Problems

Problems

2.1 Estimate the kinetic energy of the ocean liner Queen

Mary 2, with mass (displacement) 76 000 tons, mov-
ing at a cruising speed of 26 knots. Compare with the
change in potential energy of the Queen Mary 2 when

27

which can be checked by differentiating with respect to each coor-
dinate. We set the integration constant C = 0 so that the potential energy is zero at x = 0.

The equation of motion is mx = F = −kx, and suggested
solution can be written x(t) = x0 cos ωt, where x0 = (x0, y0, z0).
Differentiating twice with respect to t we see x = −ω2 x, which
satisfies the equation of motion if ω = √k/m.

1

lifted by a three foot tide.

The knot is a nautical unit of speed, 1 knot = 0.5144 m/s, so

The total energy is the sum of kinetic, Ekin = potential, V(x).

1 1

2mx2,

and

26 knots = 13.4 m/s. The Queen Mary 2’s kinetic energy is then
 1

E =

=

2mx2 + 2kx21

kx20cos2ωt=1

Ekin =

2mv2 =

0.5(76 × 106 kg) (13.4 m/s)2 = 6.82 GJ .

2mω2 x0 sin2 ωt+2 2kx0 ,

The change in potential energy when lifted by a three foot tide is
 )

where we have used mω2 = k and sin2 + cos2 = 1.
 This solution is not the most general since it requires the oscil-

V = mgh = (76 × 106 kg)(9.81 m/s2)(3 ft)

or about 10% of the kinetic energy.

( 0.3048 m lator to be maximally stretched at t = 0. It is easy to show that

= 663 MJ ,

1 ft x(t) = x0 cos ω(t − t0 ) is a solution for any t0. Using the identity

cos(θ − φ) = cos θ cos φ + sin θ sin φ, it is clear that this solution is a sum of sine and cosine functions.

2.2 [T] A mass m moves under the influence of a force

derived from the potential V(x) = V0 coshax, where
the properties of cosh x and other hyperbolic functions
are given in Appendix B.4.2. What is the force on
the mass? What is the frequency of small oscillations
about the origin? As the magnitude of the oscillations
grows, does their frequency increase, decrease, or stay
the same?

The force on the mass is F(x) = −dV/dx = −V0a sinh ax,

1

with sinh x = 2 (ex − e−x). For small x, the exponential function

2.4 Estimate the potential energy of the international space

station (mass 370 000 kg) relative to Earth’s surface when in orbit at a height of 350 km. Compute the veloc-
ity of the space station in a circular orbit and compare the kinetic and potential energy.

From eq. (2.20), the potential energy relative to the surface is

V(h) = −GmM⊕ + GmM⊕

R⊕ + h R⊕

= (6.67 × 10−11 N m2/kg2) (3.7 × 105 kg) (6 × 1024 kg)×
 ( )

is approximated using a Taylor expansion by

2

×

1

6.37 × 106 m

−

1

6.37 × 106 + 3.5 × 105 m

= 1.21 TJ.

ey = 1 + y +

y y32 + 6 +

...,

In a circular orbit, the gravitational force is equal to the mass times the centripetal acceleration,

so coshy = 1+y2/2+. .. and sinh y = y+y3/6+ For small oscil-

lations, i.e. when ax ≪ 1, so we only need keep the first term in the expansion of sinh ax, the force is F(x) = −V0a2 x. The equation of motion for small oscillations is

mx = −V0a2x,

mv2

r

Solving for velocity,
 √ √

= GM⊕m

r2

which is the equation for a harmonic oscillator (see eq. (2.10)) with v=
frequency ω = a √V0/m.

This frequency is independent of magnitude of oscillations. If,

GM⊕

r

=

(6.67 × 10−11 )(6 × 1024)
(6.37 × 106 + 3.5 × 105)

m/s = 7.7 km/s.

however, the amplitude of the oscillation becomes large compared
to 1/a, then the non-linear terms in the expansion of sinh x cannot

The kinetic energy is equal to
 1 1

be ignored and the equation of motion becomes Ekin =

( )

2mv2 = 2

(3.7 × 105 kg) (7.7 × 103 m/s)2 = 11 TJ.

mx = −V0a ax +

1

6a3 x3 +

,.

