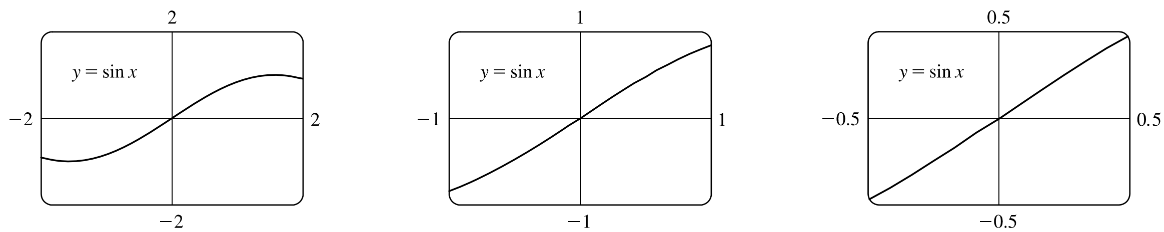
2 ☐ DERIVATIVES

2.1 Derivatives and Rates of Change

1. (a) This is just the slope of the line through two points: $m_{PQ} = \dfrac{\Delta y}{\Delta x} = \dfrac{f(x) - f(3)}{x - 3}$.

(b) This is the limit of the slope of the secant line PQ as Q approaches P: $m = \lim\limits_{x \to 3} \dfrac{f(x) - f(3)}{x - 3}$.

2. The curve looks more like a line as the viewing rectangle gets smaller.



3. (a) (i) Using Definition 1 with $f(x) = 4x - x^2$ and $P(1, 3)$,

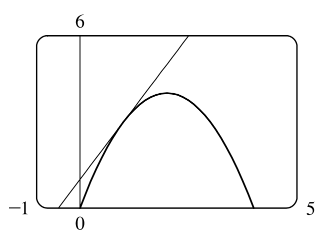
$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to 1} \frac{(4x - x^2) - 3}{x - 1} = \lim_{x \to 1} \frac{-(x^2 - 4x + 3)}{x - 1} = \lim_{x \to 1} \frac{-(x - 1)(x - 3)}{x - 1}$$
$$= \lim_{x \to 1} (3 - x) = 3 - 1 = 2$$

(ii) Using Equation 2 with $f(x) = 4x - x^2$ and $P(1, 3)$,

$$m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \lim_{h \to 0} \frac{f(1 + h) - f(1)}{h} = \lim_{h \to 0} \frac{\left[4(1 + h) - (1 + h)^2\right] - 3}{h}$$
$$= \lim_{h \to 0} \frac{4 + 4h - 1 - 2h - h^2 - 3}{h} = \lim_{h \to 0} \frac{-h^2 + 2h}{h} = \lim_{h \to 0} \frac{h(-h + 2)}{h} = \lim_{h \to 0} (-h + 2) = 2$$

(b) An equation of the tangent line is $y - f(a) = f'(a)(x - a) \quad \Rightarrow \quad y - f(1) = f'(1)(x - 1) \quad \Rightarrow \quad y - 3 = 2(x - 1)$, or $y = 2x + 1$.

(c)



The graph of $y = 2x + 1$ is tangent to the graph of $y = 4x - x^2$ at the point $(1, 3)$. Now zoom in toward the point $(1, 3)$ until the parabola and the tangent line are indistiguishable.

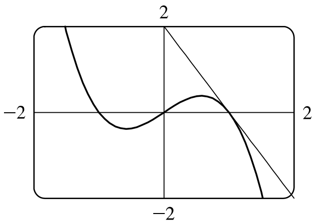
4. (a) (i) Using Definition 1 with $f(x) = x - x^3$ and $P(1, 0)$,

$$m = \lim_{x \to 1} \frac{f(x) - 0}{x - 1} = \lim_{x \to 1} \frac{x - x^3}{x - 1} = \lim_{x \to 1} \frac{x(1 - x^2)}{x - 1} = \lim_{x \to 1} \frac{x(1 + x)(1 - x)}{x - 1}$$
$$= \lim_{x \to 1} [-x(1 + x)] = -1(2) = -2$$

(ii) Using Equation 2 with $f(x) = x - x^3$ and $P(1, 0)$,

$$m = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{\left[(1+h) - (1+h)^3\right] - 0}{h}$$
$$= \lim_{h\to 0} \frac{1 + h - (1 + 3h + 3h^2 + h^3)}{h} = \lim_{h\to 0} \frac{-h^3 - 3h^2 - 2h}{h} = \lim_{h\to 0} \frac{h(-h^2 - 3h - 2)}{h}$$
$$= \lim_{h\to 0} (-h^2 - 3h - 2) = -2$$

(b) An equation of the tangent line is $y - f(a) = f'(a)(x - a) \;\Rightarrow\; y - f(1) = f'(1)(x - 1) \;\Rightarrow\; y - 0 = -2(x - 1)$, or $y = -2x + 2$.

(c) 

The graph of $y = -2x + 2$ is tangent to the graph of $y = x - x^3$ at the point $(1, 0)$. Now zoom in toward the point $(1, 0)$ until the cubic and the tangent line are indistinguishable.

5. Using (1) with $f(x) = 4x - 3x^2$ and $P(2, -4)$ [we could also use (2)],

$$m = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{x\to 2} \frac{(4x - 3x^2) - (-4)}{x - 2} = \lim_{x\to 2} \frac{-3x^2 + 4x + 4}{x - 2}$$
$$= \lim_{x\to 2} \frac{(-3x - 2)(x - 2)}{x - 2} = \lim_{x\to 2} (-3x - 2) = -3(2) - 2 = -8$$

Tangent line: $y - (-4) = -8(x - 2) \;\Leftrightarrow\; y + 4 = -8x + 16 \;\Leftrightarrow\; y = -8x + 12$.

6. Using (2) with $f(x) = x^3 - 3x + 1$ and $P(2, 3)$,

$$m = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0} \frac{(2+h)^3 - 3(2+h) + 1 - 3}{h}$$
$$= \lim_{h\to 0} \frac{8 + 12h + 6h^2 + h^3 - 6 - 3h - 2}{h} = \lim_{h\to 0} \frac{9h + 6h^2 + h^3}{h} = \lim_{h\to 0} \frac{h(9 + 6h + h^2)}{h}$$
$$= \lim_{h\to 0} (9 + 6h + h^2) = 9$$

Tangent line: $y - 3 = 9(x - 2) \;\Leftrightarrow\; y - 3 = 9x - 18 \;\Leftrightarrow\; y = 9x - 15$

7. Using (1),

$$m = \lim_{x\to 1} \frac{\sqrt{x} - \sqrt{1}}{x - 1} = \lim_{x\to 1} \frac{(\sqrt{x} - 1)(\sqrt{x} + 1)}{(x - 1)(\sqrt{x} + 1)} = \lim_{x\to 1} \frac{x - 1}{(x - 1)(\sqrt{x} + 1)} = \lim_{x\to 1} \frac{1}{\sqrt{x} + 1} = \frac{1}{2}$$

Tangent line: $y - 1 = \frac{1}{2}(x - 1) \;\Leftrightarrow\; y = \frac{1}{2}x + \frac{1}{2}$

8. Using (1) with $f(x) = \dfrac{2x+1}{x+2}$ and $P(1,1)$,

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to 1} \frac{\dfrac{2x+1}{x+2} - 1}{x - 1} = \lim_{x \to 1} \frac{\dfrac{2x+1-(x+2)}{x+2}}{x-1} = \lim_{x \to 1} \frac{x-1}{(x-1)(x+2)}$$

$$= \lim_{x \to 1} \frac{1}{x+2} = \frac{1}{1+2} = \frac{1}{3}$$

Tangent line: $y - 1 = \frac{1}{3}(x-1) \quad \Leftrightarrow \quad y - 1 = \frac{1}{3}x - \frac{1}{3} \quad \Leftrightarrow \quad y = \frac{1}{3}x + \frac{2}{3}$

9. (a) Using (2) with $y = f(x) = 3 + 4x^2 - 2x^3$,

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{3 + 4(a+h)^2 - 2(a+h)^3 - (3 + 4a^2 - 2a^3)}{h}$$

$$= \lim_{h \to 0} \frac{3 + 4(a^2 + 2ah + h^2) - 2(a^3 + 3a^2h + 3ah^2 + h^3) - 3 - 4a^2 + 2a^3}{h}$$

$$= \lim_{h \to 0} \frac{3 + 4a^2 + 8ah + 4h^2 - 2a^3 - 6a^2h - 6ah^2 - 2h^3 - 3 - 4a^2 + 2a^3}{h}$$

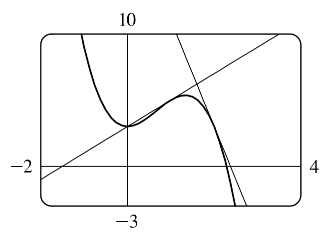
$$= \lim_{h \to 0} \frac{8ah + 4h^2 - 6a^2h - 6ah^2 - 2h^3}{h} = \lim_{h \to 0} \frac{h(8a + 4h - 6a^2 - 6ah - 2h^2)}{h}$$

$$= \lim_{h \to 0} (8a + 4h - 6a^2 - 6ah - 2h^2) = 8a - 6a^2$$

(b) At $(1,5)$: $m = 8(1) - 6(1)^2 = 2$, so an equation of the tangent line is $y - 5 = 2(x-1) \quad \Leftrightarrow \quad y = 2x + 3$.

At $(2,3)$: $m = 8(2) - 6(2)^2 = -8$, so an equation of the tangent line is $y - 3 = -8(x-2) \quad \Leftrightarrow \quad y = -8x + 19$.

(c)



10. (a) Using (1),

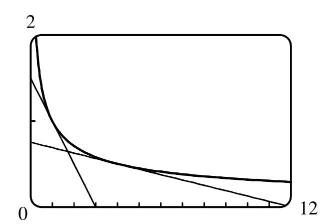
$$m = \lim_{x \to a} \frac{\dfrac{1}{\sqrt{x}} - \dfrac{1}{\sqrt{a}}}{x - a} = \lim_{x \to a} \frac{\dfrac{\sqrt{a} - \sqrt{x}}{\sqrt{ax}}}{x - a} = \lim_{x \to a} \frac{(\sqrt{a} - \sqrt{x})(\sqrt{a} + \sqrt{x})}{\sqrt{ax}\,(x-a)(\sqrt{a} + \sqrt{x})} = \lim_{x \to a} \frac{a - x}{\sqrt{ax}\,(x-a)(\sqrt{a} + \sqrt{x})}$$

$$= \lim_{x \to a} \frac{-1}{\sqrt{ax}\,(\sqrt{a} + \sqrt{x})} = \frac{-1}{\sqrt{a^2}\,(2\sqrt{a})} = -\frac{1}{2a^{3/2}} \text{ or } -\frac{1}{2}a^{-3/2} \quad [a > 0]$$

(b) At $(1,1)$: $m = -\frac{1}{2}$, so an equation of the tangent line is $y - 1 = -\frac{1}{2}(x-1) \quad \Leftrightarrow \quad y = -\frac{1}{2}x + \frac{3}{2}$.

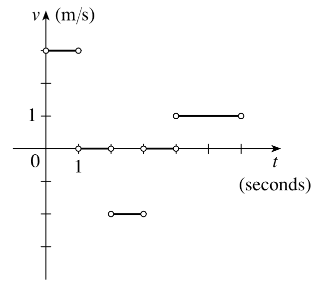
At $\left(4, \frac{1}{2}\right)$: $m = -\frac{1}{16}$, so an equation of the tangent line is $y - \frac{1}{2} = -\frac{1}{16}(x-4) \quad \Leftrightarrow \quad y = -\frac{1}{16}x + \frac{3}{4}$.

(c)



11. (a) The particle is moving to the right when s is increasing; that is, on the intervals $(0,1)$ and $(4,6)$. The particle is moving to the left when s is decreasing; that is, on the interval $(2,3)$. The particle is standing still when s is constant; that is, on the intervals $(1,2)$ and $(3,4)$.

(b) The velocity of the particle is equal to the slope of the tangent line of the graph. Note that there is no slope at the corner points on the graph. On the interval $(0, 1)$, the slope is $\frac{3-0}{1-0} = 3$. On the interval $(2, 3)$, the slope is $\frac{1-3}{3-2} = -2$. On the interval $(4, 6)$, the slope is $\frac{3-1}{6-4} = 1$.



12. (a) **Runner A** runs the entire 100-meter race at the same velocity since the slope of the position function is constant.
Runner B starts the race at a slower velocity than runner A, but finishes the race at a faster velocity.

(b) The distance between the runners is the greatest at the time when the largest vertical line segment fits between the two graphs—this appears to be somewhere between 9 and 10 seconds.

(c) The runners had the same velocity when the slopes of their respective position functions are equal—this also appears to be at about 9.5 s. Note that the answers for parts (b) and (c) must be the same for these graphs because as soon as the velocity for runner B overtakes the velocity for runner A, the distance between the runners starts to decrease.

13. Let $s(t) = 40t - 16t^2$.

$$v(2) = \lim_{t\to 2} \frac{s(t) - s(2)}{t-2} = \lim_{t\to 2} \frac{(40t - 16t^2) - 16}{t-2} = \lim_{t\to 2} \frac{-16t^2 + 40t - 16}{t-2} = \lim_{t\to 2} \frac{-8(2t^2 - 5t + 2)}{t-2}$$

$$= \lim_{t\to 2} \frac{-8(t-2)(2t-1)}{t-2} = -8 \lim_{t\to 2}(2t-1) = -8(3) = -24$$

Thus, the instantaneous velocity when $t = 2$ is -24 ft/s.

14. (a) Let $H(t) = 10t - 1.86t^2$.

$$v(1) = \lim_{h\to 0} \frac{H(1+h) - H(1)}{h} = \lim_{h\to 0} \frac{[10(1+h) - 1.86(1+h)^2] - (10 - 1.86)}{h}$$

$$= \lim_{h\to 0} \frac{10 + 10h - 1.86(1 + 2h + h^2) - 10 + 1.86}{h}$$

$$= \lim_{h\to 0} \frac{10 + 10h - 1.86 - 3.72h - 1.86h^2 - 10 + 1.86}{h}$$

$$= \lim_{h\to 0} \frac{6.28h - 1.86h^2}{h} = \lim_{h\to 0}(6.28 - 1.86h) = 6.28$$

The velocity of the rock after one second is 6.28 m/s.

(b) $$v(a) = \lim_{h\to 0} \frac{H(a+h) - H(a)}{h} = \lim_{h\to 0} \frac{[10(a+h) - 1.86(a+h)^2] - (10a - 1.86a^2)}{h}$$

$$= \lim_{h\to 0} \frac{10a + 10h - 1.86(a^2 + 2ah + h^2) - 10a + 1.86a^2}{h}$$

$$= \lim_{h\to 0} \frac{10a + 10h - 1.86a^2 - 3.72ah - 1.86h^2 - 10a + 1.86a^2}{h} = \lim_{h\to 0} \frac{10h - 3.72ah - 1.86h^2}{h}$$

$$= \lim_{h\to 0} \frac{h(10 - 3.72a - 1.86h)}{h} = \lim_{h\to 0}(10 - 3.72a - 1.86h) = 10 - 3.72a$$

The velocity of the rock when $t = a$ is $(10 - 3.72a)$ m/s.

(c) The rock will hit the surface when $H = 0 \Leftrightarrow 10t - 1.86t^2 = 0 \Leftrightarrow t(10 - 1.86t) = 0 \Leftrightarrow t = 0$ or $1.86t = 10$. The rock hits the surface when $t = 10/1.86 \approx 5.4$ s.

(d) The velocity of the rock when it hits the surface is $v\left(\frac{10}{1.86}\right) = 10 - 3.72\left(\frac{10}{1.86}\right) = 10 - 20 = -10$ m/s.

15. $$v(a) = \lim_{h\to 0} \frac{s(a+h) - s(a)}{h} = \lim_{h\to 0} \frac{\dfrac{1}{(a+h)^2} - \dfrac{1}{a^2}}{h} = \lim_{h\to 0} \frac{\dfrac{a^2 - (a+h)^2}{a^2(a+h)^2}}{h} = \lim_{h\to 0} \frac{a^2 - (a^2 + 2ah + h^2)}{ha^2(a+h)^2}$$

$$= \lim_{h\to 0} \frac{-(2ah + h^2)}{ha^2(a+h)^2} = \lim_{h\to 0} \frac{-h(2a + h)}{ha^2(a+h)^2} = \lim_{h\to 0} \frac{-(2a+h)}{a^2(a+h)^2} = \frac{-2a}{a^2 \cdot a^2} = \frac{-2}{a^3} \text{ m/s}$$

So $v\,(1) = \frac{-2}{1^3} = -2$ m/s, $v(2) = \frac{-2}{2^3} = -\frac{1}{4}$m/s, and $v(3) = \frac{-2}{3^3} = -\frac{2}{27}$ m/s.

16. (a) The average velocity between times t and $t + h$ is

$$\frac{s(t+h) - s(t)}{(t+h) - t} = \frac{\frac{1}{2}(t+h)^2 - 6(t+h) + 23 - \left(\frac{1}{2}t^2 - 6t + 23\right)}{h}$$
$$= \frac{\frac{1}{2}t^2 + th + \frac{1}{2}h^2 - 6t - 6h + 23 - \frac{1}{2}t^2 + 6t - 23}{h}$$
$$= \frac{th + \frac{1}{2}h^2 - 6h}{h} = \frac{h\left(t + \frac{1}{2}h - 6\right)}{h} = \left(t + \tfrac{1}{2}h - 6\right) \text{ ft/s}$$

(i) $[4, 8]$: $t = 4$, $h = 8 - 4 = 4$, so the average velocity is $4 + \frac{1}{2}(4) - 6 = 0$ ft/s.

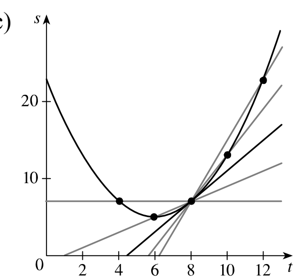
(ii) $[6, 8]$: $t = 6$, $h = 8 - 6 = 2$, so the average velocity is $6 + \frac{1}{2}(2) - 6 = 1$ ft/s.

(iii) $[8, 10]$: $t = 8$, $h = 10 - 8 = 2$, so the average velocity is $8 + \frac{1}{2}(2) - 6 = 3$ ft/s.

(iv) $[8, 12]$: $t = 8$, $h = 12 - 8 = 4$, so the average velocity is $8 + \frac{1}{2}(4) - 6 = 4$ ft/s.

(b) $v(t) = \lim_{h\to 0} \frac{s(t+h) - s(t)}{h} = \lim_{h\to 0} \left(t + \frac{1}{2}h - 6\right)$
$= t - 6$, so $v(8) = 2$ ft/s.

(c)



17. $g'(0)$ is the only negative value. The slope at $x = 4$ is smaller than the slope at $x = 2$ and both are smaller than the slope at $x = -2$. Thus, $g'(0) < 0 < g'(4) < g'(2) < g'(-2)$.

18. (a) On $[20, 60]$: $\frac{f(60) - f(20)}{60 - 20} = \frac{700 - 300}{40} = \frac{400}{40} = 10$

(b) Pick any interval that has the same y-value at its endpoints. $[0, 57]$ is such an interval since $f(0) = 600$ and $f(57) = 600$.

