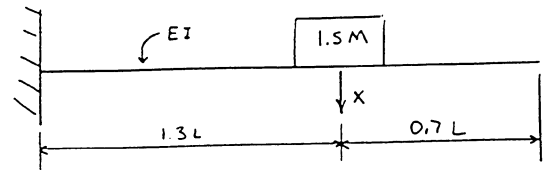
**Solutions Manual**

to accompany

# STRUCTURAL DYNAMICS
# Theory and Applications

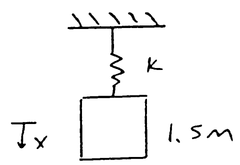
## 2.1



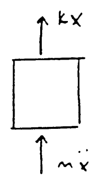

SOLUTION

d'ALEMBERT'S PRINCIPLE

$$\Sigma(\text{FORCES})_x - m\ddot{x} = 0$$




FBD




$$kx + m\ddot{x} = 0$$

$$\ddot{x} + \frac{k}{m}x = 0 \qquad \text{EQUATION OF MOTION}$$

$$k = \frac{3EI}{(1.3L)^3}$$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{3EI}{1.5M(1.3L)^3}} = 0.954\sqrt{\frac{EI}{ML^3}}$$

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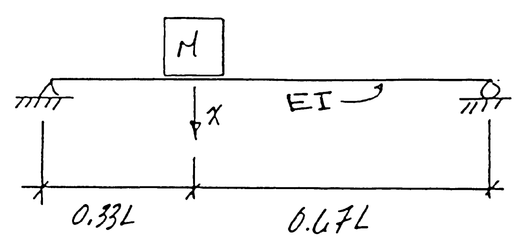
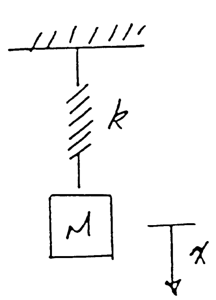
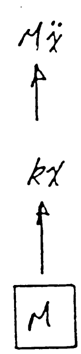
## 2.2




Solution:



## 2.2 Cont.

Equation of motion: $M\ddot{x} + kx = 0$ or $\ddot{x} + \frac{k}{M}x = 0$

$$k = \frac{6EIL}{(0.33L)(L-0.33L)[2L(0.33L)-(0.33L)^2-(0.33L)^2]}$$

$$k = \frac{61.37EI}{L^3}$$

Natural Frequency: $\omega = \sqrt{\frac{k}{M}} = 7.834\sqrt{\frac{EI}{ML^3}}$

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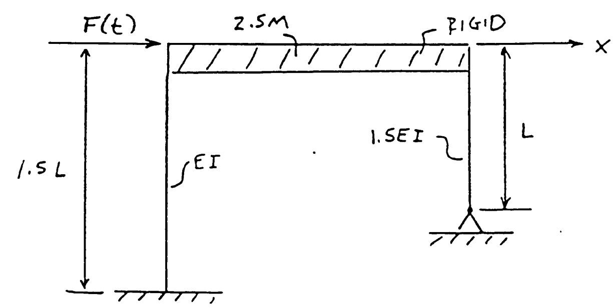
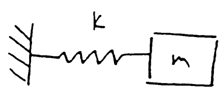
## 2.3




SOLUTION

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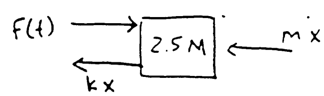



$m = 2.5M$

$\Sigma(\text{FORCES})_x - m\ddot{x} = 0$

$F(t) - kx - m\ddot{x} = 0$

$\ddot{x} + \frac{k}{m}x = \frac{F(t)}{m}$ EQUATION OF MOTION

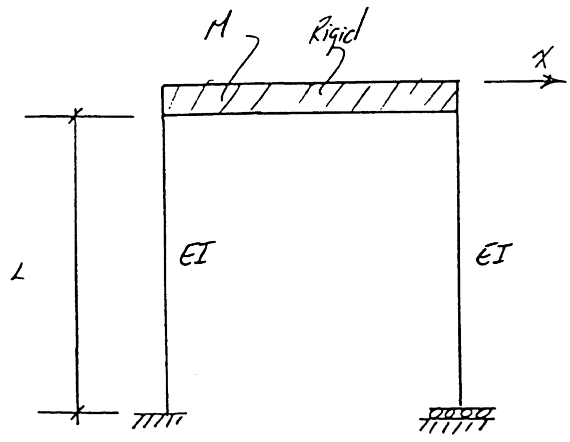
$$k = \frac{12EI}{(1.5L)^3} + \frac{3(1.5EI)}{L^3}$$

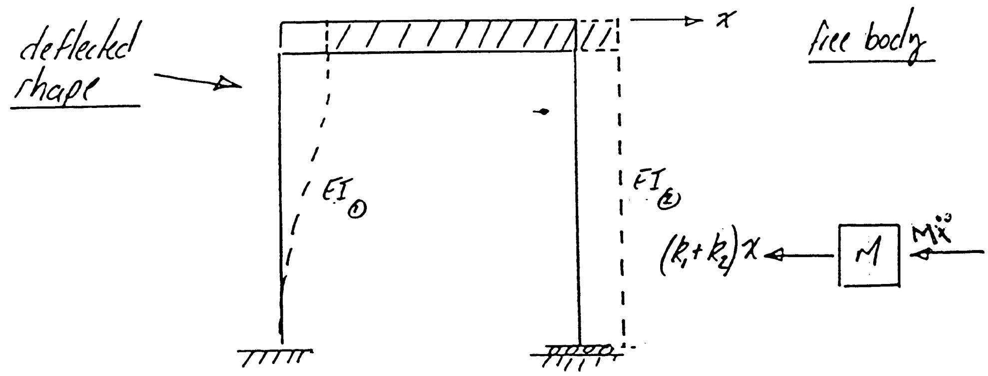
$$= \frac{12(30\times10^6)(150)}{(1.5\times12\times12.0)^3} + \frac{3(1.5)(30\times10^6)(150)}{(12.0\times12)^3}$$

$$= 12,140 \text{ LB/IN}$$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{12,140 \text{ LB/IN}}{2.5(1.0 \text{ LB}\cdot\text{sec}^2/\text{IN})}} = 69.7 \text{ rad/sec}$$

## 2.4




equation of motion: $\boxed{M\ddot{x} + kx = 0 \quad \text{or} \quad \ddot{x} + \frac{k}{M}x = 0}$ ANS

$k_1 = \frac{12EI}{L^3}$

$k_2 = 0$

natural freq. : $\boxed{\omega = \sqrt{\frac{k}{M}} = \left(\frac{12EI}{ML^3}\right)^{1/2}}$ ANS

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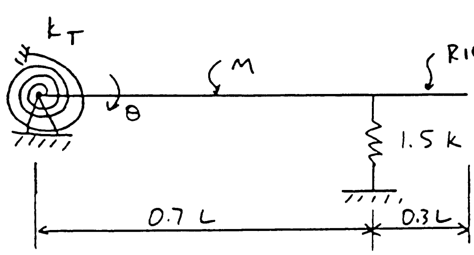
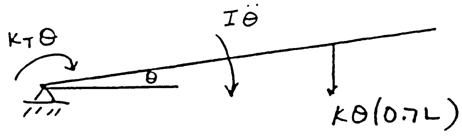
## 2.5



## 2.5 Cont.

SOLUTION




$\Delta = 0.7L\sin\theta \approx 0.7L\theta$ FOR SMALL $\theta$

$I = \dfrac{mL^2}{3}$ (ABOUT PIVOT PT)

$\Sigma M - I\ddot{\theta} = 0$

$k_T\theta + k(0.7L)\Delta + I\ddot{\theta} = 0$

$k_T\theta + k(0.7L)^2\theta + I\ddot{\theta} = 0$

$\dfrac{mL^2}{3}\ddot{\theta} + k(0.7L)^2\theta + k_T\theta = 0$

$\ddot{\theta} + \dfrac{3[k(0.7L)^2 + k_T]}{mL^2}\theta = 0$ EQUATION OF MOTION

$\omega = \sqrt{\dfrac{3k(0.7L)^2 + 3k_T}{mL^2}}$ NATURAL FREQUENCY

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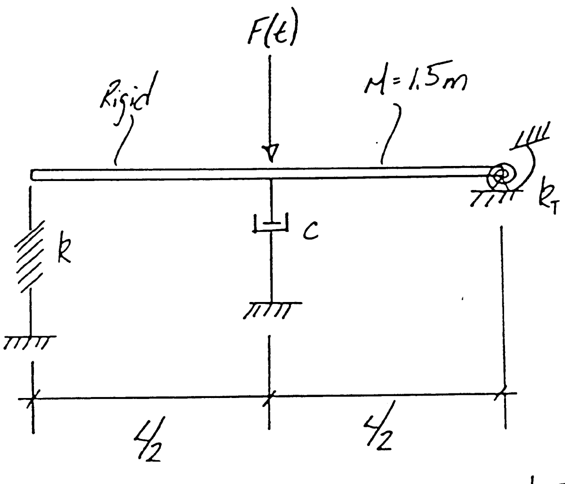
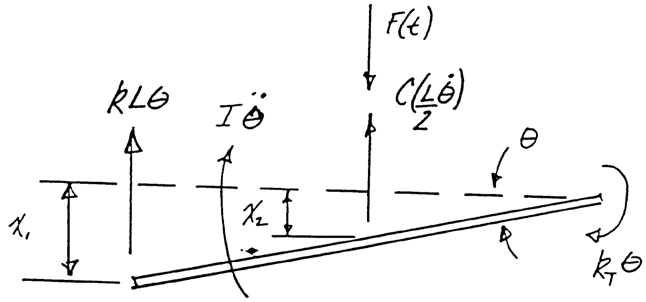
## 2.6




Solution:

free body




$x_1 = L\sin\theta$ small disp $= L\theta$

$x_2 = \frac{L}{2}\sin\theta$ small disp $= \frac{L}{2}\theta$

$\dot{x}_2 = \frac{L}{2}\dot{\theta}$

## 2.6 Cont.

$$I = \frac{ML^2}{3} = \frac{1.5mL^2}{3} = \frac{mL^2}{2}$$

equation of motion:

$$k_T\theta + kL\theta(L) + I\ddot{\theta} + c\left(\frac{L}{2}\dot{\theta}\right)\left(\frac{L}{2}\right) = F(t)\left(\frac{L}{2}\right)$$

$$k_T\theta + kL^2\theta + I\ddot{\theta} + \frac{CL^2}{4}\dot{\theta} = F(t)\left(\frac{L}{2}\right)$$

$$I\ddot{\theta} + \frac{CL^2}{4}\dot{\theta} + (kL^2 + k_T)\theta = F(t)\left(\frac{L}{2}\right)$$

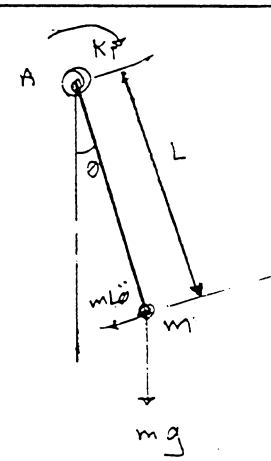
$$\left(\frac{mL^2}{2}\right)\ddot{\theta} + \frac{CL^2}{4}\theta + (kL^2 + k_T)\theta = F(t)\left(\frac{L}{2}\right)$$

$$\boxed{\ddot{\theta} + \frac{c}{2m}\theta + \frac{2(kL^2 + k_T)}{mL^2}\theta = F(t)\left(\frac{1}{mL}\right)}$$ ANS

natural frequency:

$$\boxed{\omega = \sqrt{\frac{2(kL^2 + k_T)}{mL^2}}}$$ ANS

## 2.7




$$\Sigma M_A = I_A\,\alpha$$

$$mL\ddot{\theta}(L) + mgL\sin\theta + K_t\theta = 0$$

$$mL^2\ddot{\theta} + (mgL\sin\theta + K_t\theta) = 0$$

## 2.7 Cont.

For small values of $\theta$

$\sin\theta = \theta$

$$mL^2\ddot{\theta} + (mgL + k_t)\theta = 0$$

$$\ddot{\theta} + \left(\frac{g}{L} + \frac{k_t}{mL^2}\right)\theta = 0$$

$$\omega = \sqrt{\frac{g}{L} + \frac{k_t}{mL^2}}$$

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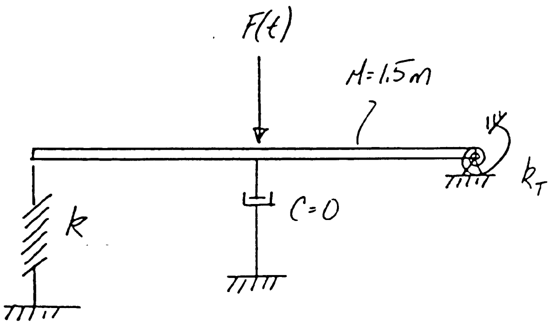
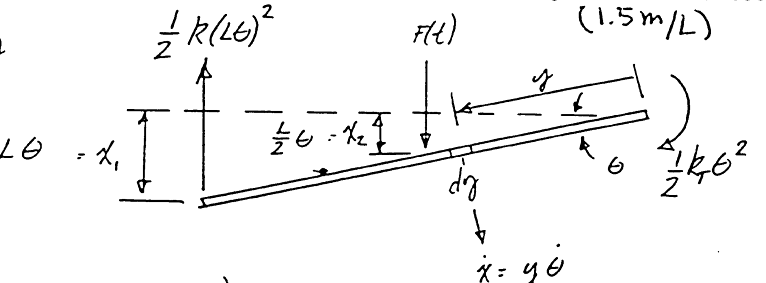
## 2.8




Assume a conservative system (i.e. no damping)

Solution:




Kinetic Energy $\left(\frac{1}{2}mv^2\right)$

$$\frac{1}{2}\int_0^L \frac{1.5m}{L}(y\dot{\theta})^2\,dy = \frac{3}{4}\int_0^L \frac{m}{L}\dot{\theta}^2 y^2\,dy = \frac{3}{4}\left(\frac{m}{3L}\dot{\theta}^2 y^3\right)\Big|_0^L$$

$$\boxed{T = \frac{mL^2\dot{\theta}^2}{4}}$$

2.8 Cont.

Potential Energy

$$V = \frac{1}{2}k(L\theta)^2 + \frac{1}{2}k_T\theta^2 - F(t)\left(\frac{L}{2}\right)\theta$$