Thus, the kinetic energy is 9.1 times the potential energy relative to Earth’s surface.

All of the terms in the expansion have the same sign, so as the amplitude of oscillations increases, the magnitude of the force increases compared to just a simple harmonic oscillator and the frequency of the oscillations increases.

2.3 [T] An object of mass m is held near the ori-

gin by a spring, F = −kx. What is its poten-
tial energy? Show that x(t) = (x0, y0, z0)cos ωt is
a possible classical motion for the mass. What is
the energy of this solution? Is this the most gen-
eral motion? If not, give an example of another
solution.

2.5 [T] Relate eq. (2.20) to eq. (2.9) and compute g in terms

of G, M⊕, R⊕ as described in the text.

The potential yielding the r−2 force law is V(r) = −GMm/r. In the approximation that the gravitational force is constant, the potential is V(z) = mgz, where z = r − R⊕ is the height from Earth’s surface, R⊕ is Earth’s radius and z ≪ R⊕. By definition,

V(z) − V(0) = −GM⊕ m + GM⊕m.

R⊕ + z R⊕

Use a series expansion for 1/(R⊕ + z), (see eq. (B.63)),
 1 1

R⊕ + z

According to eq. (2.18), the force is the negative gradient of the
potential energy. Setting F = −kx = −∇V, the solution is giving

=

R⊕

− z

R2

+ ...,

⊕

V(x) =

1 V(z) − V(0) =

GM⊕m

(z + O(z2/R3
 ⊕))

2kx2 +C, R2 ⊕

28

Comparing this to V(z) = mgz, and setting V(0) = 0, we get to leading order in z/R⊕,

GM⊕

(a) The moon orbits the earth about once a month and has an orbital

radius of about 380 000 km, so its speed is v = 2πR/T

0.91 km/s. The moon has a mass of 7.3 × 1022 kg, so this corresponds to a kinetic energy of

g=

R2

= (6.67×10−11 Nm2/kg2)(5.97×1024 kg)
 (6.37 × 106 m)2

⊕

Ekin = = 9.81 m/s2 .

1

0.5(7.3×1022 kg)(0.91×103 m/s)2 3.0×1028 J.
2mv2 =

2.6 Make a rough estimate of the maximum hydropower

available from rainfall in the US state of Col-

(b) Large raindrops are about 5 mm in diameter and fall at ∼9 m/s,

so they have a kinetic energy of

1 ( 4πρ)( d)3

orado. Look up the average yearly rainfall and aver- Ekin = age elevation of Colorado and estimate the poten-

2 3 2

v2

( 4π)

tial energy (with respect to sea level) of all the
water that falls on Colorado over a year. How
does this compare with the US yearly total energy
consumption?

Colorado has an average yearly precipitation of ∼0.4 m. Sup-
pose this is typical of the whole state of Colorado. Colorado has a mean elevation of approximately 2 070 m and a total surface area of 270 000 km2. All the precipitation falling on the state in one year then has a total potential energy of

mgh ≈ ρVgh (1000 kg/m3)(0.4 m)(2.7 × 1011 m2)

× (9.8 m/s2)(2070 m) 2.2 × 1018 J 2.2 EJ.

This is about 2% of the total energy consumed in the US each year.

2.7 Choose your favorite local mountain. Estimate how

much energy it takes to hike to the top of the mountain
(ignoring all the local ups and downs of a typical trail).
How does your result compare to a typical day’s food
energy intake of 10 MJ, taking into account the fact that
human muscles are not 100% efficient at converting
energy into work?

Mount Washington is the tallest mountain in the State of New
Hampshire? The peak of Mount Washington is at 1 917 meters
(6288 feet) above sea level. A typical route, however, begins at
about 600 meters above sea level, so the elevation gain is roughly

1 300 m. For a hiker of 80 kg (including food, water, extra clothing, etc.), the potential energy gain on the hike is

V = mgh (80 kg)(9.8 m/s2)(1300 m) 1 MJ. (2.45)

At 20-30% muscle efficiency, this represents about one third to one
half of the useful energy output from the typical hiker’s 10 MJ/day
food energy intake. This is computed just from the elevation gain,
without including local ups and downs of the trail, horizontal dis-
tance covered, or the hike down. So don’t feel bad about consuming
a lot of high-calorie trail food next time you are on a strenuous
hike!