(c) On $[40, 60]$: $\frac{f(60) - f(40)}{60 - 40} = \frac{700 - 200}{20} = \frac{500}{20} = 25$

On $[40, 70]$: $\frac{f(70) - f(40)}{70 - 40} = \frac{900 - 200}{30} = \frac{700}{30} = 23\frac{1}{3}$

Since $25 > 23\frac{1}{3}$, the average rate of change on $[40, 60]$ is larger.

(d) $\dfrac{f(40)-f(10)}{40-10}=\dfrac{200-400}{30}=\dfrac{-200}{30}=-6\frac{2}{3}$

This value represents the slope of the line segment from $(10, f(10))$ to $(40, f(40))$.

19. (a) The tangent line at $x=50$ appears to pass through the points $(43, 200)$ and $(60, 640)$, so

$f'(50)\approx\dfrac{640-200}{60-43}=\dfrac{440}{17}\approx 26.$

(b) The tangent line at $x=10$ is steeper than the tangent line at $x=30$, so it is larger in magnitude, but less in numerical value, that is, $f'(10)<f'(30)$.

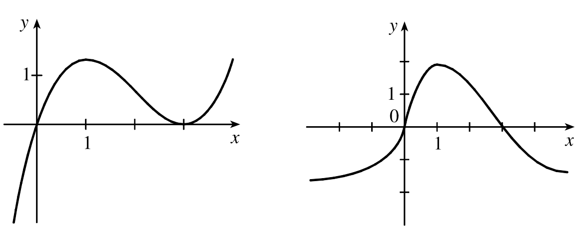
(c) The slope of the tangent line at $x=60$, $f'(60)$, is greater than the slope of the line through $(40, f(40))$ and $(80, f(80))$. So yes, $f'(60)>\dfrac{f(80)-f(40)}{80-40}$.

20. Since $g(5)=-3$, the point $(5,-3)$ is on the graph of g. Since $g'(5)=4$, the slope of the tangent line at $x=5$ is 4. Using the point-slope form of a line gives us $y-(-3)=4(x-5)$, or $y=4x-23$.

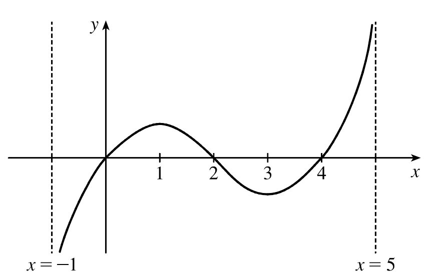
21. For the tangent line $y=4x-5$: when $x=2$, $y=4(2)-5=3$ and its slope is 4 (the coefficient of x). At the point of tangency, these values are shared with the curve $y=f(x)$; that is, $f(2)=3$ and $f'(2)=4$.

22. Since $(4,3)$ is on $y=f(x)$, $f(4)=3$. The slope of the tangent line between $(0,2)$ and $(4,3)$ is $\frac{1}{4}$, so $f'(4)=\frac{1}{4}$.

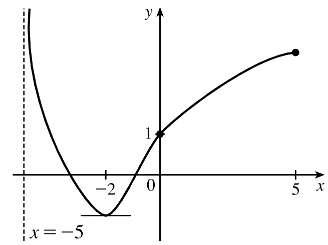
23. We begin by drawing a curve through the origin with a slope of 3 to satisfy $f(0)=0$ and $f'(0)=3$. Since $f'(1)=0$, we will round off our figure so that there is a horizontal tangent directly over $x=1$. Last, we make sure that the curve has a slope of -1 as we pass over $x=2$. Two of the many possibilities are shown.



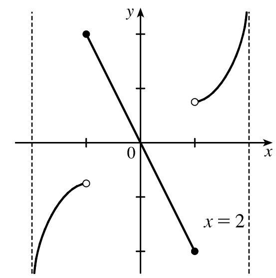
24. The condition $g(0)=g(2)=g(4)=0$ means that the graph intersects the x-axis at $(0,0)$, $(2,0)$, and $(4,0)$. The condition $g'(1)=g'(3)=0$ means that the graph has horizontal tangents at $x=1$ and $x=3$. The conditions $g'(0)=g'(4)=1$ and $g'(2)=-1$ mean that the tangents at $(0,0)$ and $(4,0)$ have slope 1, while the tangent at $(2,0)$ has slope -1. Finally, the conditions $\lim_{x\to 5^-} g(x)=\infty$ and $\lim_{x\to -1^+} g(x)=-\infty$ imply that $x=-1$ and $x=5$ are vertical asymptotes. A sample graph is shown. Note that the function shown has domain $(-1,5)$. That domain could easily be extended by drawing additional graph segments in $(-\infty,-1]$ and $[5,\infty)$ that satisfy the vertical line test.



25. We begin by drawing a curve through $(0, 1)$ with a slope of 1 to satisfy $g(0) = 1$ and $g'(0) = 1$. We round off our figure at $x = -2$ to satisfy $g'(-2) = 0$. As $x \to -5^+$, $y \to \infty$, so we draw a vertical asymptote at $x = -5$. As $x \to 5^-$, $y \to 3$, so we draw a dot at $(5, 3)$ [the dot could be open or closed].



26. We begin by drawing an odd function (symmetric with respect to the origin) through the origin with slope -2 to satisfy $f'(0) = -2$. Now draw a curve starting at $x = 1$ and increasing without bound as $x \to 2^-$ since $\lim_{x \to 2^-} f(x) = \infty$. Lastly, reflect the last curve through the origin (rotate $180°$) since f is an odd function.



27. Using (4) with $f(x) = 3x^2 - x^3$ and $a = 1$,

$$\begin{aligned} f'(1) &= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{[3(1+h)^2 - (1+h)^3] - 2}{h} \\ &= \lim_{h \to 0} \frac{(3 + 6h + 3h^2) - (1 + 3h + 3h^2 + h^3) - 2}{h} = \lim_{h \to 0} \frac{3h - h^3}{h} = \lim_{h \to 0} \frac{h(3 - h^2)}{h} \\ &= \lim_{h \to 0} (3 - h^2) = 3 - 0 = 3 \end{aligned}$$

Tangent line: $y - 2 = 3(x - 1) \quad \Leftrightarrow \quad y - 2 = 3x - 3 \quad \Leftrightarrow \quad y = 3x - 1$

28. Using (5) with $g(x) = x^4 - 2$ and $a = 1$,

$$\begin{aligned} g'(1) &= \lim_{x \to 1} \frac{g(x) - g(1)}{x - 1} = \lim_{x \to 1} \frac{(x^4 - 2) - (-1)}{x - 1} = \lim_{x \to 1} \frac{x^4 - 1}{x - 1} = \lim_{x \to 1} \frac{(x^2 + 1)(x^2 - 1)}{x - 1} \\ &= \lim_{x \to 1} \frac{(x^2 + 1)(x + 1)(x - 1)}{x - 1} = \lim_{x \to 1} [(x^2 + 1)(x + 1)] = 2(2) = 4 \end{aligned}$$

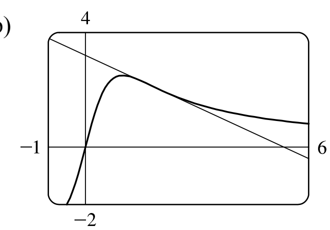
Tangent line: $y - (-1) = 4(x - 1) \quad \Leftrightarrow \quad y + 1 = 4x - 4 \quad \Leftrightarrow \quad y = 4x - 5$

29. (a) Using (4) with $F(x) = 5x/(1 + x^2)$ and the point $(2, 2)$, we have

$$\begin{aligned} F'(2) &= \lim_{h \to 0} \frac{F(2+h) - F(2)}{h} = \lim_{h \to 0} \frac{\dfrac{5(2+h)}{1 + (2+h)^2} - 2}{h} \\ &= \lim_{h \to 0} \frac{\dfrac{5h + 10}{h^2 + 4h + 5} - 2}{h} = \lim_{h \to 0} \frac{\dfrac{5h + 10 - 2(h^2 + 4h + 5)}{h^2 + 4h + 5}}{h} \\ &= \lim_{h \to 0} \frac{-2h^2 - 3h}{h(h^2 + 4h + 5)} = \lim_{h \to 0} \frac{h(-2h - 3)}{h(h^2 + 4h + 5)} = \lim_{h \to 0} \frac{-2h - 3}{h^2 + 4h + 5} = \frac{-3}{5} \end{aligned}$$

So an equation of the tangent line at $(2, 2)$ is $y - 2 = -\frac{3}{5}(x - 2)$ or $y = -\frac{3}{5}x + \frac{16}{5}$.

(b)

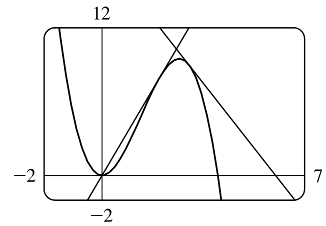


30. (a) Using (4) with $G(x) = 4x^2 - x^3$, we have

$$G'(a) = \lim_{h\to 0} \frac{G(a+h) - G(a)}{h} = \lim_{h\to 0} \frac{[4(a+h)^2 - (a+h)^3] - (4a^2 - a^3)}{h}$$
$$= \lim_{h\to 0} \frac{4a^2 + 8ah + 4h^2 - (a^3 + 3a^2h + 3ah^2 + h^3) - 4a^2 + a^3}{h}$$
$$= \lim_{h\to 0} \frac{8ah + 4h^2 - 3a^2h - 3ah^2 - h^3}{h} = \lim_{h\to 0} \frac{h(8a + 4h - 3a^2 - 3ah - h^2)}{h}$$
$$= \lim_{h\to 0}(8a + 4h - 3a^2 - 3ah - h^2) = 8a - 3a^2$$

At the point $(2, 8)$, $G'(2) = 16 - 12 = 4$, and an equation of the tangent line is $y - 8 = 4(x - 2)$, or $y = 4x$. At the point $(3, 9)$, $G'(3) = 24 - 27 = -3$, and an equation of the tangent line is $y - 9 = -3(x - 3)$, or $y = -3x + 18$.

(b)



31. Use (4) with $f(x) = 3x^2 - 4x + 1$.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{[3(a+h)^2 - 4(a+h) + 1] - (3a^2 - 4a + 1)]}{h}$$
$$= \lim_{h\to 0} \frac{3a^2 + 6ah + 3h^2 - 4a - 4h + 1 - 3a^2 + 4a - 1}{h} = \lim_{h\to 0} \frac{6ah + 3h^2 - 4h}{h}$$
$$= \lim_{h\to 0} \frac{h(6a + 3h - 4)}{h} = \lim_{h\to 0}(6a + 3h - 4) = 6a - 4$$

32. Use (4) with $f(t) = 2t^3 + t$.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{[2(a+h)^3 + (a+h)] - (2a^3 + a)}{h}$$
$$= \lim_{h\to 0} \frac{2a^3 + 6a^2h + 6ah^2 + 2h^3 + a + h - 2a^3 - a}{h} = \lim_{h\to 0} \frac{6a^2h + 6ah^2 + 2h^3 + h}{h}$$
$$= \lim_{h\to 0} \frac{h(6a^2 + 6ah + 2h^2 + 1)}{h} = \lim_{h\to 0}(6a^2 + 6ah + 2h^2 + 1) = 6a^2 + 1$$

33. Use (4) with $f(t) = (2t + 1)/(t + 3)$.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{\dfrac{2(a+h)+1}{(a+h)+3} - \dfrac{2a+1}{a+3}}{h}$$
$$= \lim_{h\to 0} \frac{(2a + 2h + 1)(a + 3) - (2a + 1)(a + h + 3)}{h(a + h + 3)(a + 3)}$$
$$= \lim_{h\to 0} \frac{(2a^2 + 6a + 2ah + 6h + a + 3) - (2a^2 + 2ah + 6a + a + h + 3)}{h(a + h + 3)(a + 3)}$$
$$= \lim_{h\to 0} \frac{5h}{h(a + h + 3)(a + 3)} = \lim_{h\to 0} \frac{5}{(a + h + 3)(a + 3)} = \frac{5}{(a + 3)^2}$$

34. Use (4) with $f(x) = x^{-2} = 1/x^2$.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{\dfrac{1}{(a+h)^2} - \dfrac{1}{a^2}}{h} = \lim_{h\to 0} \frac{\dfrac{a^2-(a+h)^2}{a^2(a+h)^2}}{h}$$

$$= \lim_{h\to 0} \frac{a^2-(a^2+2ah+h^2)}{ha^2(a+h)^2} = \lim_{h\to 0} \frac{-2ah-h^2}{ha^2(a+h)^2} = \lim_{h\to 0} \frac{h(-2a-h)}{ha^2(a+h)^2}$$

$$= \lim_{h\to 0} \frac{-2a-h}{a^2(a+h)^2} = \frac{-2a}{a^2(a^2)} = \frac{-2}{a^3}$$

35. Use (4) with $f(x) = \sqrt{1-2x}$.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{\sqrt{1-2(a+h)} - \sqrt{1-2a}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{1-2(a+h)} - \sqrt{1-2a}}{h} \cdot \frac{\sqrt{1-2(a+h)} + \sqrt{1-2a}}{\sqrt{1-2(a+h)} + \sqrt{1-2a}}$$

$$= \lim_{h\to 0} \frac{\left(\sqrt{1-2(a+h)}\right)^2 - \left(\sqrt{1-2a}\right)^2}{h\left(\sqrt{1-2(a+h)} + \sqrt{1-2a}\right)} = \lim_{h\to 0} \frac{(1-2a-2h)-(1-2a)}{h\left(\sqrt{1-2(a+h)} + \sqrt{1-2a}\right)}$$

$$= \lim_{h\to 0} \frac{-2h}{h\left(\sqrt{1-2(a+h)} + \sqrt{1-2a}\right)} = \lim_{h\to 0} \frac{-2}{\sqrt{1-2(a+h)} + \sqrt{1-2a}}$$

$$= \frac{-2}{\sqrt{1-2a} + \sqrt{1-2a}} = \frac{-2}{2\sqrt{1-2a}} = \frac{-1}{\sqrt{1-2a}}$$

36. Use (4) with $f(x) = \dfrac{4}{\sqrt{1-x}}$.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{\dfrac{4}{\sqrt{1-(a+h)}} - \dfrac{4}{\sqrt{1-a}}}{h}$$

$$= 4\lim_{h\to 0} \frac{\dfrac{\sqrt{1-a} - \sqrt{1-a-h}}{\sqrt{1-a-h}\,\sqrt{1-a}}}{h} = 4\lim_{h\to 0} \frac{\sqrt{1-a} - \sqrt{1-a-h}}{h\sqrt{1-a-h}\,\sqrt{1-a}}$$

$$= 4\lim_{h\to 0} \frac{\sqrt{1-a} - \sqrt{1-a-h}}{h\sqrt{1-a-h}\,\sqrt{1-a}} \cdot \frac{\sqrt{1-a} + \sqrt{1-a-h}}{\sqrt{1-a} + \sqrt{1-a-h}} = 4\lim_{h\to 0} \frac{(\sqrt{1-a})^2 - (\sqrt{1-a-h})^2}{h\sqrt{1-a-h}\,\sqrt{1-a}(\sqrt{1-a} + \sqrt{1-a-h})}$$

$$= 4\lim_{h\to 0} \frac{(1-a)-(1-a-h)}{h\sqrt{1-a-h}\,\sqrt{1-a}(\sqrt{1-a} + \sqrt{1-a-h})} = 4\lim_{h\to 0} \frac{h}{h\sqrt{1-a-h}\,\sqrt{1-a}(\sqrt{1-a} + \sqrt{1-a-h})}$$

$$= 4\lim_{h\to 0} \frac{1}{\sqrt{1-a-h}\,\sqrt{1-a}(\sqrt{1-a} + \sqrt{1-a-h})} = 4 \cdot \frac{1}{\sqrt{1-a}\,\sqrt{1-a}(\sqrt{1-a} + \sqrt{1-a})}$$

$$= \frac{4}{(1-a)(2\sqrt{1-a})} = \frac{2}{(1-a)^1(1-a)^{1/2}} = \frac{2}{(1-a)^{3/2}}$$

37. By (4), $\displaystyle\lim_{h\to 0} \frac{\sqrt{9+h}-3}{h} = f'(9)$, where $f(x) = \sqrt{x}$ and $a = 9$.

38. By (4), $\lim\limits_{h\to 0} \dfrac{2^{3+h}-8}{h} = f'(3)$, where $f(x) = 2^x$ and $a = 3$.

39. By Equation 5, $\lim\limits_{x\to 2} \dfrac{x^6-64}{x-2} = f'(2)$, where $f(x) = x^6$ and $a = 2$.

40. By Equation 5, $\lim\limits_{x\to 1/4} \dfrac{\dfrac{1}{x}-4}{x-\dfrac{1}{4}} = f'(4)$, where $f(x) = \dfrac{1}{x}$ and $a = \dfrac{1}{4}$.

41. By (4), $\lim\limits_{h\to 0} \dfrac{\cos(\pi+h)+1}{h} = f'(\pi)$, where $f(x) = \cos x$ and $a = \pi$.

Or: By (4), $\lim\limits_{h\to 0} \dfrac{\cos(\pi+h)+1}{h} = f'(0)$, where $f(x) = \cos(\pi + x)$ and $a = 0$.

42. By Equation 5, $\lim\limits_{\theta\to \pi/6} \dfrac{\sin\theta - \frac{1}{2}}{\theta - \dfrac{\pi}{6}} = f'\left(\dfrac{\pi}{6}\right)$, where $f(\theta) = \sin\theta$ and $a = \dfrac{\pi}{6}$.

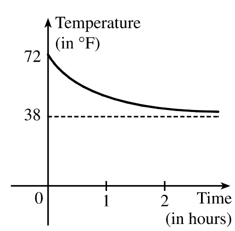
43. $$v(4) = f'(4) = \lim_{h\to 0} \frac{f(4+h)-f(4)}{h} = \lim_{h\to 0} \frac{\left[80(4+h)-6(4+h)^2\right]-\left[80(4)-6(4)^2\right]}{h}$$
$$= \lim_{h\to 0} \frac{(320+80h-96-48h-6h^2)-(320-96)}{h} = \lim_{h\to 0} \frac{32h-6h^2}{h}$$
$$= \lim_{h\to 0} \frac{h(32-6h)}{h} = \lim_{h\to 0}(32-6h) = 32 \text{ m/s}$$

The speed when $t = 4$ is $|32| = 32$ m/s.

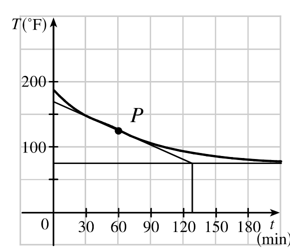
44. $$v(4) = f'(4) = \lim_{h\to 0} \frac{f(4+h)-f(4)}{h} = \lim_{h\to 0} \frac{\left(10+\dfrac{45}{4+h+1}\right)-\left(10+\dfrac{45}{4+1}\right)}{h} = \lim_{h\to 0} \frac{\dfrac{45}{5+h}-9}{h}$$
$$= \lim_{h\to 0} \frac{45-9(5+h)}{h(5+h)} = \lim_{h\to 0} \frac{-9h}{h(5+h)} = \lim_{h\to 0} \frac{-9}{5+h} = -\frac{9}{5} \text{ m/s.}$$

The speed when $t = 4$ is $\left|-\frac{9}{5}\right| = \frac{9}{5}$ m/s.

45. The sketch shows the graph for a room temperature of 72° and a refrigerator temperature of 38°. The initial rate of change is greater in magnitude than the rate of change after an hour.