TOTAL WORK $(T+V) = \text{constant}$

$$\frac{mL^2\dot{\theta}^2}{4} + \frac{1}{2}k(L\theta)^2 + \frac{1}{2}k_T\theta^2 - F(t)\left(\frac{L}{2}\right)\theta = \text{constant}$$

$$\frac{d(T+V)}{d\theta} = 0 = \frac{mL^2}{2}\ddot{\theta}\cancel{\dot{\theta}} + kL^2\cancel{\dot{\theta}}\theta + k_T\cancel{\dot{\theta}}\theta - F(t)\left(\frac{L}{2}\right)\cancel{\dot{\theta}}$$

equation of motion:

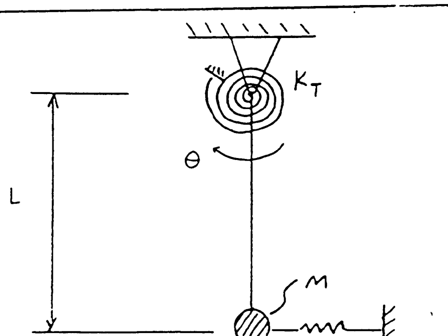
$$\frac{mL^2}{2}\ddot{\theta} + (kL^2 + k_T)\theta = F(t)\left(\frac{L}{2}\right)$$

$$\boxed{\ddot{\theta} + \frac{2(kL^2+k_T)}{mL^2}\theta = F(t)\left(\frac{1}{mL}\right)}$$ ANS

natural frequency:

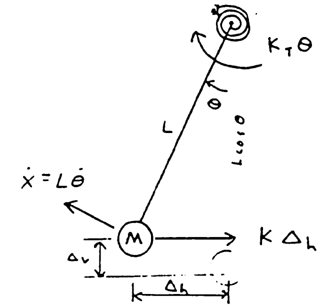
$$\boxed{\omega = \sqrt{\frac{2(kL^2+k_T)}{mL^2}}}$$ ANS

2.9



## 2.9 Cont.

SOLUTION




$\Delta_h = L\sin\theta \doteq L\theta$

$\Delta_v = L - L\cos\theta = L(1-\cos\theta)$

KINETIC ENERGY

$$T = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}m(L\dot{\theta})^2$$

POTENTIAL ENERGY

$$V = mg\Delta_v + \frac{1}{2}K\Delta_h^2 + \frac{1}{2}K_T\theta^2$$

$$= mgL(1-\cos\theta) + \frac{1}{2}K(L\theta)^2 + \frac{1}{2}K_T\theta^2$$

ENERGY METHOD

$$T + V = \text{CONSTANT}$$

$$\frac{d}{dt}(T+V) = 0$$

$$mL^2\dot{\theta}\ddot{\theta} + mgL(\sin\theta)\dot{\theta} + KL^2\theta\dot{\theta} + K_T\theta\dot{\theta} = 0$$

$$mL^2\ddot{\theta} + mgL\sin\theta + KL^2\theta + K_T\theta = 0$$

$$\ddot{\theta} + \left(\frac{MgL + KL^2 + K_T}{ML^2}\right)\theta = 0 \qquad \text{EQUATION OF MOTION}$$

$$\omega = \sqrt{\frac{MgL + KL^2 + K_T}{ML^2}} \qquad \text{NATURAL FREQUENCY}$$

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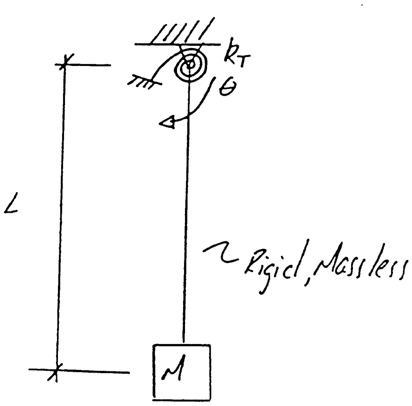
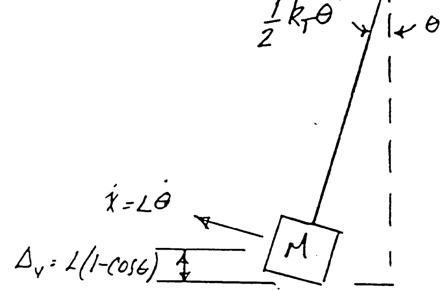
## 2.10




Solution:



## 2.10 Cont.

Kinetic Energy (T) $\frac{1}{2}mv^2$

$$\frac{1}{2}M(L\dot{\theta})^2$$

Potential energy: (V)

$$MgL(1-\cos\theta) + \frac{1}{2}k_T\theta^2$$

TOTAL WORK: (T+V)

$$\frac{1}{2}ML^2\dot{\theta}^2 + \frac{1}{2}k_T\theta^2 + MgL(1-\cos\theta) = \text{constant}$$

$$\frac{d(T+V)}{d\theta} = 0 = ML^2\ddot{\theta}\cancel{\dot{\theta}} + k_T\cancel{\dot{\theta}}\theta + MgL\,\overset{\theta}{\cancel{\sin\theta}}\,\cancel{\dot{\theta}}$$

$$ML^2\ddot{\theta} + k_T\theta + MgL\theta = 0$$

$$ML^2\ddot{\theta} + (k_T + MgL)\theta = 0$$

$$\boxed{\ddot{\theta} + \left(\frac{k_T + MgL}{ML^2}\right)\theta = 0}$$ ANS

natural frequency:

$$\boxed{\omega = \sqrt{\frac{k_T + MgL}{ML^2}}}$$ ANS

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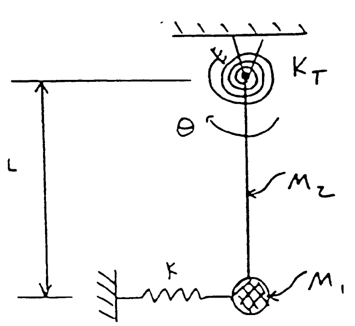
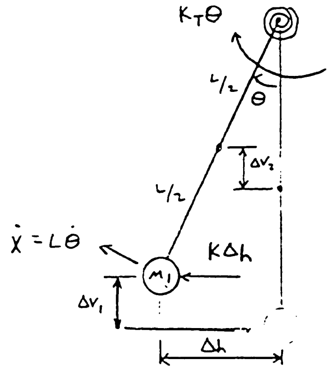
## 2.11



## 2.11 Cont.

SOLUTION




$$\Delta h = L\sin\theta \approx L\theta$$
$$\Delta V_1 = L(1-\cos\theta)$$
$$\Delta V_2 = \frac{L}{2}(1-\cos\theta)$$

$$T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}I_o\dot{\theta}^2$$
$$= \frac{1}{2}m_1(L\dot{\theta})^2 + \frac{1}{2}\left(\frac{1}{3}m_2L^2\right)\dot{\theta}^2$$
$$= \frac{1}{2}m_1L^2\dot{\theta}^2 + \frac{1}{6}m_2L^2\dot{\theta}^2$$
$$V = m_1 g\,\Delta V_1 + m_2 g\,\Delta V_2 + \frac{1}{2}k\,\Delta h^2 + \frac{1}{2}k_T\theta^2$$
$$= m_1 g L(1-\cos\theta) + m_2 g \tfrac{L}{2}(1-\cos\theta) + \frac{1}{2}k(L\theta)^2 + \frac{1}{2}k_T\theta^2$$

ENERGY METHOD

$$T + V = \text{CONSTANT}$$
$$\frac{d}{dt}(T+V) = 0$$

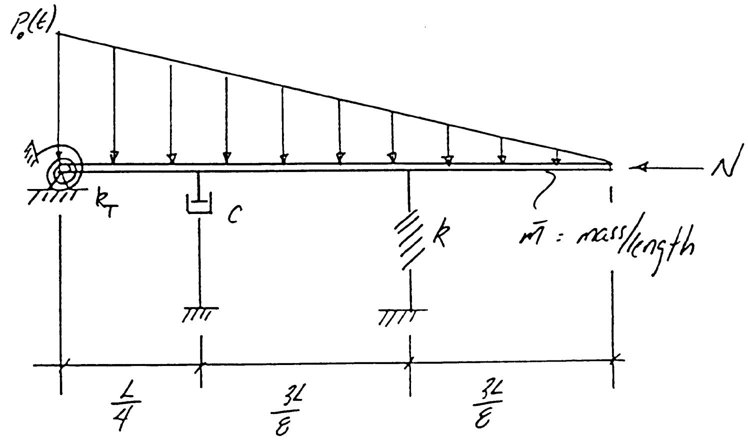
$$m_1L^2\dot{\theta}\ddot{\theta} + \frac{1}{3}m_2L^2\dot{\theta}\ddot{\theta} + m_1 g L(\sin\theta)\dot{\theta} + m_2 g\frac{L}{2}(\sin\theta)\dot{\theta} + kL^2\theta\dot{\theta} + k_T\theta\dot{\theta} = 0$$

$$m_1L^2\ddot{\theta} + \frac{1}{3}m_2L^2\ddot{\theta} + m_1gL\theta + m_2g\frac{L}{2}\theta + kL^2\theta + k_T\theta = 0$$

$$\ddot{\theta} + \frac{m_1gL + \frac{1}{2}m_2gL + kL^2 + k_T}{m_1L^2 + \frac{1}{3}m_2L^2}\,\theta = 0$$ EQUATION OF MOTION

$$\omega = \sqrt{\frac{m_1gL + \frac{1}{2}m_2gL + kL^2 + k_T}{m_1L^2 + \frac{1}{3}m_2L^2}}$$ NATURAL FREQUENCY

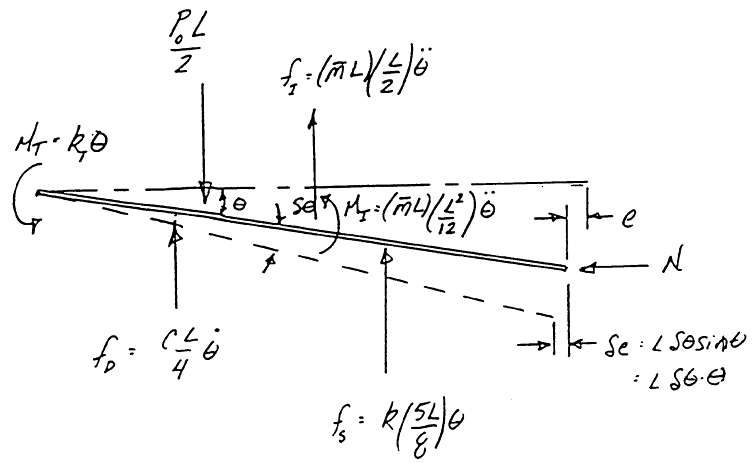
## 2.12



## 2.12 Cont.

Solution:

free body




equation of motion:

$$-f_s\left(\frac{5L}{8}\delta\theta\right) - f_D\left(\frac{L}{4}\delta\theta\right) - M_T\,\delta\theta + N\,\delta e + \frac{P_oL}{2}\left(\frac{L}{3}\delta\theta\right)$$

$$\Longrightarrow \quad - f_I\left(\frac{L}{2}\delta\theta\right) - M_I\,\delta\theta = 0$$

$$-\frac{25}{64}L^2k\theta\cancel{\delta\theta} - \frac{cL^2}{16}\dot{\theta}\cancel{\delta\theta} - k_T\theta\cancel{\delta\theta} + NL\theta\cancel{\delta\theta} + \frac{P_oL^2}{6}\cancel{\delta\theta}$$

$$\Longrightarrow \quad - \frac{\bar{m}L^3}{4}\ddot{\theta}\cancel{\delta\theta} - \frac{\bar{m}L^3}{12}\ddot{\theta}\cancel{\delta\theta} = 0$$

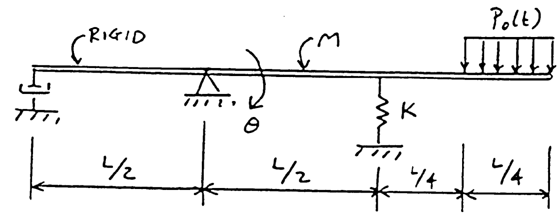
$$\Longrightarrow -\frac{\bar{m}L^3}{3}\ddot{\theta} - \frac{cL^2}{16}\dot{\theta} - \left(\frac{25}{64}kL^2 + k_T - NL\right)\theta + \frac{P_oL^2}{6} = 0$$

$$\boxed{\frac{\bar{m}L^3}{3}\ddot{\theta} + \frac{cL^2}{16}\dot{\theta} + \left(\frac{25}{64}kL^2 + k_T - NL\right)\theta = \frac{P_oL^2}{6}} \quad \underline{\text{ANS}}$$

Natural frequency: $\omega = \sqrt{\frac{k}{m}}$

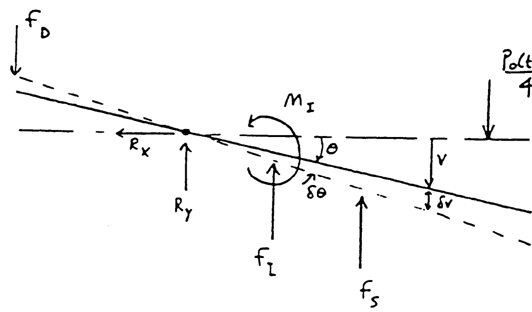
$$\boxed{\omega = \sqrt{\frac{\frac{25}{64}kL^2 + k_T - NL}{\frac{\bar{m}L^3}{3}}}} \quad \underline{\text{ANS}}$$

2.13




SOLUTION

FBD




FORCES

$$f_S = K\left(\tfrac{L}{2}\right)\theta \qquad f_{P_o} = \frac{P_o L}{4} f(t)$$
$$f_D = c\left(\tfrac{L}{2}\right)\dot{\theta}$$
$$f_I = m\left(\tfrac{L}{4}\right)\ddot{\theta} \qquad M_I = I\ddot{\theta} = \frac{mL^2}{12}\ddot{\theta}$$