2.8 Use any means at your disposal (including the inter-

net) to justify estimates of the following (an order-
of-magnitude estimate is sufficient): (a) The kinetic
energy of the Moon in its motion around Earth. (b)
The kinetic energy of a raindrop before it hits the

0.5(1000 kg/m3) (2.5 × 10−3 m)3)(9 m/s)2

3

∼ 3 mJ.

(c) Lake Powell has a volume of approximately 33 km3 and the

Glen Canyon Dam has a height of Z = 220 m. Treating the lake

as a box of area A and volume V = ZA, its potential energy is

∫ Z

V = ρgA dzz = ρgAZ2/2 = (1/2)ρgVZ

0

∼ (0.5)(1000 kg/m3)(9.8 m/s2)(33 × 109 m3 )(220 m)
∼ 36 PJ .

(d) The cross sectional area of a person on a bike is around 0.5 m2

with a drag coefficient around 1. The energy output is mostly to

counteract air resistance. The energy lost is

1

2ρaircdAv2d=0.5(1.2kg/m3)(0.5m2)
 × (4.2 m/s)2(1.5 × 104 m) 79 kJ .

Assuming an efficiency of ∼25% for the human body, around 340 kJ will will have to be expended

(e) Assume the runner has a cross-sectional area of approximately

0.5 m3 and a drag coefficient around 1. Good runners finish in

around 10 s, so they have a speed of 10 m/s. The total energy lost to air resistance will be (1/2)cd ρairAv2d ∼ 3 kJ.

(f) Estimate that a climber and gear have a mass of approximately

90 kg. The top of Mt. Everest is 8 848 m above sea level, so the climber will have

mgh (90 kg)(9.8 m/s2)(8.9 × 103 m) 7.8 MJ
of gravitational potential energy with respect to sea level.

2.9 Verify the claim that conversion of the potential energy

at the top of a 15 m hill to kinetic energy would
increase the speed of the Toyota Camry from 62 to

74 mph.

A Camry has a mass of 1800 kg, so it gains 265 kJ of kinetic
energy when its elevation falls by 15 m. Conservation of energy
requires that

1

2mv2 +mgh=constant.

Using this, the final velocity if it starts at 62 mi/hr is
 √

vf = v2 + 2gh = 32.6 m/s = 73 mi/hr.

ground. (c) The potential energy of the water in Lake
Powell (relative to the level of the Colorado river 2.10

directly beneath the dam). (d) Energy needed for you
to power a bicycle 15km at 15 km/h on flat terrain.

(e) The energy lost to air resistance by an Olympic
athlete running a 100 m race. (f) The gravitational
potential energy of a climber at the top of Mount
Everest.

i

Consider a collision between two objects of differ-
ent masses M, m moving in one dimension with ini-
tial speeds V, −v. In the center-of-mass frame, the
total momentum is zero before and after the colli-
sion. In an elastic collision both energy and momen-
tum are conserved. Compute the velocities of the two
objects after an elastic collision in the center-of-mass
frame. Show that in the limit where m/M → 0, the

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more massive object remains at rest in the center-of-
mass frame (V = 0 before and after the collision),
and the less massive object simply bounces off the
more massive object (like a tennis ball off a concrete
wall).

In the center-of-mass frame the objects have velocities V′ =

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and at 10 000 m has a potential energy of

V = mgh = (70000 kg)(9.8 m/s2)(10000 m) = 6.86 GJ.

The density of air at 10 000 m is approximately e−10/8.5 = 31% of
the density at sea level ρair = 0.31(1.17 kg/m3) 0.36 kg/m3.
Traveling at v = 800 km/h for d = 2 000 km at 10 000 m, the plane
loses

1

V +u and v′ = v+u, where −u is the velocity of the center-of-mass Eair = viewed from the original frame. The total momentum of the two

2ρaircdAv2d

0.5(0.36 kg/m3)(0.03)

objects in the center-of-mass vanishes: ptot = M(V+u)+m(−v+u) = 0, which determines the center-of-mass velocity

mv − MV

× (12 m2)(222 m/s)2(2000 × 103 m) 6.4 GJ .

due to air resistance. The total energy used by the plane over
the flight is approximately 15.3 GJ. This is 306 MJ/passenger or

u=

and

m+M

m

153 kJ/passenger-km. A typical car uses around 210 MJ of mechan-
ical energy over the 340 km trip between New York and Boston.
With 2 passengers, this is 310 kJ/passenger-km, or nearly twice the
energy usage of the airplane. So, before engine efficiencies are con-

V′ = V + u =

v′ = −v + u = −

m+M

M

m+M

(v + V)

(v + V) 2.13

sidered, automobiles use roughly twice as much energy over a long
trip.

