Instructor's Solutions Manual

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Thomas' Calculus Early Transcendentals

Fourteenth Edition

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CHAPTER 1 FUNCTIONS

1.1 FUNCTIONS AND THEIR GRAPHS

1. domain $= (-\infty, \infty)$; range $= [1, \infty)$

2. domain $= [0, \infty)$; range $= (-\infty, 1]$

3. domain $= [-2, \infty)$; y in range and $y = \sqrt{5x+10} \geq 0 \Rightarrow y$ can be any positive real number $\Rightarrow$ range $= [0, \infty)$.

4. domain $= (-\infty, 0] \cup [3, \infty)$; y in range and $y = \sqrt{x^2 - 3x} \geq 0 \Rightarrow y$ can be any positive real number $\Rightarrow$ range $= [0, \infty)$.

5. domain $= (-\infty, 3) \cup (3, \infty)$; y in range and $y = \frac{4}{3-t}$, now if $t < 3 \Rightarrow 3 - t > 0 \Rightarrow \frac{4}{3-t} > 0$, or if $t > 3 \Rightarrow 3 - t < 0 \Rightarrow \frac{4}{3-t} < 0 \Rightarrow y$ can be any nonzero real number $\Rightarrow$ range $= (-\infty, 0) \cup (0, \infty)$.

6. domain $= (-\infty, -4) \cup (-4, 4) \cup (4, \infty)$; y in range and $y = \frac{2}{t^2-16}$, now if $t < -4 \Rightarrow t^2 - 16 > 0 \Rightarrow \frac{2}{t^2-16} > 0$, or if $-4 < t < 4 \Rightarrow -16 \leq t^2 - 16 < 0 \Rightarrow -\frac{2}{16} \geq \frac{2}{t^2-16}$, or if $t > 4 \Rightarrow t^2 - 16 > 0 \Rightarrow \frac{2}{t^2-16} > 0 \Rightarrow y$ can be any nonzero real number $\Rightarrow$ range $= (-\infty, -\frac{1}{8}] \cup (0, \infty)$.

7. (a) Not the graph of a function of x since it fails the vertical line test.
   (b) Is the graph of a function of x since any vertical line intersects the graph at most once.

8. (a) Not the graph of a function of x since it fails the vertical line test.
   (b) Not the graph of a function of x since it fails the vertical line test.

9. base $= x$; $(\text{height})^2 + \left(\frac{x}{2}\right)^2 = x^2 \Rightarrow$ height $= \frac{\sqrt{3}}{2}x$; area is $a(x) = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(x)\left(\frac{\sqrt{3}}{2}x\right) = \frac{\sqrt{3}}{4}x^2$; perimeter is $p(x) = x + x + x = 3x$.

10. $s =$ side length $\Rightarrow s^2 + s^2 = d^2 \Rightarrow s = \frac{d}{\sqrt{2}}$; and area is $a = s^2 \Rightarrow a = \frac{1}{2}d^2$

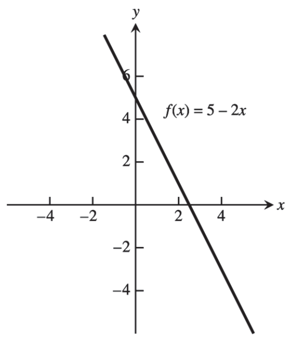
11. Let $D =$ diagonal length of a face of the cube and $\ell =$ the length of an edge. Then $\ell^2 + D^2 = d^2$ and $D^2 = 2\ell^2 \Rightarrow 3\ell^2 = d^2 \Rightarrow \ell = \frac{d}{\sqrt{3}}$. The surface area is $6\ell^2 = \frac{6d^2}{3} = 2d^2$ and the volume is $\ell^3 = \left(\frac{d^2}{3}\right)^{3/2} = \frac{d^3}{3\sqrt{3}}$.

12. The coordinates of P are $\left(x, \sqrt{x}\right)$ so the slope of the line joining P to the origin is $m = \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}}\ (x > 0)$. Thus, $\left(x, \sqrt{x}\right) = \left(\frac{1}{m^2}, \frac{1}{m}\right)$.

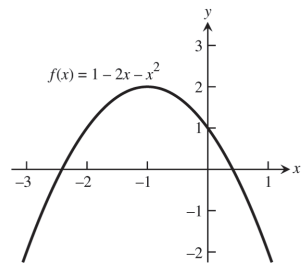
13. $2x + 4y = 5 \Rightarrow y = -\frac{1}{2}x + \frac{5}{4}$; $L = \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + (-\frac{1}{2}x + \frac{5}{4})^2} = \sqrt{x^2 + \frac{1}{4}x^2 - \frac{5}{4}x + \frac{25}{16}}$
$= \sqrt{\frac{5}{4}x^2 - \frac{5}{4}x + \frac{25}{16}} = \sqrt{\frac{20x^2 - 20x + 25}{16}} = \frac{\sqrt{20x^2 - 20x + 25}}{4}$

14. $y = \sqrt{x-3} \Rightarrow y^2 + 3 = x$; $L = \sqrt{(x-4)^2 + (y-0)^2} = \sqrt{(y^2 + 3 - 4)^2 + y^2} = \sqrt{(y^2 - 1)^2 + y^2}$
$= \sqrt{y^4 - 2y^2 + 1 + y^2} = \sqrt{y^4 - y^2 + 1}$

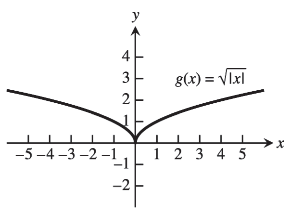
15. The domain is $(-\infty, \infty)$.



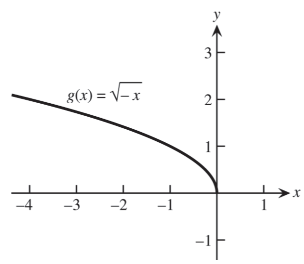
16. The domain is $(-\infty, \infty)$.



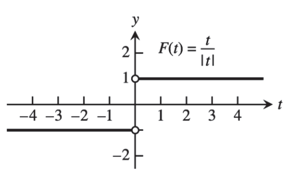
17. The domain is $(-\infty, \infty)$.



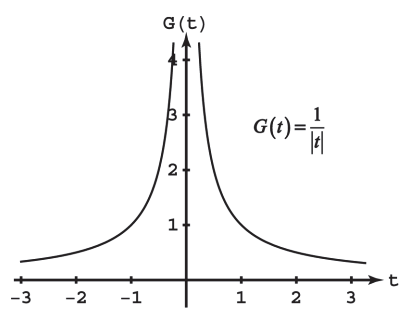
18. The domain is $(-\infty, 0]$.



19. The domain is $(-\infty, 0) \cup (0, \infty)$.



20. The domain is $(-\infty, 0) \cup (0, \infty)$.

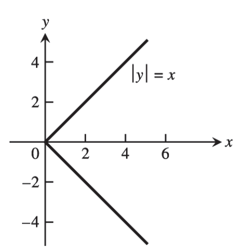


21. The domain is $(-\infty, -5) \cup (-5, -3] \cup [3, 5) \cup (5, \infty)$

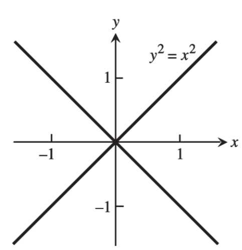
22. The range is $[5, \infty)$.

23. Neither graph passes the vertical line test

(a)

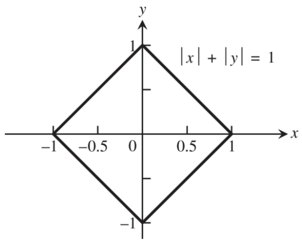


(b)

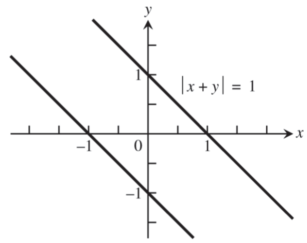


24. Neither graph passes the vertical line test

(a)



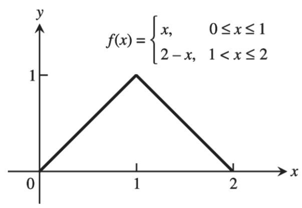
(b)



$$|x+y|=1 \Leftrightarrow \left\{\begin{matrix} x+y=1 \\ \text{or} \\ x+y=-1 \end{matrix}\right\} \Leftrightarrow \left\{\begin{matrix} y=1-x \\ \text{or} \\ y=-1-x \end{matrix}\right\}$$

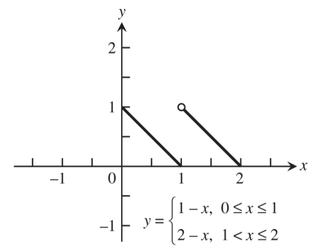
25.

x	0	1	2
y	0	1	0

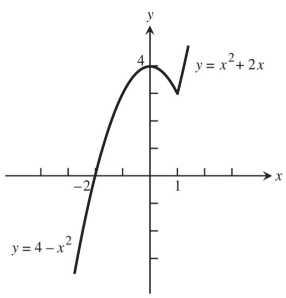


26.

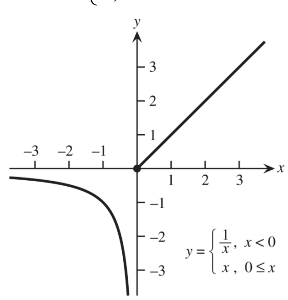
x	0	1	2
y	1	0	0



27. $F(x)=\begin{cases} 4-x^2, & x\le 1 \\ x^2+2x, & x>1 \end{cases}$



28. $G(x)=\begin{cases} \frac{1}{x}, & x<0 \\ x, & 0\le x \end{cases}$



29. (a) Line through (0, 0) and (1, 1): $y=x$; Line through (1, 1) and (2, 0): $y=-x+2$

$$f(x)=\begin{cases} x, & 0\le x\le 1 \\ -x+2, & 1<x\le 2 \end{cases}$$

(b) $f(x)=\begin{cases} 2, & 0\le x<1 \\ 0, & 1\le x<2 \\ 2, & 2\le x<3 \\ 0, & 3\le x\le 4 \end{cases}$

30. (a) Line through (0, 2) and (2, 0): $y=-x+2$

Line through (2, 1) and (5, 0): $m=\frac{0-1}{5-2}=\frac{-1}{3}=-\frac{1}{3}$, so $y=-\frac{1}{3}(x-2)+1=-\frac{1}{3}x+\frac{5}{3}$

$$f(x)=\begin{cases} -x+2, & 0<x\le 2 \\ -\frac{1}{3}x+\frac{5}{3}, & 2<x\le 5 \end{cases}$$

(b) Line through $(-1, 0)$ and $(0, -3)$: $m = \frac{-3-0}{0-(-1)} = -3$, so $y = -3x - 3$

Line through $(0, 3)$ and $(2, -1)$: $m = \frac{-1-3}{2-0} = \frac{-4}{2} = -2$, so $y = -2x + 3$

$$f(x) = \begin{cases} -3x - 3, & -1 < x \le 0 \\ -2x + 3, & 0 < x \le 2 \end{cases}$$

31. (a) Line through $(-1, 1)$ and $(0, 0)$: $y = -x$
Line through $(0, 1)$ and $(1, 1)$: $y = 1$
Line through $(1, 1)$ and $(3, 0)$: $m = \frac{0-1}{3-1} = \frac{-1}{2} = -\frac{1}{2}$, so $y = -\frac{1}{2}(x-1) + 1 = -\frac{1}{2}x + \frac{3}{2}$

$$f(x) = \begin{cases} -x & -1 \le x < 0 \\ 1 & 0 < x \le 1 \\ -\frac{1}{2}x + \frac{3}{2} & 1 < x < 3 \end{cases}$$

(b) Line through $(-2, -1)$ and $(0, 0)$: $y = \frac{1}{2}x$
Line through $(0, 2)$ and $(1, 0)$: $y = -2x + 2$
Line through $(1, -1)$ and $(3, -1)$: $y = -1$

$$f(x) = \begin{cases} \frac{1}{2}x & -2 \le x \le 0 \\ -2x + 2 & 0 < x \le 1 \\ -1 & 1 < x \le 3 \end{cases}$$

32. (a) Line through $\left(\frac{T}{2}, 0\right)$ and $(T, 1)$: $m = \frac{1-0}{T-(T/2)} = \frac{2}{T}$, so $y = \frac{2}{T}\left(x - \frac{T}{2}\right) + 0 = \frac{2}{T}x - 1$

$$f(x) = \begin{cases} 0, & 0 \le x \le \frac{T}{2} \\ \frac{2}{T}x - 1, & \frac{T}{2} < x \le T \end{cases}$$

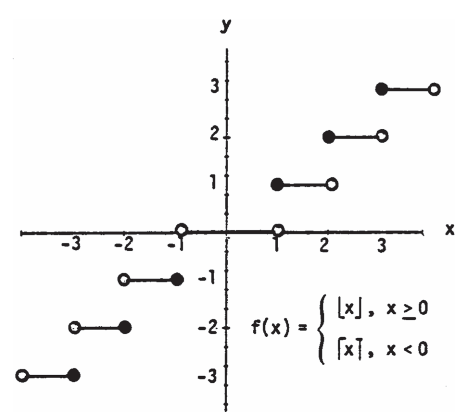
(b) $$f(x) = \begin{cases} A, & 0 \le x < \frac{T}{2} \\ -A, & \frac{T}{2} \le x < T \\ A, & T \le x < \frac{3T}{2} \\ -A, & \frac{3T}{2} \le x \le 2T \end{cases}$$

33. (a) $\lfloor x \rfloor = 0$ for $x \in [0, 1)$ (b) $\lceil x \rceil = 0$ for $x \in (-1, 0]$

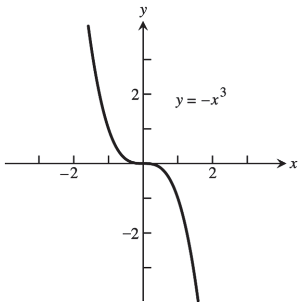
34. $\lfloor x \rfloor = \lceil x \rceil$ only when x is an integer.

35. For any real number x, $n \le x \le n+1$, where n is an integer. Now: $n \le x \le n+1 \Rightarrow -(n+1) \le -x \le -n$. By definition: $\lceil -x \rceil = -n$ and $\lfloor x \rfloor = n \Rightarrow -\lfloor x \rfloor = -n$. So $\lceil -x \rceil = -\lfloor x \rfloor$ for all real x.

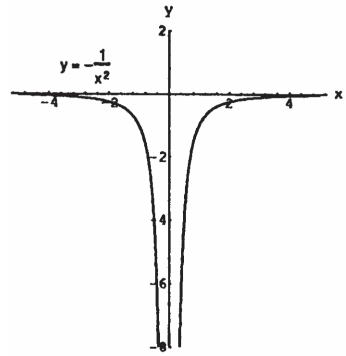
36. To find $f(x)$ you delete the decimal or fractional portion of x, leaving only the integer part.



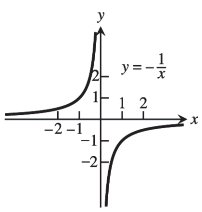
37. Symmetric about the origin
Dec: $-\infty < x < \infty$
Inc: nowhere



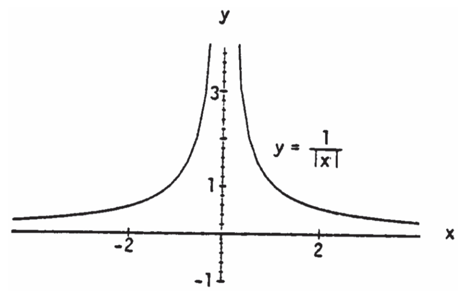
38. Symmetric about the y-axis
Dec: $-\infty < x < 0$
Inc: $0 < x < \infty$



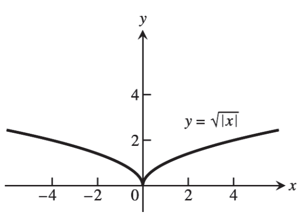
39. Symmetric about the origin
Dec: nowhere
Inc: $-\infty < x < 0$
$0 < x < \infty$



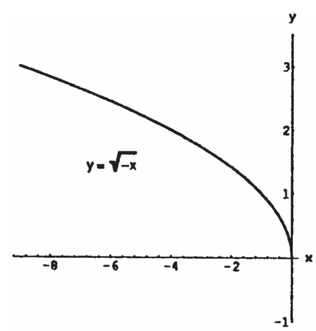
40. Symmetric about the y-axis
Dec: $0 < x < \infty$
Inc: $-\infty < x < 0$



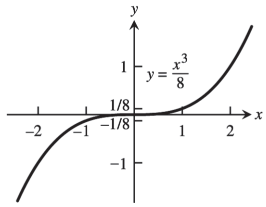
41. Symmetric about the y-axis
Dec: $-\infty < x \le 0$
Inc: $0 \le x < \infty$



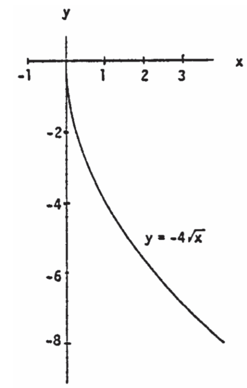
42. No symmetry
Dec: $-\infty < x \le 0$
Inc: nowhere



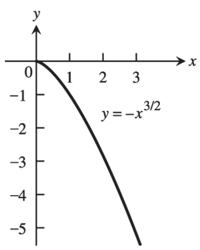
43. Symmetric about the origin
Dec: nowhere
Inc: $-\infty < x < \infty$



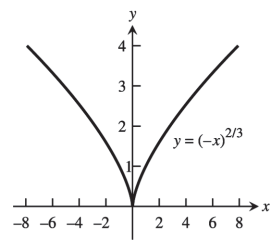
44. No symmetry
Dec: $0 \le x < \infty$
Inc: nowhere



45. No symmetry
Dec: $0 \le x < \infty$
Inc: nowhere



46. Symmetric about the y-axis
Dec: $-\infty < x \le 0$
Inc: $0 \le x < \infty$



47. Since a horizontal line not through the origin is symmetric with respect to the y-axis, but not with respect to the origin, the function is even.

48. $f(x) = x^{-5} = \frac{1}{x^5}$ and $f(-x) = (-x)^{-5} = \frac{1}{(-x)^5} = -\left(\frac{1}{x^5}\right) = -f(x)$. Thus the function is odd.

49. Since $f(x) = x^2 + 1 = (-x)^2 + 1 = f(-x)$. The function is even.

50. Since $[f(x) = x^2 + x] \ne [f(-x) = (-x)^2 - x]$ and $[f(x) = x^2 + x] \ne [-f(x) = -(x)^2 - x]$ the function is neither even nor odd.

51. Since $g(x) = x^3 + x$, $g(-x) = -x^3 - x = -(x^3 + x) = -g(x)$. So the function is odd.

52. $g(x) = x^4 + 3x^2 - 1 = (-x)^4 + 3(-x)^2 - 1 = g(-x)$, thus the function is even.

53. $g(x) = \frac{1}{x^2 - 1} = \frac{1}{(-x)^2 - 1} = g(-x)$. Thus the function is even.

54. $g(x) = \frac{x}{x^2 - 1}$; $g(-x) = -\frac{x}{x^2 - 1} = -g(x)$. So the function is odd.

55. $h(t) = \frac{1}{t-1}$; $h(-t) = \frac{1}{-t-1}$; $-h(t) = \frac{1}{1-t}$. Since $h(t) \ne -h(t)$ and $h(t) \ne h(-t)$, the function is neither even nor odd.

56. Since $|t^3| = |(-t)^3|, h(t) = h(-t)$ and the function is even.

57. $h(t) = 2t + 1$, $h(-t) = -2t + 1$. So $h(t) \neq h(-t)$. $-h(t) = -2t - 1$, so $h(t) \neq -h(t)$. The function is neither even nor odd.

58. $h(t) = 2|t| + 1$ and $h(-t) = 2|-t| + 1 = 2|t| + 1$. So $h(t) = h(-t)$ and the function is even.

59. $g(x) = \sin 2x$; $g(-x) = -\sin 2x = -g(x)$. So the function is odd.

60. $g(x) = \sin x^2$; $g(-x) = \sin x^2 = g(x)$. So the function is even.

61. $g(x) = \cos 3x$; $g(-x) = \cos 3x = g(x)$. So the function is even.

62. $g(x) = 1 + \cos x$; $g(-x) = 1 + \cos x = g(x)$. So the function is even.

63. $s = kt \Rightarrow 25 = k(75) \Rightarrow k = \frac{1}{3} \Rightarrow s = \frac{1}{3}t$; $60 = \frac{1}{3}t \Rightarrow t = 180$

64. $K = c\,v^2 \Rightarrow 12960 = c(18)^2 \Rightarrow c = 40 \Rightarrow K = 40v^2$; $K = 40(10)^2 = 4000$ joules

65. $r = \frac{k}{s} \Rightarrow 6 = \frac{k}{4} \Rightarrow k = 24 \Rightarrow r = \frac{24}{s}$; $10 = \frac{24}{s} \Rightarrow s = \frac{12}{5}$

66. $P = \frac{k}{V} \Rightarrow 14.7 = \frac{k}{1000} \Rightarrow k = 14700 \Rightarrow P = \frac{14700}{V}$; $23.4 = \frac{14700}{V} \Rightarrow V = \frac{24500}{39} \approx 628.2 \text{ in}^3$

67. $V = f(x) = x(14 - 2x)(22 - 2x) = 4x^3 - 72x^2 + 308x$; $0 < x < 7$.

68. (a) Let h = height of the triangle. Since the triangle is isosceles, $\left(\overline{AB}\right)^2 + \left(\overline{AB}\right)^2 = 2^2 \Rightarrow \overline{AB} = \sqrt{2}$. So, $h^2 + 1^2 = \left(\sqrt{2}\right)^2 \Rightarrow h = 1 \Rightarrow B$ is at $(0, 1) \Rightarrow$ slope of $AB = -1 \Rightarrow$ The equation of AB is $y = f(x) = -x + 1$; $x \in [0, 1]$.
(b) $A(x) = 2xy = 2x(-x + 1) = -2x^2 + 2x$; $x \in [0, 1]$.

69. (a) Graph h because it is an even function and rises less rapidly than does Graph g.
(b) Graph f because it is an odd function.
(c) Graph g because it is an even function and rises more rapidly than does Graph h.

70. (a) Graph f because it is linear.
(b) Graph g because it contains (0, 1).
(c) Graph h because it is a nonlinear odd function.

71. (a) From the graph, $\frac{x}{2} > 1 + \frac{4}{x} \Rightarrow x \in (-2, 0) \cup (4, \infty)$

(b) $\frac{x}{2} > 1 + \frac{4}{x} \Rightarrow \frac{x}{2} - 1 - \frac{4}{x} > 0$

$x > 0$: $\frac{x}{2} - 1 - \frac{4}{x} > 0 \Rightarrow \frac{x^2 - 2x - 8}{2x} > 0 \Rightarrow \frac{(x-4)(x+2)}{2x} > 0$

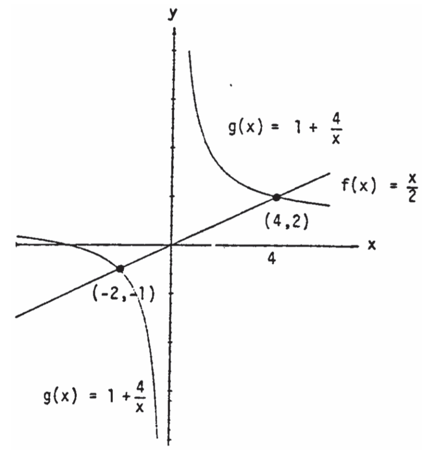
$\Rightarrow x > 4$ since x is positive;

$x < 0$: $\frac{x}{2} - 1 - \frac{4}{x} > 0 \Rightarrow \frac{x^2 - 2x - 8}{2x} < 0 \Rightarrow \frac{(x-4)(x+2)}{2x} < 0$

$\Rightarrow x < -2$ since x is negative;

sign of $(x-4)(x+2)$

Solution interval: $(-2, 0) \cup (4, \infty)$



72. (a) From the graph, $\frac{3}{x-1} < \frac{2}{x+1} \Rightarrow x \in (-\infty, -5) \cup (-1, 1)$

(b) Case $x < -1$: $\frac{3}{x-1} < \frac{2}{x+1} \Rightarrow \frac{3(x+1)}{x-1} > 2$

$\Rightarrow 3x + 3 < 2x - 2 \Rightarrow x < -5$.

Thus, $x \in (-\infty, -5)$ solves the inequality.

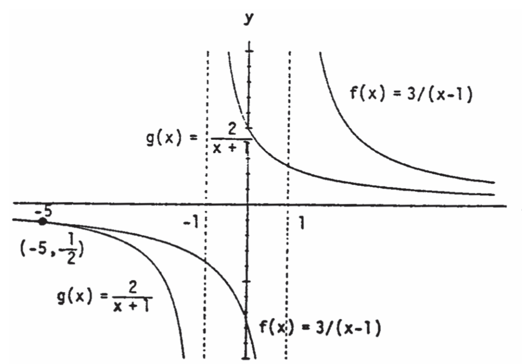
Case $-1 < x < 1$: $\frac{3}{x-1} < \frac{2}{x+1} \Rightarrow \frac{3(x+1)}{x-1} < 2$

$\Rightarrow 3x + 3 > 2x - 2 \Rightarrow x > -5$ which is true if $x > -1$. Thus, $x \in (-1, 1)$ solves the inequality.

Case $1 < x$: $\frac{3}{x-1} < \frac{2}{x+1} \Rightarrow 3x + 3 < 2x - 2 \Rightarrow x < -5$

which is never true if $1 < x$, so no solution here.

In conclusion, $x \in (-\infty, -5) \cup (-1, 1)$.



73. A curve symmetric about the x-axis will not pass the vertical line test because the points (x, y) and $(x, -y)$ lie on the same vertical line. The graph of the function $y = f(x) = 0$ is the x-axis, a horizontal line for which there is a single y-value, 0, for any x.

74. price $= 40 + 5x$, quantity $= 300 - 25x \Rightarrow R(x) = (40 + 5x)(300 - 25x)$

75. $x^2 + x^2 = h^2 \Rightarrow x = \frac{h}{\sqrt{2}} = \frac{\sqrt{2}h}{2}$; cost $= 5(2x) + 10h \Rightarrow C(h) = 10\left(\frac{\sqrt{2}h}{2}\right) + 10h = 5h\left(\sqrt{2} + 2\right)$

76. (a) Note that 2 mi = 10,560 ft, so there are $\sqrt{800^2 + x^2}$ feet of river cable at \$180 per foot and $(10{,}560 - x)$ feet of land cable at \$100 per foot. The cost is $C(x) = 180\sqrt{800^2 + x^2} + 100(10{,}560 - x)$.

(b) $C(0) = \$1{,}200{,}000$

$C(500) \approx \$1{,}175{,}812$

$C(1000) \approx \$1{,}186{,}512$

$C(1500) \approx \$1{,}212{,}000$

$C(2000) \approx \$1{,}243{,}732$

$C(2500) \approx \$1{,}278{,}479$

$C(3000) \approx \$1{,}314{,}870$

Values beyond this are all larger. It would appear that the least expensive location is less than 2000 feet from the point P.

1.2 COMBINING FUNCTIONS; SHIFTING AND SCALING GRAPHS

1. $D_f: -\infty < x < \infty, D_g: x \geq 1 \Rightarrow D_{f+g} = D_{fg}: x \geq 1$. $R_f: -\infty < y < \infty$, $R_g: y \geq 0$, $R_{f+g}: y \geq 1$, $R_{fg}: y \geq 0$

2. $D_f: x+1 \geq 0 \Rightarrow x \geq -1$, $D_g: x-1 \geq 0 \Rightarrow x \geq 1$. Therefore $D_{f+g} = D_{fg}: x \geq 1$.
$R_f = R_g: y \geq 0$, $R_{f+g}: y \geq \sqrt{2}$, $R_{fg}: y \geq 0$

3. $D_f: -\infty < x < \infty$, $D_g: -\infty < x < \infty$, $D_{f/g}: -\infty < x < \infty$, $D_{g/f}: -\infty < x < \infty$, $R_f: y = 2$, $R_g: y \geq 1$, $R_{f/g}: 0 < y \leq 2$, $R_{g/f}: \frac{1}{2} \leq y < \infty$

4. $D_f: -\infty < x < \infty$, $D_g: x \geq 0$, $D_{f/g}: x \geq 0$, $D_{g/f}: x \geq 0$; $R_f: y = 1$, $R_g: y \geq 1$, $R_{f/g}: 0 < y \leq 1$, $R_{g/f}: 1 \leq y < \infty$

5. (a) 2 (b) 22 (c) $x^2 + 2$
(d) $(x+5)^2 - 3 = x^2 + 10x + 22$ (e) 5 (f) -2
(g) $x + 10$ (h) $(x^2 - 3)^2 - 3 = x^4 - 6x^2 + 6$

6. (a) $-\frac{1}{3}$ (b) 2 (c) $\frac{1}{x+1} - 1 = \frac{-x}{x+1}$
(d) $\frac{1}{x}$ (e) 0 (f) $\frac{3}{4}$
(g) $x - 2$ (h) $\frac{1}{\frac{1}{x+1} + 1} = \frac{1}{\frac{x+2}{x+1}} = \frac{x+1}{x+2}$

7. $(f \circ g \circ h)(x) = f(g(h(x))) = f(g(4-x)) = f(3(4-x)) = f(12-3x) = (12-3x) + 1 = 13 - 3x$

8. $(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x^2)) = f(2(x^2) - 1) = f(2x^2 - 1) = 3(2x^2 - 1) + 4 = 6x^2 + 1$

9. $(f \circ g \circ h)(x) = f(g(h(x))) = f\left(g\left(\frac{1}{x}\right)\right) = f\left(\frac{1}{\frac{1}{x} + 4}\right) = f\left(\frac{x}{1+4x}\right) = \sqrt{\frac{x}{1+4x} + 1} = \sqrt{\frac{5x+1}{1+4x}}$

10. $(f \circ g \circ h)(x) = f(g(h(x))) = f\left(g\left(\sqrt{2-x}\right)\right) = f\left(\frac{\left(\sqrt{2-x}\right)^2}{\left(\sqrt{2-x}\right)^2 + 1}\right) = f\left(\frac{2-x}{3-x}\right) = \frac{\frac{2-x}{3-x} + 2}{3 - \frac{2-x}{3-x}} = \frac{8-3x}{7-2x}$

11. (a) $(f \circ g)(x)$ (b) $(j \circ g)(x)$ (c) $(g \circ g)(x)$
(d) $(j \circ j)(x)$ (e) $(g \circ h \circ f)(x)$ (f) $(h \circ j \circ f)(x)$

12. (a) $(f \circ j)(x)$ (b) $(g \circ h)(x)$ (c) $(h \circ h)(x)$
(d) $(f \circ f)(x)$ (e) $(j \circ g \circ f)(x)$ (f) $(g \circ f \circ h)(x)$

13.

	$g(x)$	$f(x)$	$(f \circ g)(x)$
(a)	$x - 7$	$\sqrt{x}$	$\sqrt{x-7}$
(b)	$x + 2$	$3x$	$3(x+2) = 3x + 6$
(c)	x^2	$\sqrt{x-5}$	$\sqrt{x^2 - 5}$
(d)	$\frac{x}{x-1}$	$\frac{x}{x-1}$	$\frac{\frac{x}{x-1}}{\frac{x}{x-1} - 1} = \frac{x}{x - (x-1)} = x$
(e)	$\frac{1}{x-1}$	$1 + \frac{1}{x}$	x

(f) $\frac{1}{x}$ $\frac{1}{x}$ x

14. (a) $(f\circ g)(x) = |g(x)| = \frac{1}{|x-1|}$.
(b) $(f\circ g)(x) = \frac{g(x)-1}{g(x)} = \frac{x}{x+1} \Rightarrow 1 - \frac{1}{g(x)} = \frac{x}{x+1} \Rightarrow 1 - \frac{x}{x+1} = \frac{1}{g(x)} \Rightarrow \frac{1}{x+1} = \frac{1}{g(x)}$, so $g(x) = x+1$.
(c) Since $(f\circ g)(x) = \sqrt{g(x)} = |x|$, $g(x) = x^2$.
(d) Since $(f\circ g)(x) = f\left(\sqrt{x}\right) = |x|$, $f(x) = x^2$. (Note that the domain of the composition is $[0, \infty)$.)

The completed table is shown. Note that the absolute value sign in part (d) is optional.

$g(x)$	$f(x)$	$(f\circ g)(x)$
$\frac{1}{x-1}$	$\|x\|$	$\frac{1}{\|x-1\|}$
$x+1$	$\frac{x-1}{x}$	$\frac{x}{x+1}$
x^2	$\sqrt{x}$	$\|x\|$
$\sqrt{x}$	x^2	$\|x\|$

15. (a) $f(g(-1)) = f(1) = 1$ (b) $g(f(0)) = g(-2) = 2$ (c) $f(f(-1)) = f(0) = -2$
(d) $g(g(2)) = g(0) = 0$ (e) $g(f(-2)) = g(1) = -1$ (f) $f(g(1)) = f(-1) = 0$

16. (a) $f(g(0)) = f(-1) = 2-(-1) = 3$, where $g(0) = 0-1 = -1$
(b) $g(f(3)) = g(-1) = -(-1) = 1$, where $f(3) = 2-3 = -1$
(c) $g(g(-1)) = g(1) = 1-1 = 0$, where $g(-1) = -(-1) = 1$
(d) $f(f(2)) = f(0) = 2-0 = 2$, where $f(2) = 2-2 = 0$
(e) $g(f(0)) = g(2) = 2-1 = 1$, where $f(0) = 2-0 = 2$
(f) $f\left(g\left(\frac{1}{2}\right)\right) = f\left(-\frac{1}{2}\right) = 2-\left(-\frac{1}{2}\right) = \frac{5}{2}$, where $g\left(\frac{1}{2}\right) = \frac{1}{2} - 1 = -\frac{1}{2}$

17. (a) $(f\circ g)(x) = f(g(x)) = \sqrt{\frac{1}{x}+1} = \sqrt{\frac{1+x}{x}}$
$(g\circ f)(x) = g(f(x)) = \frac{1}{\sqrt{x+1}}$
(b) Domain $(f\circ g)$: $(-\infty, -1] \cup (0, \infty)$, domain $(g\circ f)$: $(-1, \infty)$
(c) Range $(f\circ g)$: $(1, \infty)$, range $(g\circ f)$: $(0, \infty)$

18. (a) $(f\circ g)(x) = f(g(x)) = 1 - 2\sqrt{x} + x$
$(g\circ f)(x) = g(f(x)) = 1 - |x|$
(b) Domain $(f\circ g)$: $[0, \infty)$, domain $(g\circ f)$: $(-\infty, \infty)$
(c) Range $(f\circ g)$: $(0, \infty)$, range $(g\circ f)$: $(-\infty, 1]$

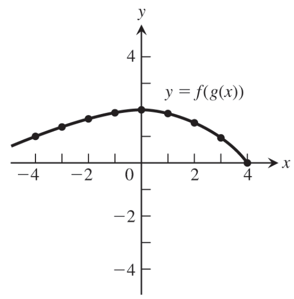
19. $(f\circ g)(x) = x \Rightarrow f(g(x)) = x \Rightarrow \frac{g(x)}{g(x)-2} = x \Rightarrow g(x) = (g(x)-2)x = x\cdot g(x) - 2x$
$\Rightarrow g(x) - x\cdot g(x) = -2x \Rightarrow g(x) = -\frac{2x}{1-x} = \frac{2x}{x-1}$

20. $(f\circ g)(x) = x+2 \Rightarrow f(g(x)) = x+2 \Rightarrow 2(g(x))^3 - 4 = x+2 \Rightarrow (g(x))^3 = \frac{x+6}{2} \Rightarrow g(x) = \sqrt[3]{\frac{x+6}{2}}$

21. $V = V(s) = V(s(t)) = V(2t-3)$
$= (2t-3)^2 + 2(2t-3) + 3$
$= 4t^2 - 8t + 6$

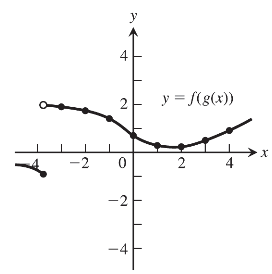
22. (a)

x	–4	–3	–2	–1	0	1	2	3	4
$g(x)$	–2	–1	–0.5	–0.2	0	0.2	0.5	1	2
$f(g(x))$	1	1.3	1.6	1.8	2	1.8	1.5	1	0



(b)

x	–4	–3	–2	–1	0	1	2	3	4
$g(x)$	1.5	0.3	–0.7	–1.5	–2.4	–2.8	–3	–2.7	–2
$f(g(x))$	–0.8	1.9	1.7	1.5	0.7	0.3	0.2	0.5	0.9



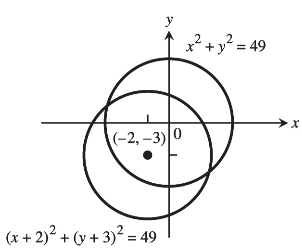
23. (a) $y = -(x+7)^2$ (b) $y = -(x-4)^2$

24. (a) $y = x^2 + 3$ (b) $y = x^2 - 5$

25. (a) Position 4 (b) Position 1 (c) Position 2 (d) Position 3

26. (a) $y = -(x-1)^2 + 4$ (b) $y = -(x+2)^2 + 3$ (c) $y = -(x+4)^2 - 1$ (d) $y = -(x-2)^2$

27.



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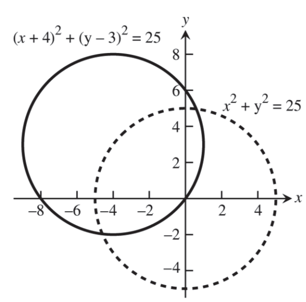

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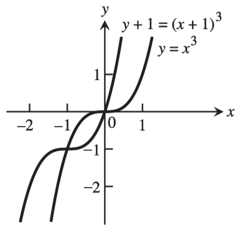

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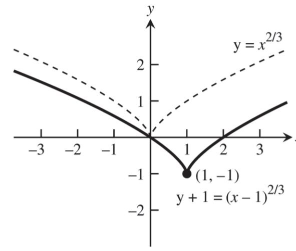

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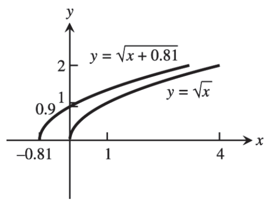

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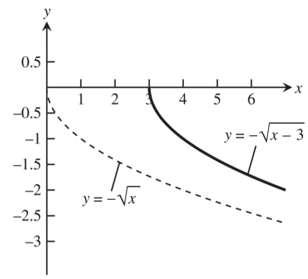

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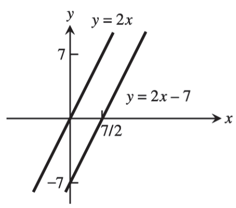

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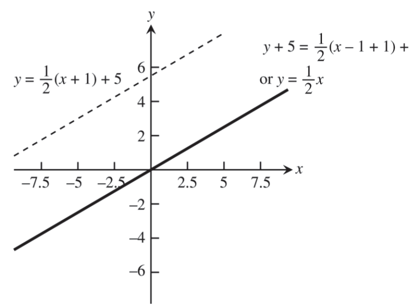

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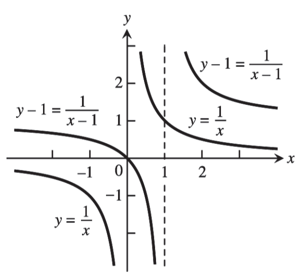

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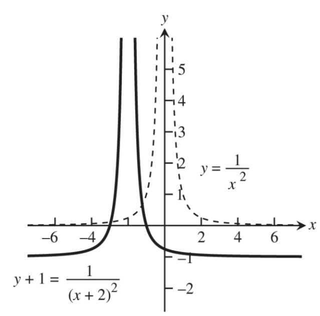

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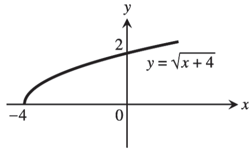

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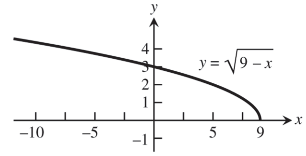

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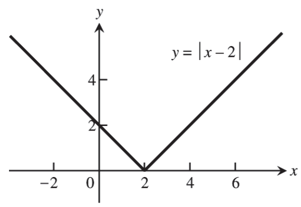

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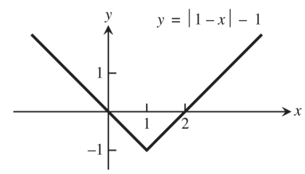

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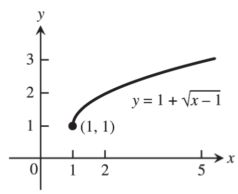

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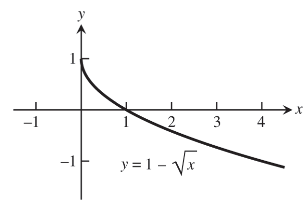

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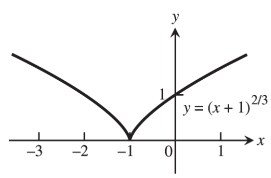

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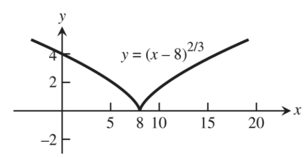

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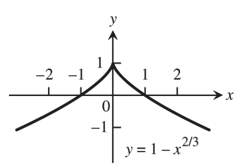

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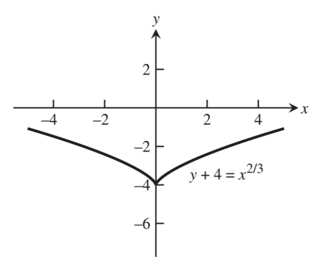

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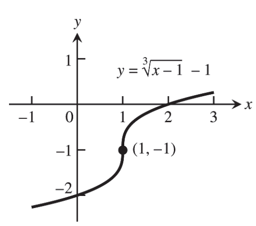

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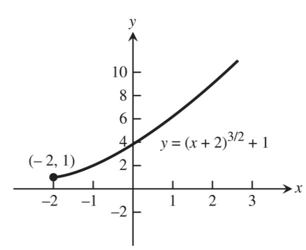

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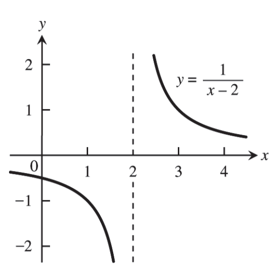

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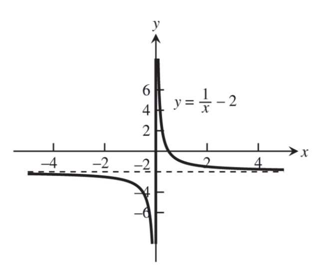

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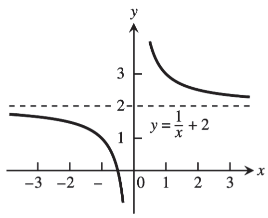

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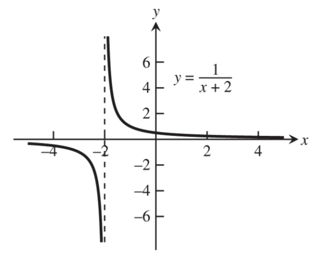

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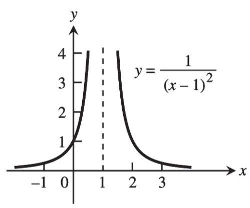

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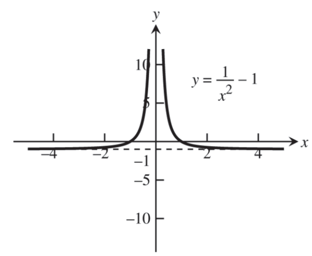

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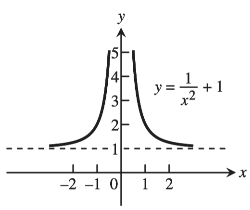

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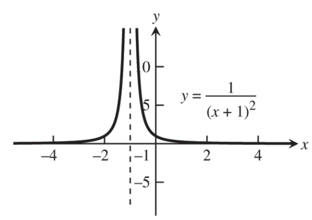
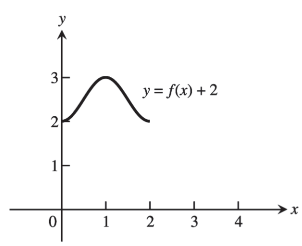
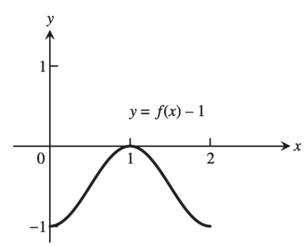


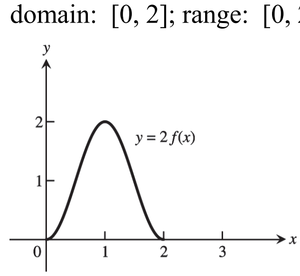
57. (a) domain: [0, 2]; range: [2, 3]