46. The slope of the tangent (that is, the rate of change of temperature with respect to time) at $t = 1$ h seems to be about $\dfrac{75 - 168}{132 - 0} \approx -0.7\,°\text{F/min}$.



47. (a) (i) $[1.0, 2.0]$: $\dfrac{C(2)-C(1)}{2-1} = \dfrac{0.018-0.033}{1} = -0.015\ \dfrac{\text{g/dL}}{\text{h}}$

(ii) $[1.5, 2.0]$: $\dfrac{C(2)-C(1.5)}{2-1.5} = \dfrac{0.018-0.024}{0.5} = \dfrac{-0.006}{0.5} = -0.012\ \dfrac{\text{g/dL}}{\text{h}}$

(iii) $[2.0, 2.5]$: $\dfrac{C(2.5)-C(2)}{2.5-2} = \dfrac{0.012-0.018}{0.5} = \dfrac{-0.006}{0.5} = -0.012\ \dfrac{\text{g/dL}}{\text{h}}$

(iv) $[2.0, 3.0]$: $\dfrac{C(3)-C(2)}{3-2} = \dfrac{0.007-0.018}{1} = -0.011\ \dfrac{\text{g/dL}}{\text{h}}$

(b) We estimate the instantaneous rate of change at $t = 2$ by averaging the average rates of change for $[1.5, 2.0]$ and $[2.0, 2.5]$: $\dfrac{-0.012+(-0.012)}{2} = -0.012\ \dfrac{\text{g/dL}}{\text{h}}$. After 2 hours, the BAC is decreasing at a rate of $0.012\ (\text{g/dL})/\text{h}$.

48. (a) (i) $[2006, 2008]$: $\dfrac{N(2008)-N(2006)}{2008-2006} = \dfrac{16{,}680-12{,}440}{2} = \dfrac{4240}{2} = 2120$ locations/year

(ii) $[2008, 2010]$: $\dfrac{N(2010)-N(2008)}{2010-2008} = \dfrac{16{,}858-16{,}680}{2} = \dfrac{178}{2} = 89$ locations/year.

The rate of growth decreased over the period from 2006 to 2010.

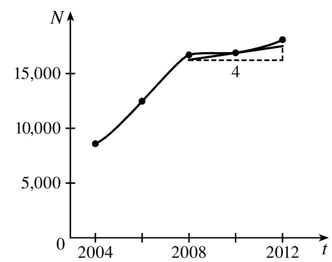
(b) $[2010, 2012]$: $\dfrac{N(2012)-N(2010)}{2012-2010} = \dfrac{18{,}066-16{,}858}{2} = \dfrac{1208}{2} = 604$ locations/year.

Using that value and the value from part (a)(ii), we have $\dfrac{89+604}{2} = \dfrac{693}{2} = 346.5$ locations/year.

(c) The tangent segment has endpoints $(2008, 16{,}250)$ and $(2012, 17{,}500)$.

An estimate of the instantaneous rate of growth in 2010 is

$\dfrac{17{,}500-16{,}250}{2012-2008} = \dfrac{1250}{4} = 312.5$ locations/year.



49. (a) $[1990, 2005]$: $\dfrac{84{,}077-66{,}533}{2005-1990} = \dfrac{17{,}544}{15} = 1169.6$ thousands of barrels per day per year. This means that oil consumption rose by an average of 1169.6 thousands of barrels per day each year from 1990 to 2005.

(b) $[1995, 2000]$: $\dfrac{76{,}784-70{,}099}{2000-1995} = \dfrac{6685}{5} = 1337$

$[2000, 2005]$: $\dfrac{84{,}077-76{,}784}{2005-2000} = \dfrac{7293}{5} = 1458.6$

An estimate of the instantaneous rate of change in 2000 is $\frac{1}{2}(1337 + 1458.6) = 1397.8$ thousands of barrels per day per year.

50. (a) (i) $[4, 11]$: $\dfrac{V(11)-V(4)}{11-4} = \dfrac{9.4-53}{7} = \dfrac{-43.6}{7} \approx -6.23\ \dfrac{\text{RNA copies/mL}}{\text{day}}$

(ii) $[8, 11]$: $\dfrac{V(11)-V(8)}{11-8} = \dfrac{9.4-18}{3} = \dfrac{-8.6}{3} \approx -2.87\ \dfrac{\text{RNA copies/mL}}{\text{day}}$

(iii) $[11, 15]$: $\dfrac{V(15) - V(11)}{15 - 11} = \dfrac{5.2 - 9.4}{4} = \dfrac{-4.2}{4} = -1.05 \ \dfrac{\text{RNA copies/mL}}{\text{day}}$

(iv) $[11, 22]$: $\dfrac{V(22) - V(11)}{22 - 11} = \dfrac{3.6 - 9.4}{11} = \dfrac{-5.8}{11} \approx -0.53 \ \dfrac{\text{RNA copies/mL}}{\text{day}}$

(b) An estimate of $V'(11)$ is the average of the answers from part (a)(ii) and (iii).

$V'(11) \approx \frac{1}{2}\left[-2.87 + (-1.05)\right] = -1.96 \ \dfrac{\text{RNA copies/mL}}{\text{day}}$.

$V'(11)$ measures the instantaneous rate of change of patient 303's viral load 11 days after ABT-538 treatment began.

51. (a) (i) $\dfrac{\Delta C}{\Delta x} = \dfrac{C(105) - C(100)}{105 - 100} = \dfrac{6601.25 - 6500}{5} = \$20.25/\text{unit}.$

(ii) $\dfrac{\Delta C}{\Delta x} = \dfrac{C(101) - C(100)}{101 - 100} = \dfrac{6520.05 - 6500}{1} = \$20.05/\text{unit}.$

(b) $$\frac{C(100 + h) - C(100)}{h} = \frac{\left[5000 + 10(100 + h) + 0.05(100 + h)^2\right] - 6500}{h} = \frac{20h + 0.05h^2}{h}$$
$$= 20 + 0.05h,\ h \neq 0$$

So the instantaneous rate of change is $\displaystyle\lim_{h \to 0} \frac{C(100 + h) - C(100)}{h} = \lim_{h \to 0}(20 + 0.05h) = \$20/\text{unit}.$

52. $$\Delta V = V(t + h) - V(t) = 100{,}000\left(1 - \frac{t + h}{60}\right)^2 - 100{,}000\left(1 - \frac{t}{60}\right)^2$$
$$= 100{,}000\left[\left(1 - \frac{t + h}{30} + \frac{(t + h)^2}{3600}\right) - \left(1 - \frac{t}{30} + \frac{t^2}{3600}\right)\right] = 100{,}000\left(-\frac{h}{30} + \frac{2th}{3600} + \frac{h^2}{3600}\right)$$
$$= \frac{100{,}000}{3600} h\,(-120 + 2t + h) = \frac{250}{9} h\,(-120 + 2t + h)$$

Dividing ΔV by h and then letting $h \to 0$, we see that the instantaneous rate of change is $\frac{500}{9}\,(t - 60)$ gal/min.

t	Flow rate (gal/min)	Water remaining $V(t)$ (gal)
0	$-3333.\overline{3}$	$100,000$
10	$-2777.\overline{7}$	$69,444.\overline{4}$
20	$-2222.\overline{2}$	$44,444.\overline{4}$
30	$-1666.\overline{6}$	$25,000$
40	$-1111.\overline{1}$	$11,111.\overline{1}$
50	$-555.\overline{5}$	$2,777.\overline{7}$
60	0	0

The magnitude of the flow rate is greatest at the beginning and gradually decreases to 0.

53. (a) $f'(x)$ is the rate of change of the production cost with respect to the number of ounces of gold produced. Its units are dollars per ounce.

(b) After 800 ounces of gold have been produced, the rate at which the production cost is increasing is \$17/ounce. So the cost of producing the 800th (or 801st) ounce is about \$17.

(c) In the short term, the values of $f'(x)$ will decrease because more efficient use is made of start-up costs as x increases. But eventually $f'(x)$ might increase due to large-scale operations.

54. (a) $f'(5)$ is the rate of growth of the bacteria population when $t = 5$ hours. Its units are bacteria per hour.

(b) With unlimited space and nutrients, f' should increase as t increases; so $f'(5) < f'(10)$. If the supply of nutrients is limited, the growth rate slows down at some point in time, and the opposite may be true.

55. (a) $H'(58)$ is the rate at which the daily heating cost changes with respect to temperature when the outside temperature is $58\,^\circ\mathrm{F}$. The units are dollars/$\,^\circ\mathrm{F}$.

(b) If the outside temperature increases, the building should require less heating, so we would expect $H'(58)$ to be negative.

56. (a) $f'(8)$ is the rate of change of the quantity of coffee sold with respect to the price per pound when the price is \$8 per pound. The units for $f'(8)$ are pounds/(dollars/pound).

(b) $f'(8)$ is negative since the quantity of coffee sold will decrease as the price charged for it increases. People are generally less willing to buy a product when its price increases.

57. (a) $S'(T)$ is the rate at which the oxygen solubility changes with respect to the water temperature. Its units are $(\mathrm{mg/L})/^\circ\mathrm{C}$.

(b) For $T = 16^\circ\mathrm{C}$, it appears that the tangent line to the curve goes through the points $(0, 14)$ and $(32, 6)$. So $S'(16) \approx \dfrac{6-14}{32-0} = -\dfrac{8}{32} = -0.25\ (\mathrm{mg/L})/^\circ\mathrm{C}$. This means that as the temperature increases past $16^\circ\mathrm{C}$, the oxygen solubility is decreasing at a rate of $0.25\ (\mathrm{mg/L})/^\circ\mathrm{C}$.

58. (a) $S'(T)$ is the rate of change of the maximum sustainable speed of Coho salmon with respect to the temperature. Its units are $(\mathrm{cm/s})/^\circ\mathrm{C}$.

(b) For $T = 15^\circ\mathrm{C}$, it appears the tangent line to the curve goes through the points $(10, 25)$ and $(20, 32)$. So $S'(15) \approx \dfrac{32-25}{20-10} = 0.7\ (\mathrm{cm/s})/^\circ\mathrm{C}$. This tells us that at $T = 15^\circ\mathrm{C}$, the maximum sustainable speed of Coho salmon is changing at a rate of $0.7\ (\mathrm{cm/s})/^\circ\mathrm{C}$. In a similar fashion for $T = 25^\circ\mathrm{C}$, we can use the points $(20, 35)$ and $(25, 25)$ to obtain $S'(25) \approx \dfrac{25-35}{25-20} = -2\ (\mathrm{cm/s})/^\circ\mathrm{C}$. As it gets warmer than $20^\circ\mathrm{C}$, the maximum sustainable speed decreases rapidly.

59. Since $f(x) = x\sin(1/x)$ when $x \neq 0$ and $f(0) = 0$, we have

$f'(0) = \lim\limits_{h\to 0} \dfrac{f(0+h)-f(0)}{h} = \lim\limits_{h\to 0} \dfrac{h\sin(1/h)-0}{h} = \lim\limits_{h\to 0} \sin(1/h)$. This limit does not exist since $\sin(1/h)$ takes the values -1 and 1 on any interval containing 0. (Compare with Example 1.5.4.)

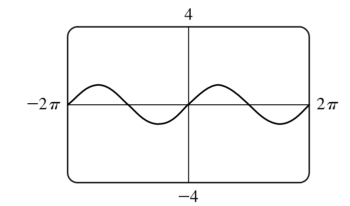
60. Since $f(x) = x^2\sin(1/x)$ when $x \neq 0$ and $f(0) = 0$, we have

$f'(0) = \lim\limits_{h\to 0} \dfrac{f(0+h)-f(0)}{h} = \lim\limits_{h\to 0} \dfrac{h^2\sin(1/h)-0}{h} = \lim\limits_{h\to 0} h\sin(1/h)$. Since $-1 \leq \sin\dfrac{1}{h} \leq 1$, we have

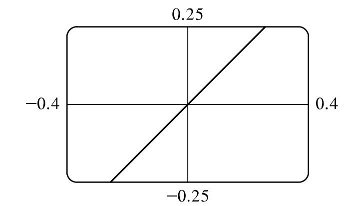
$-|h| \leq |h|\sin\dfrac{1}{h} \leq |h| \quad\Rightarrow\quad -|h| \leq h\sin\dfrac{1}{h} \leq |h|$. Because $\lim\limits_{h\to 0}(-|h|) = 0$ and $\lim\limits_{h\to 0}|h| = 0$, we know that

$\lim\limits_{h\to 0}\left(h\sin\dfrac{1}{h}\right) = 0$ by the Squeeze Theorem. Thus, $f'(0) = 0$.

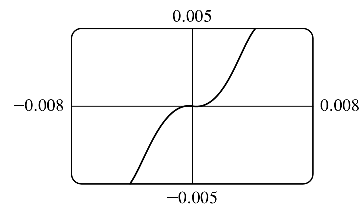
61. (a) The slope at the origin appears to be 1.



(b) The slope at the origin still appears to be 1.



(c) Yes, the slope at the origin now appears to be 0.



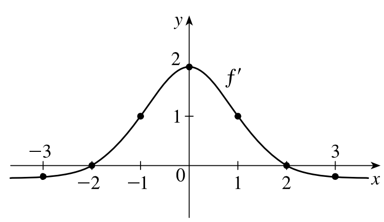
2.2 The Derivative as a Function

1. It appears that f is an odd function, so f' will be an even function—that is, $f'(-a) = f'(a)$.

(a) $f'(-3) \approx -0.2$

(b) $f'(-2) \approx 0$ (c) $f'(-1) \approx 1$ (d) $f'(0) \approx 2$

(e) $f'(1) \approx 1$ (f) $f'(2) \approx 0$ (g) $f'(3) \approx -0.2$



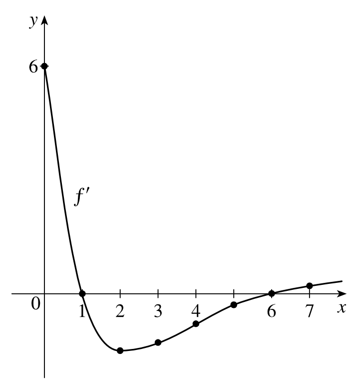
2. Your answers may vary depending on your estimates.

(a) *Note:* By estimating the slopes of tangent lines on the graph of f, it appears that $f'(0) \approx 6$.

(b) $f'(1) \approx 0$

(c) $f'(2) \approx -1.5$ (d) $f'(3) \approx -1.3$ (e) $f'(4) \approx -0.8$

(f) $f'(5) \approx -0.3$ (g) $f'(6) \approx 0$ (h) $f'(7) \approx 0.2$



3. (a)$' =$ II, since from left to right, the slopes of the tangents to graph (a) start out negative, become 0, then positive, then 0, then negative again. The actual function values in graph II follow the same pattern.

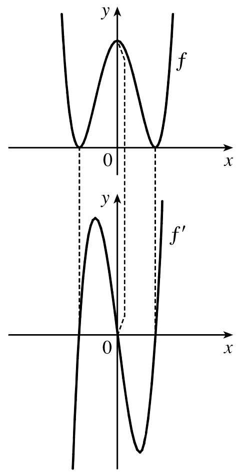
(b)$' =$ IV, since from left to right, the slopes of the tangents to graph (b) start out at a fixed positive quantity, then suddenly become negative, then positive again. The discontinuities in graph IV indicate sudden changes in the slopes of the tangents.

(c)$' =$ I, since the slopes of the tangents to graph (c) are negative for $x < 0$ and positive for $x > 0$, as are the function values of graph I.

(d)$'$ = III, since from left to right, the slopes of the tangents to graph (d) are positive, then 0, then negative, then 0, then positive, then 0, then negative again, and the function values in graph III follow the same pattern.

Hints for Exercises 4 – 11: First plot x-intercepts on the graph of f' for any horizontal tangents on the graph of f. Look for any corners on the graph of f — there will be a discontinuity on the graph of f'. On any interval where f has a tangent with positive (or negative) slope, the graph of f' will be positive (or negative). If the graph of the function is linear, the graph of f' will be a horizontal line.

4.



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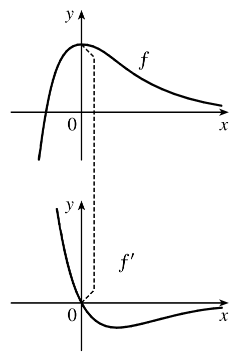

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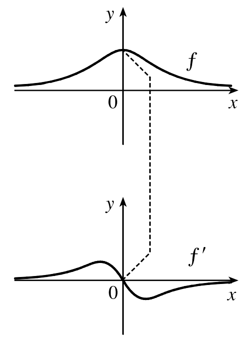

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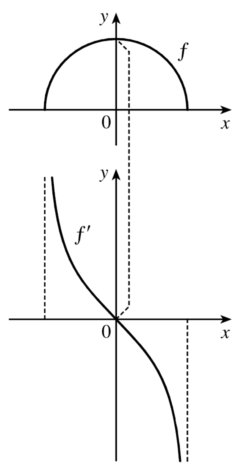

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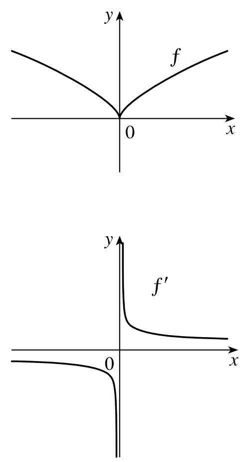

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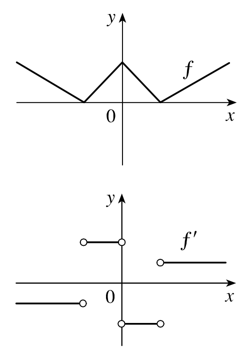

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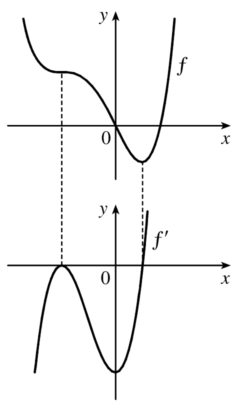
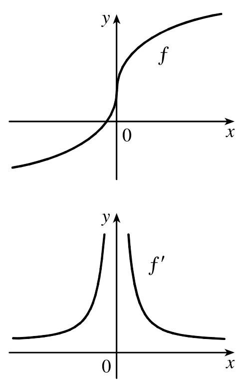
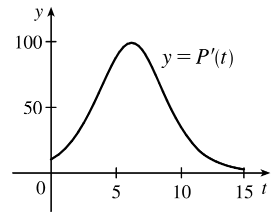


11.

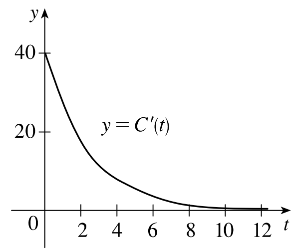


12. The slopes of the tangent lines on the graph of $y = P(t)$ are always positive, so the y-values of $y = P'(t)$ are always positive. These values start out relatively small and keep increasing, reaching a maximum at about $t = 6$. Then the y-values of $y = P'(t)$ decrease and get close to zero. The graph of P' tells us that the yeast culture grows most rapidly after 6 hours and then the growth rate declines.



13. (a) $C'(t)$ is the instantaneous rate of change of percentage of full capacity with respect to elapsed time in hours.