In the American game of baseball, a pitcher throws
a baseball, which is a round sphere of diameter b =

are the objects’ velocities before the collision in the center-of-
mass. If energy and momentum are both conserved in the collision,
the objects’ must reverse their center-of-mass velocities after the
collision,

m

0.075 m, a distance of 18.4 m (60.5 feet), to a batter,
who tries to hit the ball as far as he can. A baseball
has a mass close to 0.15 kg. A radar gun measures the
speed of a baseball at the time it reaches the batter at

V′

f

v′

f

= −V′ = − (v + V)

m+M

M

= −v′ = (v + V) .

m+M

44.7 m/s (100 mph). The drag coefficient cd of a base-
ball is about 0.3. Give a semi-quantitative estimate of
the speed of the ball when it left the pitcher’s hand

In the limit m/M → 0, V′ → 0 and V′ → 0, the heavier object

f

begins and remains at rest, while v′ = −v′ = (v + V), the lighter

f

object bounces off.

2.11 As we explain in §34.2.1, the density of air decreases

with altitude roughly as ρ ∼ ρ0e−z/H , where z is the
height above the Earth’s surface and H 8.5 km near
the surface. Compute the ratio of air resistance losses
for an airplane traveling at 750 km/h at an altitude of
12000 m compared to an altitude of 2000 m. What hap-
pens to this ratio as the speed of the airplane changes?

by (a) assuming that the ball’s speed is never too dif-
ferent from 100 mph to compute roughly how long it
takes to go from the pitcher to the batter, (b) using (a)
to estimate the energy lost to air resistance, and (c)
using (b) to estimate the original kinetic energy and
velocity.

The ball has an effective area of A = πR2 = 0.00442 m2. For air
at sea level and 25◦C, ρair = 1.17 kg/m3. The energy loss rate is

dE

How do automobile air resistance losses at 2000 m compare to losses at sea level?

dt

=−1 −0.5(0.3)(0.00442 m2)

2cdAρairv3× (1.17 kg/m3)(44.7 m/s)3 −69 J/s.

If the ball’s speed does not differ significantly from 100 mph, it

Air resistance losses are linearly proportional to the density
of air, so the ratio of loses at 12 000 m compared to 2000 m is

takes (a) t = (18.4 m)/(44.7 m/s) 0.41 sec for the ball to reach
the batter. The total amount of energy lost is (b) ΔE =dE ×t

dt

r = e−Δz/H e−10/8.5 31%. Although the losses (dEair /dt) are
proportional to speed cubed, the speed cancels out in the ratio of
losses at two different air densities, so the ratio is independent of
the airplane’s speed. Similarly for a car at 2000 m compared to sea

level r e−2/8.5 79%.

2.12 Consider an airplane with mass 70000 kg, cross-

sectional area 12 m2, and drag coefficient 0.03. Esti-
mate the energy needed to get the plane moving at
800 km/h and lift the plane to 10 000 m, and estimate
air-resistance losses for a flight of 2000 km using the
formula in the previous problem. Do a rough com-
parison of the energy used per person to do a similar
trip in an automobile, assuming that the plane car-
ries 50 passengers and the automobile carries two
people.

For simplicity, assume that acceleration occurs quickly so that air resistance during acceleration can be neglected. The plane requires a total kinetic energy of

1

−28 J.

At 100 mph, the ball has a kinetic energy of

1

Ekinf = 150 J.

2mv2

(c) The initial kinetic energy of the ball is just the difference of this and the energy loss,

Ekini = 150 J + 28 J = 178 J.

This corresponds to a velocity of

√

√ 2(178 J)

v= Ekinim m/s 49 m/s 110 mph.