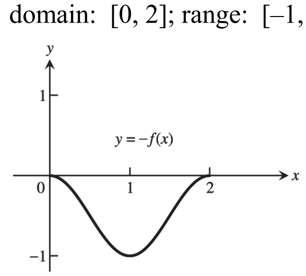
(b) domain: [0, 2]; range: [–1, 0]



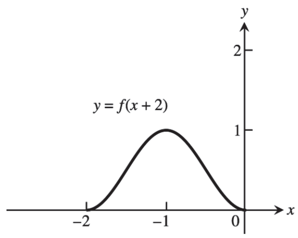
(c) domain: [0, 2]; range: [0, 2]



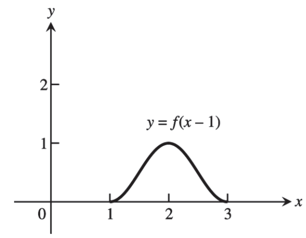
(d) domain: [0, 2]; range: [–1, 0]



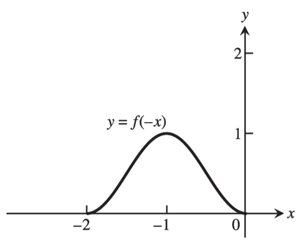
(e) domain: [–2, 0]; range: [0, 1]



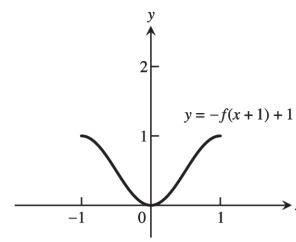
(f) domain: [1, 3]; range: [0,1]



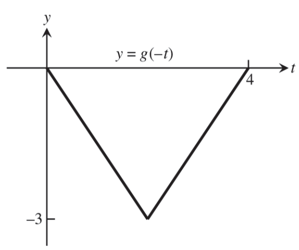
(g) domain: [–2, 0]; range: [0, 1]



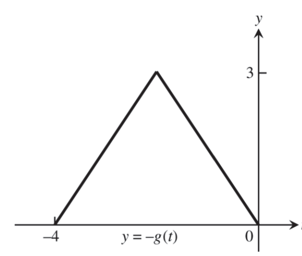
(h) domain: [–1, 1]; range: [0, 1]



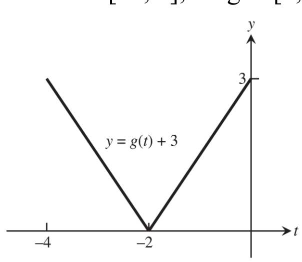
58. (a) domain: [0, 4]; range: [–3, 0]



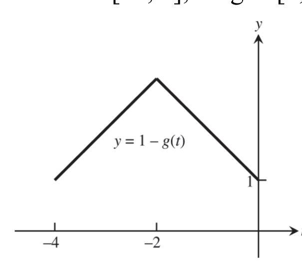
(b) domain: [–4, 0]; range: [0, 3]



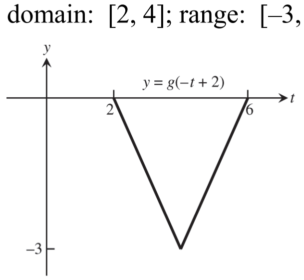
(c) domain: [–4, 0]; range: [0, 3]



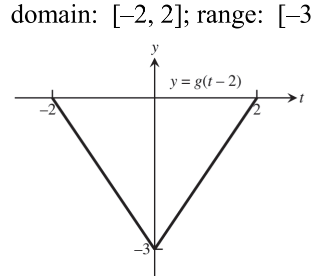
(d) domain: [–4, 0]; range: [1, 4]



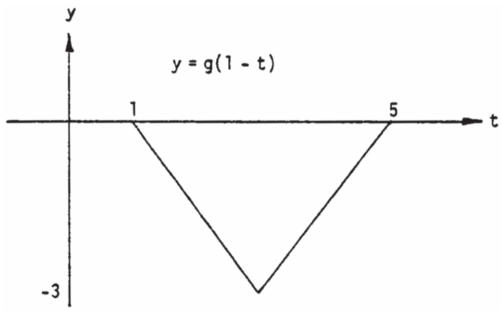
(e) domain: [2, 4]; range: [–3, 0]



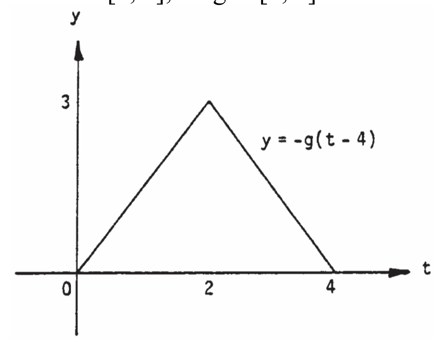
(f) domain: [–2, 2]; range: [–3, 0]



(g) domain: [1, 5]; range: [–3, 0]



(h) domain: [0, 4]; range: [0, 3]



59. $y = 3x^2 - 3$

60. $y = (2x)^2 - 1 = 4x^2 - 1$

61. $y = \frac{1}{2}\left(1 + \frac{1}{x^2}\right) = \frac{1}{2} + \frac{1}{2x^2}$

62. $y = 1 + \frac{1}{(x/3)^2} = 1 + \frac{9}{x^2}$

63. $y = \sqrt{4x+1}$

64. $y = 3\sqrt{x+1}$

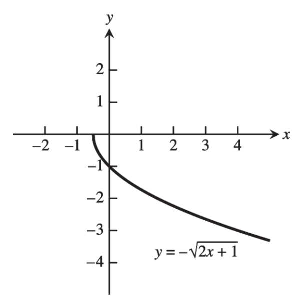
65. $y = \sqrt{4 - \left(\frac{x}{2}\right)^2} = \frac{1}{2}\sqrt{16 - x^2}$

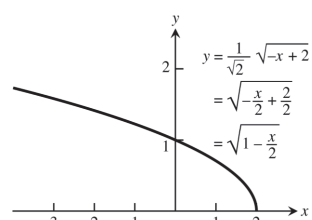
66. $y = \frac{1}{3}\sqrt{4 - x^2}$

67. $y = 1 - (3x)^3 = 1 - 27x^3$

68. $y = 1 - \left(\frac{x}{2}\right)^3 = 1 - \frac{x^3}{8}$

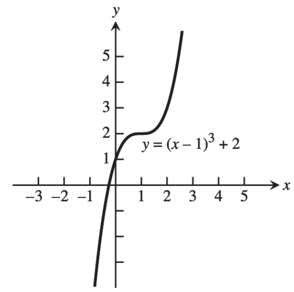
69. Let $y = -\sqrt{2x+1} = f(x)$ and let $g(x) = x^{1/2}$, $h(x) = \left(x + \frac{1}{2}\right)^{1/2}$, $i(x) = \sqrt{2}\left(x + \frac{1}{2}\right)^{1/2}$, and $j(x) = -\left[\sqrt{2}\left(x + \frac{1}{2}\right)^{1/2}\right] = f(x)$. The graph of $h(x)$ is the graph of $g(x)$ shifted left $\frac{1}{2}$ unit; the graph of $i(x)$ is the graph of $h(x)$ stretched vertically by a factor of $\sqrt{2}$; and the graph of $j(x) = f(x)$ is the graph of $i(x)$ reflected across the x-axis.



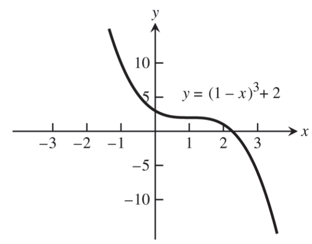


70. Let $y = \sqrt{1 - \frac{x}{2}} = f(x)$. Let $g(x) = (-x)^{1/2}$, $h(x) = (-x+2)^{1/2}$, and $i(x) = \frac{1}{\sqrt{2}}(-x+2)^{1/2} = \sqrt{1 - \frac{x}{2}} = f(x)$. The graph of $g(x)$ is the graph of $y = \sqrt{x}$ reflected across the x-axis. The graph of $h(x)$ is the graph of $g(x)$ shifted right two units. And the graph of $i(x)$ is the graph of $h(x)$ compressed vertically by a factor of $\sqrt{2}$.

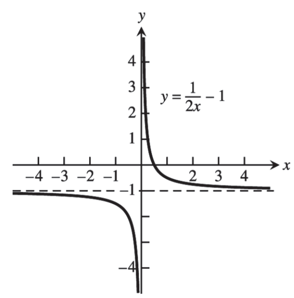
71. $y = f(x) = x^3$. Shift $f(x)$ one unit right followed by a shift two units up to get $g(x) = (x-1)^3 + 2$.



72. $y = (1-x)^3 + 2 = -[(x-1)^3 + (-2)] = f(x)$.
Let $g(x) = x^3$, $h(x) = (x-1)^3$, $i(x) = (x-1)^3 + (-2)$, and $j(x) = -[(x-1)^3 + (-2)]$. The graph of $h(x)$ is the graph of $g(x)$ shifted right one unit; the graph of $i(x)$ is the graph of $h(x)$ shifted down two units; and the graph of $f(x)$ is the graph of $i(x)$ reflected across the x-axis.

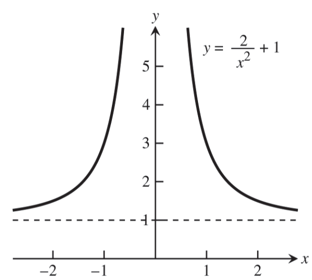


73. Compress the graph of $f(x) = \frac{1}{x}$ horizontally by a factor of 2 to get $g(x) = \frac{1}{2x}$. Then shift $g(x)$ vertically down 1 unit to get $h(x) = \frac{1}{2x} - 1$.

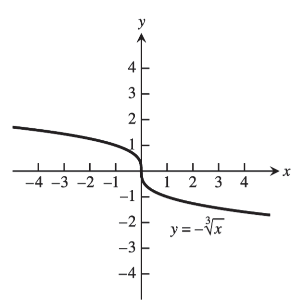


74. Let $f(x) = \frac{1}{x^2}$ and $g(x) = \frac{2}{x^2} + 1 = \frac{1}{\left(\frac{x^2}{2}\right)} + 1$

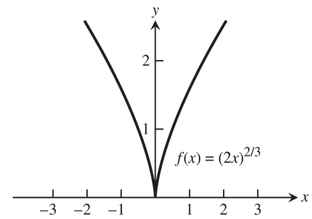
$= \frac{1}{\left(x/\sqrt{2}\right)^2} + 1 = \frac{1}{\left[\left(1/\sqrt{2}\right)x\right]^2} + 1$. Since $\sqrt{2} \approx 1.4$, we see that the graph of $f(x)$ stretched horizontally by a factor of 1.4 and shifted up 1 unit is the graph of $g(x)$.



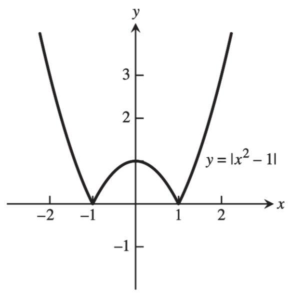
75. Reflect the graph of $y = f(x) = \sqrt[3]{x}$ across the x-axis to get $g(x) = -\sqrt[3]{x}$.



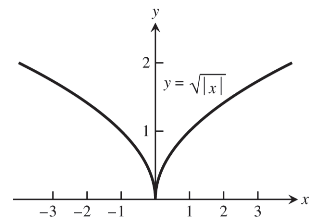
76. $y = f(x) = (-2x)^{2/3} = [(-1)(2)x]^{2/3} = (-1)^{2/3}(2x)^{2/3}$
$= (2x)^{2/3}$. So the graph of $f(x)$ is the graph of
$g(x) = x^{2/3}$ compressed horizontally by a factor of 2.



77.



78.



79. (a) $(fg)(-x) = f(-x)g(-x) = f(x)(-g(x)) = -(fg)(x)$, odd

(b) $\left(\frac{f}{g}\right)(-x) = \frac{f(-x)}{g(-x)} = \frac{f(x)}{-g(x)} = -\left(\frac{f}{g}\right)(x)$, odd

(c) $\left(\frac{g}{f}\right)(-x) = \frac{g(-x)}{f(-x)} = \frac{-g(x)}{f(x)} = -\left(\frac{g}{f}\right)(x)$, odd

(d) $f^2(-x) = f(-x)f(-x) = f(x)f(x) = f^2(x)$, even

(e) $g^2(-x) = (g(-x))^2 = (-g(x))^2 = g^2(x)$, even

(f) $(f \circ g)(-x) = f(g(-x)) = f(-g(x)) = f(g(x)) = (f \circ g)(x)$, even

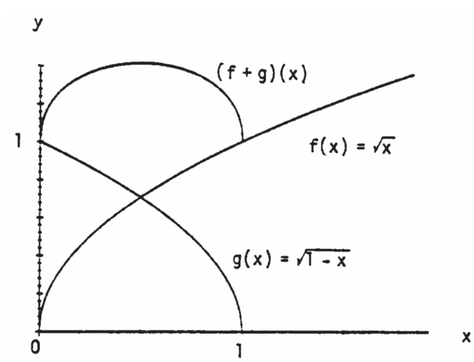
(g) $(g \circ f)(-x) = g(f(-x)) = g(f(x)) = (g \circ f)(x)$, even

(h) $(f \circ f)(-x) = f(f(-x)) = f(f(x)) = (f \circ f)(x)$, even

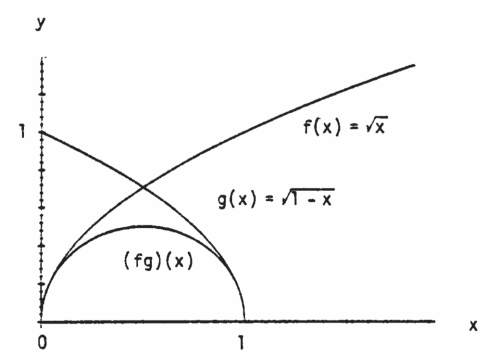
(i) $(g \circ g)(-x) = g(g(-x)) = g(-g(x)) = -g(g(x)) = -(g \circ g)(x)$, odd

80. Yes, $f(x) = 0$ is both even and odd since $f(-x) = 0 = f(x)$ and $f(-x) = 0 = -f(x)$.

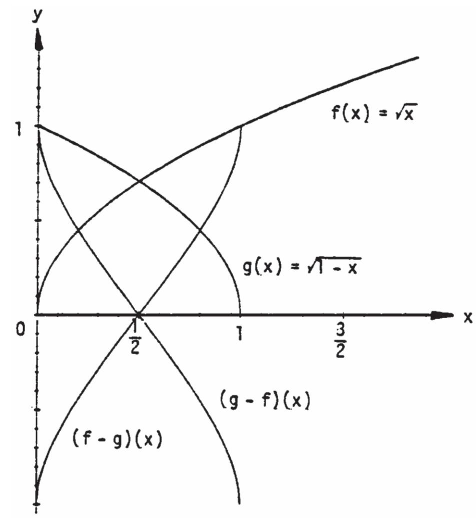
81. (a)



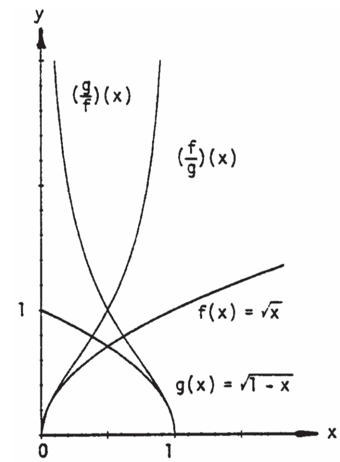
(b)



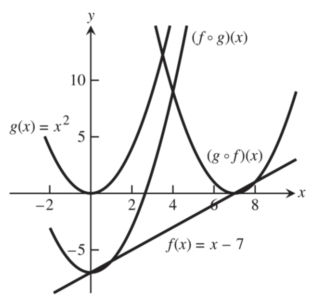
(c)



(d)



82.



1.3 TRIGONOMETRIC FUNCTIONS

1. (a) $s = r\theta = (10)\left(\frac{4\pi}{5}\right) = 8\pi$ m (b) $s = r\theta = (10)(110°)\left(\frac{\pi}{180°}\right) = \frac{110\pi}{18} = \frac{55\pi}{9}$ m

2. $\theta = \frac{s}{r} = \frac{10\pi}{8} = \frac{5\pi}{4}$ radians and $\frac{5\pi}{4}\left(\frac{180°}{\pi}\right) = 225°$

3. $\theta = 80° \Rightarrow \theta = 80°\left(\frac{\pi}{180°}\right) = \frac{4\pi}{9} \Rightarrow s = (6)\left(\frac{4\pi}{9}\right) = 8.4$ in. (since the diameter = 12 in. $\Rightarrow$ radius = 6 in.)

4. $d = 1$ meter $\Rightarrow r = 50$ cm $\Rightarrow \theta = \frac{s}{r} = \frac{30}{50} = 0.6$ rad or $0.6\left(\frac{180°}{\pi}\right) \approx 34°$

5.

θ	$-\pi$	$-\frac{2\pi}{3}$	0	$\frac{\pi}{2}$	$\frac{3\pi}{4}$
$\sin\theta$	0	$-\frac{\sqrt{3}}{2}$	0	1	$\frac{1}{\sqrt{2}}$
$\cos\theta$	-1	$-\frac{1}{2}$	1	0	$-\frac{1}{\sqrt{2}}$
$\tan\theta$	0	$\sqrt{3}$	0	und.	-1
$\cot\theta$	und.	$\frac{1}{\sqrt{3}}$	und.	0	-1
$\sec\theta$	-1	-2	1	und.	$-\sqrt{2}$
$\csc\theta$	und.	$-\frac{2}{\sqrt{3}}$	und.	1	$\sqrt{2}$

6.

θ	$-\frac{3\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{5\pi}{6}$
$\sin\theta$	1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
$\cos\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$
$\tan\theta$	und.	$-\sqrt{3}$	$-\frac{1}{\sqrt{3}}$	1	$-\frac{1}{\sqrt{3}}$
$\cot\theta$	0	$-\frac{1}{\sqrt{3}}$	$-\sqrt{3}$	1	$-\sqrt{3}$
$\sec\theta$	und.	2	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	$-\frac{2}{\sqrt{3}}$
$\csc\theta$	1	$-\frac{2}{\sqrt{3}}$	-2	$\sqrt{2}$	2

7. $\cos x = -\frac{4}{5}$, $\tan x = -\frac{3}{4}$

8. $\sin x = \frac{2}{\sqrt{5}}$, $\cos x = \frac{1}{\sqrt{5}}$

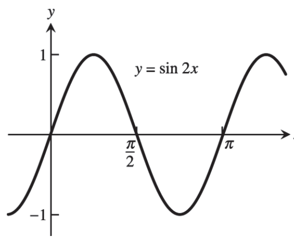
9. $\sin x = -\frac{\sqrt{8}}{3}$, $\tan x = -\sqrt{8}$

10. $\sin x = \frac{12}{13}$, $\tan x = -\frac{12}{5}$

11. $\sin x = -\frac{1}{\sqrt{5}}$, $\cos x = -\frac{2}{\sqrt{5}}$

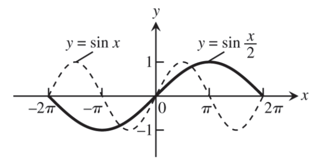
12. $\cos x = -\frac{\sqrt{3}}{2}$, $\tan x = \frac{1}{\sqrt{3}}$

13.



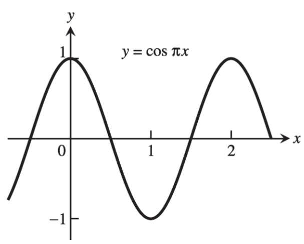
period $= \pi$

14.



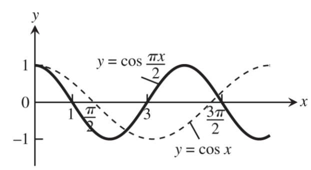
period $= 4\pi$

15.



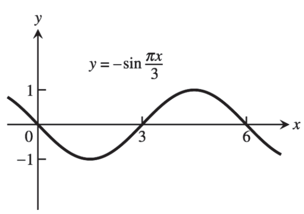
period $= 2$

16.



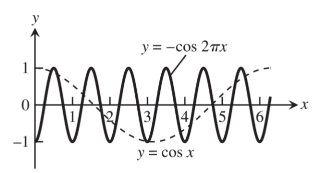
period $= 4$

17.



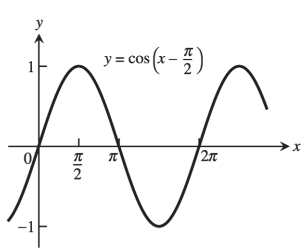
period $= 6$

18.



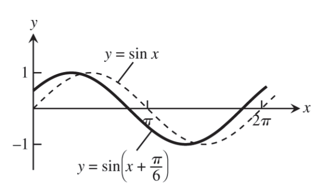
period $= 1$

19.



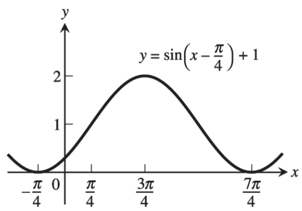
period $= 2\pi$

20.



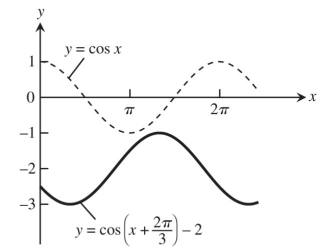
period $= 2\pi$

21.



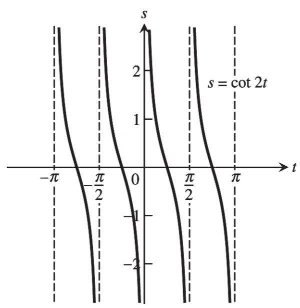
period = 2π

22.

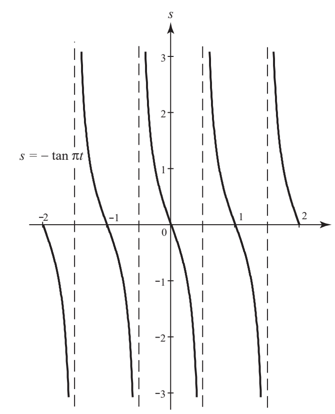


period = 2π

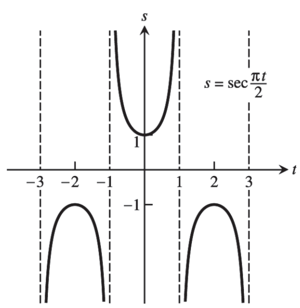
23. period = $\frac{\pi}{2}$, symmetric about the origin



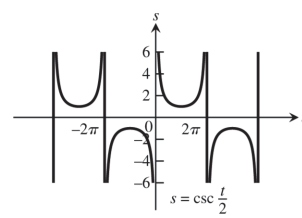
24. period = 1, symmetric about the origin



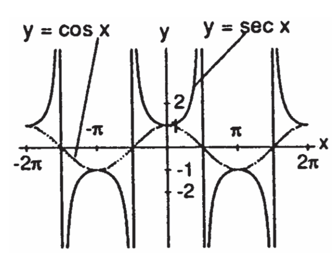
25. period = 4, symmetric about the s-axis



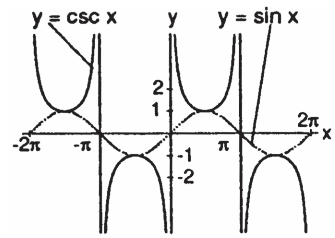
26. period = 4π, symmetric about the origin



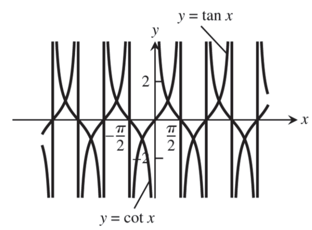
27. (a) Cos x and sec x are positive for x in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$; and cos x and sec x are negative for x in the intervals $\left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right)$ and $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$. Sec x is undefined when cos x is 0. The range of sec x is $(-\infty, -1] \cup [1, \infty)$; the range of cos x is $[-1, 1]$.



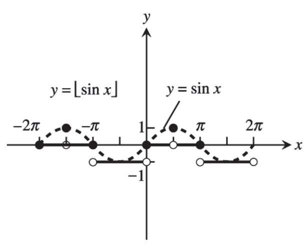
(b) Sin x and csc x are positive for x in the intervals $\left(-\frac{3\pi}{2}, -\pi\right)$ and $(0, \pi)$; and sin x and csc x are negative for x in the intervals $(-\pi, 0)$ and $\left(\pi, \frac{3\pi}{2}\right)$. Csc x is undefined when sin x is 0. The range of csc x is $(-\infty, -1] \cup [1, \infty)$; the range of sin x is $[-1, 1]$.



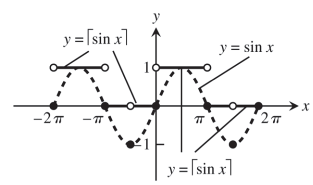
28. Since $\cot x = \frac{1}{\tan x}$, cot x is undefined when $\tan x = 0$ and is zero when tan x is undefined. As tan x approaches zero through positive values, cot x approaches infinity. Also, cot x approaches negative infinity as tan x approaches zero through negative values.



29. $D: -\infty < x < \infty;\ R:\ y = -1, 0, 1$



30. $D: -\infty < x < \infty;\ R:\ y = -1, 0, 1$



31. $\cos\left(x - \frac{\pi}{2}\right) = \cos x \cos\left(-\frac{\pi}{2}\right) - \sin x \sin\left(-\frac{\pi}{2}\right) = (\cos x)(0) - (\sin x)(-1) = \sin x$

32. $\cos\left(x + \frac{\pi}{2}\right) = \cos x \cos\left(\frac{\pi}{2}\right) - \sin x \sin\left(\frac{\pi}{2}\right) = (\cos x)(0) - (\sin x)(1) = -\sin x$

33. $\sin\left(x + \frac{\pi}{2}\right) = \sin x \cos\left(\frac{\pi}{2}\right) + \cos x \sin\left(\frac{\pi}{2}\right) = (\sin x)(0) + (\cos x)(1) = \cos x$

34. $\sin\left(x - \frac{\pi}{2}\right) = \sin x \cos\left(-\frac{\pi}{2}\right) + \cos x \sin\left(-\frac{\pi}{2}\right) = (\sin x)(0) + (\cos x)(-1) = -\cos x$

35. $\cos(A - B) = \cos(A + (-B)) = \cos A \cos(-B) - \sin A \sin(-B) = \cos A \cos B - \sin A(-\sin B)$
$= \cos A \cos B + \sin A \sin B$

36. $\sin(A - B) = \sin(A + (-B)) = \sin A \cos(-B) + \cos A \sin(-B) = \sin A \cos B + \cos A(-\sin B)$
$= \sin A \cos B - \cos A \sin B$

37. If $B = A$, $A - B = 0 \Rightarrow \cos(A - B) = \cos 0 = 1$. Also $\cos(A - B) = \cos(A - A) = \cos A \cos A + \sin A \sin A$ $= \cos^2 A + \sin^2 A$. Therefore, $\cos^2 A + \sin^2 A = 1$.

38. If $B = 2\pi$, then $\cos(A + 2\pi) = \cos A \cos 2\pi - \sin A \sin 2\pi = (\cos A)(1) - (\sin A)(0) = \cos A$ and $\sin(A + 2\pi) = \sin A \cos 2\pi + \cos A \sin 2\pi = (\sin A)(1) + (\cos A)(0) = \sin A$. The result agrees with the fact that the cosine and sine functions have period 2π.

39. $\cos(\pi + x) = \cos \pi \cos x - \sin \pi \sin x = (-1)(\cos x) - (0)(\sin x) = -\cos x$

40. $\sin(2\pi - x) = \sin 2\pi \cos(-x) + \cos(2\pi)\sin(-x) = (0)(\cos(-x)) + (1)(\sin(-x)) = -\sin x$

41. $\sin\left(\frac{3\pi}{2} - x\right) = \sin\left(\frac{3\pi}{2}\right)\cos(-x) + \cos\left(\frac{3\pi}{2}\right)\sin(-x) = (-1)(\cos x) + (0)(\sin(-x)) = -\cos x$

42. $\cos\left(\frac{3\pi}{2} + x\right) = \cos\left(\frac{3\pi}{2}\right)\cos x - \sin\left(\frac{3\pi}{2}\right)\sin x = (0)(\cos x) - (-1)(\sin x) = \sin x$

43. $\sin\frac{7\pi}{12} = \sin\left(\frac{\pi}{4} + \frac{\pi}{3}\right) = \sin\frac{\pi}{4}\cos\frac{\pi}{3} + \cos\frac{\pi}{4}\sin\frac{\pi}{3} = \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{6}+\sqrt{2}}{4}$

44. $\cos\frac{11\pi}{12} = \cos\left(\frac{\pi}{4} + \frac{2\pi}{3}\right) = \cos\frac{\pi}{4}\cos\frac{2\pi}{3} - \sin\frac{\pi}{4}\sin\frac{2\pi}{3} = \left(\frac{\sqrt{2}}{2}\right)\left(-\frac{1}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = -\frac{\sqrt{2}+\sqrt{6}}{4}$

45. $\cos\frac{\pi}{12} = \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \cos\frac{\pi}{3}\cos\left(-\frac{\pi}{4}\right) - \sin\frac{\pi}{3}\sin\left(-\frac{\pi}{4}\right) = \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{\sqrt{3}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right) = \frac{1+\sqrt{3}}{2\sqrt{2}}$

46. $\sin\frac{5\pi}{12} = \sin\left(\frac{2\pi}{3} - \frac{\pi}{4}\right) = \sin\left(\frac{2\pi}{3}\right)\cos\left(-\frac{\pi}{4}\right) + \cos\left(\frac{2\pi}{3}\right)\sin\left(-\frac{\pi}{4}\right) = \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(-\frac{1}{2}\right)\left(-\frac{\sqrt{2}}{2}\right) = \frac{1+\sqrt{3}}{2\sqrt{2}}$

47. $\cos^2\frac{\pi}{8} = \frac{1+\cos\left(\frac{2\pi}{8}\right)}{2} = \frac{1+\frac{\sqrt{2}}{2}}{2} = \frac{2+\sqrt{2}}{4}$

48. $\cos^2\frac{5\pi}{12} = \frac{1+\cos\left(\frac{10\pi}{12}\right)}{2} = \frac{1+\left(-\frac{\sqrt{3}}{2}\right)}{2} = \frac{2-\sqrt{3}}{4}$

49. $\sin^2\frac{\pi}{12} = \frac{1-\cos\left(\frac{2\pi}{12}\right)}{2} = \frac{1-\frac{\sqrt{3}}{2}}{2} = \frac{2-\sqrt{3}}{4}$

50. $\sin^2\frac{3\pi}{8} = \frac{1-\cos\left(\frac{6\pi}{8}\right)}{2} = \frac{1-\left(-\frac{\sqrt{2}}{2}\right)}{2} = \frac{2+\sqrt{2}}{4}$

51. $\sin^2\theta = \frac{3}{4} \Rightarrow \sin\theta = \pm\frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

52. $\sin^2\theta = \cos^2\theta \Rightarrow \frac{\sin^2\theta}{\cos^2\theta} = \frac{\cos^2\theta}{\cos^2\theta} \Rightarrow \tan^2\theta = 1 \Rightarrow \tan\theta = \pm 1 \Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

53. $\sin 2\theta - \cos\theta = 0 \Rightarrow 2\sin\theta\cos\theta - \cos\theta = 0 \Rightarrow \cos\theta(2\sin\theta - 1) = 0 \Rightarrow \cos\theta = 0$ or $2\sin\theta - 1 = 0$
$\Rightarrow \cos\theta = 0$ or $\sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}$, or $\theta = \frac{\pi}{6}, \frac{5\pi}{6} \Rightarrow \theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}$

54. $\cos 2\theta + \cos\theta = 0 \Rightarrow 2\cos^2\theta - 1 + \cos\theta = 0 \Rightarrow 2\cos^2\theta + \cos\theta - 1 = 0 \Rightarrow (\cos\theta + 1)(2\cos\theta - 1) = 0$
$\Rightarrow \cos\theta + 1 = 0$ or $2\cos\theta - 1 = 0 \Rightarrow \cos\theta = -1$ or $\cos\theta = \frac{1}{2} \Rightarrow \theta = \pi$ or $\theta = \frac{\pi}{3}, \frac{5\pi}{3} \Rightarrow \theta = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$

55. $\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A\cos B + \cos A\cos B}{\cos A\cos B - \sin A\sin B} = \frac{\frac{\sin A\cos B}{\cos A\cos B} + \frac{\cos A\sin B}{\cos A\cos B}}{\frac{\cos A\cos B}{\cos A\cos B} - \frac{\sin A\sin B}{\cos A\cos B}} = \frac{\tan A + \tan B}{1 - \tan A\tan B}$

56. $\tan(A-B) = \frac{\sin(A-B)}{\cos(A-B)} = \frac{\sin A\cos B - \cos A\cos B}{\cos A\cos B + \sin A\sin B} = \frac{\frac{\sin A\cos B}{\cos A\cos B} - \frac{\cos A\sin B}{\cos A\cos B}}{\frac{\cos A\cos B}{\cos A\cos B} + \frac{\sin A\sin B}{\cos A\cos B}} = \frac{\tan A - \tan B}{1 + \tan A\tan B}$

57. According to the figure in the text, we have the following: By the law of cosines, $c^2 = a^2 + b^2 - 2ab\cos\theta$ $= 1^2 + 1^2 - 2\cos(A-B) = 2 - 2\cos(A-B)$. By distance formula, $c^2 = (\cos A - \cos B)^2 + (\sin A - \sin B)^2$ $= \cos^2 A - 2\cos A\cos B + \cos^2 B + \sin^2 A - 2\sin A\sin B + \sin^2 B = 2 - 2(\cos A\cos B + \sin A\sin B)$. Thus $c^2 = 2 - 2\cos(A-B) = 2 - 2(\cos A\cos B + \sin A\sin B) \Rightarrow \cos(A-B) = \cos A\cos B + \sin A\sin B$.

58. (a) $\cos(A-B)=\cos A\cos B+\sin A\sin B$
$\sin\theta=\cos\left(\frac{\pi}{2}-\theta\right)$ and $\cos\theta=\sin\left(\frac{\pi}{2}-\theta\right)$
Let $\theta=A+B$
$\sin(A+B)=\cos\left[\frac{\pi}{2}-(A+B)\right]=\cos\left[\left(\frac{\pi}{2}-A\right)-B\right]=\cos\left(\frac{\pi}{2}-A\right)\cos B+\sin\left(\frac{\pi}{2}-A\right)\sin B$
$=\sin A\cos B+\cos A\sin B$
(b) $\cos(A-B)=\cos A\cos B+\sin A\sin B$
$\cos(A-(-B))=\cos A\cos(-B)+\sin A\sin(-B)$
$\Rightarrow\cos(A+B)=\cos A\cos(-B)+\sin A\sin(-B)=\cos A\cos B+\sin A(-\sin B)=\cos A\cos B-\sin A\sin B$
Because the cosine function is even and the sine functions is odd.

59. $c^2=a^2+b^2-2ab\cos C=2^2+3^2-2(2)(3)\cos(60°)=4+9-12\cos(60°)=13-12\left(\frac{1}{2}\right)=7.$
Thus, $c=\sqrt{7}\approx 2.65$.

60. $c^2=a^2+b^2-2ab\cos C=2^2+3^2-2(2)(3)\cos(40°)=13-12\cos(40°)$. Thus, $c=\sqrt{13-12\cos 40°}\approx 1.951$.

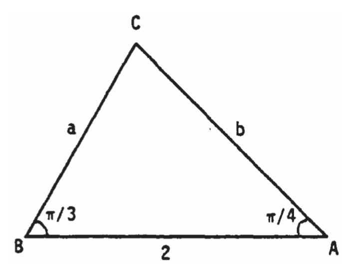
61. From the figures in the text, we see that $\sin B=\frac{h}{c}$. If C is an acute angle, then $\sin C=\frac{h}{b}$. On the other hand, if C is obtuse (as in the figure on the right in the text), then $\sin C=\sin(\pi-C)=\frac{h}{b}$. Thus, in either case, $h=b\sin C=c\sin B\Rightarrow ah=ab\sin C=ac\sin B$.

By the law of cosines, $\cos C=\frac{a^2+b^2-c^2}{2ab}$ and $\cos B=\frac{a^2+c^2-b^2}{2ac}$. Moreover, since the sum of the interior angles of triangle is π, we have $\sin A=\sin(\pi-(B+C))=\sin(B+C)=\sin B\cos C+\cos B\sin C$
$=\left(\frac{h}{c}\right)\left[\frac{a^2+b^2-c^2}{2ab}\right]+\left[\frac{a^2+c^2-b^2}{2ac}\right]\left(\frac{h}{b}\right)=\left(\frac{h}{2abc}\right)(2a^2+b^2-c^2+c^2-b^2)=\frac{ah}{bc}\Rightarrow ah=bc\sin A.$

Combining our results we have $ah=ab\sin C$, $ah=ac\sin B$, and $ah=bc\sin A$. Dividing by abc gives
$\frac{h}{bc}=\underbrace{\frac{\sin A}{a}=\frac{\sin C}{c}=\frac{\sin B}{b}}_{\text{law of sines}}.$

62. By the law of sines, $\frac{\sin A}{2}=\frac{\sin B}{3}=\frac{\sqrt{3}/2}{c}$. By Exercise 59 we know that $c=\sqrt{7}$. Thus $\sin B=\frac{3\sqrt{3}}{2\sqrt{7}}\simeq 0.982$.

63. From the figure at the right and the law of cosines,
$b^2=a^2+2^2-2(2a)\cos B$
$=a^2+4-4a\left(\frac{1}{2}\right)=a^2-2a+4.$



Applying the law of sines to the figure, $\frac{\sin A}{a}=\frac{\sin B}{b}$
$\Rightarrow\frac{\sqrt{2}/2}{a}=\frac{\sqrt{3}/2}{b}\Rightarrow b=\sqrt{\frac{3}{2}}\,a$. Thus, combining results,
$a^2-2a+4=b^2=\frac{3}{2}a^2\Rightarrow 0=\frac{1}{2}a^2+2a-4\Rightarrow 0=a^2+4a-8$. From the quadratic formula and the fact that $a>0$, we have $a=\frac{-4+\sqrt{4^2-4(1)(-8)}}{2}=\frac{4\sqrt{3}-4}{2}\simeq 1.464.$

64. $\tan\gamma = \frac{h}{c} \Rightarrow c = \frac{h}{\tan\gamma}$

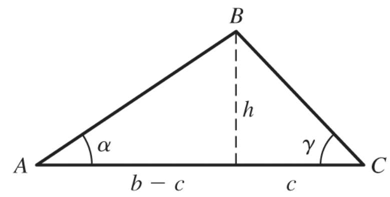
$\tan\alpha = \frac{h}{b-c} = \frac{h}{b-\frac{h}{\tan\gamma}} = \frac{h\tan\gamma}{b\tan\gamma - h} \Rightarrow$

$b\tan\alpha\tan\gamma - h\tan\alpha = h\tan\gamma \Rightarrow$

$b\tan\alpha\tan\gamma = h\tan\alpha + h\tan\gamma \Rightarrow$

$b\tan\alpha\tan\gamma = h(\tan\alpha + \tan\gamma) \Rightarrow$

$h = \frac{b\tan\alpha\tan\gamma}{\tan\alpha + \tan\gamma}$

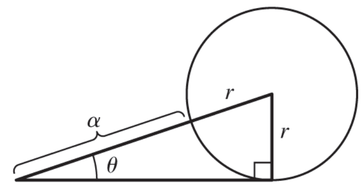


65. $\sin\theta = \frac{r}{\alpha + r}$

$\Rightarrow \alpha\sin\theta + r\sin\theta = r$

$\Rightarrow \alpha\sin\theta = r - r\sin\theta = r(1-\sin\theta)$

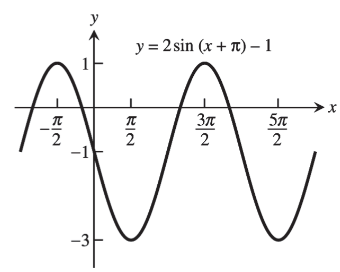
$\Rightarrow\ r = \frac{\alpha\sin\theta}{1-\sin\theta}$



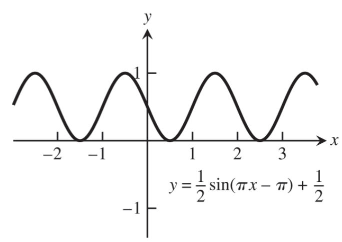
66. (a) The graphs of $y = \sin x$ and $y = x$ nearly coincide when x is near the origin (when the calculator is in radians mode).

(b) In degree mode, when x is near zero degrees the sine of x is much closer to zero than x itself. The curves look like intersecting straight lines near the origin when the calculator is in degree mode.

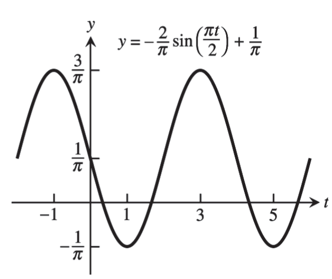
67. $A = 2,\ B = 2\pi,\ C = -\pi,\ D = -1$



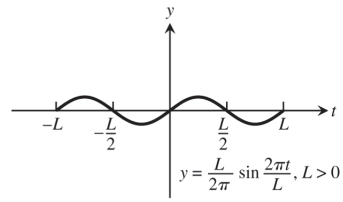
68. $A = \frac{1}{2},\ B = 2,\ C = 1,\ D = \frac{1}{2}$



69. $A = -\frac{2}{\pi},\ B = 4,\ C = 0,\ D = \frac{1}{\pi}$



70. $A = \frac{L}{2\pi}, B = L, C = 0, D = 0$



71–74. Example CAS commands:

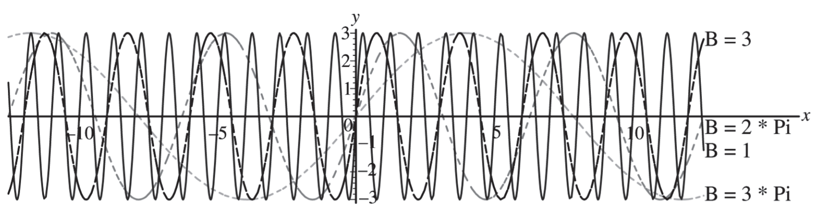
<u>Maple:</u>

```
f :=x -> A*sin((2*Pi/B)*(x-C))+D1;
A:=3; C:=0; D1:=0;
f_list := [seq(f(x), B=[1,3,2*Pi,5*Pi])];
plot(f_list, x=-4*Pi..4*Pi, scaling=constrained,
    color=[red,blue,green,cyan], linestyle=[1,3,4,7],
    legend=["B=1", "B=3","B=2*Pi","B=3*Pi"],
    title="#71 (Section 1.3)");
```

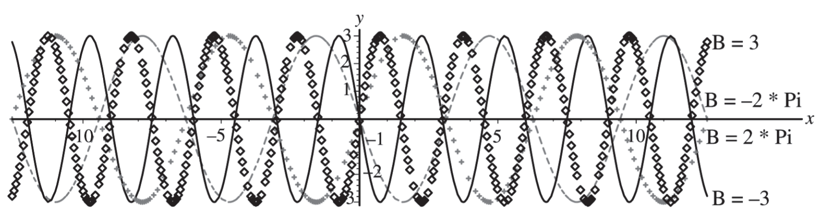
<u>Mathematica:</u>

```
Clear[a, b, c, d, f, x]
f[x_]:=a Sin[2π/b (x − c)] + d
Plot[f[x]/.{a → 3, b → 1, c → 0, d → 0}, {x, −4π, 4π}]
```

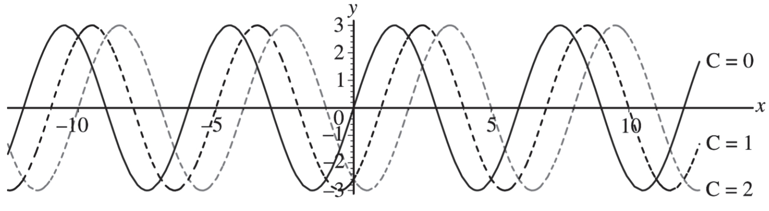
71. (a) The graph stretches horizontally.



(b) The period remains the same: period $= |B|$. The graph has a horizontal shift of $\frac{1}{2}$ period.



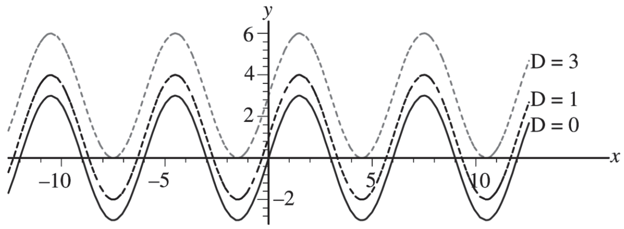
72. (a) The graph is shifted right C units.