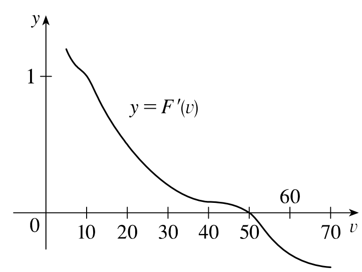
(b) The graph of $C'(t)$ tells us that the rate of change of percentage of full capacity is decreasing and approaching 0.



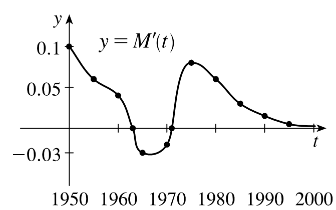
14. (a) $F'(v)$ is the instantaneous rate of change of fuel economy with respect to speed.

(b) Graphs will vary depending on estimates of F', but will change from positive to negative at about $v = 50$.

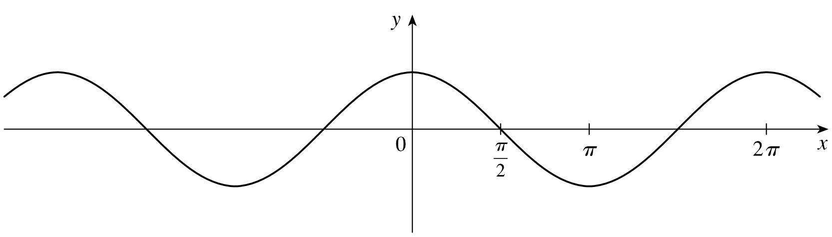
(c) To save on gas, drive at the speed where F is a maximum and F' is 0, which is about 50 mi/h.



15. It appears that there are horizontal tangents on the graph of M for $t = 1963$ and $t = 1971$. Thus, there are zeros for those values of t on the graph of M'. The derivative is negative for the years 1963 to 1971.



16.

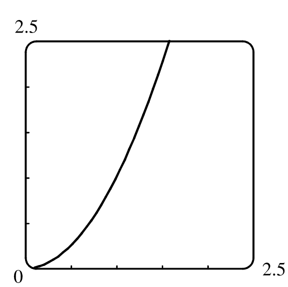


The graph of the derivative looks like the graph of the cosine function.

17. (a) By zooming in, we estimate that $f'(0) = 0$, $f'\left(\frac{1}{2}\right) = 1$, $f'(1) = 2$, and $f'(2) = 4$.

(b) By symmetry, $f'(-x) = -f'(x)$. So $f'\left(-\frac{1}{2}\right) = -1$, $f'(-1) = -2$, and $f'(-2) = -4$.

(c) It appears that $f'(x)$ is twice the value of x, so we guess that $f'(x) = 2x$.



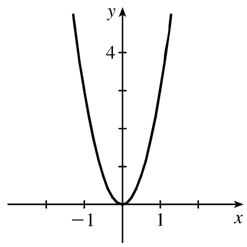
(d) $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \dfrac{(x+h)^2 - x^2}{h}$

$$= \lim_{h \to 0} \frac{\left(x^2 + 2hx + h^2\right) - x^2}{h} = \lim_{h \to 0} \frac{2hx + h^2}{h} = \lim_{h \to 0} \frac{h(2x+h)}{h} = \lim_{h \to 0} (2x + h) = 2x$$

18. (a) By zooming in, we estimate that $f'(0) = 0$, $f'\left(\frac{1}{2}\right) \approx 0.75$, $f'(1) \approx 3$, $f'(2) \approx 12$, and $f'(3) \approx 27$.

(b) By symmetry, $f'(-x) = f'(x)$. So $f'\left(-\frac{1}{2}\right) \approx 0.75$, $f'(-1) \approx 3$, $f'(-2) \approx 12$, and $f'(-3) \approx 27$.

(c)



(d) Since $f'(0) = 0$, it appears that f' may have the form $f'(x) = ax^2$. Using $f'(1) = 3$, we have $a = 3$, so $f'(x) = 3x^2$.

(e) $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \dfrac{(x+h)^3 - x^3}{h} = \lim_{h \to 0} \dfrac{(x^3 + 3x^2h + 3xh^2 + h^3) - x^3}{h}$

$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h} = \lim_{h \to 0} \frac{h(3x^2 + 3xh + h^2)}{h} = \lim_{h \to 0} (3x^2 + 3xh + h^2) = 3x^2$$

19. $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \dfrac{[3(x+h) - 8] - (3x - 8)}{h} = \lim_{h \to 0} \dfrac{3x + 3h - 8 - 3x + 8}{h}$

$$= \lim_{h \to 0} \frac{3h}{h} = \lim_{h \to 0} 3 = 3$$

Domain of f = domain of $f' = \mathbb{R}$.

20. $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \dfrac{[m(x+h) + b] - (mx + b)}{h} = \lim_{h \to 0} \dfrac{mx + mh + b - mx - b}{h}$

$$= \lim_{h \to 0} \frac{mh}{h} = \lim_{h \to 0} m = m$$

Domain of f = domain of $f' = \mathbb{R}$.

21. $f'(t) = \lim_{h \to 0} \dfrac{f(t+h) - f(t)}{h} = \lim_{h \to 0} \dfrac{\left[2.5(t+h)^2 + 6(t+h)\right] - \left(2.5t^2 + 6t\right)}{h}$

$$= \lim_{h \to 0} \frac{2.5(t^2 + 2th + h^2) + 6t + 6h - 2.5t^2 - 6t}{h} = \lim_{h \to 0} \frac{2.5t^2 + 5th + 2.5h^2 + 6h - 2.5t^2}{h}$$

$$= \lim_{h \to 0} \frac{5th + 2.5h^2 + 6h}{h} = \lim_{h \to 0} \frac{h\,(5t + 2.5h + 6)}{h} = \lim_{h \to 0} (5t + 2.5h + 6)$$

$$= 5t + 6$$

Domain of f = domain of $f' = \mathbb{R}$.

22. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{\left[4+8(x+h)-5(x+h)^2\right]-(4+8x-5x^2)}{h}$

$= \lim_{h\to 0} \dfrac{4+8x+8h-5(x^2+2xh+h^2)-4-8x+5x^2}{h} = \lim_{h\to 0} \dfrac{8h-5x^2-10xh-5h^2+5x^2}{h}$

$= \lim_{h\to 0} \dfrac{8h-10xh-5h^2}{h} = \lim_{h\to 0} \dfrac{h(8-10x-5h)}{h} = \lim_{h\to 0} (8-10x-5h)$

$= 8-10x$

Domain of f = domain of $f' = \mathbb{R}$.

23. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{[(x+h)^2-2(x+h)^3]-(x^2-2x^3)}{h}$

$= \lim_{h\to 0} \dfrac{x^2+2xh+h^2-2x^3-6x^2h-6xh^2-2h^3-x^2+2x^3}{h}$

$= \lim_{h\to 0} \dfrac{2xh+h^2-6x^2h-6xh^2-2h^3}{h} = \lim_{h\to 0} \dfrac{h(2x+h-6x^2-6xh-2h^2)}{h}$

$= \lim_{h\to 0} (2x+h-6x^2-6xh-2h^2) = 2x-6x^2$

Domain of f = domain of $f' = \mathbb{R}$.

24. $g'(t) = \lim_{h\to 0} \dfrac{g(t+h)-g(t)}{h} = \lim_{h\to 0} \dfrac{\dfrac{1}{\sqrt{t+h}} - \dfrac{1}{\sqrt{t}}}{h} = \lim_{h\to 0} \dfrac{\dfrac{\sqrt{t}-\sqrt{t+h}}{\sqrt{t+h}\,\sqrt{t}}}{h} = \lim_{h\to 0} \left(\dfrac{\sqrt{t}-\sqrt{t+h}}{h\,\sqrt{t+h}\,\sqrt{t}} \cdot \dfrac{\sqrt{t}+\sqrt{t+h}}{\sqrt{t}+\sqrt{t+h}} \right)$

$= \lim_{h\to 0} \dfrac{t-(t+h)}{h\,\sqrt{t+h}\,\sqrt{t}\left(\sqrt{t}+\sqrt{t+h}\right)} = \lim_{h\to 0} \dfrac{-h}{h\,\sqrt{t+h}\,\sqrt{t}\left(\sqrt{t}+\sqrt{t+h}\right)} = \lim_{h\to 0} \dfrac{-1}{\sqrt{t+h}\,\sqrt{t}\left(\sqrt{t}+\sqrt{t+h}\right)}$

$= \dfrac{-1}{\sqrt{t}\,\sqrt{t}\left(\sqrt{t}+\sqrt{t}\right)} = \dfrac{-1}{t\left(2\sqrt{t}\right)} = -\dfrac{1}{2t^{3/2}}$

Domain of g = domain of $g' = (0, \infty)$.

25. $g'(x) = \lim_{h\to 0} \dfrac{g(x+h)-g(x)}{h} = \lim_{h\to 0} \dfrac{\sqrt{9-(x+h)}-\sqrt{9-x}}{h} \left[\dfrac{\sqrt{9-(x+h)}+\sqrt{9-x}}{\sqrt{9-(x+h)}+\sqrt{9-x}} \right]$

$= \lim_{h\to 0} \dfrac{[9-(x+h)]-(9-x)}{h\left[\sqrt{9-(x+h)}+\sqrt{9-x}\right]} = \lim_{h\to 0} \dfrac{-h}{h\left[\sqrt{9-(x+h)}+\sqrt{9-x}\right]}$

$= \lim_{h\to 0} \dfrac{-1}{\sqrt{9-(x+h)}+\sqrt{9-x}} = \dfrac{-1}{2\sqrt{9-x}}$

Domain of $g = (-\infty, 9]$, domain of $g' = (-\infty, 9)$.

26. $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{\dfrac{(x+h)^2-1}{2(x+h)-3} - \dfrac{x^2-1}{2x-3}}{h}$

$$= \lim_{h\to 0} \frac{\dfrac{[(x+h)^2-1](2x-3)-[2(x+h)-3](x^2-1)}{[2(x+h)-3](2x-3)}}{h}$$

$$= \lim_{h\to 0} \frac{(x^2+2xh+h^2-1)(2x-3)-(2x+2h-3)(x^2-1)}{h[2(x+h)-3](2x-3)}$$

$$= \lim_{h\to 0} \frac{(2x^3+4x^2h+2xh^2-2x-3x^2-6xh-3h^2+3)-(2x^3+2x^2h-3x^2-2x-2h+3)}{h(2x+2h-3)(2x-3)}$$

$$= \lim_{h\to 0} \frac{4x^2h+2xh^2-6xh-3h^2-2x^2h+2h}{h(2x+2h-3)(2x-3)} = \lim_{h\to 0} \frac{h(2x^2+2xh-6x-3h+2)}{h(2x+2h-3)(2x-3)}$$

$$= \lim_{h\to 0} \frac{2x^2+2xh-6x-3h+2}{(2x+2h-3)(2x-3)} = \frac{2x^2-6x+2}{(2x-3)^2}$$

Domain of f = domain of $f' = (-\infty, \frac{3}{2}) \cup (\frac{3}{2}, \infty)$.

27. $G'(t) = \lim\limits_{h\to 0} \dfrac{G(t+h)-G(t)}{h} = \lim\limits_{h\to 0} \dfrac{\dfrac{1-2(t+h)}{3+(t+h)} - \dfrac{1-2t}{3+t}}{h}$

$$= \lim_{h\to 0} \frac{\dfrac{[1-2(t+h)](3+t)-[3+(t+h)](1-2t)}{[3+(t+h)](3+t)}}{h}$$

$$= \lim_{h\to 0} \frac{3+t-6t-2t^2-6h-2ht-(3-6t+t-2t^2+h-2ht)}{h[3+(t+h)](3+t)} = \lim_{h\to 0} \frac{-6h-h}{h(3+t+h)(3+t)}$$

$$= \lim_{h\to 0} \frac{-7h}{h(3+t+h)(3+t)} = \lim_{h\to 0} \frac{-7}{(3+t+h)(3+t)} = \frac{-7}{(3+t)^2}$$

Domain of G = domain of $G' = (-\infty, -3) \cup (-3, \infty)$.

28. $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{(x+h)^{3/2}-x^{3/2}}{h} = \lim\limits_{h\to 0} \dfrac{[(x+h)^{3/2}-x^{3/2}][(x+h)^{3/2}+x^{3/2}]}{h\,[(x+h)^{3/2}+x^{3/2}]}$

$$= \lim_{h\to 0} \frac{(x+h)^3-x^3}{h[(x+h)^{3/2}+x^{3/2}]} = \lim_{h\to 0} \frac{x^3+3x^2h+3xh^2+h^3-x^3}{h[(x+h)^{3/2}+x^{3/2}]} = \lim_{h\to 0} \frac{h\left(3x^2+3xh+h^2\right)}{h[(x+h)^{3/2}+x^{3/2}]}$$

$$= \lim_{h\to 0} \frac{3x^2+3xh+h^2}{(x+h)^{3/2}+x^{3/2}} = \frac{3x^2}{2x^{3/2}} = \tfrac{3}{2}x^{1/2}$$

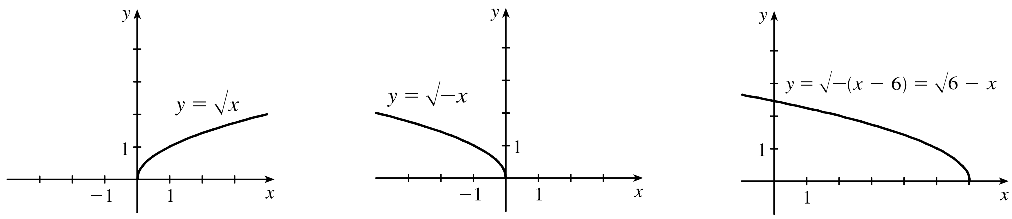
Domain of f = domain of $f' = [0, \infty)$. Strictly speaking, the domain of f' is $(0, \infty)$ because the limit that defines $f'(0)$ does not exist (as a two-sided limit). But the right-hand derivative (in the sense of Exercise 62) does exist at 0, so in that sense one could regard the domain of f' to be $[0, \infty)$.

29. $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{(x+h)^4-x^4}{h} = \lim\limits_{h\to 0} \dfrac{\left(x^4+4x^3h+6x^2h^2+4xh^3+h^4\right)-x^4}{h}$

$$= \lim_{h\to 0} \frac{4x^3h+6x^2h^2+4xh^3+h^4}{h} = \lim_{h\to 0} \left(4x^3+6x^2h+4xh^2+h^3\right) = 4x^3$$

Domain of f = domain of $f' = \mathbb{R}$.

30. (a)

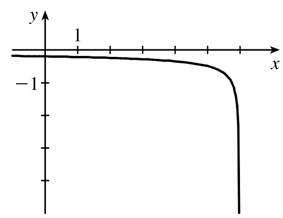


(b) Note that the third graph in part (a) has small negative values for its slope, f'; but as $x \to 6^-$, $f' \to -\infty$. See the graph in part (d).

(c)
$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{\sqrt{6-(x+h)}-\sqrt{6-x}}{h}\left[\frac{\sqrt{6-(x+h)}+\sqrt{6-x}}{\sqrt{6-(x+h)}+\sqrt{6-x}}\right] \\ &= \lim_{h\to 0}\frac{[6-(x+h)]-(6-x)}{h\left[\sqrt{6-(x+h)}+\sqrt{6-x}\right]} = \lim_{h\to 0}\frac{-h}{h\left(\sqrt{6-x-h}+\sqrt{6-x}\right)} \\ &= \lim_{h\to 0}\frac{-1}{\sqrt{6-x-h}+\sqrt{6-x}} = \frac{-1}{2\sqrt{6-x}} \end{aligned}$$

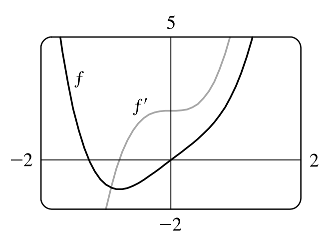
Domain of $f = (-\infty, 6]$, domain of $f' = (-\infty, 6)$.

(d)



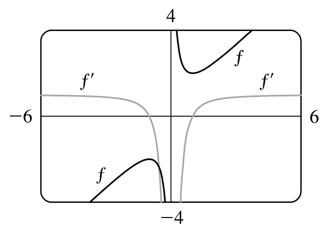
31. (a)
$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{[(x+h)^4+2(x+h)]-(x^4+2x)}{h} \\ &= \lim_{h\to 0}\frac{x^4+4x^3h+6x^2h^2+4xh^3+h^4+2x+2h-x^4-2x}{h} \\ &= \lim_{h\to 0}\frac{4x^3h+6x^2h^2+4xh^3+h^4+2h}{h} = \lim_{h\to 0}\frac{h(4x^3+6x^2h+4xh^2+h^3+2)}{h} \\ &= \lim_{h\to 0}(4x^3+6x^2h+4xh^2+h^3+2) = 4x^3+2 \end{aligned}$$

(b) Notice that $f'(x) = 0$ when f has a horizontal tangent, $f'(x)$ is positive when the tangents have positive slope, and $f'(x)$ is negative when the tangents have negative slope.



32. (a)
$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{[(x+h)+1/(x+h)]-(x+1/x)}{h} = \lim_{h\to 0}\frac{\dfrac{(x+h)^2+1}{x+h}-\dfrac{x^2+1}{x}}{h} \\ &= \lim_{h\to 0}\frac{x[(x+h)^2+1]-(x+h)(x^2+1)}{h(x+h)x} = \lim_{h\to 0}\frac{(x^3+2hx^2+xh^2+x)-(x^3+x+hx^2+h)}{h(x+h)x} \\ &= \lim_{h\to 0}\frac{hx^2+xh^2-h}{h(x+h)x} = \lim_{h\to 0}\frac{h(x^2+xh-1)}{h(x+h)x} = \lim_{h\to 0}\frac{x^2+xh-1}{(x+h)x} = \frac{x^2-1}{x^2},\ \text{or}\ 1-\frac{1}{x^2} \end{aligned}$$

(b) Notice that $f'(x) = 0$ when f has a horizontal tangent, $f'(x)$ is positive when the tangents have positive slope, and $f'(x)$ is negative when the tangents have negative slope. Both functions are discontinuous at $x = 0$.



33. (a) $U'(t)$ is the rate at which the unemployment rate is changing with respect to time. Its units are percent unemployed per year.

(b) To find $U'(t)$, we use $\lim\limits_{h\to 0} \dfrac{U(t+h)-U(t)}{h} \approx \dfrac{U(t+h)-U(t)}{h}$ for small values of h.

For 2003: $U'(2003) \approx \dfrac{U(2004)-U(2003)}{2004-2003} = \dfrac{5.5-6.0}{1} = -0.5$

For 2004: We estimate $U'(2004)$ by using $h=-1$ and $h=1$, and then average the two results to obtain a final estimate.

$h=-1 \quad\Rightarrow\quad U'(2004) \approx \dfrac{U(2003)-U(2004)}{2003-2004} = \dfrac{6.0-5.5}{-1} = -0.5;$

$h=1 \quad\Rightarrow\quad U'(2004) \approx \dfrac{U(2005)-U(2004)}{2005-2004} = \dfrac{5.1-5.5}{1} = -0.4.$

So we estimate that $U'(2004) \approx \frac{1}{2}[-0.5+(-0.4)] = -0.45$. Other values for $U'(t)$ are calculated in a similar fashion.

t	2003	2004	2005	2006	2007	2008	2009	2010	2011	2012
$U'(t)$	-0.50	-0.45	-0.45	-0.25	0.60	2.35	1.90	-0.20	-0.75	-0.80

34. (a) $N'(t)$ is the rate at which the number of minimally invasive cosmetic surgery procedures performed in the United States is changing with respect to time. Its units are thousands of surgeries per year.