(0.15 kg)

2.14 Estimate the power output of an elite cyclist pedaling

a bicycle on a flat road at 14 m/s. Assume all energy
is lost to air resistance, the cross-sectional area of the
cyclist and the bicycle is 0.4 m2, and the drag coeffi-
cient is 0.75. Now estimate the power output of the
same elite cyclist pedaling a bicycle up a hill with slope

Ekin =

2mv2 =

0.5(70000 kg)(800 × 103 m/3600 s)2 = 1.73 GJ ,

8% at 5 m/s. Compute the air resistance assuming the

30

drag coefficient times cross-sectional area is cd A =

0.45 m2 (rider is in a less aerodynamic position). Com-
pute the ratio of the power output to potential energy
gain. Assume that the mass of the rider plus bicycle is

90 kg.

On the flat road, the power output of the cyclist is equal to the power lost due to air resistance,

the lab frame, we add a velocity +v to everything, so that the cylin-
der once again is traveling at a velocity of +v and the air molecule has a velocity of +2v.

To find cd , we must now compute the rate of energy loss of the
cylinder. In time dt, the front end of the cylinder passes through a
volume Avdt of air, corresponding to a mass of dm = Aρvdt. This
mass is accelerated from rest to a velocity of 2v. So, the column of
air gains an energy of

1

P=

1

(0.5)(1.2 kg/m3)(0.75)(0.4 m2)(14 m/s)3

dEair =

490 W.

2dmvair =

2Aρv3dt .

2ρcdAv3

The power output of the cyclist on the slope is equal to the power lost due to air resistance and due to the rate of change of potential energy,

P=Pair +Pg.
The contribution from air resistance is

From conservation of energy, the cylinder loses an equal amount of energy, so the power output due to air resistance is

)

(dE

= −2Aρv3 = −1

dt 2cdAρv3 ,

cyl

where cd = 4.

Pair =

1

2ρcdAv3 =

2.17

(0.5)(1.2 kg/m3)(0.45 m2)(5 m/s)3 = 34 W.

One way to estimate the effective area (see eq. (2.31))
of an object is to measure its limiting velocity v∞falling in air. Explain how this works and find the

The contribution from gravitational potential energy

Pg = mgdz (90 kg)(9.8 m/s)(5 m/s) sin(arctan(8/100) 350 W

dt

so the total power output of the cyclist on the slope is
 P = 380 W.

2.15 Compare the rate of power lost to air resistance for the

following two vehicles at 60 km/h and 120 km/h: (a)
General Motors EV1 with cd A 0.37 m2, (b) Hummer

H2 with cd A 2.45 m2.

expression for Aeff as a function of m (the mass of the object), v∞, g, and the density of air. The drag coeffi-
cient of a soccer ball (radius 11 cm, mass 0.43 kg) is cd ≈ 0.25. What is its limiting velocity?

Because the force due to air resistance increases with velocity,
if an object is allowed to fall in air, there is some velocity at which
the upward force due to air resistance is equal in magnitude to the
downward gravitational force. When this happens, there is no net
force on the object, so the object continues falling at this velocity,
which is v∞. Thus,

1

(a). At 60 km/h (16.67 m/s)
 dEloss 1

mg =

Solving for Aeff ,

2ρairAeffv∞ .

2mg

= Aeff =

dt 2ρcdAv3

(0.5)(1.2 kg/m3)(0.37 m2)(16.7 m/s)3 = 1.03 kW

2

ρairv

∞

The effective area of the soccer ball is Aeff = cd A = cd πr2 =

At 120 km/h, the power is 8 times as high, or 8.2 kW.

(b). The power for the Hummer H2 scales with the cd A, so the H2

loses energy at a rate of (2.45/.37)(1.03 kW) = 6.8 kW at

0.0095 m2. The limiting velocity is
 √ √

2mg 2(0.43 kg)(9.8 m/s2)

60 km/h and (2.45/.37)(8.2 kW) = 54 kW at 120 km/h. In both v∞ =

ρairAeff cases, the H2 loses energy at a rate 6.6 times that of the EV1.