PRINCIPLE OF VIRTUAL DISPLACEMENTS

$\delta W = 0$

IN CALCULATING THE VIRTUAL WORK, A QUANTITY IS POSITIVE WHEN THE FORCE ACTS IN THE SAME DIRECTION AS THE VIRTUAL DISPLACEMENT.

$$-f_S\left(\tfrac{L}{2}\delta\theta\right) - f_D\left(\tfrac{L}{2}\delta\theta\right) - f_I\left(\tfrac{L}{4}\delta\theta\right) - M_I(\delta\theta) + f_{P_o}\left(\tfrac{7}{8}L\,\delta\theta\right) = 0$$

$$-K\frac{L^2}{4}\theta\delta\theta - c\frac{L^2}{4}\dot{\theta}\delta\theta - m\frac{L^2}{16}\ddot{\theta}\delta\theta - \frac{mL^2}{12}\ddot{\theta}\delta\theta + \frac{7P_o(t)L^2}{32}\delta\theta = 0$$

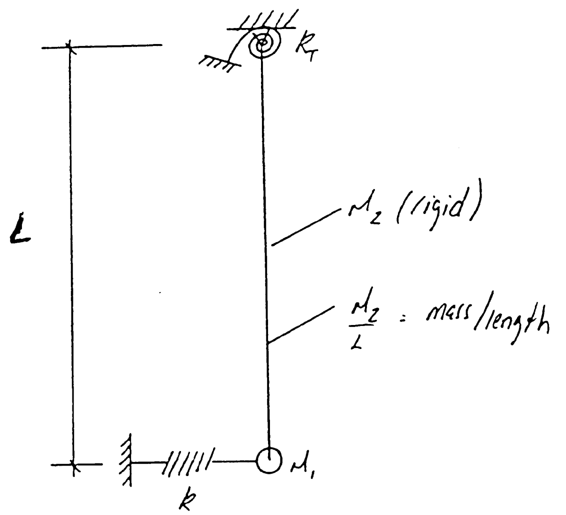
$$\left[\left(\frac{mL^2}{16} + \frac{mL^2}{12}\right)\ddot{\theta} + \left(\frac{cL^2}{4}\right)\dot{\theta} + \left(\frac{KL^2}{4}\right)\theta\right]\delta\theta = \frac{7P_o(t)L^2}{32}\delta\theta$$

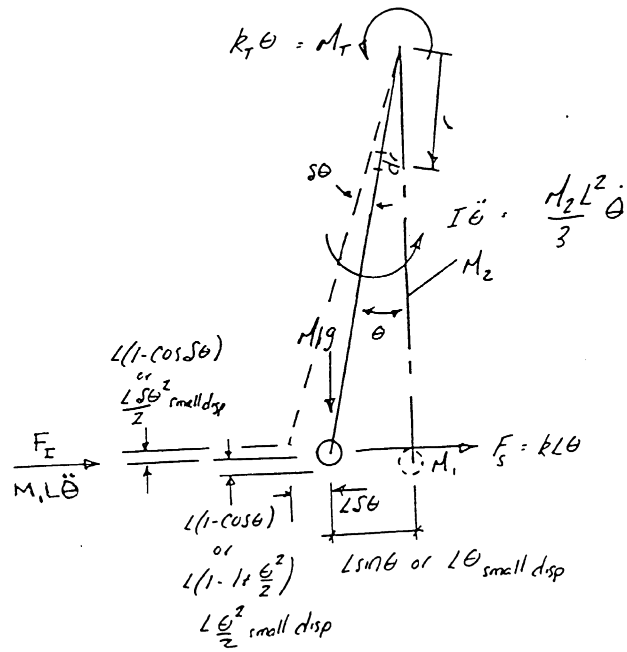
SINCE $\delta\theta \neq 0$

$$\frac{7mL^2}{48}\ddot{\theta} + \frac{cL^2}{4}\dot{\theta} + \frac{KL^2}{4}\theta = \frac{7P_o(t)L^2}{32}$$ EQUATION OF MOTION

$$\omega = \sqrt{\frac{\frac{KL^2}{4}}{\frac{7mL^2}{48}}} = \sqrt{\frac{12K}{7m}}$$ RAD/SEC NATURAL FREQUENCY

## 2.14




$$-M_T\delta\theta - F_s L\delta\theta - F_I L\delta\theta - I\ddot{\theta}\delta\theta - M_1 g L\left((1-\cos(\theta+\delta\theta))-(1-\cos\theta)\right) - \int_0^L \frac{M_2}{L} g r\left((1-\cos(\theta+\delta\theta))-(1-\cos\theta)\right)dr = 0$$

$$-k_T\theta\delta\theta - kL^2\theta\delta\theta - M_1L^2\ddot{\theta}\delta\theta - \frac{M_2L^2}{3}\ddot{\theta}\delta\theta - M_1 g L\underbrace{\left((1-\cos(\theta+\delta\theta))-(1-\cos\theta)\right)}_{x} - \int_0^L \frac{M_2}{L} g r\underbrace{\left((1-\cos(\theta+\delta\theta))-(1-\cos\theta)\right)}_{x}dr = 0$$

trigonometric Identities

$x = (1-\cos(\theta+\delta\theta)) - (1-\cos\theta)$

$x = 1-\cos(\theta+\delta\theta) - 1 + \cos\theta$

$x = \cos\theta - \cos(\theta+\delta\theta)$

$x = \cos\theta - (\cos\theta\cos\delta\theta - \sin\theta\sin\delta\theta)$

$x = \cos\theta - \cos\theta\cos\delta\theta + \sin\theta\sin\delta\theta$

small displacements $\cos\theta = 1$, $\cos\delta\theta = 1$, $\sin\theta = \theta$, $\sin\delta\theta = \delta\theta$ so, $x = \theta\delta\theta$

$\therefore$

$$k_T\theta\delta\theta + kL^2\theta\delta\theta + M_1L^2\ddot{\theta}\delta\theta + \frac{M_2L^2}{3}\ddot{\theta}\delta\theta + M_1 g L\theta\delta\theta + \underbrace{\int_0^L \frac{M_2}{L} g r\,\theta\delta\theta\,dr}_{} = 0$$

$$\frac{M_2 g}{L}\theta\delta\theta\left.\frac{r^2}{2}\right|_0^L = \frac{M_2 g}{L}\theta\delta\theta\left(\frac{L^2}{2}\right) = \frac{M_2 g L}{2}\theta\delta\theta$$

## 2.14 Cont.

$$k_T\theta\cancel{\delta\theta} + kL^2\theta\cancel{\delta\theta} + M_1L^2\ddot{\theta}\cancel{\delta\theta} + \frac{M_2L^2}{3}\ddot{\theta}\cancel{\delta\theta} + M_1gL\theta\cancel{\delta\theta} + \frac{M_2gL}{2}\theta\cancel{\delta\theta} = 0$$

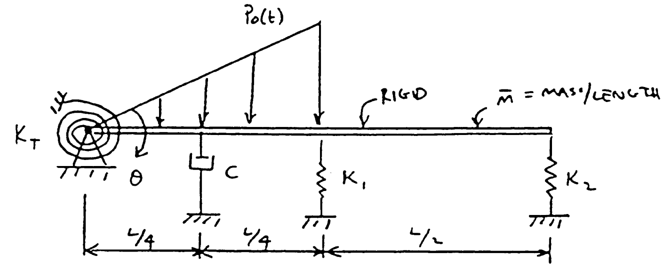
equation of motion:

$$\boxed{\left(\frac{M_2L^2}{3} + M_1L^2\right)\ddot{\theta} + \left(kL^2 + k_T + M_1gL + \frac{M_2gL}{2}\right)\theta = 0}$$ ANS

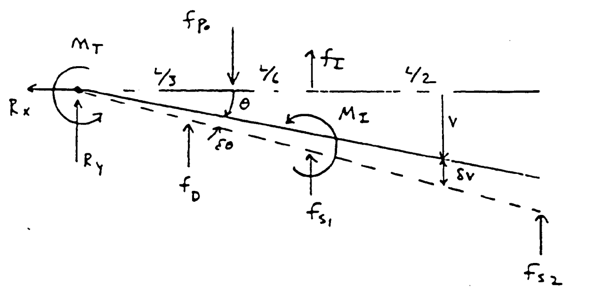
natural frequency:

$$\boxed{\omega = \sqrt{\frac{kL^2 + k_T + M_1gL + \frac{M_2gL}{2}}{\frac{M_2L^2}{3} + M_1L^2}}}$$ ANS

## 2.15




SOLUTION




PRINCIPLE OF VIRTUAL DISPLACEMENTS

$$\delta W = 0$$

$$-f_{s_1}\left(\tfrac{L}{2}\delta\theta\right) - f_{s_2}(L\delta\theta) - f_D\left(\tfrac{L}{4}\delta\theta\right) - f_I\left(\tfrac{L}{2}\delta\theta\right) - M_I\delta\theta - M_T\delta\theta + f_{P_0}\left(\tfrac{L}{3}\delta\theta\right) = 0$$

FORCES

$$f_{s_1} = k_1\left(\tfrac{L}{2}\theta\right) \qquad M_I = I\ddot{\theta} = \frac{(\bar{m}L)L^2}{12}\ddot{\theta}$$

$$f_{s_2} = k_2(L\theta) \qquad M_T = k_T\theta$$

$$f_D = c\left(\tfrac{L}{4}\dot{\theta}\right) \qquad f_{P_0} = \frac{P_0L}{4}f(t)$$

$$f_I = (\bar{m}L)\left(\tfrac{L}{2}\ddot{\theta}\right)$$

## 2.15 cont.

SUBSTITUTING INTO VIRTUAL WORK EQUATION:

$$-k_1\frac{L^2}{4}\theta\delta\theta - k_2 L^2\theta\delta\theta - c\frac{L^2}{16}\dot\theta\delta\theta - \bar m\frac{L^3}{4}\ddot\theta\delta\theta - \frac{\bar m L^3}{12}\ddot\theta\delta\theta - k_T\theta\delta\theta + \frac{P_0(t)L^2}{12}\delta\theta = 0$$

$$\left[\left(\frac{\bar m L^3}{4} + \frac{\bar m L^3}{12}\right)\ddot\theta + \left(\frac{cL^2}{16}\right)\dot\theta + \left(\frac{k_1 L^2}{4} + k_2 L^2 + k_T\right)\theta\right]\delta\theta = \frac{P_0(t)L^2}{12}\delta\theta$$

SINCE $\delta\theta \neq 0$

$$\left(\frac{\bar m L^3}{3}\right)\ddot\theta + \left(\frac{cL^2}{16}\right)\dot\theta + \left(\frac{k_1 L^2}{4} + k_2 L^2 + k_T\right)\theta = \frac{P_0(t)L^2}{12}$$

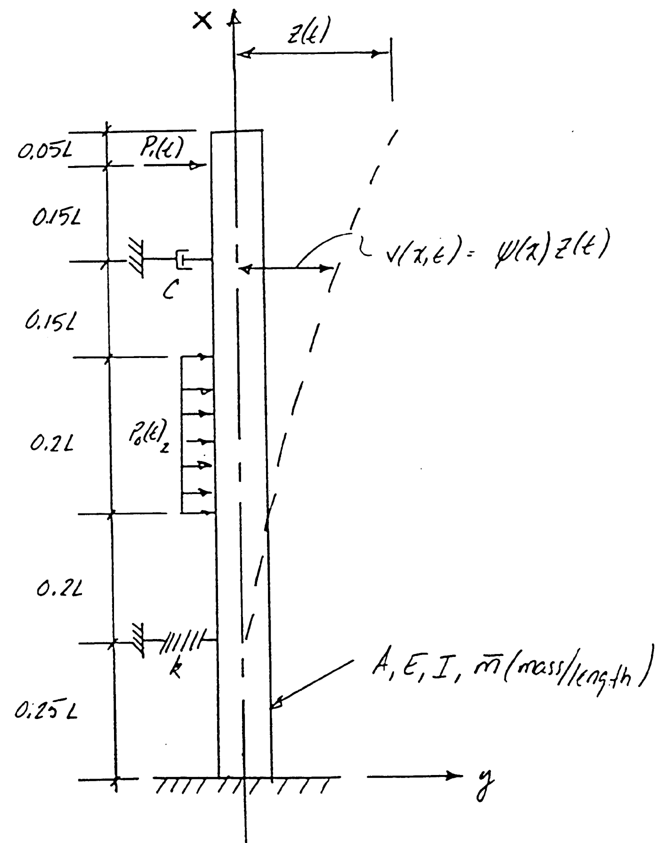
EQUATION OF MOTION

$$\omega = \sqrt{\frac{\frac{k_1 L^2}{4} + k_2 L^2 + k_T}{\frac{\bar m L^3}{3}}}$$

NATURAL FREQUENCY

---

## 2.16




Flexible Column

Assume deflected shape:

$$\psi(x) = \left(\frac{x}{L}\right)^2\left(\frac{3}{2} - \frac{x}{2L}\right)$$

Use the deflection @ the top of the structure as the generalized coordinate

## 2.16 Cont.

*Solution:*

No consentrated mass $= 0$

$$m^* = \int m(x)\psi^2(x)dx + \sum_i \cancel{M_i\psi^2(x_i)}$$

$$= \bar{m}\int_0^L \left[\left(\frac{x}{L}\right)^2\left(\frac{3}{2} - \frac{x}{2L}\right)\right]^2 dx$$

$$= \bar{m}\int_0^L \frac{9x^4}{4L^4} - \frac{3x^5}{2L^5} + \frac{x^6}{4L^6}\, dx$$

$$= \bar{m}\left| \frac{9x^5}{20L^4} - \frac{x^6}{4L^5} + \frac{x^7}{28L^6}\right|_0^L$$

$$= \bar{m}\left(\frac{9L^5}{20L^4} - \frac{L^6}{4L^5} + \frac{L^7}{28L^6}\right)$$

$$\boxed{m^* = 0.23571\,\bar{m}L}$$

$\to 0$ (no distrib. dampers)

$$c^* = \int \cancel{c(x)\psi^2(x)dx} + \sum_i c_i\psi^2(x_i)$$

$$= c\left(\frac{9x_i^4}{4L^4} - \frac{3x_i^5}{2L^5} + \frac{x_i^6}{4L^6}\right) \quad \text{where } x_i = 0.8L$$

$$= c\left(\frac{9(0.8L)^4}{4L^4} - \frac{3(0.8L)^5}{2L^5} + \frac{(0.8L)^6}{4L^6}\right)$$

$$\boxed{c^* = 0.49562\;c}$$

$$k^* = \int \cancel{k(x)\psi^2(x)dx} + \int EI(x)(\psi''(x))^2dx + \sum_i k_i\psi^2(x_i)$$