(b) To find $N'(t)$, we use $\lim\limits_{h\to 0} \dfrac{N(t+h)-N(t)}{h} \approx \dfrac{N(t+h)-N(t)}{h}$ for small values of h.

For 2000: $N'(2000) \approx \dfrac{N(2002)-N(2000)}{2002-2000} = \dfrac{4897-5500}{2} = -301.5$

For 2002: We estimate $N'(2002)$ by using $h=-2$ and $h=2$, and then average the two results to obtain a final estimate.

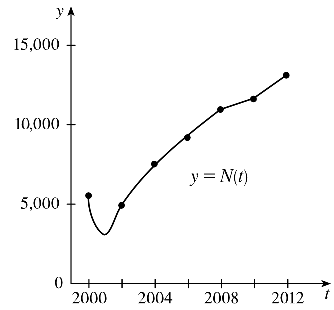
$h=-2 \quad\Rightarrow\quad N'(2002) \approx \dfrac{N(2000)-N(2002)}{2000-2002} = \dfrac{5500-4897}{-2} = -301.5$

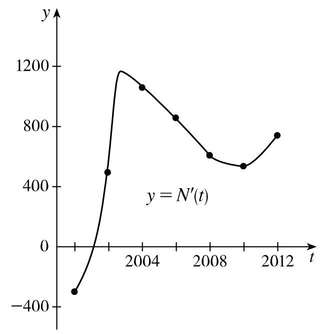
$h=2 \quad\Rightarrow\quad N'(2002) \approx \dfrac{N(2004)-N(2002)}{2004-2002} = \dfrac{7470-4897}{2} = 1286.5$

So we estimate that $N'(2002) \approx \frac{1}{2}[-301.5+1286.5] = 492.5$.

t	2000	2002	2004	2006	2008	2010	2012
$N'(t)$	-301.5	492.5	1060.25	856.75	605.75	534.5	737

(c)

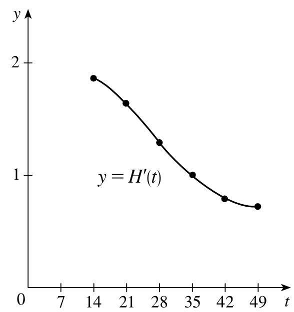




(d) We could get more accurate values for $N'(t)$ by obtaining data for more values of t.

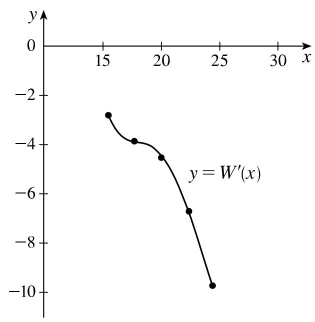
35. As in Exercise 33, we use one-sided difference quotients for the first and last values, and average two difference quotients for all other values.

t	14	21	28	35	42	49
$H(t)$	41	54	64	72	78	83
$H'(t)$	$\frac{13}{7}$	$\frac{23}{14}$	$\frac{18}{14}$	$\frac{14}{14}$	$\frac{11}{14}$	$\frac{5}{7}$



36. As in Exercise 33, we use one-sided difference quotients for the first and last values, and average two difference quotients for all other values. The units for $W'(x)$ are grams per degree (g/°C).

x	15.5	17.7	20.0	22.4	24.4
$W(x)$	37.2	31.0	19.8	9.7	−9.8
$W'(x)$	−2.82	−3.87	−4.53	−6.73	−9.75



37. (a) dP/dt is the rate at which the percentage of the city's electrical power produced by solar panels changes with respect to time t, measured in percentage points per year.

(b) 2 years after January 1, 2000 (January 1, 2002), the percentage of electrical power produced by solar panels was increasing at a rate of 3.5 percentage points per year.

38. dN/dp is the rate at which the number of people who travel by car to another state for a vacation changes with respect to the price of gasoline. If the price of gasoline goes up, we would expect fewer people to travel, so we would expect dN/dp to be negative.

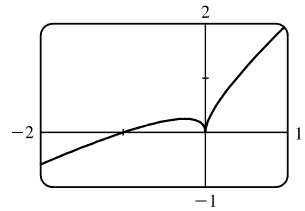
39. f is not differentiable at $x = -4$, because the graph has a corner there, and at $x = 0$, because there is a discontinuity there.

40. f is not differentiable at $x = -1$, because there is a discontinuity there, and at $x = 2$, because the graph has a corner there.

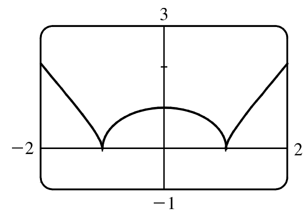
41. f is not differentiable at $x = 1$, because f is not defined there, and at $x = 5$, because the graph has a vertical tangent there.

42. f is not differentiable at $x = -2$ and $x = 3$, because the graph has corners there, and at $x = 1$, because there is a discontinuity there.

43. As we zoom in toward $(-1, 0)$, the curve appears more and more like a straight line, so $f(x) = x + \sqrt{|x|}$ is differentiable at $x = -1$. But no matter how much we zoom in toward the origin, the curve doesn't straighten out—we can't eliminate the sharp point (a cusp). So f is not differentiable at $x = 0$.



44. As we zoom in toward $(0, 1)$, the curve appears more and more like a straight line, so $g(x) = (x^2 - 1)^{2/3}$ is differentiable at $x = 0$. But no matter how much we zoom in toward $(1, 0)$ or $(-1, 0)$, the curve doesn't straighten out—we can't eliminate the sharp point (a cusp). So g is not differentiable at $x = \pm 1$.



45. Call the curve with the positive y-intercept g and the other curve h. Notice that g has a maximum (horizontal tangent) at $x = 0$, but $h \neq 0$, so h cannot be the derivative of g. Also notice that where g is positive, h is increasing. Thus, $h = f$ and $g = f'$. Now $f'(-1)$ is negative since f' is below the x-axis there and $f''(1)$ is positive since f is concave upward at $x = 1$. Therefore, $f''(1)$ is greater than $f'(-1)$.

46. Call the curve with the smallest positive x-intercept g and the other curve h. Notice that where g is positive in the first quadrant, h is increasing. Thus, $h = f$ and $g = f'$. Now $f'(-1)$ is positive since f' is above the x-axis there and $f''(1)$ appears to be zero since f has an inflection point at $x = 1$. Therefore, $f'(1)$ is greater than $f''(-1)$.

47. $a = f, b = f', c = f''$. We can see this because where a has a horizontal tangent, $b = 0$, and where b has a horizontal tangent, $c = 0$. We can immediately see that c can be neither f nor f', since at the points where c has a horizontal tangent, neither a nor b is equal to 0.

48. Where d has horizontal tangents, only c is 0, so $d' = c$. c has negative tangents for $x < 0$ and b is the only graph that is negative for $x < 0$, so $c' = b$. b has positive tangents on $\mathbb{R}$ (except at $x = 0$), and the only graph that is positive on the same domain is a, so $b' = a$. We conclude that $d = f$, $c = f'$, $b = f''$, and $a = f'''$.

49. We can immediately see that a is the graph of the acceleration function, since at the points where a has a horizontal tangent, neither c nor b is equal to 0. Next, we note that $a = 0$ at the point where b has a horizontal tangent, so b must be the graph of the velocity function, and hence, $b' = a$. We conclude that c is the graph of the position function.

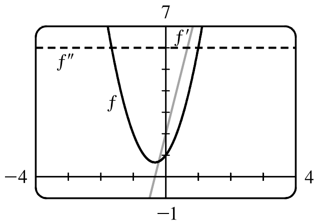
50. a must be the jerk since none of the graphs are 0 at its high and low points. a is 0 where b has a maximum, so $b' = a$. b is 0 where c has a maximum, so $c' = b$. We conclude that d is the position function, c is the velocity, b is the acceleration, and a is the jerk.

51. $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[3(x+h)^2 + 2(x+h) + 1] - (3x^2 + 2x + 1)}{h}$$
$$= \lim_{h \to 0} \frac{(3x^2 + 6xh + 3h^2 + 2x + 2h + 1) - (3x^2 + 2x + 1)}{h} = \lim_{h \to 0} \frac{6xh + 3h^2 + 2h}{h}$$
$$= \lim_{h \to 0} \frac{h(6x + 3h + 2)}{h} = \lim_{h \to 0} (6x + 3h + 2) = 6x + 2$$

[continued]

$$f''(x) = \lim_{h\to 0}\frac{f'(x+h)-f'(x)}{h} = \lim_{h\to 0}\frac{[6(x+h)+2]-(6x+2)}{h} = \lim_{h\to 0}\frac{(6x+6h+2)-(6x+2)}{h}$$

$$= \lim_{h\to 0}\frac{6h}{h} = \lim_{h\to 0} 6 = 6$$



We see from the graph that our answers are reasonable because the graph of f' is that of a linear function and the graph of f'' is that of a constant function.

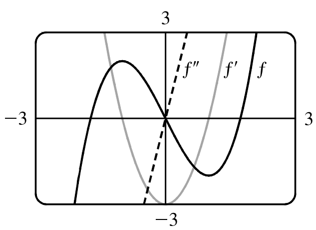
52. $$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{[(x+h)^3-3(x+h)]-(x^3-3x)}{h}$$

$$= \lim_{h\to 0}\frac{(x^3+3x^2h+3xh^2+h^3-3x-3h)-(x^3-3x)}{h} = \lim_{h\to 0}\frac{3x^2h+3xh^2+h^3-3h}{h}$$

$$= \lim_{h\to 0}\frac{h(3x^2+3xh+h^2-3)}{h} = \lim_{h\to 0}(3x^2+3xh+h^2-3) = 3x^2-3$$

$$f''(x) = \lim_{h\to 0}\frac{f'(x+h)-f'(x)}{h} = \lim_{h\to 0}\frac{[3(x+h)^2-3]-(3x^2-3)}{h} = \lim_{h\to 0}\frac{(3x^2+6xh+3h^2-3)-(3x^2-3)}{h}$$

$$= \lim_{h\to 0}\frac{6xh+3h^2}{h} = \lim_{h\to 0}\frac{h(6x+3h)}{h} = \lim_{h\to 0}(6x+3h) = 6x$$



We see from the graph that our answers are reasonable because the graph of f' is that of an even function (f is an odd function) and the graph of f'' is that of an odd function. Furthermore, $f' = 0$ when f has a horizontal tangent and $f'' = 0$ when f' has a horizontal tangent.

53. $$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\left[2(x+h)^2-(x+h)^3\right]-(2x^2-x^3)}{h}$$

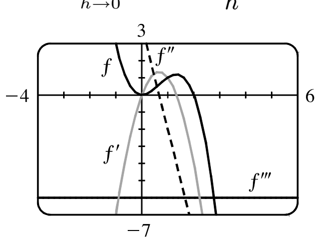
$$= \lim_{h\to 0}\frac{h(4x+2h-3x^2-3xh-h^2)}{h} = \lim_{h\to 0}(4x+2h-3x^2-3xh-h^2) = 4x-3x^2$$

$$f''(x) = \lim_{h\to 0}\frac{f'(x+h)-f'(x)}{h} = \lim_{h\to 0}\frac{\left[4(x+h)-3(x+h)^2\right]-(4x-3x^2)}{h} = \lim_{h\to 0}\frac{h(4-6x-3h)}{h}$$

$$= \lim_{h\to 0}(4-6x-3h) = 4-6x$$

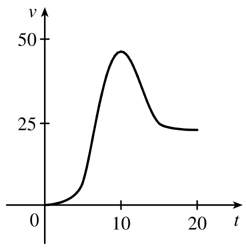
$$f'''(x) = \lim_{h\to 0}\frac{f''(x+h)-f''(x)}{h} = \lim_{h\to 0}\frac{[4-6(x+h)]-(4-6x)}{h} = \lim_{h\to 0}\frac{-6h}{h} = \lim_{h\to 0}(-6) = -6$$

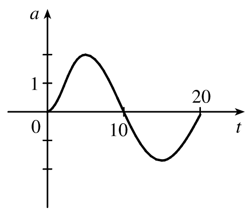
$$f^{(4)}(x) = \lim_{h\to 0}\frac{f'''(x+h)-f'''(x)}{h} = \lim_{h\to 0}\frac{-6-(-6)}{h} = \lim_{h\to 0}\frac{0}{h} = \lim_{h\to 0}(0) = 0$$



The graphs are consistent with the geometric interpretations of the derivatives because f' has zeros where f has a local minimum and a local maximum, f'' has a zero where f' has a local maximum, and f''' is a constant function equal to the slope of f''.

54. (a) Since we estimate the velocity to be a maximum at $t = 10$, the acceleration is 0 at $t = 10$.





(b) Drawing a tangent line at $t = 10$ on the graph of a, a appears to decrease by 10 ft/s^2 over a period of 20 s. So at $t = 10$ s, the jerk is approximately $-10/20 = -0.5\ (\text{ft/s}^2)/\text{s}$ or ft/s^3.

55. (a) Note that we have factored $x - a$ as the difference of two cubes in the third step.

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{x^{1/3} - a^{1/3}}{x - a} = \lim_{x \to a} \frac{x^{1/3} - a^{1/3}}{(x^{1/3} - a^{1/3})(x^{2/3} + x^{1/3}a^{1/3} + a^{2/3})}$$
$$= \lim_{x \to a} \frac{1}{x^{2/3} + x^{1/3}a^{1/3} + a^{2/3}} = \frac{1}{3a^{2/3}} \text{ or } \tfrac{1}{3}a^{-2/3}$$

(b) $f'(0) = \lim\limits_{h \to 0} \dfrac{f(0 + h) - f(0)}{h} = \lim\limits_{h \to 0} \dfrac{\sqrt[3]{h} - 0}{h} = \lim\limits_{h \to 0} \dfrac{1}{h^{2/3}}$. This function increases without bound, so the limit does not exist, and therefore $f'(0)$ does not exist.

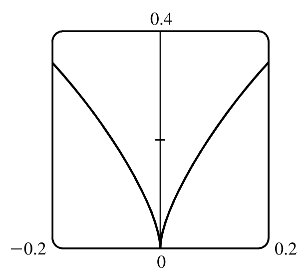
(c) $\lim\limits_{x \to 0} |f'(x)| = \lim\limits_{x \to 0} \dfrac{1}{3x^{2/3}} = \infty$ and f is continuous at $x = 0$ (root function), so f has a vertical tangent at $x = 0$.

56. (a) $g'(0) = \lim\limits_{x \to 0} \dfrac{g(x) - g(0)}{x - 0} = \lim\limits_{x \to 0} \dfrac{x^{2/3} - 0}{x} = \lim\limits_{x \to 0} \dfrac{1}{x^{1/3}}$, which does not exist.

(b) $$g'(a) = \lim_{x \to a} \frac{g(x) - g(a)}{x - a} = \lim_{x \to a} \frac{x^{2/3} - a^{2/3}}{x - a} = \lim_{x \to a} \frac{(x^{1/3} - a^{1/3})(x^{1/3} + a^{1/3})}{(x^{1/3} - a^{1/3})(x^{2/3} + x^{1/3}a^{1/3} + a^{2/3})}$$
$$= \lim_{x \to a} \frac{x^{1/3} + a^{1/3}}{x^{2/3} + x^{1/3}a^{1/3} + a^{2/3}} = \frac{2a^{1/3}}{3a^{2/3}} = \frac{2}{3a^{1/3}} \text{ or } \tfrac{2}{3}a^{-1/3}$$

(c) $g(x) = x^{2/3}$ is continuous at $x = 0$ and $\lim\limits_{x \to 0} |g'(x)| = \lim\limits_{x \to 0} \dfrac{2}{3\,|x|^{1/3}} = \infty$. This shows that g has a vertical tangent line at $x = 0$.

(d)



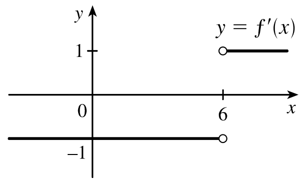
57. $f(x) = |x - 6| = \begin{cases} x - 6 & \text{if } x - 6 \geq 6 \\ -(x - 6) & \text{if } x - 6 < 0 \end{cases} = \begin{cases} x - 6 & \text{if } x \geq 6 \\ 6 - x & \text{if } x < 6 \end{cases}$

So the right-hand limit is $\lim\limits_{x \to 6^+} \dfrac{f(x) - f(6)}{x - 6} = \lim\limits_{x \to 6^+} \dfrac{|x - 6| - 0}{x - 6} = \lim\limits_{x \to 6^+} \dfrac{x - 6}{x - 6} = \lim\limits_{x \to 6^+} 1 = 1$, and the left-hand limit is $\lim\limits_{x \to 6^-} \dfrac{f(x) - f(6)}{x - 6} = \lim\limits_{x \to 6^-} \dfrac{|x - 6| - 0}{x - 6} = \lim\limits_{x \to 6^-} \dfrac{6 - x}{x - 6} = \lim\limits_{x \to 6^-} (-1) = -1$. Since these limits are not equal,

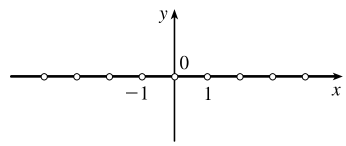
$f'(6) = \lim_{x \to 6} \dfrac{f(x) - f(6)}{x - 6}$ does not exist and f is not differentiable at 6.

However, a formula for f' is $f'(x) = \begin{cases} 1 & \text{if } x > 6 \\ -1 & \text{if } x < 6 \end{cases}$

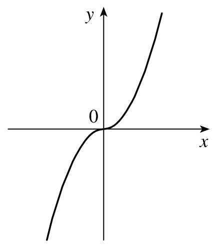
Another way of writing the formula is $f'(x) = \dfrac{x - 6}{|x - 6|}$.



58. $f(x) = [\![x]\!]$ is not continuous at any integer n, so f is not differentiable at n by the contrapositive of Theorem 4. If a is not an integer, then f is constant on an open interval containing a, so $f'(a) = 0$. Thus, $f'(x) = 0$, x not an integer.



59. (a) $f(x) = x\,|x| = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases}$



(b) Since $f(x) = x^2$ for $x \geq 0$, we have $f'(x) = 2x$ for $x > 0$. [See Exercise 17(d).] Similarly, since $f(x) = -x^2$ for $x < 0$, we have $f'(x) = -2x$ for $x < 0$. At $x = 0$, we have

$f'(0) = \lim_{x \to 0} \dfrac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \dfrac{x\,|x|}{x} = \lim_{x \to 0} |x| = 0.$

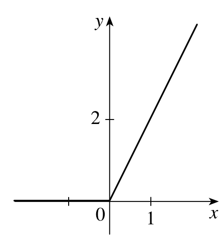
So f is differentiable at 0. Thus, f is differentiable for all x.

(c) From part (b), we have $f'(x) = \left\{ \begin{matrix} 2x & \text{if } x \geq 0 \\ -2x & \text{if } x < 0 \end{matrix} \right\} = 2\,|x|$.

60. (a) $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

so $g(x) = x + |x| = \begin{cases} 2x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$.

Graph the line $y = 2x$ for $x \geq 0$ and graph $y = 0$ (the x-axis) for $x < 0$.