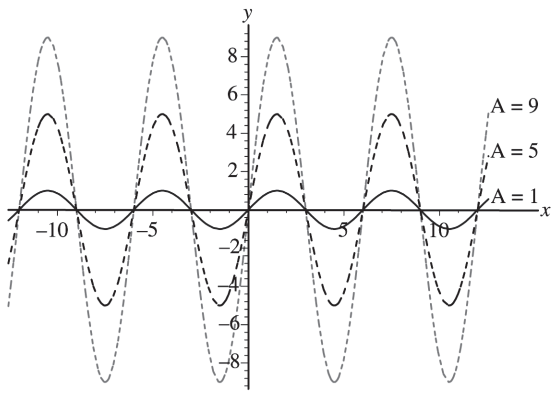
(b) The graph is shifted left C units.

(c) A shift of $\pm$ one period will produce no apparent shift. $|C| = 6$

73. (a) The graph shifts upwards $|D|$ units for $D > 0$
(b) The graph shifts down $|D|$ units for $D < 0$.



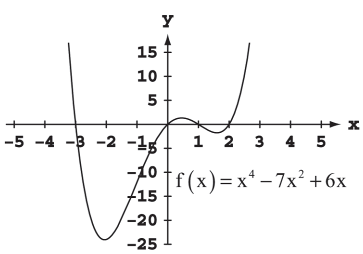
74. (a) The graph stretches $|A|$ units. (b) For $A < 0$, the graph is inverted.



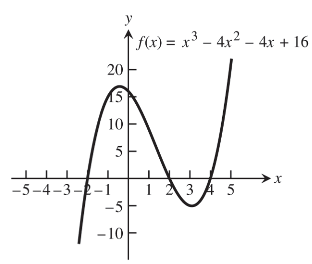
1.4 GRAPHING WITH SOFTWARE

1–4. The most appropriate viewing window displays the maxima, minima, intercepts, and end behavior of the graphs and has little unused space.

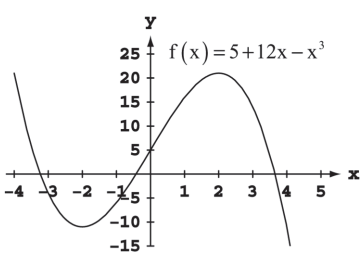
1. d.



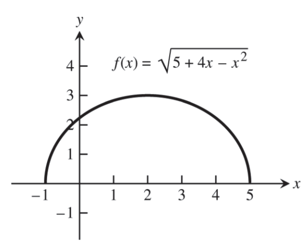
2. c.



3. d.

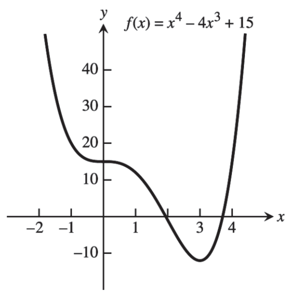


4. b.

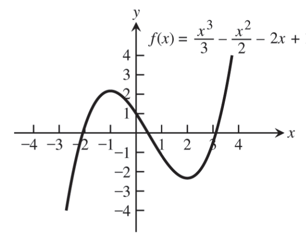


5–30. For any display there are many appropriate display widows. The graphs given as answers in Exercises 5–30 are not unique in appearance.

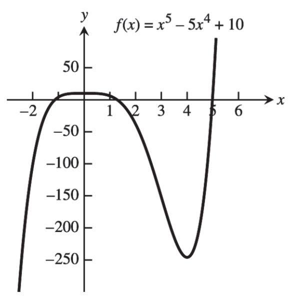
5. $[-2, 5]$ by $[-15, 40]$



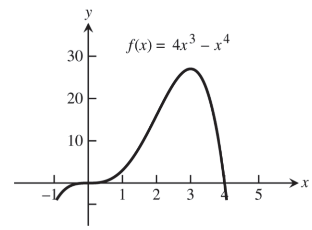
6. $[-4, 4]$ by $[-4, 4]$



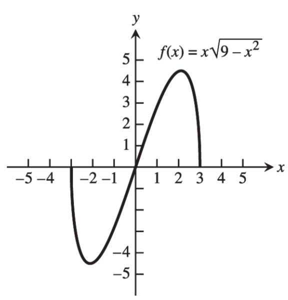
7. $[-2, 6]$ by $[-250, 50]$



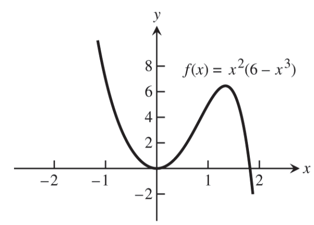
8. $[-1, 5]$ by $[-5, 30]$



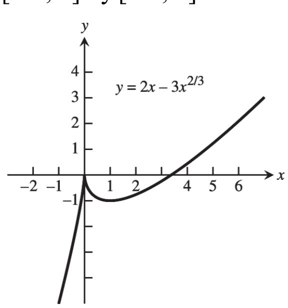
9. $[-4, 4]$ by $[-5, 5]$



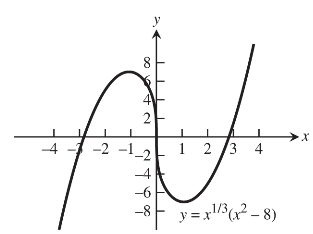
10. $[-2, 2]$ by $[-2, 8]$



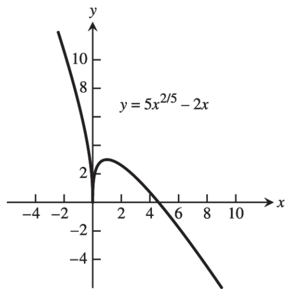
11. $[-2, 6]$ by $[-5, 4]$



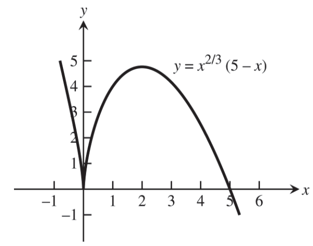
12. $[-4, 4]$ by $[-8, 8]$



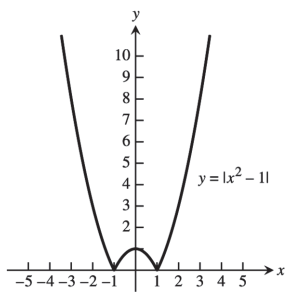
13. [−1, 6] by [−1, 4]



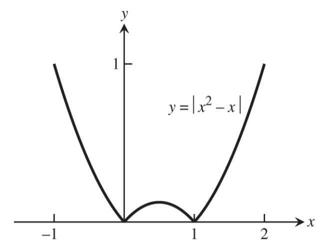
14. [−1, 6] by [−1, 5]



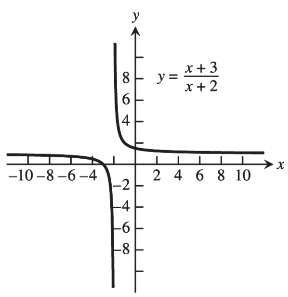
15. [−3, 3] by [0, 10]



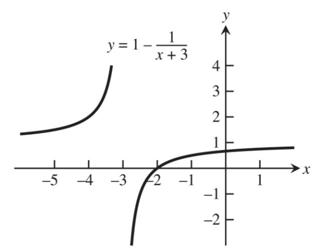
16. [−1, 2] by [0, 1]



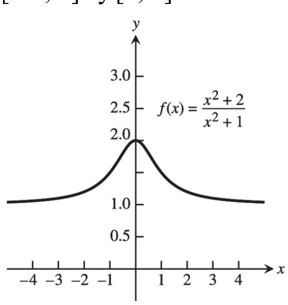
17. [−5, 1] by [−5, 5]



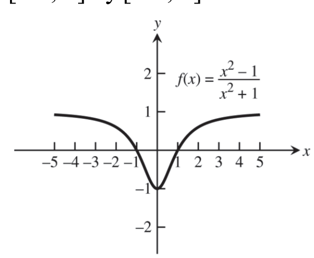
18. [−5, 1] by [−2, 4]



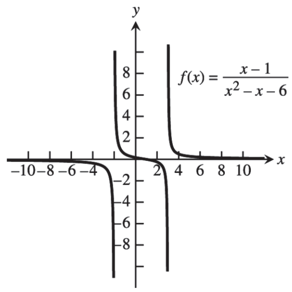
19. [−4, 4] by [0, 3]



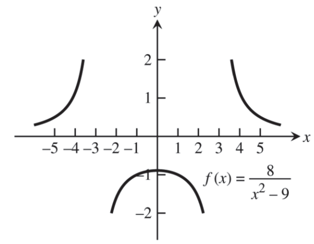
20. [−5, 5] by [−2, 2]



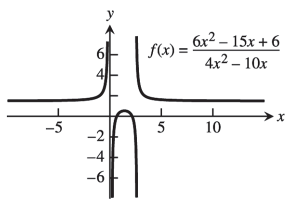
21. $[-10, 10]$ by $[-6, 6]$



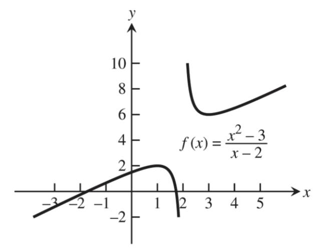
22. $[-5, 5]$ by $[-2, 2]$



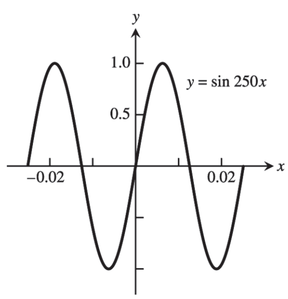
23. $[-6, 10]$ by $[-6, 6]$



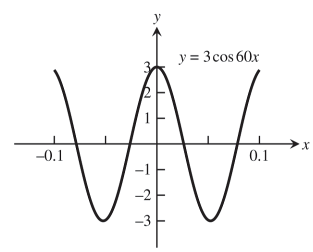
24. $[-3, 5]$ by $[-2, 10]$



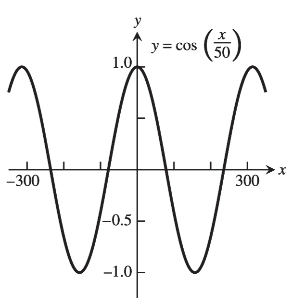
25. $[-0.03, 0.03]$ by $[-1.25, 1.25]$



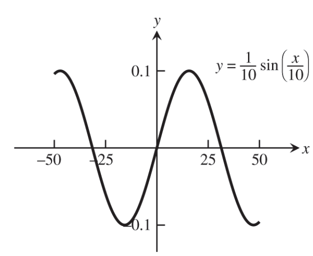
26. $[-0.1, 0.1]$ by $[-3, 3]$



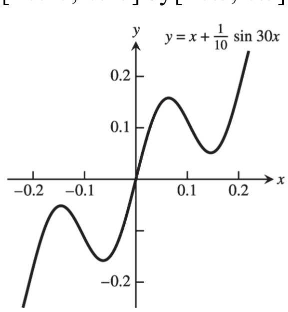
27. $[-300, 300]$ by $[-1.25, 1.25]$



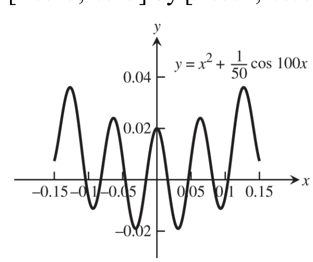
28. $[-50, 50]$ by $[-0.1, 0.1]$



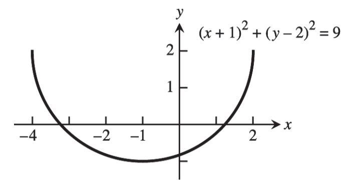
29. $[-0.25, 0.25]$ by $[-0.3, 0.3]$



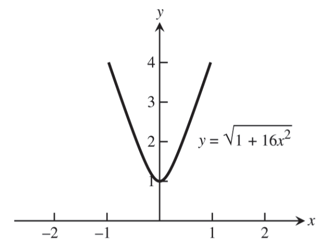
30. $[-0.15, 0.15]$ by $[-0.02, 0.05]$



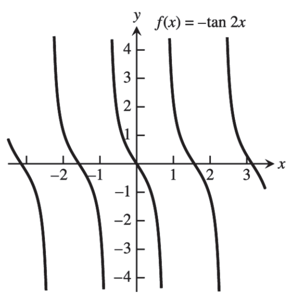
31. $x^2 + 2x = 4 + 4y - y^2 \Rightarrow y = 2 \pm \sqrt{-x^2 - 2x + 8}$. The lower half is produced by graphing $y = 2 - \sqrt{-x^2 - 2x + 8}$.



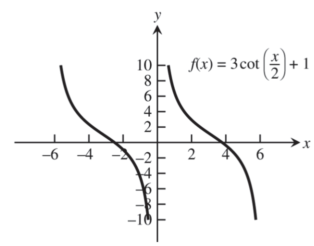
32. $y^2 - 16x^2 = 1 \Rightarrow y = \pm\sqrt{1+16x^2}$. The upper branch is produced by graphing $y = \sqrt{1+16x^2}$.



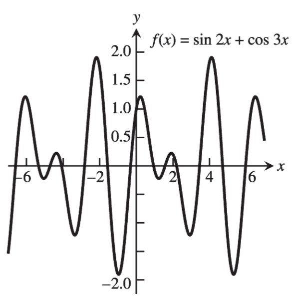
33.



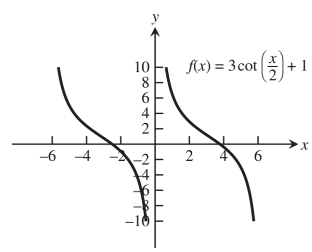
34.



35.

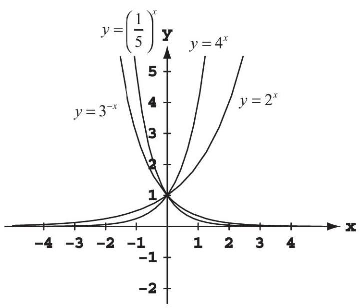


36.

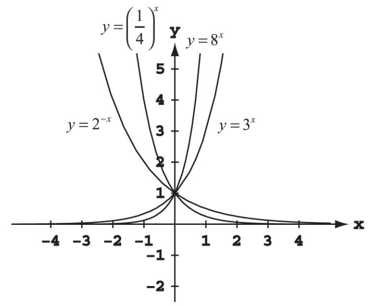


1.5 EXPONENTIAL FUNCTIONS

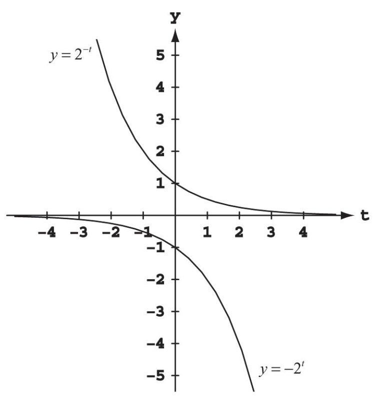
1.



2.



3.



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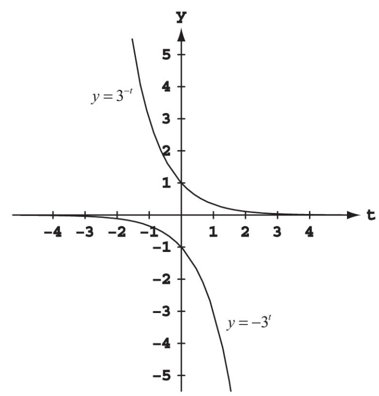
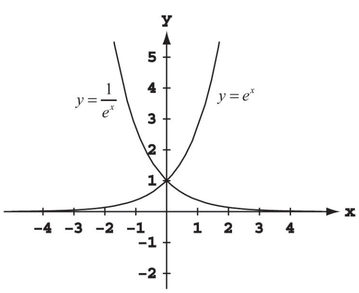
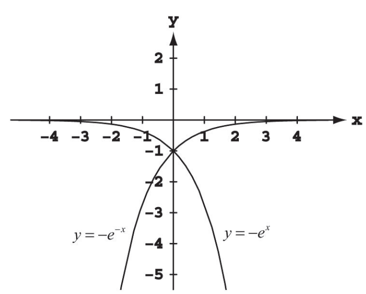


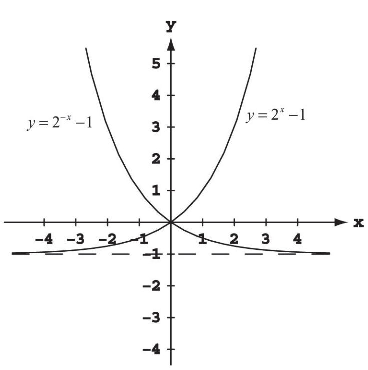
5.



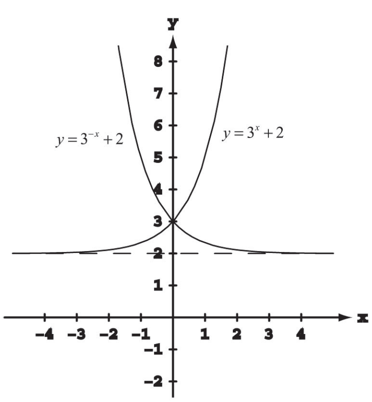
6.



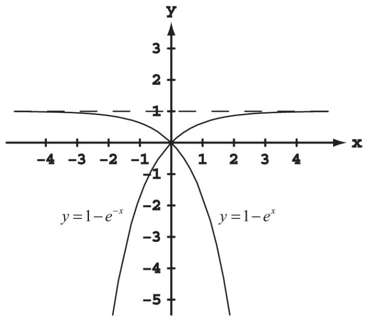
7.



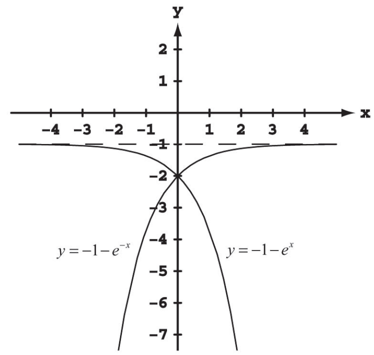
8.



9.



10.



11. $16^2 \cdot 16^{-1.75} = 16^{2+(-1.75)} = 16^{0.25} = 16^{1/4} = 2$

12. $9^{1/3} \cdot 9^{1/6} = 9^{\frac{1}{3}+\frac{1}{6}} = 9^{1/2} = 3$

13. $\frac{4^{4.2}}{4^{3.7}} = 4^{4.2-3.7} = 4^{0.5} = 4^{1/2} = 2$

14. $\frac{3^{5/3}}{3^{2/3}} = 3^{\frac{5}{3}-\frac{2}{3}} = 3^1 = 3$

15. $(25^{1/8})^4 = 25^{4/8} = 25^{1/2} = 5$

16. $\left(13^{\sqrt{2}}\right)^{\sqrt{2}/2} = 13^{2/2} = 13$

17. $2^{\sqrt{3}} \cdot 7^{\sqrt{3}} = (2 \cdot 7)^{\sqrt{3}} = 14^{\sqrt{3}}$

18. $\left(\sqrt{3}\right)^{1/2}\left(\sqrt{12}\right)^{1/2} = \left(\sqrt{3} \cdot \sqrt{12}\right)^{1/2} = \left(\sqrt{36}\right)^{1/2} = 6^{1/2}$

19. $\left(\frac{2}{\sqrt{2}}\right)^4 = \frac{2^4}{(2^{1/2})^4} = \frac{16}{2^2} = 4$

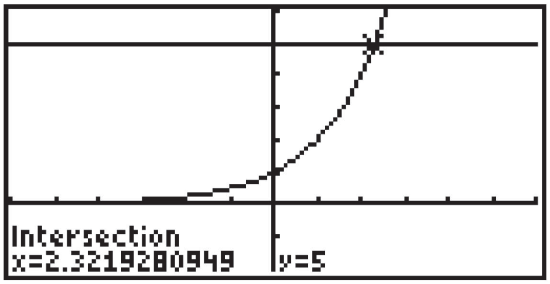
20. $\left(\frac{\sqrt{6}}{3}\right)^2 = \frac{(6^{1/2})^2}{3^2} = \frac{6}{9} = \frac{2}{3}$

21. Domain: $(-\infty, \infty)$; y in range $\Rightarrow y = \frac{1}{2+e^x}$. As x increases, e^x becomes infinitely large and y becomes a smaller and smaller positive real number. As x decreases, e^x becomes a smaller and smaller positive real number, $y < \frac{1}{2}$, and y gets arbitrarily close to $\frac{1}{2} \Rightarrow$ Range: $\left(0, \frac{1}{2}\right)$.

22. Domain: $(-\infty, \infty)$; y in range $\Rightarrow y = \cos(e^{-t})$. Since the values of e^{-t} are $(0, \infty)$ and $-1 \le \cos x \le 1 \Rightarrow$ Range: $[-1, 1]$.

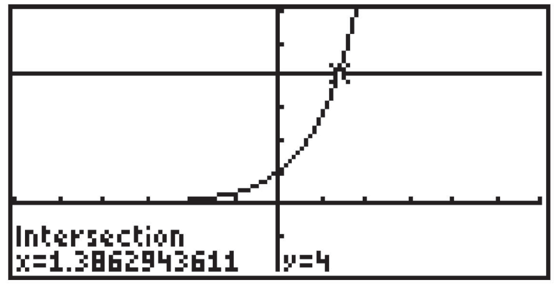
23. Domain: $(-\infty, \infty)$; y in range $\Rightarrow y = \sqrt{1+3^{-t}}$. Since the values of 3^{-t} are $(0, \infty) \Rightarrow$ Range: $(1, \infty)$.

24. If $e^{2x} = 1$, then $x = 0 \Rightarrow$ Domain: $(-\infty, 0) \cup (0, \infty)$; y in range $\Rightarrow y = \frac{3}{1-e^{2x}}$. If $x > 0$, then $1 < e^{2x} < \infty$ $\Rightarrow -\infty < y < 0$. If $x < 0$, then $0 < e^{2x} < 1 \Rightarrow 3 < y < \infty \Rightarrow$ Range: $(-\infty, 0) \cup (3, \infty)$.

25. 

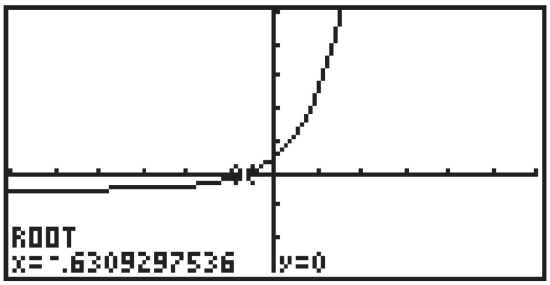
[-6, 6] by [-2, 6]

$x \approx 2.3219$

26. 

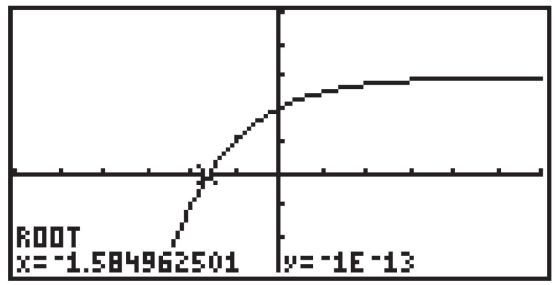
[-6, 6] by [-2, 6]

$x \approx 1.3863$

27. 

[-6, 6] by [-3, 5]

$x \approx -0.6309$

28. 

[-6, 6] by [-3, 5]

$x \approx -1.5850$

29. Let t be the number of years. Solving $500,000(1.0375)^t = 1,000,000$ graphically, we find that $t \approx 18.828$. The population will reach 1 million in about 19 years.

30. (a) The population is given by $P(t) = 6250(1.0275)^t$, where t is the number of years after 1890.
Population in 1915: $P(25) \approx 12,315$
Population in 1940: $P(50) \approx 24,265$
(b) Solving $P(t) = 50,000$ graphically, we find that $t \approx 76.651$. The population reached 50,000 about 77 years after 1890, in 1967.

31. (a) $A(t) = 6.6\left(\frac{1}{2}\right)^{t/14}$
(b) Solving $A(t) = 1$ graphically, we find that $t \approx 38$. There will be 1 gram remaining after about 38.1145 days.

32. Let t be the number of years. Solving $2300(1.60)^t = 4150$ graphically, we find that $t \approx 10.129$. It will take about 10.129 years. (If the interest is not credited to the account until the end of each year, it will take 11 years.)

33. Let A be the amount of the initial investment, and let t be the number of years. We wish to solve $A(1.0625)^t = 2A$, which is equivalent to $1.0625^t = 2$. Solving graphically, we find that $t \approx 11.433$. It will take about 11.433 years. (If the interest is credited at the end of each year, it will take 12 years.)

34. Let A be the amount of the initial investment, and let t be the number of years. We wish to solve $Ae^{0.0575t} = 3A$, which is equivalent to $e^{0.0575t} = 3$. Solving graphically, we find that $t \approx 19.106$. It will take about 19.106 years.

35. After t hours, the population is $P(t) = 2^{t/0.5}$, or equivalently, $P(t) = 2^{2t}$. After 24 hours, the population is $P(24) = 2^{48} \approx 2.815 \times 10^{14}$ bacteria.

36. (a) Each year, the number of cases is 100% − 20% = 80% of the previous year's number of cases. After t years, the number of cases will be $C(t) = 10,000(0.8)^t$. Solving $C(t) = 1000$ graphically, we find that $t \approx 10.319$. It will take 10.319 years.
(b) Solving $C(t) = 1$ graphically, we find that $t \approx 41.275$. It will take about 41.275 years.

1.6 INVERSE FUNCTIONS AND LOGARITHMS

1. Yes one-to-one, the graph passes the horizontal line test.

2. Not one-to-one, the graph fails the horizontal line test.

3. Not one-to-one since (for example) the horizontal line $y = 2$ intersects the graph twice.

4. Not one-to-one, the graph fails the horizontal line test.

5. Yes one-to-one, the graph passes the horizontal line test.

6. Yes one-to-one, the graph passes the horizontal line test.

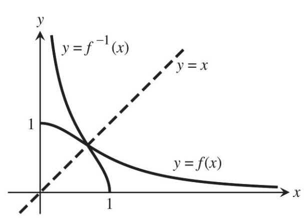
7. Not one-to-one since the horizontal line $y = 3$ intersects the graph an infinite number of times.

8. Yes one-to-one, the graph passes the horizontal line test.

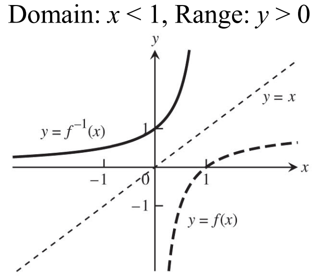
9. Yes one-to one, graph passes the horizontal line test.

10. Not one-to-one since (for example) the horizontal line $y = 1$ intersects the graph twice.

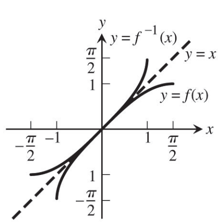
11. Domain: $0 < x \le 1$, Range: $0 \le y$



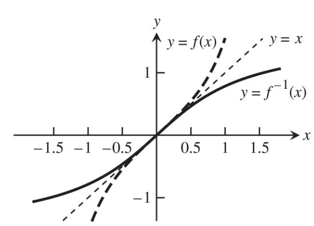
12. Domain: $x < 1$, Range: $y > 0$



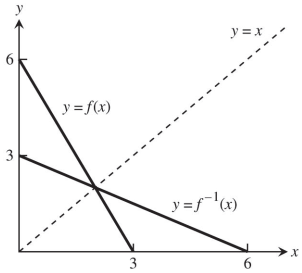
13. Domain: $-1 \le x \le 1$, Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$



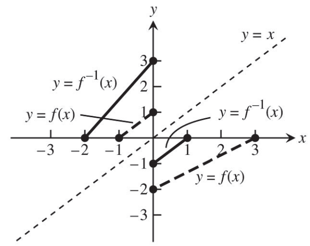
14. Domain: $-\infty < x < \infty$, Range: $-\frac{\pi}{2} < y \le \frac{\pi}{2}$



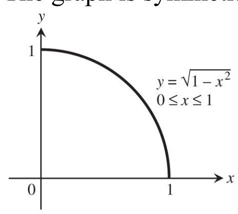
15. Domain: $0 \le x \le 6$, Range: $0 \le y \le 3$



16. Domain: $-2 \le x \le 1$, Range: $-1 \le y < 3$



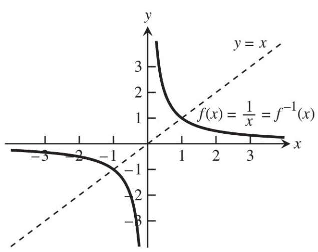
17. The graph is symmetric about $y = x$.



(b) $y = \sqrt{1-x^2} \Rightarrow y^2 = 1-x^2 \Rightarrow x^2 = 1-y^2 \Rightarrow x = \sqrt{1-y^2} \Rightarrow y = \sqrt{1-x^2} = f^{-1}(x)$

18. (a) The graph is symmetric about $y = x$.

(b) $y = \frac{1}{x} \Rightarrow x = \frac{1}{y} \Rightarrow y = \frac{1}{x} = f^{-1}(x)$



19. Step 1: $y = x^2 + 1 \Rightarrow x^2 = y - 1 \Rightarrow x = \sqrt{y-1}$
Step 2: $y = \sqrt{x-1} = f^{-1}(x)$

20. Step 1: $y = x^2 \Rightarrow x = -\sqrt{y}$, since $x \le 0$.
Step 2: $y = -\sqrt{x} = f^{-1}(x)$

21. Step 1: $y = x^3 - 1 \Rightarrow x^3 = y + 1 \Rightarrow x = (y+1)^{1/3}$
Step 2: $y = \sqrt[3]{x+1} = f^{-1}(x)$

22. Step 1: $y = x^2 - 2x + 1 \Rightarrow y = (x-1)^2 \Rightarrow \sqrt{y} = x - 1$, since $x \ge 1 \Rightarrow x = 1 + \sqrt{y}$
Step 2: $y = 1 + \sqrt{x} = f^{-1}(x)$

23. Step 1: $y = (x+1)^2 \Rightarrow \sqrt{y} = x + 1$, since $x \ge -1 \Rightarrow x = \sqrt{y} - 1$
Step 2: $y = \sqrt{x} - 1 = f^{-1}(x)$

24. Step 1: $y = x^{2/3} \Rightarrow x = y^{3/2}$

Step 2: $y = x^{3/2} = f^{-1}(x)$

25. Step 1: $y = x^5 \Rightarrow x = y^{1/5}$

Step 2: $y = \sqrt[5]{x} = f^{-1}(x)$;

Domain and Range of f^{-1}: all reals;

$f(f^{-1}(x)) = (x^{1/5})^5 = x$ and $f^{-1}(f(x)) = (x^5)^{1/5} = x$

26. Step 1: $y = x^4 \Rightarrow x = y^{1/4}$

Step 2: $y = \sqrt[4]{x} = f^{-1}(x)$;

Domain of f^{-1}: $x \geq 0$, Range of f^{-1}: $y \geq 0$;

$f(f^{-1}(x)) = (x^{1/4})^4 = x$ and $f^{-1}(f(x)) = (x^4)^{1/4} = x$

27. Step 1: $y = x^3 + 1 \Rightarrow x^3 = y - 1 \Rightarrow x = (y-1)^{1/3}$

Step 2: $y = \sqrt[3]{x-1} = f^{-1}(x)$;

Domain and Range of f^{-1}: all reals;

$f(f^{-1}(x)) = ((x-1)^{1/3})^3 + 1 = (x-1) + 1 = x$ and $f^{-1}(f(x)) = ((x^3+1)-1)^{1/3} = (x^3)^{1/3} = x$

28. Step 1: $y = \frac{1}{2}x - \frac{7}{2} \Rightarrow \frac{1}{2}x = y + \frac{7}{2} \Rightarrow x = 2y + 7$

Step 2: $y = 2x + 7 = f^{-1}(x)$;

Domain and Range of f^{-1}: all reals;

$f(f^{-1}(x)) = \frac{1}{2}(2x+7) - \frac{7}{2} = \left(x + \frac{7}{2}\right) - \frac{7}{2} = x$ and $f^{-1}(f(x)) = 2\left(\frac{1}{2}x - \frac{7}{2}\right) + 7 = (x-7) + 7 = x$

29. Step 1: $y = \frac{1}{x^2} \Rightarrow x^2 = \frac{1}{y} \Rightarrow x = \frac{1}{\sqrt{y}}$

Step 2: $y = \frac{1}{\sqrt{x}} = f^{-1}(x)$

Domain of f^{-1}: $x > 0$, Range of f^{-1}: $y > 0$;

$f(f^{-1}(x)) = \dfrac{1}{\left(\frac{1}{\sqrt{x}}\right)^2} = \dfrac{1}{\left(\frac{1}{x}\right)} = x$ and $f^{-1}(f(x)) = \dfrac{1}{\sqrt{\frac{1}{x^2}}} = \dfrac{1}{\left(\frac{1}{x}\right)} = x$ since $x > 0$.

30. Step 1: $y = \frac{1}{x^3} \Rightarrow x^3 = \dfrac{1}{y} \Rightarrow x = \dfrac{1}{y^{1/3}}$

Step 2: $y = \dfrac{1}{x^{1/3}} = \sqrt[3]{\frac{1}{x}} = f^{-1}(x)$;

Domain of f^{-1}: $x \neq 0$, Range of f^{-1}: $y \neq 0$;

$f(f^{-1}(x)) = \dfrac{1}{(x^{-1/3})^3} = \dfrac{1}{x^{-1}} = x$ and $f^{-1}(f(x)) = \left(\frac{1}{x^3}\right)^{-1/3} = \left(\frac{1}{x}\right)^{-1} = x$

31. Step 1: $y = \frac{x+3}{x-2} \Rightarrow y(x-2) = x+3 \Rightarrow xy - 2y = x+3 \Rightarrow xy - x = 2y+3 \Rightarrow x = \frac{2y+3}{y-1}$

Step 2: $y = \frac{2x+3}{x-1} = f^{-1}(x)$;

Domain of f^{-1}: $x \neq 1$, Range of f^{-1}: $y \neq 2$;

$f(f^{-1}(x)) = \frac{\left(\frac{2x+3}{x-1}\right)+3}{\left(\frac{2x+3}{x-1}\right)-2} = \frac{(2x+3)+3(x-1)}{(2x+3)-2(x-1)} = \frac{5x}{5} = x$ and $f^{-1}(f(x)) = \frac{2\left(\frac{x+3}{x-2}\right)+3}{\left(\frac{x+3}{x-2}\right)-1} = \frac{2(x+3)+3(x-2)}{(x+3)-(x-2)} = \frac{5x}{5} = x$

32. Step 1: $y = \frac{\sqrt{x}}{\sqrt{x}-3} \Rightarrow y\left(\sqrt{x}-3\right) = \sqrt{x} \Rightarrow y\sqrt{x} - 3y = \sqrt{x} \Rightarrow y\sqrt{x} - \sqrt{x} = 3y \Rightarrow x = \left(\frac{3y}{y-1}\right)^2$

Step 2: $y = \left(\frac{3x}{x-1}\right)^2 = f^{-1}(x)$;

Domain of f^{-1}: $(-\infty, 0] \cup (1, \infty)$, Range of f^{-1}: $[0, 9) \cup (9, \infty)$;

$f(f^{-1}(x)) = \frac{\sqrt{\left(\frac{3x}{x-1}\right)^2}}{\sqrt{\left(\frac{3x}{x-1}\right)^2}-3}$; If $x > 1$ or $x \le 0 \Rightarrow \frac{3x}{x-1} \ge 0 \Rightarrow \frac{\sqrt{\left(\frac{3x}{x-1}\right)^2}}{\sqrt{\left(\frac{3x}{x-1}\right)^2}-3} = \frac{\frac{3x}{x-1}}{\frac{3x}{x-1}-3} = \frac{3x}{3x-3(x-1)} = \frac{3x}{3} = x$ and

$$f^{-1}(f(x)) = \left(\frac{3\left(\frac{\sqrt{x}}{\sqrt{x}-3}\right)}{\left(\frac{\sqrt{x}}{\sqrt{x}-3}\right)-1}\right)^2 = \frac{9x}{\left(\sqrt{x}-\left(\sqrt{x}-3\right)\right)^2} = \frac{9x}{9} = x$$

33. Step 1: $y = x^2 - 2x,\ x \le 1 \Rightarrow y+1 = (x-1)^2,\ x \le 1 \Rightarrow -\sqrt{y+1} = x-1,\ x \le 1 \Rightarrow x = 1-\sqrt{y+1}$

Step 2: $y = 1-\sqrt{x+1} = f^{-1}(x)$;

Domain of f^{-1}: $[-1, \infty)$, Range of f^{-1}: $(-\infty, 1]$;

$f(f^{-1}(x)) = \left(1-\sqrt{x+1}\right)^2 - 2\left(1-\sqrt{x+1}\right) = 1-2\sqrt{x+1}+x+1-2+2\sqrt{x+1} = x$ and

$f^{-1}(f(x)) = 1-\sqrt{(x^2-2x)+1},\ x \le 1 = 1-\sqrt{(x-1)^2},\ x \le 1 = 1-|x-1| = 1-(1-x) = x$

34. Step 1: $y = (2x^3+1)^{1/5} \Rightarrow y^5 = 2x^3+1 \Rightarrow y^5 - 1 = 2x^3 \Rightarrow \frac{y^5-1}{2} = x^3 \Rightarrow x = \sqrt[3]{\frac{y^5-1}{2}}$

Step 2: $y = \sqrt[3]{\frac{x^5-1}{2}} = f^{-1}(x)$;

Domain of f^{-1}: $(-\infty, \infty)$, Range of f^{-1}: $(-\infty, \infty)$;

$f(f^{-1}(x)) = \left(2\left(\sqrt[3]{\frac{x^5-1}{2}}\right)^3+1\right)^{1/5} = \left(2\left(\frac{x^5-1}{2}\right)+1\right)^{1/5} = ((x^5-1)+1)^{1/5} = (x^5)^{1/5} = x$ and

$f^{-1}(f(x)) = \sqrt[3]{\frac{\left[(2x^3+1)^{1/5}\right]^5-1}{2}} = \sqrt[3]{\frac{(2x^3+1)-1}{2}} = \sqrt[3]{\frac{2x^3}{2}} = x$

35. $y = \frac{x+b}{x-2} = \frac{x-2+2+b}{x-2} = 1+\frac{2+b}{x-2}$, $\quad x = \frac{2y+b}{y-1}$,

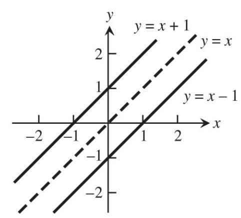
$f^{-1}(x) = \frac{2x+b}{x-1}$

36. Since $x \le b$, $x^2 - 2bx - y = 0$, $x = \frac{2b \pm \sqrt{4b^2 + 4y}}{2} = b - \sqrt{b^2 + y}$,
$x = b - \sqrt{b^2 + y}$, $f^{-1}(x) = b + \sqrt{b^2 - x}$

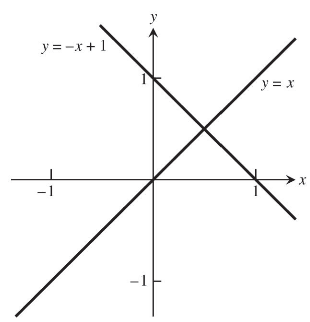
37. (a) $y = mx \Rightarrow x = \frac{1}{m}y \Rightarrow f^{-1}(x) = \frac{1}{m}x$
(b) The graph of $y = f^{-1}(x)$ is a line through the origin with slope $\frac{1}{m}$.

38. $y = mx + b \Rightarrow x = \frac{y}{m} - \frac{b}{m} \Rightarrow f^{-1}(x) = \frac{1}{m}x - \frac{b}{m}$; the graph of $f^{-1}(x)$ is a line with slope $\frac{1}{m}$ and y-intercept $-\frac{b}{m}$.

39. (a) $y = x + 1 \Rightarrow x = y - 1 \Rightarrow f^{-1}(x) = x - 1$
(b) $y = x + b \Rightarrow x = y - b \Rightarrow f^{-1}(x) = x - b$
(c) Their graphs will be parallel to one another and lie on opposite sides of the line $y = x$ equidistant from that line.



