{(x^2+1)(x-1)^2}\left[\frac{x^2-x+x^2+1}{(x^2+1)(x-1)}\right] = \frac{(2x^2-x+1)|x-1|}{\sqrt{x^2+1}(x-1)}$

59. $y = \sqrt{\frac{t}{t+1}} = \left(\frac{t}{t+1}\right)^{1/2} \Rightarrow \ln y = \frac{1}{2}[\ln t - \ln(t+1)] \Rightarrow \frac{1}{y}\frac{dy}{dt} = \frac{1}{2}\left(\frac{1}{t} - \frac{1}{t+1}\right)$
$\Rightarrow \frac{dy}{dt} = \frac{1}{2}\sqrt{\frac{t}{t+1}}\left(\frac{1}{t} - \frac{1}{t+1}\right) = \frac{1}{2}\sqrt{\frac{t}{t+1}}\left[\frac{1}{t(t+1)}\right] = \frac{1}{2\sqrt{t}(t+1)^{3/2}}$

60. $y = \sqrt{\frac{1}{t(t+1)}} = [t(t+1)]^{-1/2} \Rightarrow \ln y = \frac{1}{2}[\ln t + \ln(t+1)] \Rightarrow \frac{1}{y}\frac{dy}{dt} = -\frac{1}{2}\left(\frac{1}{t} + \frac{1}{t+1}\right)$
$\Rightarrow \frac{dy}{dt} = -\frac{1}{2}\sqrt{\frac{1}{t(t+1)}}\left[\frac{2t+1}{t(t+1)}\right] = -\frac{2t+1}{2(t^2+t)^{3/2}}$

61. $y = \sqrt{\theta+3}\,(\sin\theta) = (\theta+3)^{1/2}\sin\theta \Rightarrow \ln y = \frac{1}{2}\ln(\theta+3) + \ln(\sin\theta) \Rightarrow \frac{1}{y}\frac{dy}{d\theta} = \frac{1}{2(\theta+3)} + \frac{\cos\theta}{\sin\theta}$
$\Rightarrow \frac{dy}{d\theta} = \sqrt{\theta+3}\,(\sin\theta)\left[\frac{1}{2(\theta+3)} + \cot\theta\right]$

62. $y = (\tan\theta)\sqrt{2\theta+1} = (\tan\theta)(2\theta+1)^{1/2} \Rightarrow \ln y = \ln(\tan\theta) + \frac{1}{2}\ln(2\theta+1) \Rightarrow \frac{1}{y}\frac{dy}{d\theta} = \frac{\sec^2\theta}{\tan\theta} + \left(\frac{1}{2}\right)\left(\frac{2}{2\theta+1}\right)$
$\Rightarrow \frac{dy}{d\theta} = (\tan\theta)\sqrt{2\theta+1}\left(\frac{\sec^2\theta}{\tan\theta} + \frac{1}{2\theta+1}\right) = (\sec^2\theta)\sqrt{2\theta+1} + \frac{\tan\theta}{\sqrt{2\theta+1}}$

63. $y = t(t+1)(t+2) \Rightarrow \ln y = \ln t + \ln(t+1) + \ln(t+2) \Rightarrow \frac{1}{y}\frac{dy}{dt} = \frac{1}{t} + \frac{1}{t+1} + \frac{1}{t+2}$
$\Rightarrow \frac{dy}{dt} = t(t+1)(t+2)\left(\frac{1}{t} + \frac{1}{t+1} + \frac{1}{t+2}\right) = t(t+1)(t+2)\left[\frac{(t+1)(t+2)+t(t+2)+t(t+1)}{t(t+1)(t+2)}\right] = 3t^2+6t+2$

64. $y = \frac{1}{t(t+1)(t+2)} \Rightarrow \ln y = \ln 1 - \ln t - \ln(t+1) - \ln(t+2) \Rightarrow \frac{1}{y}\frac{dy}{dt} = -\frac{1}{t} - \frac{1}{t+1} - \frac{1}{t+2}$
$\Rightarrow \frac{dy}{dt} = \frac{1}{t(t+1)(t+2)}\left[-\frac{1}{t} - \frac{1}{t+1} - \frac{1}{t+2}\right] = \frac{-1}{t(t+1)(t+2)}\left[\frac{(t+1)(t+2)+t(t+2)+t(t+1)}{t(t+1)(t+2)}\right]$
$= -\frac{3t^2+6t+2}{(t^3+3t^2+2t)^2}$

65. $y = \frac{\theta+5}{\theta\cos\theta} \Rightarrow \ln y = \ln(\theta+5) - \ln\theta - \ln(\cos\theta) \Rightarrow \frac{1}{y}\frac{dy}{d\theta} = \frac{1}{\theta+5} - \frac{1}{\theta} + \frac{\sin\theta}{\cos\theta}$
$\Rightarrow \frac{dy}{d\theta} = \left(\frac{\theta+5}{\theta\cos\theta}\right)\left(\frac{1}{\theta+5} - \frac{1}{\theta} + \tan\theta\right)$

66. $y = \frac{\theta\sin\theta}{\sqrt{\sec\theta}} \Rightarrow \ln y = \ln\theta + \ln(\sin\theta) - \frac{1}{2}\ln(\sec\theta) \Rightarrow \frac{1}{y}\frac{dy}{d\theta} = \left[\frac{1}{\theta} + \frac{\cos\theta}{\sin\theta} - \frac{(\sec\theta)(\tan\theta)}{2\sec\theta}\right]$
$\Rightarrow \frac{dy}{d\theta} = \frac{\theta\sin\theta}{\sqrt{\sec\theta}}\left(\frac{1}{\theta} + \cot\theta - \frac{1}{2}\tan\theta\right)$

67. $y = \frac{x\sqrt{x^2+1}}{(x+1)^{2/3}} \Rightarrow \ln y = \ln x + \frac{1}{2}\ln(x^2+1) - \frac{2}{3}\ln(x+1) \Rightarrow \frac{y'}{y} = \frac{1}{x} + \frac{x}{x^2+1} - \frac{2}{3(x+1)}$
$\Rightarrow y' = \frac{x\sqrt{x^2+1}}{(x+1)^{2/3}}\left[\frac{1}{x} + \frac{x}{x^2+1} - \frac{2}{3(x+1)}\right]$

68. $y = \sqrt{\frac{(x+1)^{10}}{(2x+1)^5}} \Rightarrow \ln y = \frac{1}{2}\left[10\ln(x+1) - 5\ln(2x+1)\right] \Rightarrow \frac{y'}{y} = \frac{5}{x+1} - \frac{5}{2x+1}$
$\Rightarrow y' = \sqrt{\frac{(x+1)^{10}}{(2x+1)^5}}\left(\frac{5}{x+1} - \frac{5}{2x+1}\right)$

69. $y = \sqrt[3]{\frac{x(x-2)}{x^2+1}} \Rightarrow \ln y = \frac{1}{3}\left[\ln x + \ln(x-2) - \ln(x^2+1)\right] \Rightarrow \frac{y'}{y} = \frac{1}{3}\left(\frac{1}{x} + \frac{1}{x-2} - \frac{2x}{x^2+1}\right)$
$\Rightarrow y' = \frac{1}{3}\sqrt[3]{\frac{x(x-2)}{x^2+1}}\left(\frac{1}{x} + \frac{1}{x-2} - \frac{2x}{x^2+1}\right)$

70. $y = \sqrt[3]{\frac{x(x+1)(x-2)}{(x^2+1)(2x+3)}} \Rightarrow \ln y = \frac{1}{3}\left[\ln x + \ln(x+1) + \ln(x-2) - \ln(x^2+1) - \ln(2x+3)\right]$
$\Rightarrow y' = \frac{1}{3}\sqrt[3]{\frac{x(x+1)(x-2)}{(x^2+1)(2x+3)}}\left(\frac{1}{x} + \frac{1}{x+1} + \frac{1}{x-2} - \frac{2x}{x^2+1} - \frac{2}{2x+3}\right)$

71. $y = \ln(\cos^2\theta) \Rightarrow \frac{dy}{d\theta} = \frac{1}{\cos^2\theta}\cdot 2\cos\theta\cdot(-\sin\theta) = -2\tan\theta$

72. $y = \ln\left(3\theta e^{-\theta}\right) = \ln 3 + \ln\theta + \ln e^{-\theta} = \ln 3 + \ln\theta - \theta \Rightarrow \frac{dy}{d\theta} = \frac{1}{\theta} - 1$

73. $y = \ln(3te^{-t}) = \ln 3 + \ln t + \ln e^{-t} = \ln 3 + \ln t - t \Rightarrow \frac{dy}{dt} = \frac{1}{t} - 1 = \frac{1-t}{t}$

74. $y = \ln(2e^{-t}\sin t) = \ln 2 + \ln e^{-t} + \ln\sin t = \ln 2 - t + \ln\sin t \Rightarrow \frac{dy}{dt} = -1 + \left(\frac{1}{\sin t}\right)\frac{d}{dt}(\sin t) = -1 + \frac{\cos t}{\sin t}$
$= \frac{\cos t - \sin t}{\sin t}$

75. $y = \ln \frac{e^\theta}{1+e^\theta} = \ln e^\theta - \ln\left(1+e^\theta\right) = \theta - \ln\left(1+e^\theta\right) \Rightarrow \frac{dy}{d\theta} = 1 - \left(\frac{1}{1+e^\theta}\right)\frac{d}{d\theta}\left(1+e^\theta\right) = 1 - \frac{e^\theta}{1+e^\theta} = \frac{1}{1+e^\theta}$

76. $y = \ln \frac{\sqrt{\theta}}{1+\sqrt{\theta}} = \ln\sqrt{\theta} - \ln\left(1+\sqrt{\theta}\right) \Rightarrow \frac{dy}{d\theta} = \left(\frac{1}{\sqrt{\theta}}\right)\frac{d}{d\theta}\left(\sqrt{\theta}\right) - \left(\frac{1}{1+\sqrt{\theta}}\right)\frac{d}{d\theta}\left(1+\sqrt{\theta}\right)$
$= \left(\frac{1}{\sqrt{\theta}}\right)\left(\frac{1}{2\sqrt{\theta}}\right) - \left(\frac{1}{1+\sqrt{\theta}}\right)\left(\frac{1}{2\sqrt{\theta}}\right) = \frac{\left(1+\sqrt{\theta}\right)-\sqrt{\theta}}{2\theta\left(1+\sqrt{\theta}\right)} = \frac{1}{2\theta\left(1+\sqrt{\theta}\right)} = \frac{1}{2\theta\left(1+\theta^{1/2}\right)}$

77. $y = e^{(\cos t + \ln t)} = e^{\cos t} e^{\ln t} = te^{\cos t} \Rightarrow \frac{dy}{dt} = e^{\cos t} + te^{\cos t}\frac{d}{dt}(\cos t) = (1 - t\sin t)\, e^{\cos t}$

78. $y = e^{\sin t}\left(\ln t^2 + 1\right) \Rightarrow \frac{dy}{dt} = e^{\sin t}(\cos t)\left(\ln t^2+1\right) + \frac{2}{t}e^{\sin t} = e^{\sin t}\left[\left(\ln t^2+1\right)(\cos t) + \frac{2}{t}\right]$

79. $\ln y = e^y \sin x \Rightarrow \left(\frac{1}{y}\right)y' = (y'e^y)(\sin x) + e^y\cos x \Rightarrow y'\left(\frac{1}{y} - e^y\sin x\right) = e^y\cos x$
$\Rightarrow y'\left(\frac{1-ye^y\sin x}{y}\right) = e^y\cos x \Rightarrow y' = \frac{ye^y\cos x}{1-ye^y\sin x}$

80. $\ln xy = e^{x+y} \Rightarrow \ln x + \ln y = e^{x+y} \Rightarrow \frac{1}{x} + \left(\frac{1}{y}\right)y' = (1+y')\,e^{x+y} \Rightarrow y'\left(\frac{1}{y} - e^{x+y}\right) = e^{x+y} - \frac{1}{x}$
$\Rightarrow y'\left(\frac{1-ye^{x+y}}{y}\right) = \frac{xe^{x+y}-1}{x} \Rightarrow y' = \frac{y\left(xe^{x+y}-1\right)}{x\left(1-ye^{x+y}\right)}$

81. $x^y = y^x \Rightarrow \ln x^y = \ln y^x \Rightarrow y\ln x = x\ln y \Rightarrow y\cdot\frac{1}{x} + y'\cdot\ln x = x\cdot\frac{1}{y}\cdot y' + (1)\cdot\ln y \Rightarrow \ln x\cdot y' - \frac{x}{y}\cdot y' = \ln y - \frac{y}{x}$
$\Rightarrow y' = \frac{\ln y - \frac{y}{x}}{\ln x - \frac{x}{y}} = \frac{xy\ln y - y^2}{xy\ln x - x^2} = \frac{y}{x}\left(\frac{x\ln y - y}{y\ln x - x}\right)$

82. $\tan y = e^x + \ln x \Rightarrow (\sec^2 y)\,y' = e^x + \frac{1}{x} \Rightarrow y' = \frac{(xe^x+1)\cos^2 y}{x}$

83. $y = 2^x \Rightarrow y' = 2^x \ln 2$

84. $y = 3^{-x} \Rightarrow y' = 3^{-x}(\ln 3)(-1) = -3^{-x}\ln 3$

85. $y = 5^{\sqrt{s}} \Rightarrow \frac{dy}{ds} = 5^{\sqrt{s}}(\ln 5)\left(\frac{1}{2}s^{-1/2}\right) = \left(\frac{\ln 5}{2\sqrt{s}}\right)5^{\sqrt{s}}$

86. $y = 2^{s^2} \Rightarrow \frac{dy}{ds} = 2^{s^2}(\ln 2)2s = \left(\ln 2^2\right)\left(s2^{s^2}\right) = (\ln 4)s2^{s^2}$

87. $y = x^\pi \Rightarrow y' = \pi x^{(\pi-1)}$

88. $y = t^{1-e} \Rightarrow \frac{dy}{dt} = (1-e)\,t^{-e}$

89. $y = \log_2 5\theta = \frac{\ln 5\theta}{\ln 2} \Rightarrow \frac{dy}{d\theta} = \left(\frac{1}{\ln 2}\right)\left(\frac{1}{5\theta}\right)(5) = \frac{1}{\theta\ln 2}$

90. $y = \log_3(1+\theta\ln 3) = \frac{\ln(1+\theta\ln 3)}{\ln 3} \Rightarrow \frac{dy}{d\theta} = \left(\frac{1}{\ln 3}\right)\left(\frac{1}{1+\theta\ln 3}\right)(\ln 3) = \frac{1}{1+\theta\ln 3}$

91. $y = \log_4 x + \log_4 x^2 = \frac{\ln x}{\ln 4} + \frac{\ln x^2}{\ln 4} = \frac{\ln x}{\ln 4} + 2\,\frac{\ln x}{\ln 4} = 3\,\frac{\ln x}{\ln 4} \Rightarrow y' = \frac{3}{x\ln 4}$

92. $y = \log_{25} e^x - \log_5\sqrt{x} = \frac{x\ln e}{\ln 25} - \frac{\ln x}{2\ln 5} = \frac{x}{2\ln 5} - \frac{\ln x}{2\ln 5} = \left(\frac{1}{2\ln 5}\right)(x - \ln x) \Rightarrow y' = \left(\frac{1}{2\ln 5}\right)\left(1 - \frac{1}{x}\right) = \frac{x-1}{2x\ln 5}$

93. $y = \log_2 r\cdot\log_4 r = \left(\frac{\ln r}{\ln 2}\right)\left(\frac{\ln r}{\ln 4}\right) = \frac{\ln^2 r}{(\ln 2)(\ln 4)} \Rightarrow \frac{dy}{dr} = \left[\frac{1}{(\ln 2)(\ln 4)}\right](2\ln r)\left(\frac{1}{r}\right) = \frac{2\ln r}{r(\ln 2)(\ln 4)}$

94. $y = \log_3 r\cdot\log_9 r = \left(\frac{\ln r}{\ln 3}\right)\left(\frac{\ln r}{\ln 9}\right) = \frac{\ln^2 r}{(\ln 3)(\ln 9)} \Rightarrow \frac{dy}{dr} = \left[\frac{1}{(\ln 3)(\ln 9)}\right](2\ln r)\left(\frac{1}{r}\right) = \frac{2\ln r}{r(\ln 3)(\ln 9)}$

95. $y = \log_3\left(\left(\frac{x+1}{x-1}\right)^{\ln 3}\right) = \frac{\ln\left(\frac{x+1}{x-1}\right)^{\ln 3}}{\ln 3} = \frac{(\ln 3)\ln\left(\frac{x+1}{x-1}\right)}{\ln 3} = \ln\left(\frac{x+1}{x-1}\right) = \ln(x+1) - \ln(x-1)$
$\Rightarrow \frac{dy}{dx} = \frac{1}{x+1} - \frac{1}{x-1} = \frac{-2}{(x+1)(x-1)}$

96. $y = \log_5\sqrt{\left(\frac{7x}{3x+2}\right)^{\ln 5}} = \log_5\left(\frac{7x}{3x+2}\right)^{(\ln 5)/2} = \frac{\ln\left(\frac{7x}{3x+2}\right)^{(\ln 5)/2}}{\ln 5} = \left(\frac{\ln 5}{2}\right)\left[\frac{\ln\left(\frac{7x}{3x+2}\right)}{\ln 5}\right] = \frac{1}{2}\ln\left(\frac{7x}{3x+2}\right)$
$= \frac{1}{2}\ln 7x - \frac{1}{2}\ln(3x+2) \Rightarrow \frac{dy}{dx} = \frac{7}{2\cdot 7x} - \frac{3}{2\cdot(3x+2)} = \frac{(3x+2)-3x}{2x(3x+2)} = \frac{1}{x(3x+2)}$

97. $y = \theta\sin(\log_7\theta) = \theta\sin\left(\frac{\ln\theta}{\ln 7}\right) \Rightarrow \frac{dy}{d\theta} = \sin\left(\frac{\ln\theta}{\ln 7}\right) + \theta\left[\cos\left(\frac{\ln\theta}{\ln 7}\right)\right]\left(\frac{1}{\theta\ln 7}\right) = \sin(\log_7\theta) + \frac{1}{\ln 7}\cos(\log_7\theta)$

98. $y = \log_7\left(\frac{\sin\theta\cos\theta}{e^{\theta}2^{\theta}}\right) = \frac{\ln(\sin\theta) + \ln(\cos\theta) - \ln e^{\theta} - \ln 2^{\theta}}{\ln 7} = \frac{\ln(\sin\theta) + \ln(\cos\theta) - \theta - \theta\ln 2}{\ln 7}$
$\Rightarrow \frac{dy}{d\theta} = \frac{\cos\theta}{(\sin\theta)(\ln 7)} - \frac{\sin\theta}{(\cos\theta)(\ln 7)} - \frac{1}{\ln 7} - \frac{\ln 2}{\ln 7} = \left(\frac{1}{\ln 7}\right)(\cot\theta - \tan\theta - 1 - \ln 2)$

99. $y = \log_5 e^x = \frac{\ln e^x}{\ln 5} = \frac{x}{\ln 5} \Rightarrow y' = \frac{1}{\ln 5}$

100. $y = \log_2\left(\frac{x^2e^2}{2\sqrt{x+1}}\right) = \frac{\ln x^2 + \ln e^2 - \ln 2 - \ln\sqrt{x+1}}{\ln 2} = \frac{2\ln x + 2 - \ln 2 - \frac{1}{2}\ln(x+1)}{\ln 2}$
$\Rightarrow y' = \frac{2}{x\ln 2} - \frac{1}{2(\ln 2)(x+1)} = \frac{4(x+1)-x}{2x(x+1)(\ln 2)} = \frac{3x+4}{2x(x+1)\ln 2}$

101. $y = 3^{\log_2 t} = 3^{(\ln t)/(\ln 2)} \Rightarrow \frac{dy}{dt} = \left[3^{(\ln t)/(\ln 2)}(\ln 3)\right]\left(\frac{1}{t\ln 2}\right) = \frac{1}{t}(\log_2 3)\,3^{\log_2 t}$

102. $y = 3\log_8(\log_2 t) = \frac{3\ln(\log_2 t)}{\ln 8} = \frac{3\ln\left(\frac{\ln t}{\ln 2}\right)}{\ln 8} \Rightarrow \frac{dy}{dt} = \left(\frac{3}{\ln 8}\right)\left[\frac{1}{(\ln t)/(\ln 2)}\right]\left(\frac{1}{t\ln 2}\right) = \frac{3}{t(\ln t)(\ln 8)} = \frac{1}{t(\ln t)(\ln 2)}$

103. $y = \log_2\left(8t^{\ln 2}\right) = \frac{\ln 8 + \ln\left(t^{\ln 2}\right)}{\ln 2} = \frac{3\ln 2 + (\ln 2)(\ln t)}{\ln 2} = 3 + \ln t \Rightarrow \frac{dy}{dt} = \frac{1}{t}$

104. $y = t\log_3\left(e^{(\sin t)(\ln 3)}\right) = \frac{t\ln\left(\left(e^{\ln 3}\right)^{\sin t}\right)}{\ln 3} = \frac{t\ln\left(3^{\sin t}\right)}{\ln 3} = \frac{t(\sin t)(\ln 3)}{\ln 3} = t\sin t \Rightarrow \frac{dy}{dt} = \sin t + t\cos t$

105. $y = (x+1)^x \Rightarrow \ln y = \ln(x+1)^x = x\ln(x+1) \Rightarrow \frac{y'}{y} = \ln(x+1) + x\cdot\frac{1}{(x+1)} \Rightarrow y' = (x+1)^x\left[\frac{x}{x+1} + \ln(x+1)\right]$

106. $y = x^{(x+1)} \Rightarrow \ln y = \ln x^{(x+1)} = (x+1)\ln x \Rightarrow \frac{y'}{y} = \ln x + (x+1)\left(\frac{1}{x}\right) = \ln x + 1 + \frac{1}{x} \Rightarrow y' = x^{(x+1)}\left(1 + \frac{1}{x} + \ln x\right)$