No distributed spring stiffness $= 0$

$$\psi(x) = \left(\frac{x}{L}\right)^2\left(\frac{3}{2} - \frac{x}{2L}\right) = \frac{3x^2}{2L^2} - \frac{x^3}{2L^3}$$

$$\psi'(x) = \frac{3x}{L^2} - \frac{3x^2}{2L^3}$$

$$\psi''(x) = \frac{3}{L^2} - \frac{3x}{L^3}$$

## 2.16 Cont.

$$k^* = EI\int_0^L \left(\frac{3}{L^2} - \frac{3x}{L^3}\right)^2 dx + k\left(\frac{9x_i^4}{4L^4} - \frac{3x_i^5}{2L^5} + \frac{x_i^6}{4L^6}\right)$$

where $x_i = 0.25L$

$$= EI\int_0^L \left(\frac{9}{L^4} - \frac{18x}{L^5} + \frac{9x^2}{L^6}\right)dx + k\left(\frac{9(0.25L)^4}{4L^4} - \frac{3(0.25L)^5}{2L^5} + \frac{(0.25L)^6}{4L^6}\right)$$

$$= EI\left|\frac{9x}{L^4} - \frac{9x^2}{L^5} + \frac{3x^3}{L^6}\right|_0^L + (0.00739\ k)$$

$$= EI\left(\frac{9L}{L^4} - \frac{9L^2}{L^5} + \frac{3L^3}{L^6}\right) + (0.00314k)$$

$$= EI\left(\frac{9}{L^3} - \frac{9}{L^3} + \frac{3}{L^3}\right) + (0.00314\ k)$$

$$\boxed{k^* = \frac{3EI}{L^3} + 0.00739k}$$

$$p^*(t) = \int p(x,t)\psi(x)dx + \sum_i P_i\,\psi(x_i)$$

$$= P_0\int_{0.45L}^{0.65L} \frac{3x^2}{2L^2} - \frac{x^3}{2L^3}\,dx + P_1\left(\frac{3x_i^2}{2L^2} - \frac{x_i^3}{2L^3}\right)$$

where $x_i = 0.95L$

$$= P_0\left|\frac{x^3}{2L^2} - \frac{x^4}{8L^3}\right|_{0.45L}^{0.65L} + P_1\left(\frac{3(0.95L)^2}{2L^2} - \frac{(0.95L)^3}{2L^3}\right)$$

$$= P_0\left(\frac{(0.65L)^3}{2L^2} - \frac{(0.65L)^4}{8L^3} - \frac{(0.45L)^3}{2L^2} + \frac{(0.45L)^4}{8L^3}\right) + (0.92506\ P_1)$$

$$\boxed{p^*(t) = 0.07456\ P_0 L + 0.92506\ P_1}$$

## 2.16 Cont.

natural frequency:

$$m^* \ddot{z} + c^* \dot{z} + k^* z = p^*(t)$$

$$(0.23571\, \bar{m} L)\ddot{z} + (0.49562\, c)\dot{z} + \left(\frac{3EI}{L^3} + 0.00739\, k\right) z$$

equation of motion $\longrightarrow$

$$= 0.07456\, P_o L + 0.92506\, P_1$$

$$\boxed{\omega = \sqrt{\frac{\frac{3EI}{L^3} + 0.00739\, k}{0.23571\, \bar{m} L}}}$$ ANS

find critical Load $N_{cr}$

$$k^*_{cr} = N_{cr} \int (\psi'(x))^2 dx$$

$$= N_{cr} \int_0^L \left(\frac{3x}{L^2} - \frac{3x^2}{2L^3}\right)^2 dx$$

$$= N_{cr} \int_0^L \left(\frac{9x^2}{L^4} - \frac{9x^3}{L^5} + \frac{9x^4}{4L^6}\right) dx$$

$$= N_{cr} \left| \frac{3x^3}{L^4} - \frac{9x^4}{4L^5} + \frac{9x^5}{20L^6} \right|_0^L$$

$$= N_{cr} \left( \frac{3L^3}{L^4} - \frac{9L^4}{4L^5} + \frac{9L^5}{20L^6} \right)$$

$$\boxed{k^*_{cr} = N_{cr}\left(\frac{1.2}{L}\right) = \frac{1.2 N_{cr}}{L}}$$ ANS

$$\boxed{k^* - k^*_{cr} = \frac{3EI}{L^3} + 0.00739\, k - \frac{1.2 N_{cr}}{L} = 0}$$

## 2.16 Cont.

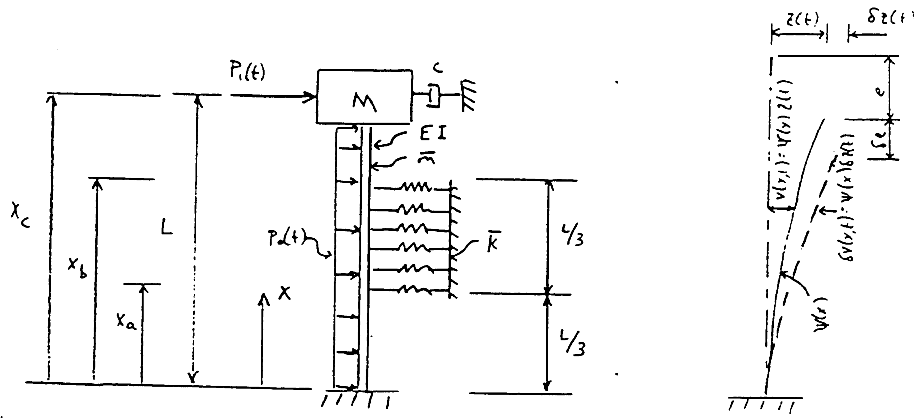
$$\frac{1.2 N_{cr}}{L} = \frac{3EI}{L^3} + 0.00739\,k$$

$$N_{cr} = \left(\frac{3EI}{L^3} + 0.00739\,k\right)\left(\frac{L}{1.2}\right)$$

$$\boxed{N_{cr} = \frac{2.5EI}{L^2} + 0.00616\,kL} \quad \text{ANS}$$

---

## 2.17




SOLUTION

CHECK BOUNDARY CONDITIONS

$$v(0,t) = v'(0,t) = 0$$

$$\psi(x) = \frac{3x^2}{L^2} - \frac{x^3}{L^3} \rightarrow \psi(0) = 0$$

$$\psi'(x) = \frac{6x}{L^2} - \frac{3x^2}{L^3} \rightarrow \psi'(0) = 0$$

ALL KINEMATIC BOUNDARY CONDITIONS SATISFIED.

HELPFUL RELATIONSHIPS IN COMPUTING VIRTUAL WORK EXPRESSIONS

$$v'(x,t) = \psi'(x)z(t) \qquad \delta v(x,t) = \psi(x)\delta z(t)$$

$$v''(x,t) = \psi''(x)z(t) \qquad \delta v'(x,t) = \psi'(x)\delta z(t)$$

$$\dot v(x,t) = \psi(x)\dot z(t) \qquad \delta \dot v(x,t) = \psi(x)\delta \dot z(t)$$

$$\ddot v(x,t) = \psi(x)\ddot z(t) \qquad \delta \ddot v(x,t) = \psi(x)\delta \ddot z(t)$$

INERTIA

$$\delta W_{INERTIA} = -\int_0^L \bar{m}\,\ddot v(x,t)\,\delta v(x,t)\,dx - M\ddot v(x_c,t)\,\delta v(x_c,t)$$

$$= -\left[\int_0^L \bar{m}\,\psi(x)^2\,dx + M\psi(x_c)^2\right]\ddot z\,\delta z$$

DAMPING

$$\delta W_{DAMPING} = -c\dot v(x_c,t)\,\delta v(x_c,t)$$

$$= -c\,\psi(x_c)^2\,\dot z\,\delta z$$

TRANSVERSE LOADS

$$\delta W_p = \int_0^L p_0(x,t)\,\delta v(x,t)\,dx + P_1(t)\,\delta v(x_c,t)$$

$$= \left[\int_0^L p_0(x,t)\psi(x)\,dx + P_1(t)\psi(x_c)\right]\delta z$$

## 2.17 Cont.

AXIAL LOAD

$$\delta W_N = N \delta e$$

$$e = \frac{1}{2}\int_0^L (x')^2 dx \quad \text{AND} \quad \delta e = \frac{1}{2}\int_0^L v'\delta v' dx$$

$$\delta W_N = \int_0^L N[\psi'(x)]^2 dx \quad z\delta z$$

SPRINGS

$$\delta W_{SPRING} = -\int_{x_a}^{x_b} \bar{k}\, v(x,t)\,\delta v(x,t)\,dx$$

$$= -\left[\int_{x_a}^{x_b} \bar{k}\, \psi(x)^2 dx\right] z\delta z$$

BENDING

$$\delta W_{BENDING} = -\int_0^L EI(x)\; v''(x,t)\,\delta v''(x,t)\,dx$$

$$= -\left[\int_0^L EI(x)\,[\psi''(x)]^2 dx\right] z\delta z$$

VIRTUAL WORK EQUATION

$$\delta W = 0$$

$$-\left[\int_0^L \bar{m}\,\psi(x)^2 dx + M\psi(x_c)^2\right]\ddot{z}\delta z - \left[c\,\psi(x_c)^2\right]\dot{z}\delta z - \left[\int_{x_a}^{x_b}\bar{k}\,\psi(x)^2 dx + \int_0^L EI(x)[\psi''(x)]^2 dx\right] z\delta z$$

$$+\left[\int_0^L N\,[\psi'(x)]^2 dx\right] z\delta z + \left[\int_0^L p_0(x,t)\,\psi(x)\,dx + P_1(t)\,\psi(x_c)\right]\delta z = 0$$

REARRANGING:

$$\left[\int_0^L \bar{m}\,\psi(x)^2 dx + M\psi(x_c)^2\right]\ddot{z}\delta z + \left[c\,\psi(x_c)^2\right]\dot{z}\delta z + \left[\int_{x_a}^{x_b}\bar{k}\,\psi(x)^2 dx + \int_0^L EI(x)[\psi''(x)]^2 dx\right.$$

$$\left. - \int_0^L N[\psi'(x)]^2 dx\right] z\delta z - \left[\int_0^L p_0(x,t)\,\psi(x)\,dx + P_1(t)\,\psi(x_c)\right]\delta z = 0$$

SIMPLIFYING THE FORM

$$\left[m^*\ddot{z} + c^*\dot{z} + k^* z - k_G^* z - p^*(t)\right]\delta z = 0$$

SINCE $\delta z \neq 0$

$$m^*\ddot{z} + c^*\dot{z} + (k^* - k_G^*)\,z = p^*(t)$$

GENERALIZED PARAMETERS

$$m^* = \int_0^L \bar{m}\,\psi(x)^2 dx + M\psi(x_c)^2$$

$$= \bar{m}\int_0^L \left(3\left(\tfrac{x}{L}\right)^2 - \left(\tfrac{x}{L}\right)^3\right)^2 dx + M\left[3\left(\tfrac{x_c}{L}\right)^2 - \left(\tfrac{x_c}{L}\right)^3\right]^2$$

$$= \bar{m}\int_0^L \left[9\frac{x^4}{L^4} - 2(3)\left(\frac{x^2}{L^2}\right)\left(\frac{x^3}{L^3}\right) + \frac{x^6}{L^6}\right] dx + M[3(1) - 1]^2$$

$$= \bar{m}\int_0^L \left(\frac{9}{L^4}x^4 - \frac{6}{L^5}x^5 + \frac{x^6}{L^6}\right) dx + M(4)$$

$$= \bar{m}\left[\frac{9}{5L^4}x^5 - \frac{x^6}{L^6} + \frac{x^7}{7L^6}\right]_0^L + 4M$$

$$= \bar{m}\left(\frac{9L^5}{5L^4} - 1 + \frac{L^7}{7L^6}\right) + 4M$$

$$= \bar{m}\left(\frac{9}{5}L - 1 + \frac{L}{7}\right) + 4M$$

$$= 1.943\,\bar{m}L - \bar{m} + 4M$$

## 2.17 Cont.

$$
\begin{aligned}
c^* &= c\,\psi(x_c)^2 \\
&= c\left[3\left(\tfrac{x_c}{L}\right)^2 - \left(\tfrac{x_c}{L}\right)^3\right]^2 \\
&= c\left[3\left(\tfrac{L}{L}\right)^2 - \left(\tfrac{L}{L}\right)^3\right]^2 \\
&= 4c
\end{aligned}
$$

$$
\begin{aligned}
k^* &= \int_{x_a}^{x_b} \bar{k}\,\psi(x)^2\,dx + \int EI(x)\left[\psi''(x)\right]^2 dx \\
&= \bar{k}\int_{L/3}^{2L/3}\left(\frac{9}{L^4}x^4 - \frac{6}{L^5}x^5 + \frac{x^6}{L^6}\right)dx + EI\int_0^L\left[\frac{6}{L^2} - \frac{6x}{L^3}\right]^2 dx \\
&= \bar{k}\left[\frac{9x^5}{5L^4} - \frac{x^6}{L^6} + \frac{x^7}{7L^6}\right]_{L/3}^{2L/3} + EI\int_0^L\left(\frac{36}{L^4} - \frac{72x}{L^5} + \frac{36x^2}{L^6}\right)dx \\
&= \bar{k}\left[\left[\frac{9}{5L^4}\left(\frac{2L}{3}\right)^5 - \frac{1}{L^6}\left(\frac{2L}{3}\right)^6 + \frac{1}{7L^6}\left(\frac{2L}{3}\right)^7\right] - \left[\frac{9}{5L^4}\left(\frac{L}{3}\right)^5 - \frac{1}{L^6}\left(\frac{L}{3}\right)^6 + \frac{1}{7L^6}\left(\frac{L}{3}\right)^7\right]\right] \\
&\quad + EI\left[\frac{36}{L^4}x - \frac{36x^2}{L^5} + \frac{12x^3}{L^6}\right]_0^L \\
&= \bar{k}\left[(0.2370L - 0.08779 + 0.008361L) - (0.007407L - 0.001372 + 6.532\times10^{-5}L)\right] \\
&\quad + EI\left(\frac{36}{L^3} - \frac{36}{L^3} + \frac{12}{L^3}\right) \\
&= 12\frac{EI}{L^3} + (0.2379L - 0.08642)\bar{k}
\end{aligned}
$$

$$
k_G^* = N\int_0^L\left[\psi'(x)\right]^2 dx = 0 \quad \text{SINCE } N = 0
$$

(NEGLECTING SELF-WEIGHT)

$$
\begin{aligned}
p^*(t) &= \int_0^L p_o(x,t)\,\psi(x)\,dx + P_1(t)\,\psi(x_c) \\
&= p_o\int_0^L\left(\frac{3x^2}{L^2} - \frac{x^3}{L^3}\right)dx + P_1(t)\left[3\left(\tfrac{x_c}{L}\right)^2 - \left(\tfrac{x_c}{L}\right)^3\right] \\
&= p_o\left[\frac{x^3}{L^2} - \frac{x^4}{4L^3}\right]_0^L + P_1(t)\left[3 - 1\right] \\
&= p_o\left[L - \frac{L}{4}\right] + 2P_1(t) \\
&= 0.75\,p_o(t)\,L + 2P_1(t)
\end{aligned}
$$