40. (a) $y = -x + 1 \Rightarrow x = -y + 1 \Rightarrow f^{-1}(x) = 1 - x$; the lines intersect at a right angle
(b) $y = -x + b \Rightarrow x = -y + b \Rightarrow f^{-1}(x) = b - x$; the lines intersect at a right angle
(c) Such a function is its own inverse



41. (a) $\ln 0.75 = \ln\frac{3}{4} = \ln 3 - \ln 4 = \ln 3 - \ln 2^2 = \ln 3 - 2\ln 2$
(b) $\ln\frac{4}{9} = \ln 4 - \ln 9 = \ln 2^2 - \ln 3^2 = 2\ln 2 - 2\ln 3$
(c) $\ln\frac{1}{2} = \ln 1 - \ln 2 = -\ln 2$
(d) $\ln\sqrt[3]{9} = \frac{1}{3}\ln 9 = \frac{1}{3}\ln 3^2 = \frac{2}{3}\ln 3$
(e) $\ln 3\sqrt{2} = \ln 3 + \ln 2^{1/2} = \ln 3 + \frac{1}{2}\ln 2$
(f) $\ln\sqrt{13.5} = \frac{1}{2}\ln 13.5 = \frac{1}{2}\ln\frac{27}{2} = \frac{1}{2}(\ln 3^3 - \ln 2) = \frac{1}{2}(3\ln 3 - \ln 2)$

42. (a) $\ln\frac{1}{125} = \ln 1 - 3\ln 5 = -3\ln 5$
(b) $\ln 9.8 = \ln\frac{49}{5} = \ln 7^2 - \ln 5 = 2\ln 7 - \ln 5$
(c) $\ln 7\sqrt{7} = \ln 7^{3/2} = \frac{3}{2}\ln 7$
(d) $\ln 1225 = \ln 35^2 = 2\ln 35 = 2\ln 5 + 2\ln 7$
(e) $\ln 0.056 - \ln\frac{7}{125} = \ln 7 - \ln 5^3 = \ln 7 - 3\ln 5$
(f) $\frac{\ln 35 + \ln\frac{1}{7}}{\ln 25} = \frac{\ln 5 + \ln 7 - \ln 7}{2\ln 5} = \frac{1}{2}$

43. (a) $\ln \sin\theta - \ln\left(\frac{\sin\theta}{5}\right) = \ln\left(\frac{\sin\theta}{\left(\frac{\sin\theta}{5}\right)}\right) = \ln 5$ (b) $\ln(3x^2-9x)+\ln\left(\frac{1}{3x}\right) = \ln\left(\frac{3x^2-9x}{3x}\right) = \ln(x-3)$

(c) $\frac{1}{2}\ln(4t^4) - \ln 2 = \ln\sqrt{4t^4} - \ln 2 = \ln 2t^2 - \ln 2 = \ln\left(\frac{2t^2}{2}\right) = \ln(t^2)$

44. (a) $\ln \sec\theta + \ln\cos\theta = \ln[(\sec\theta)(\cos\theta)] = \ln 1 = 0$

(b) $\ln(8x+4) - \ln 2^2 = \ln(8x+4) - \ln 4 = \ln\left(\frac{8x+4}{4}\right) = \ln(2x+1)$

(c) $3\ln\sqrt[3]{t^2-1} - \ln(t+1) = 3\ln(t^2-1)^{1/3} - \ln(t+1) = 3\left(\frac{1}{3}\right)\ln(t^2-1) - \ln(t+1) = \ln\left(\frac{(t+1)(t-1)}{(t+1)}\right) = \ln(t-1)$

45. (a) $e^{\ln 7.2} = 7.2$ (b) $e^{-\ln x^2} = \frac{1}{e^{\ln x^2}} = \frac{1}{x^2}$ (c) $e^{\ln x - \ln y} = e^{\ln(x/y)} = \frac{x}{y}$

46. (a) $e^{\ln(x^2+y^2)} = x^2+y^2$ (b) $e^{-\ln 0.3} = \frac{1}{e^{\ln 0.3}} = \frac{1}{0.3}$ (c) $e^{\ln \pi x - \ln 2} = e^{\ln(\pi x/2)} = \frac{\pi x}{2}$

47. (a) $2\ln\sqrt{e} = 2\ln e^{1/2} = (2)\left(\frac{1}{2}\right)\ln e = 1$ (b) $\ln(\ln e^e) = \ln(e\ln e) = \ln e = 1$

(c) $\ln e^{(-x^2-y^2)} = (-x^2-y^2)\ln e = -x^2-y^2$

48. (a) $\ln\left(e^{\sec\theta}\right) = (\sec\theta)(\ln e) = \sec\theta$ (b) $\ln e^{(e^x)} = (e^x)(\ln e) = e^x$

(c) $\ln(e^{2\ln x}) = \ln\left(e^{\ln x^2}\right) = \ln x^2 = 2\ln x$

49. $\ln y = 2t+4 \Rightarrow e^{\ln y} = e^{2t+4} \Rightarrow y = e^{2t+4}$

50. $\ln y = -t+5 \Rightarrow e^{\ln y} = e^{-t+5} \Rightarrow y = e^{-t+5}$

51. $\ln(y-40) = 5t \Rightarrow e^{\ln(y-40)} = e^{5t} \Rightarrow y-40 = e^{5t} \Rightarrow y = e^{5t}+40$

52. $\ln(1-2y) = t \Rightarrow e^{\ln(1-2y)} = e^t \Rightarrow 1-2y = e^t \Rightarrow -2y = e^t - 1 \Rightarrow y = -\left(\frac{e^t-1}{2}\right)$

53. $\ln(y-1) - \ln 2 = x + \ln x \Rightarrow \ln(y-1) - \ln 2 - \ln x = x \Rightarrow \ln\left(\frac{y-1}{2x}\right) = x \Rightarrow e^{\ln\left(\frac{y-1}{2x}\right)} = e^x \Rightarrow \frac{y-1}{2x} = e^x$

$\Rightarrow y-1 = 2xe^x \Rightarrow y = 2xe^x+1$

54. $\ln(y^2-1) - \ln(y+1) = \ln(\sin x) \Rightarrow \ln\left(\frac{y^2-1}{y+1}\right) = \ln(\sin x) \Rightarrow \ln(y-1) = \ln(\sin x) \Rightarrow e^{\ln(y-1)} = e^{\ln(\sin x)}$

$\Rightarrow y-1 = \sin x \Rightarrow y = \sin x + 1$

55. (a) $e^{2k} = 4 \Rightarrow \ln e^{2k} = \ln 4 \Rightarrow 2k\ln e = \ln 2^2 \Rightarrow 2k = 2\ln 2 \Rightarrow k = \ln 2$

(b) $100e^{10k} = 200 \Rightarrow e^{10k} = 2 \Rightarrow \ln e^{10k} = \ln 2 \Rightarrow 10k\ln e = \ln 2 \Rightarrow 10k = \ln 2 \Rightarrow k = \frac{\ln 2}{10}$

(c) $e^{k/1000} = a \Rightarrow \ln e^{k/1000} = \ln a \Rightarrow \frac{k}{1000}\ln e = \ln a \Rightarrow \frac{k}{1000} = \ln a \Rightarrow k = 1000\ln a$

56. (a) $e^{5k}=\frac{1}{4}\Rightarrow \ln e^{5k}=\ln 4^{-1}\Rightarrow 5k\ln e=-\ln 4\Rightarrow 5k=-\ln 4\Rightarrow k=-\frac{\ln 4}{5}$

(b) $80e^{k}=1\Rightarrow e^{k}=80^{-1}\Rightarrow \ln e^{k}=\ln 80^{-1}\Rightarrow k\ln e=-\ln 80\Rightarrow k=-\ln 80$

(c) $e^{(\ln 0.8)k}=0.8\Rightarrow (e^{\ln 0.8})^{k}=0.8\Rightarrow (0.8)^{k}=0.8\Rightarrow k=1$

57. (a) $e^{-0.3t}=27\Rightarrow \ln e^{-0.3t}=\ln 3^{3}\Rightarrow (-0.3t)\ln e=3\ln 3\Rightarrow -0.3t=3\ln 3\Rightarrow t=-10\ln 3$

(b) $e^{kt}=\frac{1}{2}\Rightarrow \ln e^{kt}=\ln 2^{-1}=kt\ln e=-\ln 2\Rightarrow t=-\frac{\ln 2}{k}$

(c) $e^{(\ln 0.2)t}=0.4\Rightarrow (e^{\ln 0.2})^{t}=0.4\Rightarrow 0.2^{t}=0.4\Rightarrow \ln 0.2^{t}=\ln 0.4\Rightarrow t\ln 0.2=\ln 0.4\Rightarrow t=\frac{\ln 0.4}{\ln 0.2}$

58. (a) $e^{-0.01t}=1000\Rightarrow \ln e^{-0.01t}=\ln 1000\Rightarrow (-0.01t)\ln e=\ln 1000\Rightarrow -0.01t=\ln 1000\Rightarrow t=-100\ln 1000$

(b) $e^{kt}=\frac{1}{10}\Rightarrow \ln e^{kt}=\ln 10^{-1}=kt\ln e=-\ln 10\Rightarrow kt=-\ln 10\Rightarrow t=-\frac{\ln 10}{k}$

(c) $e^{(\ln 2)t}=\frac{1}{2}\Rightarrow (e^{\ln 2})^{t}=2^{-1}\Rightarrow 2^{t}=2^{-1}\Rightarrow t=-1$

59. $e^{\sqrt{t}}=x^{2}\Rightarrow \ln e^{\sqrt{t}}=\ln x^{2}\Rightarrow \sqrt{t}=2\ln x\Rightarrow t=4(\ln x)^{2}$

60. $e^{x^{2}}e^{2x+1}=e^{t}\Rightarrow e^{x^{2}+2x+1}=e^{t}\Rightarrow \ln e^{x^{2}+2x+1}=\ln e^{t}\Rightarrow t=x^{2}+2x+1$

61. $e^{2t}-3e^{t}=(e^{t})^{2}-3e^{t}=e^{t}(e^{t}-3)=0\Rightarrow e^{t}-3=0\Rightarrow e^{t}=3\Rightarrow \ln e^{t}=\ln 3\Rightarrow t=\ln 3$

62. $e^{-2t}+6=5e^{-t}\Rightarrow (e^{-t})^{2}-5e^{-t}+6=0\Rightarrow (e^{-t}-3)(e^{-t}-2)=0\Rightarrow e^{-t}-3=0$ or $e^{-t}-2=0\Rightarrow e^{-t}=3$ or $e^{-t}=2\Rightarrow \ln e^{-t}=\ln 3$ or $\ln e^{-t}=\ln 2\Rightarrow -t=\ln 3$ or $-t=\ln 2\Rightarrow t=-\ln 3$ or $t=-\ln 2$

63. $\ln\left(\frac{t}{t-1}\right)=2\Rightarrow e^{\ln\left(\frac{t}{t-1}\right)}=e^{2}\Rightarrow \frac{t}{t-1}=e^{2}\Rightarrow t=e^{2}t-e^{2}\Rightarrow e^{2}=e^{2}t-t=(e^{2}-1)t\Rightarrow t=\frac{e^{2}}{e^{2}-1}$

64. $\ln(t-2)=\ln 8-\ln t\Rightarrow \ln(t-2)+\ln t=\ln 8\Rightarrow \ln((t-2)t)=\ln 8\Rightarrow e^{\ln(t^{2}-2t)}=e^{\ln 8}\Rightarrow t^{2}-2t=8\Rightarrow t^{2}-2t-8=(t-4)(t+2)=0\Rightarrow t-4=0\Rightarrow t=4$

65. (a) $5^{\log_5 7}=7$ (b) $8^{\log_8 \sqrt{2}}=\sqrt{2}$ (c) $1.3^{\log_3 75}=75$

(d) $\log_4 16=\log_4 4^{2}=2\log_4 4=2\cdot 1=2$

(e) $\log_3 \sqrt{3}=\log_3 3^{1/2}=\frac{1}{2}\log_3 3=\frac{1}{2}\cdot 1=\frac{1}{2}=0.5$

(f) $\log_4\left(\frac{1}{4}\right)=\log_4 4^{-1}=-1\log_4 4=-1\cdot 1=-1$

66. (a) $2^{\log_2 3}=3$ (b) $10^{\log_{10}(1/2)}=\frac{1}{2}$ (c) $\pi^{\log_\pi 7}=7$

(d) $\log_{11} 121=\log_{11} 11^{2}=2\log_{11} 11=2\cdot 1=2$

(e) $\log_{121} 11=\log_{121} 121^{1/2}=\left(\frac{1}{2}\right)\log_{121} 121=\left(\frac{1}{2}\right)\cdot 1=\frac{1}{2}$

(f) $\log_3\left(\frac{1}{9}\right)=\log_3 3^{-2}=-2\log_3 3=-2\cdot 1=-2$

67. (a) Let $z = \log_4 x \Rightarrow 4^z = x \Rightarrow 2^{2z} = x \Rightarrow (2^z)^2 = x \Rightarrow 2^z = \sqrt{x}$

(b) Let $z = \log_3 x \Rightarrow 3^z = x \Rightarrow (3^z)^2 = x^2 \Rightarrow 3^{2z} = x^2 \Rightarrow 9^z = x^2$

(c) $\log_2(e^{(\ln 2)\sin x}) = \log_2 2^{\sin x} = \sin x$

68. (a) Let $z = \log_5(3x^2) \Rightarrow 5^z = 3x^2 \Rightarrow 25^z = 9x^4$

(b) $\log_e(e^x) = x$

(c) $\log_4(2^{e^x \sin x}) = \log_4 4^{(e^x \sin x)/2} = \frac{e^x \sin x}{2}$

69. (a) $\frac{\log_2 x}{\log_3 x} = \frac{\ln x}{\ln 2} \div \frac{\ln x}{\ln 3} = \frac{\ln x}{\ln 2} \cdot \frac{\ln 3}{\ln x} = \frac{\ln 3}{\ln 2}$

(b) $\frac{\log_2 x}{\log_8 x} = \frac{\ln x}{\ln 2} \div \frac{\ln x}{\ln 8} = \frac{\ln x}{\ln 2} \cdot \frac{\ln 8}{\ln x} = \frac{3\ln 2}{\ln 2} = 3$

(c) $\frac{\ln_x a}{\ln_{x^2} a} = \frac{\ln a}{\ln x} \div \frac{\ln a}{\ln x^2} = \frac{\ln a}{\ln x} \cdot \frac{\ln x^2}{\ln a} = \frac{2\ln x}{\ln x} = 2$

70. (a) $\frac{\log_9 x}{\log_3 x} = \frac{\ln x}{\ln 9} \div \frac{\ln x}{\ln 3} = \frac{\ln x}{2\ln 3} \cdot \frac{\ln 3}{\ln x} = \frac{1}{2}$

(b) $\frac{\log_{\sqrt{10}} x}{\log_{\sqrt{2}} x} = \frac{\ln x}{\ln\sqrt{10}} \div \frac{\ln x}{\ln\sqrt{2}} = \frac{\ln x}{\left(\frac{1}{2}\right)\ln 10} \cdot \frac{\left(\frac{1}{2}\right)\ln 2}{\ln x} = \frac{\ln 2}{\ln 10}$

(c) $\frac{\log_a b}{\log_b a} = \frac{\ln b}{\ln a} \div \frac{\ln a}{\ln b} = \frac{\ln b}{\ln a} \cdot \frac{\ln b}{\ln a} = \left(\frac{\ln b}{\ln a}\right)^2$

71. (a) $-\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $-\frac{\pi}{3}$

72. (a) $\frac{\pi}{3}$ (b) $\frac{3\pi}{4}$ (c) $\frac{\pi}{6}$

73. (a) $\arccos(-1) = \pi$ since $\cos(\pi) = -1$ and $0 \le \pi \le \pi$.

(b) $\arccos(0) = \frac{\pi}{2}$ since $\cos\left(\frac{\pi}{2}\right) = 0$ and $0 \le \frac{\pi}{2} \le \pi$.

74. (a) $\arcsin(-1) = -\frac{\pi}{2}$ since $\sin\left(-\frac{\pi}{2}\right) = -1$ and $-\frac{\pi}{2} \le -\frac{\pi}{2} \le \frac{\pi}{2}$.

(b) $\arcsin\left(-\frac{1}{\sqrt{2}}\right) = -\frac{\pi}{4}$ since $\sin\left(-\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$ and $-\frac{\pi}{2} \le -\frac{\pi}{4} \le \frac{\pi}{2}$.

75. The function $g(x)$ is also one-to-one. The reasoning: $f(x)$ is one-to-one means that if $x_1 \ne x_2$ then $f(x_1) \ne f(x_2)$, so $-f(x_1) \ne -f(x_2)$ and therefore $g(x_1) \ne g(x_2)$. Therefore $g(x)$ is one-to-one as well.

76. The function $h(x)$ is also one-to-one. The reasoning: $f(x)$ is one-to-one means that if $x_1 \ne x_2$ then $f(x_1) \ne f(x_2)$, so $\frac{1}{f(x_1)} \ne \frac{1}{f(x_2)}$, and therefore $h(x_1) \ne h(x_2)$.

77. The composition is one-to-one also. The reasoning: If $x_1 \ne x_2$ then $g(x_1) \ne g(x_2)$ because g is one-to-one. Since $g(x_1) \ne g(x_2)$, we also have $f(g(x_1)) \ne f(g(x_2))$ because f is one-to-one; thus, $f \circ g$ is one-to-one because $x_1 \ne x_2 \Rightarrow f(g(x_1)) \ne f(g(x_2))$.

78. Yes, g must be one-to-one. If g were not one-to-one, there would exist numbers $x_1 \neq x_2$ in the domain of g with $g(x_1) = g(x_2)$. For these numbers we would also have $f(g(x_1)) = f(g(x_2))$, contradicting the assumption that $f \circ g$ is one-to-one.

79. (a) $y = \frac{100}{1+2^{-x}} \to 1+2^{-x} = \frac{100}{y} \to 2^{-x} = \frac{100}{y} - 1 \to \log_2(2^{-x}) = \log_2\left(\frac{100}{y} - 1\right) \to -x = \log_2\left(\frac{100}{y} - 1\right)$

$x = -\log_2\left(\frac{100}{y} - 1\right) = -\log_2\left(\frac{100-y}{y}\right) = \log_2\left(\frac{y}{100-y}\right)$.

Interchange x and y: $y = \log_2\left(\frac{x}{100-x}\right) \to f^{-1}(x) = \log_2\left(\frac{x}{100-x}\right)$

Verify.

$(f \circ f^{-1})(x) = f\left(\log_2\left(\frac{x}{100-x}\right)\right) = \frac{100}{1+2^{\log_2\left(\frac{x}{100-x}\right)}} = \frac{100}{1+2^{\log_2\left(\frac{100-x}{x}\right)}} = \frac{100}{1+\frac{100-x}{x}} = \frac{100}{x+100-x} = \frac{100x}{100} = x$

$(f^{-1} \circ f)(x) = f^{-1}\left(\frac{100}{1+2^{-x}}\right) = \log_2\left(\frac{\frac{100}{1+2^{-x}}}{100-\frac{100}{1+2^{-x}}}\right) = \log_2\left(\frac{100}{100(1+2^{-x})-100}\right) = \log_2\left(\frac{1}{2^{-x}}\right) = \log_2(2^x) = x$

(b) $y = \frac{50}{1+1.1^{-x}} \to 1+1.1^{-x} = \frac{50}{y} \to 1.1^{-x} = \frac{50}{y} - 1 \to \log_{1.1}(1.1^{-x}) = \log_{1.1}\left(\frac{50}{y} - 1\right) \to -x = \log_{1.1}\left(\frac{50}{y} - 1\right)$

$x = -\log_{1.1}\left(\frac{50}{y} - 1\right) = -\log_{1.1}\left(\frac{50-y}{y}\right) = \log_{1.1}\left(\frac{y}{50-y}\right)$.

Interchange x and y: $y = \log_{1.1}\left(\frac{x}{50-x}\right) \to f^{-1}(x) = \log_{1.1}\left(\frac{x}{50-x}\right)$

Verify.

$(f \circ f^{-1})(x) = f\left(\log_{1.1}\left(\frac{x}{50-x}\right)\right) = \frac{50}{1+1.1^{\log_{1.1}\left(\frac{x}{50-x}\right)}} = \frac{50}{1+1.1^{\log_{1.1}\left(\frac{50-x}{x}\right)}} = \frac{50}{1+\frac{50-x}{x}} = \frac{50x}{x+50-x} = \frac{50x}{50} = x$

$(f^{-1} \circ f)(x) = f^{-1}\left(\frac{50}{1+1.1^{-x}}\right) = \log_{1.1}\left(\frac{\frac{50}{1+1.1^{-x}}}{50-\frac{50}{1+1.1^{-x}}}\right) = \log_{1.1}\left(\frac{50}{50(1+1.1^{-x})-50}\right) = \log_{1.1}\left(\frac{1}{1.1^{-x}}\right) = \log_{1.1}(1.1^x) = x$

(c) $y = \frac{e^x-1}{e^x+1} \Rightarrow (e^x+1)y = e^x - 1 \Rightarrow e^x y + y = e^x - 1 \Rightarrow y+1 = e^x - e^x y = (1-y)e^x \Rightarrow e^x = \frac{y+1}{1-y} \Rightarrow$

$\ln e^x = \ln\left(\frac{y+1}{1-y}\right) \Rightarrow x = \ln\left(\frac{y+1}{1-y}\right)$.

Interchange x and y: $y = \ln\left(\frac{x+1}{1-x}\right) \Rightarrow f^{-1}(x) = \ln\left(\frac{x+1}{1-x}\right)$.

Verify.

$(f \circ f^{-1})(x) = f\left(\ln\left(\frac{x+1}{1-x}\right)\right) = \frac{e^{\ln\left(\frac{x+1}{1-x}\right)}-1}{e^{\ln\left(\frac{x+1}{1-x}\right)}+1} = \frac{\frac{x+1}{1-x} - \frac{1-x}{1-x}}{\frac{x+1}{1-x} + \frac{1-x}{1-x}} = \frac{\frac{x+1-1+x}{1-x}}{\frac{x+1+1-x}{1-x}} = \frac{2x}{1-x} \cdot \frac{1-x}{2} = x.$

$(f^{-1} \circ f)(x) = f^{-1}\left(\frac{e^x-1}{e^x+1}\right) = \ln\left(\frac{\frac{e^x-1}{e^x+1}+1}{1-\frac{e^x-1}{e^x+1}}\right) = \ln\left(\frac{e^x-1+(e^x+1)}{e^x+1-(e^x-1)}\right) = \ln\left(\frac{2e^x}{2}\right) = \ln e^x = x$

(d) $y = \frac{\ln x}{2-\ln x} \Rightarrow (2-\ln x)y = \ln x \Rightarrow 2y - y\ln x = \ln x \Rightarrow 2y = y\ln x + \ln x \Rightarrow 2y = (y+1)\ln x \Rightarrow \ln x = \frac{2y}{y+1} \Rightarrow$

$e^{\ln x} = e^{\frac{2y}{y+1}} \Rightarrow x = e^{\frac{2y}{y+1}}$

Interchange x and y: $y = e^{\frac{2x}{x+1}} \Rightarrow f^{-1}(x) = e^{\frac{2x}{x+1}}$.

Verify.

$$(f\circ f^{-1})(x)=f\left(e^{\frac{2x}{x+1}}\right)=\frac{\ln\left(e^{\frac{2x}{x+1}}\right)}{2-\ln\left(e^{\frac{2x}{x+1}}\right)}=\frac{\frac{2x}{x+1}}{2-\frac{2x}{x+1}}\cdot\frac{x+1}{x+1}=\frac{2x}{2(x+1)-2x}=\frac{2x}{2}=x.$$

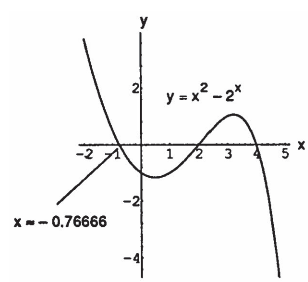
$$(f^{-1}\circ f)(x)=f^{-1}\left(\frac{\ln x}{2-\ln x}\right)=e^{\left[\frac{2\left(\frac{\ln x}{2-\ln x}\right)}{\left(\frac{\ln x}{2-\ln x}\right)+1}\right]}=e^{\left[\frac{2\ln x}{\ln x+(2-\ln x)}\right]}=e^{\frac{2\ln x}{2}}=e^{\ln x}=x$$

80. $\sin^{-1}(1)+\cos^{-1}(1)=\frac{\pi}{2}+0=\frac{\pi}{2}$; $\sin^{-1}(0)+\cos^{-1}(0)=0+\frac{\pi}{2}=\frac{\pi}{2}$; and $\sin^{-1}(-1)+\cos^{-1}(-1)=-\frac{\pi}{2}+\pi=\frac{\pi}{2}$.
If $x\in(-1,0)$ and $x=-a$, then $\sin^{-1}(x)+\cos^{-1}(x)=\sin^{-1}(-a)+\cos^{-1}(-a)=-\sin^{-1}a+(\pi-\cos^{-1}a)$
$=\pi-(\sin^{-1}a+\cos^{-1}a)=\pi-\frac{\pi}{2}=\frac{\pi}{2}$ from Equations (3) and (4) in the text.

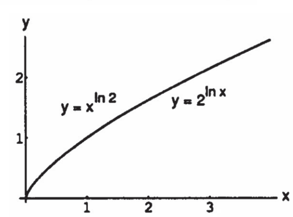
81. (a) Begin with $y=\ln x$ and reduce the y-value by 3 $\Rightarrow y=\ln x-3$.
(b) Begin with $y=\ln x$ and replace x with $x-1\Rightarrow y=\ln(x-1)$.
(c) Begin with $y=\ln x$, replace x with $x+1$, and increase the y-value by 3 $\Rightarrow y=\ln(x+1)+3$.
(d) Begin with $y=\ln x$, reduce the y-value by 4, and replace x with $x-2\Rightarrow y=\ln(x-2)-4$.
(e) Begin with $y=\ln x$ and replace x with $-x\Rightarrow y=\ln(-x)$.
(f) Begin with $y=\ln x$ and switch x and $y\Rightarrow x=\ln y$ or $y=e^x$.

82. (a) Begin with $y=\ln x$ and multiply the y-value by 2 $\Rightarrow y=2\ln x$.
(b) Begin with $y=\ln x$ and replace x with $\frac{x}{3}\Rightarrow y=\ln\left(\frac{x}{3}\right)$.
(c) Begin with $y=\ln x$ and multiply the y-value by $\frac{1}{4}\Rightarrow y=\frac{1}{4}\ln x$.
(d) Begin with $y=\ln x$ and replace x with $2x\Rightarrow y=\ln 2x$.

83. From zooming in on the graph at the right, we estimate the third root to be $x\approx-0.76666$.



84. The functions $f(x)=x^{\ln 2}$ and $g(x)=2^{\ln x}$ appear to have identical graphs for $x>0$. This is no accident, because $x^{\ln 2}=e^{\ln 2\cdot\ln x}=(e^{\ln 2})^{\ln x}=2^{\ln x}$.



85. (a) Amount $=8\left(\frac{1}{2}\right)^{t/12}$
(b) $8\left(\frac{1}{2}\right)^{t/12}=1\rightarrow\left(\frac{1}{2}\right)^{t/12}=\frac{1}{8}\rightarrow\left(\frac{1}{2}\right)^{t/12}=\left(\frac{1}{2}\right)^3\rightarrow\frac{t}{12}=3\rightarrow t=36$
There will be 1 gram remaining after 36 hours.

86. $500(1.0475)^t=1000\rightarrow1.0475^t=2\rightarrow\ln(1.0475^t)=\ln(2)\rightarrow t\ln(1.0475)=\ln(2)\rightarrow t=\frac{\ln(2)}{\ln(1.0475)}\approx14.936$
It will take about 14.936 years. (If the interest is paid at the end of each year, it will take 15 years.)

87. $375,000(1.0225)^t = 1,000,000 \to 1.0225^t = \frac{8}{3} \to \ln(1.0225^t) = \ln\left(\frac{8}{3}\right) \to t\ln(1.0225) = \ln\left(\frac{8}{3}\right)$

$\to t = \frac{\ln\left(\frac{8}{3}\right)}{\ln(1.0225)} \approx 44.081$

It will take about 44.081 years.

88. $y = y_0 e^{-0.18t}$ represents the decay equation; solving $(0.9)y_0 = y_0 e^{-0.18t} \Rightarrow t = \frac{\ln(0.9)}{-0.18} \approx 0.585$ days

CHAPTER 1 PRACTICE EXERCISES

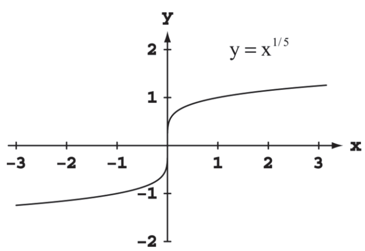
1. The area is $A = \pi r^2$ and the circumference is $C = 2\pi r$. Thus, $r = \frac{C}{2\pi} \Rightarrow A = \pi\left(\frac{C}{2\pi}\right)^2 = \frac{C^2}{4\pi}$.

2. The surface area is $S = 4\pi r^2 \Rightarrow r = \left(\frac{S}{4\pi}\right)^{1/2}$. The volume is $V = \frac{4}{3}\pi r^3 \Rightarrow r = \sqrt[3]{\frac{3V}{4\pi}}$. Substitution into the formula for surface area gives $S = 4\pi r^2 = 4\pi\left(\frac{3V}{4\pi}\right)^{2/3}$.

3. The coordinates of a point on the parabola are (x, x^2). The angle of inclination θ joining this point to the origin satisfies the equation $\tan\theta = \frac{x^2}{x} = x$. Thus the point has coordinates $(x, x^2) = (\tan\theta, \tan^2\theta)$.

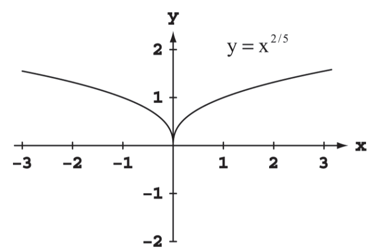
4. $\tan\theta = \frac{\text{rise}}{\text{run}} = \frac{h}{500} \Rightarrow h = 500\tan\theta$ ft.

5.



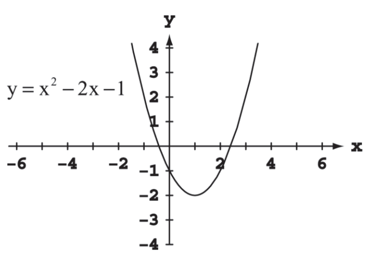
Symmetric about the origin.

6.



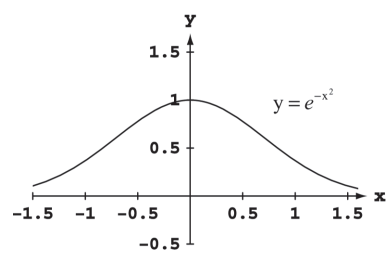
Symmetric about the y-axis.

7.



Neither

8.



Symmetric about the y-axis.

9. $y(-x) = (-x)^2 + 1 = x^2 + 1 = y(x)$. Even.

10. $y(-x) = (-x)^5 - (-x)^3 - (-x) = -x^5 + x^3 + x = -y(x)$. Odd.

11. $y(-x) = 1 - \cos(-x) = 1 - \cos x = y(x)$. Even.

12. $y(-x)=\sec(-x)\tan(-x)=\frac{\sin(-x)}{\cos^2(-x)}=\frac{-\sin x}{\cos^2 x}=-\sec x\tan x=-y(x)$. Odd.

13. $y(-x)=\frac{(-x)^4+1}{(-x)^3-2(-x)}=\frac{x^4+1}{-x^3+2x}=-\frac{x^4+1}{x^3-2x}=-y(x)$. Odd.

14. $y(-x)=(-x)-\sin(-x)=(-x)+\sin x=-(x-\sin x)=-y(x)$. Odd.

15. $y(-x)=-x+\cos(-x)=-x+\cos x$. Neither even nor odd.

16. $y(-x)=(-x)\cos(-x)=-x\cos x=-y(x)$. Odd.

17. Since f and g are odd $\Rightarrow f(-x)=-f(x)$ and $g(-x)=-g(x)$.
    (a) $(f\cdot g)(-x)=f(-x)g(-x)=[-f(x)][-g(x)]=f(x)g(x)=(f\cdot g)(x)\Rightarrow f\cdot g$ is even.
    (b) $f^3(-x)=f(-x)f(-x)f(-x)=[-f(x)][-f(x)][-f(x)]=-f(x)\cdot f(x)\cdot f(x)=-f^3(x)\Rightarrow f^3$ is odd.
    (c) $f(\sin(-x))=f(-\sin(x))=-f(\sin(x))\Rightarrow f(\sin(x))$ is odd.
    (d) $g(\sec(-x))=g(\sec(x))\Rightarrow g(\sec(x))$ is even.
    (e) $|g(-x)|=|-g(x)|=|g(x)|\Rightarrow |g|$ is even.

18. Let $f(a-x)=f(a+x)$ and define $g(x)=f(x+a)$. Then $g(-x)=f((-x)+a)=f(a-x)=f(a+x)=f(x+a)=g(x)\Rightarrow g(x)=f(x+a)$ is even.

19. (a) The function is defined for all values of x, so the domain is $(-\infty,\infty)$.
    (b) Since $|x|$ attains all nonnegative values, the range is $[-2,\infty)$.

20. (a) Since the square root requires $1-x\ge 0$, the domain is $(-\infty,1]$.
    (b) Since $\sqrt{1-x}$ attains all nonnegative values, the range is $[-2,\infty)$.

21. (a) Since the square root requires $16-x^2\ge 0$, the domain is $[-4,\ 4]$.
    (b) For values of x in the domain, $0\le 16-x^2\le 16$, so $0\le\sqrt{16-x^2}\le 4$. The range is $[0,4]$.

22. (a) The function is defined for all values of x, so the domain is $(-\infty,\infty)$.
    (b) Since 3^{2-x} attains all positive values, the range is $(1,\ \infty)$.

23. (a) The function is defined for all values of x, so the domain is $(-\infty,\infty)$.
    (b) Since $2e^{-x}$ attains all positive values, the range is $(-3,\infty)$.

24. (a) The function is equivalent to $y=\tan 2x$, so we require $2x\ne\frac{k\pi}{2}$ for odd integers k. The domain is given by $x\ne\frac{k\pi}{4}$ for odd integers k.
    (b) Since the tangent function attains all values, the range is $(-\infty,\infty)$.

25. (a) The function is defined for all values of x, so the domain is $(-\infty,\infty)$.
    (b) The sine function attains values from -1 to 1, so $-2\le 2\sin(3x+\pi)\le 2$ and hence $-3\le 2\sin(3x+\pi)-1\le 1$. The range is $[-3,\ 1]$.

26. (a) The function is defined for all values of x, so the domain is $(-\infty,\ \infty)$.
    (b) The function is equivalent to $y=\sqrt[5]{x^2}$, which attains all nonnegative values. The range is $[0,\infty)$.

27. (a) The logarithm requires $x-3>0$, so the domain is $(3,\infty)$.
    (b) The logarithm attains all real values, so the range is $(-\infty,\infty)$.

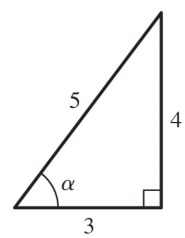
28. (a) The function is defined for all values of x, so the domain is $(-\infty, \infty)$.
(b) The cube root attains all real values, so the range is $(-\infty, \infty)$.

29. $y = 5 - \sqrt{(x-3)(x+1)}$ so the domain $= (-\infty, -1] \cup [3, \infty)$; $\sqrt{(x-3)(x+1)} \geq 0$ and can be any positive number, so the range $= (-\infty, 5]$.

30. $y = 2 + \frac{3x^2}{x^2+4}$ so the domain $= (-\infty, \infty)$; $0 \leq \frac{3x^2}{x^2+4} < 3$ so the range $= [2, 5)$.

31. $y = 4\sin\left(\frac{1}{x}\right)$ so the domain $= (-\infty, 0) \cup (0, \infty)$; if $\frac{2}{3\pi} \leq x \leq \frac{2}{\pi}$, then $-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$, so the range $= [-4, 4]$.

32. $y = 3\cos x + 4\sin x$ so the domain $= (-\infty, \infty)$;
and $\sqrt{3^2 + 4^2} = 5$ so $3\cos x + 4\sin x = 5\left(\frac{3}{5}\cos x + \frac{4}{5}\sin x\right)$
$= 5(\cos\alpha\cos x + \sin\alpha\sin x) = 5\cos(\alpha - x)$, where and
$-1 \leq \cos(\alpha - x) \leq 1$ so the range $= [-5, 5]$.



33. (a) Increasing because volume increases as radius increases.
(b) Neither, since the greatest integer function is composed of horizontal (constant) line segments.
(c) Decreasing because as the height increases, the atmospheric pressure decreases.
(d) Increasing because the kinetic (motion) energy increases as the particles velocity increases.

34. (a) Increasing on $[2, \infty)$
(b) Increasing on $[-1, \infty)$
(c) Increasing on $(-\infty, \infty)$
(d) Increasing on $\left[\frac{1}{2}, \infty\right)$

35. (a) The function is defined for $-4 \leq x \leq 4$, so the domain is $[-4, 4]$.
(b) The function is equivalent to $y = \sqrt{|x|}, -4 \leq x \leq 4$, which attains values from 0 to 2 for x in the domain. The range is $[0, 2]$.

36. (a) The function is defined for $-2 \leq x \leq 2$, so the domain is $[-2, 2]$.
(b) The range is $[-1, 1]$.

37. First piece: Line through (0, 1) and (1, 0). $m = \frac{0-1}{1-0} = \frac{-1}{1} = -1 \Rightarrow y = -x + 1 = 1 - x$
Second piece: Line through (1, 1) and (2, 0). $m = \frac{0-1}{2-1} = \frac{-1}{1} = -1 \Rightarrow y = -(x-1) + 1 = -x + 2 = 2 - x$
$$f(x) = \begin{cases} 1 - x, & 0 \leq x < 1 \\ 2 - x, & 1 \leq x \leq 2 \end{cases}$$

38. First piece: Line through (0, 0) and (2, 5). $m = \frac{5-0}{2-0} = \frac{5}{2} \Rightarrow y = \frac{5}{2}x$
Second piece: Line through (2, 5) and (4, 0). $m = \frac{0-5}{4-2} = \frac{-5}{2} = -\frac{5}{2} \Rightarrow y = -\frac{5}{2}(x-2) + 5 = -\frac{5}{2}x + 10 = 10 - \frac{5x}{2}$
$$f(x) = \begin{cases} \frac{5}{2}x, & 0 \leq x < 2 \\ 10 - \frac{5x}{2}, & 2 \leq x \leq 4 \end{cases}$$ (Note: $x = 2$ can be included on either piece.)

39. (a) $(f \circ g)(-1) = f(g(-1)) = f\left(\frac{1}{\sqrt{-1+2}}\right) = f(1) = \frac{1}{1} = 1$
(b) $(g \circ f)(2) = g(f(2)) = g\left(\frac{1}{2}\right) = \frac{1}{\sqrt{\frac{1}{2}+2}} = \frac{1}{\sqrt{2.5}}$ or $\sqrt{\frac{2}{5}}$

(c) $(f \circ f)(x) = f(f(x)) = f\left(\frac{1}{x}\right) = \frac{1}{1/x} = x,\ x \neq 0$

(d) $(g \circ g)(x) = g(g(x)) = g\left(\frac{1}{\sqrt{x+2}}\right) = \frac{1}{\sqrt{\frac{1}{\sqrt{x+2}}+2}} = \frac{\sqrt[4]{x+2}}{\sqrt{1+2\sqrt{x+2}}}$

40. (a) $(f \circ g)(-1) = f(g(-1)) = f\left(\sqrt[3]{-1+1}\right) = f(0) = 2 - 0 = 2$

(b) $(g \circ f)(2) = f(g(2)) = g(2-2) = g(0) = \sqrt[3]{0+1} = 1$

(c) $(f \circ f)(x) = f(f(x)) = f(2-x) = 2-(2-x) = x$

(d) $(g \circ g)(x) = g(g(x)) = g\left(\sqrt[3]{x+1}\right) = \sqrt[3]{\sqrt[3]{x+1}+1}$

41. (a) $(f \circ g)(x) = f(g(x)) = f\left(\sqrt{x+2}\right) = 2 - \left(\sqrt{x+2}\right)^2 = -x,\ x \geq -2.$

$(g \circ f)(x) = g(f(x)) = g(2-x^2) = \sqrt{\left(2-x^2\right)+2} = \sqrt{4-x^2}$

(b) Domain of $f \circ g$: $[-2, \infty)$.
Domain of $g \circ f$: $[-2, 2]$.

(c) Range of $f \circ g$: $(-\infty, 2]$.
Range of $g \circ f$: $[0, 2]$.

42. (a) $(f \circ g)(x) = f(g(x)) = f\left(\sqrt{1-x}\right) = \sqrt{\sqrt{1-x}} = \sqrt[4]{1-x}.$

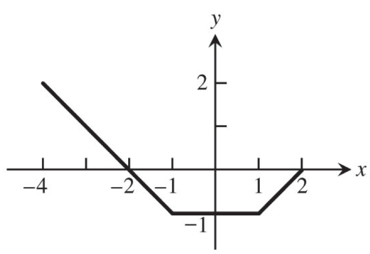
$(g \circ f)(x) = g(f(x)) = g\left(\sqrt{x}\right) = \sqrt{1-\sqrt{x}}$

(b) Domain of $f \circ g$: $(-\infty, 1]$.
Domain of $g \circ f$: $[0, 1]$.

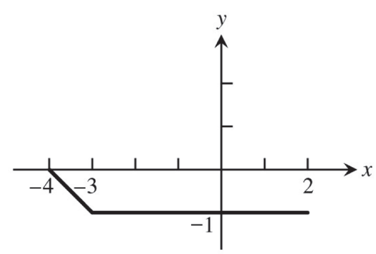
(c) Range of $f \circ g$: $[0, \infty)$.
Range of $g \circ f$: $[0, 1]$.

43.

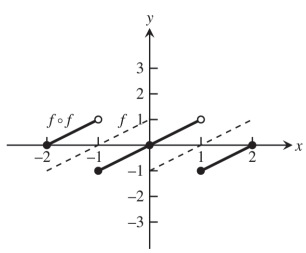
$y = f(x)$



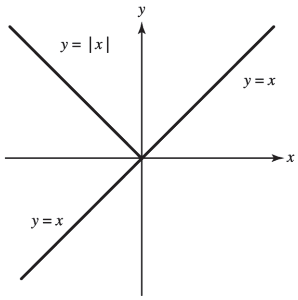
$y = (f \circ f)(x)$



44.

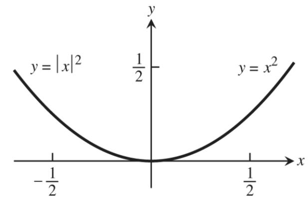


45.



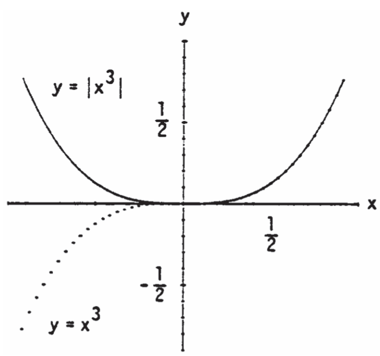
The graph of $f_2(x) = f_1(|x|)$ is the same as the graph of $f_1(x)$ to the right of the y-axis. The graph of $f_2(x)$ to the left of the y-axis is the reflection of $y = f_1(x), x \geq 0$ across the y-axis.

46.



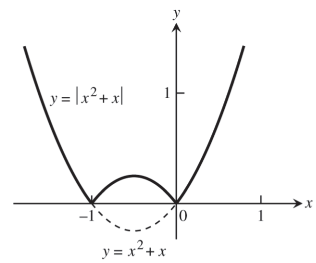
It does not change the graph.

47.



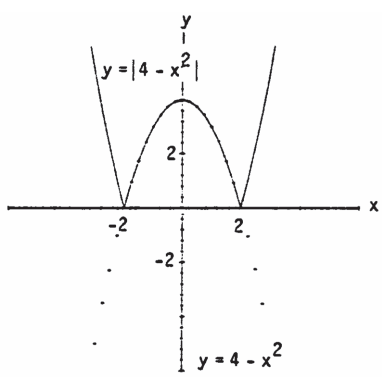
Whenever $g_1(x)$ is positive, the graph of $y = g_2(x) = |g_1(x)|$ is the same as the graph of $y = g_1(x)$. When $g_1(x)$ is negative, the graph of $y = g_2(x)$ is the reflection of the graph of $y = g_1(x)$ across the x-axis.

48.



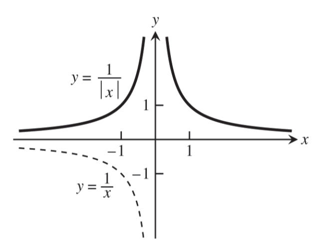
Whenever $g_1(x)$ is positive, the graph of $y = g_2(x) = |g_1(x)|$ is the same as the graph of $y = g_1(x)$. When $g_1(x)$ is negative, the graph of $y = g_2(x)$ is the reflection of the graph of $y = g_1(x)$ across the x-axis.

49.



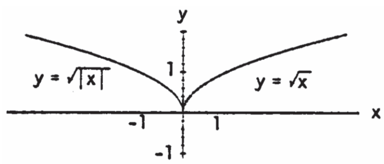
Whenever $g_1(x)$ is positive, the graph of $y = g_2(x) = |g_1(x)|$ is the same as graph of $y = g_1(x)$. When $g_1(x)$ is negative, the graph of $y = g_2(x)$ is the reflection of the graph of $y = g_1(x)$ across the x-axis.

50.



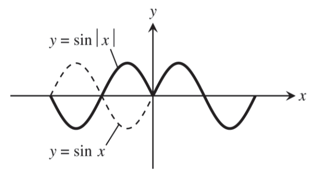
The graph of $f_2(x) = f_1(|x|)$ is the same as the graph of $f_1(x)$ to the right of the y-axis. The graph of $f_2(x)$ to the left of the y-axis is the reflection of $y = f_1(x), x \geq 0$ across the y-axis.

51.



The graph of $f_2(x) = f_1(|x|)$ is the same as the graph of $f_1(x)$ to the right of the y-axis. The graph of $f_2(x)$ to the left of the y-axis is the reflection of $y = f_1(x),\ x \geq 0$ across the y-axis.

52.

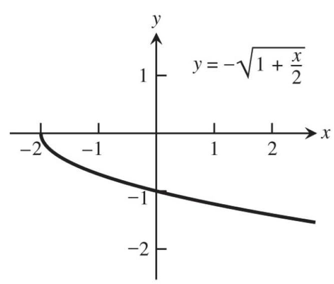


The graph of $f_2(x) = f_1(|x|)$ is the same as the graph of $f_1(x)$ to the right of the y-axis. The graph of $f_2(x)$ to the left of the y-axis is the reflection of $y = f_1(x),\ x \geq 0$ across the y-axis.

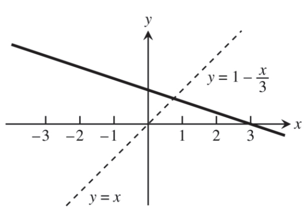
53. (a) $y = g(x-3) + \frac{1}{2}$
(b) $y = g\left(x + \frac{2}{3}\right) - 2$
(c) $y = g(-x)$
(d) $y = -g(x)$
(e) $y = 5 \cdot g(x)$
(f) $y = g(5x)$

54. (a) Shift the graph of f right 5 units
(b) Horizontally compress the graph of f by a factor of 4
(c) Horizontally compress the graph of f by a factor of 3 and then reflect the graph about the y-axis.
(d) Horizontally compress the graph of f by a factor of 2 and then shift the graph left $\frac{1}{2}$ unit.
(e) Horizontally stretch the graph of f by a factor of 3 and then shift the graph down 4 units.
(f) Vertically stretch the graph of f by a factor of 3, then reflect the graph about the x-axis, and finally shift the graph up $\frac{1}{4}$ unit.

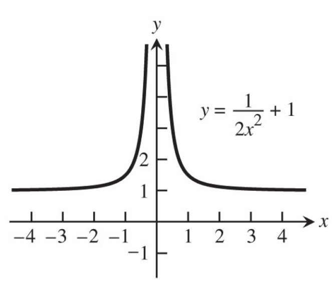
55. Reflection of the graph of $y = \sqrt{x}$ about the x-axis followed by a horizontal compression by a factor of $\frac{1}{2}$ then a shift left 2 units.



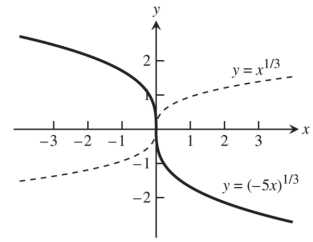
56. Reflect the graph of $y = x$ about the x-axis, followed by a vertical compression of the graph by a factor of 3, then shift the graph up 1 unit.



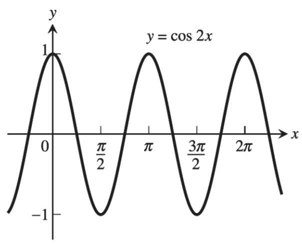
57. Vertical compression of the graph of $y = \frac{1}{x^2}$ by a factor of 2, then shift the graph up 1 unit.



58. Reflect the graph of $y = x^{1/3}$ about the y-axis, then compress the graph horizontally by a factor of 5.

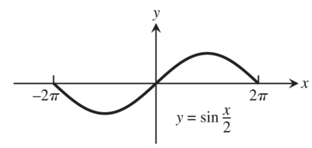


59.



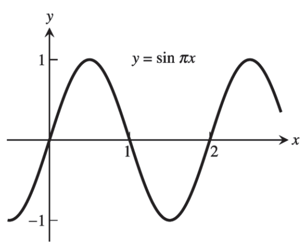
period $= \pi$

60.