107. $y = \left(\sqrt{t}\right)^t = \left(t^{1/2}\right)^t = t^{t/2} \Rightarrow \ln y = \ln t^{t/2} = \left(\frac{t}{2}\right)\ln t \Rightarrow \frac{1}{y}\frac{dy}{dt} = \left(\frac{1}{2}\right)(\ln t) + \left(\frac{t}{2}\right)\left(\frac{1}{t}\right) = \frac{\ln t}{2} + \frac{1}{2}$
$\Rightarrow \frac{dy}{dt} = \left(\sqrt{t}\right)^t\left(\frac{\ln t}{2} + \frac{1}{2}\right)$

108. $y = t^{\sqrt{t}} = t^{\left(t^{1/2}\right)} \Rightarrow \ln y = \ln t^{\left(t^{1/2}\right)} = \left(t^{1/2}\right)(\ln t) \Rightarrow \frac{1}{y}\frac{dy}{dt} = \left(\frac{1}{2}t^{-1/2}\right)(\ln t) + t^{1/2}\left(\frac{1}{t}\right) = \frac{\ln t + 2}{2\sqrt{t}}$
$\Rightarrow \frac{dy}{dt} = \left(\frac{\ln t + 2}{2\sqrt{t}}\right)t^{\sqrt{t}}$

109. $y = (\sin x)^x \Rightarrow \ln y = \ln(\sin x)^x = x\ln(\sin x) \Rightarrow \frac{y'}{y} = \ln(\sin x) + x\left(\frac{\cos x}{\sin x}\right) \Rightarrow y' = (\sin x)^x[\ln(\sin x) + x\cot x]$

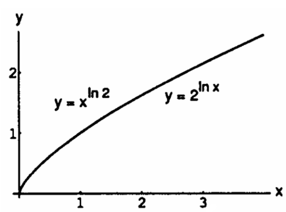
110. $y = x^{\sin x} \Rightarrow \ln y = \ln x^{\sin x} = (\sin x)(\ln x) \Rightarrow \frac{y'}{y} = (\cos x)(\ln x) + (\sin x)\left(\frac{1}{x}\right) = \frac{\sin x + x(\ln x)(\cos x)}{x}$
$\Rightarrow y' = x^{\sin x}\left[\frac{\sin x + x(\ln x)(\cos x)}{x}\right]$

111. $y = x^{\ln x}, x > 0 \Rightarrow \ln y = (\ln x)^2 \Rightarrow \frac{y'}{y} = 2(\ln x)\left(\frac{1}{x}\right) \Rightarrow y' = \left(x^{\ln x}\right)\left(\frac{\ln x^2}{x}\right)$

112. $y = (\ln x)^{\ln x} \Rightarrow \ln y = (\ln x)\ln(\ln x) \Rightarrow \frac{y'}{y} = \left(\frac{1}{x}\right)\ln(\ln x) + (\ln x)\left(\frac{1}{\ln x}\right)\frac{d}{dx}(\ln x) = \frac{\ln(\ln x)}{x} + \frac{1}{x}$
$\Rightarrow y' = \left(\frac{\ln(\ln x)+1}{x}\right)(\ln x)^{\ln x}$

113. (a) Begin with $y = \ln x$ and reduce the y-value by 3 $\Rightarrow y = \ln x - 3$.
(b) Begin with $y = \ln x$ and replace x with $x - 1 \Rightarrow y = \ln(x - 1)$.
(c) Begin with $y = \ln x$, replace x with $x + 1$, and increase the y-value by 3 $\Rightarrow y = \ln(x + 1) + 3$.
(d) Begin with $y = \ln x$, reduce the y-value by 4, and replace x with $x - 2 \Rightarrow y = \ln(x - 2) - 4$.
(e) Begin with $y = \ln x$ and replace x with $-x \Rightarrow y = \ln(-x)$.
(f) Begin with $y = \ln x$ and switch x and y $\Rightarrow x = \ln y$ or $y = e^x$.

114. The functions $f(x) = x^{\ln 2}$ and $g(x) = 2^{\ln x}$ appear to have identical graphs for $x > 0$. This is no accident, because $x^{\ln 2} = e^{\ln 2 \cdot \ln x} = \left(e^{\ln 2}\right)^{\ln x} = 2^{\ln x}$.



115. $y = A\sin(\ln x) + B\cos(\ln x) \Rightarrow y' = A\cos(\ln x)\cdot\frac{1}{x} - B\sin(\ln x)\cdot\frac{1}{x} = (A\cos(\ln x) - B\sin(\ln x))\cdot\frac{1}{x}$
$\Rightarrow y'' = (A\cos(\ln x) - B\sin(\ln x))\cdot\frac{-1}{x^2} + \left(-A\sin(\ln x)\cdot\frac{1}{x} - B\cos(\ln x)\cdot\frac{1}{x}\right)\cdot\frac{1}{x}$
$= (-A(\cos(\ln x) + \sin(\ln x)) + B(\sin(\ln x) - \cos(\ln x)))\cdot\frac{1}{x^2}$
$\Rightarrow x^2y'' + xy' + y = (-A(\cos(\ln x) + \sin(\ln x)) + B(\sin(\ln x) - \cos(\ln x))) + (A\cos(\ln x) - B\sin(\ln x))$
$+ (A\sin(\ln x) + B\cos(\ln x)) = 0$

116. $\lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = \lim_{n\to\infty}\left[\left(1 + \frac{1}{(n/x)}\right)^{(n/x)}\right]^x = e^x$ for any $x > 0$.

117. (a) Amount $= 8\left(\frac{1}{2}\right)^{t/12}$
(b) $8\left(\frac{1}{2}\right)^{t/12} = 1 \to \left(\frac{1}{2}\right)^{t/12} = \frac{1}{8} \to \left(\frac{1}{2}\right)^{t/12} = \left(\frac{1}{2}\right)^3 \to \frac{t}{12} = 3 \to t = 36$
There will be 1 gram remaining after 36 hours.

118. $y = y_0e^{-0.18t}$ represents the decay equation; solving $(0.9)y_0 = y_0e^{-0.18t} \Rightarrow t = \frac{\ln(0.9)}{-0.18} \approx 0.585$ days

2.11 INVERSE TRIGONOMETRIC FUNCTIONS

1. (a) $\frac{\pi}{4}$ (b) $-\frac{\pi}{3}$ (c) $\frac{\pi}{6}$
2. (a) $-\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $-\frac{\pi}{6}$
3. (a) $-\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $-\frac{\pi}{3}$
4. (a) $\frac{\pi}{6}$ (b) $-\frac{\pi}{4}$ (c) $\frac{\pi}{3}$
5. (a) $\frac{\pi}{3}$ (b) $\frac{3\pi}{4}$ (c) $\frac{\pi}{6}$
6. (a) $\frac{2\pi}{3}$ (b) $\frac{\pi}{4}$ (c) $\frac{5\pi}{6}$
7. (a) $\frac{3\pi}{4}$ (b) $\frac{\pi}{6}$ (c) $\frac{2\pi}{3}$
8. (a) $\frac{\pi}{4}$ (b) $\frac{5\pi}{6}$ (c) $\frac{\pi}{3}$
9. (a) $\frac{\pi}{4}$ (b) $-\frac{\pi}{3}$ (c) $\frac{\pi}{6}$
10. (a) $-\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $-\frac{\pi}{6}$

11. (a) $\frac{3\pi}{4}$ (b) $\frac{\pi}{6}$ (c) $\frac{2\pi}{3}$

12. (a) $\frac{\pi}{4}$ (b) $\frac{5\pi}{6}$ (c) $\frac{\pi}{3}$

13. $\lim\limits_{x \to 1^-} \sin^{-1} x = \frac{\pi}{2}$

14. $\lim\limits_{x \to -1^+} \cos^{-1} x = \pi$

15. $\lim\limits_{x \to \infty} \tan^{-1} x = \frac{\pi}{2}$

16. $\lim\limits_{x \to -\infty} \tan^{-1} x = -\frac{\pi}{2}$

17. $\lim\limits_{x \to \infty} \sec^{-1} x = \frac{\pi}{2}$

18. $\lim\limits_{x \to -\infty} \sec^{-1} x = \lim\limits_{x \to -\infty} \cos^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2}$

19. $\lim\limits_{x \to \infty} \csc^{-1} x = \lim\limits_{x \to \infty} \sin^{-1}\left(\frac{1}{x}\right) = 0$

20. $\lim\limits_{x \to -\infty} \csc^{-1} x = \lim\limits_{x \to -\infty} \sin^{-1}\left(\frac{1}{x}\right) = 0$

21. $y = \cos^{-1}(x^2) \Rightarrow \frac{dy}{dx} = -\frac{2x}{\sqrt{1-(x^2)^2}} = \frac{-2x}{\sqrt{1-x^4}}$

22. $y = \cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1} x \Rightarrow \frac{dy}{dx} = \frac{1}{|x|\sqrt{x^2-1}}$

23. $y = \sin^{-1}\sqrt{2}t \Rightarrow \frac{dy}{dt} = \frac{\sqrt{2}}{\sqrt{1-\left(\sqrt{2}t\right)^2}} = \frac{\sqrt{2}}{\sqrt{1-2t^2}}$

24. $y = \sin^{-1}(1-t) \Rightarrow \frac{dy}{dt} = \frac{-1}{\sqrt{1-(1-t)^2}} = \frac{-1}{\sqrt{2t-t^2}}$

25. $y = \sec^{-1}(2s+1) \Rightarrow \frac{dy}{ds} = \frac{2}{|2s+1|\sqrt{(2s+1)^2-1}} = \frac{2}{|2s+1|\sqrt{4s^2+4s}} = \frac{1}{|2s+1|\sqrt{s^2+s}}$

26. $y = \sec^{-1} 5s \Rightarrow \frac{dy}{ds} = \frac{5}{|5s|\sqrt{(5s)^2-1}} = \frac{1}{|s|\sqrt{25s^2-1}}$

27. $y = \csc^{-1}(x^2+1) \Rightarrow \frac{dy}{dx} = -\frac{2x}{|x^2+1|\sqrt{(x^2+1)^2-1}} = \frac{-2x}{(x^2+1)\sqrt{x^4+2x^2}}$

28. $y = \csc^{-1}\left(\frac{x}{2}\right) \Rightarrow \frac{dy}{dx} = -\frac{\left(\frac{1}{2}\right)}{\left|\frac{x}{2}\right|\sqrt{\left(\frac{x}{2}\right)^2-1}} = \frac{-1}{|x|\sqrt{\frac{x^2-4}{4}}} = \frac{-2}{|x|\sqrt{x^2-4}}$

29. $y = \sec^{-1}\left(\frac{1}{t}\right) = \cos^{-1} t \Rightarrow \frac{dy}{dt} = \frac{-1}{\sqrt{1-t^2}}$

30. $y = \sin^{-1}\left(\frac{3}{t^2}\right) = \csc^{-1}\left(\frac{t^2}{3}\right) \Rightarrow \frac{dy}{dt} = -\frac{\left(\frac{2t}{3}\right)}{\left|\frac{t^2}{3}\right|\sqrt{\left(\frac{t^2}{3}\right)^2-1}} = \frac{-2t}{t^2\sqrt{\frac{t^4-9}{9}}} = \frac{-6}{t\sqrt{t^4-9}}$

31. $y = \cot^{-1}\sqrt{t} = \cot^{-1} t^{1/2} \Rightarrow \frac{dy}{dt} = -\frac{\left(\frac{1}{2}\right)t^{-1/2}}{1+(t^{1/2})^2} = \frac{-1}{2\sqrt{t}(1+t)}$

32. $y = \cot^{-1}\sqrt{t-1} = \cot^{-1}(t-1)^{1/2} \Rightarrow \frac{dy}{dt} = -\frac{\left(\frac{1}{2}\right)(t-1)^{-1/2}}{1+[(t-1)^{1/2}]^2} = \frac{-1}{2\sqrt{t-1}(1+t-1)} = \frac{-1}{2t\sqrt{t-1}}$

33. $y = \ln(\tan^{-1} x) \Rightarrow \frac{dy}{dx} = \frac{\left(\frac{1}{1+x^2}\right)}{\tan^{-1} x} = \frac{1}{(\tan^{-1} x)(1+x^2)}$

34. $y = \tan^{-1}(\ln x) \Rightarrow \frac{dy}{dx} = \frac{\left(\frac{1}{x}\right)}{1+(\ln x)^2} = \frac{1}{x[1+(\ln x)^2]}$

35. $y = \csc^{-1}(e^t) \Rightarrow \frac{dy}{dt} = -\frac{e^t}{|e^t|\sqrt{(e^t)^2-1}} = \frac{-1}{\sqrt{e^{2t}-1}}$

36. $y = \cos^{-1}(e^{-t}) \Rightarrow \frac{dy}{dt} = -\frac{-e^{-t}}{\sqrt{1-(e^{-t})^2}} = \frac{e^{-t}}{\sqrt{1-e^{-2t}}}$

37. $y = s\sqrt{1-s^2} + \cos^{-1} s = s\,(1-s^2)^{1/2} + \cos^{-1} s \Rightarrow \frac{dy}{ds} = (1-s^2)^{1/2} + s\left(\frac{1}{2}\right)(1-s^2)^{-1/2}(-2s) - \frac{1}{\sqrt{1-s^2}}$
$= \sqrt{1-s^2} - \frac{s^2}{\sqrt{1-s^2}} - \frac{1}{\sqrt{1-s^2}} = \sqrt{1-s^2} - \frac{s^2+1}{\sqrt{1-s^2}} = \frac{1-s^2-s^2-1}{\sqrt{1-s^2}} = \frac{-2s^2}{\sqrt{1-s^2}}$

38. $y = \sqrt{s^2-1} - \sec^{-1} s = (s^2-1)^{1/2} - \sec^{-1} s \Rightarrow \frac{dy}{dx} = \left(\frac{1}{2}\right)(s^2-1)^{-1/2}(2s) - \frac{1}{|s|\sqrt{s^2-1}} = \frac{s}{\sqrt{s^2-1}} - \frac{1}{|s|\sqrt{s^2-1}}$
$= \frac{s\,|s|-1}{|s|\sqrt{s^2-1}}$

39. $y = \tan^{-1}\sqrt{x^2-1} + \csc^{-1} x = \tan^{-1}(x^2-1)^{1/2} + \csc^{-1} x \Rightarrow \frac{dy}{dx} = \frac{\left(\frac{1}{2}\right)(x^2-1)^{-1/2}(2x)}{1+\left[(x^2-1)^{1/2}\right]^2} - \frac{1}{|x|\sqrt{x^2-1}}$
$= \frac{1}{x\sqrt{x^2-1}} - \frac{1}{|x|\sqrt{x^2-1}} = 0$, for $x > 1$

40. $y = \cot^{-1}\left(\frac{1}{x}\right) - \tan^{-1} x = \frac{\pi}{2} - \tan^{-1}(x^{-1}) - \tan^{-1} x \Rightarrow \frac{dy}{dx} = 0 - \frac{-x^{-2}}{1+(x^{-1})^2} - \frac{1}{1+x^2} = \frac{1}{x^2+1} - \frac{1}{1+x^2} = 0$

41. $y = x\sin^{-1} x + \sqrt{1-x^2} = x\sin^{-1} x + (1-x^2)^{1/2} \Rightarrow \frac{dy}{dx} = \sin^{-1} x + x\left(\frac{1}{\sqrt{1-x^2}}\right) + \left(\frac{1}{2}\right)(1-x^2)^{-1/2}(-2x)$
$= \sin^{-1} x + \frac{x}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}} = \sin^{-1} x$

42. $y = \ln(x^2+4) - x\tan^{-1}\left(\frac{x}{2}\right) \Rightarrow \frac{dy}{dx} = \frac{2x}{x^2+4} - \tan^{-1}\left(\frac{x}{2}\right) - x\left[\frac{\left(\frac{1}{2}\right)}{1+\left(\frac{x}{2}\right)^2}\right] = \frac{2x}{x^2+4} - \tan^{-1}\left(\frac{x}{2}\right) - \frac{2x}{4+x^2} = -\tan^{-1}\left(\frac{x}{2}\right)$

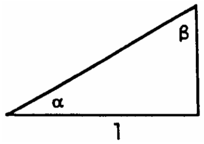
43. The angle α is the large angle between the wall and the right end of the blackboard minus the small angle between the left end of the blackboard and the wall $\Rightarrow \alpha = \cot^{-1}\left(\frac{x}{15}\right) - \cot^{-1}\left(\frac{x}{3}\right)$.

44. (a) From the symmetry of the diagram, we see that $\pi - \sec^{-1} x$ is the vertical distance from the graph of $y = \sec^{-1} x$ to the line $y = \pi$ and this distance is the same as the height of $y = \sec^{-1} x$ above the x-axis at $-x$; i.e., $\pi - \sec^{-1} x = \sec^{-1}(-x)$.
(b) $\cos^{-1}(-x) = \pi - \cos^{-1} x$, where $-1 \le x \le 1 \Rightarrow \cos^{-1}\left(-\frac{1}{x}\right) = \pi - \cos^{-1}\left(\frac{1}{x}\right)$, where $x \ge 1$ or $x \le -1$
$\Rightarrow \sec^{-1}(-x) = \pi - \sec^{-1} x$

45. If $x = 1$: $\sin^{-1}(1) + \cos^{-1}(1) = \frac{\pi}{2} + 0 = \frac{\pi}{2}$.
If $x = 0$: $\sin^{-1}(0) + \cos^{-1}(0) = 0 + \frac{\pi}{2} = \frac{\pi}{2}$.
If $x = -1$: $\sin^{-1}(-1) + \cos^{-1}(-1) = -\frac{\pi}{2} + \pi = \frac{\pi}{2}$.
The identity $\sin^{-1}(x) + \cos^{-1}(x) = \frac{\pi}{2}$ has been established for x in $(0, 1)$, by Figure 1.6.7. So now if x is in $(-1, 0)$, note that $-x$ is in $(0, 1)$, and we have that

$$\begin{aligned}
\sin^{-1}(x) + \cos^{-1}(x) &= -\sin^{-1}(-x) + \cos^{-1}(x) && \text{since } \sin^{-1} \text{ is odd} \\
&= -\sin^{-1}(-x) + \pi - \cos^{-1}(-x) && \text{by Eq. 3, Section 1.6} \\
&= -(\sin^{-1}(-x) + \cos^{-1}(-x)) + \pi \\
&= -\frac{\pi}{2} + \pi \\
&= \frac{\pi}{2}
\end{aligned}$$

This establishes the identity for all x in $[-1, 1]$.

46. 

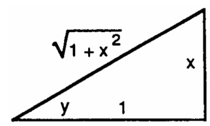
x $\Rightarrow \tan\alpha = x$ and $\tan\beta = \frac{1}{x} \Rightarrow \frac{\pi}{2} = \alpha + \beta = \tan^{-1} x + \tan^{-1}\frac{1}{x}$.

47. (a) Defined; there is an angle whose tangent is 2.
    (b) Not defined; there is no angle whose cosine is 2.
    (c) Not defined; there is no angle whose sine is $\sqrt{2}$.

48. (a) Not defined; there is no angle whose secant is 0.
    (b) Defined; there is an angle whose cotangent is $-\frac{1}{2}$.
    (c) Not defined; there is no angle whose cosecant is $\frac{1}{2}$.

49. $\csc^{-1} u = \frac{\pi}{2} - \sec^{-1} u \Rightarrow \frac{d}{dx}\left(\csc^{-1} u\right) = \frac{d}{dx}\left(\frac{\pi}{2} - \sec^{-1} u\right) = 0 - \frac{\frac{du}{dx}}{|u|\sqrt{u^2-1}} = -\frac{\frac{du}{dx}}{|u|\sqrt{u^2-1}}$, $|u| > 1$

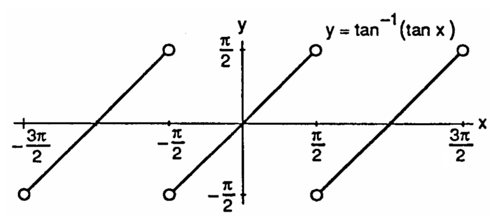
50. $y = \tan^{-1} x \Rightarrow \tan y = x \Rightarrow \frac{d}{dx}(\tan y) = \frac{d}{dx}(x)$
    $\Rightarrow (\sec^2 y)\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{\left(\sqrt{1+x^2}\right)^2}$
    $= \frac{1}{1+x^2}$, as indicated by the triangle



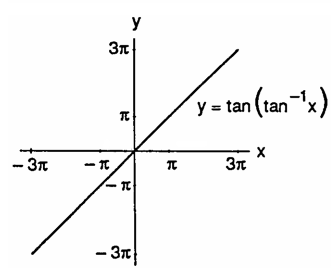
51. $f(x) = \sec x \Rightarrow f'(x) = \sec x \tan x \Rightarrow \left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\frac{df}{dx}\right|_{x=f^{-1}(b)}} = \frac{1}{\sec(\sec^{-1} b)\tan(\sec^{-1} b)} = \frac{1}{b\left(\pm\sqrt{b^2-1}\right)}$.
    Since the slope of $\sec^{-1} x$ is always positive, we the right sign by writing $\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}$.

52. $\cot^{-1} u = \frac{\pi}{2} - \tan^{-1} u \Rightarrow \frac{d}{dx}\left(\cot^{-1} u\right) = \frac{d}{dx}\left(\frac{\pi}{2} - \tan^{-1} u\right) = 0 - \frac{\frac{du}{dx}}{1+u^2} = -\frac{\frac{du}{dx}}{1+u^2}$

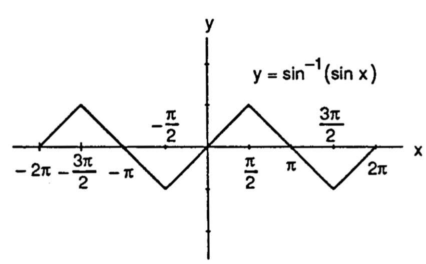
53. (a) Domain: all real numbers except those having the form $\frac{\pi}{2} + k\pi$ where k is an integer.
    Range: $-\frac{\pi}{2} < y < \frac{\pi}{2}$



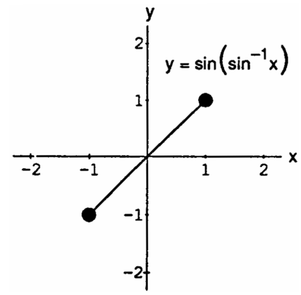
    (b) Domain: $-\infty < x < \infty$; Range: $-\infty < y < \infty$
    The graph of $y = \tan^{-1}(\tan x)$ is periodic, the graph of $y = \tan\left(\tan^{-1} x\right) = x$ for $-\infty \le x < \infty$.