EQUATION OF MOTION

$$
\left[1.943\,\bar{m}L - \bar{m} + 4M\right]\ddot{z} + \left[4c\right]\dot{z} + \left[12\frac{EI}{L^3} + 0.2379L\bar{k} - 0.08642\bar{k}\right]z = 0.75\,p_o(t)\,L + 2P_1(t)
$$

NATURAL FREQUENCY

$$
\omega = \sqrt{\frac{k^* - k_G^*}{m^*}} = \sqrt{\frac{12\frac{EI}{L^3} + 0.2379\bar{k}L - 0.08642\bar{k}}{1.943\,\bar{m}L - \bar{m} + 4m}} \quad \text{RAD/SEC}
$$

## 2.17 Cont.

DOWNWARD LOAD N APPLIED:

$$K_G^* = N \int_0^L [\psi'(x)]^2 \, dx$$

$$= N \int_0^L \left( \frac{36}{L^4} x^2 - \frac{36}{L^5} x^3 + \frac{9}{L^6} x^4 \right) dx$$

$$= N \left[ \frac{12}{L} - \frac{9}{L} + \frac{9}{5L} \right]$$

$$= 4.8 \frac{N}{L}$$

CALCULATE COMBINED STIFFNESS:

$$k^* - k_G^* = 12 \frac{EI}{L^3} + 0.2379 L \bar{k} - 0.08642 \bar{k} - 4.8 \frac{N}{L}$$

CALCULATE $N_{CR}$

$$K^* - K_{CR}^* = 0$$

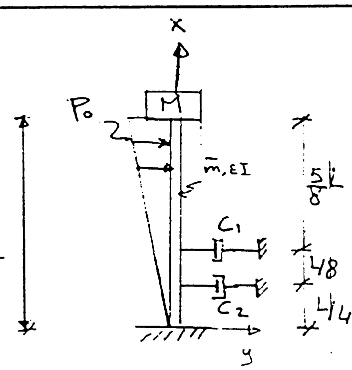
$$12 \frac{EI}{L^3} + 0.2379 L \bar{k} - 0.08642 \bar{k} - 4.8 \frac{N_{CR}}{L} = 0$$

$$N_{CR} = 2.5 \frac{EI}{L^2} + 0.04956 \bar{k} L^2 - 1.80 \times 10^{-2} \bar{k} L$$

---

## 2.18

$$\phi(x) = \frac{x^2}{L^2}$$




$$M^* = \int_0^L \bar{m} \, \phi^2(x) \, dx + M [\phi(x)]^2 \Big|_{x=L}$$

$$= \bar{m} \int_0^L \frac{x^4}{L^4} \, dx + M$$

$$= \bar{m} \frac{x^5}{5L^4} \Bigg|_0^L + M$$

$$\boxed{M^* = \frac{\bar{m} L}{5} + M}$$

## 2.18 Cont.

$$C^* = C_1 \left[\phi(x)\right]^2_{x=\frac{3}{8}L} + C_2 \left[\phi(x)\right]^2_{x=\frac{L}{4}}$$

$$= C_1 \left.\frac{x^4}{L^4}\right|_{x=\frac{3}{8}L} + C_2 \left.\frac{x^4}{L^4}\right|_{x=\frac{L}{4}}$$

$$\boxed{C^* = \frac{81}{4096} C_1 + \frac{1}{256} C_2}$$

$$K^* = \int_0^L EI \left[\phi''(x)\right]^2 dx$$

$$\phi(x) = \frac{x^2}{L^2} \qquad \phi'' = \frac{2}{L^2} \qquad \left[\phi''\right]^2 = \frac{4}{L^4}$$

$$K^* = EI \int_0^L \frac{4}{L^4}\, dx = \frac{4EI}{L^4}\, x\Big|_0^L = \frac{4EI}{L^3}$$

note compare with cantilever $k^* = \frac{3EI}{L^3}$ this is because $\phi(x)$ is not accurate enough to describe the deformed shape (i.e. $\phi''$ is constant for the whole beam)

$$F^*(t) = \int_0^L \frac{P_0 x}{L} f(t) \frac{x^2}{L^2}\, dx$$

$$= \frac{P_0}{L^3} f(t) \int_0^L x^3\, dx = \frac{P_0 f(t)}{L^3} \left|\frac{x^4}{4}\right|_0^L$$

$$= \frac{P_0 L}{4} f(t)$$

$$K_g^* = N \int_0^L \left(\frac{2x}{L^2}\right)^2 dx$$

$$= N \int_0^L \frac{4x^2}{L^4}\, dx = \frac{4N}{L^4} \left|\frac{x^3}{3}\right|_0^L$$

$$K_g^* = \frac{4}{3} \frac{N}{L}$$

$$\overline{K}^* = K^* - K_g^* \qquad \frac{4EI}{L^3} - \frac{4}{3}\frac{N}{L}$$

## 2.18 Cont.

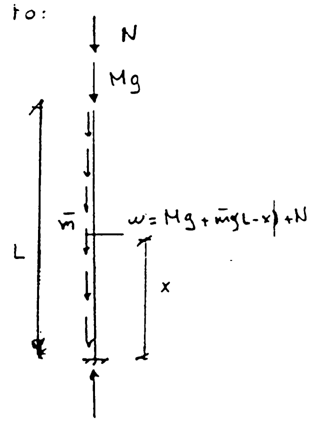
for buckling load $\bar{K}^* = 0$

$$\frac{4EI}{L^3} = \frac{4}{3}\frac{N_{cr}}{L}$$

$$\boxed{N_{cr} = \frac{3EI}{L^2}}$$

Note 1 for cantilever beam $N_{cr} = \frac{\pi^2 EI}{4L^2} = 2.46\frac{EI}{L^2}$

Note 2 if we consider the o.w of the beam & M at the top the $K_G$ is equals to:




$$K_G = \int_0^L [Mg + \bar{m}g(L-x) + N]\frac{4x^2}{L^4}\,dx$$

$$= \frac{4}{3}\left(\frac{N}{L} + \frac{Mg}{L} + \bar{m}g\right) - \int_0^L \bar{m}g\frac{4x^3}{L^4}\,dx$$

$$= \frac{4}{3}\left(\frac{N}{L} + \frac{Mg}{L} + \bar{m}g\right) - \bar{m}g$$

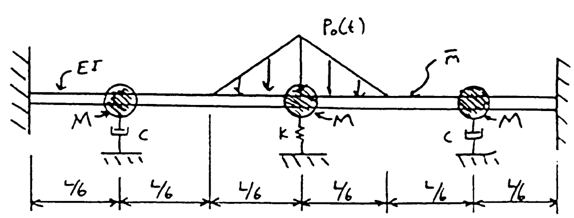
$$= \frac{4}{3}\frac{N}{L} + \frac{4}{3}\frac{Mg}{L} + \frac{\bar{m}g}{3}$$

$$\frac{4}{3}\frac{N}{L} + \frac{4}{3}\frac{Mg}{L} + \frac{\bar{m}g}{3} = \frac{4EI}{L^3}$$

$$\frac{4}{3}\frac{N}{L} = \frac{4EI}{L^3} - \frac{4}{3}\frac{Mg}{L} - \frac{\bar{m}g}{3}$$

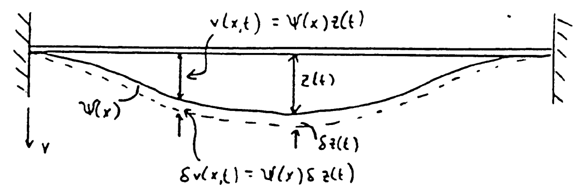
$$\boxed{N_{cr} = \frac{3EI}{L^2} - Mg - \frac{\bar{m}gL}{4}}$$

## 2.19



## 2.19 Cont.

SOLUTION




CHECK BOUNDARY CONDITIONS

$$v(0,t) = v(L,t) = v'(0,t) = v'(\tfrac{L}{2},t) = v'(L,t) = 0$$

$$\psi(x) = \frac{16x^2}{L^2} - \frac{32x^3}{L^3} + \frac{16x^4}{L^4} \qquad \psi(0)=0 \quad \psi(L) = 16-32+16 = 0$$

$$\psi'(x) = \frac{32x}{L^2} - \frac{96x^2}{L^3} + \frac{64x^3}{L^4} \qquad \psi'(0) = 0 \quad \psi'(L) = \frac{32}{L} - \frac{96}{L} + \frac{64}{L} = 0$$

$$\psi'(\tfrac{L}{2}) = \frac{16}{L} - \frac{24}{L} + \frac{8}{L} = 0$$

EVALUATE THE GENERALIZED PARAMETERS

MASS
$$m^* = \bar{m}\int_0^L \psi(x)^2\,dx + M\psi^2(\tfrac{L}{6}) + M\psi^2(\tfrac{L}{2}) + M\psi^2(\tfrac{5L}{6})$$

$$= \bar{m}\int_0^L \left[\frac{256x^4}{L^4} - \frac{1024x^5}{L^5} + \frac{1536x^6}{L^6} - \frac{1024x^7}{L^7} + \frac{256x^8}{L^8}\right]dx$$

$$+ 256M\left[\frac{(\frac{L}{6})^4}{L^4} - \frac{4(\frac{L}{6})^5}{L^5} + \frac{6(\frac{L}{6})^6}{L^6} - \frac{4(\frac{L}{6})^7}{L^7} + \frac{(\frac{L}{6})^8}{L^8}\right]$$

$$+ 256M\left[\frac{(\frac{L}{2})^4}{L^4} - \frac{4(\frac{L}{2})^5}{L^5} + \frac{6(\frac{L}{2})^6}{L^6} - \frac{4(\frac{L}{2})^7}{L^7} + \frac{(\frac{L}{2})^8}{L^8}\right]$$

$$+ 256M\left[\frac{(\frac{5L}{6})^4}{L^4} - \frac{4(\frac{5L}{6})^5}{L^5} + \frac{6(\frac{5L}{6})^6}{L^6} - \frac{4(\frac{5L}{6})^7}{L^7} + \frac{(\frac{5L}{6})^8}{L^8}\right]$$

$$m^* = 0.40635\,\bar{m}L + 1.1905\,M$$

DAMPING
$$c^* = c\psi^2(\tfrac{L}{6}) + c\psi^2(\tfrac{5L}{6})$$

$$= 256c\left[\frac{(\frac{L}{6})^4}{L^4} - \frac{4(\frac{L}{6})^5}{L^5} + \frac{6(\frac{L}{6})^6}{L^6} - \frac{4(\frac{L}{6})^7}{L^7} + \frac{(\frac{L}{6})^8}{L^8}\right]$$

$$+ 256c\left[\frac{(\frac{5L}{6})^4}{L^4} - \frac{4(\frac{5L}{6})^5}{L^5} + \frac{6(\frac{5L}{6})^6}{L^6} - \frac{4(\frac{5L}{6})^7}{L^7} + \frac{(\frac{5L}{6})^8}{L^8}\right]$$

$$= 0.1905\,c$$

STIFFNESS
$$k^* = \int_0^L EI(x)\left[\psi''(x)\right]^2 dx + k\psi^2(\tfrac{L}{2})$$

$$= \int_0^L EI(x)\left[\frac{32}{L^2} - \frac{192x}{L^3} + \frac{192x^2}{L^4}\right]^2 dx + 256k\left[\frac{(\frac{L}{2})^4}{L^4} - \frac{4(\frac{L}{2})^5}{L^5} + \frac{6(\frac{L}{2})^6}{L^6} - \frac{4(\frac{L}{2})^7}{L^7} + \frac{(\frac{L}{2})^8}{L^8}\right]$$

$$= EI\int_0^L\left(\frac{1024}{L^4} - \frac{12288x}{L^5} + \frac{49152x^2}{L^6} - \frac{73728x^3}{L^7} + \frac{36864x^4}{L^8}\right)dx + k(1)$$

$$= EI\left[\frac{1024}{L^3} - \frac{12288}{2L^3} + \frac{49152}{3L^3} - \frac{73728}{4L^3} + \frac{36864}{5L^3}\right] + k$$

$$= 204.8\,\frac{EI}{L^3} + k$$

# 2.19 Cont.

FORCE

$$p^*(t) = \int_{L/3}^{L/2} \left(6p_o \frac{x}{L} - 2p_o\right) \psi(x)\, dx + \int_{L/2}^{2L/3} \left(-6p_o \frac{x}{L} + 4p_o\right) \psi(x)\, dx$$