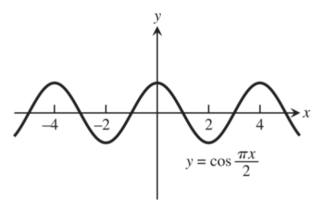
period $= 4\pi$

61.



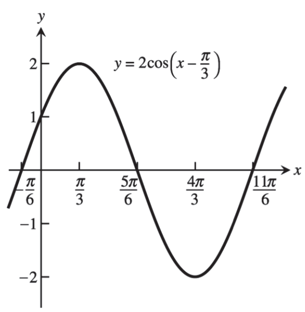
period $= 2$

62.



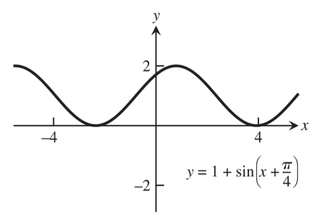
period $= 4$

63.



period $= 2\pi$

64.



period $= 2\pi$

65. (a) $\sin B = \sin \frac{\pi}{3} = \frac{b}{c} = \frac{b}{2} \Rightarrow b = 2\sin\frac{\pi}{3} = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$. By the theorem of Pythagoras, $a^2 + b^2 = c^2 \Rightarrow a = \sqrt{c^2 - b^2} = \sqrt{4-3} = 1.$

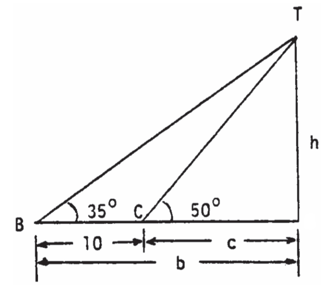
(b) $\sin B = \sin \frac{\pi}{3} = \frac{b}{c} = \frac{2}{c} \Rightarrow c = \frac{2}{\sin\frac{\pi}{3}} = \frac{2}{\left(\frac{\sqrt{3}}{2}\right)} = \frac{4}{\sqrt{3}}$. Thus, $a = \sqrt{c^2 - b^2} = \sqrt{\left(\frac{4}{\sqrt{3}}\right)^2 - (2)^2} = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}}.$

66. (a) $\sin A = \frac{a}{c} \Rightarrow a = c\sin A$ (b) $\tan A = \frac{a}{b} \Rightarrow a = b\tan A$

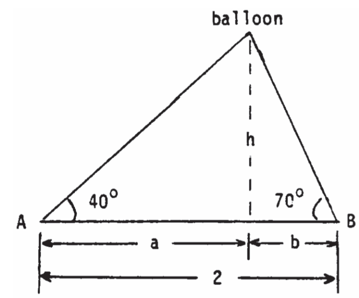
67. (a) $\tan B = \frac{b}{a} \Rightarrow a = \frac{b}{\tan B}$ (b) $\sin A = \frac{a}{c} \Rightarrow c = \frac{a}{\sin A}$

68. (a) $\sin A = \frac{a}{c}$ (b) $\sin A = \frac{a}{c} = \frac{\sqrt{c^2 - b^2}}{c}$

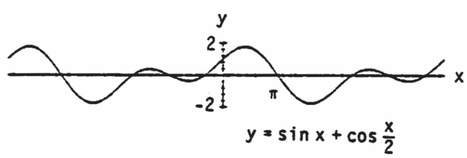
69. Let h = height of vertical pole, and let b and c denote the distances of points B and C from the base of the pole, measured along the flat ground, respectively. Then, $\tan 50° = \frac{h}{c}$, $\tan 35° = \frac{h}{b}$, and $b - c = 10$.
Thus, $h = c\tan 50°$ and $h = b\tan 35° = (c+10)\tan 35°$
$\Rightarrow c\tan 50° = (c+10)\tan 35°$
$\Rightarrow c(\tan 50° - \tan 35°) = 10\tan 35°$
$\Rightarrow c = \frac{10\tan 35°}{\tan 50° - \tan 35°} \Rightarrow h = c\tan 50°$
$= \frac{10\tan 35°\tan 50°}{\tan 50° - \tan 35°} \approx 16.98$ m.



70. Let h = height of balloon above ground. From the figure at the right, $\tan 40° = \frac{h}{a}$, $\tan 70° = \frac{h}{b}$, and $a + b = 2$. Thus, $h = b\tan 70° \Rightarrow h = (2-a)\tan 70°$ and $h = a\tan 40° \Rightarrow (2-a)\tan 70° = a\tan 40°$
$\Rightarrow a(\tan 40° + \tan 70°) = 2\tan 70°$
$\Rightarrow a = \frac{2\tan 70°}{\tan 40° + \tan 70°} \Rightarrow h = a\tan 40°$
$= \frac{2\tan 70°\tan 40°}{\tan 40° + \tan 70°} \approx 1.3$ km.



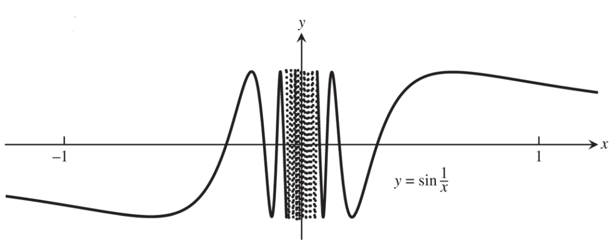
71. (a)



(b) The period appears to be 4π.

(c) $f(x+4\pi) = \sin(x+4\pi) + \cos\left(\frac{x+4\pi}{2}\right) = \sin(x+2\pi) + \cos\left(\frac{x}{2} + 2\pi\right) = \sin x + \cos\frac{x}{2}$ since the period of sine and cosine is 2π. Thus, $f(x)$ has period 4π.

72. (a)



(b) $D = (-\infty, 0) \cup (0, \infty)$; $R = [-1, 1]$

(c) f is not periodic. For suppose f has period p. Then $f\left(\frac{1}{2\pi} + kp\right) = f\left(\frac{1}{2\pi}\right) = \sin 2\pi = 0$ for all integers k. Choose k so large that $\frac{1}{2\pi} + kp > \frac{1}{\pi} \Rightarrow 0 < \frac{1}{(1/(2\pi)) + kp} < \pi$. But then $f\left(\frac{1}{2\pi} + kp\right) = \sin\left(\frac{1}{(1/(2\pi)) + kp}\right) > 0$ which is a contradiction. Thus f has no period, as claimed.

73. (a) D: $-\infty < x < \infty$

(b) D: $x > 0$

74. (a) $D = (-\infty, 0) \cup (0, \infty)$ (b) $D = (-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

75. (a) D: $-3 \le x \le 3$ (b) D: $0 \le x \le 4$

76. (a) $D = [-1, 1)$ (b) $D = [-1, 1]$

77. $(f \circ g)(x) = \ln(4 - x^2)$ and domain: $-2 < x < 2$;

$(g \circ f)(x) = 4 - (\ln x)^2$ and domain: $x > 0$;

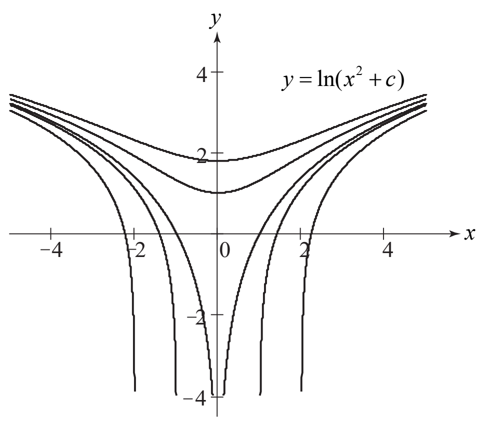
$(f \circ f)(x) = \ln(\ln x)$ and domain: $x > 1$;

$(g \circ g)(x) = -x^4 + 8x^2 - 12$ and domain: $-\infty < x < \infty$.

78. (a) Even (b) Neither even nor odd (c) Neither even nor odd (d) Even

79. This requires Grapher Technology.

80. For $c > 0$, $D = (-\infty, \infty)$
For $c \le 0$, $D = (-\infty, -\sqrt{c}) \cup (\sqrt{c}, \infty)$

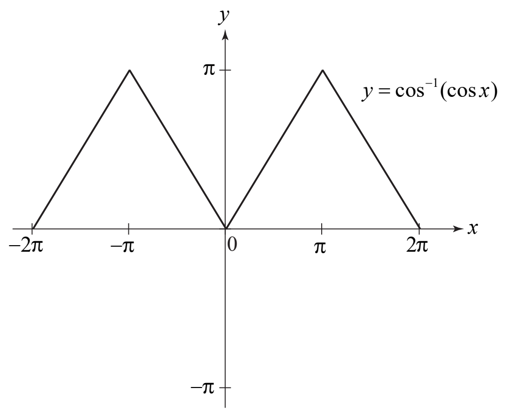


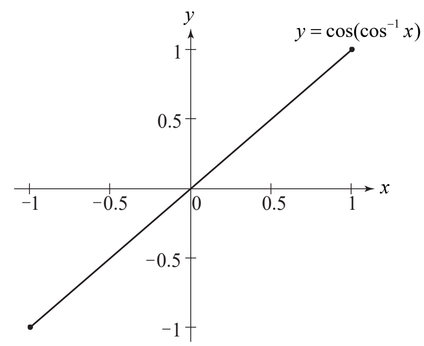
81. This requires Grapher Technology.

82. For large values of x, $y = a^x$ has the largest values; $y = \log_a x$ has the smallest.

83. (a) $D: (-\infty, \infty)$ $R: \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ (b) $D: [-1, 1]$ $R: [-1, 1]$

84. (a) $D: -\infty < x < \infty;\ R: 0 \le y \le \pi$ (b) $D: -1 \le x \le 1;\ R: -1 \le y \le 1$



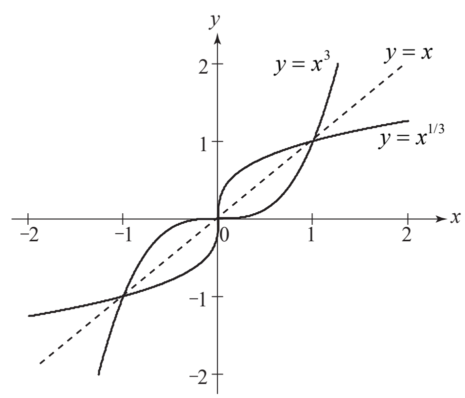


85. (a) No (b) Yes

86. Answers depend on the view screen used. For $[15, 17] \times [5 \cdot 10^6, 10^7]$ it appears that $e^x > 10^7$ for $x \ge 16.128$.

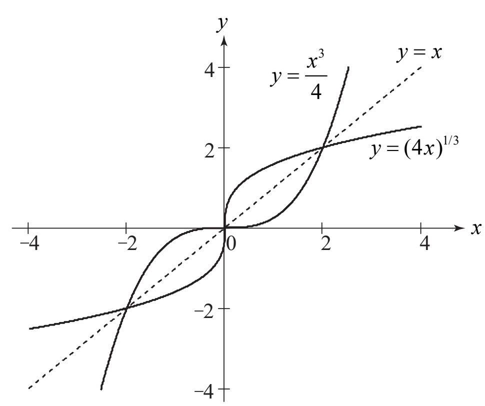
87. (a) $f(g(x)) = \left(\sqrt[3]{x}\right)^3 = x,\ g(f(x)) = \sqrt[3]{x^3} = x$

(b)



88. (a) $h(k(x)) = \frac{1}{4}\left((4x)^{1/3}\right)^3 = x,\ k(h(x)) = \left(4 \cdot \frac{x^3}{4}\right)^{1/3} = x$

(b)



CHAPTER 1 ADDITIONAL AND ADVANCED EXERCISES

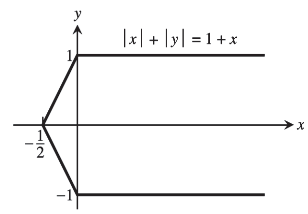
1. There are (infinitely) many such function pairs. For example, $f(x) = 3x$ and $g(x) = 4x$ satisfy $f(g(x)) = f(4x) = 3(4x) = 12x = 4(3x) = g(3x) = g(f(x))$.

2. Yes, there are many such function pairs. For example, if $g(x) = (2x+3)^3$ and $f(x) = x^{1/3}$, then $(f \circ g)(x) = f(g(x)) = f((2x+3)^3) = ((2x+3)^3)^{1/3} = 2x+3$.

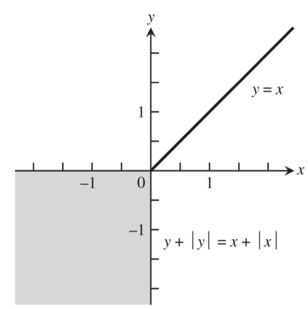
3. If f is odd and defined at x, then $f(-x) = -f(x)$. Thus $g(-x) = f(-x) - 2 = -f(x) - 2$ whereas $-g(x) = -(f(x) - 2) = -f(x) + 2$. Then g cannot be odd because $g(-x) = -g(x) \Rightarrow -f(x) - 2 = -f(x) + 2$ $\Rightarrow 4 = 0$, which is a contradiction. Also, $g(x)$ is not even unless $f(x) = 0$ for all x. On the other hand, if f is even, then $g(x) = f(x) - 2$ is also even: $g(-x) = f(-x) - 2 = f(x) - 2 = g(x)$.

4. If g is odd and $g(0)$ is defined, then $g(0) = g(-0) = -g(0)$. Therefore, $2g(0) = 0 \Rightarrow g(0) = 0$.

5. For (x, y) in the 1st quadrant, $|x| + |y| = 1 + x$ $\Leftrightarrow x + y = 1 + x \Leftrightarrow y = 1$. For (x, y) in the 2nd quadrant, $|x| + |y| = x + 1 \Leftrightarrow -x + y = x + 1$ $\Leftrightarrow y = 2x + 1$. In the 3rd quadrant, $|x| + |y| = x + 1$ $\Leftrightarrow -x - y = x + 1 \Leftrightarrow y = -2x - 1$. In the 4th quadrant, $|x| + |y| = x + 1 \Leftrightarrow x + (-y) = x + 1$ $\Leftrightarrow y = -1$. The graph is given at the right.

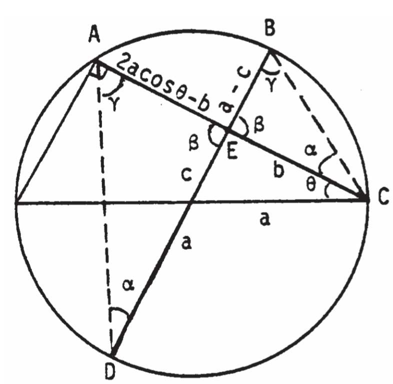


6. We use reasoning similar to Exercise 5.
   (1) 1st quadrant: $y + |y| = x + |x|$
   $\Leftrightarrow 2y = 2x \Leftrightarrow y = x$.
   (2) 2nd quadrant: $y + |y| = x + |x|$
   $\Leftrightarrow 2y = x + (-x) = 0 \Leftrightarrow y = 0$.
   (3) 3rd quadrant: $y + |y| = x + |x|$
   $\Leftrightarrow y + (-y) = x + (-x) \Leftrightarrow 0 = 0$
   $\Rightarrow$ <u>all</u> points in the 3rd quadrant satisfy the equation.
   (4) 4th quadrant: $y + |y| = x + |x|$
   $\Leftrightarrow y + (-y) = 2x \Leftrightarrow 0 = x$. Combining these results we have the graph given at the right:



7. (a) $\sin^2 x + \cos^2 x = 1 \Rightarrow \sin^2 x = 1 - \cos^2 x = (1 - \cos x)(1 + \cos x) \Rightarrow (1 - \cos x) = \frac{\sin^2 x}{1 + \cos x} \Rightarrow \frac{1 - \cos x}{\sin x} = \frac{\sin x}{1 + \cos x}$
   (b) Using the definition of the tangent function and the double angle formulas, we have
   $$\tan^2\left(\frac{x}{2}\right) = \frac{\sin^2\left(\frac{x}{2}\right)}{\cos^2\left(\frac{x}{2}\right)} = \frac{\frac{1 - \cos\left(2\left(\frac{x}{2}\right)\right)}{2}}{\frac{1 + \cos\left(2\left(\frac{x}{2}\right)\right)}{2}} = \frac{1 - \cos x}{1 + \cos x}.$$

8. The angles labeled γ in the accompanying figure are equal since both angles subtend arc CD. Similarly, the two angles labeled α are equal since they both subtend arc AB. Thus, triangles AED and BEC are similar which implies $\frac{a - c}{b} = \frac{2a\cos\theta - b}{a + c}$
   $\Rightarrow (a - c)(a + c) = b(2a\cos\theta - b)$
   $\Rightarrow a^2 - c^2 = 2ab\cos\theta - b^2$
   $\Rightarrow c^2 - a^2 + b^2 - 2ab\cos\theta$.



9. As in the proof of the law of sines of Section 1.3, Exercise 61, $ah = bc\sin A = ab\sin C = ac\sin B$ $\Rightarrow$ the area of $ABC = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}ah = \frac{1}{2}bc\sin A = \frac{1}{2}ab\sin C = \frac{1}{2}ac\sin B$.

10. As in Section 1.3, Exercise 61, $(\text{Area of } ABC)^2 = \frac{1}{4}(\text{base})^2(\text{height})^2 = \frac{1}{4}a^2h^2 = \frac{1}{4}a^2b^2\sin^2 C$ $= \frac{1}{4}a^2b^2(1-\cos^2 C)$. By the law of cosines, $c^2 = a^2+b^2-2ab\cos C \Rightarrow \cos C = \frac{a^2+b^2-c^2}{2ab}$. Thus,

$$(\text{area of } ABC)^2 = \frac{1}{4}a^2b^2(1-\cos^2 C) = \frac{1}{4}a^2b^2\left(1-\left(\frac{a^2+b^2-c^2}{2ab}\right)^2\right) = \frac{a^2b^2}{4}\left(1-\frac{(a^2+b^2-c^2)^2}{4a^2b^2}\right)$$
$$= \frac{1}{16}\left(4a^2b^2-(a^2+b^2-c^2)^2\right) = \frac{1}{16}[(2ab+(a^2+b^2-c^2))\,(2ab-(a^2+b^2-c^2))]$$
$$= \frac{1}{16}[((a+b)^2-c^2)(c^2-(a-b)^2)] = \frac{1}{16}[((a+b)+c)((a+b)-c)(c+(a-b))(c-(a-b))]$$
$$= \left[\left(\frac{a+b+c}{2}\right)\left(\frac{-a+b+c}{2}\right)\left(\frac{a-b+c}{2}\right)\left(\frac{a+b-c}{2}\right)\right] = s(s-a)(s-b)(s-c), \text{ where } s = \frac{a+b+c}{2}.$$

Therefore, the area of ABC equals $\sqrt{s(s-a)(s-b)(s-c)}$.

11. If f is even and odd, then $f(-x) = -f(x)$ and $f(-x) = f(x) \Rightarrow f(x) = -f(x)$ for all x in the domain of f. Thus $2f(x) = 0 \Rightarrow f(x) = 0$.

12. (a) As suggested, let $E(x) = \frac{f(x)+f(-x)}{2} \Rightarrow E(-x) = \frac{f(-x)+f(-(-x))}{2} = \frac{f(x)+f(-x)}{2} = E(x) \Rightarrow E$ is an even function. Define $O(x) = f(x) - E(x) = f(x) - \frac{f(x)+f(-x)}{2} = \frac{f(x)-f(-x)}{2}$. Then $O(-x) = \frac{f(-x)-f(-(-x))}{2}$ $= \frac{f(-x)-f(x)}{2} = -\left(\frac{f(x)-f(-x)}{2}\right) = -O(x) \Rightarrow O$ is an odd function $\Rightarrow f(x) = E(x)+O(x)$ is the sum of an even and an odd function.

(b) Part (a) shows that $f(x) = E(x)+O(x)$ is the sum of an even and an odd function. If also $f(x) = E_1(x)+O_1(x)$, where E_1 is even and O_1 is odd, then $f(x)-f(x) = 0$ $= (E_1(x)+O_1(x))-(E(x)+O(x))$. Thus, $E(x)-E_1(x) = O_1(x)-O(x)$ for all x in the domain of f (which is the same as the domain of $E-E_1$ and $O-O_1$). Now $(E-E_1)(-x) = E(-x)-E_1(-x) = E(x)-E_1(x)$ (since E and E_1 are even) $= (E-E_1)(x) \Rightarrow E-E_1$ is even. Likewise, $(O_1-O)(-x) = O_1(-x)-O(-x)$ $= -O_1(x)-(-O(x))$ (since O and O_1 are odd) $= -(O_1(x)-O(x)) = -(O_1-O)(x) \Rightarrow O_1-O$ is odd. Therefore, $E-E_1$ and O_1-O are both even and odd so they must be zero at each x in the domain of f by Exercise 11. That is, $E_1 = E$ and $O_1 = O$, so the decomposition of f found in part (a) is unique.

13. $y = ax^2+bx+c = a\left(x^2+\frac{b}{a}x+\frac{b^2}{4a^2}\right)-\frac{b^2}{4a}+c = a\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a}+c$

(a) If $a > 0$ the graph is a parabola that opens upward. Increasing a causes a vertical stretching and a shift of the vertex toward the y-axis and upward. If $a < 0$ the graph is a parabola that opens downward. Decreasing a causes a vertical stretching and a shift of the vertex toward the y-axis and downward.

(b) If $a > 0$ the graph is a parabola that opens upward. If also $b > 0$, then increasing b causes a shift of the graph downward to the left; if $b < 0$, then decreasing b causes a shift of the graph downward and to the right. If $a < 0$ the graph is a parabola that opens downward. If $b > 0$, increasing b shifts the graph upward to the right. If $b < 0$, decreasing b shifts the graph upward to the left.

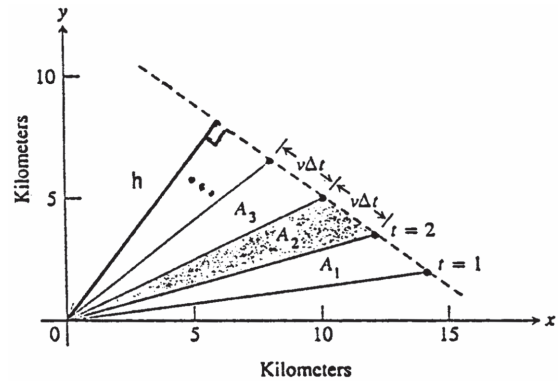
(c) Changing c (for fixed a and b) by Δc shifts the graph upward Δc units if $\Delta c > 0$, and downward $-\Delta c$ units if $\Delta c < 0$.

14. (a) If $a > 0$, the graph rises to the right of the vertical line $x = -b$ and falls to the left. If $a < 0$, the graph falls to the right of the line $x = -b$ and rises to the left. If $a = 0$, the graph reduces to the horizontal line $y = c$. As $|a|$ increases, the slope at any given point $x = x_0$ increases in magnitude and the graph becomes steeper. As $|a|$ decreases, the slope at x_0 decreases in magnitude and the graph rises or falls more gradually.

(b) Increasing b shifts the graph to the left; decreasing b shifts it to the right.

(c) Increasing c shifts the graph upward; decreasing c shifts it downward.

15. Each of the triangles pictured has the same base $b = v\Delta t = v(1\text{ sec})$. Moreover, the height of each triangle is the same value h. Thus $\frac{1}{2}$ (base)(height) $= \frac{1}{2}bh = A_1 = A_2 = A_3 = \dots$. In conclusion, the object sweeps out equal areas in each one second interval.



16. (a) Using the midpoint formula, the coordinates of P are $\left(\frac{a+0}{2}, \frac{b+0}{2}\right) = \left(\frac{a}{2}, \frac{b}{2}\right)$. Thus the slope of $\overline{OP} = \frac{\Delta y}{\Delta x} = \frac{b/2}{a/2} = \frac{b}{a}$.

(b) The slope of $\overline{AB} = \frac{b-0}{0-a} = -\frac{b}{a}$. The line segments $\overline{AB}$ and $\overline{OP}$ are perpendicular when the product of their slopes is $-1 = \left(\frac{b}{a}\right)\left(-\frac{b}{a}\right) = -\frac{b^2}{a^2}$. Thus, $b^2 = a^2 \Rightarrow a = b$ (since both are positive). Therefore, $\overline{AB}$ is perpendicular to $\overline{OP}$ when $a = b$.

17. From the figure we see that $0 \le \theta \le \frac{\pi}{2}$ and $AB = AD = 1$. From trigonometry we have the following: $\sin\theta = \frac{EB}{AB} = EB$, $\cos\theta = \frac{AE}{AB} = AE$, $\tan\theta = \frac{CD}{AD} = CD$, and $\tan\theta = \frac{EB}{AE} = \frac{\sin\theta}{\cos\theta}$. We can see that:
area $\Delta AEB <$ area sector $\widehat{DB} <$ area $\Delta ADC \Rightarrow \frac{1}{2}(AE)(EB) < \frac{1}{2}(AD)^2\theta < \frac{1}{2}(AD)(CD)$
$\Rightarrow \frac{1}{2}\sin\theta\cos\theta < \frac{1}{2}(1)^2\theta < \frac{1}{2}(1)(\tan\theta) \Rightarrow \frac{1}{2}\sin\theta\cos\theta < \frac{1}{2}\theta < \frac{1}{2}\frac{\sin\theta}{\cos\theta}$

18. $(f \circ g)(x) = f(g(x)) = a(cx+d)+b = acx+ad+b$ and $(g \circ f)(x) = g(f(x)) = c(ax+b)+d = acx+cb+d$
Thus $(f \circ g)(x) = (g \circ f)(x) \Rightarrow acx+ad+b = acx+bc+d \Rightarrow ad+b = bc+d$. Note that $f(d) = ad+b$ and $g(b) = cb+d$, thus $(f \circ g)(x) = (g \circ f)(x)$ if $f(d) = g(b)$.

19. (a) The expression $a(b^{c-x})+d$ is defined for all values of x, so the domain is $(-\infty, \infty)$. Since b^{c-x} attains all positive values, the range is (d, ∞) if $a > 0$ and the range is $(-d, \infty)$ if $a < 0$.

(b) The expression $a\log_b(x-c)+d$ is defined when $x - c > 0$, so the domain is (c, ∞). Since $a\log_b(x-c)+d$ attains every real value for some value of x, the range is $(-\infty, \infty)$.

20. (a) Suppose $f(x_1) = f(x_2)$. Then:

$$\frac{ax_1+b}{cx_1+d} = \frac{ax_2+b}{cx_2+d}$$
$$(ax_1+b)(cx_2+d) = (ax_2+b)(cx_1+d)$$
$$acx_1x_2 + adx_1 + bcx_2 + bd = acx_1x_2 + adx_2 + bcx_1 + bd$$
$$adx_1 + bcx_2 = adx_2 + bcx_1$$
$$(ad-bc)x_1 = (ad-bc)x_2$$

Since $ad - bc \ne 0$, this means that $x_1 = x_2$.

(b) $y = \frac{ax+b}{cx+d}$

$cxy + dy = ax + b$

$(cy - a)x = -dy + b$

$x = \frac{-dy+b}{cy-a}$

Interchange x and y.

$y = \dfrac{-dx+b}{cx-a}$

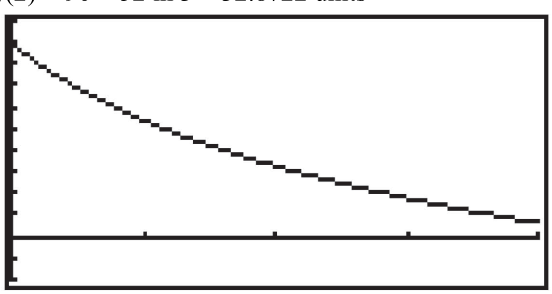
$f^{-1}(x) = \frac{-dx+b}{cx-a}$

21. (a) $y = 100{,}000 - 10{,}000x,\ 0 \le x \le 10$

(b)
$$y = 55{,}000$$
$$100{,}000 - 10{,}000x = 55{,}000$$
$$-10{,}000x = 55{,}000$$
$$x = 4.5$$

The value is \$55,000 after 4.5 years.

22. (a) $f(0) = 90$ units
(b) $f(2) = 90 - 52 \ln 3 \approx 32.8722$ units
(c)



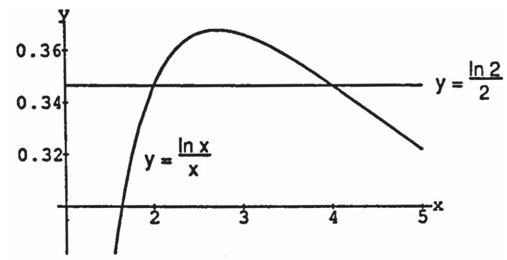
[0, 4] by [-20, 100]

23. $1500(1.08)^t = 5000 \rightarrow 1.08^t = \frac{5000}{1500} = \frac{10}{3} \rightarrow \ln(1.08)^t = \ln\frac{10}{3} \rightarrow t\ln 1.08 = \ln\frac{10}{3} \rightarrow t = \frac{\ln(10/3)}{\ln 1.08} \approx 15.6439$
It will take about 15.6439 years. (If the bank only pays interest at the end of the year, it will take 16 years.)

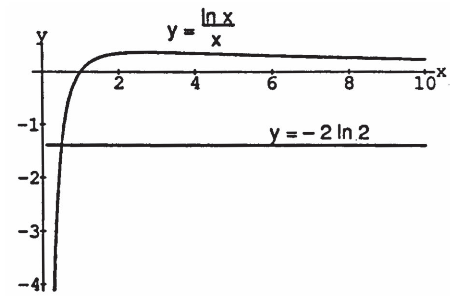
24. $A(t) = A_0 e^{rt};\ A(t) = 2A_0 \Rightarrow 2A_0 = A_0 e^{rt} \Rightarrow e^{rt} = 2 \Rightarrow rt = \ln 2 \Rightarrow t = \frac{\ln 2}{r} \Rightarrow t \approx \frac{0.7}{r} = \frac{70}{100r} = \frac{70}{(r\%)}$

25. $\ln x^{(x^x)} = x^x \ln x$ and $\ln(x^x)^x = x\ln(x^x) = x^2 \ln x$; then, $x^x \ln x = x^2 \ln x \Rightarrow x^x = x^2 \Rightarrow x\ln x = 2\ln x \Rightarrow x = 2.$
Therefore, $x^{(x^x)} = (x^x)^x$ when $x = 2$.

26. (a) No, there are two intersections: one at $x = 2$ and the other at $x = 4$.



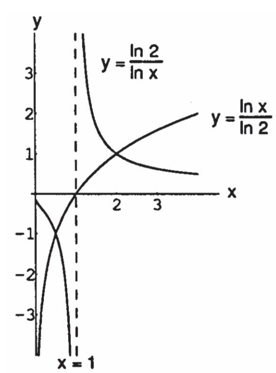
(b) Yes, because there is only one intersection.



27. $\frac{\log_4 x}{\log_2 x} = \frac{\left(\frac{\ln x}{\ln 4}\right)}{\left(\frac{\ln x}{\ln 2}\right)} = \frac{\ln x}{\ln 4} \cdot \frac{\ln 2}{\ln x} = \frac{\ln 2}{\ln 4} = \frac{\ln 2}{2 \ln 2} = \frac{1}{2}$

28. (a) $f(x) = \frac{\ln 2}{\ln x}$, $g(x) = \frac{\ln x}{\ln 2}$

(b) f is negative when g is negative, positive when g is positive, and undefined when $g = 0$; the values of f decrease as those of g increase.



CHAPTER 2 LIMITS AND CONTINUITY

2.1 RATES OF CHANGE AND TANGENTS TO CURVES

1. (a) $\frac{\Delta f}{\Delta x} = \frac{f(3)-f(2)}{3-2} = \frac{28-9}{1} = 19$ (b) $\frac{\Delta f}{\Delta x} = \frac{f(1)-f(-1)}{1-(-1)} = \frac{2-0}{2} = 1$

2. (a) $\frac{\Delta g}{\Delta x} = \frac{g(3)-g(1)}{3-1} = \frac{3-(-1)}{2} = 2$ (b) $\frac{\Delta g}{\Delta x} = \frac{g(4)-g(-2)}{4-(-2)} = \frac{8-8}{6} = 0$

3. (a) $\frac{\Delta h}{\Delta t} = \frac{h\left(\frac{3\pi}{4}\right)-h\left(\frac{\pi}{4}\right)}{\frac{3\pi}{4}-\frac{\pi}{4}} = \frac{-1-1}{\frac{\pi}{2}} = -\frac{4}{\pi}$ (b) $\frac{\Delta h}{\Delta t} = \frac{h\left(\frac{\pi}{2}\right)-h\left(\frac{\pi}{6}\right)}{\frac{\pi}{2}-\frac{\pi}{6}} = \frac{0-\sqrt{3}}{\frac{\pi}{3}} = \frac{-3\sqrt{3}}{\pi}$

4. (a) $\frac{\Delta g}{\Delta t} = \frac{g(\pi)-g(0)}{\pi-0} = \frac{(2-1)-(2+1)}{\pi-0} = -\frac{2}{\pi}$ (b) $\frac{\Delta g}{\Delta t} = \frac{g(\pi)-g(-\pi)}{\pi-(-\pi)} = \frac{(2-1)-(2-1)}{2\pi} = 0$

5. $\frac{\Delta R}{\Delta \theta} = \frac{R(2)-R(0)}{2-0} = \frac{\sqrt{8+1}-\sqrt{1}}{2} = \frac{3-1}{2} = 1$

6. $\frac{\Delta P}{\Delta \theta} = \frac{P(2)-P(1)}{2-1} = \frac{(8-16+10)-(1-4+5)}{1} = 2-2 = 0$

7. (a) $\frac{\Delta y}{\Delta x} = \frac{((2+h)^2-5)-(2^2-5)}{h} = \frac{4+4h+h^2-5+1}{h} = \frac{4h+h^2}{h} = 4+h.$ As $h \to 0,\ 4+h \to 4 \Rightarrow$ at $P(2,-1)$ the slope is 4.
   (b) $y-(-1) = 4(x-2) \Rightarrow y+1 = 4x-8 \Rightarrow y = 4x-9$

8. (a) $\frac{\Delta y}{\Delta x} = \frac{(7-(2+h)^2)-(7-2^2)}{h} = \frac{7-4-4h-h^2-3}{h} = \frac{-4h-h^2}{h} = -4-h.$ As $h \to 0, -4-h \to -4 \Rightarrow$ at $P(2,3)$ the slope is -4.
   (b) $y-3 = (-4)(x-2) \Rightarrow y-3 = -4x+8 \Rightarrow y = -4x+11$

9. (a) $\frac{\Delta y}{\Delta x} = \frac{((2+h)^2-2(2+h)-3)-(2^2-2(2)-3)}{h} = \frac{4+4h+h^2-4-2h-3-(-3)}{h} = \frac{2h+h^2}{h} = 2+h.$ As $h \to 0,\ 2+h \to 2 \Rightarrow$ at $P(2,-3)$ the slope is 2.
   (b) $y-(-3) = 2(x-2) \Rightarrow y+3 = 2x-4 \Rightarrow y = 2x-7.$

10. (a) $\frac{\Delta y}{\Delta x} = \frac{((1+h)^2-4(1+h))-(1^2-4(1))}{h} = \frac{1+2h+h^2-4-4h-(-3)}{h} = \frac{h^2-2h}{h} = h-2.$ As $h \to 0,\ h-2 \to -2 \Rightarrow$ at $P(1,-3)$ the slope is -2.
    (b) $y-(-3) = (-2)(x-1) \Rightarrow y+3 = -2x+2 \Rightarrow y = -2x-1.$

11. (a) $\frac{\Delta y}{\Delta x} = \frac{(2+h)^3-2^3}{h} = \frac{8+12h+4h^2+h^3-8}{h} = \frac{12h+4h^2+h^3}{h} = 12+4h+h^2.$ As $h \to 0,\ 12+4h+h^2 \to 12, \Rightarrow$ at $P(2,8)$ the slope is 12.
    (b) $y-8 = 12(x-2) \Rightarrow y-8 = 12x-24 \Rightarrow y = 12x-16.$

12. (a) $\frac{\Delta y}{\Delta x} = \frac{2-(1+h)^3-(2-1^3)}{h} = \frac{2-1-3h-3h^2-h^3-1}{h} = \frac{-3h-3h^2-h^3}{h} = -3-3h-h^2.$ As $h \to 0,\ -3-3h-h^2 \to -3, \Rightarrow$ at $P(1,1)$ the slope is -3.
    (b) $y-1 = (-3)(x-1) \Rightarrow y-1 = -3x+3 \Rightarrow y = -3x+4.$

13. (a) $\frac{\Delta y}{\Delta x} = \frac{(1+h)^3-12(1+h)-(1^3-12(1))}{h} = \frac{1+3h+3h^2+h^3-12-12h-(-11)}{h} = \frac{-9h+3h^2+h^3}{h} = -9+3h+h^2.$
As $h \to 0$, $-9+3h+h^2 \to -9 \Rightarrow$ at $P(1,-11)$ the slope is -9.
(b) $y-(-11) = (-9)(x-1) \Rightarrow y+11 = -9x+9 \Rightarrow y = -9x-2.$

14. (a) $\frac{\Delta y}{\Delta x} = \frac{(2+h)^3-3(2+h)^2+4-(2^3-3(2)^2+4)}{h} = \frac{8+12h+6h^2+h^3-12-12h-3h^2+4-0}{h} = \frac{3h^2+h^3}{h} = 3h+h^2.$
As $h \to 0$, $3h+h^2 \to 0 \Rightarrow$ at $P(2, 0)$ the slope is 0.
(b) $y-0 = 0(x-2) \Rightarrow y = 0.$

15. (a) $\frac{\Delta y}{\Delta x} = \frac{\frac{1}{-2+h}-\frac{1}{-2}}{h} = \frac{2+(-2+h)}{2(-2+h)}\cdot\frac{1}{h} = \frac{1}{2(-2+h)}.$
As $h \to 0$, $\frac{1}{2(-2+h)} \to \frac{-1}{4}$, $\Rightarrow$ at $P\left(-2, \frac{-1}{2}\right)$ the slope is $\frac{-1}{4}$.
(b) $y-\left(\frac{-1}{2}\right) = \frac{-1}{4}(x-(-2)) \Rightarrow y+\frac{1}{2} = \frac{-1}{4}x-\frac{1}{2} \Rightarrow y = \frac{-1}{4}x-1$

16. (a) $\frac{\Delta y}{\Delta x} = \frac{\frac{(4+h)}{2-(4+h)}-\frac{4}{2-4}}{h} = \left(\frac{4+h}{-2-h}+\frac{2}{1}\right)\cdot\frac{1}{h} = \frac{4+h+2(-2-h)}{-2-h}\cdot\frac{1}{h} = \frac{-1}{-2-h} = \frac{1}{2+h}.$
As $h \to 0$, $\frac{1}{2+h} \to \frac{1}{2}$, $\Rightarrow$ at $P(4,-2)$ the slope is $\frac{1}{2}$.
(b) $y-(-2) = \frac{1}{2}(x-4) \Rightarrow y+2 = \frac{1}{2}x-2 \Rightarrow y = \frac{1}{2}x-4$

17. (a) $\frac{\Delta y}{\Delta x} = \frac{\sqrt{4+h}-\sqrt{4}}{h} = \frac{\sqrt{4+h}-2}{h}\cdot\frac{\sqrt{4+h}+2}{\sqrt{4+h}+2} = \frac{(4+h)-4}{h(\sqrt{4+h}+2)} = \frac{1}{\sqrt{4+h}+2}.$
As $h \to 0$, $\frac{1}{\sqrt{4+h}+2} \to \frac{1}{\sqrt{4}+2} = \frac{1}{4}$, $\Rightarrow$ at $P(4,2)$ the slope is $\frac{1}{4}$.
(b) $y-2 = \frac{1}{4}(x-4) \Rightarrow y-2 = \frac{1}{4}x-1 \Rightarrow y = \frac{1}{4}x+1$

18. (a) $\frac{\Delta y}{\Delta x} = \frac{\sqrt{7-(-2+h)}-\sqrt{7-(-2)}}{h} = \frac{\sqrt{9-h}-3}{h} = \frac{\sqrt{9-h}-3}{h}\cdot\frac{\sqrt{9-h}+3}{\sqrt{9-h}+3} = \frac{(9-h)-9}{h(\sqrt{9-h}+3)} = \frac{-1}{\sqrt{9-h}+3}.$
As $h \to 0$, $\frac{-1}{\sqrt{9-h}+3} \to \frac{-1}{\sqrt{9}+3} = \frac{-1}{6}$, $\Rightarrow$ at $P(-2,3)$ the slope is $\frac{-1}{6}$.
(b) $y-3 = \frac{-1}{6}(x-(-2)) \Rightarrow y-3 = \frac{-1}{6}x-\frac{1}{3} \Rightarrow y = \frac{-1}{6}x+\frac{8}{3}$

19. (a)

Q	Slope of $PQ = \frac{\Delta p}{\Delta t}$
$Q_1(10,225)$	$\frac{650-225}{20-10} = 42.5$ m/sec
$Q_2(14,375)$	$\frac{650-375}{20-14} = 45.83$ m/sec
$Q_3(16.5,475)$	$\frac{650-475}{20-16.5} = 50.00$ m/sec
$Q_4(18,550)$	$\frac{650-550}{20-18} = 50.00$ m/sec

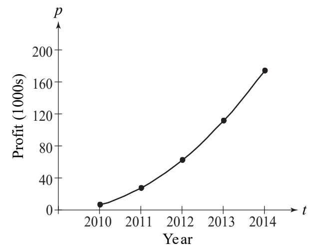
(b) At $t = 20$, the sportscar was traveling approximately 50 m/sec or 180 km/h.

20. (a)

Q	Slope of $PQ = \frac{\Delta p}{\Delta t}$
$Q_1(5,20)$	$\frac{80-20}{10-5} = 12$ m/sec
$Q_2(7,39)$	$\frac{80-39}{10-7} = 13.7$ m/sec
$Q_3(8.5,58)$	$\frac{80-58}{10-8.5} = 14.7$ m/sec
$Q_4(9.5,72)$	$\frac{80-72}{10-9.5} = 16$ m/sec

(b) Approximately 16 m/sec

21. (a)



(b) $\frac{\Delta p}{\Delta t} = \frac{174-62}{2014-2012} = \frac{112}{2} = 56$ thousand dollars per year

(c) The average rate of change from 2011 to 2012 is $\frac{\Delta p}{\Delta t} = \frac{62-27}{2012-2011} = 35$ thousand dollars per year.

The average rate of change from 2012 to 2013 is $\frac{\Delta p}{\Delta t} = \frac{111-62}{2013-2012} = 49$ thousand dollars per year.

So, the rate at which profits were changing in 2012 is approximately $\frac{1}{2}(35+49) = 42$ thousand dollars per year.

22. (a) $F(x) = (x+2)/(x-2)$

x	1.2	1.1	1.01	1.001	1.0001	1
$F(x)$	-4.0	$-3.\overline{4}$	$-3.\overline{04}$	$-3.\overline{004}$	$-3.\overline{0004}$	-3

$\frac{\Delta F}{\Delta x} = \frac{-4.0-(-3)}{1.2-1} = -5.0$; $\frac{\Delta F}{\Delta x} = \frac{-3.\overline{4}-(-3)}{1.1-1} = -4.\overline{4}$;

$\frac{\Delta F}{\Delta x} = \frac{-3.\overline{04}-(-3)}{1.01-1} = -4.\overline{04}$; $\frac{\Delta F}{\Delta x} = \frac{-3.\overline{004}-(-3)}{1.001-1} = -4.\overline{004}$;

$\frac{\Delta F}{\Delta x} = \frac{-3.\overline{0004}-(-3)}{1.0001-1} = -4.\overline{0004}$;

(b) The rate of change of $F(x)$ at $x = 1$ is -4.

23. (a) $\frac{\Delta g}{\Delta x} = \frac{g(2)-g(1)}{2-1} = \frac{\sqrt{2}-1}{2-1} \approx 0.414213$ $\frac{\Delta g}{\Delta x} = \frac{g(1.5)-g(1)}{1.5-1} = \frac{\sqrt{1.5}-1}{0.5} \approx 0.449489$

$\frac{\Delta g}{\Delta x} = \frac{g(1+h)-g(1)}{(1+h)-1} = \frac{\sqrt{1+h}-1}{h}$

(b) $g(x) = \sqrt{x}$

$1+h$	1.1	1.01	1.001	1.0001	1.00001	1.000001
$\sqrt{1+h}$	1.04880	1.004987	1.0004998	1.0000499	1.000005	1.0000005
$\left(\sqrt{1+h}-1\right)/h$	0.4880	0.4987	0.4998	0.499	0.5	0.5

(c) The rate of change of $g(x)$ at $x = 1$ is 0.5.

(d) The calculator gives $\lim_{h\to 0} \frac{\sqrt{1+h}-1}{h} = \frac{1}{2}$.