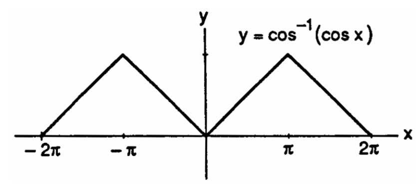
54. (a) Domain: $-\infty < x < \infty$; Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$



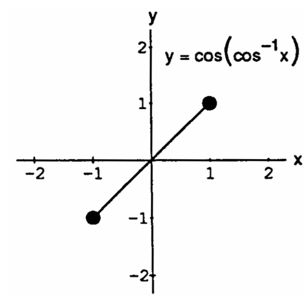
(b) Domain: $-1 \le x \le 1$; Range: $-1 \le y \le 1$
The graph of $y = \sin^{-1}(\sin x)$ is periodic; the graph of $y = \sin(\sin^{-1} x) = x$ for $-1 \le x \le 1$.



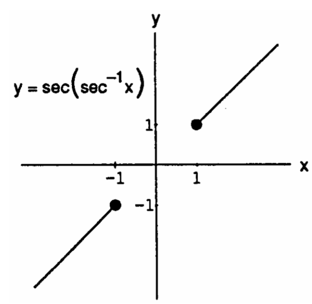
55. (a) Domain: $-\infty < x < \infty$; Range: $0 \le y \le \pi$



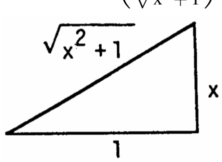
(b) Domain: $-1 \le x \le 1$; Range: $-1 \le y \le 1$
The graph of $y = \cos^{-1}(\cos x)$ is periodic; the graph of $y = \cos(\cos^{-1} x) = x$ for $-1 \le x \le 1$.

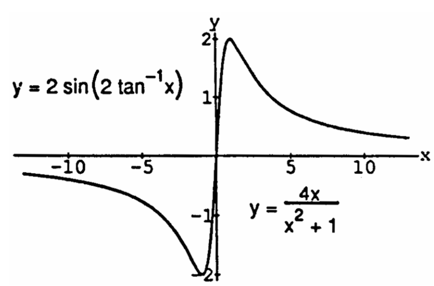


56. Since the domain of $\sec^{-1} x$ is $(-\infty, -1] \cup [1, \infty)$, we have $\sec(\sec^{-1} x) = x$ for $|x| \ge 1$. The graph of $y = \sec(\sec^{-1} x)$ is the line $y = x$ with the open line segment from $(-1, -1)$ to $(1, 1)$ removed.

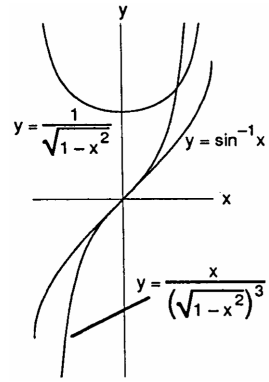


57. The graphs are identical for $y = 2\sin(2\tan^{-1} x)$
$= 4[\sin(\tan^{-1} x)][\cos(\tan^{-1} x)] = 4\left(\frac{x}{\sqrt{x^2+1}}\right)\left(\frac{1}{\sqrt{x^2+1}}\right)$
$= \frac{4x}{x^2+1}$ from the triangle





58. The values of f increase over the interval $[-1, 1]$ because $f' > 0$, and the graph of f steepens as the values of f' increase towards the ends of the interval. The graph of f is concave down to the left of the origin where $f'' < 0$, and concave up to the right of the origin where $f'' > 0$. There is an inflection point at $x = 0$ where $f'' = 0$ and f' has a local minimum value.



2.12 RELATED RATES

1. $A = \pi r^2 \Rightarrow \frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

2. $S = 4\pi r^2 \Rightarrow \frac{dS}{dt} = 8\pi r \frac{dr}{dt}$

3. (a) $V = \pi r^2 h \Rightarrow \frac{dV}{dt} = \pi r^2 \frac{dh}{dt}$
(b) $V = \pi r^2 h \Rightarrow \frac{dV}{dt} = 2\pi rh \frac{dr}{dt}$
(c) $V = \pi r^2 h \Rightarrow \frac{dV}{dt} = \pi r^2 \frac{dh}{dt} + 2\pi rh \frac{dr}{dt}$

4. (a) $V = \frac{1}{3}\pi r^2 h \Rightarrow \frac{dV}{dt} = \frac{1}{3}\pi r^2 \frac{dh}{dt}$
(b) $V = \frac{1}{3}\pi r^2 h \Rightarrow \frac{dV}{dt} = \frac{2}{3}\pi rh \frac{dr}{dt}$
(c) $\frac{dV}{dt} = \frac{1}{3}\pi r^2 \frac{dh}{dt} + \frac{2}{3}\pi rh \frac{dr}{dt}$

5. (a) $\frac{dV}{dt} = 1$ volt/sec
(b) $\frac{dI}{dt} = -\frac{1}{3}$ amp/sec
(c) $\frac{dV}{dt} = R\left(\frac{dI}{dt}\right) + I\left(\frac{dR}{dt}\right) \Rightarrow \frac{dR}{dt} = \frac{1}{I}\left(\frac{dV}{dt} - R\frac{dI}{dt}\right) \Rightarrow \frac{dR}{dt} = \frac{1}{I}\left(\frac{dV}{dt} - \frac{V}{I}\frac{dI}{dt}\right)$
(d) $\frac{dR}{dt} = \frac{1}{2}\left[1 - \frac{12}{2}\left(-\frac{1}{3}\right)\right] = \left(\frac{1}{2}\right)(3) = \frac{3}{2}$ ohms/sec, R is increasing

6. (a) $P = RI^2 \Rightarrow \frac{dP}{dt} = I^2 \frac{dR}{dt} + 2RI \frac{dI}{dt}$
(b) $P = RI^2 \Rightarrow 0 = \frac{dP}{dt} = I^2 \frac{dR}{dt} + 2RI \frac{dI}{dt} \Rightarrow \frac{dR}{dt} = -\frac{2RI}{I^2}\frac{dI}{dt} = -\frac{2\left(\frac{P}{I^2}\right)}{I}\frac{dI}{dt} = -\frac{2P}{I^3}\frac{dI}{dt}$

7. (a) $s = \sqrt{x^2 + y^2} = (x^2 + y^2)^{1/2} \Rightarrow \frac{ds}{dt} = \frac{x}{\sqrt{x^2+y^2}}\frac{dx}{dt}$
(b) $s = \sqrt{x^2 + y^2} = (x^2 + y^2)^{1/2} \Rightarrow \frac{ds}{dt} = \frac{x}{\sqrt{x^2+y^2}}\frac{dx}{dt} + \frac{y}{\sqrt{x^2+y^2}}\frac{dy}{dt}$
(c) $s = \sqrt{x^2 + y^2} \Rightarrow s^2 = x^2 + y^2 \Rightarrow 2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \Rightarrow 2s \cdot 0 = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \Rightarrow \frac{dx}{dt} = -\frac{y}{x}\frac{dy}{dt}$

8. (a) $s = \sqrt{x^2 + y^2 + z^2} \Rightarrow s^2 = x^2 + y^2 + z^2 \Rightarrow 2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} + 2z\frac{dz}{dt}$
$\Rightarrow \frac{ds}{dt} = \frac{x}{\sqrt{x^2+y^2+z^2}}\frac{dx}{dt} + \frac{y}{\sqrt{x^2+y^2+z^2}}\frac{dy}{dt} + \frac{z}{\sqrt{x^2+y^2+z^2}}\frac{dz}{dt}$
(b) From part (a) with $\frac{dx}{dt} = 0 \Rightarrow \frac{ds}{dt} = \frac{y}{\sqrt{x^2+y^2+z^2}}\frac{dy}{dt} + \frac{z}{\sqrt{x^2+y^2+z^2}}\frac{dz}{dt}$
(c) From part (a) with $\frac{ds}{dt} = 0 \Rightarrow 0 = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} + 2z\frac{dz}{dt} \Rightarrow \frac{dx}{dt} + \frac{y}{x}\frac{dy}{dt} + \frac{z}{x}\frac{dz}{dt} = 0$

9. (a) $A = \frac{1}{2}ab\sin\theta \Rightarrow \frac{dA}{dt} = \frac{1}{2}ab\cos\theta\frac{d\theta}{dt}$
(b) $A = \frac{1}{2}ab\sin\theta \Rightarrow \frac{dA}{dt} = \frac{1}{2}ab\cos\theta\frac{d\theta}{dt} + \frac{1}{2}b\sin\theta\frac{da}{dt}$
(c) $A = \frac{1}{2}ab\sin\theta \Rightarrow \frac{dA}{dt} = \frac{1}{2}ab\cos\theta\frac{d\theta}{dt} + \frac{1}{2}b\sin\theta\frac{da}{dt} + \frac{1}{2}a\sin\theta\frac{db}{dt}$

10. Given $A = \pi r^2$, $\frac{dr}{dt} = 0.01$ cm/sec, and $r = 50$ cm. Since $\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$, then $\left.\frac{dA}{dt}\right|_{r=50} = 2\pi(50)\left(\frac{1}{100}\right) = \pi$ cm²/min.

11. Given $\frac{d\ell}{dt} = -2$ cm/sec, $\frac{dw}{dt} = 2$ cm/sec, $\ell = 12$ cm and $w = 5$ cm.
(a) $A = \ell w \Rightarrow \frac{dA}{dt} = \ell\frac{dw}{dt} + w\frac{d\ell}{dt} \Rightarrow \frac{dA}{dt} = 12(2) + 5(-2) = 14$ cm²/sec, increasing

(b) $P = 2\ell + 2w \Rightarrow \frac{dP}{dt} = 2\,\frac{d\ell}{dt} + 2\,\frac{dw}{dt} = 2(-2) + 2(2) = 0$ cm/sec, constant

(c) $D = \sqrt{w^2 + \ell^2} = (w^2 + \ell^2)^{1/2} \Rightarrow \frac{dD}{dt} = \frac{1}{2}(w^2 + \ell^2)^{-1/2}\left(2w\,\frac{dw}{dt} + 2\ell\,\frac{d\ell}{dt}\right) \Rightarrow \frac{dD}{dt} = \frac{w\,\frac{dw}{dt} + \ell\,\frac{d\ell}{dt}}{\sqrt{w^2 + \ell^2}}$
$= \frac{(5)(2)\ +(12)(-2)}{\sqrt{25 + 144}} = -\frac{14}{13}$ cm/sec, decreasing

12. (a) $V = xyz \Rightarrow \frac{dV}{dt} = yz\,\frac{dx}{dt} + xz\,\frac{dy}{dt} + xy\,\frac{dz}{dt} \Rightarrow \left.\frac{dV}{dt}\right|_{(4,3,2)} = (3)(2)(1) + (4)(2)(-2) + (4)(3)(1) = 2\ m^3/\text{sec}$

(b) $S = 2xy + 2xz + 2yz \Rightarrow \frac{dS}{dt} = (2y + 2z)\,\frac{dx}{dt} + (2x + 2z)\,\frac{dy}{dt} + (2x + 2y)\,\frac{dz}{dt}$
$\Rightarrow \left.\frac{dS}{dt}\right|_{(4,3,2)} = (10)(1) + (12)(-2) + (14)(1) = 0\ m^2/\text{sec}$

(c) $\ell = \sqrt{x^2 + y^2 + z^2} = (x^2 + y^2 + z^2)^{1/2} \Rightarrow \frac{d\ell}{dt} = \frac{x}{\sqrt{x^2+y^2+z^2}}\,\frac{dx}{dt} + \frac{y}{\sqrt{x^2+y^2+z^2}}\,\frac{dy}{dt} + \frac{z}{\sqrt{x^2+y^2+z^2}}\,\frac{dz}{dt}$
$\Rightarrow \left.\frac{d\ell}{dt}\right|_{(4,3,2)} = \left(\frac{4}{\sqrt{29}}\right)(1) + \left(\frac{3}{\sqrt{29}}\right)(-2) + \left(\frac{2}{\sqrt{29}}\right)(1) = 0$ m/sec

13. Given: $\frac{dx}{dt} = 5$ ft/sec, the ladder is 13 ft long, and $x = 12$, $y = 5$ at the instant of time

(a) Since $x^2 + y^2 = 169 \Rightarrow \frac{dy}{dt} = -\frac{x}{y}\,\frac{dx}{dt} = -\left(\frac{12}{5}\right)(5) = -12$ ft/sec, the ladder is sliding down the wall

(b) The area of the triangle formed by the ladder and walls is $A = \frac{1}{2}xy \Rightarrow \frac{dA}{dt} = \left(\frac{1}{2}\right)\left(x\,\frac{dy}{dt} + y\,\frac{dx}{dt}\right)$. The area is changing at $\frac{1}{2}[12(-12) + 5(5)] = -\frac{119}{2} = -59.5\ \text{ft}^2/\text{sec}$.

(c) $\cos\theta = \frac{x}{13} \Rightarrow -\sin\theta\,\frac{d\theta}{dt} = \frac{1}{13}\cdot\frac{dx}{dt} \Rightarrow \frac{d\theta}{dt} = -\frac{1}{13\sin\theta}\cdot\frac{dx}{dt} = -\left(\frac{1}{5}\right)(5) = -1$ rad/sec

14. $s^2 = y^2 + x^2 \Rightarrow 2s\,\frac{ds}{dt} = 2x\,\frac{dx}{dt} + 2y\,\frac{dy}{dt} \Rightarrow \frac{ds}{dt} = \frac{1}{s}\left(x\,\frac{dx}{dt} + y\,\frac{dy}{dt}\right) \Rightarrow \frac{ds}{dt} = \frac{1}{\sqrt{169}}[5(-442) + 12(-481)] = -614$ knots

15. Let s represent the distance between the girl and the kite and x represents the horizontal distance between the girl and kite $\Rightarrow s^2 = (300)^2 + x^2 \Rightarrow \frac{ds}{dt} = \frac{x}{s}\,\frac{dx}{dt} = \frac{400(25)}{500} = 20$ ft/sec.

16. When the diameter is 3.8 in., the radius is 1.9 in. and $\frac{dr}{dt} = \frac{1}{3000}$ in/min. Also $V = 6\pi r^2 \Rightarrow \frac{dV}{dt} = 12\pi r\,\frac{dr}{dt}$
$\Rightarrow \frac{dV}{dt} = 12\pi(1.9)\left(\frac{1}{3000}\right) = 0.0076\pi$. The volume is changing at about $0.0239\ \text{in}^3/\text{min}$.

17. $V = \frac{1}{3}\pi r^2 h$, $h = \frac{3}{8}(2r) = \frac{3r}{4} \Rightarrow r = \frac{4h}{3} \Rightarrow V = \frac{1}{3}\pi\left(\frac{4h}{3}\right)^2 h = \frac{16\pi h^3}{27} \Rightarrow \frac{dV}{dt} = \frac{16\pi h^2}{9}\,\frac{dh}{dt}$

(a) $\left.\frac{dh}{dt}\right|_{h=4} = \left(\frac{9}{16\pi 4^2}\right)(10) = \frac{90}{256\pi} \approx 0.1119$ m/sec $= 11.19$ cm/sec

(b) $r = \frac{4h}{3} \Rightarrow \frac{dr}{dt} = \frac{4}{3}\,\frac{dh}{dt} = \frac{4}{3}\left(\frac{90}{256\pi}\right) = \frac{15}{32\pi} \approx 0.1492$ m/sec $= 14.92$ cm/sec

18. (a) $V = \frac{1}{3}\pi r^2 h$ and $r = \frac{15h}{2} \Rightarrow V = \frac{1}{3}\pi\left(\frac{15h}{2}\right)^2 h = \frac{75\pi h^3}{4} \Rightarrow \frac{dV}{dt} = \frac{225\pi h^2}{4}\,\frac{dh}{dt} \Rightarrow \left.\frac{dh}{dt}\right|_{h=5} = \frac{4(-50)}{225\pi(5)^2} = \frac{-8}{225\pi}$
≈ -0.0113 m/min $= -1.13$ cm/min

(b) $r = \frac{15h}{2} \Rightarrow \frac{dr}{dt} = \frac{15}{2}\,\frac{dh}{dt} \Rightarrow \left.\frac{dr}{dt}\right|_{h=5} = \left(\frac{15}{2}\right)\left(\frac{-8}{225\pi}\right) = \frac{-4}{15\pi} \approx -0.0849$ m/sec $= -8.49$ cm/sec

19. (a) $V = \frac{\pi}{3}y^2(3R - y) \Rightarrow \frac{dV}{dt} = \frac{\pi}{3}[2y(3R - y) + y^2(-1)]\,\frac{dy}{dt} \Rightarrow \frac{dy}{dt} = \left[\frac{\pi}{3}(6Ry - 3y^2)\right]^{-1}\frac{dV}{dt} \Rightarrow$ at $R = 13$ and $y = 8$ we have $\frac{dy}{dt} = \frac{1}{144\pi}(-6) = \frac{-1}{24\pi}$ m/min

(b) The hemisphere is on the circle $r^2 + (13 - y)^2 = 169 \Rightarrow r = \sqrt{26y - y^2}$ m

(c) $r = (26y - y^2)^{1/2} \Rightarrow \frac{dr}{dt} = \frac{1}{2}(26y - y^2)^{-1/2}(26 - 2y)\,\frac{dy}{dt} \Rightarrow \frac{dr}{dt} = \frac{13 - y}{\sqrt{26y - y^2}}\,\frac{dy}{dt} \Rightarrow \left.\frac{dr}{dt}\right|_{y=8} = \frac{13 - 8}{\sqrt{26\cdot 8 - 64}}\left(\frac{-1}{24\pi}\right)$
$= \frac{-5}{288\pi}$ m/min

20. If $V = \frac{4}{3}\pi r^3$, $S = 4\pi r^2$, and $\frac{dV}{dt} = kS = 4k\pi r^2$, then $\frac{dV}{dt} = 4\pi r^2\,\frac{dr}{dt} \Rightarrow 4k\pi r^2 = 4\pi r^2\,\frac{dr}{dt} \Rightarrow \frac{dr}{dt} = k$, a constant. Therefore, the radius is increasing at a constant rate.

21. If $V = \frac{4}{3}\pi r^3$, r = 5, and $\frac{dV}{dt} = 100\pi$ ft^3/min, then $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = 1$ ft/min. Then $S = 4\pi r^2 \Rightarrow \frac{dS}{dt} = 8\pi r \frac{dr}{dt} = 8\pi(5)(1) = 40\pi$ ft^2/min, the rate at which the surface area is increasing.

22. Let s represent the length of the rope and x the horizontal distance of the boat from the dock.
(a) We have $s^2 = x^2 + 36 \Rightarrow \frac{dx}{dt} = \frac{s}{x}\frac{ds}{dt} = \frac{s}{\sqrt{s^2-36}}\frac{ds}{dt}$. Therefore, the boat is approaching the dock at $\left.\frac{dx}{dt}\right|_{s=10} = \frac{10}{\sqrt{10^2-36}}(-2) = -2.5$ ft/sec.
(b) $\cos\theta = \frac{6}{r} \Rightarrow -\sin\theta\frac{d\theta}{dt} = -\frac{6}{r^2}\frac{dr}{dt} \Rightarrow \frac{d\theta}{dt} = \frac{6}{r^2\sin\theta}\frac{dr}{dt}$. Thus, r = 10, x = 8, and $\sin\theta = \frac{8}{10}$ $\Rightarrow \frac{d\theta}{dt} = \frac{6}{10^2\left(\frac{8}{10}\right)}\cdot(-2) = -\frac{3}{20}$ rad/sec

23. Let s represent the distance between the bicycle and balloon, h the height of the balloon and x the horizontal distance between the balloon and the bicycle. The relationship between the variables is $s^2 = h^2 + x^2$ $\Rightarrow \frac{ds}{dt} = \frac{1}{s}\left(h\frac{dh}{dt} + x\frac{dx}{dt}\right) \Rightarrow \frac{ds}{dt} = \frac{1}{85}[68(1) + 51(17)] = 11$ ft/sec.

24. (a) Let h be the height of the coffee in the pot. Since the radius of the pot is 3, the volume of the coffee is $V = 9\pi h \Rightarrow \frac{dV}{dt} = 9\pi\frac{dh}{dt} \Rightarrow$ the rate the coffee is rising is $\frac{dh}{dt} = \frac{1}{9\pi}\frac{dV}{dt} = \frac{10}{9\pi}$ in/min.
(b) Let h be the height of the coffee in the pot. From the figure, the radius of the filter $r = \frac{h}{2} \Rightarrow V = \frac{1}{3}\pi r^2 h = \frac{\pi h^3}{12}$, the volume of the filter. The rate the coffee is falling is $\frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt} = \frac{4}{25\pi}(-10) = -\frac{8}{5\pi}$ in/min.

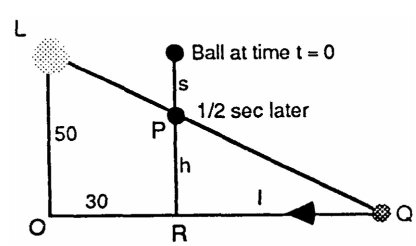
25. Let P(x, y) represent a point on the curve $y = x^2$ and θ the angle of inclination of a line containing P and the origin. Consequently, $\tan\theta = \frac{y}{x} \Rightarrow \tan\theta = \frac{x^2}{x} = x \Rightarrow \sec^2\theta\frac{d\theta}{dt} = \frac{dx}{dt} \Rightarrow \frac{d\theta}{dt} = \cos^2\theta\frac{dx}{dt}$. Since $\frac{dx}{dt} = 10$ m/sec and $\left.\cos^2\theta\right|_{x=3} = \frac{x^2}{y^2+x^2} = \frac{3^2}{9^2+3^2} = \frac{1}{10}$, we have $\left.\frac{d\theta}{dt}\right|_{x=3} = 1$ rad/sec.