$$= 2p_o \left[ \int_{L/3}^{L/2} \left(\frac{3x}{L} - 1\right)\left(\frac{16x^2}{L^2} - \frac{32x^3}{L^3} + \frac{16x^4}{L^4}\right) dx + \int_{L/2}^{2L/3} \left(-\frac{3x}{L} + 2\right)\left(\frac{16x^2}{L^2} - \frac{32x^3}{L^3} + \frac{16x^4}{L^4}\right) dx \right.$$

$$= 2p_o \left[ \int_{L/3}^{L/2} \left(-\frac{16x^2}{L^2} + \frac{80x^3}{L^3} - \frac{112x^4}{L^4} + \frac{48x^5}{L^5}\right) dx + \int_{L/2}^{2L/3} \left(\frac{32x^2}{L^2} - \frac{112x^3}{L^3} + \frac{128x^4}{L^4} - \frac{48x^5}{L^5}\right) dx \right.$$

$$= 2p_o \left( \left[ -\frac{16x^3}{3L^2} + \frac{20x^4}{L^3} - \frac{112x^5}{5L^4} + \frac{8x^6}{L^5} \right]_{L/3}^{L/2} + \left[ \frac{32x^3}{3L^2} - \frac{28x^4}{4L^3} + \frac{128x^5}{5L^4} - \frac{8x^6}{L^5} \right]_{L/2}^{2L/3} \right.$$

$$= 2p_o \left( -\frac{16(\frac{L}{2})^3}{3L^2} + \frac{20(\frac{L}{2})^4}{L^3} - \frac{112(\frac{L}{2})^5}{5L^4} + \frac{8(\frac{L}{2})^6}{L^5} + \frac{16(\frac{L}{3})^3}{3L^2} - \frac{20(\frac{L}{3})^4}{L^3} + \frac{112(\frac{L}{3})^5}{5L^4} - \frac{8(\frac{L}{3})^6}{L^5} \right.$$

$$\left. + \frac{32(\frac{2L}{3})^3}{3L^2} - \frac{28(\frac{2L}{3})^4}{4L^3} + \frac{128(\frac{2L}{3})^5}{5L^4} - \frac{8(\frac{2L}{3})^6}{L^5} - \frac{32(\frac{L}{2})^3}{3L^2} + \frac{28(\frac{L}{2})^3}{4L^3} - \frac{128(\frac{L}{2})^5}{5L^4} + \frac{8(\frac{L}{2})^6}{L^5} \right)$$

$$= 6.7069\, p_o(t)\, L$$

EQUATION OF MOTION

$$(0.40635\bar{m}L + 1.1905M)\ddot{z} + (0.1905c)\dot{z} + \left(204.8\frac{EI}{L^3} + k\right) z = 6.7069 p_o(t) L$$

NATURAL FREQUENCY

$$\omega = \sqrt{\frac{204.8\frac{EI}{L^3} + k}{0.40635\bar{m}L + 1.1905M}}$$

GENERALIZED GEOMETRIC STIFFNESS

$$K_G^* = N \int_0^L [\psi'(x)]^2\, dx$$

$$= N \int_0^L \left(32\frac{x}{L^2} - 96\frac{x^2}{L^3} + 64\frac{x^3}{L^4}\right)^2 dx$$

$$= 1024N \int_0^L \left(\frac{x^2}{L^4} - \frac{6x^3}{L^5} + \frac{13x^4}{L^6} - \frac{12x^5}{L^7} + \frac{4x^6}{L^8}\right) dx$$

$$= 1024N \left[\frac{1}{3L} - \frac{6}{4L} + \frac{13}{5L} - \frac{2}{L} + \frac{4}{7L}\right]$$

$$= 4.8762\frac{N}{L}$$

COMBINED STIFFNESS

$$K^* - K_G^* = 204.8\frac{EI}{L^3} + k - 4.8762\frac{N}{L}$$

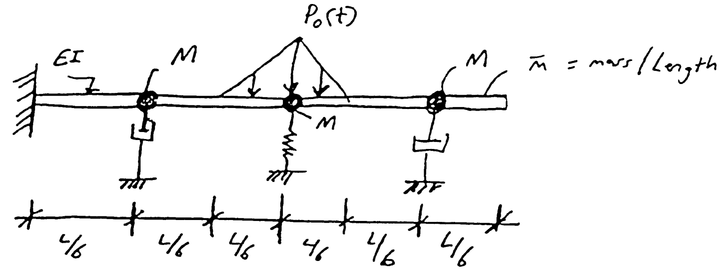
CRITICAL BUCKLING LOAD, $N_{CR}$

$$K^* - K_G^* = 0$$

$$204.8\frac{EI}{L^3} + k - 4.8762\frac{N_{CR}}{L} = 0$$

$$N_{CR} = 42\frac{EI}{L^2} + 0.20508\, k\, L$$

## 2.20




$\psi_{(x)} = 3\left(\frac{x}{L}\right)^2 - \left(\frac{x}{L}\right)^3$

Use deflection at the free end as the generalized coordinate

SOLUTION:

$$\psi(x) = \frac{3x^2}{L^2} - \frac{x^3}{L^3} \qquad [\psi(x)]^2 = \frac{9x^4}{L^4} - \frac{6x^5}{L^5} + \frac{x^6}{L^6}$$

$$\psi'(x) = \frac{6x}{L^2} - \frac{3x^2}{L^3} \qquad [\psi'(x)]^2 = \frac{36x^2}{L^4} - \frac{36x^3}{L^5} + \frac{9x^4}{L^6}$$

$$\psi''(x) = \frac{6}{L^2} - \frac{6x}{L^3} \qquad [\psi''(x)]^2 = \frac{36}{L^4} - \frac{72x}{L^5} + \frac{36x^2}{L^6}$$

$$m^* = \int_0^L m(x)\psi^2(x)\,dx + \sum M_i \psi^2(x_i)$$

$$m^* = \bar{m}\int_0^L \left(\frac{9x^4}{L^4} - \frac{6x^5}{L^5} + \frac{x^6}{L^6}\right)dx + M\left[\frac{9x^4}{L^4} - \frac{6x^5}{L^5} + \frac{x^6}{L^6}\right]_{L/6,\ L/2,\ 5L/6}$$

$$m^* = \bar{m}\left[\frac{9x^5}{5L^4} - \frac{x^6}{L^5} + \frac{x^7}{7L^6}\right]_0^L + M\left[\frac{9\left(\frac{L}{6}\right)^4}{L^4} - \frac{6\left(\frac{L}{6}\right)^5}{L^5} + \frac{\left(\frac{L}{6}\right)^6}{L^6}\right]$$

$$+ M\left[\frac{9\left(\frac{L}{2}\right)^4}{L^4} - \frac{6\left(\frac{L}{2}\right)^5}{L^5} + \frac{\left(\frac{L}{2}\right)^6}{L^6}\right]$$

$$+ M\left[\frac{9\left(\frac{5L}{6}\right)^4}{L^4} - \frac{6\left(\frac{5L}{6}\right)^5}{L^5} + \frac{\left(\frac{5L}{6}\right)^6}{L^6}\right]$$

$$\boxed{m^* = .943\,\bar{m}L + 2.66\,M}$$

$$c^* = \sum c\psi^2(x) = c\left[\frac{9x^4}{L^4} - \frac{6x^5}{L^5} + \frac{x^6}{L^6}\right]_{L/6,\ 5L/6}$$

$$= c\left[\frac{9\left(\frac{L}{6}\right)^4}{L^4} - \frac{6\left(\frac{L}{6}\right)^5}{L^5} + \frac{\left(\frac{L}{6}\right)^6}{L^6}\right]$$

$$+ c\left[\frac{9\left(\frac{5L}{6}\right)^4}{L^4} - \frac{6\left(\frac{5L}{6}\right)^5}{L^5} + \frac{\left(\frac{5L}{6}\right)^6}{L^6}\right]$$

$$\boxed{c^* = 2.27c}$$

## 2.20 Cont.

$$k^* = \int EI(x)\left[\psi''(x)\right]^2 dx + \sum k_i \psi^2(x_i)$$

$$= EI\int_0^L \left(\frac{36}{L^4} - \frac{72x}{L^5} + \frac{36x^2}{L^6}\right) dx + k\left[\frac{9x^4}{L^4} - \frac{6x^5}{L^5} + \frac{x^6}{L^6}\right]^{L/2}$$

$$= EI\left[\frac{36x}{L^4} - \frac{72x^2}{2L^5} + \frac{36x^3}{3L^6}\right]_0^L + k\left[\frac{9\left(\frac{L}{2}\right)^4}{L^4} - \frac{6\left(\frac{L}{2}\right)^5}{L^5} + \frac{\left(\frac{L}{2}\right)^6}{L^6}\right]$$

$$= EI\left[\frac{36}{L^3} - \frac{36}{L^3} + \frac{12}{L^3}\right] + k(.391)$$

$$\boxed{k^* = \frac{12EI}{L^3} + .391K}$$

$$p^*(t) = \int p(x,t)\psi(x)\,dx + \sum p_i \cancel{\psi}(x_i) \quad 0$$

$$= \int_{L/3}^{2L/3} p_0(t)\left[\frac{3x^2}{L^2} - \frac{x^3}{L^3}\right] dx$$

$$= p_0\left[\frac{x^3}{L^2} - \frac{x^4}{4L^3}\right]_{L/3}^{2L/3}$$

$$= p_0\left[\left(\frac{\left(\frac{2L}{3}\right)^3}{L^2} - \frac{\left(\frac{2L}{3}\right)^4}{4L^3}\right) - \left(\frac{\left(\frac{L}{3}\right)^3}{L^2} - \frac{\left(\frac{L}{3}\right)^4}{4L^3}\right)\right]$$

$$\boxed{p^*(t) = .213\, p_0 L}$$

Eq. of Motion

$$(.943\bar{m}L + 2.66M)\ddot{z} + (2.27c)\dot{z} + \left(\frac{12EI}{L^3} + .391K\right)z$$

$$= .213\, p_0 L$$

Natural Frequency

$$\omega = \sqrt{\frac{k^*}{m^*}} = \sqrt{\frac{\left(\frac{12EI}{L^3}\right) + .391K}{.943\bar{m}L + 2.66M}}$$

$$k_G^* = N\int \left(\psi'(x)\right)^2 dx$$

$$= N\int_0^L \left[\frac{36x^2}{L^4} - \frac{36x^3}{L^5} + \frac{9x^4}{L^6}\right]_0^L$$

$$= N\left[\frac{12x^3}{L^4} - \frac{9x^4}{L^5} + \frac{9x^5}{5L^6}\right]_0^L$$

$$= N\left[\frac{12}{L} - \frac{9}{L} + \frac{1.8}{L}\right] \Rightarrow k_G^* = 4.8\frac{N}{L}$$

## 2.20 Cont.

Combined Stiffness

$$k^* - kG^* = \frac{12EI}{L^3} + .391k - \frac{4.8 N_{cr}}{L} = 0$$

$$\boxed{N_{cr} = 2.5 \frac{EI}{L^2} + .08145\, kL}$$

## 3.1

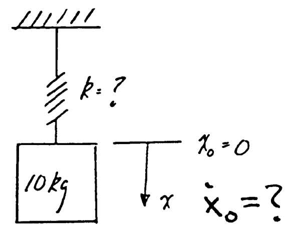
$$mass = (50 lb\ weight)\left(\frac{1}{32.2\ ft/sec^2}\right) = 1.553\ lb\,sec^2/ft \quad or \quad 0.1294\ lb\,sec^2/in$$

$$\omega = \sqrt{\frac{k}{m}} = \left(\frac{10\ lb/in}{0.1294\ lb\,sec^2/in}\right)^{1/2} = 8.791\ rad/sec$$

$$\boxed{f = \frac{\omega}{2\pi} = \frac{8.791\ rad/sec}{2\pi} = 1.399\ cps}$$

---

## 3.2




T (period) = 0.2 sec

X (amplitude) = 60mm

a) find the spring constant k

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\left(\frac{k}{m}\right)^{1/2}}$$

$$0.2\ sec = \frac{2\pi}{\left(\frac{k}{10kg}\right)^{1/2}}$$

$$0.2\ sec = \frac{2\pi\,(10kg)^{1/2}}{k^{1/2}}$$

$$\boxed{k = \left(\frac{2\pi(10kg)^{1/2}}{0.2\ sec}\right)^2 = 9869.6\ kg/sec^2\left(\frac{N\,sec^2/m}{1kg}\right) = 9869.6\ N/m}$$

$$1N = 1\ kg\ m/sec^2$$

$$1kg = 1N\,sec^2/m$$

## 3.2 Cont.

b) find initial velocity: $\dot{x}_o$

$$X = \left(x_o^2 + \left(\frac{\dot{x}_o}{\omega}\right)^2\right)^{1/2} \qquad \omega = \sqrt{\frac{k}{m}} = 31.42 \text{ rad/sec}$$

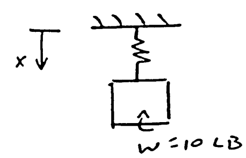
$$0.06\text{m} = \left(0 + \left(\frac{\dot{x}_o}{31.42}\right)^2\right)^{1/2}$$

$$\frac{\dot{x}_o^2}{(31.42)^2} = 0.0036$$

$$\boxed{\dot{x}_o = 1.89 \text{ m/sec}}$$

---

## 3.3




$x_o = 0.25$ IN

$\dot{x}_o = 0$

SOLUTION

$$\ddot{x} + \frac{k}{m} x = 0$$

$$K = \frac{W}{x} = \frac{10 \text{ LB}}{0.25 \text{ IN}} = 40 \text{ LB/IN}$$

$$\omega = \sqrt{\frac{k}{m}}$$

$$\omega = \sqrt{\frac{(40 \text{ LB/IN})}{10 \text{ LB}/32.2 \text{ FT/s}^2 (12 \text{ IN/FT})}}$$

$$\boxed{\omega = 39.31 \text{ RAD/SEC}}$$

---

## 3.4

$$\ddot{x}_{max} = 50 \text{ m/sec}^2$$

$f = 100$ Hz or 100 cps $\qquad \omega = 2\pi f = 628.32$ rad/sec

## 3.4 Cont.

a) determine the amplitude of vibration

$x(t) = X \sin(\omega t + \phi)$

$\dot{x}(t) = X\omega \cos(\omega t + \phi)$

$\ddot{x}(t) = -X\omega^2 \sin(\omega t + \phi)$

$\ddot{x}(t)_{max}$ when $\sin(\omega t + \phi) = 1$

$\ddot{x}(t)_{max} = -X\omega^2$

$50 \text{ in}/\text{sec}^2 = -X\omega^2$

$$\boxed{|X| = \frac{50 \text{ in}/\text{sec}^2}{(628.32 \text{ rad}/\text{sec})^2} = 1.2666 \times 10^{-4} \text{ in}}$$

b) determine max velocity of the mass

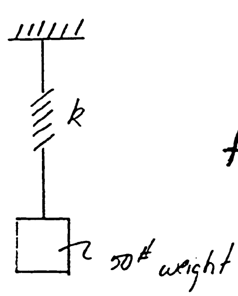
$\dot{x}(t) = X\omega \cos(\omega t + \phi)$

$\dot{x}(t)_{max}$ when $\cos(\omega t + \phi) = 1$

$$\boxed{\dot{x}(t)_{max} = X\omega = (1.2666 \times 10^{-4} \text{ in})(628.32 \text{ rad}/\text{sec}) = 0.0796 \text{ in}/\text{sec}}$$

---

## 3.5




$f = 400$ cpm

## 3.5 Cont.

Solution:

$$\omega = 2\pi f = 2513.27 \text{ rad/min or } 41.888 \text{ rad/sec}$$

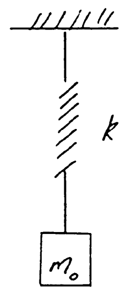
$$\omega = \sqrt{\frac{k}{m}} \qquad m = 50\#/32.2 \text{ ft/sec}^2 = 1.5528 \text{ \#sec}^2\text{/ft}$$

$$41.88 \text{ rad/sec} = \left(\frac{k}{0.1294 \text{ \#sec}^2\text{/in}}\right)^{1/2}$$

$$\boxed{k = 227.04 \text{ \#/in}}$$

---

## 3.6




$T_o = 0.35$ sec

when a 2lb mass is added to the system the period increases by 10 percent.