24. (a) i) $\frac{f(3)-f(2)}{3-2} = \frac{\frac{1}{3}-\frac{1}{2}}{1} = \frac{\frac{-1}{6}}{1} = -\frac{1}{6}$

ii) $\frac{f(T)-f(2)}{T-2} = \frac{\frac{1}{T}-\frac{1}{2}}{T-2} = \frac{\frac{2}{2T}-\frac{T}{2T}}{T-2} = \frac{2-T}{2T(T-2)} = \frac{2-T}{-2T(2-T)} = -\frac{1}{2T}, T \neq 2$

(b)

T	2.1	2.01	2.001	2.0001	2.00001	2.000001
$f(T)$	0.476190	0.497512	0.499750	0.4999750	0.499997	0.499999
$(f(T)-f(2))/(T-2)$	-0.2381	-0.2488	-0.2500	-0.2500	-0.2500	-0.2500

(c) The table indicates the rate of change is -0.25 at $t = 2$.

(d) $\lim_{T\to 2}\left(\frac{1}{-2T}\right) = -\frac{1}{4}$

NOTE: Answers will vary in Exercises 25 and 26.

25. (a) $[0, 1]$: $\frac{\Delta s}{\Delta t} = \frac{15-0}{1-0} = 15$ mph; $[1, 2.5]$: $\frac{\Delta s}{\Delta t} = \frac{20-15}{2.5-1} = \frac{10}{3}$ mph; $[2.5, 3.5]$: $\frac{\Delta s}{\Delta t} = \frac{30-20}{3.5-2.5} = 10$ mph

(b) At $P\left(\frac{1}{2}, 7.5\right)$: Since the portion of the graph from $t = 0$ to $t = 1$ is nearly linear, the instantaneous rate of change will be almost the same as the average rate of change, thus the instantaneous speed at $t = \frac{1}{2}$ is $\frac{15-7.5}{1-0.5} = 15$ mi/hr. At $P(2, 20)$: Since the portion of the graph from $t = 2$ to $t = 2.5$ is nearly linear, the instantaneous rate of change will be nearly the same as the average rate of change, thus $v = \frac{20-20}{2.5-2} = 0$ mi/hr. For values of t less than 2, we have

Q	Slope of $PQ = \frac{\Delta s}{\Delta t}$
$Q_1(1, 15)$	$\frac{15-20}{1-2} = 5$ mi/hr
$Q_2(1.5, 19)$	$\frac{19-20}{1.5-2} = 2$ mi/hr
$Q_3(1.9, 19.9)$	$\frac{19.9-20}{1.9-2} = 1$ mi/hr

Thus, it appears that the instantaneous speed at $t = 2$ is 0 mi/hr.
At $P(3, 22)$:

Q	Slope of $PQ = \frac{\Delta s}{\Delta t}$
$Q_1(4, 35)$	$\frac{35-22}{4-3} = 13$ mi/hr
$Q_2(3.5, 30)$	$\frac{30-22}{3.5-3} = 16$ mi/hr
$Q_3(3.1, 23)$	$\frac{23-22}{3.1-3} = 10$ mi/hr

Q	Slope of $PQ = \frac{\Delta s}{\Delta t}$
$Q_1(2, 20)$	$\frac{20-22}{2-3} = 2$ mi/hr
$Q_2(2.5, 20)$	$\frac{20-22}{2.5-3} = 4$ mi/hr
$Q_3(2.9, 21.6)$	$\frac{21.6-22}{2.9-3} = 4$ mi/hr

Thus, it appears that the instantaneous speed at $t = 3$ is about 7 mi/hr.

(c) It appears that the curve is increasing the fastest at $t = 3.5$. Thus for $P(3.5, 30)$

Q	Slope of $PQ = \frac{\Delta s}{\Delta t}$
$Q_1(4, 35)$	$\frac{35-30}{4-3.5} = 10$ mi/hr
$Q_2(3.75, 34)$	$\frac{34-30}{3.75-3.5} = 16$ mi/hr
$Q_3(3.6, 32)$	$\frac{32-30}{3.6-3.5} = 20$ mi/hr

Q	Slope of $PQ = \frac{\Delta s}{\Delta t}$
$Q_1(3, 22)$	$\frac{22-30}{3-3.5} = 16$ mi/hr
$Q_2(3.25, 25)$	$\frac{25-30}{3.25-3.5} = 20$ mi/hr
$Q_3(3.4, 28)$	$\frac{28-30}{3.4-3.5} = 20$ mi/hr

Thus, it appears that the instantaneous speed at $t = 3.5$ is about 20 mi/hr.

26. (a) $[0, 3]$: $\frac{\Delta A}{\Delta t} = \frac{10-15}{3-0} \approx -1.67 \frac{\text{gal}}{\text{day}}$; $[0, 5]$: $\frac{\Delta A}{\Delta t} = \frac{3.9-15}{5-0} \approx -2.2 \frac{\text{gal}}{\text{day}}$; $[7, 10]$: $\frac{\Delta A}{\Delta t} = \frac{0-1.4}{10-7} \approx -0.5 \frac{\text{gal}}{\text{day}}$

(b) At $P(1, 14)$:

Q	Slope of $PQ = \frac{\Delta A}{\Delta t}$
$Q_1(2, 12.2)$	$\frac{12.2-14}{2-1} = -1.8$ gal/day
$Q_2(1.5, 13.2)$	$\frac{13.2-14}{1.5-1} = -1.6$ gal/day
$Q_3(1.1, 13.85)$	$\frac{13.85-14}{1.1-1} = -1.5$ gal/day

Q	Slope of $PQ = \frac{\Delta A}{\Delta t}$
$Q_1(0, 15)$	$\frac{15-14}{0-1} = -1$ gal/day
$Q_2(0.5, 14.6)$	$\frac{14.6-14}{0.5-1} = -1.2$ gal/day
$Q_3(0.9, 14.86)$	$\frac{14.86-14}{0.9-1} = -1.4$ gal/day

Thus, it appears that the instantaneous rate of consumption at $t = 1$ is about -1.45 gal/day.
At $P(4, 6)$:

Q	Slope of $PQ = \frac{\Delta A}{\Delta t}$
$Q_1(5, 3.9)$	$\frac{3.9-6}{5-4} = -2.1$ gal/day
$Q_2(4.5, 4.8)$	$\frac{4.8-6}{4.5-4} = -2.4$ gal/day
$Q_3(4.1, 5.7)$	$\frac{5.7-6}{4.1-4} = -3$ gal/day

Q	Slope of $PQ = \frac{\Delta A}{\Delta t}$
$Q_1(3, 10)$	$\frac{10-6}{3-4} = -4$ gal/day
$Q_2(3.5, 7.8)$	$\frac{7.8-6}{3.5-4} = -3.6$ gal/day
$Q_3(3.9, 6.3)$	$\frac{6.3-6}{3.9-4} = -3$ gal/day

Thus, it appears that the instantaneous rate of consumption at $t = 1$ is -3 gal/day.

(*solution continues on next page*)

At $P(8,1)$:

Q	Slope of $PQ = \frac{\Delta A}{\Delta t}$
$Q_1(9, 0.5)$	$\frac{0.5-1}{9-8} = -0.5$ gal/day
$Q_2(8.5, 0.7)$	$\frac{0.7-1}{8.5-8} = -0.6$ gal/day
$Q_3(8.1, 0.95)$	$\frac{0.95-1}{8.1-8} = -0.5$ gal/day

Q	Slope of $PQ = \frac{\Delta A}{\Delta t}$
$Q_1(7, 1.4)$	$\frac{1.4-1}{7-8} = -0.6$ gal/day
$Q_2(7.5, 1.3)$	$\frac{1.3-1}{7.5-8} = -0.6$ gal/day
$Q_3(7.9, 1.04)$	$\frac{1.04-1}{7.9-8} = -0.6$ gal/day

Thus, it appears that the instantaneous rate of consumption at $t = 1$ is -0.55 gal/day.

(c) It appears that the curve (the consumption) is decreasing the fastest at $t = 3.5$. Thus for $P(3.5, 7.8)$

Q	Slope of $PQ = \frac{\Delta A}{\Delta t}$
$Q_1(4.5, 4.8)$	$\frac{4.8-7.8}{4.5-3.5} = -3$ gal/day
$Q_2(4, 6)$	$\frac{6-7.8}{4-3.5} = -3.6$ gal/day
$Q_3(3.6, 7.4)$	$\frac{7.4-7.8}{3.6-3.5} = -4$ gal/day

Q	Slope of $PQ = \frac{\Delta s}{\Delta t}$
$Q_1(2.5, 11.2)$	$\frac{11.2-7.8}{2.5-3.5} = -3.4$ gal/day
$Q_2(3, 10)$	$\frac{10-7.8}{3-3.5} = -4.4$ gal/day
$Q_3(3.4, 8.2)$	$\frac{8.2-7.8}{3.4-3.5} = -4$ gal/day

Thus, it appears that the rate of consumption at $t = 3.5$ is about -4 gal/day.

2.2 LIMIT OF A FUNCTION AND LIMIT LAWS

1. (a) Does not exist. As x approaches 1 from the right, $g(x)$ approaches 0. As x approaches 1 from the left, $g(x)$ approaches 1. There is no single number L that all the values $g(x)$ get arbitrarily close to as $x \to 1$.
   (b) 1 (c) 0 (d) 0.5

2. (a) 0
   (b) -1
   (c) Does not exist. As t approaches 0 from the left, $f(t)$ approaches -1. As t approaches 0 from the right, $f(t)$ approaches 1. There is no single number L that $f(t)$ gets arbitrarily close to as $t \to 0$.
   (d) -1

3. (a) True (b) True (c) False
   (d) False (e) False (f) True
   (g) True (h) False (i) True
   (j) True (k) False

4. (a) False (b) False (c) True
   (d) True (e) True (f) True
   (g) False (h) True (i) False

5. $\lim_{x\to 0} \frac{x}{|x|}$ does not exist because $\frac{x}{|x|} = \frac{x}{x} = 1$ if $x > 0$ and $\frac{x}{|x|} = \frac{x}{-x} = -1$ if $x < 0$. As x approaches 0 from the left, $\frac{x}{|x|}$ approaches -1. As x approaches 0 from the right, $\frac{x}{|x|}$ approaches 1. There is no single number L that all the function values get arbitrarily close to as $x \to 0$.

6. As x approaches 1 from the left, the values of $\frac{1}{x-1}$ become increasingly large and negative. As x approaches 1 from the right, the values become increasingly large and positive. There is no number L that all the function values get arbitrarily close to as $x \to 1$, so $\lim_{x\to 1} \frac{1}{x-1}$ does not exist.

7. Nothing can be said about $f(x)$ because the existence of a limit as $x \to x_0$ does not depend on how the function is defined at x_0. In order for a limit to exist, $f(x)$ must be arbitrarily close to a single real number L when x is close enough to x_0. That is, the existence of a limit depends on the values of $f(x)$ for x near x_0, not on the definition of $f(x)$ at x_0 itself.

8. Nothing can be said. In order for $\lim_{x\to 0} f(x)$ to exist, $f(x)$ must close to a single value for x near 0 regardless of the value $f(0)$ itself.

9. No, the definition does not require that f be defined at $x=1$ in order for a limiting value to exist there. If $f(1)$ is defined, it can be any real number, so we can conclude nothing about $f(1)$ from $\lim_{x\to 1} f(x) = 5$.

10. No, because the existence of a limit depends on the values of $f(x)$ when x is near 1, not on $f(1)$ itself. If $\lim_{x\to 1} f(x)$ exists, its value may be some number other than $f(1) = 5$. We can conclude nothing about $\lim_{x\to 1} f(x)$, whether it exists or what its value is if it does exist, from knowing the value of $f(1)$ alone.

11. $\lim_{x\to -3} (x^2 - 13) = (-3)^2 - 13 = 9 - 13 = -4$

12. $\lim_{x\to 2} (-x^2 + 5x - 2) = -(2)^2 + 5(2) - 2 = -4 + 10 - 2 = 4$

13. $\lim_{t\to 6} 8(t-5)(t-7) = 8(6-5)(6-7) = -8$

14. $\lim_{x\to -2} (x^3 - 2x^2 + 4x + 8) = (-2)^3 - 2(-2)^2 + 4(-2) + 8 = -8 - 8 - 8 + 8 = -16$

15. $\lim_{x\to 2} \frac{2x+5}{11-x^3} = \frac{2(2)+5}{11-(2)^3} = \frac{9}{3} = 3$

16. $\lim_{t\to 2/3} (8-3s)(2s-1) = \left(8 - 5\left(\frac{2}{3}\right)\right)\left(2\left(\frac{2}{3}\right) - 1\right) = (8-2)\left(\left(\frac{4}{3}\right) - 1\right) = (6)\left(\frac{1}{3}\right) = 2$

17. $\lim_{x\to -1/2} 4x(3x+4)^2 = 4\left(-\frac{1}{2}\right)\left(3\left(-\frac{1}{2}\right) + 4\right)^2 = (-2)\left(-\frac{3}{2} + 4\right)^2 = (-2)\left(\frac{5}{2}\right)^2 = -\frac{25}{2}$

18. $\lim_{y\to 2} \frac{y+2}{y^2+5y+6} = \frac{2+2}{(2)^2+5(2)+6} = \frac{4}{4+10+6} = \frac{4}{20} = \frac{1}{5}$

19. $\lim_{y\to -3} (5-y)^{4/3} = [5-(-3)]^{4/3} = (8)^{4/3} = \left((8)^{1/3}\right)^4 = 2^4 = 16$

20. $\lim_{z\to 4} \sqrt{z^2 - 10} = \sqrt{4^2 - 10} = \sqrt{16-10} = \sqrt{6}$

21. $\lim_{h\to 0} \frac{3}{\sqrt{3h+1}+1} = \frac{3}{\sqrt{3(0)+1}+1} = \frac{3}{\sqrt{1}+1} = \frac{3}{2}$

22. $\lim_{h\to 0} \frac{\sqrt{5h+4}-2}{h} = \lim_{h\to 0} \frac{\sqrt{5h+4}-2}{h} \cdot \frac{\sqrt{5h+4}+2}{\sqrt{5h+4}+2} = \lim_{h\to 0} \frac{(5h+4)-4}{h\left(\sqrt{5h+4}+2\right)} = \lim_{h\to 0} \frac{5h}{h\left(\sqrt{5h+4}+2\right)} = \lim_{h\to 0} \frac{5}{\sqrt{5h+4}+2} = \frac{5}{\sqrt{4}+2} = \frac{5}{4}$

23. $\lim_{x\to 5} \frac{x-5}{x^2-25} = \lim_{x\to 5} \frac{x-5}{(x+5)(x-5)} = \lim_{x\to 5} \frac{1}{x+5} = \frac{1}{5+5} = \frac{1}{10}$

24. $\lim_{x\to -3} \frac{x+3}{x^2+4x+3} = \lim_{x\to -3} \frac{x+3}{(x+3)(x+1)} = \lim_{x\to -3} \frac{1}{x+1} = \frac{1}{-3+1} = -\frac{1}{2}$

25. $\lim_{x\to -5} \frac{x^2+3x-10}{x+5} = \lim_{x\to -5} \frac{(x+5)(x-2)}{x+5} = \lim_{x\to -5} (x-2) = -5-2 = -7$

26. $\lim_{x\to 2} \frac{x^2-7x-10}{x-2} = \lim_{x\to 2} \frac{(x-5)(x-2)}{x-2} = \lim_{x\to 2} (x-5) = 2-5 = -3$

27. $\lim_{t\to 1} \frac{t^2+t-2}{t^2-1} = \lim_{t\to 1} \frac{(t+2)(t-1)}{(t-1)(t+1)} = \lim_{t\to 1} \frac{t+2}{t+1} = \frac{1+2}{1+1} = \frac{3}{2}$

28. $\lim_{t\to -1} \frac{t^2+3t+2}{t^2-t-2} = \lim_{t\to -1} \frac{(t+2)(t+1)}{(t-2)(t+1)} = \lim_{t\to -1} \frac{t+2}{t-2} = \frac{-1+2}{-1-2} = -\frac{1}{3}$

29. $\lim_{x\to -2} \frac{-2x-4}{x^3+2x^2} = \lim_{x\to -2} \frac{-2(x+2)}{x^2(x+2)} = \lim_{x\to -2} \frac{-2}{x^2} = \frac{-2}{4} = -\frac{1}{2}$

30. $\lim_{y\to 0} \frac{5y^3+8y^2}{3y^4-16y^2} = \lim_{y\to 0} \frac{y^2(5y+8)}{y^2(3y^2-16)} = \lim_{y\to 0} \frac{5y+8}{3y^2-16} = \frac{8}{-16} = -\frac{1}{2}$

31. $\lim_{x\to 1} \frac{x^{-1}-1}{x-1} = \lim_{x\to 1} \frac{\frac{1-x}{x}}{x-1} = \lim_{x\to 1} \left(\frac{1-x}{x}\cdot\frac{1}{x-1}\right) = \lim_{x\to 1} \ -\frac{1}{x} = -1$

32. $\lim_{x\to 0} \frac{\frac{1}{x-1}+\frac{1}{x+1}}{x} = \lim_{x\to 0} \frac{\frac{(x+1)+(x-1)}{(x-1)(x+1)}}{x} = \lim_{x\to 0} \left(\frac{2x}{(x-1)(x+1)}\cdot\frac{1}{x}\right) = \lim_{x\to 0} \frac{2}{(x-1)(x+1)} = \frac{2}{-1} = -2$

33. $\lim_{u\to 1} \frac{u^4-1}{u^3-1} = \lim_{u\to 1} \frac{(u^2+1)(u+1)(u-1)}{(u^2+u+1)(u-1)} = \lim_{u\to 1} \frac{(u^2+1)(u+1)}{u^2+u+1} = \frac{(1+1)(1+1)}{1+1+1} = \frac{4}{3}$

34. $\lim_{v\to 2} \frac{v^3-8}{v^4-16} = \lim_{v\to 2} \frac{(v-2)(v^2+2v+4)}{(v-2)(v+2)(v^2+4)} = \lim_{v\to 2} \frac{v^2+2v+4}{(v+2)(v^2+4)} = \frac{4+4+4}{(4)(8)} = \frac{12}{32} = \frac{3}{8}$

35. $\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9} = \lim_{x\to 9} \frac{\sqrt{x}-3}{(\sqrt{x}-3)(\sqrt{x}+3)} = \lim_{x\to 9} \frac{1}{\sqrt{x}+3} = \frac{1}{\sqrt{9}+3} = \frac{1}{6}$

36. $\lim_{x\to 4} \frac{4x-x^2}{2-\sqrt{x}} = \lim_{x\to 4} \frac{x(4-x)}{2-\sqrt{x}} = \lim_{x\to 4} \frac{x(2+\sqrt{x})(2-\sqrt{x})}{2-\sqrt{x}} = \lim_{x\to 4} x\left(2+\sqrt{x}\right) = 4(2+2) = 16$

37. $\lim_{x\to 1} \frac{x-1}{\sqrt{x+3}-2} = \lim_{x\to 1} \frac{(x-1)\left(\sqrt{x+3}+2\right)}{\left(\sqrt{x+3}-2\right)\left(\sqrt{x+3}+2\right)} = \lim_{x\to 1} \frac{(x-1)\left(\sqrt{x+3}+2\right)}{(x+3)-4} = \lim_{x\to 1}\left(\sqrt{x+3}+2\right) = \sqrt{4}+2 = 4$

38. $\lim_{x\to -1} \frac{\sqrt{x^2+8}-3}{x+1} = \lim_{x\to -1} \frac{\left(\sqrt{x^2+8}-3\right)\left(\sqrt{x^2+8}+3\right)}{(x+1)\left(\sqrt{x^2+8}+3\right)} = \lim_{x\to -1} \frac{(x^2+8)-9}{(x+1)\left(\sqrt{x^2+8}+3\right)} = \lim_{x\to -1} \frac{(x+1)(x-1)}{(x+1)\left(\sqrt{x^2+8}+3\right)}$

$= \lim_{x\to -1} \frac{x-1}{\sqrt{x^2+8}+3} = \frac{-2}{3+3} = -\frac{1}{3}$

39. $\lim_{x\to 2} \frac{\sqrt{x^2+12}-4}{x-2} = \lim_{x\to 2} \frac{\left(\sqrt{x^2+12}-4\right)\left(\sqrt{x^2+12}+4\right)}{(x-2)\left(\sqrt{x^2+12}+4\right)} = \lim_{x\to 2} \frac{(x^2+12)-16}{(x-2)\left(\sqrt{x^2+12}+4\right)} = \lim_{x\to 2} \frac{(x-2)(x+2)}{(x-2)\left(\sqrt{x^2+12}+4\right)}$

$= \lim_{x\to 2} \frac{x+2}{\sqrt{x^2+12}+4} = \frac{4}{\sqrt{16}+4} = \frac{1}{2}$

40. $\lim_{x\to -2} \frac{x+2}{\sqrt{x^2+5}-3} = \lim_{x\to -2} \frac{(x+2)\left(\sqrt{x^2+5}+3\right)}{\left(\sqrt{x^2+5}-3\right)\left(\sqrt{x^2+5}+3\right)} = \lim_{x\to -2} \frac{(x+2)\left(\sqrt{x^2+5}+3\right)}{(x^2+5)-9} = \lim_{x\to -2} \frac{(x+2)\left(\sqrt{x^2+5}+3\right)}{(x+2)(x-2)}$
$= \lim_{x\to -2} \frac{\sqrt{x^2+5}+3}{x-2} = \frac{\sqrt{9}+3}{-4} = -\frac{3}{2}$

41. $\lim_{x\to -3} \frac{2-\sqrt{x^2-5}}{x+3} = \lim_{x\to -3} \frac{\left(2-\sqrt{x^2-5}\right)\left(2+\sqrt{x^2-5}\right)}{(x+3)\left(2+\sqrt{x^2-5}\right)} = \lim_{x\to -3} \frac{4-(x^2-5)}{(x+3)\left(2+\sqrt{x^2-5}\right)} = \lim_{x\to -3} \frac{9-x^2}{(x+3)\left(2+\sqrt{x^2-5}\right)}$
$= \lim_{x\to -3} \frac{(3-x)(3+x)}{(x+3)\left(2+\sqrt{x^2-5}\right)} = \lim_{x\to -3} \frac{3-x}{2+\sqrt{x^2-5}} = \frac{6}{2+\sqrt{4}} = \frac{3}{2}$

42. $\lim_{x\to 4} \frac{4-x}{5-\sqrt{x^2+9}} = \lim_{x\to 4} \frac{(4-x)\left(5+\sqrt{x^2+9}\right)}{\left(5-\sqrt{x^2+9}\right)\left(5+\sqrt{x^2+9}\right)} = \lim_{x\to 4} \frac{(4-x)\left(5+\sqrt{x^2+9}\right)}{25-(x^2+9)} = \lim_{x\to 4} \frac{(4-x)\left(5+\sqrt{x^2+9}\right)}{16-x^2}$
$= \lim_{x\to 4} \frac{(4-x)\left(5+\sqrt{x^2+9}\right)}{(4-x)(4+x)} = \lim_{x\to 4} \frac{5+\sqrt{x^2+9}}{4+x} = \frac{5+\sqrt{25}}{8} = \frac{5}{4}$

43. $\lim_{x\to 0} (2\sin x - 1) = 2\sin 0 - 1 = 0 - 1 = -1$

44. $\lim_{x\to 0} \sin^2 x = \left(\lim_{x\to 0} \sin x\right)^2 = (\sin 0)^2 = 0^2 = 0$

45. $\lim_{x\to 0} \sec x = \lim_{x\to 0} \frac{1}{\cos x} = \frac{1}{\cos 0} = \frac{1}{1} = 1$

46. $\lim_{x\to 0} \tan x = \lim_{x\to 0} \frac{\sin x}{\cos x} = \frac{\sin 0}{\cos 0} = \frac{0}{1} = 0$

47. $\lim_{x\to 0} \frac{1+x+\sin x}{3\cos x} = \frac{1+0+\sin 0}{3\cos 0} = \frac{1+0+0}{3} = \frac{1}{3}$

48. $\lim_{x\to 0} (x^2-1)(2-\cos x) = (0^2-1)(2-\cos 0) = (-1)(2-1) = (-1)(1) = -1$

49. $\lim_{x\to -\pi} \sqrt{x+4}\cos(x+\pi) = \lim_{x\to -\pi} \sqrt{x+4} \cdot \lim_{x\to -\pi} \cos(x+\pi) = \sqrt{-\pi+4} \cdot \cos 0 = \sqrt{4-\pi} \cdot 1 = \sqrt{4-\pi}$

50. $\lim_{x\to 0} \sqrt{7+\sec^2 x} = \sqrt{\lim_{x\to 0}(7+\sec^2 x)} = \sqrt{7+\lim_{x\to 0} \sec^2 x} = \sqrt{7+\sec^2 0} = \sqrt{7+(1)^2} = 2\sqrt{2}$

51. (a) quotient rule
(b) difference and power rules
(c) sum and constant multiple rules

52. (a) quotient rule
(b) power and product rules
(c) difference and constant multiple rules

53. (a) $\lim_{x\to c} f(x)g(x) = \left[\lim_{x\to c} f(x)\right]\left[\lim_{x\to c} g(x)\right] = (5)(-2) = -10$
(b) $\lim_{x\to c} 2f(x)g(x) = 2\left[\lim_{x\to c} f(x)\right]\left[\lim_{x\to c} g(x)\right] = 2(5)(-2) = -20$
(c) $\lim_{x\to c} [f(x)+3g(x)] = \lim_{x\to c} f(x) + 3\lim_{x\to c} g(x) = 5+3(-2) = -1$
(d) $\lim_{x\to c} \frac{f(x)}{f(x)-g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} f(x) - \lim_{x\to c} g(x)} = \frac{5}{5-(-2)} = \frac{5}{7}$

54. (a) $\lim_{x\to 4}[g(x)+3] = \lim_{x\to 4} g(x) + \lim_{x\to 4} 3 = -3+3 = 0$

(b) $\lim_{x\to 4} xf(x) = \lim_{x\to 4} x \cdot \lim_{x\to 4} f(x) = (4)(0) = 0$

(c) $\lim_{x\to 4}[g(x)]^2 = \left[\lim_{x\to 4} g(x)\right]^2 = [-3]^2 = 9$

(d) $\lim_{x\to 4} \frac{g(x)}{f(x)-1} = \frac{\lim_{x\to 4} g(x)}{\lim_{x\to 4} f(x) - \lim_{x\to 4} 1} = \frac{-3}{0-1} = 3$

55. (a) $\lim_{x\to b}[f(x)+g(x)] = \lim_{x\to b} f(x) + \lim_{x\to b} g(x) = 7+(-3) = 4$

(b) $\lim_{x\to b} f(x)\cdot g(x) = \left[\lim_{x\to b} f(x)\right]\left[\lim_{x\to b} g(x)\right] = (7)(-3) = -21$

(c) $\lim_{x\to b} 4g(x) = \left[\lim_{x\to b} 4\right]\left[\lim_{x\to b} g(x)\right] = (4)(-3) = -12$

(d) $\lim_{x\to b} f(x)/g(x) = \lim_{x\to b} f(x)/\lim_{x\to b} g(x) = \frac{7}{-3} = -\frac{7}{3}$

56. (a) $\lim_{x\to -2}[p(x)+r(x)+s(x)] = \lim_{x\to -2} p(x) + \lim_{x\to -2} r(x) + \lim_{x\to -2} s(x) = 4+0+(-3) = 1$

(b) $\lim_{x\to -2} p(x)\cdot r(x)\cdot s(x) = \left[\lim_{x\to -2} p(x)\right]\left[\lim_{x\to -2} r(x)\right]\left[\lim_{x\to -2} s(x)\right] = (4)(0)(-3) = 0$

(c) $\lim_{x\to -2}[-4p(x)+5r(x)]/s(x) = \left[-4\lim_{x\to -2} p(x) + 5\lim_{x\to -2} r(x)\right] \Big/ \lim_{x\to -2} s(x) = [-4(4)+5(0)]/-3 = \frac{16}{3}$

57. $\lim_{h\to 0} \frac{(1+h)^2-1^2}{h} = \lim_{h\to 0} \frac{1+2h+h^2-1}{h} = \lim_{h\to 0} \frac{h(2+h)}{h} = \lim_{h\to 0}(2+h) = 2$

58. $\lim_{h\to 0} \frac{(-2+h)^2-(-2)^2}{h} = \lim_{h\to 0} \frac{4-4h+h^2-4}{h} = \lim_{h\to 0} \frac{h(h-4)}{h} = \lim_{h\to 0}(h-4) = -4$

59. $\lim_{h\to 0} \frac{[3(2+h)-4]-[3(2)-4]}{h} = \lim_{h\to 0} \frac{3h}{h} = 3$

60. $\lim_{h\to 0} \frac{\left(\frac{1}{-2+h}\right)-\left(\frac{1}{-2}\right)}{h} = \lim_{h\to 0} \frac{\frac{-2}{-2+h}-1}{-2h} = \lim_{h\to 0} \frac{-2-(-2+h)}{-2h(-2+h)} = \lim_{h\to 0} \frac{-h}{h(4-2h)} = -\frac{1}{4}$

61. $\lim_{h\to 0} \frac{\sqrt{7+h}-\sqrt{7}}{h} = \lim_{h\to 0} \frac{\left(\sqrt{7+h}-\sqrt{7}\right)\left(\sqrt{7+h}+\sqrt{7}\right)}{h\left(\sqrt{7+h}+\sqrt{7}\right)} = \lim_{h\to 0} \frac{(7+h)-7}{h\left(\sqrt{7+h}+\sqrt{7}\right)} = \lim_{h\to 0} \frac{h}{h\left(\sqrt{7+h}+\sqrt{7}\right)} = \lim_{h\to 0} \frac{1}{\sqrt{7+h}+\sqrt{7}} = \frac{1}{2\sqrt{7}}$

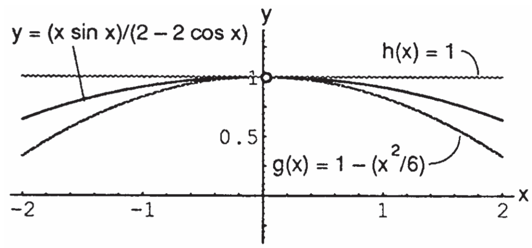
62. $\lim_{h\to 0} \frac{\sqrt{3(0+h)+1}-\sqrt{3(0)+1}}{h} = \lim_{h\to 0} \frac{\left(\sqrt{3h+1}-1\right)\left(\sqrt{3h+1}+1\right)}{h\left(\sqrt{3h+1}+1\right)} = \lim_{h\to 0} \frac{(3h+1)-1}{h\left(\sqrt{3h+1}+1\right)} = \lim_{h\to 0} \frac{3h}{h\left(\sqrt{3h+1}+1\right)} = \lim_{h\to 0} \frac{3}{\sqrt{3h+1}+1} = \frac{3}{2}$

63. $\lim_{x\to 0} \sqrt{5-2x^2} = \sqrt{5-2(0)^2} = \sqrt{5}$ and $\lim_{x\to 0} \sqrt{5-x^2} = \sqrt{5-(0)^2} = \sqrt{5}$; by the sandwich theorem, $\lim_{x\to 0} f(x) = \sqrt{5}$

64. $\lim_{x\to 0}(2-x^2) = 2-0 = 2$ and $\lim_{x\to 0} 2\cos x = 2(1) = 2$; by the sandwich theorem, $\lim_{x\to 0} g(x) = 2$

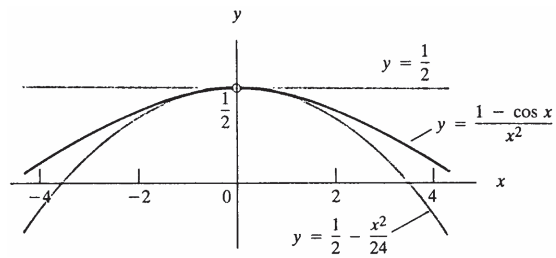
65. (a) $\lim_{x\to 0}\left(1-\frac{x^2}{6}\right) = 1-\frac{0}{6} = 1$ and $\lim_{x\to 0} 1 = 1$; by the sandwich theorem, $\lim_{x\to 0} \frac{x\sin x}{2-2\cos x} = 1$

(b) For $x \neq 0$, $y = (x \sin x)/(2 - 2\cos x)$ lies between the other two graphs in the figure, and the graphs converge as $x \to 0$.



66. (a) $\lim\limits_{x\to 0}\left(\frac{1}{2} - \frac{x^2}{24}\right) = \lim\limits_{x\to 0} \frac{1}{2} - \lim\limits_{x\to 0} \frac{x^2}{24} = \frac{1}{2} - 0 = \frac{1}{2}$ and $\lim\limits_{x\to 0} \frac{1}{2} = \frac{1}{2}$; by the sandwich theorem, $\lim\limits_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$.

(b) For all $x \neq 0$, the graph of $f(x) = (1 - \cos x)/x^2$ lies between the line $y = \frac{1}{2}$ and the parabola $y = \frac{1}{2} - x^2/24$, and the graphs converge as $x \to 0$.



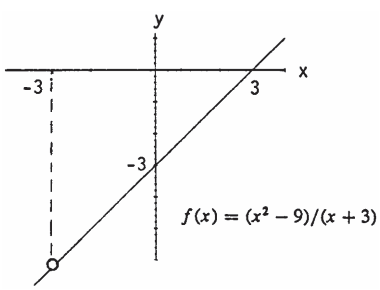
67. (a) $f(x) = (x^2 - 9)/(x + 3)$

x	−3.1	−3.01	−3.001	−3.0001	−3.00001	−3.000001
$f(x)$	−6.1	−6.01	−6.001	−6.0001	−6.00001	−6.000001

x	−2.9	−2.99	−2.999	−2.9999	−2.99999	−2.999999
$f(x)$	−5.9	−5.99	−5.999	−5.9999	−5.99999	−5.999999

The estimate is $\lim\limits_{x\to -3} f(x) = -6$.

(b)

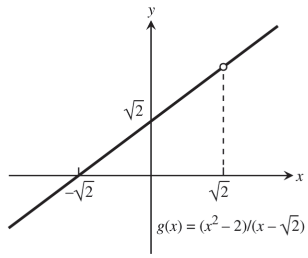


(c) $f(x) = \frac{x^2-9}{x+3} = \frac{(x+3)(x-3)}{x+3} = x - 3$ if $x \neq -3$, and $\lim\limits_{x\to -3} (x - 3) = -3 - 3 = -6$.

68. (a) $g(x) = (x^2 - 2)/\left(x - \sqrt{2}\right)$

x	1.4	1.41	1.414	1.4142	1.41421	1.414213
$g(x)$	2.81421	2.82421	2.82821	2.828413	2.828423	2.828426

(b)



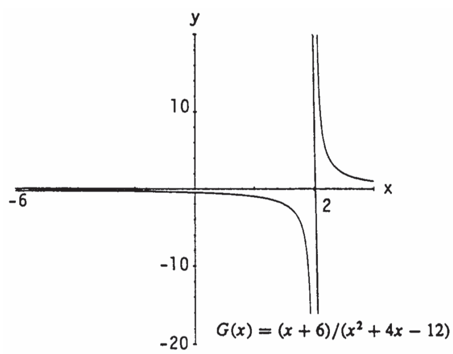
(c) $g(x)=\frac{x^2-2}{x-\sqrt{2}}=\frac{\left(x+\sqrt{2}\right)\left(x-\sqrt{2}\right)}{\left(x-\sqrt{2}\right)}=x+\sqrt{2}$ if $x\neq\sqrt{2}$, and $\lim\limits_{x\to\sqrt{2}}\left(x+\sqrt{2}\right)=\sqrt{2}+\sqrt{2}=2\sqrt{2}$.

69. (a) $G(x)=(x+6)/(x^2+4x-12)$

x	−5.9	−5.99	−5.999	−5.9999	−5.99999	−5.999999
$G(x)$	−.126582	−.1251564	−.1250156	−.1250015	−.1250001	−.1250000

x	−6.1	−6.01	−6.001	−6.0001	−6.00001	−6.000001
$G(x)$	−.123456	−.124843	−.124984	−.124998	−.124999	−.124999

(b)



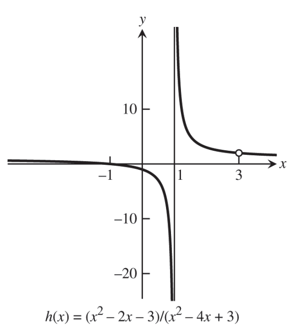
(c) $G(x)=\frac{x+6}{(x^2+4x-12)}=\frac{x+6}{(x+6)(x-2)}=\frac{1}{x-2}$ if $x\neq-6$, and $\lim\limits_{x\to-6}\frac{1}{x-2}=\frac{1}{-6-2}=-\frac{1}{8}=-0.125$.

70. (a) $h(x)=(x^2-2x-3)/(x^2-4x+3)$

x	2.9	2.99	2.999	2.9999	2.99999	2.999999
$h(x)$	2.052631	2.005025	2.000500	2.000050	2.000005	2.0000005

x	3.1	3.01	3.001	3.0001	3.00001	3.000001
$h(x)$	1.952380	1.995024	1.999500	1.999950	1.999995	1.999999

(b)



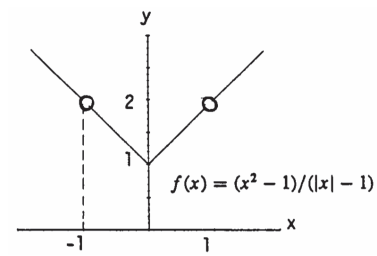
(c) $h(x)=\frac{x^2-2x-3}{x^2-4x+3}=\frac{(x-3)(x+1)}{(x-3)(x-1)}=\frac{x+1}{x-1}$ if $x\neq3$, and $\lim\limits_{x\to3}\frac{x+1}{x-1}=\frac{3+1}{3-1}=\frac{4}{2}=2$.

71. (a) $f(x) = (x^2 - 1)/(|x| - 1)$

x	-1.1	-1.01	-1.001	-1.0001	-1.00001	-1.000001
$f(x)$	2.1	2.01	2.001	2.0001	2.00001	2.000001

x	$-.9$	$-.99$	$-.999$	$-.9999$	$-.99999$	$-.999999$
$f(x)$	1.9	1.99	1.999	1.9999	1.99999	1.999999

(b)



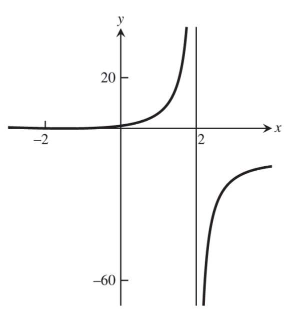
(c) $f(x) = \frac{x^2-1}{|x|-1} = \begin{cases} \frac{(x+1)(x-1)}{x-1} = x+1, & x \ge 0 \text{ and } x \ne 1 \\ \frac{(x+1)(x-1)}{-(x+1)} = 1-x, & x < 0 \text{ and } x \ne -1 \end{cases}$, and $\lim_{x \to -1} (1-x) = 1-(-1) = 2.$

72. (a) $F(x) = (x^2 + 3x + 2)/(2 - |x|)$

x	-2.1	-2.01	-2.001	-2.0001	-2.00001	-2.000001
$F(x)$	-1.1	-1.01	-1.001	-1.0001	-1.00001	-1.000001

x	-1.9	-1.99	-1.999	-1.9999	-1.99999	-1.999999
$F(x)$	$-.9$	$-.99$	$-.999$	$-.9999$	$-.99999$	$-.999999$

(b)



(c) $F(x) = \frac{x^2+3x+2}{2-|x|} = \begin{cases} \frac{(x+2)(x+1)}{2-x}, & x \ge 0 \\ \frac{(x+2)(x+1)}{2+x} = x+1, & x < 0 \text{ and } x \ne -2 \end{cases}$, and $\lim_{x \to -2} (x+1) = -2+1 = -1.$

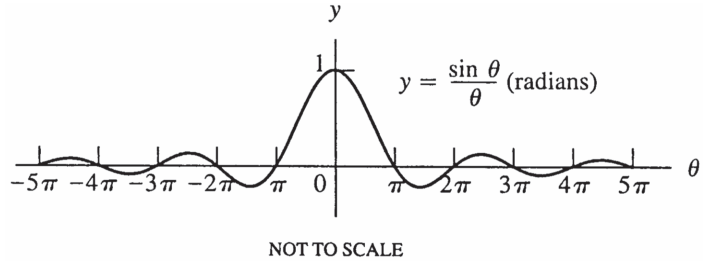
73. (a) $g(\theta) = (\sin\theta)/\theta$

θ	.1	.01	.001	.0001	.00001	.000001
$g(\theta)$	.998334	.999983	.999999	.999999	.999999	.999999

θ	$-.1$	$-.01$	$-.001$	$-.0001$	$-.00001$	$-.000001$
$g(\theta)$	.998334	.999983	.999999	.999999	.999999	.999999

$\lim_{\theta \to 0} g(\theta) = 1$

(b)



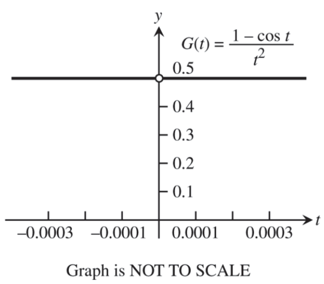
74. (a) $G(t) = (1 - \cos t)/t^2$

t	.1	.01	.001	.0001	.00001	.000001
$G(t)$	.499583	.499995	.499999	.5	.5	.5

t	−.1	−.01	−.001	−.0001	−.00001	−.000001
$G(t)$	.499583	.499995	.499999	.5	.5	.5

$\lim_{t \to 0} G(t) = 0.5$

(b)



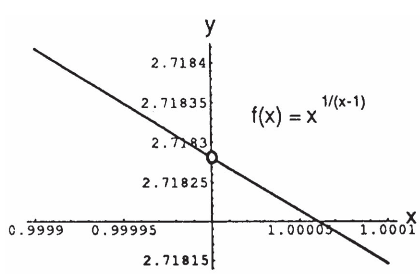
75. (a) $f(x) = x^{1/(1-x)}$

x	.9	.99	.999	.9999	.99999	.999999
$f(x)$	.348678	.366032	.367695	.367861	.367877	.367879

x	1.1	1.01	1.001	1.0001	1.00001	1.000001
$f(x)$	.385543	.369711	.368063	.367897	.367881	.367878

$\lim_{x \to 1} f(x) \approx 0.36788$

(b)



Graph is NOT TO SCALE. Also, the intersection of the axes is not the origin: the axes intersect at the point (1, 2.71820).

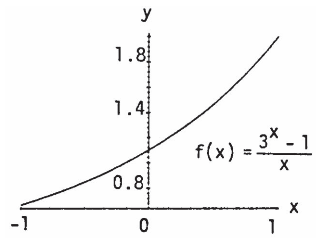
76. (a) $f(x) = (3^x - 1)/x$

x	.1	.01	.001	.0001	.00001	.000001
$f(x)$	1.161231	1.104669	1.099215	1.098672	1.098618	1.098612

x	–.1	–.01	–.001	–.0001	–.00001	–.000001
$f(x)$	1.040415	1.092599	1.098009	1.098551	1.098606	1.098611

$\lim_{x\to 1} f(x) \approx 1.0986$

(b)



77. $\lim_{x\to c} f(x)$ exists at those points c where $\lim_{x\to c} x^4 = \lim_{x\to c} x^2$. Thus, $c^4 = c^2 \Rightarrow c^2(1-c^2) = 0 \Rightarrow c = 0, 1$, or -1. Moreover, $\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2 = 0$ and $\lim_{x\to -1} f(x) = \lim_{x\to 1} f(x) = 1$.

78. Nothing can be concluded about the values of f, g, and h at $x = 2$. Yes, $f(2)$ could be 0. Since the conditions of the sandwich theorem are satisfied, $\lim_{x\to 2} f(x) = -5 \neq 0$.

79. $1 = \lim_{x\to 4} \frac{f(x)-5}{x-2} = \frac{\lim_{x\to 4} f(x) - \lim_{x\to 4} 5}{\lim_{x\to 4} x - \lim_{x\to 4} 2} = \frac{\lim_{x\to 4} f(x) - 5}{4-2} \Rightarrow \lim_{x\to 4} f(x) - 5 = 2(1) \Rightarrow \lim_{x\to 4} f(x) = 2 + 5 = 7.$

80. (a) $1 = \lim_{x\to -2} \frac{f(x)}{x^2} = \frac{\lim_{x\to -2} f(x)}{\lim_{x\to -2} x^2} = \frac{\lim_{x\to -2} f(x)}{4} \Rightarrow \lim_{x\to -2} f(x) = 4.$

(b) $1 = \lim_{x\to -2} \frac{f(x)}{x^2} = \left[\lim_{x\to -2} \frac{f(x)}{x}\right]\left[\lim_{x\to -2} \frac{1}{x}\right] = \left[\lim_{x\to -2} \frac{f(x)}{x}\right]\left(\frac{1}{-2}\right) \Rightarrow \lim_{x\to -2} \frac{f(x)}{x} = -2.$

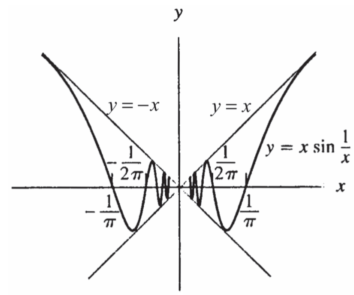
81. (a) $0 = 3 \cdot 0 = \left[\lim_{x\to 2} \frac{f(x)-5}{x-2}\right]\left[\lim_{x\to 2} (x-2)\right] = \lim_{x\to 2} \left[\left(\frac{f(x)-5}{x-2}\right)(x-2)\right] = \lim_{x\to 2} [f(x) - 5]$
$= \lim_{x\to 2} f(x) - 5 \Rightarrow \lim_{x\to 2} f(x) = 5.$

(b) $0 = 4 \cdot 0 = \left[\lim_{x\to 2} \frac{f(x)-5}{x-2}\right]\left[\lim_{x\to 2} (x-2)\right] \Rightarrow \lim_{x\to 2} f(x) = 5$ as in part (a).