26. $y = (-x)^{1/2}$ and $\tan\theta = \frac{y}{x} \Rightarrow \tan\theta = \frac{(-x)^{1/2}}{x} \Rightarrow \sec^2\theta\frac{d\theta}{dt} = \frac{\left(\frac{1}{2}\right)(-x)^{-1/2}(-1)x - (-x)^{1/2}(1)}{x^2}\frac{dx}{dt}$
$\Rightarrow \frac{d\theta}{dt} = \left(\frac{\frac{-x}{2\sqrt{-x}} - \sqrt{-x}}{x^2}\right)(\cos^2\theta)\left(\frac{dx}{dt}\right)$. Now, $\tan\theta = \frac{2}{-4} = -\frac{1}{2} \Rightarrow \cos\theta = -\frac{2}{\sqrt{5}} \Rightarrow \cos^2\theta = \frac{4}{5}$. Then $\frac{d\theta}{dt} = \left(\frac{\frac{4}{4}-2}{16}\right)\left(\frac{4}{5}\right)(-8) = \frac{2}{5}$ rad/sec.

27. The distance from the origin is $s = \sqrt{x^2+y^2}$ and we wish to find $\left.\frac{ds}{dt}\right|_{(5,12)}$
$= \frac{1}{2}(x^2+y^2)^{-1/2}\left.\left(2x\frac{dx}{dt} + 2y\frac{dy}{dt}\right)\right|_{(5,12)} = \frac{(5)(-1)+(12)(-5)}{\sqrt{25+144}} = -5$ m/sec

28. When s represents the length of the shadow and x the distance of the man from the streetlight, then $s = \frac{3}{5}x$. If I represents the distance of the tip of the shadow from the streetlight, then $I = s + x \Rightarrow \frac{dI}{dt} = \frac{ds}{dt} + \frac{dx}{dt}$ (which is velocity not speed) $\Rightarrow \left|\frac{dI}{dt}\right| = \left|\frac{3}{5}\frac{dx}{dt} + \frac{dx}{dt}\right| = \left|\frac{8}{5}\right|\left|\frac{dx}{dt}\right| = \frac{8}{5}|-5| = 8$ ft/sec, the speed the tip of the shadow is moving along the ground. $\frac{ds}{dt} = \frac{3}{5}\frac{dx}{dt} = \frac{3}{5}(-5) = -3$ ft/sec, so the length of the shadow is decreasing at a rate of 3 ft/sec.

29. Let $s = 16t^2$ represent the distance the ball has fallen, h the distance between the ball and the ground, and I the distance between the shadow and the point directly beneath the ball. Accordingly, s + h = 50 and since the triangle LOQ and triangle PRQ are similar we have $I = \frac{30h}{50-h} \Rightarrow h = 50 - 16t^2$ and $I = \frac{30(50-16t^2)}{50-(50-16t^2)}$ $= \frac{1500}{16t^2} - 30 \Rightarrow \frac{dI}{dt} = -\frac{1500}{8t^3} \Rightarrow \left.\frac{dI}{dt}\right|_{t=\frac{1}{2}} = -1500$ ft/sec.



30. Let s = distance of car from foot of perpendicular in the textbook diagram $\Rightarrow$ $\tan\theta = \frac{s}{132}$ $\Rightarrow$ $\sec^2\theta\,\frac{d\theta}{dt} = \frac{1}{132}\,\frac{ds}{dt}$ $\Rightarrow$ $\frac{d\theta}{dt} = \frac{\cos^2\theta}{132}\,\frac{ds}{dt}$; $\frac{ds}{dt} = -264$ and $\theta = 0$ $\Rightarrow$ $\frac{d\theta}{dt} = -2$ rad/sec. A half second later the car has traveled 132 ft right of the perpendicular $\Rightarrow$ $|\theta| = \frac{\pi}{4}$, $\cos^2\theta = \frac{1}{2}$, and $\frac{ds}{dt} = 264$ (since s increases) $\Rightarrow$ $\frac{d\theta}{dt} = \frac{\left(\frac{1}{2}\right)}{132}(264) = 1$ rad/sec.

31. The volume of the ice is $V = \frac{4}{3}\pi r^3 - \frac{4}{3}\pi 4^3$ $\Rightarrow$ $\frac{dV}{dt} = 4\pi r^2\,\frac{dr}{dt}$ $\Rightarrow$ $\left.\frac{dr}{dt}\right|_{r=6} = \frac{-5}{72\pi}$ in./min when $\frac{dV}{dt} = -10$ in^3/min, the thickness of the ice is decreasing at $\frac{5}{72\pi}$ in/min. The surface area is $S = 4\pi r^2$ $\Rightarrow$ $\frac{dS}{dt} = 8\pi r\,\frac{dr}{dt}$ $\Rightarrow$ $\left.\frac{dS}{dt}\right|_{r=6} = 48\pi\left(\frac{-5}{72\pi}\right) = -\frac{10}{3}$ in^2/min, the outer surface area of the ice is decreasing at $\frac{10}{3}$ in^2/min.

32. Let s represent the horizontal distance between the car and plane while r is the line-of-sight distance between the car and plane $\Rightarrow$ $9 + s^2 = r^2$ $\Rightarrow$ $\frac{ds}{dt} = \frac{r}{\sqrt{r^2-9}}\,\frac{dr}{dt}$ $\Rightarrow$ $\left.\frac{ds}{dt}\right|_{r=5} = \frac{5}{\sqrt{16}}(-160) = -200$ mph $\Rightarrow$ speed of plane + speed of car = 200 mph $\Rightarrow$ the speed of the car is 80 mph.

33. When x represents the length of the shadow, then $\tan\theta = \frac{80}{x}$ $\Rightarrow$ $\sec^2\theta\,\frac{d\theta}{dt} = -\frac{80}{x^2}\,\frac{dx}{dt}$ $\Rightarrow$ $\frac{dx}{dt} = \frac{-x^2\sec^2\theta}{80}\,\frac{d\theta}{dt}$.
We are given that $\frac{d\theta}{dt} = 0.27° = \frac{3\pi}{2000}$ rad/min. At $x = 60$, $\cos\theta = \frac{3}{5}$ $\Rightarrow$
$\left|\frac{dx}{dt}\right| = \left|\frac{-x^2\sec^2\theta}{80}\,\frac{d\theta}{dt}\right|\Bigg|_{\left(\frac{d\theta}{dt} = \frac{3\pi}{2000}\text{ and }\sec\theta = \frac{5}{3}\right)} = \frac{3\pi}{16}$ ft/min ≈ 0.589 ft/min ≈ 7.1 in./min.

34. Let a represent the distance between point O and ship A, b the distance between point O and ship B, and D the distance between the ships. By the Law of Cosines, $D^2 = a^2 + b^2 - 2ab\cos 120°$ $\Rightarrow$ $\frac{dD}{dt} = \frac{1}{2D}\left[2a\,\frac{da}{dt} + 2b\,\frac{db}{dt} + a\,\frac{db}{dt} + b\,\frac{da}{dt}\right]$. When $a = 5$, $\frac{da}{dt} = 14$, $b = 3$, and $\frac{db}{dt} = 21$, then $\frac{dD}{dt} = \frac{413}{2D}$ where $D = 7$. The ships are moving $\frac{dD}{dt} = 29.5$ knots apart.

2.13 LINEARIZATION AND DIFFERENTIALS

1. $f(x) = x^3 - 2x + 3$ $\Rightarrow$ $f'(x) = 3x^2 - 2$ $\Rightarrow$ $L(x) = f'(2)(x-2) + f(2) = 10(x-2) + 7$ $\Rightarrow$ $L(x) = 10x - 13$ at $x = 2$

2. $f(x) = \sqrt{x^2+9} = (x^2+9)^{1/2}$ $\Rightarrow$ $f'(x) = \left(\frac{1}{2}\right)(x^2+9)^{-1/2}(2x) = \frac{x}{\sqrt{x^2+9}}$ $\Rightarrow$ $L(x) = f'(-4)(x+4) + f(-4) = -\frac{4}{5}(x+4) + 5$ $\Rightarrow$ $L(x) = -\frac{4}{5}x + \frac{9}{5}$ at $x = -4$

3. $f(x) = x + \frac{1}{x}$ $\Rightarrow$ $f'(x) = 1 - x^{-2}$ $\Rightarrow$ $L(x) = f(1) + f'(1)(x-1) = 2 + 0(x-1) = 2$

4. $f(x) = x^{1/3}$ $\Rightarrow$ $f'(x) = \frac{1}{3x^{2/3}}$ $\Rightarrow$ $L(x) = f'(-8)(x-(-8)) + f(-8) = \frac{1}{12}(x+8) - 2$ $\Rightarrow$ $L(x) = \frac{1}{12}x - \frac{4}{3}$

5. $f(x) = \tan x \Rightarrow f'(x) = \sec^2 x \Rightarrow L(x) = f(\pi) + f'(\pi)(x-\pi) = 0 + 1(x-\pi) = x - \pi$

6. (a) $f(x) = \sin x \Rightarrow f'(x) = \cos x \Rightarrow L(x) = f(0) + f'(0)(x-0) = x \Rightarrow L(x) = x$
(b) $f(x) = \cos x \Rightarrow f'(x) = -\sin x \Rightarrow L(x) = f(0) + f'(0)(x-0) = 1 \Rightarrow L(x) = 1$
(c) $f(x) = \tan x \Rightarrow f'(x) = \sec^2 x \Rightarrow L(x) = f(0) + f'(0)(x-0) = x \Rightarrow L(x) = x$
(d) $f(x) = e^x \Rightarrow f'(x) = e^x \Rightarrow L(x) = f(0) + f'(0)(x-0) = x + 1 \Rightarrow L(x) = x + 1$
(e) $f(x) = \ln(1+x) \Rightarrow f'(x) = \frac{1}{1+x} \Rightarrow L(x) = f(0) + f'(0)(x-0) = x \Rightarrow L(x) = x$

7. $f(x) = x^2 + 2x$ $\Rightarrow$ $f'(x) = 2x + 2$ $\Rightarrow$ $L(x) = f'(0)(x-0) + f(0) = 2(x-0) + 0$ $\Rightarrow$ $L(x) = 2x$ at $x = 0$

8. $f(x) = x^{-1}$ $\Rightarrow$ $f'(x) = -x^{-2}$ $\Rightarrow$ $L(x) = f'(1)(x-1) + f(1) = (-1)(x-1) + 1$ $\Rightarrow$ $L(x) = -x + 2$ at $x = 1$

9. $f(x) = 2x^2 + 4x - 3$ $\Rightarrow$ $f'(x) = 4x + 4$ $\Rightarrow$ $L(x) = f'(-1)(x+1) + f(-1) = 0(x+1) + (-5)$ $\Rightarrow$ $L(x) = -5$ at $x = -1$

10. $f(x) = 1 + x \Rightarrow f'(x) = 1 \Rightarrow L(x) = f'(8)(x-8) + f(8) = 1(x-8) + 9 \Rightarrow L(x) = x + 1$ at $x = 8$

11. $f(x) = \sqrt[3]{x} = x^{1/3} \Rightarrow f'(x) = \left(\frac{1}{3}\right) x^{-2/3} \Rightarrow L(x) = f'(8)(x-8) + f(8) = \frac{1}{12}(x-8) + 2 \Rightarrow L(x) = \frac{1}{12}x + \frac{4}{3}$ at $x = 8$

12. $f(x) = \frac{x}{x+1} \Rightarrow f'(x) = \frac{(1)(x+1)-(1)(x)}{(x+1)^2} = \frac{1}{(x+1)^2} \Rightarrow L(x) = f'(1)(x-1) + f(1) = \frac{1}{4}(x-1) + \frac{1}{2}$
$\Rightarrow L(x) = \frac{1}{4}x + \frac{1}{4}$ at $x = 1$

13. $f(x) = e^{-x} \Rightarrow f'(x) = -e^{-x} \Rightarrow L(x) = f(0) + f'(0)(x-0) = -x + 1$

14. $f(x) = \sin^{-1}x \Rightarrow f'(x) = \frac{1}{\sqrt{1-x^2}} \Rightarrow L(x) = f(0) + f'(0)(x-0) = x$

15. $f'(x) = k(1+x)^{k-1}$. We have $f(0) = 1$ and $f'(0) = k$. $L(x) = f(0) + f'(0)(x-0) = 1 + k(x-0) = 1 + kx$

16. (a) $f(x) = (1-x)^6 = [1+(-x)]^6 \approx 1 + 6(-x) = 1 - 6x$
(b) $f(x) = \frac{2}{1-x} = 2[1+(-x)]^{-1} \approx 2[1+(-1)(-x)] = 2 + 2x$
(c) $f(x) = (1+x)^{-1/2} \approx 1 + \left(-\frac{1}{2}\right)x = 1 - \frac{x}{2}$
(d) $f(x) = \sqrt{2+x^2} = \sqrt{2}\left(1 + \frac{x^2}{2}\right)^{1/2} \approx \sqrt{2}\left(1 + \frac{1}{2}\frac{x^2}{2}\right) = \sqrt{2}\left(1 + \frac{x^2}{4}\right)$
(e) $f(x) = (4+3x)^{1/3} = 4^{1/3}\left(1 + \frac{3x}{4}\right)^{1/3} \approx 4^{1/3}\left(1 + \frac{1}{3}\frac{3x}{4}\right) = 4^{1/3}\left(1 + \frac{x}{4}\right)$
(f) $f(x) = \left(1 - \frac{1}{2+x}\right)^{2/3} = \left[1 + \left(-\frac{1}{2+x}\right)\right]^{2/3} \approx 1 + \frac{2}{3}\left(-\frac{1}{2+x}\right) = 1 - \frac{2}{6+3x}$

17. (a) $(1.0002)^{50} = (1 + 0.0002)^{50} \approx 1 + 50(0.0002) = 1 + .01 = 1.01$
(b) $\sqrt[3]{1.009} = (1 + 0.009)^{1/3} \approx 1 + \left(\frac{1}{3}\right)(0.009) = 1 + 0.003 = 1.003$

18. $f(x) = \sqrt{x+1} + \sin x = (x+1)^{1/2} + \sin x \Rightarrow f'(x) = \left(\frac{1}{2}\right)(x+1)^{-1/2} + \cos x \Rightarrow L_f(x) = f'(0)(x-0) + f(0)$
$= \frac{3}{2}(x-0) + 1 \Rightarrow L_f(x) = \frac{3}{2}x + 1$, the linearization of $f(x)$; $g(x) = \sqrt{x+1} = (x+1)^{1/2} \Rightarrow g'(x)$
$= \left(\frac{1}{2}\right)(x+1)^{-1/2} \Rightarrow L_g(x) = g'(0)(x-0) + g(0) = \frac{1}{2}(x-0) + 1 \Rightarrow L_g(x) = \frac{1}{2}x + 1$, the linearization of $g(x)$;
$h(x) = \sin x \Rightarrow h'(x) = \cos x \Rightarrow L_h(x) = h'(0)(x-0) + h(0) = (1)(x-0) + 0 \Rightarrow L_h(x) = x$, the linearization of $h(x)$. $L_f(x) = L_g(x) + L_h(x)$ implies that the linearization of a sum is equal to the sum of the linearizations.

19. $y = x^3 - 3\sqrt{x} = x^3 - 3x^{1/2} \Rightarrow dy = \left(3x^2 - \frac{3}{2}x^{-1/2}\right)dx \Rightarrow dy = \left(3x^2 - \frac{3}{2\sqrt{x}}\right)dx$

20. $y = \frac{2x}{1+x^2} \Rightarrow dy = \left(\frac{(2)(1+x^2) - (2x)(2x)}{(1+x^2)^2}\right)dx = \frac{2-2x^2}{(1+x^2)^2}\,dx$

21. $2y^{3/2} + xy - x = 0 \Rightarrow 3y^{1/2}\,dy + y\,dx + x\,dy - dx = 0 \Rightarrow \left(3y^{1/2} + x\right)dy = (1-y)\,dx \Rightarrow dy = \frac{1-y}{3\sqrt{y}+x}\,dx$

22. $xy^2 - 4x^{3/2} - y = 0 \Rightarrow y^2\,dx + 2xy\,dy - 6x^{1/2}\,dx - dy = 0 \Rightarrow (2xy - 1)\,dy = \left(6x^{1/2} - y^2\right)dx \Rightarrow dy = \frac{6\sqrt{x} - y^2}{2xy - 1}\,dx$

23. $y = \sin\left(5\sqrt{x}\right) = \sin\left(5x^{1/2}\right) \Rightarrow dy = \left(\cos\left(5x^{1/2}\right)\right)\left(\frac{5}{2}x^{-1/2}\right)dx \Rightarrow dy = \frac{5\cos\left(5\sqrt{x}\right)}{2\sqrt{x}}\,dx$

24. $y = 4\tan\left(\frac{x^3}{3}\right) \Rightarrow dy = 4\left(\sec^2\left(\frac{x^3}{3}\right)\right)(x^2)\,dx \Rightarrow dy = 4x^2\sec^2\left(\frac{x^3}{3}\right)dx$

25. $y = e^{\sqrt{x}} \Rightarrow dy = \frac{e^{\sqrt{x}}}{2\sqrt{x}}\,dx$

26. $y = x\,e^{-x} \Rightarrow dy = \left(-x\,e^{-x} + e^{-x}\right)dx = (1-x)\,e^{-x}\,dx$

27. $y = \ln(1+x^2) \Rightarrow dy = \frac{2x}{1+x^2}\,dx$

28. $y = \sec^{-1}(e^{-x}) \Rightarrow dy = \frac{1}{e^{-x}\sqrt{(e^{-x})^2-1}} \cdot (-e^{-x})\,dx = \frac{-1}{\sqrt{\left(\frac{1}{e^x}\right)^2-1}}\,dx = \frac{-e^x}{\sqrt{1-e^{2x}}}\,dx$

29. $f(x) = x^2+2x,\ x_0 = 1,\ dx = 0.1 \ \Rightarrow\ f'(x) = 2x+2$
   (a) $\Delta f = f(x_0+dx) - f(x_0) = f(1.1) - f(1) = 3.41 - 3 = 0.41$
   (b) $df = f'(x_0)\,dx = [2(1)+2](0.1) = 0.4$
   (c) $|\Delta f - df| = |0.41 - 0.4| = 0.01$

30. $f(x) = x^3 - x,\ x_0 = 1,\ dx = 0.1 \ \Rightarrow\ f'(x) = 3x^2 - 1$
   (a) $\Delta f = f(x_0+dx) - f(x_0) = f(1.1) - f(1) = .231$
   (b) $df = f'(x_0)\,dx = [3(1)^2 - 1](.1) = .2$
   (c) $|\Delta f - df| = |.231 - .2| = .031$

31. $f(x) = x^{-1},\ x_0 = 0.5,\ dx = 0.1 \ \Rightarrow\ f'(x) = -x^{-2}$
   (a) $\Delta f = f(x_0+dx) - f(x_0) = f(.6) - f(.5) = -\frac{1}{3}$
   (b) $df = f'(x_0)\,dx = (-4)\left(\frac{1}{10}\right) = -\frac{2}{5}$
   (c) $|\Delta f - df| = \left|-\frac{1}{3} + \frac{2}{5}\right| = \frac{1}{15}$

32. $f(x) = x^3 - 2x + 3,\ x_0 = 2,\ dx = 0.1 \ \Rightarrow\ f'(x) = 3x^2 - 2$
   (a) $\Delta f = f(x_0+dx) - f(x_0) = f(2.1) - f(2) = 1.061$
   (b) $df = f'(x_0)\,dx = (10)(0.10) = 1$
   (c) $|\Delta f - df| = |1.061 - 1| = .061$

33. $V = \frac{4}{3}\pi r^3 \ \Rightarrow\ dV = 4\pi r_0^2\,dr$

34. $S = 6x^2 \ \Rightarrow\ dS = 12x_0\,dx$

35. $V = \pi r^2 h$, height constant $\ \Rightarrow\ dV = 2\pi r_0 h\,dr$

36. $S = \pi r\sqrt{r^2+h^2} = \pi r\left(r^2+h^2\right)^{1/2}$, h constant $\ \Rightarrow\ \frac{dS}{dr} = \pi\left(r^2+h^2\right)^{1/2} + \pi r \cdot r\left(r^2+h^2\right)^{-1/2}$
$\Rightarrow\ \frac{dS}{dr} = \frac{\pi\left(r^2+h^2\right)+\pi r^2}{\sqrt{r^2+h^2}} \ \Rightarrow\ dS = \frac{\pi\left(2r_0^2+h^2\right)}{\sqrt{r_0^2+h^2}}\,dr$, h constant

37. Given $r = 2$ m, $dr = .02$ m
   (a) $A = \pi r^2 \ \Rightarrow\ dA = 2\pi r\,dr = 2\pi(2)(.02) = .08\pi\ m^2$
   (b) $\left(\frac{.08\pi}{4\pi}\right)(100\%) = 2\%$

38. $C = 2\pi r$ and $dC = 2$ in. $\Rightarrow\ dC = 2\pi\,dr \ \Rightarrow\ dr = \frac{1}{\pi} \ \Rightarrow$ the diameter grew about $\frac{2}{\pi}$ in.; $A = \pi r^2 \ \Rightarrow\ dA = 2\pi r\,dr$
$= 2\pi(5)\left(\frac{1}{\pi}\right) = 10\ \text{in.}^2$

39. The volume of a cylinder is $V = \pi r^2 h$. When h is held fixed, we have $\frac{dV}{dr} = 2\pi rh$, and so $dV = 2\pi rh\,dr$. For $h = 30$ in., $r = 6$ in., and $dr = 0.5$ in., the volume of the material in the shell is approximately $dV = 2\pi rh\,dr = 2\pi(6)(30)(0.5)$
$= 180\pi \approx 565.5\ \text{in}^3$.