Solution: $2lb \rightarrow mass = 2lb/32.2 \text{ ft/sec}^2 = 6.2112 \times 10^{-2} \text{ lb sec}^2\text{/ft}$

or

$m = 5.17598 \times 10^{-3} \text{ lb sec}^2\text{/in}$

$m_f = m_o + m$

$T_f = 0.35(1+0.1) = 0.385$

$$\omega_o = \frac{2\pi}{T_o} = \left(\frac{k}{m_o}\right)^{1/2} \quad \text{eq①}$$

$$\omega_f = \frac{2\pi}{T_f} = \left(\frac{k}{m_f}\right)^{1/2} \quad \text{eq②}$$

$$\text{eq① } k = \left(\frac{2\pi}{T_o}(m_o)^{1/2}\right)^2 \qquad \text{eq② } k = \left(\frac{2\pi}{T_f}(m_f)^{1/2}\right)^2$$

## 3.6 Cont.

equating eq ① to eq ②

$$\frac{2\pi}{T_0}(M_0)^{1/2} = \frac{2\pi}{T_f}(M_f)^{1/2}$$

$$\frac{2\pi}{0.35}(M_0)^{1/2} = \frac{2\pi}{0.385}(M_0 + 5.17598\times10^{-3})^{1/2}$$

$$(M_0)^{1/2} = \frac{0.35}{0.385}(M_0 + 5.17598\times10^{-3})^{1/2}$$

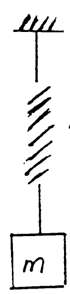
$$M_0 = \left(\frac{0.35}{0.385}\right)^2 (M_0 + 5.17598\times10^{-3})$$

$$0.1736\, M_0 = 4.27767\times10^{-3}$$

$$\boxed{\begin{array}{l} M_0 = 2.464\times10^{-2}\ \text{lb sec}^2/\text{in} \\ k = 7.94\ \#/\text{in} \end{array}}$$

---

## 3.7



$k = 200\ N/m$

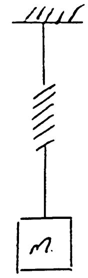
$f = 50\,Hz$

$$f = \frac{\omega}{2\pi} \Rightarrow \omega = 50\,Hz(2\pi) = 314.16\ \text{rad/sec}$$

$$\omega = \sqrt{\frac{k}{m}}$$

$$\boxed{M = \frac{k}{\omega^2} = \frac{200\ \text{kg m/sec}^2\,\text{rad}}{(314.16\ \text{rad/sec})^2} = 2.026\times10^{-3}\ \text{kg} \quad \text{or} \quad 2.026\ g}$$

## 3.8



$\dot{x}(t)_{max} = 10$ in/sec

$T = 1.5$ sec

$x_o = 3.0$ in

a) determine the amplitude of free vibration

$\dot{x}(t) = X\omega\cos(\omega t + \phi)$

$\dot{x}(t)_{max}$ when $\cos(\omega t + \phi) = 1$

$\dot{x}(t)_{max} = X\omega$ where $\omega = \frac{2\pi}{T} = 4.1888$ rad/se

$10$ in/sec $= X(4.1888$ rad/sec$)$

$\boxed{X = 2.387 \text{ in}}$

b) determine the maximum acceleration of the mass

$\ddot{x}(t) = -X\omega^2\sin(\omega t + \phi)$

$\ddot{x}(t)_{max}$ when $\sin(\omega t + \phi) = 1$

$\boxed{\ddot{x}(t)_{max} = -X\omega^2 = -41.888 \text{ in/sec}^2 \text{ or } [41.888 \text{ in/sec}^2] \text{ magnitude}}$

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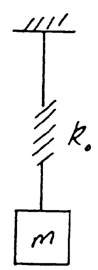
## 3.9



$f_o = 10$ Hz

$k_f = k_o + 5$ lb/in

$f_f = 10(1+.25) = 12.5$ Hz

## 3.9 Cont.

a) determine the spring constant for the original system ($k_o$)

$$f_o = \frac{\omega_o}{2\pi} \Rightarrow \omega_o = f_o 2\pi = 10(2\pi) = 20\pi = \sqrt{\frac{k_o}{m}}$$

$$m = \frac{k_o}{(20\pi)^2} \quad eq①$$

$$f_f = \frac{\omega_f}{2\pi} \Rightarrow \omega_f = 12.5(2\pi) = 25\pi = \sqrt{\frac{k_f}{m}}$$

$$m = \frac{k_f}{(25\pi)^2} = \frac{k_o + 5\ lb/in}{(25\pi)^2} \quad eq②.$$

equating eq① to eq②

$$\frac{k_o}{(20\pi)^2} = \frac{k_o + 5}{(25\pi)^2} \Rightarrow 0.5625 k_o = 5$$

$$\boxed{k_o = 8.89\ lb/in}$$

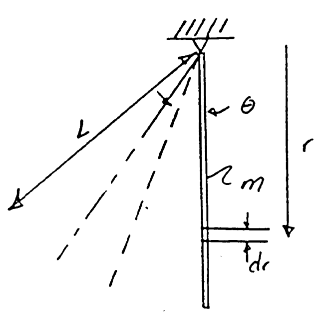
b) determine the mass of the original system

$$m = \frac{k_o}{(20\pi)^2} = \frac{8.89\ lb/in}{(20\pi\ rad/sec)^2} = 2.25186 \times 10^{-3}\ lb\,sec^2/in$$

or

$$\boxed{m = 2.7021 \times 10^{-2}\ lb\,sec^2/ft}$$

## 3.10




$T = 0.75\ sec$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{3/4} = \frac{8\pi}{3}$$

## 3.10 Cont.

$$T = \frac{2\pi}{\omega} = 2\pi\left(\frac{2L}{3g}\right)^{1/2}$$

$$\frac{T^2}{4\pi^2} = \frac{2L}{3g} \Longrightarrow L = \frac{3gT^2}{8\pi^2} = \frac{3g(3/4)^2}{8\pi^2}$$

$$\boxed{L = \frac{27g}{128\pi^2}} = 8.5 \text{ in}$$

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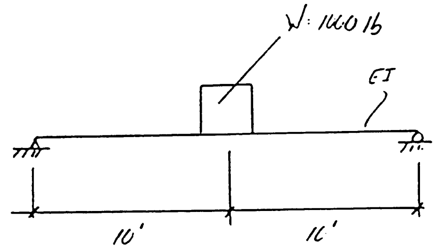
## 3.11

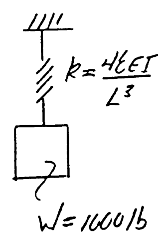



$E = 29{,}000$ ksi
$I = 1830 \text{ in}^4$
$E = 29 \times 10^6$ psi

$x_0 = 1.5 \text{ in}$
$V_0 = \dot{x}_0 = 4.5 \text{ in/sec}$

a) determine the natural period of vibration

$$k = \frac{48EI}{L^3}$$




$$m = 1600\text{ lb} / 32.2 \text{ ft/sec}^2 = 31.056 \text{ lb sec}^2/\text{ft}$$
$$= 2.588 \text{ lb sec}^2/\text{in}$$

$$\omega = \left(\frac{k}{m}\right)^{1/2} = \left(\frac{48EI}{mL^3}\right)^{1/2}$$

$$= \left(\frac{48(29\times10^6 \text{ lb/in}^2)(1830 \text{ in}^4)}{(2.588 \text{ lb sec}^2/\text{in})(240 \text{ in})^3}\right)^{1/2} = 266.84 \text{ rad/sec}$$

$$\boxed{T = \frac{2\pi}{\omega} = 0.02355 \text{ sec}}$$

## 3.11 Cont.

Energy Method

(kinetic energy) "T"

$$T = \frac{1}{2} m \dot{v}^2 = \frac{1}{2} \int_0^L \frac{m}{L} (r(\sin\dot{\theta}))^2 \, dr$$

$$= \frac{1}{2} \left[ \frac{m r^3}{3L} \sin\dot{\theta} \right]_0^L = \frac{1}{2} \frac{m L^2}{3} (\sin\dot{\theta})^2$$

$\sin\dot{\theta} = \dot{\theta}$

$$= \frac{m L^2}{6} \dot{\theta}^2$$

(potential energy) "V"

$$V = \int_0^L \frac{m}{L} g \, r (1-\cos\theta) \, dr = \left[ \frac{m g r^2}{2L} (1-\cos\theta) \right]_0^L$$

$$= \frac{m g L}{2} (1-\cos\theta)$$

$$\frac{d(T+V)}{d\theta} = 0 = \frac{m L^2}{3} \dot{\theta} \ddot{\theta} + \frac{m g L}{2} \sin\theta \, \dot{\theta}$$

$\sin\theta = \theta$ small disp.

$$\frac{m L^2}{3} \ddot{\theta} + \frac{m g L}{2} \theta = 0$$

$$\ddot{\theta} + \left( \frac{3}{2} \frac{m g L}{m L^2} \right) \theta = 0$$

$$\ddot{\theta} + \left( \frac{3g}{2L} \right) \theta = 0$$

(equation of motion)

$$\boxed{\omega = \sqrt{\frac{3g}{2L}}}$$

**3.11 Cont.**

b) determine max displacement

$$x(t) = X \sin(\omega t + \phi)$$

$$x(t)_{max} = X = \left((x_o)^2 + \left(\frac{\dot{x}_o}{\omega}\right)^2\right)^{1/2}$$

$$\boxed{x(t)_{max} = X = \left((1.5)^2 + \left(\frac{4.5}{264.84}\right)^2\right)^{1/2} = 1.5 \text{ in}}$$

c) determine max velocity

$$\dot{x}(t) = X\omega \cos(\omega t + \phi)$$

$$\boxed{\dot{x}(t)_{max} = X\omega = (1.5)(266.84) = 400.26 \text{ in/sec}}$$

d) determine the max. acceleration of the mass

$$\ddot{x}(t) = -X\omega^2 \sin(\omega t + \phi)$$

$$\ddot{x}(t)_{max} = -X\omega^2 = -(1.5)(266.84)^2 = -106805.38 \text{ in/sec}^2$$

or

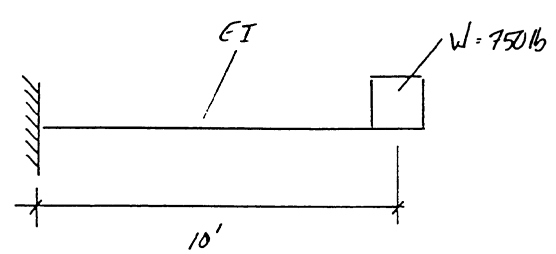
$106805.38$ in/sec² magnitude

e) determine phase angle

$$\boxed{\phi = \tan^{-1}\left(\frac{x_o \omega}{\dot{x}_o}\right) = 89.36°}$$

---

**3.12**




$E = 30{,}000$ ksi
$= 30 \times 10^6$ psi
$I = 2500 \text{ in}^4$

$x_o = 2.0$ in

$V_o = \dot{x}_o = 3.0$ in/sec

## 3.12 Cont.

a) determine the natural frequency of free vibration

$$k = \frac{3EI}{L^3} \qquad m = \frac{750\ lb}{32.2\ ft/sec^2} = 23.29\ lb\ sec^2/ft$$

$$\omega = \left(\frac{k}{m}\right)^{1/2} = \left(\frac{3(30\times10^6)(2520)}{(1.941)(120)^3}\right)^{1/2} = 259\ rad/sec \qquad = 1.941\ lb\ sec^2/in$$

$$\boxed{f = \frac{\omega}{2\pi} = 41.22\ Hz}$$

b) determine the max displacement, velocity, & acceleration of the mass.