82. (a) $0 = 1 \cdot 0 = \left[\lim_{x\to 0} \frac{f(x)}{x^2}\right]\left[\lim_{x\to 0} x\right]^2 = \left[\lim_{x\to 0} \frac{f(x)}{x^2}\right]\left[\lim_{x\to 0} x^2\right] = \lim_{x\to 0} \left[\frac{f(x)}{x^2} \cdot x^2\right] = \lim_{x\to 0} f(x).$
That is, $\lim_{x\to 0} f(x) = 0.$

(b) $0 = 1 \cdot 0 = \left[\lim_{x\to 0} \frac{f(x)}{x^2}\right]\left[\lim_{x\to 0} x\right] = \lim_{x\to 0} \left[\frac{f(x)}{x^2} \cdot x\right] = \lim_{x\to 0} \frac{f(x)}{x}$. That is, $\lim_{x\to 0} \frac{f(x)}{x} = 0.$

83. (a) $\lim_{x\to 0} x\sin\frac{1}{x} = 0$

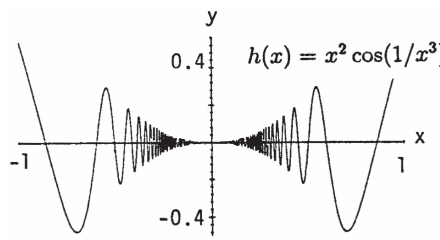


(b) $-1 \le \sin\frac{1}{x} \le 1$ for $x \ne 0$:

$x > 0 \Rightarrow -x \le x\sin\frac{1}{x} \le x \Rightarrow \lim_{x\to 0} x\sin\frac{1}{x} = 0$ by the sandwich theorem;

$x < 0 \Rightarrow -x \ge x\sin\frac{1}{x} \ge x \Rightarrow \lim_{x\to 0} x\sin\frac{1}{x} = 0$ by the sandwich theorem.

84. (a) $\lim_{x\to 0} x^2\cos\left(\frac{1}{x^3}\right) = 0$



(b) $-1 \le \cos\left(\frac{1}{x^3}\right) \le 1$ for $x \ne 0 \Rightarrow -x^2 \le x^2\cos\left(\frac{1}{x^3}\right) \le x^2 \Rightarrow \lim_{x\to 0} x^2\cos\left(\frac{1}{x^3}\right) = 0$ by the sandwich theorem since $\lim_{x\to 0} x^2 = 0$.

85–90. Example CAS commands:

Maple:

```
f := x -> (x^4 - 16)/(x - 2);
x0 := 2;
plot( f(x), x = x0-1..x0+1, color = black,
      title = "Section 2.2, #85(a)" );
limit( f(x), x = x0 );
```

In Exercise 87, note that the standard cube root, x^(1/3), is not defined for x<0 in many CASs. This can be overcome in Maple by entering the function as f := x -> (surd(x+1, 3) − 1)/x.

Mathematica: (assigned function and values for x0 and h may vary)

```
Clear[f, x]
f[x_]:=(x^3 - x^2 - 5x - 3)/(x + 1)^2
x0= -1; h = 0.1;
Plot[f[x],{x, x0 - h, x0 + h}]
Limit[f[x], x → x0]
```

2.3 THE PRECISE DEFINITION OF A LIMIT

1. 

Step 1: $|x-5| < \delta \Rightarrow -\delta < x-5 < \delta \Rightarrow -\delta+5 < x < \delta+5$

Step 2: $\delta + 5 = 7 \Rightarrow \delta = 2$, or $-\delta + 5 = 1 \Rightarrow \delta = 4$.

The value of δ which assures $|x-5| < \delta \Rightarrow 1 < x < 7$ is the smaller value, $\delta = 2$.

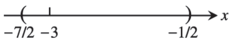
2.

Step 1: $|x-2|<\delta \Rightarrow -\delta<x-2<\delta \Rightarrow -\delta+2<x<\delta+2$

Step 2: $-\delta+2=1 \Rightarrow \delta=1$, or $\delta+2=7 \Rightarrow \delta=5$.

The value of δ which assures $|x-2|<\delta \Rightarrow 1<x<7$ is the smaller value, $\delta=1$.

3.

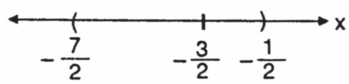


Step 1: $|x-(-3)|<\delta \Rightarrow -\delta<x+3<\delta \Rightarrow -\delta-3<x<\delta-3$

Step 2: $-\delta-3=-\frac{7}{2} \Rightarrow \delta=\frac{1}{2}$, or $\delta-3=-\frac{1}{2} \Rightarrow \delta=\frac{5}{2}$.

The value of δ which assures $|x-(-3)|<\delta \Rightarrow -\frac{7}{2}<x<-\frac{1}{2}$ is the smaller value, $\delta=\frac{1}{2}$.

4.

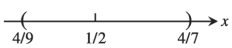


Step 1: $\left|x-\left(-\frac{3}{2}\right)\right|<\delta \Rightarrow -\delta<x+\frac{3}{2}<\delta \Rightarrow -\delta-\frac{3}{2}<x<\delta-\frac{3}{2}$

Step 2: $-\delta-\frac{3}{2}=-\frac{7}{2} \Rightarrow \delta=2$, or $\delta-\frac{3}{2}=-\frac{1}{2} \Rightarrow \delta=1$.

The value of δ which assures $\left|x-\left(-\frac{3}{2}\right)\right|<\delta \Rightarrow -\frac{7}{2}<x<-\frac{1}{2}$ is the smaller value, $\delta=1$.

5.

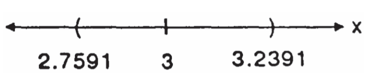


Step 1: $\left|x-\frac{1}{2}\right|<\delta \Rightarrow -\delta<x-\frac{1}{2}<\delta \Rightarrow -\delta+\frac{1}{2}<x<\delta+\frac{1}{2}$

Step 2: $-\delta+\frac{1}{2}=\frac{4}{9} \Rightarrow \delta=\frac{1}{18}$, or $\delta+\frac{1}{2}=\frac{4}{7} \Rightarrow \delta=\frac{1}{14}$.

The value of δ which assures $\left|x-\frac{1}{2}\right|<\delta \Rightarrow \frac{4}{9}<x<\frac{4}{7}$ is the smaller value, $\delta=\frac{1}{18}$.

6.



Step 1: $|x-3|<\delta \Rightarrow -\delta<x-3<\delta \Rightarrow -\delta+3<x<\delta+3$

Step 2: $-\delta+3=2.7591 \Rightarrow \delta=0.2409$, or $\delta+3=3.2391 \Rightarrow \delta=0.2391$.

The value of δ which assures $|x-3|<\delta \Rightarrow 2.7591<x<3.2391$ is the smaller value, $\delta=0.2391$.

7. Step 1: $|x-5|<\delta \Rightarrow -\delta<x-5<\delta \Rightarrow -\delta+5<x<\delta+5$

Step 2: From the graph, $-\delta+5=4.9 \Rightarrow \delta=0.1$, or $\delta+5=5.1 \Rightarrow \delta=0.1$; thus $\delta=0.1$ in either case.

8. Step 1: $|x-(-3)|<\delta \Rightarrow -\delta<x+3<\delta \Rightarrow -\delta-3<x<\delta-3$

Step 2: From the graph, $-\delta-3=-3.1 \Rightarrow \delta=0.1$, or $\delta-3=-2.9 \Rightarrow \delta=0.1$; thus $\delta=0.1$.

9. Step 1: $|x-1|<\delta \Rightarrow -\delta<x-1<\delta \Rightarrow -\delta+1<x<\delta+1$

Step 2: From the graph, $-\delta+1=\frac{9}{16} \Rightarrow \delta=\frac{7}{16}$, or $\delta+1=\frac{25}{16} \Rightarrow \delta=\frac{9}{16}$; thus $\delta=\frac{7}{16}$.

10. Step 1: $|x-3|<\delta \Rightarrow -\delta<x-3<\delta \Rightarrow -\delta+3<x<\delta+3$

Step 2: From the graph, $-\delta+3=2.61 \Rightarrow \delta=0.39$, or $\delta+3=3.41 \Rightarrow \delta=0.41$; thus $\delta=0.39$.

11. Step 1: $|x-2|<\delta \Rightarrow -\delta < x-2 < \delta \Rightarrow -\delta+2 < x < \delta+2$
Step 2: From the graph, $-\delta+2=\sqrt{3} \Rightarrow \delta = 2-\sqrt{3} \approx 0.2679$, or $\delta+2=\sqrt{5} \Rightarrow \delta=\sqrt{5}-2 \approx 0.2361$; thus $\delta=\sqrt{5}-2$.

12. Step 1: $|x-(-1)|<\delta \Rightarrow -\delta < x+1 < \delta \Rightarrow -\delta-1 < x < \delta-1$
Step 2: From the graph, $-\delta-1=-\frac{\sqrt{5}}{2} \Rightarrow \delta=\frac{\sqrt{5}-2}{2} \approx 0.118$ or $\delta-1=-\frac{\sqrt{3}}{2} \Rightarrow \delta=\frac{2-\sqrt{3}}{2} \approx 0.1340$; thus $\delta=\frac{\sqrt{5}-2}{2}$.

13. Step 1: $|x-(-1)|<\delta \Rightarrow -\delta < x+1 < \delta \Rightarrow -\delta-1 < x < \delta-1$
Step 2: From the graph, $-\delta-1=-\frac{16}{9} \Rightarrow \delta=\frac{7}{9} \approx 0.77$, or $\delta-1=-\frac{16}{25} \Rightarrow \frac{9}{25}=0.36$; thus $\delta=\frac{9}{25}=0.36$.

14. Step 1: $\left|x-\frac{1}{2}\right|<\delta \Rightarrow -\delta < x-\frac{1}{2} < \delta \Rightarrow -\delta+\frac{1}{2} < x < \delta+\frac{1}{2}$
Step 2: From the graph, $-\delta+\frac{1}{2}=\frac{1}{2.01} \Rightarrow \delta=\frac{1}{2}-\frac{1}{2.01} \approx 0.00248$, or $\delta+\frac{1}{2}=\frac{1}{1.99} \Rightarrow \delta=\frac{1}{1.99}-\frac{1}{2} \approx 0.00251$; thus $\delta=0.00248$.

15. Step 1: $|(x+1)-5|<0.01 \Rightarrow |x-4|<0.01 \Rightarrow -0.01 < x-4 < 0.01 \Rightarrow 3.99 < x < 4.01$
Step 2: $|x-4|<\delta \Rightarrow -\delta < x-4 < \delta \Rightarrow -\delta+4 < x < \delta+4 \Rightarrow \delta=0.01$.

16. Step 1: $|(2x-2)-(-6)|<0.02 \Rightarrow |2x+4|<0.02 \Rightarrow -0.02 < 2x+4 < 0.02$
$\Rightarrow -4.02 < 2x < -3.98 \Rightarrow -2.01 < x < -1.99$
Step 2: $|x-(-2)|<\delta \Rightarrow -\delta < x+2 < \delta \Rightarrow -\delta-2 < x < \delta-2 \Rightarrow \delta=0.01$.

17. Step 1: $\left|\sqrt{x+1}-1\right|<0.1 \Rightarrow -0.1 < \sqrt{x+1}-1 < 0.1 \Rightarrow 0.9 < \sqrt{x+1} < 1.1 \Rightarrow 0.81 < x+1 < 1.21$
$\Rightarrow -0.19 < x < 0.21$
Step 2: $|x-0|<\delta \Rightarrow -\delta < x < \delta$. Then, $-\delta=-0.19 \Rightarrow \delta=0.19$ or $\delta=0.21$; thus, $\delta=0.19$.

18. Step 1: $\left|\sqrt{x}-\frac{1}{2}\right|<0.1 \Rightarrow -0.1 < \sqrt{x}-\frac{1}{2} < 0.1 \Rightarrow 0.4 < \sqrt{x} < 0.6 \Rightarrow 0.16 < x < 0.36$
Step 2: $\left|x-\frac{1}{4}\right|<\delta \Rightarrow -\delta < x-\frac{1}{4} < \delta \Rightarrow$
Then $-\delta+\frac{1}{4}=0.16 \Rightarrow \delta=0.09$ or $\delta+\frac{1}{4}=0.36 \Rightarrow \delta=0.11$; thus $\delta=0.09$.

19. Step 1: $\left|\sqrt{19-x}-3\right|<1 \Rightarrow -1 < \sqrt{19-x}-3 < 1 \Rightarrow 2 < \sqrt{19-x} < 4 \Rightarrow 4 < 19-x < 16$
$\Rightarrow -4 > x-19 > -16 \Rightarrow 15 > x > 3$ or $3 < x < 15$
Step 2: $|x-10|<\delta \Rightarrow -\delta < x-10 < \delta \Rightarrow -\delta+10 < x < \delta+10$.
Then $-\delta+10=3 \Rightarrow \delta=7$, or $\delta+10=15 \Rightarrow \delta=5$; thus $\delta=5$.

20. Step 1: $\left|\sqrt{x-7}-4\right|<1 \Rightarrow -1 < \sqrt{x-7}-4 < 1 \Rightarrow 3 < \sqrt{x-7} < 5 \Rightarrow 9 < x-7 < 25 \Rightarrow 16 < x < 32$
Step 2: $|x-23|<\delta \Rightarrow -\delta < x-23 < \delta \Rightarrow -\delta+23 < x < \delta+23$.
Then $-\delta+23=16 \Rightarrow \delta=7$, or $\delta+23=32 \Rightarrow \delta=9$; thus $\delta=7$.

21. Step 1: $\left|\frac{1}{x}-\frac{1}{4}\right|<0.05 \Rightarrow -0.05 < \frac{1}{x}-\frac{1}{4} < 0.05 \Rightarrow 0.2 < \frac{1}{x} < 0.3 \Rightarrow \frac{10}{2} > x > \frac{10}{3}$ or $\frac{10}{3} < x < 5$.
Step 2: $|x-4|<\delta \Rightarrow -\delta < x-4 < \delta \Rightarrow -\delta+4 < x < \delta+4$.
Then $-\delta+4=\frac{10}{3}$ or $\delta=\frac{2}{3}$, or $\delta+4=5$ or $\delta=1$; thus $\delta=\frac{2}{3}$.

22. Step 1: $\left|x^2-3\right|<0.1 \Rightarrow -0.1<x^2-3<0.1 \Rightarrow 2.9<x^2<3.1 \Rightarrow \sqrt{2.9}<x<\sqrt{3.1}$
Step 2: $\left|x-\sqrt{3}\right|<\delta \Rightarrow -\delta<x-\sqrt{3}<\delta \Rightarrow -\delta+\sqrt{3}<x<\delta+\sqrt{3}$.
Then $-\delta+\sqrt{3}=\sqrt{2.9} \Rightarrow \delta=\sqrt{3}-\sqrt{2.9}\approx 0.0291$, or $\delta+\sqrt{3}=\sqrt{3.1} \Rightarrow \delta=\sqrt{3.1}-\sqrt{3}\approx 0.0286$; thus $\delta=0.0286$

23. Step 1: $\left|x^2-4\right|<0.5 \Rightarrow -0.5<x^2-4<0.5 \Rightarrow 3.5<x^2<4.5 \Rightarrow \sqrt{3.5}<|x|<\sqrt{4.5} \Rightarrow -\sqrt{4.5}<x<-\sqrt{3.5}$, for x near -2.
Step 2: $|x-(-2)|<\delta \Rightarrow -\delta<x+2<\delta \Rightarrow -\delta-2<x<\delta-2$.
Then $-\delta-2=-\sqrt{4.5} \Rightarrow \delta=\sqrt{4.5}-2\approx 0.1213$, or $\delta-2=-\sqrt{3.5} \Rightarrow \delta=2-\sqrt{3.5}\approx 0.1292$; thus $\delta=\sqrt{4.5}-2\approx 0.12$.

24. Step 1: $\left|\frac{1}{x}-(-1)\right|<0.1 \Rightarrow -0.1<\frac{1}{x}+1<0.1 \Rightarrow -\frac{11}{10}<\frac{1}{x}<-\frac{9}{10} \Rightarrow -\frac{10}{11}>x>-\frac{10}{9}$ or $-\frac{10}{9}<x<-\frac{10}{11}$.
Step 2: $|x-(-1)|<\delta \Rightarrow -\delta<x+1<\delta \Rightarrow -\delta-1<x<\delta-1$.
Then $-\delta-1=-\frac{10}{9} \Rightarrow \delta=\frac{1}{9}$, or $\delta-1=-\frac{10}{11} \Rightarrow \delta=\frac{1}{11}$; thus $\delta=\frac{1}{11}$.

25. Step 1: $\left|(x^2-5)-11\right|<1 \Rightarrow \left|x^2-16\right|<1 \Rightarrow -1<x^2-16<1 \Rightarrow 15<x^2<17 \Rightarrow \sqrt{15}<x<\sqrt{17}$.
Step 2: $|x-4|<\delta \Rightarrow -\delta<x-4<\delta \Rightarrow -\delta+4<x<\delta+4$.
Then $-\delta+4=\sqrt{15} \Rightarrow \delta=4-\sqrt{15}\approx 0.1270$, or $\delta+4=\sqrt{17} \Rightarrow \delta=\sqrt{17}-4\approx 0.1231$; thus $\delta=\sqrt{17}-4\approx 0.12$.

26. Step 1: $\left|\frac{120}{x}-5\right|<1 \Rightarrow -1<\frac{120}{x}-5<1 \Rightarrow 4<\frac{120}{x}<6 \Rightarrow \frac{1}{4}>\frac{x}{120}>\frac{1}{6} \Rightarrow 30>x>20$ or $20<x<30$.
Step 2: $|x-24|<\delta \Rightarrow -\delta<x-24<\delta \Rightarrow -\delta+24<x<\delta+24$.
Then $-\delta+24=20 \Rightarrow \delta=4$, or $\delta+24=30 \Rightarrow \delta=6$; thus $\Rightarrow \delta=4$.

27. Step 1: $|mx-2m|<0.03 \Rightarrow -0.03<mx-2m<0.03 \Rightarrow -0.03+2m<mx<0.03+2m \Rightarrow 2-\frac{0.03}{m}<x<2+\frac{0.03}{m}$.
Step 2: $|x-2|<\delta \Rightarrow -\delta<x-2<\delta \Rightarrow -\delta+2<x<\delta+2$.
Then $-\delta+2=2-\frac{0.03}{m} \Rightarrow \delta=\frac{0.03}{m}$, or $\delta+2=2+\frac{0.03}{m} \Rightarrow \delta=\frac{0.03}{m}$. In either case, $\delta=\frac{0.03}{m}$.

28. Step 1: $|mx-3m|<c \Rightarrow -c<mx-3m<c \Rightarrow -c+3m<mx<c+3m \Rightarrow 3-\frac{c}{m}<x<3+\frac{c}{m}$
Step 2: $|x-3|<\delta \Rightarrow -\delta<x-3<\delta \Rightarrow -\delta+3<x<\delta+3$.
Then $-\delta+3=3-\frac{c}{m} \Rightarrow \delta=\frac{c}{m}$, or $\delta+3=3+\frac{c}{m} \Rightarrow \delta=\frac{c}{m}$. In either case, $\delta=\frac{c}{m}$.

29. Step 1: $\left|(mx+b)-\left(\frac{m}{2}+b\right)\right|<c \Rightarrow -c<mx-\frac{m}{2}<c \Rightarrow -c+\frac{m}{2}<mx<c+\frac{m}{2} \Rightarrow \frac{1}{2}-\frac{c}{m}<x<\frac{1}{2}+\frac{c}{m}$.
Step 2: $\left|x-\frac{1}{2}\right|<\delta \Rightarrow -\delta<x-\frac{1}{2}<\delta \Rightarrow -\delta+\frac{1}{2}<x<\delta+\frac{1}{2}$.
Then $-\delta+\frac{1}{2}=\frac{1}{2}-\frac{c}{m} \Rightarrow \delta=\frac{c}{m}$, or $\delta+\frac{1}{2}=\frac{1}{2}+\frac{c}{m} \Rightarrow \delta=\frac{c}{m}$. In either case, $\delta=\frac{c}{m}$.

30. Step 1: $|(mx+b)-(m+b)|<0.05 \Rightarrow -0.05<mx-m<0.05 \Rightarrow -0.05+m<mx<0.05+m$
$\Rightarrow 1-\frac{0.05}{m}<x<1+\frac{0.05}{m}$.
Step 2: $|x-1|<\delta \Rightarrow -\delta<x-1<\delta \Rightarrow -\delta+1<x<\delta+1$.
Then $-\delta+1=1-\frac{0.05}{m} \Rightarrow \delta=\frac{0.05}{m}$, or $\delta+1=1+\frac{0.05}{m} \Rightarrow \delta=\frac{0.05}{m}$. In either case, $\delta=\frac{0.05}{m}$.

31. $\lim_{x\to 3}(3-2x)=3-2(3)=-3$
Step 1: $|(3-2x)-(-3)|<0.02 \Rightarrow -0.02<6-2x<0.02 \Rightarrow -6.02<-2x<-5.98 \Rightarrow 3.01>x>2.99$ or $2.99<x<3.01$.

Step 2: $0<|x-3|<\delta \Rightarrow -\delta < x-3 < \delta \Rightarrow -\delta+3 < x < \delta+3$.
Then $-\delta+3 = 2.99 \Rightarrow \delta = 0.01$, or $\delta+3 = 3.01 \Rightarrow \delta = 0.01$; thus $\delta = 0.01$.

32. $\lim_{x \to -1} (-3x-2) = (-3)(-1)-2 = 1$
Step 1: $|(-3x-2)-1| < 0.03 \Rightarrow -0.03 < -3x-3 < 0.03 \Rightarrow 0.01 > x+1 > -0.01 \Rightarrow -1.01 < x < -0.99$.
Step 2: $|x-(-1)| < \delta \Rightarrow -\delta < x+1 < \delta \Rightarrow -\delta-1 < x < \delta-1$.
Then $-\delta-1 = -1.01 \Rightarrow \delta = 0.01$, or $\delta-1 = -0.99 \Rightarrow \delta = 0.01$; thus $\delta = 0.01$.

33. $\lim_{x \to 2} \frac{x^2-4}{x-2} = \lim_{x \to 2} \frac{(x+2)(x-2)}{(x-2)} = \lim_{x \to 2} (x+2) = 2+2 = 4,\ x \neq 2$
Step 1: $\left|\left(\frac{x^2-4}{x-2}\right)-4\right| < 0.05 \Rightarrow -0.05 < \frac{(x+2)(x-2)}{(x-2)} - 4 < 0.05 \Rightarrow 3.95 < x+2 < 4.05,\ x \neq 2$
$\Rightarrow 1.95 < x < 2.05,\ x \neq 2$.
Step 2: $|x-2| < \delta \Rightarrow -\delta < x-2 < \delta \Rightarrow -\delta+2 < x < \delta+2$.
Then $-\delta+2 = 1.95 \Rightarrow \delta = 0.05$, or $\delta+2 = 2.05 \Rightarrow \delta = 0.05$; thus $\delta = 0.05$.

34. $\lim_{x \to -5} \frac{x^2+6x+5}{x+5} = \lim_{x \to -5} \frac{(x+5)(x+1)}{(x+5)} = \lim_{x \to -5} (x+1) = -4,\ x \neq -5$.
Step 1: $\left|\left(\frac{x^2+6x+5}{x+5}\right)-(-4)\right| < 0.05 \Rightarrow -0.05 < \frac{(x+5)(x+1)}{(x+5)} + 4 < 0.05 \Rightarrow -4.05 < x+1 < -3.95,\ x \neq -5$
$\Rightarrow -5.05 < x < -4.95,\ x \neq -5$.
Step 2: $|x-(-5)| < \delta \Rightarrow -\delta < x+5 < \delta \Rightarrow -\delta-5 < x < \delta-5$.
Then $-\delta-5 = -5.05 \Rightarrow \delta = 0.05$, or $\delta-5 = -4.95 \Rightarrow \delta = 0.05$; thus $\delta = 0.05$.

35. $\lim_{x \to -3} \sqrt{1-5x} = \sqrt{1-5(-3)} = \sqrt{16} = 4$
Step 1: $\left|\sqrt{1-5x}-4\right| < 0.5 \Rightarrow -0.5 < \sqrt{1-5x} - 4 < 0.5 \Rightarrow 3.5 < \sqrt{1-5x} < 4.5 \Rightarrow 12.25 < 1-5x < 20.25$
$\Rightarrow 11.25 < -5x < 19.25 \Rightarrow -3.85 < x < 2.25$.
Step 2: $|x-(-3)| < \delta \Rightarrow -\delta < x+3 < \delta \Rightarrow -\delta-3 < x < \delta-3$.
Then $-\delta-3 = -3.85 \Rightarrow \delta = 0.85$, or $\delta-3 = -2.25 \Rightarrow 0.75$; thus $\delta = 0.75$.

36. $\lim_{x \to 2} \frac{4}{x} = \frac{4}{2} = 2$
Step 1: $\left|\frac{4}{x}-2\right| < 0.4 \Rightarrow -0.4 < \frac{4}{x}-2 < 0.4 \Rightarrow 1.6 < \frac{4}{x} < 2.4 \Rightarrow \frac{10}{16} > \frac{x}{4} > \frac{10}{24} \Rightarrow \frac{10}{4} > x > \frac{10}{6}$ or $\frac{5}{3} < x < \frac{5}{2}$.
Step 2: $|x-2| < \delta \Rightarrow -\delta < x-2 < \delta \Rightarrow -\delta+2 < x < \delta+2$.
Then $-\delta+2 = \frac{5}{3} \Rightarrow \delta = \frac{1}{3}$, or $\delta+2 = \frac{5}{2} \Rightarrow \delta = \frac{1}{2}$; thus $\delta = \frac{1}{3}$.

37. Step 1: $|(9-x)-5| < \epsilon \Rightarrow -\epsilon < 4-x < \epsilon \Rightarrow -\epsilon-4 < -x < \epsilon-4 \Rightarrow \epsilon+4 > x > 4-\epsilon \Rightarrow 4-\epsilon < x < 4+\epsilon$.
Step 2: $|x-4| < \delta \Rightarrow -\delta < x-4 < \delta \Rightarrow -\delta+4 < x < \delta+4$.
Then $-\delta+4 = -\epsilon+4 \Rightarrow \delta = \epsilon$, or $\delta+4 = \epsilon+4 \Rightarrow \delta = \epsilon$. Thus choose $\delta = \epsilon$.

38. Step 1: $|(3x-7)-2| < \epsilon \Rightarrow -\epsilon < 3x-9 < \epsilon \Rightarrow 9-\epsilon < 3x < 9+\epsilon \Rightarrow 3-\frac{\epsilon}{3} < x < 3+\frac{\epsilon}{3}$.
Step 2: $|x-3| < \delta \Rightarrow -\delta < x-3 < \delta \Rightarrow -\delta+3 < x < \delta+3$.
Then $-\delta+3 = 3-\frac{\epsilon}{3} \Rightarrow \delta = \frac{\epsilon}{3}$, or $\delta+3 = 3+\frac{\epsilon}{3} \Rightarrow \delta = \frac{\epsilon}{3}$. Thus choose $\delta = \frac{\epsilon}{3}$.

39. Step 1: $\left|\sqrt{x-5}-2\right| < \epsilon \Rightarrow -\epsilon < \sqrt{x-5} - 2 < \epsilon \Rightarrow 2-\epsilon < \sqrt{x-5} < 2+\epsilon \Rightarrow (2-\epsilon)^2 < x-5 < (2+\epsilon)^2$
$\Rightarrow (2-\epsilon)^2 + 5 < x < (2+\epsilon)^2 + 5$.
Step 2: $|x-9| < \delta \Rightarrow -\delta < x-9 < \delta \Rightarrow -\delta+9 < x < \delta+9$.
Then $-\delta+9 = \epsilon^2 - 4\epsilon + 9 \Rightarrow \delta = 4\epsilon - \epsilon^2$, or $\delta+9 = \epsilon^2 + 4\epsilon + 9 \Rightarrow \delta = 4\epsilon + \epsilon^2$. Thus choose the smaller distance, $\delta = 4\epsilon - \epsilon^2$.

40. Step 1: $\left|\sqrt{4-x}-2\right|<\epsilon \Rightarrow -\epsilon<\sqrt{4-x}-2<\epsilon \Rightarrow 2-\epsilon<\sqrt{4-x}<2+\epsilon \Rightarrow (2-\epsilon)^2<4-x<(2+\epsilon)^2$
$\Rightarrow -(2+\epsilon)^2<x-4<-(2-\epsilon)^2 \Rightarrow -(2+\epsilon)^2+4<x<-(2-\epsilon)^2+4.$
Step 2: $|x-0|<\delta \Rightarrow -\delta<x<\delta.$
Then $-\delta=-(2+\epsilon)^2+4=-\epsilon^2-4\epsilon \Rightarrow \delta=4\epsilon+\epsilon^2$, or $\delta=-(2-\epsilon)^2+4=4\epsilon-\epsilon^2$. Thus choose the smaller distance, $\delta=4\epsilon-\epsilon^2$.

41. Step 1: For $x\neq 1$, $\left|x^2-1\right|<\epsilon \Rightarrow -\epsilon<x^2-1<\epsilon \Rightarrow 1-\epsilon<x^2<1+\epsilon \Rightarrow \sqrt{1-\epsilon}<|x|<\sqrt{1+\epsilon}$
$\Rightarrow \sqrt{1-\epsilon}<x<\sqrt{1+\epsilon}$ near $x=1$.
Step 2: $|x-1|<\delta \Rightarrow -\delta<x-1<\delta \Rightarrow -\delta+1<x<\delta+1.$
Then $-\delta+1=\sqrt{1-\epsilon} \Rightarrow \delta=1-\sqrt{1-\epsilon}$, or $\delta+1=\sqrt{1+\epsilon} \Rightarrow \delta=\sqrt{1+\epsilon}-1$. Choose $\delta=\min\left\{1-\sqrt{1-\epsilon}, \sqrt{1+\epsilon}-1\right\}$, that is, the smaller of the two distances.

42. Step 1: For $x\neq -2$, $\left|x^2-4\right|<\epsilon \Rightarrow -\epsilon<x^2-4<\epsilon \Rightarrow 4-\epsilon<x^2<4+\epsilon \Rightarrow \sqrt{4-\epsilon}<|x|<\sqrt{4+\epsilon} \Rightarrow -\sqrt{4+\epsilon}<x$ $<-\sqrt{4-\epsilon}$ near $x=-2$.
Step 2: $|x-(-2)|<\delta \Rightarrow -\delta<x+2<\delta \Rightarrow -\delta-2<x<\delta-2.$
Then $-\delta-2=-\sqrt{4+\epsilon} \Rightarrow \delta=\sqrt{4+\epsilon}-2$, or $\delta-2=-\sqrt{4-\epsilon} \Rightarrow \delta=2-\sqrt{4-\epsilon}$. Choose $\delta=\min\left\{\sqrt{4+\epsilon}-2, 2-\sqrt{4-\epsilon}\right\}$.

43. Step 1: $\left|\frac{1}{x}-1\right|<\epsilon \Rightarrow -\epsilon<\frac{1}{x}-1<\epsilon \Rightarrow 1-\epsilon<\frac{1}{x}<1+\epsilon \Rightarrow \frac{1}{1+\epsilon}<x<\frac{1}{1-\epsilon}.$
Step 2: $|x-1|<\delta \Rightarrow -\delta<x-1<\delta \Rightarrow 1-\delta<x<1+\delta.$
Then $1-\delta=\frac{1}{1+\epsilon} \Rightarrow \delta=1-\frac{1}{1+\epsilon}=\frac{\epsilon}{1+\epsilon}$, or $1+\delta=\frac{1}{1-\epsilon} \Rightarrow \delta=\frac{1}{1-\epsilon}-1=\frac{\epsilon}{1-\epsilon}$.
Choose $\delta=\frac{\epsilon}{1+\epsilon}$, the smaller of the two distances.

44. Step 1: $\left|\frac{1}{x^2}-\frac{1}{3}\right|<\epsilon \Rightarrow -\epsilon<\frac{1}{x^2}-\frac{1}{3}<\epsilon \Rightarrow \frac{1}{3}-\epsilon<\frac{1}{x^2}<\frac{1}{3}+\epsilon \Rightarrow \frac{1-3\epsilon}{3}<\frac{1}{x^2}<\frac{1+3\epsilon}{3}$
$\Rightarrow \frac{3}{1-3\epsilon}>x^2>\frac{3}{1+3\epsilon} \Rightarrow \sqrt{\frac{3}{1+3\epsilon}}<|x|<\sqrt{\frac{3}{1-3\epsilon}}$, or $\sqrt{\frac{3}{1+3\epsilon}}<x<\sqrt{\frac{3}{1-3\epsilon}}$ for x near $\sqrt{3}$.
Step 2: $\left|x-\sqrt{3}\right|<\delta \Rightarrow -\delta<x-\sqrt{3}<\delta \Rightarrow \sqrt{3}-\delta<x<\sqrt{3}+\delta.$
Then $\sqrt{3}-\delta=\sqrt{\frac{3}{1+3\epsilon}} \Rightarrow \delta=\sqrt{3}-\sqrt{\frac{3}{1+3\epsilon}}$, or $\sqrt{3}+\delta=\sqrt{\frac{3}{1-3\epsilon}} \Rightarrow \delta=\sqrt{\frac{3}{1-3\epsilon}}-\sqrt{3}$.
Choose $\delta=\min\left\{\sqrt{3}-\sqrt{\frac{3}{1+3\epsilon}}, \sqrt{\frac{3}{1-3\epsilon}}-\sqrt{3}\right\}$.

45. Step 1: $\left|\left(\frac{x^2-9}{x+3}\right)-(-6)\right|<\epsilon \Rightarrow -\epsilon<(x-3)+6<\epsilon,\ x\neq -3 \Rightarrow -\epsilon<x+3<\epsilon \Rightarrow -\epsilon-3<x<\epsilon-3.$
Step 2: $|x-(-3)|<\delta \Rightarrow -\delta<x+3<\delta \Rightarrow -\delta-3<x<\delta-3.$
Then $-\delta-3=-\epsilon-3 \Rightarrow \delta=\epsilon$, or $\delta-3=\epsilon-3 \Rightarrow \delta=\epsilon$. Choose $\delta=\epsilon$.

46. Step 1: $\left|\left(\frac{x^2-1}{x-1}\right)-2\right|<\epsilon \Rightarrow -\epsilon<(x+1)-2<\epsilon,\ x\neq 1 \Rightarrow 1-\epsilon<x<1+\epsilon.$
Step 2: $|x-1|<\delta \Rightarrow -\delta<x-1<\delta \Rightarrow 1-\delta<x<1+\delta.$
Then $1-\delta=1-\epsilon \Rightarrow \delta=\epsilon$, or $1+\delta=1+\epsilon \Rightarrow \delta=\epsilon$. Choose $\delta=\epsilon$.

47. Step 1: $x<1$: $|(4-2x)-2|<\epsilon \Rightarrow 0<2-2x<\epsilon$ since $x<1$. Thus, $1-\frac{\epsilon}{2}<x<0$;
$x\geq 1$: $|(6x-4)-2|<\epsilon \Rightarrow 0\leq 6x-6<\epsilon$ since $x\geq 1$. Thus, $1\leq x<1+\frac{\epsilon}{6}$.
Step 2: $|x-1|<\delta \Rightarrow -\delta<x-1<\delta \Rightarrow 1-\delta<x<1+\delta.$
Then $1-\delta=1-\frac{\epsilon}{2} \Rightarrow \delta=\frac{\epsilon}{2}$, or $1+\delta=1+\frac{\epsilon}{6} \Rightarrow \delta=\frac{\epsilon}{6}$. Choose $\delta=\frac{\epsilon}{6}$.

48. Step 1: $x < 0: |2x - 0| < \epsilon \Rightarrow -\epsilon < 2x < 0 \Rightarrow -\frac{\epsilon}{2} < x < 0;$
$x \geq 0: \left|\frac{x}{2} - 0\right| < \epsilon \Rightarrow 0 \leq x < 2\epsilon.$
Step 2: $|x - 0| < \delta \Rightarrow -\delta < x < \delta.$
Then $-\delta = -\frac{\epsilon}{2} \Rightarrow \delta = \frac{\epsilon}{2}$, or $\delta = 2\epsilon \Rightarrow \delta = 2\epsilon$. Choose $\delta = \frac{\epsilon}{2}$.

49. By the figure, $-x \leq x \sin\frac{1}{x} \leq x$ for all $x > 0$ and $-x \geq x \sin\frac{1}{x} \geq x$ for $x < 0$. Since $\lim\limits_{x\to 0}(-x) = \lim\limits_{x\to 0} x = 0$, then by the sandwich theorem, in either case, $\lim\limits_{x\to 0} x \sin\frac{1}{x} = 0$.

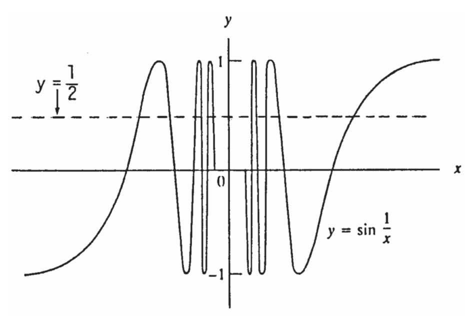
50. By the figure, $-x^2 \leq x^2 \sin\frac{1}{x} \leq x^2$ for all x except possibly at $x = 0$. Since $\lim\limits_{x\to 0}(-x^2) = \lim\limits_{x\to 0} x^2 = 0$, then by the sandwich theorem, $\lim\limits_{x\to 0} x^2 \sin\frac{1}{x} = 0$.

51. As x approaches the value 0, the values of $g(x)$ approach k. Thus for every number $\epsilon > 0$, there exists a $\delta > 0$ such that $0 < |x - 0| < \delta \Rightarrow |g(x) - k| < \epsilon$.

52. Write $x = h + c$. Then $0 < |x - c| < \delta \Leftrightarrow -\delta < x - c < \delta,\ x \neq c \Leftrightarrow -\delta < (h + c) - c < \delta,\ h + c \neq c \Leftrightarrow -\delta < h < \delta,$ $h \neq 0 \Leftrightarrow 0 < |h - 0| < \delta.$
Thus, $\lim\limits_{x\to c} f(x) = L \Leftrightarrow$ for any $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - c| < \delta \Leftrightarrow$ $|f(h + c) - L| < \epsilon$ whenever $0 < |h - 0| < \delta \Leftrightarrow \lim\limits_{h\to 0} f(h + c) = L.$

53. Let $f(x) = x^2$. The function values do get closer to -1 as x approaches 0, but $\lim\limits_{x\to 0} f(x) = 0$, not -1. The function $f(x) = x^2$ never gets <u>arbitrarily</u> <u>close</u> to -1 for x near 0.

54. Let $f(x) = \sin x$, $L = \frac{1}{2}$, and $x_0 = 0$. There exists a value of x (namely $x = \frac{\pi}{6}$) for which $\left|\sin x - \frac{1}{2}\right| < \epsilon$ for any given $\epsilon > 0$. However, $\lim\limits_{x\to 0} \sin x = 0$, not $\frac{1}{2}$. The wrong statement does not require x to be arbitrarily close to x_0. As another example, let $g(x) = \sin\frac{1}{x}$, $L = \frac{1}{2}$, and $x_0 = 0$. We can choose infinitely many values of x near 0 such that $\sin\frac{1}{x} = \frac{1}{2}$ as you can see from the accompanying figure. However, $\lim\limits_{x\to 0} \sin\frac{1}{x}$ fails to exist. The wrong statement does not require <u>all</u> values of x arbitrarily close to $x_0 = 0$ to lie within $\epsilon > 0$ of $L = \frac{1}{2}$. Again you can see from the figure that there are also infinitely many values of x near 0 such that $\sin\frac{1}{x} = 0$. If we choose $\epsilon < \frac{1}{4}$ we cannot satisfy the inequality $\left|\sin\frac{1}{x} - \frac{1}{2}\right| < \epsilon$ for all values of x sufficiently near $x_0 = 0$.



55. $|A - 9| \leq 0.01 \Rightarrow -0.01 \leq \pi\left(\frac{x}{2}\right)^2 - 9 \leq 0.01 \Rightarrow 8.99 \leq \frac{\pi x^2}{4} \leq 9.01 \Rightarrow \frac{4}{\pi}(8.99) \leq x^2 \leq \frac{4}{\pi}(9.01)$
$\Rightarrow 2\sqrt{\frac{8.99}{\pi}} \leq x \leq 2\sqrt{\frac{9.01}{\pi}}$ or $3.384 \leq x \leq 3.387$. To be safe, the left endpoint was rounded up and the right endpoint was rounded down.

56. $V = RI \Rightarrow \frac{V}{R} = I \Rightarrow \left|\frac{V}{R} - 5\right| \le 0.1 \Rightarrow -0.1 \le \frac{120}{R} - 5 \le 0.1 \Rightarrow 4.9 \le \frac{120}{R} \le 5.1 \Rightarrow \frac{10}{49} \ge \frac{R}{120} \ge \frac{10}{51}$
$\Rightarrow \frac{(120)(10)}{51} \le R \le \frac{(120)(10)}{49} \Rightarrow 23.53 \le R \le 24.48.$

To be safe, the left endpoint was rounded up and the right endpoint was rounded down.

57. (a) $-\delta < x - 1 < 0 \Rightarrow 1 - \delta < x < 1 \Rightarrow f(x) = x$. Then $|f(x) - 2| = |x - 2| = 2 - x > 2 - 1 = 1$. That is, $|f(x) - 2| \ge 1 \ge \frac{1}{2}$ no matter how small δ is taken when $1 - \delta < x < 1 \Rightarrow \lim_{x\to 1} f(x) \ne 2$.

(b) $0 < x - 1 < \delta \Rightarrow 1 < x < 1 + \delta \Rightarrow f(x) = x + 1$. Then $|f(x) - 1| = |(x+1) - 1| = |x| = x > 1$. That is, $|f(x) - 1| \ge 1$ no matter how small δ is taken when $1 < x < 1 + \delta \Rightarrow \lim_{x\to 1} f(x) \ne 1$.

(c) $-\delta < x - 1 < 0 \Rightarrow 1 - \delta < x < 1 \Rightarrow f(x) = x$. Then $|f(x) - 1.5| = |x - 1.5| = 1.5 - x > 1.5 - 1 = 0.5$. Also, $0 < x - 1 < \delta \Rightarrow 1 < x < 1 + \delta \Rightarrow f(x) = x + 1$. Then $|f(x) - 1.5| = |(x+1) - 1.5| = |x - 0.5|$ $= x - 0.5 > 1 - 0.5 = 0.5$. Thus, no matter how small δ is taken, there exists a value of x such that $-\delta < x - 1 < \delta$ but $|f(x) - 1.5| \ge \frac{1}{2} \Rightarrow \lim_{x\to 1} f(x) \ne 1.5$.

58. (a) For $2 < x < 2 + \delta \Rightarrow h(x) = 2 \Rightarrow |h(x) - 4| = 2$. Thus for $\epsilon < 2$, $|h(x) - 4| \ge \epsilon$ whenever $2 < x < 2 + \delta$ no matter how small we choose $\delta > 0 \Rightarrow \lim_{x\to 2} h(x) \ne 4$.

(b) For $2 < x < 2 + \delta \Rightarrow h(x) = 2 \Rightarrow |h(x) - 3| = 1$. Thus for $\epsilon < 1$, $|h(x) - 3| \ge \epsilon$ whenever $2 < x < 2 + \delta$ no matter how small we choose $\delta > 0 \Rightarrow \lim_{x\to 2} h(x) \ne 3$.

(c) For $2 - \delta < x < 2 \Rightarrow h(x) = x^2$ so $|h(x) - 2| = |x^2 - 2|$. No matter how small $\delta > 0$ is chosen, x^2 is close to 4 when x is near 2 and to the left on the real line $\Rightarrow |x^2 - 2|$ will be close to 2. Thus if $\epsilon < 1$, $|h(x) - 2| \ge \epsilon$ whenever $2 - \delta < x < 2$ no matter how small we choose $\delta > 0 \Rightarrow \lim_{x\to 2} h(x) \ne 2$.