40. Let θ = angle of elevation and h = height of building. Then $h = 30\tan\theta$, so $dh = 30\sec^2\theta\, d\theta$. We want $|dh| < 0.04h$, which gives: $|30\sec^2\theta\, d\theta| < 0.04|30\tan\theta| \Rightarrow \frac{1}{\cos^2\theta}|d\theta| < \frac{0.04\sin\theta}{\cos\theta} \Rightarrow |d\theta| < 0.04\sin\theta\cos\theta \Rightarrow |d\theta| < 0.04\sin\frac{5\pi}{12}\cos\frac{5\pi}{12}$ $= 0.01$ radian. The angle should be measured with an error of less than 0.01 radian (or approximatley 0.57 degrees), which is a percentage error of approximately 0.76%.

41. $V = \pi h^3 \Rightarrow dV = 3\pi h^2\, dh$; recall that $\Delta V \approx dV$. Then $|\Delta V| \le (1\%)(V) = \frac{(1)(\pi h^3)}{100} \Rightarrow |dV| \le \frac{(1)(\pi h^3)}{100}$ $\Rightarrow |3\pi h^2\, dh| \le \frac{(1)(\pi h^3)}{100} \Rightarrow |dh| \le \frac{1}{300}h = \left(\frac{1}{3}\%\right)h$. Therefore the greatest tolerated error in the measurement of h is $\frac{1}{3}\%$.

42. (a) Let D_i represent the inside diameter. Then $V = \pi r^2 h = \pi\left(\frac{D_i}{2}\right)^2 h = \frac{\pi D_i^2 h}{4}$ and $h = 10 \Rightarrow V = \frac{5\pi D_i^2}{2} \Rightarrow$ $dV = 5\pi D_i\, dD_i$. Recall that $\Delta V \approx dV$. We want $|\Delta V| \le (1\%)(V) \Rightarrow |dV| \le \left(\frac{1}{100}\right)\left(\frac{5\pi D_i^2}{2}\right) = \frac{\pi D_i^2}{40}$ $\Rightarrow 5\pi D_i\, dD_i \le \frac{\pi D_i^2}{40} \Rightarrow \frac{dD_i}{D_i} \le 200$. The inside diameter must be measured to within 0.5%.

(b) Let D_e represent the exterior diameter, h the height and S the area of the painted surface. $S = \pi D_e h \Rightarrow dS = \pi h dD_e$ $\Rightarrow \frac{dS}{S} = \frac{dD_e}{D_e}$. Thus for small changes in exterior diameter, the approximate percentage change in the exterior diameter is equal to the approximate percentage change in the area painted, and to estimate the amount of paint required to within 5%, the tanks's exterior diameter must be measured to within 5%.

43. $V = \pi r^2 h$, h is constant $\Rightarrow dV = 2\pi rh\, dr$; recall that $\Delta V \approx dV$. We want $|\Delta V| \le \frac{1}{1000}V \Rightarrow |dV| \le \frac{\pi r^2 h}{1000}$ $\Rightarrow |2\pi rh\, dr| \le \frac{\pi r^2 h}{1000} \Rightarrow |dr| \le \frac{r}{2000} = (.05\%)r \Rightarrow$ a .05% variation in the radius can be tolerated.

44. (a) $T = 2\pi\left(\frac{L}{g}\right)^{1/2} \Rightarrow dT = 2\pi\sqrt{L}\left(-\frac{1}{2}g^{-3/2}\right)dg = -\pi\sqrt{L}\,g^{-3/2}\,dg$

(b) If g increases, then $dg > 0 \Rightarrow dT < 0$. The period T decreases and the clock ticks more frequently. Both the pendulum speed and clock speed increase.

(c) $0.001 = -\pi\sqrt{100}\left(980^{-3/2}\right)dg \Rightarrow dg \approx -0.977$ cm/sec^2 $\Rightarrow$ the new $g \approx 979$ cm/sec^2

45. The error in measurement $dx = (1\%)(10) = 0.1$ cm; $V = x^3 \Rightarrow dV = 3x^2\, dx = 3(10)^2(0.1) = 30$ cm^3 $\Rightarrow$ the percentage error in the volume calculation is $\left(\frac{30}{1000}\right)(100\%) = 3\%$

46. $A = s^2 \Rightarrow dA = 2s\, ds$; recall that $\Delta A \approx dA$. Then $|\Delta A| \le (2\%)A = \frac{2s^2}{100} = \frac{s^2}{50} \Rightarrow |dA| \le \frac{s^2}{50} \Rightarrow |2s\, ds| \le \frac{s^2}{50}$ $\Rightarrow |ds| \le \frac{s^2}{(2s)(50)} = \frac{s}{100} = (1\%)\,s \Rightarrow$ the error must be no more than 1% of the true value.

47. Given $D = 100$ cm, $dD = 1$ cm, $V = \frac{4}{3}\pi\left(\frac{D}{2}\right)^3 = \frac{\pi D^3}{6} \Rightarrow dV = \frac{\pi}{2}D^2\, dD = \frac{\pi}{2}(100)^2(1) = \frac{10^4\pi}{2}$. Then $\frac{dV}{V}(100\%)$ $= \left[\frac{\frac{10^4\pi}{2}}{\frac{10^6\pi}{6}}\right](10^2\%) = \left[\frac{\frac{10^6\pi}{2}}{\frac{10^6\pi}{6}}\right]\% = 3\%$

48. $V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi\left(\frac{D}{2}\right)^3 = \frac{\pi D^3}{6} \Rightarrow dV = \frac{\pi D^2}{2}dD$; recall that $\Delta V \approx dV$. Then $|\Delta V| \le (3\%)V = \left(\frac{3}{100}\right)\left(\frac{\pi D^3}{6}\right)$ $= \frac{\pi D^3}{200} \Rightarrow |dV| \le \frac{\pi D^3}{200} \Rightarrow \left|\frac{\pi D^2}{2}dD\right| \le \frac{\pi D^3}{200} \Rightarrow |dD| \le \frac{D}{100} = (1\%)\,D \Rightarrow$ the allowable percentage error in measuring the diameter is 1%.

49. $E(x) = f(x) - g(x) \Rightarrow E(x) = f(x) - m(x-a) - c$. Then $E(a) = 0 \Rightarrow f(a) - m(a-a) - c = 0 \Rightarrow c = f(a)$. Next we calculate m: $\lim_{x\to a}\frac{E(x)}{x-a} = 0 \Rightarrow \lim_{x\to a}\frac{f(x)-m(x-a)-c}{x-a} = 0 \Rightarrow \lim_{x\to a}\left[\frac{f(x)-f(a)}{x-a} - m\right] = 0$ (since $c = f(a)$) $\Rightarrow f'(a) - m = 0 \Rightarrow m = f'(a)$. Therefore, $g(x) = m(x-a) + c = f'(a)(x-a) + f(a)$ is the linear approximation, as claimed.

50. (a) i. $Q(a) = f(a)$ implies that $b_0 = f(a)$.

ii. Since $Q'(x) = b_1 + 2b_2(x - a)$, $Q'(a) = f'(a)$ implies that $b_1 = f'(a)$.

iii. Since $Q''(x) = 2b_2$, $Q''(a) = f''(a)$ implies that $b_2 = \frac{f''(a)}{2}$.

In summary, $b_0 = f(a)$, $b_1 = f'(a)$, and $b_2 = \frac{f''(a)}{2}$.

(b) $f(x) = (1 - x)^{-1}$

$f'(x) = -1(1 - x)^{-2}(-1) = (1 - x)^{-2}$

$f''(x) = -2(1 - x)^{-3}(-1) = 2(1 - x)^{-3}$

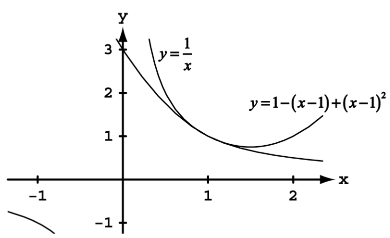
Since $f(0) = 1$, $f'(0) = 1$, and $f''(0) = 2$, the coefficients are $b_0 = 1$, $b_1 = 1$, $b_2 = \frac{2}{2} = 1$. The quadratic approximation is $Q(x) = 1 + x + x^2$.

(c) $g(x) = x^{-1}$

$g'(x) = -1x^{-2}$

$g''(x) = 2x^{-3}$

Since $g(1) = 1$, $g'(1) = -1$, and $g''(1) = 2$, the coefficients are $b_0 = 1$, $b_1 = -1$, $b_2 = \frac{2}{2} = 1$. The quadratic approximation is $Q(x) = 1 - (x - 1) + (x - 1)^2$.



[-1.35, 3.35] by [-1.25, 3.25]

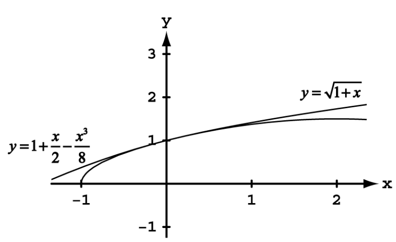
As one zooms in, the two graphs quickly become indistinguishable. They appear to be identical.

(d) $h(x) = (1 + x)^{1/2}$

$h'(x) = \frac{1}{2}(1 + x)^{-1/2}$

$h''(x) = -\frac{1}{4}(1 + x)^{-3/2}$

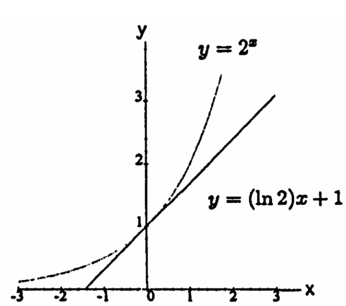
Since $h(0) = 1$, $h'(0) = \frac{1}{2}$, and $h''(0) = -\frac{1}{4}$, the coefficients are $b_0 = 1$, $b_1 = \frac{1}{2}$, $b_2 = \frac{-\frac{1}{4}}{2} = -\frac{1}{8}$. The quadratic approximation is $Q(x) = 1 + \frac{x}{2} - \frac{x^2}{8}$.

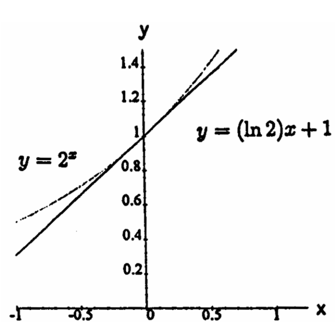


As one zooms in, the two graphs quickly become indistinguishable. They appear to be identical.

51. (a) $f(x) = 2^x \Rightarrow f'(x) = 2^x \ln 2$; $L(x) = (2^0 \ln 2)\,x + 2^0 = x \ln 2 + 1 \approx 0.69x + 1$

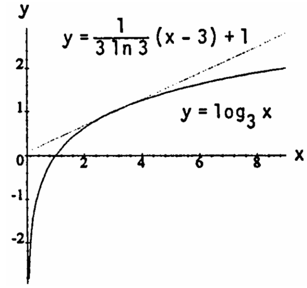
(b)

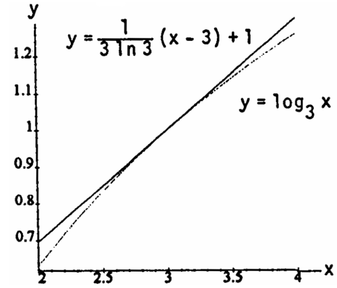




52. (a) $f(x) = \log_3 x \Rightarrow f'(x) = \frac{1}{x \ln 3}$, and $f(3) = \frac{\ln 3}{\ln 3} \Rightarrow L(x) = \frac{1}{3 \ln 3}(x-3) + \frac{\ln 3}{\ln 3} = \frac{x}{3 \ln 3} - \frac{1}{\ln 3} + 1$
$\approx 0.30x + 0.09$

(b)





53-58. Example CAS commands:

Maple:

```
with(plots):
a:= 1: f:=x -> x ∧ 3 + x ∧ 2 − 2*x;
plot(f(x), x=−1..2);
diff(f(x),x);
fp := unapply (″,x);
L:=x -> f(a) + fp(a)*(x − a);
plot({f(x), L(x)}, x=−1..2);
err:=x -> abs(f(x) − L(x));
plot(err(x), x=−1..2, title = #absolute error function#);
err(−1);
```

Mathematica: (function, x1, x2, and a may vary):

```
Clear[f, x]
{x1, x2} = {−1, 2}; a = 1;
f[x_]:=x^3 + x^2 − 2x
Plot[f[x], {x, x1, x2}]
lin[x_]=f[a] + f'[a](x − a)
Plot[{f[x], lin[x]}, {x, x1, x2}]
err[x_]=Abs[f[x] − lin[x]]
Plot[err[x], {x, x1,x 2}]
err//N
```

After reviewing the error function, plot the error function and epsilon for differing values of epsilon (eps) and delta (del)

```
eps = 0.5; del = 0.4
Plot[{err[x], eps},{x, a − del, a + del}]
```

CHAPTER 2 PRACTICE AND ADDITIONAL EXERCISES

1. $y = x^5 - 0.125x^2 + 0.25x \Rightarrow \frac{dy}{dx} = 5x^4 - 0.25x + 0.25$

2. $y = 3 - 0.7x^3 + 0.3x^7 \Rightarrow \frac{dy}{dx} = -2.1x^2 + 2.1x^6$

3. $y = x^3 - 3(x^2 + \pi^2) \Rightarrow \frac{dy}{dx} = 3x^2 - 3(2x + 0) = 3x^2 - 6x = 3x(x-2)$

4. $y = x^7 + \sqrt{7}x - \frac{1}{\pi+1} \Rightarrow \frac{dy}{dx} = 7x^6 + \sqrt{7}$

5. $y = (x+1)^2(x^2+2x) \Rightarrow \frac{dy}{dx} = (x+1)^2(2x+2) + (x^2+2x)(2(x+1)) = 2(x+1)\left[(x+1)^2 + x(x+2)\right]$
$= 2(x+1)(2x^2+4x+1)$

6. $y = (2x-5)(4-x)^{-1} \Rightarrow \frac{dy}{dx} = (2x-5)(-1)(4-x)^{-2}(-1) + (4-x)^{-1}(2) = (4-x)^{-2}\left[(2x-5) + 2(4-x)\right]$
$= 3(4-x)^{-2}$

7. $y = (\theta^2 + \sec\theta + 1)^3 \Rightarrow \frac{dy}{d\theta} = 3(\theta^2 + \sec\theta + 1)^2(2\theta + \sec\theta\tan\theta)$

8. $y = \left(-1 - \frac{\csc\theta}{2} - \frac{\theta^2}{4}\right)^2 \Rightarrow \frac{dy}{d\theta} = 2\left(-1 - \frac{\csc\theta}{2} - \frac{\theta^2}{4}\right)\left(\frac{\csc\theta\cot\theta}{2} - \frac{\theta}{2}\right) = \left(-1 - \frac{\csc\theta}{2} - \frac{\theta^2}{4}\right)(\csc\theta\cot\theta - \theta)$

9. $s = \frac{\sqrt{t}}{1+\sqrt{t}} \Rightarrow \frac{ds}{dt} = \frac{(1+\sqrt{t})\cdot\frac{1}{2\sqrt{t}} - \sqrt{t}\left(\frac{1}{2\sqrt{t}}\right)}{(1+\sqrt{t})^2} = \frac{(1+\sqrt{t}) - \sqrt{t}}{2\sqrt{t}(1+\sqrt{t})^2} = \frac{1}{2\sqrt{t}(1+\sqrt{t})^2}$

10. $s = \frac{1}{\sqrt{t}-1} \Rightarrow \frac{ds}{dt} = \frac{(\sqrt{t}-1)(0) - 1\left(\frac{1}{2\sqrt{t}}\right)}{(\sqrt{t}-1)^2} = \frac{-1}{2\sqrt{t}(\sqrt{t}-1)^2}$

11. $y = 2\tan^2 x - \sec^2 x \Rightarrow \frac{dy}{dx} = (4\tan x)(\sec^2 x) - (2\sec x)(\sec x\tan x) = 2\sec^2 x\tan x$

12. $y = \frac{1}{\sin^2 x} - \frac{2}{\sin x} = \csc^2 x - 2\csc x \Rightarrow \frac{dy}{dx} = (2\csc x)(-\csc x\cot x) - 2(-\csc x\cot x) = (2\csc x\cot x)(1 - \csc x)$

13. $s = \cos^4(1-2t) \Rightarrow \frac{ds}{dt} = 4\cos^3(1-2t)(-\sin(1-2t))(-2) = 8\cos^3(1-2t)\sin(1-2t)$

14. $s = \cot^3\left(\frac{2}{t}\right) \Rightarrow \frac{ds}{dt} = 3\cot^2\left(\frac{2}{t}\right)\left(-\csc^2\left(\frac{2}{t}\right)\right)\left(\frac{-2}{t^2}\right) = \frac{6}{t^2}\cot^2\left(\frac{2}{t}\right)\csc^2\left(\frac{2}{t}\right)$

15. $s = (\sec t + \tan t)^5 \Rightarrow \frac{ds}{dt} = 5(\sec t + \tan t)^4(\sec t\tan t + \sec^2 t) = 5(\sec t)(\sec t + \tan t)^5$

16. $s = \csc^5(1 - t + 3t^2) \Rightarrow \frac{ds}{dt} = 5\csc^4(1 - t + 3t^2)(-\csc(1 - t + 3t^2)\cot(1 - t + 3t^2))(-1 + 6t)$
$= -5(6t-1)\csc^5(1 - t + 3t^2)\cot(1 - t + 3t^2)$

17. $r = \sqrt{2\theta\sin\theta} = (2\theta\sin\theta)^{1/2} \Rightarrow \frac{dr}{d\theta} = \frac{1}{2}(2\theta\sin\theta)^{-1/2}(2\theta\cos\theta + 2\sin\theta) = \frac{\theta\cos\theta + \sin\theta}{\sqrt{2\theta\sin\theta}}$

18. $r = 2\theta\sqrt{\cos\theta} = 2\theta(\cos\theta)^{1/2} \Rightarrow \frac{dr}{d\theta} = 2\theta\left(\frac{1}{2}\right)(\cos\theta)^{-1/2}(-\sin\theta) + 2(\cos\theta)^{1/2} = \frac{-\theta\sin\theta}{\sqrt{\cos\theta}} + 2\sqrt{\cos\theta}$
$= \frac{2\cos\theta - \theta\sin\theta}{\sqrt{\cos\theta}}$

19. $r = \sin\sqrt{2\theta} = \sin(2\theta)^{1/2} \Rightarrow \frac{dr}{d\theta} = \cos(2\theta)^{1/2}\left(\frac{1}{2}(2\theta)^{-1/2}(2)\right) = \frac{\cos\sqrt{2\theta}}{\sqrt{2\theta}}$

20. $r = \sin\left(\theta + \sqrt{\theta+1}\right) \Rightarrow \frac{dr}{d\theta} = \cos\left(\theta + \sqrt{\theta+1}\right)\left(1 + \frac{1}{2\sqrt{\theta+1}}\right) = \frac{2\sqrt{\theta+1}+1}{2\sqrt{\theta+1}}\cos\left(\theta + \sqrt{\theta+1}\right)$

21. $y = \frac{1}{2}x^2\csc\frac{2}{x} \Rightarrow \frac{dy}{dx} = \frac{1}{2}x^2\left(-\csc\frac{2}{x}\cot\frac{2}{x}\right)\left(\frac{-2}{x^2}\right) + \left(\csc\frac{2}{x}\right)\left(\frac{1}{2}\cdot 2x\right) = \csc\frac{2}{x}\cot\frac{2}{x} + x\csc\frac{2}{x}$

22. $y = 2\sqrt{x}\sin\sqrt{x} \Rightarrow \frac{dy}{dx} = 2\sqrt{x}\left(\cos\sqrt{x}\right)\left(\frac{1}{2\sqrt{x}}\right) + \left(\sin\sqrt{x}\right)\left(\frac{2}{2\sqrt{x}}\right) = \cos\sqrt{x} + \frac{\sin\sqrt{x}}{\sqrt{x}}$

23. $y = x^{-1/2} \sec (2x)^2 \Rightarrow \frac{dy}{dx} = x^{-1/2} \sec (2x)^2 \tan (2x)^2 (2(2x) \cdot 2) + \sec (2x)^2 \left(-\frac{1}{2} x^{-3/2}\right)$
$= 8x^{1/2} \sec (2x)^2 \tan (2x)^2 - \frac{1}{2} x^{-3/2} \sec (2x)^2 = \frac{1}{2} x^{1/2} \sec (2x)^2 \left[16 \tan (2x)^2 - x^{-2}\right]$ or $\frac{1}{2x^{3/2}} \sec(2x)^2 \left[16x^2 \tan(2x)^2 - 1\right]$

24. $y = \sqrt{x} \csc (x+1)^3 = x^{1/2} \csc (x+1)^3$
$\Rightarrow \frac{dy}{dx} = x^{1/2} \left(-\csc (x+1)^3 \cot (x+1)^3\right) \left(3(x+1)^2\right) + \csc (x+1)^3 \left(\frac{1}{2} x^{-1/2}\right)$
$= -3\sqrt{x} (x+1)^2 \csc (x+1)^3 \cot (x+1)^3 + \frac{\csc (x+1)^3}{2\sqrt{x}} = \frac{1}{2} \sqrt{x} \csc (x+1)^3 \left[\frac{1}{x} - 6(x+1)^2 \cot (x+1)^3\right]$
or $\frac{1}{2\sqrt{x}} \csc(x+1)^3 [1 - 6x(x+1)^2 \cot (x+1)^3]$

25. $y = 5 \cot x^2 \Rightarrow \frac{dy}{dx} = 5\left(-\csc^2 x^2\right)(2x) = -10x \csc^2 (x^2)$