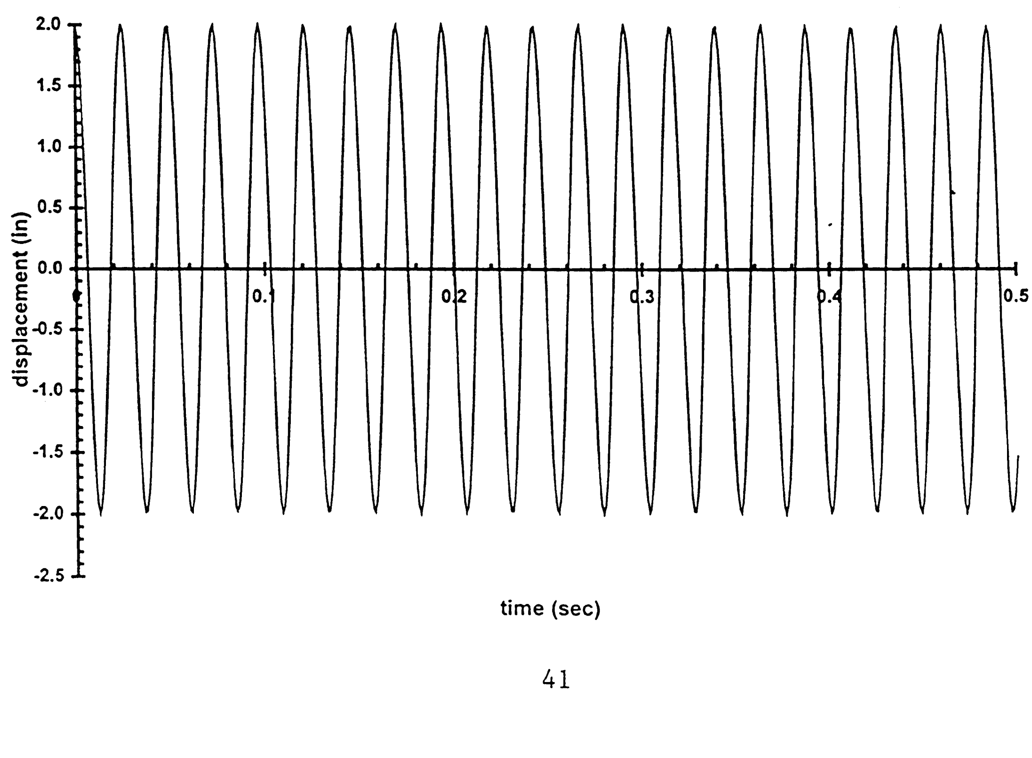
$$x(t)_{max} = X = \left((x_0)^2 + \left(\frac{\dot{x}_0}{\omega}\right)^2\right)^{1/2} = 2\ in$$

$$\dot{x}(t)_{max} = X\omega = 518\ in/sec$$

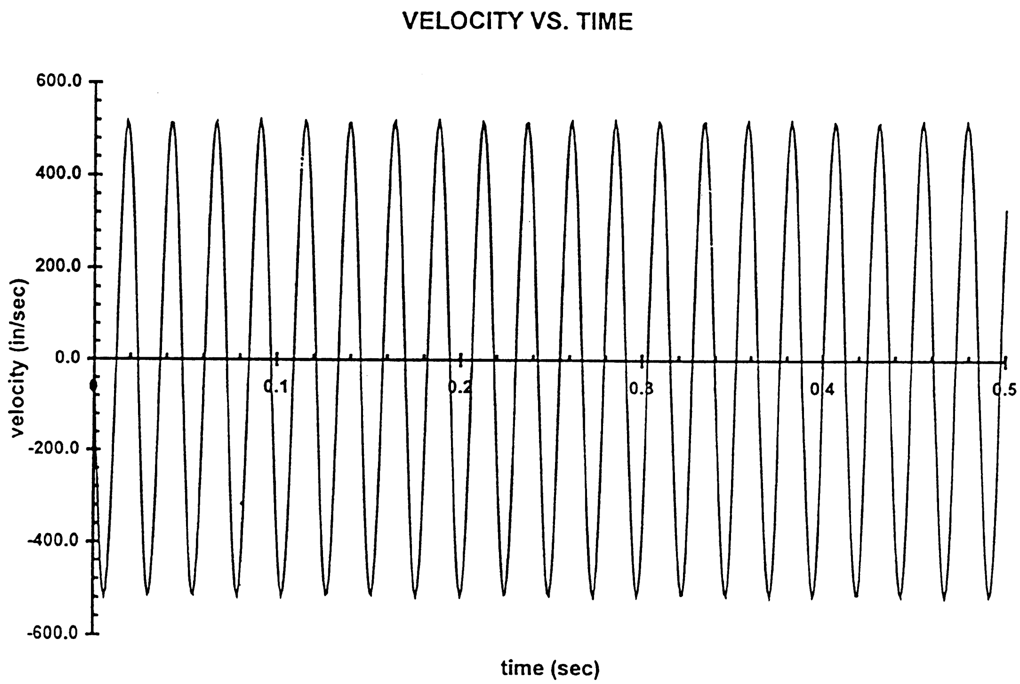
$$\ddot{x}(t)_{max} = X\omega^2 = 134162\ in/sec^2$$

c) determine phase angle

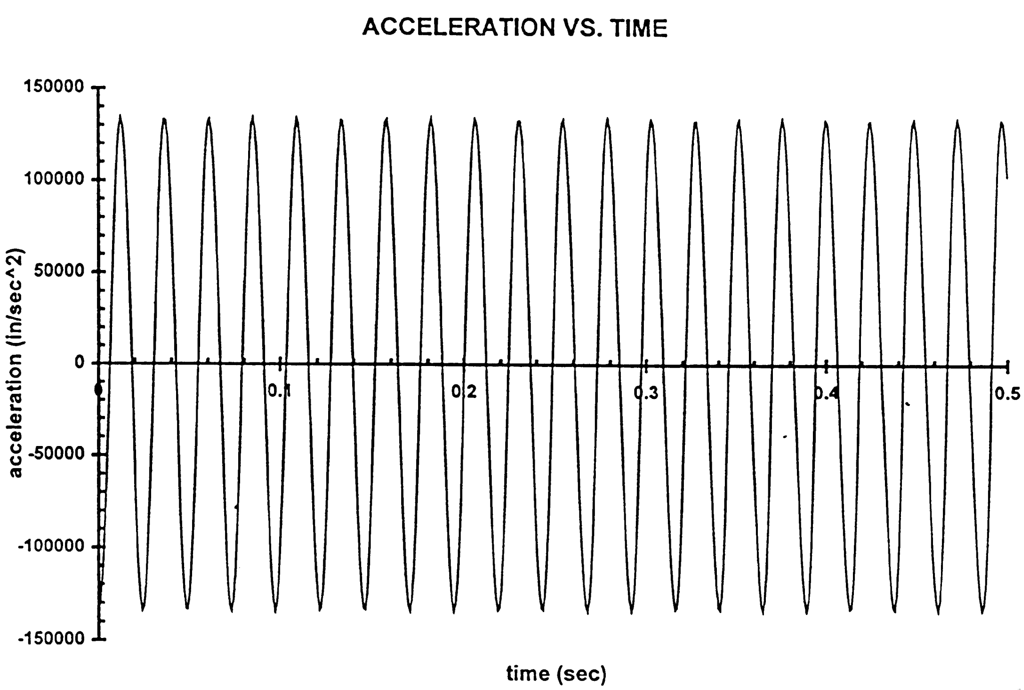
$$\phi = \tan^{-1}\left(\frac{x_0\omega}{\dot{x}_0}\right) = 89.67^\circ$$



3.12 Cont.

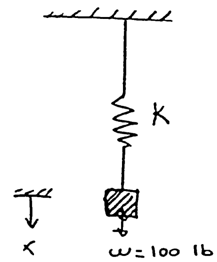


VELOCITY VS. TIME
600.0
400.0
200.0
0.0
-200.0
-400.0
-600.0
velocity (in/sec)
0.1
0.2
0.3
0.4
0.5
time (sec)

ACCELERATION VS. TIME
150000
100000
50000
0
-50000
-100000
-150000
acceleration (in/sec^2)
0.1
0.2
0.3
0.4
0.5
time (sec)

## 3.13




$$M = \frac{100}{386.4} = 0.258 \quad \frac{lb \cdot s^2}{in}$$

$$y_0 = 0$$

$$T = 0.15 = \frac{2\pi}{\omega}$$

$$\therefore \omega = \frac{2\pi}{.15} = 41.88 \text{ rad/sec}$$

$$41.88 = \sqrt{\frac{K}{m}} = \sqrt{\frac{K}{.258}}$$

$$\boxed{K = 452.7 \quad lb/in} \quad (a)$$

$$X = \sqrt{y_0^2 + \left(\frac{V}{\omega}\right)^2}$$

$$\overline{1.5}^2 = \frac{V^2}{(41.88)^2}$$

$$\boxed{V_0 = 62.82 \quad in/sec} \quad (b)$$

$$\phi = \tan^{-1} \frac{y_0}{v_0/\omega} = 0$$

$$\therefore X = 1.5 \sin 41.88t$$

$$\dot{X} = 1.5\,(41.88) \cos 41.88t$$

$$\ddot{X} = -1.5\,(41.88)^2 \sin 41.88t$$

at $t = 0.41$ sec (c)

| | | |
|---|---|---|
| $X = -1.49$ in | $\dot{X} = -677$ in/sec | $\ddot{X} = 2615.6$ in/sec$^2$ |

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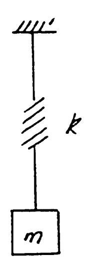
## 3.14




$x_0 = 0$

$V_0 = \dot{x}_0 = 2$ in/sec

$T = 0.3$ sec

50 lb weight (mass = 0.1294 lb sec$^2$/in)

## 3.14 Cont.

a) determine the spring constant (k)

$$\omega = \left(\frac{k}{m}\right)^{1/2} \; ; \quad T = \frac{2\pi}{\omega} = 0.3\ sec \Rightarrow \omega = \frac{20\pi}{3}$$

$$\boxed{k = \omega^2 m = \left(\frac{20\pi}{3}\right)^2 (0.1294) = 56.76\ lb/in}$$

b) determine the amplitude of vibration

$$\boxed{X = \left(\left(x_0\right)^2 + \left(\frac{\dot{x}_0}{\omega}\right)^2\right)^{1/2} = \left(\left(\frac{6}{20\pi}\right)^2\right)^{1/2} = \frac{6}{20\pi} = 9.549 \times 10^{-2}\ in}$$

c) determine the velocity & acceleration of the mass @ t = 0.5 sec

$\phi = 0$

$\dot{x}(t) = X\omega \cos \omega t$

$$\boxed{\dot{x}(0.5\,sec) = \frac{6}{20\pi}\left(\frac{20\pi}{3}\right)\cos\left(\frac{20\pi}{3}\right)(0.5) = -1.0\ in/sec}$$

$\ddot{x}(t) = -X\omega^2 \sin \omega t$

$$\boxed{\ddot{x}(0.5\,sec) = -\left(\frac{6}{20\pi}\right)\left(\frac{20\pi}{3}\right)^2 \sin\left(\frac{20\pi}{3}\right)(0.5) = 36.28\ in/sec^2}$$

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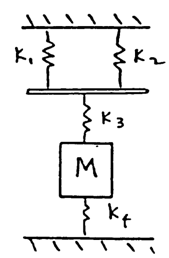
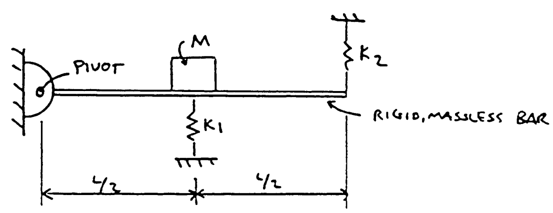
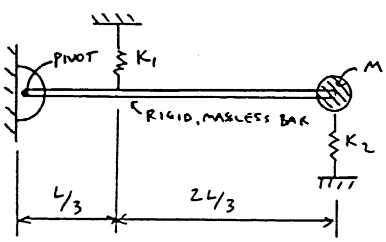
## 3.15



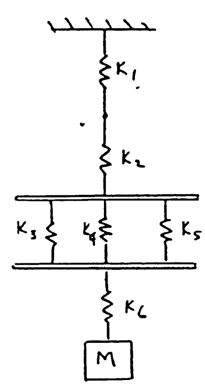

(A)




(B)




(C)

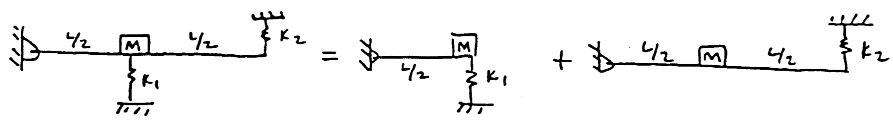



(D)

## 3.15 Cont.

SYSTEM (B)

(A) EQUIVALENT STIFFNESS, $k_e$



$$k_e = k_1\left(\frac{L/2}{L/2}\right)^2 + k_2\left(\frac{L}{L/2}\right)^2$$

$$k_e = k_1 + 4k_2$$

(B) EQUATION OF MOTION

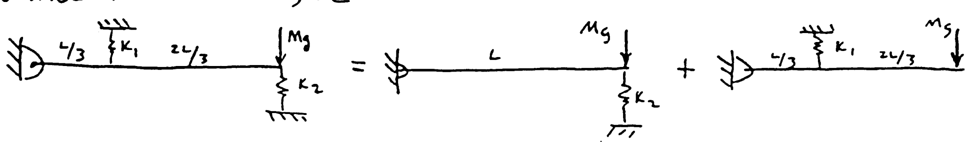
$$\ddot{x} + \frac{k_1 + 4k_2}{m}x = 0$$

(C) NATURAL FREQUENCY

$$\omega = \sqrt{\frac{k_1 + 4k_2}{M}}$$

SYSTEM (C)

(A) EQUIVALENT STIFFNESS, $k_e$



$$k_e = k_2\left(\frac{L}{L}\right)^2 + k_1\left(\frac{L/3}{L}\right)^2$$

$$k_e = k_2 + \frac{k_1}{9}$$

(B) EQUATION OF MOTION

$$\ddot{x} + \frac{k_1/9 + k_2}{m}x = 0$$

(C) NATURAL FREQUENCY

$$\omega = \sqrt{\frac{k_1/9 + k_2}{M}}$$

SYSTEM (D)

$$\frac{1}{ke_1} = \frac{1}{k_1} + \frac{1}{k_2}$$

$$\frac{1}{ke_1} = \frac{k_1 + k_2}{k_1 k_2}$$

$$ke_1 = \frac{k_1 k_2}{k_1 + k_2}$$

$$ke_2 = k_3 + k_4 + k_5$$