(b) g is not differentiable at $x = 0$ because the graph has a corner there, but is differentiable at all other values; that is, g is differentiable on $(-\infty, 0) \cup (0, \infty)$.

(c) $g(x) = \begin{cases} 2x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \quad \Rightarrow \quad g'(x) = \begin{cases} 2 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$

Another way of writing the formula is $g'(x) = 1 + \operatorname{sgn} x$ for $x \neq 0$.

61. (a) If f is even, then

$$\begin{aligned} f'(-x) &= \lim_{h \to 0} \frac{f(-x + h) - f(-x)}{h} = \lim_{h \to 0} \frac{f[-(x - h)] - f(-x)}{h} \\ &= \lim_{h \to 0} \frac{f(x - h) - f(x)}{h} = -\lim_{h \to 0} \frac{f(x - h) - f(x)}{-h} \qquad [\text{let } \Delta x = -h] \\ &= -\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = -f'(x) \end{aligned}$$

Therefore, f' is odd.

(b) If f is odd, then

$$f'(-x) = \lim_{h\to 0}\frac{f(-x+h)-f(-x)}{h} = \lim_{h\to 0}\frac{f[-(x-h)]-f(-x)}{h}$$
$$= \lim_{h\to 0}\frac{-f(x-h)+f(x)}{h} = \lim_{h\to 0}\frac{f(x-h)-f(x)}{-h} \qquad [\text{let } \Delta x = -h]$$
$$= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = f'(x)$$

Therefore, f' is even.

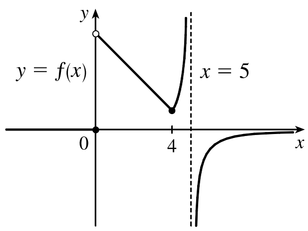
62. (a) $f'_-(4) = \lim_{h\to 0^-}\frac{f(4+h)-f(4)}{h} = \lim_{h\to 0^-}\frac{5-(4+h)-1}{h}$

$= \lim_{h\to 0^-}\frac{-h}{h} = -1$

and

$f'_+(4) = \lim_{h\to 0^+}\frac{f(4+h)-f(4)}{h} = \lim_{h\to 0^+}\frac{\frac{1}{5-(4+h)}-1}{h}$

$= \lim_{h\to 0^+}\frac{1-(1-h)}{h(1-h)} = \lim_{h\to 0^+}\frac{1}{1-h} = 1$

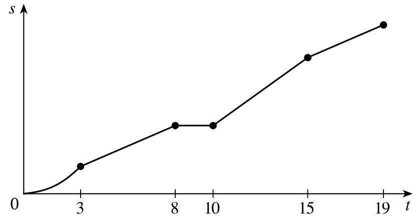
(b) 

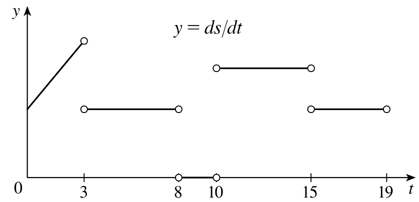
(c) $f(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 5-x & \text{if } 0 < x < 4 \\ 1/(5-x) & \text{if } x \ge 4 \end{cases}$

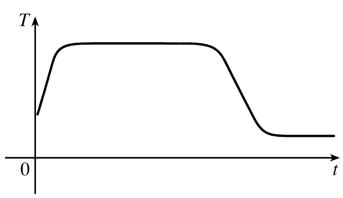
At 4 we have $\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-}(5-x) = 1$ and $\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+}\frac{1}{5-x} = 1$, so $\lim_{x\to 4} f(x) = 1 = f(4)$ and f is continuous at 4. Since $f(5)$ is not defined, f is discontinuous at 5. These expressions show that f is continuous on the intervals $(-\infty, 0)$, $(0, 4)$, $(4, 5)$ and $(5, \infty)$. Since $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(5-x) = 5 \ne 0 = \lim_{x\to 0^-} f(x)$, $\lim_{x\to 0} f(x)$ does not exist, so f is discontinuous (and therefore not differentiable) at 0.

(d) From (a), f is not differentiable at 4 since $f'_-(4) \ne f'_+(4)$, and from (c), f is not differentiable at 0 or 5.

63. These graphs are idealizations conveying the spirit of the problem. In reality, changes in speed are not instantaneous, so the graph in (a) would not have corners and the graph in (b) would be continuous.

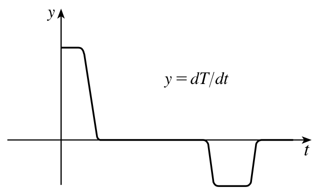
(a) 

(b) 

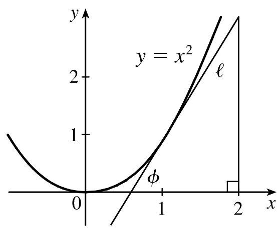
64. (a) 

(b) The initial temperature of the water is close to room temperature because of the water that was in the pipes. When the water from the hot water tank starts coming out, dT/dt is large and positive as T increases to the temperature of the water in the tank. In the next phase, $dT/dt = 0$ as the water comes out at a constant, high temperature. After some time, dT/dt becomes small and negative as the contents of the hot water tank are exhausted. Finally, when the hot water has run out, dT/dt is once again 0 as the water maintains its (cold) temperature.

(c)



65.



In the right triangle in the diagram, let Δy be the side opposite angle ϕ and Δx the side adjacent to angle ϕ. Then the slope of the tangent line ℓ is $m = \Delta y/\Delta x = \tan\phi$. Note that $0 < \phi < \frac{\pi}{2}$. We know (see Exercise 17) that the derivative of $f(x) = x^2$ is $f'(x) = 2x$. So the slope of the tangent to the curve at the point $(1, 1)$ is 2. Thus, ϕ is the angle between 0 and $\frac{\pi}{2}$ whose tangent is 2; that is, $\phi = \tan^{-1} 2 \approx 63°$.

2.3 Differentiation Formulas

1. $f(x) = 2^{40}$ is a constant function, so its derivative is 0, that is, $f'(x) = 0$.

2. $f(x) = \pi^2$ is a constant function, so its derivative is 0, that is, $f'(x) = 0$.

3. $f(x) = 5.2x + 2.3 \quad\Rightarrow\quad f'(x) = 5.2(1) + 0 = 5.2$

4. $g(x) = \frac{7}{4}x^2 - 3x + 12 \quad\Rightarrow\quad g'(x) = \frac{7}{4}(2x) - 3(1) + 0 = \frac{7}{2}x - 3$

5. $f(t) = 2t^3 - 3t^2 - 4t \quad\Rightarrow\quad f'(t) = 2(3t^2) - 3(2t) - 4(1) = 6t^2 - 6t - 4$

6. $f(t) = 1.4t^5 - 2.5t^2 + 6.7 \quad\Rightarrow\quad f'(t) = 1.4(5t^4) - 2.5(2t) + 0 = 7t^4 - 5t$

7. $g(x) = x^2(1 - 2x) = x^2 - 2x^3 \quad\Rightarrow\quad g'(x) = 2x - 2(3x^2) = 2x - 6x^2$

8. $H(u) = (3u - 1)(u + 2) = 3u^2 + 5u - 2 \quad\Rightarrow\quad H'(u) = 3(2u) + 5(1) - 0 = 6u + 5$

9. $g(t) = 2t^{-3/4} \quad\Rightarrow\quad g'(t) = 2\left(-\frac{3}{4}t^{-7/4}\right) = -\frac{3}{2}t^{-7/4}$

10. $B(y) = cy^{-6} \quad\Rightarrow\quad B'(y) = c(-6y^{-7}) = -6cy^{-7}$

11. $F(r) = \dfrac{5}{r^3} = 5r^{-3} \quad\Rightarrow\quad F'(r) = 5(-3r^{-4}) = -15r^{-4} = -\dfrac{15}{r^4}$

12. $y = x^{5/3} - x^{2/3} \quad\Rightarrow\quad y' = \frac{5}{3}x^{2/3} - \frac{2}{3}x^{-1/3}$

13. $S(p) = \sqrt{p} - p = p^{1/2} - p \;\Rightarrow\; S'(p) = \frac{1}{2}p^{-1/2} - 1$ or $\dfrac{1}{2\sqrt{p}} - 1$

14. $y = \sqrt[3]{x}\,(2+x) = 2x^{1/3} + x^{4/3} \;\Rightarrow\; y' = 2\left(\frac{1}{3}x^{-2/3}\right) + \frac{4}{3}x^{1/3} = \frac{2}{3}x^{-2/3} + \frac{4}{3}x^{1/3}$ or $\dfrac{2}{3\sqrt[3]{x^2}} + \dfrac{4}{3}\sqrt[3]{x}$

15. $R(a) = (3a+1)^2 = 9a^2 + 6a + 1 \;\Rightarrow\; R'(a) = 9(2a) + 6(1) + 0 = 18a + 6$

16. $S(R) = 4\pi R^2 \;\Rightarrow\; S'(R) = 4\pi(2R) = 8\pi R$

17. $y = \dfrac{x^2+4x+3}{\sqrt{x}} = x^{3/2} + 4x^{1/2} + 3x^{-1/2} \;\Rightarrow$

$y' = \frac{3}{2}x^{1/2} + 4\left(\frac{1}{2}\right)x^{-1/2} + 3\left(-\frac{1}{2}\right)x^{-3/2} = \frac{3}{2}\sqrt{x} + \dfrac{2}{\sqrt{x}} - \dfrac{3}{2x\sqrt{x}}$ $\left[\text{note that } x^{3/2} = x^{2/2}\cdot x^{1/2} = x\sqrt{x}\right]$

The last expression can be written as $\dfrac{3x^2}{2x\sqrt{x}} + \dfrac{4x}{2x\sqrt{x}} - \dfrac{3}{2x\sqrt{x}} = \dfrac{3x^2+4x-3}{2x\sqrt{x}}$.

18. $y = \dfrac{\sqrt{x}+x}{x^2} = \dfrac{\sqrt{x}}{x^2} + \dfrac{x}{x^2} = x^{1/2-2} + x^{1-2} = x^{-3/2} + x^{-1} \;\Rightarrow\; y' = -\frac{3}{2}x^{-5/2} + (-1x^{-2}) = -\frac{3}{2}x^{-5/2} - x^{-2}$

19. $G(q) = \left(1+q^{-1}\right)^2 = 1 + 2q^{-1} + q^{-2} \;\Rightarrow\; G'(q) = 0 + 2(-1q^{-2}) + (-2q^{-3}) = -2q^{-2} - 2q^{-3}$

20. $G(t) = \sqrt{5t} + \dfrac{\sqrt{7}}{t} = \sqrt{5}\,t^{1/2} + \sqrt{7}\,t^{-1} \;\Rightarrow\; G'(t) = \sqrt{5}\left(\frac{1}{2}t^{-1/2}\right) + \sqrt{7}\left(-1t^{-2}\right) = \dfrac{\sqrt{5}}{2\sqrt{t}} - \dfrac{\sqrt{7}}{t^2}$

21. $u = \left(\dfrac{1}{t} - \dfrac{1}{\sqrt{t}}\right)^2 = \dfrac{1}{t^2} - \dfrac{2}{t^{3/2}} + \dfrac{1}{t} = t^{-2} - 2t^{-3/2} + t^{-1} \;\Rightarrow$

$u' = -2t^{-3} - 2\left(-\frac{3}{2}\right)t^{-5/2} + (-1)t^{-2} = -\dfrac{2}{t^3} + \dfrac{3}{t^{5/2}} - \dfrac{1}{t^2} = -\dfrac{2}{t^3} + \dfrac{3}{t^2\sqrt{t}} - \dfrac{1}{t^2}$

22. $D(t) = \dfrac{1+16t^2}{(4t)^3} = \dfrac{1+16t^2}{64t^3} = \frac{1}{64}t^{-3} + \frac{1}{4}t^{-1} \;\Rightarrow$

$D'(t) = \frac{1}{64}(-3t^{-4}) + \frac{1}{4}(-1t^{-2}) = -\frac{3}{64}t^{-4} - \frac{1}{4}t^{-2}$ or $-\dfrac{3}{64t^4} - \dfrac{1}{4t^2}$

23. Product Rule: $f(x) = (1+2x^2)(x-x^2) \;\Rightarrow$

$$f'(x) = (1+2x^2)(1-2x) + (x-x^2)(4x) = 1 - 2x + 2x^2 - 4x^3 + 4x^2 - 4x^3 = 1 - 2x + 6x^2 - 8x^3.$$

Multiplying first: $f(x) = (1+2x^2)(x-x^2) = x - x^2 + 2x^3 - 2x^4 \;\Rightarrow\; f'(x) = 1 - 2x + 6x^2 - 8x^3$ (equivalent).

24. Quotient Rule: $F(x) = \dfrac{x^4 - 5x^3 + \sqrt{x}}{x^2} = \dfrac{x^4 - 5x^3 + x^{1/2}}{x^2} \;\Rightarrow$

$$F'(x) = \frac{x^2(4x^3 - 15x^2 + \frac{1}{2}x^{-1/2}) - (x^4 - 5x^3 + x^{1/2})(2x)}{(x^2)^2} = \frac{4x^5 - 15x^4 + \frac{1}{2}x^{3/2} - 2x^5 + 10x^4 - 2x^{3/2}}{x^4}$$

$$= \frac{2x^5 - 5x^4 - \frac{3}{2}x^{3/2}}{x^4} = 2x - 5 - \tfrac{3}{2}x^{-5/2}$$

Simplifying first: $F(x) = \dfrac{x^4 - 5x^3 + \sqrt{x}}{x^2} = x^2 - 5x + x^{-3/2} \;\Rightarrow\; F'(x) = 2x - 5 - \frac{3}{2}x^{-5/2}$ (equivalent).

For this problem, simplifying first seems to be the better method.

25. $f(x) = (5x^2 - 2)(x^3 + 3x) \stackrel{\text{PR}}{\Rightarrow}$

$$f'(x) = (5x^2 - 2)(3x^2 + 3) + (x^3 + 3x)(10x) = 15x^4 + 9x^2 - 6 + 10x^4 + 30x^2 = 25x^4 + 39x^2 - 6$$

26. $B(u) = (u^3 + 1)(2u^2 - 4u - 1) \stackrel{\text{PR}}{\Rightarrow}$

$$\begin{aligned} B'(u) &= (u^3 + 1)(4u - 4) + (2u^2 - 4u - 1)(3u^2) \\ &= 4u^4 - 4u^3 + 4u - 4 + 6u^4 - 12u^3 - 3u^2 = 10u^4 - 16u^3 - 3u^2 + 4u - 4 \end{aligned}$$

27. $F(y) = \left(\dfrac{1}{y^2} - \dfrac{3}{y^4}\right)(y + 5y^3) = (y^{-2} - 3y^{-4})(y + 5y^3) \stackrel{\text{PR}}{\Rightarrow}$

$$\begin{aligned} F'(y) &= (y^{-2} - 3y^{-4})(1 + 15y^2) + (y + 5y^3)(-2y^{-3} + 12y^{-5}) \\ &= (y^{-2} + 15 - 3y^{-4} - 45y^{-2}) + (-2y^{-2} + 12y^{-4} - 10 + 60y^{-2}) \\ &= 5 + 14y^{-2} + 9y^{-4} \text{ or } 5 + 14/y^2 + 9/y^4 \end{aligned}$$

28. $J(v) = (v^3 - 2v)(v^{-4} + v^{-2}) \stackrel{\text{PR}}{\Rightarrow}$

$$\begin{aligned} J'(v) &= (v^3 - 2v)(-4v^{-5} - 2v^{-3}) + (v^{-4} + v^{-2})(3v^2 - 2) \\ &= -4v^{-2} - 2v^0 + 8v^{-4} + 4v^{-2} + 3v^{-2} - 2v^{-4} + 3v^0 - 2v^{-2} = 1 + v^{-2} + 6v^{-4} \end{aligned}$$

29. $g(x) = \dfrac{1 + 2x}{3 - 4x} \stackrel{\text{QR}}{\Rightarrow} g'(x) = \dfrac{(3 - 4x)(2) - (1 + 2x)(-4)}{(3 - 4x)^2} = \dfrac{6 - 8x + 4 + 8x}{(3 - 4x)^2} = \dfrac{10}{(3 - 4x)^2}$

30. $h(t) = \dfrac{6t + 1}{6t - 1} \stackrel{\text{QR}}{\Rightarrow} h'(t) = \dfrac{(6t - 1)(6) - (6t + 1)(6)}{(6t - 1)^2} = \dfrac{36t - 6 - 36t - 6}{(6t - 1)^2} = -\dfrac{12}{(6t - 1)^2}$

31. $y = \dfrac{x^2 + 1}{x^3 - 1} \stackrel{\text{QR}}{\Rightarrow}$

$$y' = \frac{(x^3 - 1)(2x) - (x^2 + 1)(3x^2)}{(x^3 - 1)^2} = \frac{x\left[(x^3 - 1)(2) - (x^2 + 1)(3x)\right]}{(x^3 - 1)^2} = \frac{x(2x^3 - 2 - 3x^3 - 3x)}{(x^3 - 1)^2} = \frac{x(-x^3 - 3x - 2)}{(x^3 - 1)^2}$$

32. $y = \dfrac{1}{t^3 + 2t^2 - 1} \stackrel{\text{QR}}{\Rightarrow} y' = \dfrac{(t^3 + 2t^2 - 1)(0) - 1(3t^2 + 4t)}{(t^3 + 2t^2 - 1)^2} = -\dfrac{3t^2 + 4t}{(t^3 + 2t^2 - 1)^2}$

33. $y = \dfrac{t^3 + 3t}{t^2 - 4t + 3} \stackrel{\text{QR}}{\Rightarrow}$

$$\begin{aligned} y' &= \frac{(t^2 - 4t + 3)(3t^2 + 3) - (t^3 + 3t)(2t - 4)}{(t^2 - 4t + 3)^2} \\ &= \frac{3t^4 + 3t^2 - 12t^3 - 12t + 9t^2 + 9 - (2t^4 - 4t^3 + 6t^2 - 12t)}{(t^2 - 4t + 3)^2} = \frac{t^4 - 8t^3 + 6t^2 + 9}{(t^2 - 4t + 3)^2} \end{aligned}$$

34. $y = \dfrac{(u + 2)^2}{1 - u} = \dfrac{u^2 + 4u + 4}{1 - u} \stackrel{\text{QR}}{\Rightarrow}$

$$y' = \frac{(1 - u)(2u + 4) - (u^2 + 4u + 4)(-1)}{(1 - u)^2} = \frac{2u + 4 - 2u^2 - 4u + u^2 + 4u + 4}{(1 - u)^2} = \frac{-u^2 + 2u + 8}{(1 - u)^2}$$

35. $y = \dfrac{s-\sqrt{s}}{s^2} = \dfrac{s}{s^2} - \dfrac{\sqrt{s}}{s^2} = s^{-1} - s^{-3/2} \quad\Rightarrow\quad y' = -s^{-2} + \frac{3}{2}s^{-5/2} = \dfrac{-1}{s^2} + \dfrac{3}{2s^{5/2}} = \dfrac{3-2\sqrt{s}}{2s^{5/2}}$

36. $y = \dfrac{\sqrt{x}}{2+x} \quad\overset{\text{QR}}{\Rightarrow}$

$$y' = \frac{(2+x)\left(\dfrac{1}{2\sqrt{x}}\right) - \sqrt{x}\,(1)}{(2+x)^2} = \frac{\dfrac{1}{\sqrt{x}} + \dfrac{\sqrt{x}}{2} - \sqrt{x}}{(2+x)^2} = \frac{\dfrac{2+x-2x}{2\sqrt{x}}}{(2+x)^2} = \frac{2-x}{2\sqrt{x}(2+x)^2}$$