27 m/s . (1.2 kg/m3)(0.0095 m2)

2.16 [T] Consider an idealized cylinder of cross-sectional

area A moving along its axis through an idealized
diffuse gas of air molecules with vanishing initial
velocity. Assume that the air molecules are pointlike
and do not interact with one another. Compute the
velocity that each air molecule acquires after a col-
lision with the cylinder, assuming that the cylinder
is much more massive than the air molecules. [Hint:
assume that energy is conserved in the reference frame
of the moving cylinder and use the result of Prob-
lem 2.10.] Use this result to show that the drag coeffi-
cient of the cylinder in this idealized approximation is

2.18 If the vehicle used as an example in this chapter accel-

erates to 50 km/h between each stoplight, find the max-
imum distance between stoplights for which the energy
used to accelerate the vehicle exceeds the energy lost to
air resistance. (You may assume that the time for accel-
eration is negligible in this calculation.) How does the
result change if the vehicle travels at 100 km/h between
lights? Why?

The vehicle in this chapter has A = 2.7 m2, cd = 1/3, and m = 1800 kg. The total energy lost to friction after a distance d at a velocity v is

1

Elost =

cd = 4.

2ρcdAv2d.

Suppose the cylinder is traveling at a velocity v. In the cylinder’s
rest frame, the air molecules are traveling at a velocity −v per-
pendicular to the surface of the cylinder. Taking the limit that the
cylinder is infinitely more massive than the air molecules, this is
the center of mass frame. In this limit, a collision may change the

Setting this equal to the kinetic energy and solving for d to get the maximum distance for which the kinetic energy exceeds the energy lost due to air resistance,

1 1

direction of the air molecule but not the speed. Since the velocity 2mv2 = of the air molecules is perpendicular to the end of the cylinder, the

final velocity must be +v in the cylinder rest frame. Going back to d=

2ρcdAv2d

m 1 800 kg

ρcdA (1.2 kg/m3)(1/3)(2.7 m2)

1.67 km.

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The result is the same if the vehicle travels at 100 km/h. Both
the kinetic energy and the energy loss are proportional to v2 so the
distance at which they are equal is independent of the velocity.

2.19 Estimate the rotational kinetic energy in a spinning yo-

yo (a plastic toy that you can assume is a cylinder of diameter 5.7 cm and mass 52 g, which rotates at 100 Hz). Compare to the gravitational potential energy of the yo-yo at height of 0.75 m.

The moment of inertia for a cylinder of mass m, length z and radius R rotating around its axis is (see Example 2.4),

1 1

Icylinder =

2ρπzR4 = 2mR2.

(2.46)

The energy of the rotating yo-yo is E =1 2Iω2 andω=2πν=

(2π s−1/Hz)(100 Hz) 630 s−1, so

1

Erot =

2Iω2

(0.25)(0.052 kg)(0.0285 m)2 (630 s−1)2 4.2 J .

(2.47)

Its gravitational potential energy at a height of 0.75 m is mgh

0.38 J, almost a factor of ten smaller than its rotational kinetic energy at 10 Hz. (Most of the energy is imparted in the initial “throw,” and not from gravitational potential energy.)

2.20 Verify the assertion (see Example 2.3) that Ekin =

−12V for the Moon in a circular orbit around
Earth. [Hint: the magnitude of the centripetal
acceleration for circular motion (2.35) can be rewritten
a = v2/r.]

Assume throughout that M⊕ ≫ mmoon, so we may take Earth
to be at rest. For a circular orbit, the centripetal acceleration is
a = v2/r. The acceleration due to gravity is GM⊕/r2 where M⊕is Earth’s mass, which is much greater than that of the moon. The
gravitational potential energy of the moon is V = −GM⊕ m/r

The centripetal acceleration and gravitational acceleration are
identical, so

v2 GM⊕

= GM⊕

r r2

or v2 =

, so

r

(

1

Ekin = −GM⊕m) =−1

2mv2 =−2 r 2V.

2.21 Estimate Earth’s kinetic energy of rotation (the moment

of inertia of a uniform sphere is2 5MR2).

Earth has a mass of 5.972 × 1024 kg and a mean radius of

6 371 km. It rotates about its axis once per sidereal day (23.9345 h), giving it an angular velocity of ω = 2π/(1 sidereal day) = 7.292 × 10−5 s−1. Its kinetic energy of rotation is

2

E=

MR

5

ω2 = (0.2)(5.972 × 1024 kg)(6 371 × 103 m)2

× (7.292 × 10−5 rad/s)2 = 2.578 × 1029 J .