26. $y = x^2 \cot 5x \Rightarrow \frac{dy}{dx} = x^2 \left(-\csc^2 5x\right)(5) + (\cot 5x)(2x) = -5x^2 \csc^2 5x + 2x \cot 5x$

27. $y = x^2 \sin^2 (2x^2) \Rightarrow \frac{dy}{dx} = x^2 \left(2 \sin (2x^2)\right) \left(\cos (2x^2)\right) (4x) + \sin^2 (2x^2) (2x) = 8x^3 \sin (2x^2) \cos (2x^2) + 2x \sin^2 (2x^2)$

28. $y = x^{-2} \sin^2 (x^3) \Rightarrow \frac{dy}{dx} = x^{-2} \left(2 \sin (x^3)\right) \left(\cos (x^3)\right) (3x^2) + \sin^2 (x^3) \left(-2x^{-3}\right) = 6 \sin (x^3) \cos (x^3) - 2x^{-3} \sin^2 (x^3)$

29. $s = \left(\frac{4t}{t+1}\right)^{-2} \Rightarrow \frac{ds}{dt} = -2\left(\frac{4t}{t+1}\right)^{-3} \left(\frac{(t+1)(4) - (4t)(1)}{(t+1)^2}\right) = -2\left(\frac{4t}{t+1}\right)^{-3} \frac{4}{(t+1)^2} = -\frac{(t+1)}{8t^3}$

30. $s = \frac{-1}{15(15t-1)^3} = -\frac{1}{15} (15t-1)^{-3} \Rightarrow \frac{ds}{dt} = -\frac{1}{15} (-3)(15t-1)^{-4}(15) = \frac{3}{(15t-1)^4}$

31. $y = \left(\frac{\sqrt{x}}{x+1}\right)^2 \Rightarrow \frac{dy}{dx} = 2\left(\frac{\sqrt{x}}{x+1}\right) \cdot \frac{(x+1)\left(\frac{1}{2\sqrt{x}}\right) - \left(\sqrt{x}\right)(1)}{(x+1)^2} = \frac{(x+1) - 2x}{(x+1)^3} = \frac{1-x}{(x+1)^3}$

32. $y = \left(\frac{2\sqrt{x}}{2\sqrt{x}+1}\right)^2 \Rightarrow \frac{dy}{dx} = 2\left(\frac{2\sqrt{x}}{2\sqrt{x}+1}\right) \left(\frac{(2\sqrt{x}+1)\left(\frac{1}{\sqrt{x}}\right) - (2\sqrt{x})\left(\frac{1}{\sqrt{x}}\right)}{(2\sqrt{x}+1)^2}\right) = \frac{4\sqrt{x}\left(\frac{1}{\sqrt{x}}\right)}{(2\sqrt{x}+1)^3} = \frac{4}{(2\sqrt{x}+1)^3}$

33. $y = \sqrt{\frac{x^2+x}{x^2}} = \left(1 + \frac{1}{x}\right)^{1/2} \Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(1 + \frac{1}{x}\right)^{-1/2} \left(-\frac{1}{x^2}\right) = -\frac{1}{2x^2 \sqrt{1 + \frac{1}{x}}}$

34. $y = 4x\sqrt{x + \sqrt{x}} = 4x\left(x + x^{1/2}\right)^{1/2} \Rightarrow \frac{dy}{dx} = 4x\left(\frac{1}{2}\right)\left(x + x^{1/2}\right)^{-1/2} \left(1 + \frac{1}{2} x^{-1/2}\right) + \left(x + x^{1/2}\right)^{1/2}(4)$
$= \left(x + \sqrt{x}\right)^{-1/2} \left[2x\left(1 + \frac{1}{2\sqrt{x}}\right) + 4\left(x + \sqrt{x}\right)\right] = \left(x + \sqrt{x}\right)^{-1/2} \left(2x + \sqrt{x} + 4x + 4\sqrt{x}\right) = \frac{6x + 5\sqrt{x}}{\sqrt{x + \sqrt{x}}}$

35. $r = \left(\frac{\sin \theta}{\cos \theta - 1}\right)^2 \Rightarrow \frac{dr}{d\theta} = 2\left(\frac{\sin \theta}{\cos \theta - 1}\right) \left[\frac{(\cos \theta - 1)(\cos \theta) - (\sin \theta)(-\sin \theta)}{(\cos \theta - 1)^2}\right]$
$= 2\left(\frac{\sin \theta}{\cos \theta - 1}\right) \left(\frac{\cos^2 \theta - \cos \theta + \sin^2 \theta}{(\cos \theta - 1)^2}\right) = \frac{(2 \sin \theta)(1 - \cos \theta)}{(\cos \theta - 1)^3} = \frac{-2 \sin \theta}{(\cos \theta - 1)^2}$

36. $r = \left(\frac{\sin \theta + 1}{1 - \cos \theta}\right)^2 \Rightarrow \frac{dr}{d\theta} = 2\left(\frac{\sin \theta + 1}{1 - \cos \theta}\right) \left[\frac{(1 - \cos \theta)(\cos \theta) - (\sin \theta + 1)(\sin \theta)}{(1 - \cos \theta)^2}\right]$
$= \frac{2(\sin \theta + 1)}{(1 - \cos \theta)^3} \left(\cos \theta - \cos^2 \theta - \sin^2 \theta - \sin \theta\right) = \frac{2(\sin \theta + 1)(\cos \theta - \sin \theta - 1)}{(1 - \cos \theta)^3}$

37. $y = (2x+1)\sqrt{2x+1} = (2x+1)^{3/2} \Rightarrow \frac{dy}{dx} = \frac{3}{2}(2x+1)^{1/2}(2) = 3\sqrt{2x+1}$

38. $y = 20(3x-4)^{1/4}(3x-4)^{-1/5} = 20(3x-4)^{1/20} \Rightarrow \frac{dy}{dx} = 20\left(\frac{1}{20}\right)(3x-4)^{-19/20}(3) = \frac{3}{(3x-4)^{19/20}}$

39. $y = 3\left(5x^2 + \sin 2x\right)^{-3/2} \Rightarrow \frac{dy}{dx} = 3\left(-\frac{3}{2}\right)\left(5x^2 + \sin 2x\right)^{-5/2}[10x + (\cos 2x)(2)] = \frac{-9(5x + \cos 2x)}{(5x^2 + \sin 2x)^{5/2}}$

40. $y = \left(3 + \cos^3 3x\right)^{-1/3} \Rightarrow \frac{dy}{dx} = -\frac{1}{3}\left(3 + \cos^3 3x\right)^{-4/3}\left(3\cos^2 3x\right)(-\sin 3x)(3) = \frac{3\cos^2 3x \sin 3x}{(3 + \cos^3 3x)^{4/3}}$

41. $y = 10e^{-x/5} \Rightarrow \frac{dy}{dx} = (10)\left(-\frac{1}{5}\right)e^{-x/5} = -2e^{-x/5}$

42. $y = \sqrt{2}\, e^{\sqrt{2}x} \Rightarrow \frac{dy}{dx} = \left(\sqrt{2}\right)\left(\sqrt{2}\right)e^{\sqrt{2}x} = 2e^{\sqrt{2}x}$

43. $y = \frac{1}{4}xe^{4x} - \frac{1}{16}e^{4x} \Rightarrow \frac{dy}{dx} = \frac{1}{4}\left[x\left(4e^{4x}\right) + e^{4x}(1)\right] - \frac{1}{16}\left(4e^{4x}\right) = xe^{4x} + \frac{1}{4}e^{4x} - \frac{1}{4}e^{4x} = xe^{4x}$

44. $y = x^2e^{-2/x} = x^2e^{-2x^{-1}} \Rightarrow \frac{dy}{dx} = x^2\left[\left(2x^{-2}\right)e^{-2x^{-1}}\right] + e^{-2x^{-1}}(2x) = (2 + 2x)e^{-2x^{-1}} = 2e^{-2/x}(1 + x)$

45. $y = \ln\left(\sin^2\theta\right) \Rightarrow \frac{dy}{d\theta} = \frac{2(\sin\theta)(\cos\theta)}{\sin^2\theta} = \frac{2\cos\theta}{\sin\theta} = 2\cot\theta$

46. $y = \ln\left(\sec^2\theta\right) \Rightarrow \frac{dy}{d\theta} = \frac{2(\sec\theta)(\sec\theta\tan\theta)}{\sec^2\theta} = 2\tan\theta$

47. $y = \log_2\left(\frac{x^2}{2}\right) = \frac{\ln\left(\frac{x^2}{2}\right)}{\ln 2} \Rightarrow \frac{dy}{dx} = \frac{1}{\ln 2}\left(\frac{x}{\left(\frac{x^2}{2}\right)}\right) = \frac{2}{(\ln 2)x}$

48. $y = \log_5(3x - 7) = \frac{\ln(3x-7)}{\ln 5} \Rightarrow \frac{dy}{dx} = \left(\frac{1}{\ln 5}\right)\left(\frac{3}{3x-7}\right) = \frac{3}{(\ln 5)(3x-7)}$

49. $y = 8^{-t} \Rightarrow \frac{dy}{dt} = 8^{-t}(\ln 8)(-1) = -8^{-t}(\ln 8)$

50. $y = 9^{2t} \Rightarrow \frac{dy}{dt} = 9^{2t}(\ln 9)(2) = 9^{2t}(2\ln 9)$

51. $y = 5x^{3.6} \Rightarrow \frac{dy}{dx} = 5(3.6)x^{2.6} = 18x^{2.6}$

52. $y = \sqrt{2}\, x^{-\sqrt{2}} \Rightarrow \frac{dy}{dx} = \left(\sqrt{2}\right)\left(-\sqrt{2}\right)x^{\left(-\sqrt{2}-1\right)} = -2x^{\left(-\sqrt{2}-1\right)}$

53. $y = (x + 2)^{x+2} \Rightarrow \ln y = \ln(x + 2)^{x+2} = (x + 2)\ln(x + 2) \Rightarrow \frac{y'}{y} = (x + 2)\left(\frac{1}{x+2}\right) + (1)\ln(x + 2)$
$\Rightarrow \frac{dy}{dx} = (x + 2)^{x+2}[\ln(x + 2) + 1]$

54. $y = 2(\ln x)^{x/2} \Rightarrow \ln y = \ln\left[2(\ln x)^{x/2}\right] = \ln(2) + \left(\frac{x}{2}\right)\ln(\ln x) \Rightarrow \frac{y'}{y} = 0 + \left(\frac{x}{2}\right)\left[\frac{\left(\frac{1}{x}\right)}{\ln x}\right] + (\ln(\ln x))\left(\frac{1}{2}\right)$
$\Rightarrow y' = \left[\frac{1}{2\ln x} + \left(\frac{1}{2}\right)\ln(\ln x)\right]2(\ln x)^{x/2} = (\ln x)^{x/2}\left[\ln(\ln x) + \frac{1}{\ln x}\right]$

55. $y = \sin^{-1}\sqrt{1 - u^2} = \sin^{-1}\left(1 - u^2\right)^{1/2} \Rightarrow \frac{dy}{du} = \frac{\frac{1}{2}\left(1 - u^2\right)^{-1/2}(-2u)}{\sqrt{1 - \left[\left(1 - u^2\right)^{1/2}\right]^2}} = \frac{-u}{\sqrt{1 - u^2}\sqrt{1 - (1 - u^2)}} = \frac{-u}{|u|\sqrt{1 - u^2}}$
$= \frac{-u}{u\sqrt{1 - u^2}} = \frac{-1}{\sqrt{1 - u^2}},\ 0 < u < 1$

56. $y = \sin^{-1}\left(\frac{1}{\sqrt{v}}\right) = \sin^{-1}v^{-1/2} \Rightarrow \frac{dy}{dv} = \frac{-\frac{1}{2}v^{-3/2}}{\sqrt{1 - \left(v^{-1/2}\right)^2}} = \frac{-1}{2v^{3/2}\sqrt{1 - v^{-1}}} = \frac{-1}{2v^{3/2}\sqrt{\frac{v-1}{v}}} = \frac{-\sqrt{v}}{2v^{3/2}\sqrt{v - 1}}$
$= \frac{-1}{2v\sqrt{v - 1}}$

57. $y = \ln\left(\cos^{-1}x\right) \Rightarrow y' = \frac{\left(\frac{-1}{\sqrt{1 - x^2}}\right)}{\cos^{-1}x} = \frac{-1}{\sqrt{1 - x^2}\cos^{-1}x}$

58. $y = z\cos^{-1} z - \sqrt{1-z^2} = z\cos^{-1} z - (1-z^2)^{1/2} \Rightarrow \frac{dy}{dz} = \cos^{-1} z - \frac{z}{\sqrt{1-z^2}} - \left(\frac{1}{2}\right)(1-z^2)^{-1/2}(-2z)$
$= \cos^{-1} z - \frac{z}{\sqrt{1-z^2}} + \frac{z}{\sqrt{1-z^2}} = \cos^{-1} z$

59. $y = t\tan^{-1} t - \left(\frac{1}{2}\right)\ln t \Rightarrow \frac{dy}{dt} = \tan^{-1} t + t\left(\frac{1}{1+t^2}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{t}\right) = \tan^{-1} t + \frac{t}{1+t^2} - \frac{1}{2t}$

60. $y = (1+t^2)\cot^{-1} 2t \Rightarrow \frac{dy}{dt} = 2t\cot^{-1} 2t + (1+t^2)\left(\frac{-2}{1+4t^2}\right)$

61. $y = z\sec^{-1} z - \sqrt{z^2-1} = z\sec^{-1} z - (z^2-1)^{1/2} \Rightarrow \frac{dy}{dz} = z\left(\frac{1}{|z|\sqrt{z^2-1}}\right) + (\sec^{-1} z)(1) - \frac{1}{2}(z^2-1)^{-1/2}(2z)$
$= \frac{z}{|z|\sqrt{z^2-1}} - \frac{z}{\sqrt{z^2-1}} + \sec^{-1} z = \frac{1-z}{\sqrt{z^2-1}} + \sec^{-1} z,\ z > 1$

62. $y = 2\sqrt{x-1}\sec^{-1}\sqrt{x} = 2(x-1)^{1/2}\sec^{-1}\left(x^{1/2}\right)$
$\Rightarrow \frac{dy}{dx} = 2\left[\left(\frac{1}{2}\right)(x-1)^{-1/2}\sec^{-1}\left(x^{1/2}\right) + (x-1)^{1/2}\left(\frac{\left(\frac{1}{2}\right)x^{-1/2}}{\sqrt{x}\sqrt{x-1}}\right)\right] = 2\left(\frac{\sec^{-1}\sqrt{x}}{2\sqrt{x-1}} + \frac{1}{2x}\right) = \frac{\sec^{-1}\sqrt{x}}{\sqrt{x-1}} + \frac{1}{x}$

63. $y = \csc^{-1}(\sec\theta) \Rightarrow \frac{dy}{d\theta} = \frac{-\sec\theta\tan\theta}{|\sec\theta|\sqrt{\sec^2\theta-1}} = -\frac{\tan\theta}{|\tan\theta|} = -1,\ 0 < \theta < \frac{\pi}{2}$

64. $y = (1+x^2)e^{\tan^{-1} x} \Rightarrow y' = 2xe^{\tan^{-1} x} + (1+x^2)\left(\frac{e^{\tan^{-1} x}}{1+x^2}\right) = 2xe^{\tan^{-1} x} + e^{\tan^{-1} x}$

65. $xy + 2x + 3y = 1 \Rightarrow (xy' + y) + 2 + 3y' = 0 \Rightarrow xy' + 3y' = -2 - y \Rightarrow y'(x+3) = -2 - y \Rightarrow y' = -\frac{y+2}{x+3}$

66. $x^2 + xy + y^2 - 5x = 2 \Rightarrow 2x + \left(x\frac{dy}{dx} + y\right) + 2y\frac{dy}{dx} - 5 = 0 \Rightarrow x\frac{dy}{dx} + 2y\frac{dy}{dx} = 5 - 2x - y \Rightarrow \frac{dy}{dx}(x+2y)$
$= 5 - 2x - y \Rightarrow \frac{dy}{dx} = \frac{5-2x-y}{x+2y}$

67. $x^3 + 4xy - 3y^{4/3} = 2x \Rightarrow 3x^2 + \left(4x\frac{dy}{dx} + 4y\right) - 4y^{1/3}\frac{dy}{dx} = 2 \Rightarrow 4x\frac{dy}{dx} - 4y^{1/3}\frac{dy}{dx} = 2 - 3x^2 - 4y$
$\Rightarrow \frac{dy}{dx}\left(4x - 4y^{1/3}\right) = 2 - 3x^2 - 4y \Rightarrow \frac{dy}{dx} = \frac{2-3x^2-4y}{4x-4y^{1/3}}$

68. $5x^{4/5} + 10y^{6/5} = 15 \Rightarrow 4x^{-1/5} + 12y^{1/5}\frac{dy}{dx} = 0 \Rightarrow 12y^{1/5}\frac{dy}{dx} = -4x^{-1/5} \Rightarrow \frac{dy}{dx} = -\frac{1}{3}x^{-1/5}y^{-1/5} = -\frac{1}{3(xy)^{1/5}}$

69. $(xy)^{1/2} = 1 \Rightarrow \frac{1}{2}(xy)^{-1/2}\left(x\frac{dy}{dx} + y\right) = 0 \Rightarrow x^{1/2}y^{-1/2}\frac{dy}{dx} = -x^{-1/2}y^{1/2} \Rightarrow \frac{dy}{dx} = -x^{-1}y \Rightarrow \frac{dy}{dx} = -\frac{y}{x}$

70. $x^2y^2 = 1 \Rightarrow x^2\left(2y\frac{dy}{dx}\right) + y^2(2x) = 0 \Rightarrow 2x^2y\frac{dy}{dx} = -2xy^2 \Rightarrow \frac{dy}{dx} = -\frac{y}{x}$

71. $y^2 = \frac{x}{x+1} \Rightarrow 2y\frac{dy}{dx} = \frac{(x+1)(1)-(x)(1)}{(x+1)^2} \Rightarrow \frac{dy}{dx} = \frac{1}{2y(x+1)^2}$

72. $y^2 = \left(\frac{1+x}{1-x}\right)^{1/2} \Rightarrow y^4 = \frac{1+x}{1-x} \Rightarrow 4y^3\frac{dy}{dx} = \frac{(1-x)(1)-(1+x)(-1)}{(1-x)^2} \Rightarrow \frac{dy}{dx} = \frac{1}{2y^3(1-x)^2}$

73. $e^{x+2y} = 1 \Rightarrow e^{x+2y}\left(1 + 2\frac{dy}{dx}\right) = 0 \Rightarrow \frac{dy}{dx} = -\frac{1}{2}$

74. $y^2 = 2e^{-1/x} \Rightarrow 2y\frac{dy}{dx} = 2e^{-1/x}\frac{d}{dx}\left(-x^{-1}\right) = \frac{2e^{-1/x}}{x^2} \Rightarrow \frac{dy}{dx} = \frac{e^{-1/x}}{y\,x^2}$

75. $\ln\left(\frac{x}{y}\right) = 1 \Rightarrow \frac{1}{x/y}\frac{d}{dx}\left(\frac{x}{y}\right) = 0 \Rightarrow \frac{y(1) - x\frac{dy}{dx}}{y^2} = 0 \Rightarrow \frac{dy}{dx} = \frac{y}{x}$

76. $x\sin^{-1}y = 1 + x^2 \Rightarrow y = \sin(x^{-1} + x) \Rightarrow \frac{dy}{dx} = \cos(x^{-1} + x)\frac{d}{dx}(x^{-1} + x) = (1 - x^{-2})\cos(x^{-1} + x)$
$= \left(\frac{x^2-1}{x}\right)\cos\left(\frac{x^2+1}{x}\right)$

77. $y\,e^{\tan^{-1}x} = 2 \Rightarrow y = 2e^{-\tan^{-1}x} \Rightarrow \frac{dy}{dx} = 2e^{-\tan^{-1}x}\frac{d}{dx}(-\tan^{-1}x) = -2e^{-\tan^{-1}x}\left(\frac{1}{1+x^2}\right) = -\frac{2e^{-\tan^{-1}x}}{1+x^2}$

78. $x^y = \sqrt{2} \Rightarrow \ln(x^y) = \ln\left(2^{1/2}\right) \Rightarrow y\ln x = \frac{\ln 2}{2} \Rightarrow \frac{d}{dx}(y\ln x) = 0 \Rightarrow y\left(\frac{1}{x}\right) + \ln x\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{y}{x\ln x}$

79. $p^3 + 4pq - 3q^2 = 2 \Rightarrow 3p^2\frac{dp}{dq} + 4\left(p + q\frac{dp}{dq}\right) - 6q = 0 \Rightarrow 3p^2\frac{dp}{dq} + 4q\frac{dp}{dq} = 6q - 4p \Rightarrow \frac{dp}{dq}(3p^2 + 4q) = 6q - 4p$
$\Rightarrow \frac{dp}{dq} = \frac{6q - 4p}{3p^2 + 4q}$

80. $q = (5p^2 + 2p)^{-3/2} \Rightarrow 1 = -\frac{3}{2}(5p^2 + 2p)^{-5/2}\left(10p\frac{dp}{dq} + 2\frac{dp}{dq}\right) \Rightarrow -\frac{2}{3}(5p^2 + 2p)^{5/2} = \frac{dp}{dq}(10p + 2)$
$\Rightarrow \frac{dp}{dq} = -\frac{(5p^2 + 2p)^{5/2}}{3(5p + 1)}$

81. $r\cos 2s + \sin^2 s = \pi \Rightarrow r(-\sin 2s)(2) + (\cos 2s)\left(\frac{dr}{ds}\right) + 2\sin s\cos s = 0 \Rightarrow \frac{dr}{ds}(\cos 2s) = 2r\sin 2s - 2\sin s\cos s$
$\Rightarrow \frac{dr}{ds} = \frac{2r\sin 2s - \sin 2s}{\cos 2s} = \frac{(2r - 1)(\sin 2s)}{\cos 2s} = (2r - 1)(\tan 2s)$