37. $f(t) = \dfrac{\sqrt[3]{t}}{t-3} \quad\overset{\text{QR}}{\Rightarrow}$

$$f'(t) = \frac{(t-3)\left(\frac{1}{3}t^{-2/3}\right) - t^{1/3}(1)}{(t-3)^2} = \frac{\frac{1}{3}t^{1/3} - t^{-2/3} - t^{1/3}}{(t-3)^2} = \frac{-\frac{2}{3}t^{1/3} - t^{-2/3}}{(t-3)^2} = \frac{\dfrac{-2t}{3t^{2/3}} - \dfrac{3}{3t^{2/3}}}{(t-3)^2} = \frac{-2t-3}{3t^{2/3}(t-3)^2}$$

38. $y = \dfrac{cx}{1+cx} \quad\Rightarrow\quad y' = \dfrac{(1+cx)(c) - (cx)(c)}{(1+cx)^2} = \dfrac{c + c^2x - c^2x}{(1+cx)^2} = \dfrac{c}{(1+cx)^2}$

39. $F(x) = \dfrac{2x^5 + x^4 - 6x}{x^3} = 2x^2 + x - 6x^{-2} \quad\Rightarrow\quad F'(x) = 4x + 1 + 12x^{-3} = 4x + 1 + \dfrac{12}{x^3}$ or $\dfrac{4x^4 + x^3 + 12}{x^3}$

40. $A(v) = v^{2/3}(2v^2 + 1 - v^{-2}) = 2v^{8/3} + v^{2/3} - v^{-4/3} \quad\Rightarrow$

$$A'(v) = \tfrac{16}{3}v^{5/3} + \tfrac{2}{3}v^{-1/3} + \tfrac{4}{3}v^{-7/3} = \tfrac{2}{3}v^{-7/3}(8v^{12/3} + v^{6/3} + 2) = \frac{2(8v^4 + v^2 + 2)}{3v^{7/3}}$$

41. $G(y) = \dfrac{B}{Ay^3 + B} \quad\overset{\text{QR}}{\Rightarrow}\quad G'(y) = \dfrac{(Ay^3+B)(0) - B(3Ay^2)}{(Ay^3+B)^2} = -\dfrac{3ABy^2}{(Ay^3+B)^2}$

42. $F(t) = \dfrac{At}{Bt^2 + Ct^3} = \dfrac{A}{Bt + Ct^2} \quad\overset{\text{QR}}{\Rightarrow}$

$$F'(t) = \frac{(Bt+Ct^2)(0) - A(B+2Ct)}{(Bt+Ct^2)^2} = \frac{-A(B+2Ct)}{(t)^2(B+Ct)^2} = -\frac{A(B+2Ct)}{t^2(B+Ct)^2}$$

43. $f(x) = \dfrac{x}{x + c/x} \quad\Rightarrow\quad f'(x) = \dfrac{(x+c/x)(1) - x(1-c/x^2)}{\left(x + \dfrac{c}{x}\right)^2} = \dfrac{x + c/x - x + c/x}{\left(\dfrac{x^2+c}{x}\right)^2} = \dfrac{2c/x}{\dfrac{(x^2+c)^2}{x^2}} \cdot \dfrac{x^2}{x^2} = \dfrac{2cx}{(x^2+c)^2}$

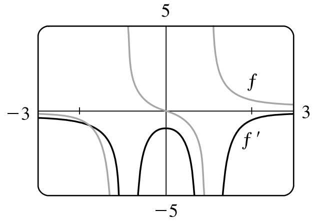
44. $f(x) = \dfrac{ax+b}{cx+d} \quad\Rightarrow\quad f'(x) = \dfrac{(cx+d)(a) - (ax+b)(c)}{(cx+d)^2} = \dfrac{acx + ad - acx - bc}{(cx+d)^2} = \dfrac{ad-bc}{(cx+d)^2}$

45. $P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0 \quad\Rightarrow\quad P'(x) = na_nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + 2a_2x + a_1$

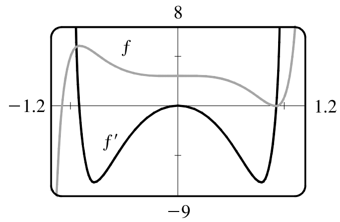
46. $f(x) = \dfrac{x}{x^2-1} \quad\Rightarrow$

$f'(x) = \dfrac{(x^2-1)1 - x(2x)}{(x^2-1)^2} = \dfrac{-x^2-1}{(x^2-1)^2} = -\dfrac{x^2+1}{(x^2-1)^2}$

Notice that the slopes of all tangents to f are negative and $f'(x) < 0$ always.

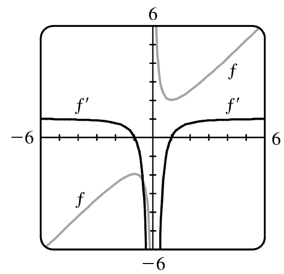


47. $f(x) = 3x^{15} - 5x^3 + 3 \quad \Rightarrow \quad f'(x) = 45x^{14} - 15x^2$.



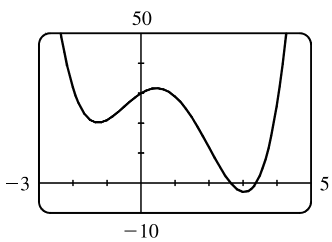
Notice that $f'(x) = 0$ when f has a horizontal tangent, f' is positive when f is increasing, and f' is negative when f is decreasing.

48. $f(x) = x + 1/x = x + x^{-1} \quad \Rightarrow \quad f'(x) = 1 - x^{-2} = 1 - 1/x^2$.

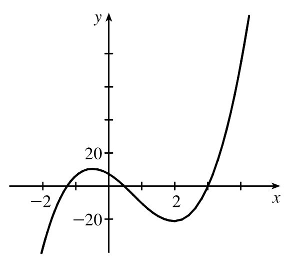


Notice that $f'(x) = 0$ when f has a horizontal tangent, f' is positive when f is increasing, and f' is negative when f is decreasing.

49. (a)

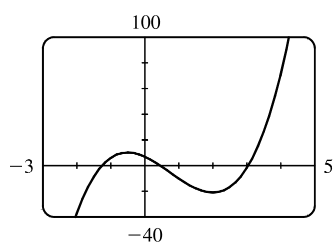


(b) From the graph in part (a), it appears that f' is zero at $x_1 \approx -1.25$, $x_2 \approx 0.5$, and $x_3 \approx 3$. The slopes are negative (so f' is negative) on $(-\infty, x_1)$ and (x_2, x_3). The slopes are positive (so f' is positive) on (x_1, x_2) and (x_3, ∞).

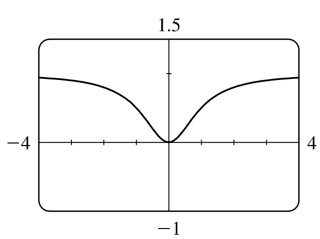


(c) $f(x) = x^4 - 3x^3 - 6x^2 + 7x + 30 \quad \Rightarrow$

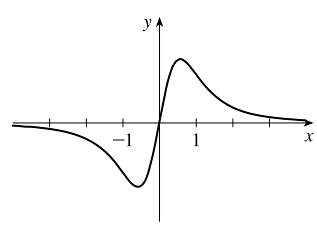
$f'(x) = 4x^3 - 9x^2 - 12x + 7$



50. (a)



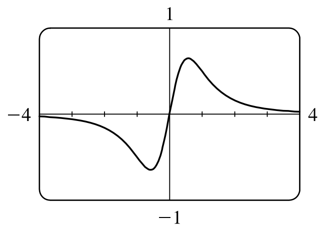
(b)



From the graph in part (a), it appears that g' is zero at $x = 0$. The slopes are negative (so g' is negative) on $(-\infty, 0)$. The slopes are positive (so g' is positive) on $(0, \infty)$.

(c) $g(x) = \dfrac{x^2}{x^2+1} \quad\Rightarrow$

$$g'(x) = \frac{(x^2+1)(2x) - x^2(2x)}{(x^2+1)^2} = \frac{2x}{(x^2+1)^2}$$



51. $y = \dfrac{2x}{x+1} \quad\Rightarrow\quad y' = \dfrac{(x+1)(2) - (2x)(1)}{(x+1)^2} = \dfrac{2}{(x+1)^2}$.

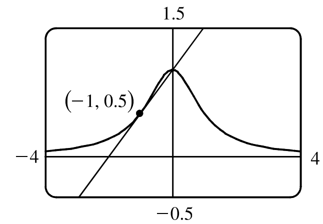
At $(1,1)$, $y' = \frac{1}{2}$, and an equation of the tangent line is $y - 1 = \frac{1}{2}(x-1)$, or $y = \frac{1}{2}x + \frac{1}{2}$.

52. $y = 2x^3 - x^2 + 2 \quad\Rightarrow\quad y' = 6x^2 - 2x$. At $(1,3)$, $y' = 6(1)^2 - 2(1) = 4$ and an equation of the tangent line is $y - 3 = 4(x-1)$ or $y = 4x - 1$.

53. (a) $y = f(x) = \dfrac{1}{1+x^2} \quad\Rightarrow$

$f'(x) = \dfrac{(1+x^2)(0) - 1(2x)}{(1+x^2)^2} = \dfrac{-2x}{(1+x^2)^2}$. So the slope of the tangent line at the point $\left(-1, \frac{1}{2}\right)$ is $f'(-1) = \dfrac{2}{2^2} = \frac{1}{2}$ and its equation is $y - \frac{1}{2} = \frac{1}{2}(x+1)$ or $y = \frac{1}{2}x + 1$.

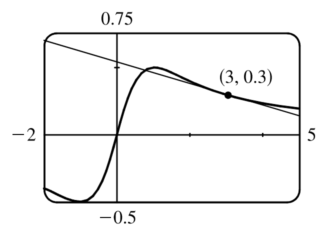
(b)



54. (a) $y = f(x) = \dfrac{x}{1+x^2} \quad\Rightarrow$

$f'(x) = \dfrac{(1+x^2)1 - x(2x)}{(1+x^2)^2} = \dfrac{1-x^2}{(1+x^2)^2}$. So the slope of the tangent line at the point $(3, 0.3)$ is $f'(3) = \frac{-8}{100}$ and its equation is $y - 0.3 = -0.08(x-3)$ or $y = -0.08x + 0.54$.

(b)



55. $y = x + \sqrt{x} \quad\Rightarrow\quad y' = 1 + \frac{1}{2}x^{-1/2} = 1 + 1/(2\sqrt{x})$. At $(1,2)$, $y' = \frac{3}{2}$, and an equation of the tangent line is $y - 2 = \frac{3}{2}(x-1)$, or $y = \frac{3}{2}x + \frac{1}{2}$. The slope of the normal line is $-\frac{2}{3}$, so an equation of the normal line is $y - 2 = -\frac{2}{3}(x-1)$, or $y = -\frac{2}{3}x + \frac{8}{3}$.

56. $y^2 = x^3 \quad\Rightarrow\quad y = x^{3/2}$ [since x and y are positive at $(1,1)$] $\quad\Rightarrow\quad y' = \frac{3}{2}x^{1/2}$. At $(1,1)$, $y' = \frac{3}{2}$ and an equation of the tangent line is $y - 1 = \frac{3}{2}(x-1)$ or $y = \frac{3}{2}x - \frac{1}{2}$. The slope of the normal line is $-\frac{2}{3}$ $\left(\text{the negative reciprocal of } \frac{3}{2}\right)$ and an equation of the normal line is $y - 1 = -\frac{2}{3}(x-1)$ or $y = -\frac{2}{3}x + \frac{5}{3}$.

57. $y = \dfrac{3x+1}{x^2+1} \quad\Rightarrow\quad y' = \dfrac{(x^2+1)(3) - (3x+1)(2x)}{(x^2+1)^2}$. At $(1,2)$, $y' = \dfrac{6-8}{2^2} = -\dfrac{1}{2}$, and an equation of the tangent line is $y - 2 = -\frac{1}{2}(x-1)$, or $y = -\frac{1}{2}x + \frac{5}{2}$. The slope of the normal line is 2, so an equation of the normal line is $y - 2 = 2(x-1)$, or $y = 2x$.

58. $y = \dfrac{\sqrt{x}}{x+1} \quad \Rightarrow \quad y' = \dfrac{(x+1)\left(\dfrac{1}{2\sqrt{x}}\right) - \sqrt{x}\,(1)}{(x+1)^2} = \dfrac{(x+1)-(2x)}{2\sqrt{x}\,(x+1)^2} = \dfrac{1-x}{2\sqrt{x}\,(x+1)^2}.$

At $(4, 0.4)$, $y' = \frac{-3}{100} = -0.03$, and an equation of the tangent line is $y - 0.4 = -0.03(x-4)$, or $y = -0.03x + 0.52$. The slope of the normal line is $\frac{100}{3}$, so an equation of the normal line is $y - 0.4 = \frac{100}{3}(x-4) \quad \Leftrightarrow \quad y = \frac{100}{3}x - \frac{400}{3} + \frac{2}{5} \quad \Leftrightarrow$ $y = \frac{100}{3}x - \frac{1994}{15}$.

59. $f(x) = 0.001x^5 - 0.02x^3 \quad \Rightarrow \quad f'(x) = 0.005x^4 - 0.06x^2 \quad \Rightarrow \quad f''(x) = 0.02x^3 - 0.12x$

60. $G(r) = \sqrt{r} + \sqrt[3]{r} \quad \Rightarrow \quad G'(r) = \frac{1}{2}r^{-1/2} + \frac{1}{3}r^{-2/3} \quad \Rightarrow \quad G''(r) = -\frac{1}{4}r^{-3/2} - \frac{2}{9}r^{-5/3}$

61. $f(x) = \dfrac{x^2}{1+2x} \quad \Rightarrow \quad f'(x) = \dfrac{(1+2x)(2x) - x^2(2)}{(1+2x)^2} = \dfrac{2x + 4x^2 - 2x^2}{(1+2x)^2} = \dfrac{2x^2 + 2x}{(1+2x)^2} \quad \Rightarrow$

$$f''(x) = \frac{(1+2x)^2(4x+2) - (2x^2+2x)(1+4x+4x^2)'}{[(1+2x)^2]^2} = \frac{2(1+2x)^2(2x+1) - 2x(x+1)(4+8x)}{(1+2x)^4}$$
$$= \frac{2(1+2x)[(1+2x)^2 - 4x(x+1)]}{(1+2x)^4} = \frac{2(1+4x+4x^2-4x^2-4x)}{(1+2x)^3} = \frac{2}{(1+2x)^3}$$

62. Using the Reciprocal Rule, $f(x) = \dfrac{1}{3-x} \quad \Rightarrow \quad f'(x) = -\dfrac{(3-x)'}{(3-x)^2} = -\dfrac{-1}{(3-x)^2} = \dfrac{1}{(3-x)^2} \quad \Rightarrow$

$$f''(x) = -\frac{[(3-x)^2]'}{[(3-x)^2]^2} = -\frac{(9-6x+x^2)'}{(3-x)^4} = -\frac{-6+2x}{(3-x)^4} = -\frac{-2(3-x)}{(3-x)^4} = \frac{2}{(3-x)^3}.$$

63. (a) $s = t^3 - 3t \quad \Rightarrow \quad v(t) = s'(t) = 3t^2 - 3 \quad \Rightarrow \quad a(t) = v'(t) = 6t$

(b) $a(2) = 6(2) = 12\ \text{m/s}^2$

(c) $v(t) = 3t^2 - 3 = 0$ when $t^2 = 1$, that is, $t = 1\ \ [t \geq 0]$ and $a(1) = 6\ \text{m/s}^2$.

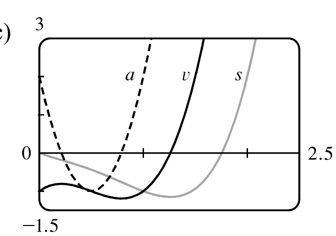
64. (a) $s = t^4 - 2t^3 + t^2 - t \quad \Rightarrow$

$v(t) = s'(t) = 4t^3 - 6t^2 + 2t - 1 \quad \Rightarrow$

$a(t) = v'(t) = 12t^2 - 12t + 2$

(b) $a(1) = 12(1)^2 - 12(1) + 2 = 2\ \text{m/s}^2$

(c)



65. $L = 0.0155A^3 - 0.372A^2 + 3.95A + 1.21 \quad \Rightarrow \quad \dfrac{dL}{dA} = 0.0465A^2 - 0.744A + 3.95$, so

$\left.\dfrac{dL}{dA}\right|_{A=12} = 0.0465(12)^2 - 0.744(12) + 3.95 = 1.718$. The derivative is the instantaneous rate of change of the length of an Alaskan rockfish with respect to its age when its age is 12 years.

66. $S(A) = 0.882A^{0.842} \quad \Rightarrow \quad S'(A) = 0.882(0.842A^{-0.158}) = 0.742644A^{-0.158}$, so

$S'(100) = 0.742644(100)^{-0.158} \approx 0.36$. The derivative is the instantaneous rate of change of the number of tree species with respect to area. Its units are number of species per square meter.

67. (a) $P = \dfrac{k}{V}$ and $P = 50$ when $V = 0.106$, so $k = PV = 50(0.106) = 5.3$. Thus, $P = \dfrac{5.3}{V}$ and $V = \dfrac{5.3}{P}$.

(b) $V = 5.3P^{-1} \quad \Rightarrow \quad \dfrac{dV}{dP} = 5.3(-1P^{-2}) = -\dfrac{5.3}{P^2}$. When $P = 50$, $\dfrac{dV}{dP} = -\dfrac{5.3}{50^2} = -0.00212$. The derivative is the instantaneous rate of change of the volume with respect to the pressure at $25\,^\circ\mathrm{C}$. Its units are $\mathrm{m}^3/\mathrm{kPa}$.

68. (a) $L = aP^2 + bP + c$, where $a \approx -0.275428$, $b \approx 19.74853$, and $c \approx -273.55234$.

(b) $\dfrac{dL}{dP} = 2aP + b$. When $P = 30$, $\dfrac{dL}{dP} \approx 3.2$, and when $P = 40$, $\dfrac{dL}{dP} \approx -2.3$. The derivative is the instantaneous rate of change of tire life with respect to pressure. Its units are (thousands of miles)$/(\mathrm{lb/in}^2)$. When $\dfrac{dL}{dP}$ is positive, tire life is increasing, and when $\dfrac{dL}{dP} < 0$, tire life is decreasing.

69. We are given that $f(5) = 1$, $f'(5) = 6$, $g(5) = -3$, and $g'(5) = 2$.

(a) $(fg)'(5) = f(5)g'(5) + g(5)f'(5) = (1)(2) + (-3)(6) = 2 - 18 = -16$

(b) $\left(\dfrac{f}{g}\right)'(5) = \dfrac{g(5)f'(5) - f(5)g'(5)}{[g(5)]^2} = \dfrac{(-3)(6) - (1)(2)}{(-3)^2} = -\dfrac{20}{9}$

(c) $\left(\dfrac{g}{f}\right)'(5) = \dfrac{f(5)g'(5) - g(5)f'(5)}{[f(5)]^2} = \dfrac{(1)(2) - (-3)(6)}{(1)^2} = 20$

70. We are given that $f(4) = 2$, $g(4) = 5$, $f'(4) = 6$, and $g'(4) = -3$.

(a) $h(x) = 3f(x) + 8g(x) \quad \Rightarrow \quad h'(x) = 3f'(x) + 8g'(x)$, so

$h'(4) = 3f'(4) + 8g'(4) = 3(6) + 8(-3) = 18 - 24 = -6$.