59. (a) For $3 - \delta < x < 3 \Rightarrow f(x) > 4.8 \Rightarrow |f(x) - 4| \ge 0.8$. Thus for $\epsilon < 0.8$, $|f(x) - 4| \ge \epsilon$ whenever $3 - \delta < x < 3$ no matter how small we choose $\delta > 0 \Rightarrow \lim_{x\to 3} f(x) \ne 4$.

(b) For $3 < x < 3 + \delta \Rightarrow f(x) < 3 \Rightarrow |f(x) - 4.8| \ge 1.8$. Thus for $\epsilon < 1.8$, $|f(x) - 4.8| \ge \epsilon$ whenever $3 < x < 3 + \delta$ no matter how small we choose $\delta > 0 \Rightarrow \lim_{x\to 3} f(x) \ne 4.8$.

(c) For $3 - \delta < x < 3 \Rightarrow f(x) > 4.8 \Rightarrow |f(x) - 3| \ge 1.8$. Again, for $\epsilon < 1.8$, $|f(x) - 3| \ge \epsilon$ whenever $3 - \delta < x < 3$ no matter how small we choose $\delta > 0 \Rightarrow \lim_{x\to 3} f(x) \ne 3$.

60. (a) No matter how small we choose $\delta > 0$, for x near -1 satisfying $-1 - \delta < x < -1 + \delta$, the values of $g(x)$ are near $1 \Rightarrow |g(x) - 2|$ is near 1. Then, for $\epsilon = \frac{1}{2}$ we have $|g(x) - 2| \ge \frac{1}{2}$ for some x satisfying $-1 - \delta < x < -1 + \delta$, or $0 < |x + 1| < \delta \Rightarrow \lim_{x\to -1} g(x) \ne 2$.

(b) Yes, $\lim_{x\to -1} g(x) = 1$ because from the graph we can find a $\delta > 0$ such that $|g(x) - 1| < \epsilon$ if $0 < |x - (-1)| < \delta$.

61–66. Example CAS commands (values of del may vary for a specified eps):

Maple:

```
f := x -> (x^4-81)/(x-3); x0 := 3;
plot( f(x), x=x0-1..x0+1, color=black,                # (a)
      title="Section 2.3, #61(a)" );
L := limit( f(x), x=x0 );                             # (b)
epsilon := 0.2;                                       # (c)
plot( [f(x), L-epsilon,L+epsilon], x=x0-0.01..x0+0.01,
      color=black, linestyle=[1,3,3], title="Section 2.3, #61(c)" );
```

```
q := fsolve( abs( f(x)-L ) = epsilon, x=x0-1..x0+1 );            # (d)
delta := abs(x0-q);
plot( [f(x), L-epsilon, L+epsilon], x=x0-delta..x0+delta, color=black, title="Section 2.3, #61(d)" );
for eps in [0.1, 0.005, 0.001 ] do                                # (e)
q := fsolve( abs( f(x)-L ) = eps, x=x0-1..x0+1 );
delta := abs(x0-q);
head := sprintf("Section 2.3, #61(e)\n epsilon = %5f, delta = %5f \n", eps, delta );
print(plot( [f(x), L-eps, L+eps], x=x0-delta..x0+delta,
color=black, linestyle=[1,3,3], title=head ));
end do:
```

Mathematica (assigned function and values for x0, eps and del may vary):

```
Clear[f, x]
y1: = L − eps; y2: = L + eps; x0 = 1;
f[x_]: = (3x^2 − (7x + 1)Sqrt[x] + 5)/(x − 1)
Plot[f[x], {x, x0 − 0.2, x0 + 0.2}]
L: = Limit[f[x], x → x0]
eps = 0.1; del = 0.2;
Plot[{f[x], y1, y2}, {x, x0 − del, x0 + del}, PlotRange → {L − 2eps, L + 2eps}]
```

2.4 ONE-SIDED LIMITS

1. (a) True (b) True (c) False (d) True
(e) True (f) True (g) False (h) False
(i) False (j) False (k) True (l) False

2. (a) True (b) False (c) False (d) True
(e) True (f) True (g) True (h) True
(i) True (j) False (k) True

3. (a) $\lim_{x\to 2^+} f(x) = \frac{2}{2} + 1 = 2,\ \lim_{x\to 2^-} f(x) = 3 - 2 = 1$
(b) No, $\lim_{x\to 2} f(x)$ does not exist because $\lim_{x\to 2^+} f(x) \neq \lim_{x\to 2^-} f(x)$
(c) $\lim_{x\to 4^-} f(x) = \frac{4}{2} + 1 = 3,\ \lim_{x\to 4^+} f(x) = \frac{4}{2} + 1 = 3$
(d) Yes, $\lim_{x\to 4} f(x) = 3$ because $3 = \lim_{x\to 4^-} f(x) = \lim_{x\to 4^+} f(x)$

4. (a) $\lim_{x\to 2^+} f(x) = \frac{2}{2} = 1,\ \lim_{x\to 2^-} f(x) = 3 - 2 = 1,\ f(2) = 2$
(b) Yes, $\lim_{x\to 2} f(x) = 1$ because $1 = \lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x)$
(c) $\lim_{x\to -1^-} f(x) = 3 - (-1) = 4,\ \lim_{x\to -1^+} f(x) = 3 - (-1) = 4$
(d) Yes, $\lim_{x\to -1} f(x) = 4$ because $4 = \lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x)$

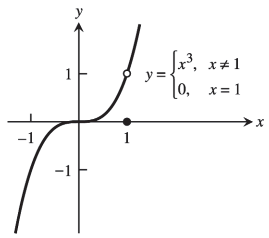
5. (a) No, $\lim_{x\to 0^+} f(x)$ does not exist since $\sin\left(\frac{1}{x}\right)$ does not approach any single value as x approaches 0
(b) $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 0 = 0$
(c) $\lim_{x\to 0} f(x)$ does not exist because $\lim_{x\to 0^+} f(x)$ does not exist

6. (a) Yes, $\lim_{x\to 0^+} g(x) = 0$ by the sandwich theorem since $-\sqrt{x} \le g(x) \le \sqrt{x}$ when $x > 0$

   (b) No, $\lim_{x\to 0^-} g(x)$ does not exist since $\sqrt{x}$ is not defined for $x < 0$

   (c) Yes, $\lim_{x\to 0} g(x) = \lim_{x\to 0^+} g(x) = 0$ since $x = 0$ is a boundary point of the domain

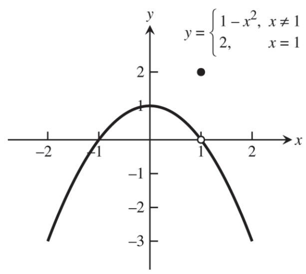
7. (a)



   (b) $\lim_{x\to 1^-} f(x) = 1 = \lim_{x\to 1^+} f(x)$

   (c) Yes, $\lim_{x\to 1} f(x) = 1$ since the right-hand and left-hand limits exist and equal 1

8. (a)



   (b) $\lim_{x\to 1^+} f(x) = 0 = \lim_{x\to 1^-} f(x)$

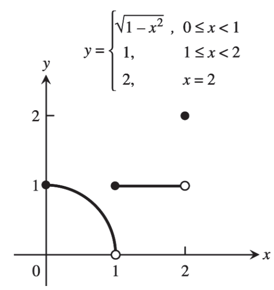
   (c) Yes, $\lim_{x\to 1} f(x) = 0$ since the right-hand and left-hand limits exist and equal 0

9. (a) domain: $0 \le x \le 2$
   range: $0 < y \le 1$ and $y = 2$

   (b) $\lim_{x\to c} f(x)$ exists for c belonging to $(0, 1) \cup (1, 2)$

   (c) $x = 2$

   (d) $x = 0$

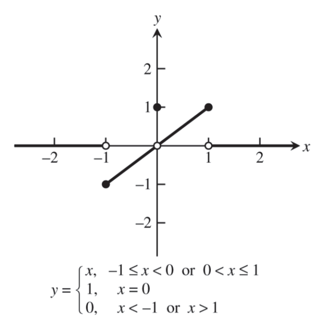


10. (a) domain: $-\infty < x < \infty$
    range: $-1 \le y \le 1$

    (b) $\lim_{x\to c} f(x)$ exists for c belonging to
    $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

    (c) none

    (d) none



11. $\lim_{x\to -0.5^-} \sqrt{\frac{x+2}{x-1}} = \sqrt{\frac{-0.5+2}{-0.5+1}} = \sqrt{\frac{3/2}{1/2}} = \sqrt{3}$

12. $\lim_{x\to 1^+} \sqrt{\frac{x-1}{x+2}} = \sqrt{\frac{1-1}{1+2}} = \sqrt{0} = 0$

13. $\lim_{x\to -2^+} \left(\frac{x}{x+1}\right)\left(\frac{2x+5}{x^2+x}\right) = \left(\frac{-2}{-2+1}\right)\left(\frac{2(-2)+5}{(-2)^2+(-2)}\right) = (2)\left(\frac{1}{2}\right) = 1$

14. $\lim_{x\to 1^-}\left(\frac{1}{x+1}\right)\left(\frac{x+6}{x}\right)\left(\frac{3-x}{7}\right)=\left(\frac{1}{1+1}\right)\left(\frac{1+6}{1}\right)\left(\frac{3-1}{7}\right)=\left(\frac{1}{2}\right)\left(\frac{7}{1}\right)\left(\frac{2}{7}\right)=1$

15. $\lim_{h\to 0^+}\frac{\sqrt{h^2+4h+5}-\sqrt{5}}{h}=\lim_{h\to 0^+}\left(\frac{\sqrt{h^2+4h+5}-\sqrt{5}}{h}\right)\left(\frac{\sqrt{h^2+4h+5}+\sqrt{5}}{\sqrt{h^2+4h+5}+\sqrt{5}}\right)=\lim_{h\to 0^+}\frac{(h^2+4h+5)-5}{h\left(\sqrt{h^2+4h+5}+\sqrt{5}\right)}$

$=\lim_{h\to 0^+}\frac{h(h+4)}{h\left(\sqrt{h^2+4h+5}+\sqrt{5}\right)}=\frac{0+4}{\sqrt{5}+\sqrt{5}}=\frac{2}{\sqrt{5}}$

16. $\lim_{h\to 0^-}\frac{\sqrt{6}-\sqrt{5h^2+11h+6}}{h}=\lim_{h\to 0^-}\left(\frac{\sqrt{6}-\sqrt{5h^2+11h+6}}{h}\right)\left(\frac{\sqrt{6}+\sqrt{5h^2+11h+6}}{\sqrt{6}+\sqrt{5h^2+11h+6}}\right)$

$=\lim_{h\to 0^-}\frac{6-(5h^2+11h+6)}{h\left(\sqrt{6}+\sqrt{5h^2+11h+6}\right)}=\lim_{h\to 0^-}\frac{-h(5h+11)}{h\left(\sqrt{6}+\sqrt{5h^2+11h+6}\right)}=\frac{-(0+11)}{\sqrt{6}+\sqrt{6}}=-\frac{11}{2\sqrt{6}}$

17. (a) $\lim_{x\to -2^+}(x+3)\frac{|x+2|}{x+2}=\lim_{x\to -2^+}(x+3)\frac{(x+2)}{(x+2)}$ $\quad(|x+2|=(x+2)\text{ for }x>-2)$

$=\lim_{x\to -2^+}(x+3)=((-2)+3)=1$

(b) $\lim_{x\to -2^-}(x+3)\frac{|x+2|}{x+2}=\lim_{x\to -2^-}(x+3)\left[\frac{-(x+2)}{(x+2)}\right]$ $\quad(|x+2|=-(x+2)\text{ for }x<-2)$

$=\lim_{x\to -2^-}(x+3)(-1)=-(-2+3)=-1$

18. (a) $\lim_{x\to 1^+}\frac{\sqrt{2x}(x-1)}{|x-1|}=\lim_{x\to -1^+}\frac{\sqrt{2x}(x-1)}{(x-1)}$ $\quad(|x-1|=x-1\text{ for }x>1)$

$=\lim_{x\to -1^+}\sqrt{2x}=\sqrt{2}$

(b) $\lim_{x\to 1^-}\frac{\sqrt{2x}(x-1)}{|x-1|}=\lim_{x\to 1^-}\frac{\sqrt{2x}(x-1)}{-(x-1)}$ $\quad(|x-1|=-(x-1)\text{ for }x<1)$

$=\lim_{x\to 1^-}-\sqrt{2x}=-\sqrt{2}$

19. (a) If $0<x<\frac{\pi}{2}$, then $\sin x>0$, so that $\lim_{x\to 0^+}\frac{|\sin x|}{\sin x}=\lim_{x\to 0^+}\frac{\sin x}{\sin x}=\lim_{x\to 0^+}1=1$

(b) If $\frac{-\pi}{2}<x<0$, then $\sin x<0$, so that $\lim_{x\to 0^-}\frac{|\sin x|}{\sin x}=\lim_{x\to 0^-}\frac{-\sin x}{\sin x}=\lim_{x\to 0^-}-1=-1$

20. (a) If $0<x<\frac{\pi}{2}$, then $\cos x<1$, so that $\lim_{x\to 0^+}\frac{1-\cos x}{|\cos x-1|}=\lim_{x\to 0^+}\frac{1-\cos x}{-(\cos x-1)}=\lim_{x\to 0^+}\frac{1-\cos x}{1-\cos x}=\lim_{x\to 0^+}1=1$

(b) If $\frac{-\pi}{2}<x<0$, then $\cos x<1$, so that $\lim_{x\to 0^-}\frac{\cos x-1}{|\cos x-1|}=\lim_{x\to 0^-}\frac{\cos x-1}{-(\cos x-1)}=\lim_{x\to 0^-}-1=-1$

21. (a) $\lim_{\theta\to 3^+}\frac{\lfloor\theta\rfloor}{\theta}=\frac{3}{3}=1$ (b) $\lim_{\theta\to 3^-}\frac{\lfloor\theta\rfloor}{\theta}=\frac{2}{3}$

22. (a) $\lim_{t\to 4^+}(t-\lfloor t\rfloor)=4-4=0$ (b) $\lim_{t\to 4^-}(t-\lfloor t\rfloor)=4-3=1$

23. $\lim_{\theta\to 0}\frac{\sin\sqrt{2}\theta}{\sqrt{2}\theta}=\lim_{x\to 0}\frac{\sin x}{x}=1$ $\quad$ (where $x=\sqrt{2}\theta$)

24. $\lim_{t\to 0}\frac{\sin kt}{t}=\lim_{t\to 0}\frac{k\sin kt}{kt}=\lim_{\theta\to 0}\frac{k\sin\theta}{\theta}=k\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=k\cdot 1=k$ $\quad$ (where $\theta=kt$)

25. $\lim_{y\to 0} \frac{\sin 3y}{4y} = \frac{1}{4} \lim_{y\to 0} \frac{3\sin 3y}{3y} = \frac{3}{4} \lim_{y\to 0} \frac{\sin 3y}{3y} = \frac{3}{4} \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = \frac{3}{4}$ (where $\theta = 3y$)

26. $\lim_{h\to 0^-} \frac{h}{\sin 3h} = \lim_{h\to 0^-} \left(\frac{1}{3}\cdot\frac{3h}{\sin 3h}\right) = \frac{1}{3} \lim_{h\to 0^-} \frac{1}{\left(\frac{\sin 3h}{3h}\right)} = \frac{1}{3}\left(\frac{1}{\lim_{\theta\to 0^-} \frac{\sin\theta}{\theta}}\right) = \frac{1}{3}\cdot 1 = \frac{1}{3}$ (where $\theta = 3h$)

27. $\lim_{x\to 0} \frac{\tan 2x}{x} = \lim_{x\to 0} \frac{\left(\frac{\sin 2x}{\cos 2x}\right)}{x} = \lim_{x\to 0} \frac{\sin 2x}{x\cos 2x} = \left(\lim_{x\to 0} \frac{1}{\cos 2x}\right)\left(\lim_{x\to 0} \frac{2\sin 2x}{2x}\right) = 1\cdot 2 = 2$

28. $\lim_{t\to 0} \frac{2t}{\tan t} = 2\lim_{t\to 0} \frac{t}{\left(\frac{\sin t}{\cos t}\right)} = 2\lim_{t\to 0} \frac{t\cos t}{\sin t} = 2\left(\lim_{t\to 0} \cos t\right)\left(\frac{1}{\lim_{t\to 0} \frac{\sin t}{t}}\right) = 2\cdot 1\cdot 1 = 2$

29. $\lim_{x\to 0} \frac{x\csc 2x}{\cos 5x} = \lim_{x\to 0} \left(\frac{x}{\sin 2x}\cdot\frac{1}{\cos 5x}\right) = \left(\frac{1}{2}\lim_{x\to 0} \frac{2x}{\sin 2x}\right)\left(\lim_{x\to 0} \frac{1}{\cos 5x}\right) = \left(\frac{1}{2}\cdot 1\right)(1) = \frac{1}{2}$

30. $\lim_{x\to 0} 6x^2(\cot x)(\csc 2x) = \lim_{x\to 0} \frac{6x^2\cos x}{\sin x\sin 2x} = \lim_{x\to 0} \left(3\cos x\cdot\frac{x}{\sin x}\cdot\frac{2x}{\sin 2x}\right) = 3\cdot 1\cdot 1 = 3$

31. $\lim_{x\to 0} \frac{x+x\cos x}{\sin x\cos x} = \lim_{x\to 0} \left(\frac{x}{\sin x\cos x} + \frac{x\cos x}{\sin x\cos x}\right) = \lim_{x\to 0} \left(\frac{x}{\sin x}\cdot\frac{1}{\cos x}\right) + \lim_{x\to 0} \frac{x}{\sin x}$

$= \lim_{x\to 0} \left(\frac{1}{\frac{\sin x}{x}}\right)\cdot\lim_{x\to 0} \left(\frac{1}{\cos x}\right) + \lim_{x\to 0} \left(\frac{1}{\frac{\sin x}{x}}\right) = (1)(1) + 1 = 2$

32. $\lim_{x\to 0} \frac{x^2-x+\sin x}{2x} = \lim_{x\to 0} \left(\frac{x}{2} - \frac{1}{2} + \frac{1}{2}\left(\frac{\sin x}{x}\right)\right) = 0 - \frac{1}{2} + \frac{1}{2}(1) = 0$

33. $\lim_{\theta\to 0} \frac{1-\cos\theta}{\sin 2\theta} = \lim_{\theta\to 0} \frac{(1-\cos\theta)(1+\cos\theta)}{(2\sin\theta\cos\theta)(1+\cos\theta)} = \lim_{\theta\to 0} \frac{1-\cos^2\theta}{(2\sin\theta\cos\theta)(1+\cos\theta)} = \lim_{\theta\to 0} \frac{\sin^2\theta}{(2\sin\theta\cos\theta)(1+\cos\theta)}$

$= \lim_{\theta\to 0} \frac{\sin\theta}{(2\cos\theta)(1+\cos\theta)} = \frac{0}{(2)(2)} = 0$

34. $\lim_{x\to 0} \frac{x-x\cos x}{\sin^2 3x} = \lim_{x\to 0} \frac{x(1-\cos x)}{\sin^2 3x} = \lim_{x\to 0} \frac{\frac{x(1-\cos x)}{9x^2}}{\frac{\sin^2 3x}{9x^2}} = \lim_{x\to 0} \frac{\frac{1-\cos x}{9x}}{\left(\frac{\sin 3x}{3x}\right)^2} = \frac{\frac{1}{9}\lim_{x\to 0}\left(\frac{1-\cos x}{x}\right)}{\lim_{x\to 0}\left(\frac{\sin 3x}{3x}\right)^2} = \frac{\frac{1}{9}(0)}{1^2} = 0$

35. $\lim_{t\to 0} \frac{\sin(1-\cos t)}{1-\cos t} = \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$ since $\theta = 1-\cos t \to 0$ as $t \to 0$

36. $\lim_{h\to 0} \frac{\sin(\sin h)}{\sin h} = \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$ since $\theta = \sin h \to 0$ as $h \to 0$

37. $\lim_{\theta\to 0} \frac{\sin\theta}{\sin 2\theta} = \lim_{\theta\to 0} \left(\frac{\sin\theta}{\sin 2\theta}\cdot\frac{2\theta}{2\theta}\right) = \frac{1}{2}\lim_{\theta\to 0} \left(\frac{\sin\theta}{\theta}\cdot\frac{2\theta}{\sin 2\theta}\right) = \frac{1}{2}\cdot 1\cdot 1 = \frac{1}{2}$

38. $\lim_{x\to 0} \frac{\sin 5x}{\sin 4x} = \lim_{x\to 0} \left(\frac{\sin 5x}{\sin 4x}\cdot\frac{4x}{5x}\cdot\frac{5}{4}\right) = \frac{5}{4}\lim_{x\to 0} \left(\frac{\sin 5x}{5x}\cdot\frac{4x}{\sin 4x}\right) = \frac{5}{4}\cdot 1\cdot 1 = \frac{5}{4}$

39. $\lim_{\theta\to 0} \theta\cos\theta = 0\cdot 1 = 0$

40. $\lim_{\theta\to 0} \sin\theta\cot 2\theta = \lim_{\theta\to 0} \sin\theta\frac{\cos 2\theta}{\sin 2\theta} = \lim_{\theta\to 0} \sin\theta\frac{\cos 2\theta}{2\sin\theta\cos\theta} = \lim_{\theta\to 0} \frac{\cos 2\theta}{2\cos\theta} = \frac{1}{2}$

41. $\lim\limits_{x\to 0} \frac{\tan 3x}{\sin 8x} = \lim\limits_{x\to 0} \left(\frac{\sin 3x}{\cos 3x} \cdot \frac{1}{\sin 8x}\right) = \lim\limits_{x\to 0} \left(\frac{\sin 3x}{\cos 3x} \cdot \frac{1}{\sin 8x} \cdot \frac{8x}{3x} \cdot \frac{3}{8}\right)$

$= \frac{3}{8} \lim\limits_{x\to 0} \left(\frac{1}{\cos 3x}\right)\left(\frac{\sin 3x}{3x}\right)\left(\frac{8x}{\sin 8x}\right) = \frac{3}{8} \cdot 1 \cdot 1 \cdot 1 = \frac{3}{8}$

42. $\lim\limits_{y\to 0} \frac{\sin 3y \cot 5y}{y \cot 4y} = \lim\limits_{y\to 0} \frac{\sin 3y \sin 4y \cos 5y}{y \cos 4y \sin 5y} = \lim\limits_{y\to 0} \left(\frac{\sin 3y}{y}\right)\left(\frac{\sin 4y}{\cos 4y}\right)\left(\frac{\cos 5y}{\sin 5y}\right)\left(\frac{3\cdot 4\cdot 5y}{3\cdot 4\cdot 5y}\right)$

$= \lim\limits_{y\to 0} \left(\frac{\sin 3y}{3y}\right)\left(\frac{\sin 4y}{4y}\right)\left(\frac{5y}{\sin 5y}\right)\left(\frac{\cos 5y}{\cos 4y}\right)\left(\frac{3\cdot 4}{5}\right) = 1\cdot 1\cdot 1\cdot 1\cdot \frac{12}{5} = \frac{12}{5}$

43. $\lim\limits_{\theta\to 0} \frac{\tan\theta}{\theta^2 \cot 3\theta} = \lim\limits_{\theta\to 0} \frac{\frac{\sin\theta}{\cos\theta}}{\theta^2 \frac{\cos 3\theta}{\sin 3\theta}} = \lim\limits_{\theta\to 0} \frac{\sin\theta \sin 3\theta}{\theta^2 \cos\theta \cos 3\theta} = \lim\limits_{\theta\to 0} \left(\frac{\sin\theta}{\theta}\right)\left(\frac{\sin 3\theta}{3\theta}\right)\left(\frac{3}{\cos\theta\cos 3\theta}\right) = (1)(1)\left(\frac{3}{1\cdot 1}\right) = 3$

44. $\lim\limits_{\theta\to 0} \frac{\theta\cot 4\theta}{\sin^2\theta\cot^2 2\theta} = \lim\limits_{\theta\to 0} \frac{\theta\frac{\cos 4\theta}{\sin 4\theta}}{\sin^2\theta\frac{\cos^2 2\theta}{\sin^2 2\theta}} = \lim\limits_{\theta\to 0} \frac{\theta\cos 4\theta\sin^2 2\theta}{\sin^2\theta\cos^2 2\theta\sin 4\theta} = \lim\limits_{\theta\to 0} \frac{\theta\cos 4\theta(2\sin\theta\cos\theta)^2}{\sin^2\theta\cos^2 2\theta\sin 4\theta} = \lim\limits_{\theta\to 0} \frac{\theta\cos 4\theta(4\sin^2\theta\cos^2\theta)}{\sin^2\theta\cos^2 2\theta\sin 4\theta}$

$= \lim\limits_{\theta\to 0} \frac{4\theta\cos 4\theta\cos^2\theta}{\cos^2 2\theta\sin 4\theta} = \lim\limits_{\theta\to 0} \left(\frac{4\theta}{\sin 4\theta}\right)\left(\frac{\cos 4\theta\cos^2\theta}{\cos^2 2\theta}\right) = \lim\limits_{\theta\to 0} \left(\frac{1}{\frac{\sin 4\theta}{4\theta}}\right)\left(\frac{\cos 4\theta\cos^2\theta}{\cos^2 2\theta}\right) = \left(\frac{1}{1}\right)\left(\frac{1\cdot 1^2}{1^2}\right) = 1$

45. $\lim\limits_{x\to 0} \frac{1-\cos 3x}{2x} = \lim\limits_{x\to 0} \frac{1-\cos 3x}{2x} \cdot \frac{1+\cos 3x}{1+\cos 3x} = \lim\limits_{x\to 0} \frac{1-\cos^2 3x}{2x(1+\cos 3x)} = \lim\limits_{x\to 0} \frac{\sin^2 3x}{2x(1+\cos 3x)} = \lim\limits_{x\to 0} \frac{3}{2} \cdot \frac{\sin 3x}{3x} \cdot \frac{\sin 3x}{1+\cos 3x}$

$= \lim\limits_{\theta\to 0} \frac{3}{2} \cdot \frac{\sin\theta}{\theta} \cdot \frac{\sin\theta}{1+\cos\theta} = \frac{3}{2}(1)\left(\frac{0}{1+1}\right) = 0$ (where $\theta = 3x$)

46. $\lim\limits_{x\to 0} \frac{\cos^2 x - \cos x}{x^2} = \lim\limits_{x\to 0} \frac{\cos x(\cos x - 1)}{x^2} = \lim\limits_{x\to 0} \frac{\cos x(\cos x - 1)}{x^2} \cdot \frac{\cos x + 1}{\cos x + 1} = \lim\limits_{x\to 0} \frac{\cos x(\cos^2 x - 1)}{x^2(\cos x + 1)} = \lim\limits_{x\to 0} \frac{\cos x\cdot(-\sin^2 x)}{x^2(\cos x + 1)}$

$= \lim\limits_{x\to 0} \left\{-\frac{\sin x}{x} \cdot \frac{\sin x}{x} \cdot \frac{\cos x}{\cos x + 1}\right\} = -(1)(1) \cdot \frac{1}{1+1} = \frac{-1}{2}$

47. Yes. If $\lim\limits_{x\to a^+} f(x) = L = \lim\limits_{x\to a^-} f(x)$, then $\lim\limits_{x\to a} f(x) = L$. If $\lim\limits_{x\to a^+} f(x) \neq \lim\limits_{x\to a^-} f(x)$, then $\lim\limits_{x\to a} f(x)$ does not exist.

48. Since $\lim\limits_{x\to c} f(x) = L$ if and only if $\lim\limits_{x\to c^+} f(x) = L$ and $\lim\limits_{x\to c^-} f(x) = L$, then $\lim\limits_{x\to c} f(x)$ can be found by calculating $\lim\limits_{x\to c^+} f(x)$.

49. If f is an odd function of x, then $f(-x) = -f(x)$. Given $\lim\limits_{x\to 0^+} f(x) = 3$, then $\lim\limits_{x\to 0^-} f(x) = -3$.

50. If f is an even function of x, then $f(-x) = f(x)$. Given $\lim\limits_{x\to 2^-} f(x) = 7$ then $\lim\limits_{x\to -2^+} f(x) = 7$. However, nothing can be said about $\lim\limits_{x\to -2^-} f(x)$ because we don't know $\lim\limits_{x\to 2^+} f(x)$.

51. $I = (5, 5+\delta) \Rightarrow 5 < x < 5+\delta$. Also, $\sqrt{x-5} < \epsilon \Rightarrow x-5 < \epsilon^2 \Rightarrow x < 5+\epsilon^2$. Choose $\delta = \epsilon^2 \Rightarrow \lim\limits_{x\to 5^+} \sqrt{x-5} = 0$.

52. $I = (4-\delta, 4) \Rightarrow 4-\delta < x < 4$. Also, $\sqrt{4-x} < \epsilon \Rightarrow 4-x < \epsilon^2 \Rightarrow x > 4-\epsilon^2$. Choose $\delta = \epsilon^2 \Rightarrow \lim\limits_{x\to 4^-} \sqrt{4-x} = 0$.

53. As $x \to 0^-$ the number x is always negative. Thus, $\left|\frac{x}{|x|} - (-1)\right| < \epsilon \Rightarrow \left|\frac{x}{-x} + 1\right| < \epsilon \Rightarrow 0 < \epsilon$ which is always true independent of the value of x. Hence we can choose any $\delta > 0$ with $-\delta < x < 0 \Rightarrow \lim_{x \to 0^-} \frac{x}{|x|} = -1$.

54. Since $x \to 2^+$ we have $x > 2$ and $|x-2| = x-2$. Then, $\left|\frac{x-2}{|x-2|} - 1\right| = \left|\frac{x-2}{x-2} - 1\right| < \epsilon \Rightarrow 0 < \epsilon$ which is always true so long as $x > 2$. Hence we can choose any $\delta > 0$, and thus $2 < x < 2 + \delta \Rightarrow \left|\frac{x-2}{|x-2|} - 1\right| < \epsilon$. Thus, $\lim_{x \to -2^+} \frac{x-2}{|x-2|} = 1$.

55. (a) $\lim_{x \to 400^+} \lfloor x \rfloor = 400$. Just observe that if $400 < x < 401$, then $\lfloor x \rfloor = 400$. Thus if we choose $\delta = 1$, we have for any number $\epsilon > 0$ that $400 < x < 400 + \delta \Rightarrow |\lfloor x \rfloor - 400| = |400 - 400| = 0 < \epsilon$.
    (b) $\lim_{x \to 400^-} \lfloor x \rfloor = 399$. Just observe that if $399 < x < 400$ then $\lfloor x \rfloor = 399$. Thus if we choose $\delta = 1$, we have for any number $\epsilon > 0$ that $400 - \delta < x < 400 \Rightarrow |\lfloor x \rfloor - 399| = |399 - 399| = 0 < \epsilon$.
    (c) Since $\lim_{x \to 400^+} \lfloor x \rfloor \neq \lim_{x \to 400^-} \lfloor x \rfloor$ we conclude that $\lim_{x \to 400} \lfloor x \rfloor$ does not exist.

56. (a) $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \sqrt{x} = \sqrt{0} = 0$; $|\sqrt{x} - 0| < \epsilon \Rightarrow -\epsilon < \sqrt{x} < \epsilon \Rightarrow 0 < x < \epsilon^2$ for x positive. Choose $\delta = \epsilon^2$ $\Rightarrow \lim_{x \to 0^+} f(x) = 0$.
    (b) $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x^2 \sin\left(\frac{1}{x}\right) = 0$ by the sandwich theorem since $-x^2 \le x^2 \sin\left(\frac{1}{x}\right) \le x^2$ for all $x \neq 0$. Since $|x^2 - 0| = |-x^2 - 0| = x^2 < \epsilon$ whenever $|x| < \sqrt{\epsilon}$, we choose $\delta = \sqrt{\epsilon}$ and obtain $\left|x^2 \sin\left(\frac{1}{x}\right) - 0\right| < \epsilon$ if $-\delta < x < 0$.
    (c) The function f has limit 0 at $x_0 = 0$ since both the right-hand and left-hand limits exist and equal 0.

2.5 CONTINUITY

1. No, discontinuous at $x = 2$, not defined at $x = 2$

2. No, discontinuous at $x = 3$, $1 = \lim_{x \to 3^-} g(x) \neq g(3) = 1.5$

3. Continuous on $[-1, 3]$

4. No, discontinuous at $x = 1$, $1.5 = \lim_{x \to 1^-} k(x) \neq \lim_{x \to 1^+} k(x) = 0$

5. (a) Yes
   (b) Yes, $\lim_{x \to -1^+} f(x) = 0$
   (c) Yes
   (d) Yes

6. (a) Yes, $f(1) = 1$
   (b) Yes, $\lim_{x \to 1} f(x) = 2$
   (c) No
   (d) No

7. (a) No
   (b) No

8. $[-1, 0) \cup (0, 1) \cup (1, 2) \cup (2, 3)$

9. $f(2) = 0$, since $\lim_{x \to 2^-} f(x) = -2(2) + 4 = 0 = \lim_{x \to 2^+} f(x)$

10. $f(1)$ should be changed to $2 = \lim_{x\to 1} f(x)$

11. Nonremovable discontinuity at $x = 1$ because $\lim_{x\to 1} f(x)$ fails to exist ($\lim_{x\to 1^-} f(x) = 1$ and $\lim_{x\to 1^+} f(x) = 0$). Removable discontinuity at $x = 0$ by assigning the number $\lim_{x\to 0} f(x) = 0$ to be the value of $f(0)$ rather than $f(0) = 1$.

12. Nonremovable discontinuity at $x = 1$ because $\lim_{x\to 1} f(x)$ fails to exist ($\lim_{x\to 1^-} f(x) = 2$ and $\lim_{x\to 1^+} f(x) = 1$). Removable discontinuity at $x = 2$ by assigning the number $\lim_{x\to 2} f(x) = 1$ to be the value of $f(2)$ rather than $f(2) = 2$.

13. Discontinuous only when $x - 2 = 0 \Rightarrow x = 2$

14. Discontinuous only when $(x+2)^2 = 0 \Rightarrow x = -2$

15. Discontinuous only when $x^2 - 4x + 3 = 0 \Rightarrow (x-3)(x-1) = 0 \Rightarrow x = 3$ or $x = 1$

16. Discontinuous only when $x^2 - 3x - 10 = 0 \Rightarrow (x-5)(x+2) = 0 \Rightarrow x = 5$ or $x = -2$

17. Continuous everywhere. ($|x-1| + \sin x$ defined for all x; limits exist and are equal to function values.)

18. Continuous everywhere. ($|x| + 1 \neq 0$ for all x; limits exist and are equal to function values.)

19. Discontinuous only at $x = 0$

20. Discontinuous at odd integer multiples of $\frac{\pi}{2}$, i.e., $x = (2n-1)\frac{\pi}{2}$, n an integer, but continuous at all other x.

21. Discontinuous when $2x$ is an integer multiple of π, i.e., $2x = n\pi$, n an integer $\Rightarrow x = \frac{n\pi}{2}$, n an integer, but continuous at all other x.

22. Discontinuous when $\frac{\pi x}{2}$ is an odd integer multiple of $\frac{\pi}{2}$, i.e., $\frac{\pi x}{2} = (2n-1)\frac{\pi}{2}$, n an integer $\Rightarrow x = 2n-1$, n an integer (i.e., x is an odd integer). Continuous everywhere else.

23. Discontinuous at odd integer multiples of $\frac{\pi}{2}$, i.e., $x = (2n-1)\frac{\pi}{2}$, n an integer, but continuous at all other x.

24. Continuous everywhere since $-1 \le \sin x \le 1 \Rightarrow 0 \le \sin^2 x \le 1 \Rightarrow 1 + \sin^2 x \ge 1$; limits exist and are equal to the function values.

25. Discontinuous when $2x + 3 < 0$ or $x < -\frac{3}{2} \Rightarrow$ continuous on the interval $\left[-\frac{3}{2}, \infty\right)$.

26. Discontinuous when $3x - 1 < 0$ or $x < \frac{1}{3} \Rightarrow$ continuous on the interval $\left[\frac{1}{3}, \infty\right)$.

27. Continuous everywhere: $(2x-1)^{1/3}$ is defined for all x; limits exist and are equal to function values.

28. Continuous everywhere: $(2-x)^{1/5}$ is defined for all x; limits exist and are equal to function values.

29. Continuous everywhere since $\lim_{x\to 3} \frac{x^2-x-6}{x-3} = \lim_{x\to 3} \frac{(x-3)(x+2)}{x-3} = \lim_{x\to 3} (x+2) = 5 = g(3)$

30. Discontinuous at $x = -2$ since $\lim_{x \to -2} f(x)$ does not exist while $f(-2) = 4$.

31. Discontinuous at $x = 1$; $\lim_{x \to 1^+} (x^2 + 2) = 3$, but $\lim_{x \to 1^-} e^x = e$, so that $\lim_{x \to 1} f(x)$ does not exist while $f(1) = e$; and $\lim_{x \to 0^-} (1 - x) = 1 = \lim_{x \to 0^+} e^x$, so that $\lim_{x \to 0} f(x) = 1 = f(0)$

32. Discontinuous at $x = \ln 2$, since $2 - e^x = 0 \Rightarrow e^x = 2 \Rightarrow \ln e^x = \ln 2 \Rightarrow x = \ln 2$

33. $\lim_{x \to \pi} \sin(x - \sin x) = \sin(\pi - \sin \pi) = \sin(\pi - 0) = \sin \pi = 0$, and function continuous at $x = \pi$.

34. $\lim_{t \to 0} \sin(\frac{\pi}{2} \cos(\tan t)) = \sin(\frac{\pi}{2} \cos(\tan(0))) = \sin\left(\frac{\pi}{2} \cos(0)\right) = \sin\left(\frac{\pi}{2}\right) = 1$, and function continuous at $t = 0$.

35. $\lim_{y \to 1} \sec(y \sec^2 y - \tan^2 y - 1) = \lim_{y \to 1} \sec(y \sec^2 y - \sec^2 y) = \lim_{y \to 1} \sec((y-1)\sec^2 y) = \sec((1-1)\sec^2 1) =$ $\sec 0 = 1$, and function continuous at $y = 1$.

36. $\lim_{x \to 0} \tan\left[\frac{\pi}{4} \cos(\sin x^{1/3})\right] = \tan\left[\frac{\pi}{4} \cos(\sin(0))\right] = \tan\left(\frac{\pi}{4} \cos(0)\right) = \tan\left(\frac{\pi}{4}\right) = 1$, and function continuous at $x = 0$.

37. $\lim_{t \to 0} \cos\left[\frac{\pi}{\sqrt{19 - 3\sec 2t}}\right] = \cos\left[\frac{\pi}{\sqrt{19 - 3\sec 0}}\right] = \cos \frac{\pi}{\sqrt{16}} = \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$, and function continuous at $t = 0$.

38. $\lim_{x \to \frac{\pi}{6}} \sqrt{\csc^2 x + 5\sqrt{3} \tan x} = \sqrt{\csc^2\left(\frac{\pi}{6}\right) + 5\sqrt{3} \tan\left(\frac{\pi}{6}\right)} = \sqrt{4 + 5\sqrt{3}\left(\frac{1}{\sqrt{3}}\right)} = \sqrt{9} = 3$, and function continuous at $x = \frac{\pi}{6}$.

39. $\lim_{x \to 0^+} \sin\left(\frac{\pi}{2} e^{\sqrt{x}}\right) = \sin\left(\frac{\pi}{2} e^0\right) = \sin\left(\frac{\pi}{2}\right) = 1$, and the function is continuous at $x = 0$.

40. $\lim_{x \to 1} \cos^{-1}\left(\ln \sqrt{x}\right) = \cos^{-1}\left(\ln \sqrt{1}\right) = \cos^{-1}(0) = \frac{\pi}{2}$, and the function is continuous at $x = 1$.

41. $g(x) = \frac{x^2 - 9}{x - 3} = \frac{(x+3)(x-3)}{(x-3)} = x + 3,\ x \neq 3 \Rightarrow g(3) = \lim_{x \to 3} (x + 3) = 6$

42. $h(t) = \frac{t^2 + 3t - 10}{t - 2} = \frac{(t+5)(t-2)}{t-2} = t + 5,\ t \neq 2 \Rightarrow h(2) = \lim_{t \to 2} (t + 5) = 7$

43. $f(s) = \frac{s^3 - 1}{s^3 - 1} = \frac{(s^2 + s + 1)(s - 1)}{(s+1)(s-1)} = \frac{s^2 + s + 1}{s + 1},\ s \neq 1 \Rightarrow f(1) = \lim_{s \to 1} \left(\frac{s^2 + s + 1}{s + 1}\right) = \frac{3}{2}$

44. $g(x) = \frac{x^2 - 16}{x^2 - 3x - 4} = \frac{(x+4)(x-4)}{(x-4)(x+1)} = \frac{x+4}{x+1},\ x \neq 4 \Rightarrow g(4) = \lim_{x \to 4} \left(\frac{x+4}{x+1}\right) = \frac{8}{5}$

45. As defined, $\lim_{x \to 3^-} f(x) = (3)^2 - 1 = 8$ and $\lim_{x \to 3^+} (2a)(3) = 6a$. For $f(x)$ to be continuous we must have $6a = 8 \Rightarrow a = \frac{4}{3}$.

46. As defined, $\lim_{x\to -2^-} g(x) = -2$ and $\lim_{x\to -2^+} g(x) = b(-2)^2 = 4b$. For $g(x)$ to be continuous we must have $4b = -2 \Rightarrow b = -\frac{1}{2}$.

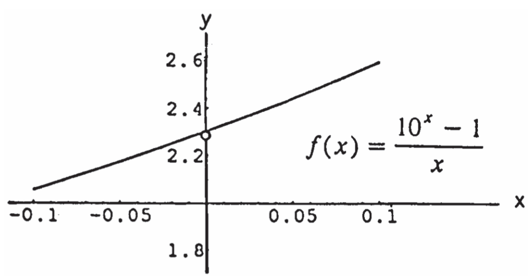
47. As defined, $\lim_{x\to 2^-} f(x) = 12$ and $\lim_{x\to 2^+} f(x) = a^2(2) - 2a = 2a^2 - 2a$. For $f(x)$ to be continuous we must have $12 = 2a^2 - 2a \Rightarrow a = 3$ or $a = -2$.

48. As defined, $\lim_{x\to 0^-} g(x) = \frac{0-b}{b+1} = \frac{-b}{b+1}$ and $\lim_{x\to 0^+} g(x) = (0)^2 + b = b$. For $g(x)$ to be continuous we must have $\frac{-b}{b+1} = b \Rightarrow b = 0$ or $b = -2$.

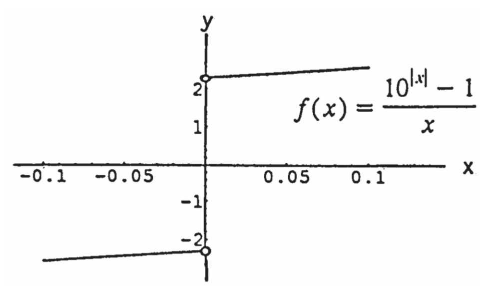
49. As defined, $\lim_{x\to -1^-} f(x) = -2$ and $\lim_{x\to -1^+} f(x) = a(-1) + b = -a + b$, and $\lim_{x\to 1^-} f(x) = a(1) + b = a + b$ and $\lim_{x\to 1^+} f(x) = 3$. For $f(x)$ to be continuous we must have $-2 = -a + b$ and $a + b = 3 \Rightarrow a = \frac{5}{2}$ and $b = \frac{1}{2}$.

50. As defined, $\lim_{x\to 0^-} g(x) = a(0) + 2b = 2b$ and $\lim_{x\to 0^+} g(x) = (0)^2 + 3a - b = 3a - b$, and $\lim_{x\to 2^-} g(x) = (2)^2 + 3a - b = 4 + 3a - b$ and $\lim_{x\to 0^+} g(x) = 3(2) - 5 = 1$. For $g(x)$ to be continuous we must have $2b = 3a - b$ and $4 + 3a - b = 1 \Rightarrow a = -\frac{3}{2}$ and $b = -\frac{3}{2}$.

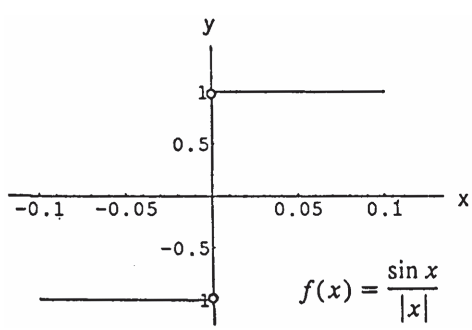
51. The function can be extended: $f(0) \approx 2.3$.



52. The function cannot be extended to be continuous at $x = 0$. If $f(0) \approx 2.3$, it will be continuous from the right. Or if $f(0) \approx -2.3$, it will be continuous from the left.



53. The function cannot be extended to be continuous at $x = 0$. If $f(0) = 1$, it will be continuous from the right. Or if $f(0) = -1$, it will be continuous from the left.



54. The function can be extended: $f(0) \approx 7.39$.