82. $2rs - r - s + s^2 = -3 \Rightarrow 2\left(r + s\frac{dr}{ds}\right) - \frac{dr}{ds} - 1 + 2s = 0 \Rightarrow \frac{dr}{ds}(2s - 1) = 1 - 2s - 2r \Rightarrow \frac{dr}{ds} = \frac{1 - 2s - 2r}{2s - 1}$

83. (a) $x^3 + y^3 = 1 \Rightarrow 3x^2 + 3y^2\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x^2}{y^2} \Rightarrow \frac{d^2y}{dx^2} = \frac{y^2(-2x) - (-x^2)\left(2y\frac{dy}{dx}\right)}{y^4}$
$\Rightarrow \frac{d^2y}{dx^2} = \frac{-2xy^2 + (2yx^2)\left(-\frac{x^2}{y^2}\right)}{y^4} = \frac{-2xy^2 - \frac{2x^4}{y}}{y^4} = \frac{-2xy^3 - 2x^4}{y^5}$

(b) $y^2 = 1 - \frac{2}{x} \Rightarrow 2y\frac{dy}{dx} = \frac{2}{x^2} \Rightarrow \frac{dy}{dx} = \frac{1}{yx^2} \Rightarrow \frac{dy}{dx} = (yx^2)^{-1} \Rightarrow \frac{d^2y}{dx^2} = -(yx^2)^{-2}\left[y(2x) + x^2\frac{dy}{dx}\right]$
$\Rightarrow \frac{d^2y}{dx^2} = \frac{-2xy - x^2\left(\frac{1}{yx^2}\right)}{y^2x^4} = \frac{-2xy^2 - 1}{y^3x^4}$

84. (a) $x^2 - y^2 = 1 \Rightarrow 2x - 2y\frac{dy}{dx} = 0 \Rightarrow -2y\frac{dy}{dx} = -2x \Rightarrow \frac{dy}{dx} = \frac{x}{y}$

(b) $\frac{dy}{dx} = \frac{x}{y} \Rightarrow \frac{d^2y}{dx^2} = \frac{y(1) - x\frac{dy}{dx}}{y^2} = \frac{y - x\left(\frac{x}{y}\right)}{y^2} = \frac{y^2 - x^2}{y^3} = \frac{-1}{y^3}$ (since $y^2 - x^2 = -1$)

85. (a) Let $h(x) = 6f(x) - g(x) \Rightarrow h'(x) = 6f'(x) - g'(x) \Rightarrow h'(1) = 6f'(1) - g'(1) = 6\left(\frac{1}{2}\right) - (-4) = 7$

(b) Let $h(x) = f(x)g^2(x) \Rightarrow h'(x) = f(x)(2g(x))g'(x) + g^2(x)f'(x) \Rightarrow h'(0) = 2f(0)g(0)g'(0) + g^2(0)f'(0)$
$= 2(1)(1)\left(\frac{1}{2}\right) + (1)^2(-3) = -2$

(c) Let $h(x) = \frac{f(x)}{g(x) + 1} \Rightarrow h'(x) = \frac{(g(x) + 1)f'(x) - f(x)g'(x)}{(g(x) + 1)^2} \Rightarrow h'(1) = \frac{(g(1) + 1)f'(1) - f(1)g'(1)}{(g(1) + 1)^2} = \frac{(5 + 1)\left(\frac{1}{2}\right) - 3(-4)}{(5 + 1)^2} = \frac{5}{12}$

(d) Let $h(x) = f(g(x)) \Rightarrow h'(x) = f'(g(x))g'(x) \Rightarrow h'(0) = f'(g(0))g'(0) = f'(1)\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$

(e) Let $h(x) = g(f(x)) \Rightarrow h'(x) = g'(f(x))f'(x) \Rightarrow h'(0) = g'(f(0))f'(0) = g'(1)f'(0) = (-4)(-3) = 12$

(f) Let $h(x) = (x + f(x))^{3/2} \Rightarrow h'(x) = \frac{3}{2}(x + f(x))^{1/2}(1 + f'(x)) \Rightarrow h'(1) = \frac{3}{2}(1 + f(1))^{1/2}(1 + f'(1))$
$= \frac{3}{2}(1 + 3)^{1/2}\left(1 + \frac{1}{2}\right) = \frac{9}{2}$

(g) Let $h(x) = f(x + g(x)) \Rightarrow h'(x) = f'(x + g(x))(1 + g'(x)) \Rightarrow h'(0) = f'(g(0))(1 + g'(0)) = f'(1)\left(1 + \frac{1}{2}\right) = \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)$
$= \frac{3}{4}$

86. (a) Let $h(x) = \sqrt{x}\, f(x) \Rightarrow h'(x) = \sqrt{x}\, f'(x) + f(x) \cdot \frac{1}{2\sqrt{x}} \Rightarrow h'(1) = \sqrt{1}\, f'(1) + f(1) \cdot \frac{1}{2\sqrt{1}} = \frac{1}{5} + (-3)\left(\frac{1}{2}\right) = -\frac{13}{10}$

(b) Let $h(x) = (f(x))^{1/2} \Rightarrow h'(x) = \frac{1}{2}(f(x))^{-1/2}(f'(x)) \Rightarrow h'(0) = \frac{1}{2}(f(0))^{-1/2} f'(0) = \frac{1}{2}(9)^{-1/2}(-2) = -\frac{1}{3}$

(c) Let $h(x) = f\left(\sqrt{x}\right) \Rightarrow h'(x) = f'\left(\sqrt{x}\right) \cdot \frac{1}{2\sqrt{x}} \Rightarrow h'(1) = f'\left(\sqrt{1}\right) \cdot \frac{1}{2\sqrt{1}} = \frac{1}{5} \cdot \frac{1}{2} = \frac{1}{10}$

(d) Let $h(x) = f(1 - 5\tan x) \Rightarrow h'(x) = f'(1 - 5\tan x)(-5\sec^2 x) \Rightarrow h'(0) = f'(1 - 5\tan 0)(-5\sec^2 0)$
$= f'(1)(-5) = \frac{1}{5}(-5) = -1$

(e) Let $h(x) = \frac{f(x)}{2+\cos x} \Rightarrow h'(x) = \frac{(2+\cos x)f'(x) - f(x)(-\sin x)}{(2+\cos x)^2} \Rightarrow h'(0) = \frac{(2+1)f'(0) - f(0)(0)}{(2+1)^2} = \frac{3(-2)}{9} = -\frac{2}{3}$

(f) Let $h(x) = 10\sin\left(\frac{\pi x}{2}\right) f^2(x) \Rightarrow h'(x) = 10\sin\left(\frac{\pi x}{2}\right)(2f(x)f'(x)) + f^2(x)\left(10\cos\left(\frac{\pi x}{2}\right)\right)\left(\frac{\pi}{2}\right)$
$\Rightarrow h'(1) = 10\sin\left(\frac{\pi}{2}\right)(2f(1)f'(1)) + f^2(1)\left(10\cos\left(\frac{\pi}{2}\right)\right)\left(\frac{\pi}{2}\right) = 20(-3)\left(\frac{1}{5}\right) + 0 = -12$

87. $x = t^2 + \pi \Rightarrow \frac{dx}{dt} = 2t$; $y = 3\sin 2x \Rightarrow \frac{dy}{dx} = 3(\cos 2x)(2) = 6\cos 2x = 6\cos(2t^2 + 2\pi) = 6\cos(2t^2)$; thus,
$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = 6\cos(2t^2) \cdot 2t \Rightarrow \left.\frac{dy}{dt}\right|_{t=0} = 6\cos(0) \cdot 0 = 0$

88. $t = (u^2 + 2u)^{1/3} \Rightarrow \frac{dt}{du} = \frac{1}{3}(u^2 + 2u)^{-2/3}(2u + 2) = \frac{2}{3}(u^2 + 2u)^{-2/3}(u + 1)$; $s = t^2 + 5t \Rightarrow \frac{ds}{dt} = 2t + 5$
$= 2(u^2 + 2u)^{1/3} + 5$; thus $\frac{ds}{du} = \frac{ds}{dt} \cdot \frac{dt}{du} = \left[2(u^2 + 2u)^{1/3} + 5\right]\left(\frac{2}{3}\right)(u^2 + 2u)^{-2/3}(u + 1)$
$\Rightarrow \left.\frac{ds}{du}\right|_{u=2} = \left[2(2^2 + 2(2))^{1/3} + 5\right]\left(\frac{2}{3}\right)(2^2 + 2(2))^{-2/3}(2 + 1) = 2\left(2 \cdot 8^{1/3} + 5\right)\left(8^{-2/3}\right) = 2(2 \cdot 2 + 5)\left(\frac{1}{4}\right) = \frac{9}{2}$

89. $\frac{dw}{ds} = \frac{dw}{dr} \cdot \frac{dr}{ds} = \left[\cos\left(e^{\sqrt{r}}\right)\left(e^{\sqrt{r}} \frac{1}{2\sqrt{r}}\right)\right]\left[3\cos\left(s + \frac{\pi}{6}\right)\right]$ at $x = 0$, $r = 3\sin\frac{\pi}{6} = \frac{3}{2}$
$\Rightarrow \frac{dw}{ds} = \cos\left(e^{\sqrt{3/2}}\right)\left(e^{\sqrt{3/2}} \frac{1}{2\sqrt{3/2}}\right)\left(3\cos\left(\frac{\pi}{6}\right)\right) = \frac{3\sqrt{3}\, e^{\sqrt{3/2}}}{4\sqrt{3/2}}\cos\left(e^{\sqrt{3/2}}\right) = \frac{3\sqrt{2}\, e^{\sqrt{3/2}}}{4}\cos\left(e^{\sqrt{3/2}}\right)$

90. $\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt}$; $\frac{dr}{d\theta} = \frac{1}{3}(\theta^2 + 7)^{-2/3}(2\theta)$; $\theta^2 e^t + \theta = 1 \Rightarrow \frac{d}{dt}(\theta^2 e^t + \theta) = \frac{d}{dt}(1) \Rightarrow \theta^2 e^t + 2\theta\frac{d\theta}{dt}e^t + \frac{d\theta}{dt} = 0$
$\Rightarrow (1 + 2\theta e^t)\frac{d\theta}{dt} = -\theta^2 e^t \Rightarrow \frac{d\theta}{dt} = -\frac{\theta^2 e^t}{1+2\theta e^t} \Rightarrow \frac{dr}{dt} = \left[\frac{2\theta}{3(\theta^2+7)^{2/3}}\right]\left[-\frac{\theta^2 e^t}{1+2\theta e^t}\right] = -\frac{2\theta^3 e^t}{3(1+2\theta e^t)(\theta^2+7)^{2/3}}$
At $t = 0$, $\theta^2 + \theta - 1 = 0 \Rightarrow \theta = \frac{-1 \pm \sqrt{5}}{2} \Rightarrow \frac{dr}{dt} = -\frac{2\left(\frac{-1\pm\sqrt{5}}{2}\right)^3}{3\left(1+\left(-1\pm\sqrt{5}\right)\right)\left(\left(\frac{-1\pm\sqrt{5}}{2}\right)^2+7\right)^{2/3}}$

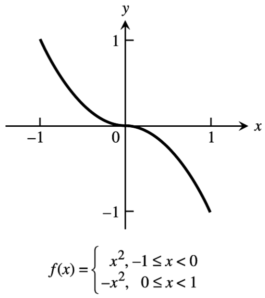
91. $y^3 + y = 2\cos x \Rightarrow 3y^2\frac{dy}{dx} + \frac{dy}{dx} = -2\sin x \Rightarrow \frac{dy}{dx}(3y^2 + 1) = -2\sin x \Rightarrow \frac{dy}{dx} = \frac{-2\sin x}{3y^2+1} \Rightarrow \left.\frac{dy}{dx}\right|_{(0,1)}$
$= \frac{-2\sin(0)}{3+1} = 0$; $\frac{d^2y}{dx^2} = \frac{(3y^2+1)(-2\cos x) - (-2\sin x)\left(6y\frac{dy}{dx}\right)}{(3y^2+1)^2} \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{(0,1)} = \frac{(3+1)(-2\cos 0) - (-2\sin 0)(6 \cdot 0)}{(3+1)^2} = -\frac{1}{2}$

92. $x^{1/3} + y^{1/3} = 4 \Rightarrow \frac{1}{3}x^{-2/3} + \frac{1}{3}y^{-2/3}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{y^{2/3}}{x^{2/3}} \Rightarrow \left.\frac{dy}{dx}\right|_{(8,8)} = -1$; $\frac{dy}{dx} = \frac{-y^{2/3}}{x^{2/3}}$
$\Rightarrow \frac{d^2y}{dx^2} = \frac{(x^{2/3})\left(-\frac{2}{3}y^{-1/3}\frac{dy}{dx}\right) - (-y^{2/3})\left(\frac{2}{3}x^{-1/3}\right)}{(x^{2/3})^2} \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{(8,8)} = \frac{(8^{2/3})\left[-\frac{2}{3}\cdot 8^{-1/3}\cdot(-1)\right] + (8^{2/3})\left(\frac{2}{3}\cdot 8^{-1/3}\right)}{8^{4/3}}$
$= \frac{\frac{1}{3}+\frac{1}{3}}{8^{2/3}} = \frac{\frac{2}{3}}{4} = \frac{1}{6}$

93. $f(t) = \frac{1}{2t+1}$ and $f(t+h) = \frac{1}{2(t+h)+1} \Rightarrow \frac{f(t+h) - f(t)}{h} = \frac{\frac{1}{2(t+h)+1} - \frac{1}{2t+1}}{h} = \frac{2t + 1 - (2t + 2h + 1)}{(2t + 2h + 1)(2t + 1)h}$
$= \frac{-2h}{(2t+2h+1)(2t+1)h} = \frac{-2}{(2t+2h+1)(2t+1)} \Rightarrow f'(t) = \lim_{h\to 0}\frac{f(t+h) - f(t)}{h} = \lim_{h\to 0}\frac{-2}{(2t+2h+1)(2t+1)} = \frac{-2}{(2t+1)^2}$

94. $g(x) = 2x^2 + 1$ and $g(x+h) = 2(x+h)^2 + 1 = 2x^2 + 4xh + 2h^2 + 1 \Rightarrow \frac{g(x+h) - g(x)}{h}$
$= \frac{(2x^2 + 4xh + 2h^2 + 1) - (2x^2 + 1)}{h} = \frac{4xh + 2h^2}{h} = 4x + 2h \Rightarrow g'(x) = \lim_{h\to 0}\frac{g(x+h) - g(x)}{h} = \lim_{h\to 0}(4x + 2h) = 4x$

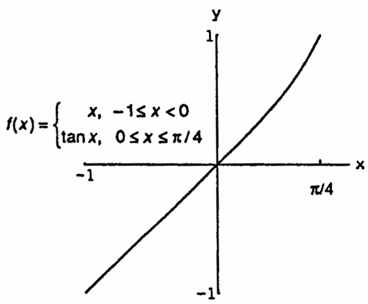
95. (a)



(b) $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} x^2 = 0$ and $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} -x^2 = 0 \Rightarrow \lim\limits_{x \to 0} f(x) = 0$. Since $\lim\limits_{x \to 0} f(x) = 0 = f(0)$ it follows that f is continuous at $x = 0$.

(c) $\lim\limits_{x \to 0^-} f'(x) = \lim\limits_{x \to 0^-} (2x) = 0$ and $\lim\limits_{x \to 0^+} f'(x) = \lim\limits_{x \to 0^+} (-2x) = 0 \Rightarrow \lim\limits_{x \to 0} f'(x) = 0$. Since this limit exists, it follows that f is differentiable at $x = 0$.

96. (a)



(b) $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} x = 0$ and $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} \tan x = 0 \Rightarrow \lim\limits_{x \to 0} f(x) = 0$. Since $\lim\limits_{x \to 0} f(x) = 0 = f(0)$, it follows that f is continuous at $x = 0$.

(c) $\lim\limits_{x \to 0^-} f'(x) = \lim\limits_{x \to 0^-} 1 = 1$ and $\lim\limits_{x \to 0^+} f'(x) = \lim\limits_{x \to 0^+} \sec^2 x = 1 \Rightarrow \lim\limits_{x \to 0} f'(x) = 1$. Since this limit exists it follows that f is differentiable at $x = 0$.

97. $y = \frac{x}{2} + \frac{1}{2x-4} = \frac{1}{2}x + (2x-4)^{-1} \Rightarrow \frac{dy}{dx} = \frac{1}{2} - 2(2x-4)^{-2}$; the slope of the tangent is $-\frac{3}{2} \Rightarrow -\frac{3}{2} = \frac{1}{2} - 2(2x-4)^{-2} \Rightarrow -2 = -2(2x-4)^{-2} \Rightarrow 1 = \frac{1}{(2x-4)^2} \Rightarrow (2x-4)^2 = 1 \Rightarrow 4x^2 - 16x + 16 = 1 \Rightarrow 4x^2 - 16x + 15 = 0 \Rightarrow (2x-5)(2x-3) = 0 \Rightarrow x = \frac{5}{2}$ or $x = \frac{3}{2} \Rightarrow \left(\frac{5}{2}, \frac{9}{4}\right)$ and $\left(\frac{3}{2}, -\frac{1}{4}\right)$ are points on the curve where the slope is $-\frac{3}{2}$.

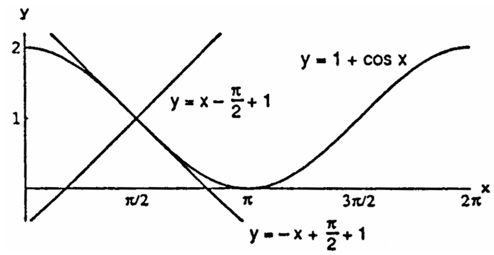
98. $y = x - e^{-x}$; $\frac{dy}{dx} = 1 + e^{-x} = 2 \Rightarrow e^{-x} = 1 \Rightarrow x = 0 \Rightarrow y = 0 - e^0 = -1$. Therefore, the curve has a tangent with a slope of 2 at the point $(0, -1)$.

99. $y = 2x^3 - 3x^2 - 12x + 20 \Rightarrow \frac{dy}{dx} = 6x^2 - 6x - 12$

(a) The tangent is perpendicular to the line $y = 1 - \frac{x}{24}$ when $\frac{dy}{dx} = -\left(\frac{1}{-\left(\frac{1}{24}\right)}\right) = 24$; $6x^2 - 6x - 12 = 24 \Rightarrow x^2 - x - 2 = 4 \Rightarrow x^2 - x - 6 = 0 \Rightarrow (x-3)(x+2) = 0 \Rightarrow x = -2$ or $x = 3 \Rightarrow (-2, 16)$ and $(3, 11)$ are points where the tangent is perpendicular to $y = 1 - \frac{x}{24}$.

(b) The tangent is parallel to the line $y = \sqrt{2} - 12x$ when $\frac{dy}{dx} = -12 \Rightarrow 6x^2 - 6x - 12 = -12 \Rightarrow x^2 - x = 0 \Rightarrow x(x-1) = 0 \Rightarrow x = 0$ or $x = 1 \Rightarrow (0, 20)$ and $(1, 7)$ are points where the tangent is parallel to $y = \sqrt{2} - 12x$.

100. $y = 1 + \cos x \Rightarrow \frac{dy}{dx} = -\sin x \Rightarrow \left.\frac{dy}{dx}\right|_{(\frac{\pi}{2},1)} = -1$
$\Rightarrow$ the tangent at $\left(\frac{\pi}{2}, 1\right)$ is the line $y - 1 = -\left(x - \frac{\pi}{2}\right)$
$\Rightarrow y = -x + \frac{\pi}{2} + 1$; the normal at $\left(\frac{\pi}{2}, 1\right)$ is
$y - 1 = (1)\left(x - \frac{\pi}{2}\right) \Rightarrow y = x - \frac{\pi}{2} + 1$



101. $y = x^2 + C \Rightarrow \frac{dy}{dx} = 2x$ and $y = x \Rightarrow \frac{dy}{dx} = 1$; the parabola is tangent to $y = x$ when $2x = 1 \Rightarrow x = \frac{1}{2} \Rightarrow y = \frac{1}{2}$; thus, $\frac{1}{2} = \left(\frac{1}{2}\right)^2 + C \Rightarrow C = \frac{1}{4}$

102. $y = x^3 \Rightarrow \frac{dy}{dx} = 3x^2 \Rightarrow \left.\frac{dy}{dx}\right|_{x=a} = 3a^2 \Rightarrow$ the tangent line at (a, a^3) is $y - a^3 = 3a^2(x - a)$. The tangent line intersects $y = x^3$ when $x^3 - a^3 = 3a^2(x - a) \Rightarrow (x - a)(x^2 + xa + a^2) = 3a^2(x - a) \Rightarrow (x - a)(x^2 + xa - 2a^2) = 0$
$\Rightarrow (x - a)^2(x + 2a) = 0 \Rightarrow x = a$ or $x = -2a$. Now $\left.\frac{dy}{dx}\right|_{x=-2a} = 3(-2a)^2 = 12a^2 = 4(3a^2)$, so the slope at $x = -2a$ is 4 times as large as the slope at (a, a^3) where $x = a$.

103. $x^2 + 2y^2 = 9 \Rightarrow 2x + 4y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{2y} \Rightarrow \left.\frac{dy}{dx}\right|_{(1,2)} = -\frac{1}{4} \Rightarrow$ the tangent line is $y = 2 - \frac{1}{4}(x - 1) = -\frac{1}{4}x + \frac{9}{4}$ and the normal line is $y = 2 + 4(x - 1) = 4x - 2$.

104. $e^x + y^2 = 2 \Rightarrow \frac{d}{dx}(e^x + y^2) = \frac{d}{dx}(2) \Rightarrow e^x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{e^x}{2y} \Rightarrow m_{tan} = \left.\frac{dy}{dx}\right|_{(0,\,1)} = -\frac{e^0}{2(1)} = -\frac{1}{2}$;
$m_\perp = -\frac{1}{m_{tan}} = 2$; tangent line: $y - 1 = -\frac{1}{2}(x - 0) \Rightarrow y = 1 - \frac{x}{2}$; normal line: $y - 1 = 2(x - 0) \Rightarrow y = 2x + 1$

105. $xy + 2x - 5y = 2 \Rightarrow \left(x\frac{dy}{dx} + y\right) + 2 - 5\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx}(x - 5) = -y - 2 \Rightarrow \frac{dy}{dx} = \frac{-y-2}{x-5} \Rightarrow \left.\frac{dy}{dx}\right|_{(3,2)} = 2$
$\Rightarrow$ the tangent line is $y = 2 + 2(x - 3) = 2x - 4$ and the normal line is $y = 2 + \frac{-1}{2}(x - 3) = -\frac{1}{2}x + \frac{7}{2}$.

106. $x + \sqrt{xy} = 6 \Rightarrow 1 + \frac{1}{2\sqrt{xy}}\left(x\frac{dy}{dx} + y\right) = 0 \Rightarrow x\frac{dy}{dx} + y = -2\sqrt{xy} \Rightarrow \frac{dy}{dx} = \frac{-2\sqrt{xy} - y}{x} \Rightarrow \left.\frac{dy}{dx}\right|_{(4,1)} = \frac{-5}{4}$
$\Rightarrow$ the tangent line is $y = 1 - \frac{5}{4}(x - 4) = -\frac{5}{4}x + 6$ and the normal line is $y = 1 + \frac{4}{5}(x - 4) = \frac{4}{5}x - \frac{11}{5}$.

107. $\lim\limits_{x \to 0} \frac{\sin x}{2x^2 - x} = \lim\limits_{x \to 0} \left[\left(\frac{\sin x}{x}\right) \cdot \frac{1}{(2x-1)}\right] = (1)\left(\frac{1}{-1}\right) = -1$

108. $\lim\limits_{x \to 0} \frac{3x - \tan 7x}{2x} = \lim\limits_{x \to 0} \left(\frac{3x}{2x} - \frac{\sin 7x}{2x \cos 7x}\right) = \frac{3}{2} - \lim\limits_{x \to 0} \left(\frac{1}{\cos 7x} \cdot \frac{\sin 7x}{7x} \cdot \frac{1}{\left(\frac{2}{7}\right)}\right) = \frac{3}{2} - \left(1 \cdot 1 \cdot \frac{7}{2}\right) = -2$

109. $\lim\limits_{r \to 0} \frac{\sin r}{\tan 2r} = \lim\limits_{r \to 0} \left(\frac{\sin r}{r} \cdot \frac{2r}{\tan 2r} \cdot \frac{1}{2}\right) = \left(\frac{1}{2}\right)(1) \lim\limits_{r \to 0} \frac{\cos 2r}{\left(\frac{\sin 2r}{2r}\right)} = \left(\frac{1}{2}\right)(1)\left(\frac{1}{1}\right) = \frac{1}{2}$

110. $\lim\limits_{\theta \to 0} \frac{\sin(\sin\theta)}{\theta} = \lim\limits_{\theta \to 0} \left(\frac{\sin(\sin\theta)}{\sin\theta}\right)\left(\frac{\sin\theta}{\theta}\right) = \lim\limits_{\theta \to 0} \frac{\sin(\sin\theta)}{\sin\theta}$. Let $x = \sin\theta$. Then $x \to 0$ as $\theta \to 0$
$\Rightarrow \lim\limits_{\theta \to 0} \frac{\sin(\sin\theta)}{\sin\theta} = \lim\limits_{x \to 0} \frac{\sin x}{x} = 1$

111. $\lim\limits_{\theta \to \left(\frac{\pi}{2}\right)^-} \frac{4\tan^2\theta + \tan\theta + 1}{\tan^2\theta + 5} = \lim\limits_{\theta \to \left(\frac{\pi}{2}\right)^-} \frac{\left(4 + \frac{1}{\tan\theta} + \frac{1}{\tan^2\theta}\right)}{\left(1 + \frac{5}{\tan^2\theta}\right)} = \frac{(4+0+0)}{(1+0)} = 4$

112. $\lim\limits_{x \to 0} \frac{x \sin x}{2 - 2\cos x} = \lim\limits_{x \to 0} \frac{x \sin x}{2(1-\cos x)} = \lim\limits_{x \to 0} \frac{x \sin x}{2\left(2\sin^2\left(\frac{x}{2}\right)\right)} = \lim\limits_{x \to 0} \left[\frac{\frac{x}{2} \cdot \frac{x}{2}}{\sin^2\left(\frac{x}{2}\right)} \cdot \frac{\sin x}{x}\right]$
$= \lim\limits_{x \to 0} \left[\frac{\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)} \cdot \frac{\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)} \cdot \frac{\sin x}{x}\right] = (1)(1)(1) = 1$

113. $y = \frac{2(x^2+1)}{\sqrt{\cos 2x}} \Rightarrow \ln y = \ln\left(\frac{2(x^2+1)}{\sqrt{\cos 2x}}\right) = \ln(2) + \ln(x^2+1) - \frac{1}{2}\ln(\cos 2x) \Rightarrow \frac{y'}{y} = 0 + \frac{2x}{x^2+1} - \left(\frac{1}{2}\right)\frac{(-2\sin 2x)}{\cos 2x}$
$\Rightarrow y' = \left(\frac{2x}{x^2+1} + \tan 2x\right) y = \frac{2(x^2+1)}{\sqrt{\cos 2x}}\left(\frac{2x}{x^2+1} + \tan 2x\right)$

114. $y = \sqrt[10]{\frac{3x+4}{2x-4}} \Rightarrow \ln y = \ln \sqrt[10]{\frac{3x+4}{2x-4}} = \frac{1}{10}\left[\ln(3x+4) - \ln(2x-4)\right] \Rightarrow \frac{y'}{y} = \frac{1}{10}\left(\frac{3}{3x+4} - \frac{2}{2x-4}\right)$
$\Rightarrow y' = \frac{1}{10}\left(\frac{3}{3x+4} - \frac{1}{x-2}\right) y = \sqrt[10]{\frac{3x+4}{2x-4}}\left(\frac{1}{10}\right)\left(\frac{3}{3x+4} - \frac{1}{x-2}\right)$

115. $y = \left[\frac{(t+1)(t-1)}{(t-2)(t+3)}\right]^5 \Rightarrow \ln y = 5\left[\ln(t+1) + \ln(t-1) - \ln(t-2) - \ln(t+3)\right] \Rightarrow \left(\frac{1}{y}\right)\left(\frac{dy}{dt}\right)$
$= 5\left(\frac{1}{t+1} + \frac{1}{t-1} - \frac{1}{t-2} - \frac{1}{t+3}\right) \Rightarrow \frac{dy}{dt} = 5\left[\frac{(t+1)(t-1)}{(t-2)(t+3)}\right]^5 \left(\frac{1}{t+1} + \frac{1}{t-1} - \frac{1}{t-2} - \frac{1}{t+3}\right)$

116. $y = \frac{2u2^u}{\sqrt{u^2+1}} \Rightarrow \ln y = \ln 2 + \ln u + u \ln 2 - \frac{1}{2}\ln(u^2+1) \Rightarrow \left(\frac{1}{y}\right)\left(\frac{dy}{du}\right) = \frac{1}{u} + \ln 2 - \frac{1}{2}\left(\frac{2u}{u^2+1}\right)$
$\Rightarrow \frac{dy}{du} = \frac{2u2^u}{\sqrt{u^2+1}}\left(\frac{1}{u} + \ln 2 - \frac{u}{u^2+1}\right)$

117. $y = (\sin\theta)^{\sqrt{\theta}} \Rightarrow \ln y = \sqrt{\theta}\,\ln(\sin\theta) \Rightarrow \left(\frac{1}{y}\right)\left(\frac{dy}{d\theta}\right) = \sqrt{\theta}\left(\frac{\cos\theta}{\sin\theta}\right) + \frac{1}{2}\theta^{-1/2}\ln(\sin\theta)$
$\Rightarrow \frac{dy}{d\theta} = (\sin\theta)^{\sqrt{\theta}}\left(\sqrt{\theta}\cot\theta + \frac{\ln(\sin\theta)}{2\sqrt{\theta}}\right)$

118. $y = (\ln x)^{1/\ln x} \Rightarrow \ln y = \left(\frac{1}{\ln x}\right)\ln(\ln x) \Rightarrow \frac{y'}{y} = \left(\frac{1}{\ln x}\right)\left(\frac{1}{\ln x}\right)\left(\frac{1}{x}\right) + \ln(\ln x)\left[\frac{-1}{(\ln x)^2}\right]\left(\frac{1}{x}\right)$
$\Rightarrow y' = (\ln x)^{1/\ln x}\left[\frac{1-\ln(\ln x)}{x(\ln x)^2}\right]$

119. (a) $S = 2\pi r^2 + 2\pi rh$ and h constant $\Rightarrow \frac{dS}{dt} = 4\pi r\frac{dr}{dt} + 2\pi h\frac{dr}{dt} = (4\pi r + 2\pi h)\frac{dr}{dt}$
(b) $S = 2\pi r^2 + 2\pi rh$ and r constant $\Rightarrow \frac{dS}{dt} = 2\pi r\frac{dh}{dt}$
(c) $S = 2\pi r^2 + 2\pi rh \Rightarrow \frac{dS}{dt} = 4\pi r\frac{dr}{dt} + 2\pi\left(r\frac{dh}{dt} + h\frac{dr}{dt}\right) = (4\pi r + 2\pi h)\frac{dr}{dt} + 2\pi r\frac{dh}{dt}$
(d) S constant $\Rightarrow \frac{dS}{dt} = 0 \Rightarrow 0 = (4\pi r + 2\pi h)\frac{dr}{dt} + 2\pi r\frac{dh}{dt} \Rightarrow (2r+h)\frac{dr}{dt} = -r\frac{dh}{dt} \Rightarrow \frac{dr}{dt} = \frac{-r}{2r+h}\frac{dh}{dt}$

120. $S = \pi r\sqrt{r^2+h^2} \Rightarrow \frac{dS}{dt} = \pi r \cdot \frac{\left(r\frac{dr}{dt} + h\frac{dh}{dt}\right)}{\sqrt{r^2+h^2}} + \pi\sqrt{r^2+h^2}\frac{dr}{dt}$;
(a) h constant $\Rightarrow \frac{dh}{dt} = 0 \Rightarrow \frac{dS}{dt} = \frac{\pi r^2\frac{dr}{dt}}{\sqrt{r^2+h^2}} + \pi\sqrt{r^2+h^2}\frac{dr}{dt} = \left[\pi\sqrt{r^2+h^2} + \frac{\pi r^2}{\sqrt{r^2+h^2}}\right]\frac{dr}{dt}$
(b) r constant $\Rightarrow \frac{dr}{dt} = 0 \Rightarrow \frac{dS}{dt} = \frac{\pi rh}{\sqrt{r^2+h^2}}\frac{dh}{dt}$
(c) In general, $\frac{dS}{dt} = \left[\pi\sqrt{r^2+h^2} + \frac{\pi r^2}{\sqrt{r^2+h^2}}\right]\frac{dr}{dt} + \frac{\pi rh}{\sqrt{r^2+h^2}}\frac{dh}{dt}$

121. $\frac{dR_1}{dt} = -1$ ohm/sec, $\frac{dR_2}{dt} = 0.5$ ohm/sec; and $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \Rightarrow \frac{-1}{R^2}\frac{dR}{dt} = \frac{-1}{R_1^2}\frac{dR_1}{dt} - \frac{1}{R_2^2}\frac{dR_2}{dt}$. Also,
$R_1 = 75$ ohms and $R_2 = 50$ ohms $\Rightarrow \frac{1}{R} = \frac{1}{75} + \frac{1}{50} \Rightarrow R = 30$ ohms. Therefore, from the derivative equation,
$\frac{-1}{(30)^2}\frac{dR}{dt} = \frac{-1}{(75)^2}(-1) - \frac{1}{(50)^2}(0.5) = \left(\frac{1}{5625} - \frac{1}{5000}\right) \Rightarrow \frac{dR}{dt} = (-900)\left(\frac{5000-5625}{5625\cdot 5000}\right) = \frac{9(625)}{50(5625)} = \frac{1}{50} = 0.02$ ohm/sec.

122. $\frac{dR}{dt} = 3$ ohms/sec and $\frac{dX}{dt} = -2$ ohms/sec; $Z = \sqrt{R^2+X^2} \Rightarrow \frac{dZ}{dt} = \frac{R\frac{dR}{dt} + X\frac{dX}{dt}}{\sqrt{R^2+X^2}}$ so that $R = 10$ ohms and
$X = 20$ ohms $\Rightarrow \frac{dZ}{dt} = \frac{(10)(3)+(20)(-2)}{\sqrt{10^2+20^2}} = \frac{-1}{\sqrt{5}} \approx -0.45$ ohm/sec.

123. Given $\frac{dx}{dt} = 10$ m/sec and $\frac{dy}{dt} = 5$ m/sec, let D be the distance from the origin $\Rightarrow D^2 = x^2 + y^2 \Rightarrow 2D\,\frac{dD}{dt}$ $= 2x\,\frac{dx}{dt} + 2y\,\frac{dy}{dt} \Rightarrow D\,\frac{dD}{dt} = x\,\frac{dx}{dt} + y\,\frac{dy}{dt}$. When $(x, y) = (3, -4)$, $D = \sqrt{3^2 + (-4)^2} = 5$ and $5\,\frac{dD}{dt} = (5)(10) + (12)(5) \Rightarrow \frac{dD}{dt} = \frac{110}{5} = 22$. Therefore, the particle is moving <u>away from</u> the origin at 22 m/sec (because the distance D is increasing).

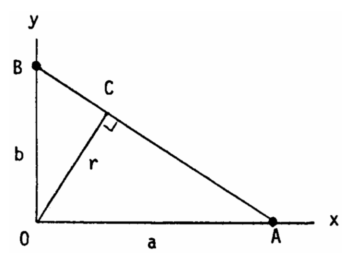
124. Let D be the distance from the origin. We are given that $\frac{dD}{dt} = 11$ units/sec. Then $D^2 = x^2 + y^2$ $= x^2 + \left(x^{3/2}\right)^2 = x^2 + x^3 \Rightarrow 2D\,\frac{dD}{dt} = 2x\,\frac{dx}{dt} + 3x^2\,\frac{dx}{dt} = x(2 + 3x)\,\frac{dx}{dt}$; $x = 3 \Rightarrow D = \sqrt{3^2 + 3^3} = 6$ and substitution in the derivative equation gives $(2)(6)(11) = (3)(2 + 9)\,\frac{dx}{dt} \Rightarrow \frac{dx}{dt} = 4$ units/sec.

125. (a) From the diagram we have $\frac{10}{h} = \frac{4}{r} \Rightarrow r = \frac{2}{5}\,h$.
(b) $V = \frac{1}{3}\,\pi r^2 h = \frac{1}{3}\,\pi\left(\frac{2}{5}\,h\right)^2 h = \frac{4\pi h^3}{75} \Rightarrow \frac{dV}{dt} = \frac{4\pi h^2}{25}\,\frac{dh}{dt}$, so $\frac{dV}{dt} = -5$ and $h = 6 \Rightarrow \frac{dh}{dt} = -\frac{125}{144\pi}$ ft/min.

126. From the sketch in the text, $s = r\theta \Rightarrow \frac{ds}{dt} = r\,\frac{d\theta}{dt} + \theta\,\frac{dr}{dt}$. Also $r = 1.2$ is constant $\Rightarrow \frac{dr}{dt} = 0$ $\Rightarrow \frac{ds}{dt} = r\,\frac{d\theta}{dt} = (1.2)\,\frac{d\theta}{dt}$. Therefore, $\frac{ds}{dt} = 6$ ft/sec and $r = 1.2$ ft $\Rightarrow \frac{d\theta}{dt} = 5$ rad/sec

127. (a) From the sketch in the text, $\frac{d\theta}{dt} = -0.6$ rad/sec and $x = \tan\theta$. Also $x = \tan\theta \Rightarrow \frac{dx}{dt} = \sec^2\theta\,\frac{d\theta}{dt}$; at point A, $x = 0 \Rightarrow \theta = 0 \Rightarrow \frac{dx}{dt} = \left(\sec^2 0\right)(-0.6) = -0.6$. Therefore the speed of the light is $0.6 = \frac{3}{5}$ km/sec when it reaches point A.
(b) $\frac{(3/5)\text{ rad}}{\text{sec}} \cdot \frac{1\text{ rev}}{2\pi\text{ rad}} \cdot \frac{60\text{ sec}}{\text{min}} = \frac{18}{\pi}$ revs/min

128. From the figure, $\frac{a}{r} = \frac{b}{BC} \Rightarrow \frac{a}{r} = \frac{b}{\sqrt{b^2 - r^2}}$. We are given that r is constant. Differentiation gives,

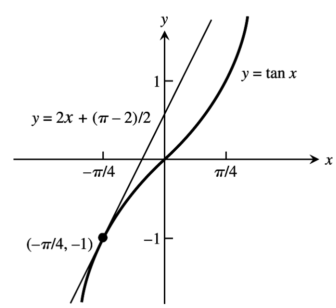


$\frac{1}{r} \cdot \frac{da}{dt} = \frac{\left(\sqrt{b^2 - r^2}\right)\left(\frac{db}{dt}\right) - (b)\left(\frac{b}{\sqrt{b^2 - r^2}}\right)\left(\frac{db}{dt}\right)}{b^2 - r^2}$. Then,
$b = 2r$ and $\frac{db}{dt} = -0.3r$

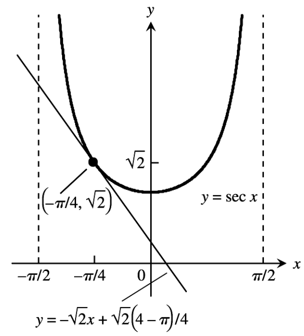
$$\Rightarrow \frac{da}{dt} = r\left[\frac{\sqrt{(2r)^2 - r^2}\,(-0.3r) - (2r)\left(\frac{2r(-0.3r)}{\sqrt{(2r)^2 - r^2}}\right)}{(2r)^2 - r^2}\right]$$

$$= \frac{\sqrt{3r^2}\,(-0.3r) + \frac{4r^2(0.3r)}{\sqrt{3r^2}}}{3r} = \frac{\left(3r^2\right)(-0.3r) + \left(4r^2\right)(0.3r)}{3\sqrt{3}\,r^2} = \frac{0.3r}{3\sqrt{3}} = \frac{r}{10\sqrt{3}}$$ m/sec. Since $\frac{da}{dt}$ is positive, the distance OA is increasing when $OB = 2r$, and B is moving toward O at the rate of $0.3r$ m/sec.

129. (a) If $f(x) = \tan x$ and $x = -\frac{\pi}{4}$, then $f'(x) = \sec^2 x$, $f\left(-\frac{\pi}{4}\right) = -1$ and $f'\left(-\frac{\pi}{4}\right) = 2$. The linearization of $f(x)$ is $L(x) = 2\left(x + \frac{\pi}{4}\right) + (-1) = 2x + \frac{\pi - 2}{2}$.



(b) If $f(x) = \sec x$ and $x = -\frac{\pi}{4}$, then $f'(x) = \sec x \tan x$, $f\left(-\frac{\pi}{4}\right) = \sqrt{2}$ and $f'\left(-\frac{\pi}{4}\right) = -\sqrt{2}$. The linearization of $f(x)$ is $L(x) = -\sqrt{2}\left(x + \frac{\pi}{4}\right) + \sqrt{2}$ $= -\sqrt{2}x + \frac{\sqrt{2}(4-\pi)}{4}$.



130. $f(x) = \sqrt{x+1} + \sin x - 0.5 = (x+1)^{1/2} + \sin x - 0.5 \Rightarrow f'(x) = \left(\frac{1}{2}\right)(x+1)^{-1/2} + \cos x$ $\Rightarrow L(x) = f'(0)(x-0) + f(0) = 1.5(x-0) + 0.5 \Rightarrow L(x) = 1.5x + 0.5$, the linearization of $f(x)$.

131. $C = 2\pi r \Rightarrow r = \frac{C}{2\pi}$, $S = 4\pi r^2 = \frac{C^2}{\pi}$, and $V = \frac{4}{3}\pi r^3 = \frac{C^3}{6\pi^2}$. It also follows that $dr = \frac{1}{2\pi}\,dC$, $dS = \frac{2C}{\pi}\,dC$ and $dV = \frac{C^2}{2\pi^2}\,dC$. Recall that $C = 10$ cm and $dC = 0.4$ cm.

(a) $dr = \frac{0.4}{2\pi} = \frac{0.2}{\pi}$ cm $\Rightarrow \left(\frac{dr}{r}\right)(100\%) = \left(\frac{0.2}{\pi}\right)\left(\frac{2\pi}{10}\right)(100\%) = (.04)(100\%) = 4\%$

(b) $dS = \frac{20}{\pi}(0.4) = \frac{8}{\pi}$ cm $\Rightarrow \left(\frac{dS}{S}\right)(100\%) = \left(\frac{8}{\pi}\right)\left(\frac{\pi}{100}\right)(100\%) = 8\%$

(c) $dV = \frac{10^2}{2\pi^2}(0.4) = \frac{20}{\pi^2}$ cm $\Rightarrow \left(\frac{dV}{V}\right)(100\%) = \left(\frac{20}{\pi^2}\right)\left(\frac{6\pi^2}{1000}\right)(100\%) = 12\%$

132. Similar triangles yield $\frac{35}{h} = \frac{15}{6} \Rightarrow h = 14$ ft. The same triangles imply that $\frac{20+a}{h} = \frac{a}{6} \Rightarrow h = 120a^{-1} + 6$ $\Rightarrow dh = -120a^{-2}\,da = -\frac{120}{a^2}\,da = \left(-\frac{120}{a^2}\right)\left(\pm\frac{1}{12}\right) = \left(-\frac{120}{15^2}\right)\left(\pm\frac{1}{12}\right) = \pm\frac{2}{45} \approx \pm .0444$ ft $= \pm 0.53$ inches.