(b) $h(x) = f(x)\,g(x) \quad \Rightarrow \quad h'(x) = f(x)\,g'(x) + g(x)\,f'(x)$, so

$h'(4) = f(4)\,g'(4) + g(4)\,f'(4) = 2(-3) + 5(6) = -6 + 30 = 24$.

(c) $h(x) = \dfrac{f(x)}{g(x)} \quad \Rightarrow \quad h'(x) = \dfrac{g(x)\,f'(x) - f(x)\,g'(x)}{[g(x)]^2}$, so

$h'(4) = \dfrac{g(4)\,f'(4) - f(4)\,g'(4)}{[g(4)]^2} = \dfrac{5(6) - 2(-3)}{5^2} = \dfrac{30 + 6}{25} = \dfrac{36}{25}$.

(d) $h(x) = \dfrac{g(x)}{f(x) + g(x)} \quad \Rightarrow$

$$h'(4) = \frac{[f(4) + g(4)]\,g'(4) - g(4)\,[f'(4) + g'(4)]}{[f(4) + g(4)]^2} = \frac{(2+5)\,(-3) - 5\,[6 + (-3)]}{(2+5)^2} = \frac{-21 - 15}{7^2} = -\frac{36}{49}$$

71. $f(x) = \sqrt{x}\,g(x) \quad \Rightarrow \quad f'(x) = \sqrt{x}\,g'(x) + g(x) \cdot \frac{1}{2}x^{-1/2}$, so $f'(4) = \sqrt{4}\,g'(4) + g(4) \cdot \dfrac{1}{2\sqrt{4}} = 2 \cdot 7 + 8 \cdot \dfrac{1}{4} = 16$.

72. $\dfrac{d}{dx}\left[\dfrac{h(x)}{x}\right] = \dfrac{xh'(x) - h(x) \cdot 1}{x^2} \quad \Rightarrow \quad \dfrac{d}{dx}\left[\dfrac{h(x)}{x}\right]_{x=2} = \dfrac{2h'(2) - h(2)}{2^2} = \dfrac{2(-3) - (4)}{4} = \dfrac{-10}{4} = -2.5$

73. (a) From the graphs of f and g, we obtain the following values: $f(1) = 2$ since the point $(1, 2)$ is on the graph of f; $g(1) = 1$ since the point $(1, 1)$ is on the graph of g; $f'(1) = 2$ since the slope of the line segment between $(0, 0)$ and $(2, 4)$ is $\dfrac{4-0}{2-0} = 2$; $g'(1) = -1$ since the slope of the line segment between $(-2, 4)$ and $(2, 0)$ is $\dfrac{0-4}{2-(-2)} = -1$.

Now $u(x) = f(x)g(x)$, so $u'(1) = f(1)g'(1) + g(1)\,f'(1) = 2\cdot(-1) + 1\cdot 2 = 0$.

(b) $v(x) = f(x)/g(x)$, so $v'(5) = \dfrac{g(5)f'(5) - f(5)g'(5)}{[g(5)]^2} = \dfrac{2\left(-\frac{1}{3}\right) - 3\cdot\frac{2}{3}}{2^2} = \dfrac{-\frac{8}{3}}{4} = -\dfrac{2}{3}$

74. (a) $P(x) = F(x)\,G(x)$, so $P'(2) = F(2)\,G'(2) + G(2)\,F'(2) = 3\cdot\frac{2}{4} + 2\cdot 0 = \frac{3}{2}$.

(b) $Q(x) = F(x)/G(x)$, so $Q'(7) = \dfrac{G(7)\,F'(7) - F(7)\,G'(7)}{[G(7)]^2} = \dfrac{1\cdot\frac{1}{4} - 5\cdot\left(-\frac{2}{3}\right)}{1^2} = \dfrac{1}{4} + \dfrac{10}{3} = \dfrac{43}{12}$

75. (a) $y = xg(x) \quad\Rightarrow\quad y' = xg'(x) + g(x)\cdot 1 = xg'(x) + g(x)$

(b) $y = \dfrac{x}{g(x)} \quad\Rightarrow\quad y' = \dfrac{g(x)\cdot 1 - xg'(x)}{[g(x)]^2} = \dfrac{g(x) - xg'(x)}{[g(x)]^2}$

(c) $y = \dfrac{g(x)}{x} \quad\Rightarrow\quad y' = \dfrac{xg'(x) - g(x)\cdot 1}{(x)^2} = \dfrac{xg'(x) - g(x)}{x^2}$

76. (a) $y = x^2 f(x) \quad\Rightarrow\quad y' = x^2 f'(x) + f(x)(2x)$

(b) $y = \dfrac{f(x)}{x^2} \quad\Rightarrow\quad y' = \dfrac{x^2 f'(x) - f(x)(2x)}{(x^2)^2} = \dfrac{xf'(x) - 2f(x)}{x^3}$

(c) $y = \dfrac{x^2}{f(x)} \quad\Rightarrow\quad y' = \dfrac{f(x)(2x) - x^2 f'(x)}{[f(x)]^2}$

(d) $y = \dfrac{1 + xf(x)}{\sqrt{x}} \quad\Rightarrow$

$$y' = \frac{\sqrt{x}\,[xf'(x) + f(x)] - [1 + xf(x)]\,\dfrac{1}{2\sqrt{x}}}{\left(\sqrt{x}\right)^2}$$

$$= \frac{x^{3/2}f'(x) + x^{1/2}f(x) - \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{1/2}f(x)}{x}\cdot\frac{2x^{1/2}}{2x^{1/2}} = \frac{xf(x) + 2x^2 f'(x) - 1}{2x^{3/2}}$$

77. The curve $y = 2x^3 + 3x^2 - 12x + 1$ has a horizontal tangent when $y' = 6x^2 + 6x - 12 = 0 \quad\Leftrightarrow\quad 6(x^2 + x - 2) = 0 \quad\Leftrightarrow$ $6(x+2)(x-1) = 0 \quad\Leftrightarrow\quad x = -2$ or $x = 1$. The points on the curve are $(-2, 21)$ and $(1, -6)$.

78. $f(x) = x^3 + 3x^2 + x + 3$ has a horizontal tangent when $f'(x) = 3x^2 + 6x + 1 = 0 \quad\Leftrightarrow$

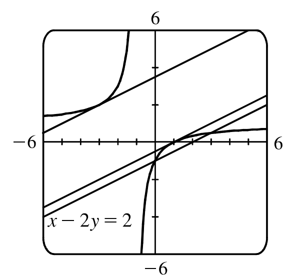
$x = \dfrac{-6 \pm \sqrt{36 - 12}}{6} = -1 \pm \frac{1}{3}\sqrt{6}$.

79. $y = 6x^3 + 5x - 3 \quad\Rightarrow\quad m = y' = 18x^2 + 5$, but $x^2 \geq 0$ for all x, so $m \geq 5$ for all x.

80. $y = x^4 + 1 \quad\Rightarrow\quad y' = 4x^3$. The slope of the line $32x - y = 15$ (or $y = 32x - 15$) is 32, so the slope of any line parallel to it is also 32. Thus, $y' = 32 \quad\Leftrightarrow\quad 4x^3 = 32 \quad\Leftrightarrow\quad x^3 = 8 \quad\Leftrightarrow\quad x = 2$, which is the x-coordinate of the point on the curve at which the slope is 32. The y-coordinate is $2^4 + 1 = 17$, so an equation of the tangent line is $y - 17 = 32(x - 2)$ or $y = 32x - 47$.

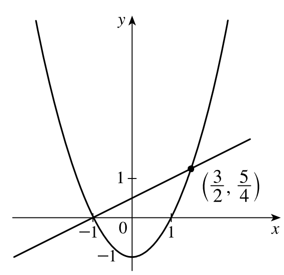
81. The slope of the line $3x - y = 15$ (or $y = 3x - 15$) is 3, so the slope of both tangent lines to the curve is 3. $y = x^3 - 3x^2 + 3x - 3 \Rightarrow y' = 3x^2 - 6x + 3 = 3(x^2 - 2x + 1) = 3(x-1)^2$. Thus, $3(x-1)^2 = 3 \Rightarrow (x-1)^2 = 1 \Rightarrow x - 1 = \pm 1 \Rightarrow x = 0$ or 2, which are the x-coordinates at which the tangent lines have slope 3. The points on the curve are $(0, -3)$ and $(2, -1)$, so the tangent line equations are $y - (-3) = 3(x - 0)$ or $y = 3x - 3$ and $y - (-1) = 3(x - 2)$ or $y = 3x - 7$.

82. $y = \dfrac{x-1}{x+1} \Rightarrow y' = \dfrac{(x+1)(1) - (x-1)(1)}{(x+1)^2} = \dfrac{2}{(x+1)^2}$. If the tangent intersects the curve when $x = a$, then its slope is $2/(a+1)^2$. But if the tangent is parallel to $x - 2y = 2$, that is, $y = \frac{1}{2}x - 1$, then its slope is $\frac{1}{2}$. Thus, $\dfrac{2}{(a+1)^2} = \dfrac{1}{2} \Rightarrow (a+1)^2 = 4 \Rightarrow a + 1 = \pm 2 \Rightarrow a = 1$ or -3. When $a = 1$, $y = 0$ and the equation of the tangent is $y - 0 = \frac{1}{2}(x - 1)$ or $y = \frac{1}{2}x - \frac{1}{2}$. When $a = -3$, $y = 2$ and the equation of the tangent is $y - 2 = \frac{1}{2}(x + 3)$ or $y = \frac{1}{2}x + \frac{7}{2}$.

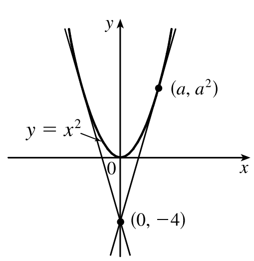


83. The slope of $y = \sqrt{x}$ is given by $y = \frac{1}{2}x^{-1/2} = \dfrac{1}{2\sqrt{x}}$. The slope of $2x + y = 1$ (or $y = -2x + 1$) is -2, so the desired normal line must have slope -2, and hence, the tangent line to the curve must have slope $\frac{1}{2}$. This occurs if $\dfrac{1}{2\sqrt{x}} = \dfrac{1}{2} \Rightarrow \sqrt{x} = 1 \Rightarrow x = 1$. When $x = 1$, $y = \sqrt{1} = 1$, and an equation of the normal line is $y - 1 = -2(x - 1)$ or $y = -2x + 3$.

84. $y = f(x) = x^2 - 1 \Rightarrow f'(x) = 2x$. So $f'(-1) = -2$, and the slope of the normal line is $\frac{1}{2}$. The equation of the normal line at $(-1, 0)$ is $y - 0 = \frac{1}{2}[x - (-1)]$ or $y = \frac{1}{2}x + \frac{1}{2}$. Substituting this into the equation of the parabola, we obtain $\frac{1}{2}x + \frac{1}{2} = x^2 - 1 \Leftrightarrow x + 1 = 2x^2 - 2 \Leftrightarrow 2x^2 - x - 3 = 0 \Leftrightarrow (2x - 3)(x + 1) = 0 \Leftrightarrow x = \frac{3}{2}$ or -1. Substituting $\frac{3}{2}$ into the equation of the normal line gives us $y = \frac{5}{4}$. Thus, the second point of intersection is $\left(\frac{3}{2}, \frac{5}{4}\right)$, as shown in the sketch.



85.



Let (a, a^2) be a point on the parabola at which the tangent line passes through the point $(0, -4)$. The tangent line has slope $2a$ and equation $y - (-4) = 2a(x - 0) \Leftrightarrow y = 2ax - 4$. Since (a, a^2) also lies on the line, $a^2 = 2a(a) - 4$, or $a^2 = 4$. So $a = \pm 2$ and the points are $(2, 4)$ and $(-2, 4)$.

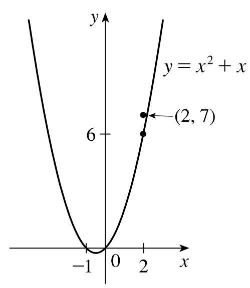
86. (a) If $y = x^2 + x$, then $y' = 2x + 1$. If the point at which a tangent meets the parabola is $(a, a^2 + a)$, then the slope of the tangent is $2a + 1$. But since it passes through $(2, -3)$, the slope must also be $\dfrac{\Delta y}{\Delta x} = \dfrac{a^2 + a + 3}{a - 2}$.

Therefore, $2a + 1 = \dfrac{a^2 + a + 3}{a - 2}$. Solving this equation for a we get $a^2 + a + 3 = 2a^2 - 3a - 2 \quad \Leftrightarrow$ $a^2 - 4a - 5 = (a - 5)(a + 1) = 0 \quad \Leftrightarrow \quad a = 5$ or -1. If $a = -1$, the point is $(-1, 0)$ and the slope is -1, so the equation is $y - 0 = (-1)(x + 1)$ or $y = -x - 1$. If $a = 5$, the point is $(5, 30)$ and the slope is 11, so the equation is $y - 30 = 11(x - 5)$ or $y = 11x - 25$.

(b) As in part (a), but using the point $(2, 7)$, we get the equation

$2a + 1 = \dfrac{a^2 + a - 7}{a - 2} \quad \Rightarrow \quad 2a^2 - 3a - 2 = a^2 + a - 7 \quad \Leftrightarrow \quad a^2 - 4a + 5 = 0.$

The last equation has no real solution (discriminant $= -16 < 0$), so there is no line through the point $(2, 7)$ that is tangent to the parabola. The diagram shows that the point $(2, 7)$ is "inside" the parabola, but tangent lines to the parabola do not pass through points inside the parabola.



87. (a) $(fgh)' = [(fg)h]' = (fg)'h + (fg)h' = (f'g + fg')h + (fg)h' = f'gh + fg'h + fgh'$

(b) Putting $f = g = h$ in part (a), we have $\dfrac{d}{dx}[f(x)]^3 = (fff)' = f'ff + ff'f + fff' = 3fff' = 3[f(x)]^2 f'(x)$.

(c) $y = (x^4 + 3x^3 + 17x + 82)^3 \quad \Rightarrow \quad y' = 3(x^4 + 3x^3 + 17x + 82)^2(4x^3 + 9x^2 + 17)$

88. (a) $f(x) = x^n \quad \Rightarrow \quad f'(x) = nx^{n-1} \quad \Rightarrow \quad f''(x) = n(n-1)x^{n-2} \quad \Rightarrow \quad \cdots \quad \Rightarrow$

$f^{(n)}(x) = n(n-1)(n-2)\cdots 2 \cdot 1x^{n-n} = n!$

(b) $f(x) = x^{-1} \quad \Rightarrow \quad f'(x) = (-1)x^{-2} \quad \Rightarrow \quad f''(x) = (-1)(-2)x^{-3} \quad \Rightarrow \quad \cdots \quad \Rightarrow$

$f^{(n)}(x) = (-1)(-2)(-3)\cdots(-n)x^{-(n+1)} = (-1)^n n! x^{-(n+1)}$ or $\dfrac{(-1)^n\, n!}{x^{n+1}}$

89. Let $P(x) = ax^2 + bx + c$. Then $P'(x) = 2ax + b$ and $P''(x) = 2a$. $P''(2) = 2 \quad \Rightarrow \quad 2a = 2 \quad \Rightarrow \quad a = 1$.
$P'(2) = 3 \quad \Rightarrow \quad 2(1)(2) + b = 3 \quad \Rightarrow \quad 4 + b = 3 \quad \Rightarrow \quad b = -1$.
$P(2) = 5 \quad \Rightarrow \quad 1(2)^2 + (-1)(2) + c = 5 \quad \Rightarrow \quad 2 + c = 5 \quad \Rightarrow \quad c = 3$. So $P(x) = x^2 - x + 3$.

90. $y = Ax^2 + Bx + C \quad \Rightarrow \quad y' = 2Ax + B \quad \Rightarrow \quad y'' = 2A$. We substitute these expressions into the equation $y'' + y' - 2y = x^2$ to get

$$(2A) + (2Ax + B) - 2(Ax^2 + Bx + C) = x^2$$
$$2A + 2Ax + B - 2Ax^2 - 2Bx - 2C = x^2$$
$$(-2A)x^2 + (2A - 2B)x + (2A + B - 2C) = (1)x^2 + (0)x + (0)$$

The coefficients of x^2 on each side must be equal, so $-2A = 1 \quad \Rightarrow \quad A = -\frac{1}{2}$. Similarly, $2A - 2B = 0 \quad \Rightarrow$ $A = B = -\frac{1}{2}$ and $2A + B - 2C = 0 \quad \Rightarrow \quad -1 - \frac{1}{2} - 2C = 0 \quad \Rightarrow \quad C = -\frac{3}{4}$.

91. $y = f(x) = ax^3 + bx^2 + cx + d \quad \Rightarrow \quad f'(x) = 3ax^2 + 2bx + c$. The point $(-2, 6)$ is on f, so $f(-2) = 6 \quad \Rightarrow$ $-8a + 4b - 2c + d = 6$ **(1)**. The point $(2, 0)$ is on f, so $f(2) = 0 \quad \Rightarrow \quad 8a + 4b + 2c + d = 0$ **(2)**. Since there are horizontal tangents at $(-2, 6)$ and $(2, 0)$, $f'(\pm 2) = 0$. $f'(-2) = 0 \quad \Rightarrow \quad 12a - 4b + c = 0$ **(3)** and $f'(2) = 0 \quad \Rightarrow$ $12a + 4b + c = 0$ **(4)**. Subtracting equation **(3)** from **(4)** gives $8b = 0 \quad \Rightarrow \quad b = 0$. Adding **(1)** and **(2)** gives $8b + 2d = 6$, so $d = 3$ since $b = 0$. From **(3)** we have $c = -12a$, so **(2)** becomes $8a + 4(0) + 2(-12a) + 3 = 0 \quad \Rightarrow \quad 3 = 16a \quad \Rightarrow$ $a = \frac{3}{16}$. Now $c = -12a = -12\left(\frac{3}{16}\right) = -\frac{9}{4}$ and the desired cubic function is $y = \frac{3}{16}x^3 - \frac{9}{4}x + 3$.

92. $y = ax^2 + bx + c \quad \Rightarrow \quad y'(x) = 2ax + b$. The parabola has slope 4 at $x = 1$ and slope -8 at $x = -1$, so $y'(1) = 4 \quad \Rightarrow$ $2a + b = 4$ **(1)** and $y'(-1) = -8 \quad \Rightarrow \quad -2a + b = -8$ **(2)**. Adding **(1)** and **(2)** gives us $2b = -4 \quad \Leftrightarrow \quad b = -2$. From **(1)**, $2a - 2 = 4 \quad \Leftrightarrow \quad a = 3$. Thus, the equation of the parabola is $y = 3x^2 - 2x + c$. Since it passes through the point $(2, 15)$, we have $15 = 3(2)^2 - 2(2) + c \quad \Rightarrow \quad c = 7$, so the equation is $y = 3x^2 - 2x + 7$.

93. If $P(t)$ denotes the population at time t and $A(t)$ the average annual income, then $T(t) = P(t)A(t)$ is the total personal income. The rate at which $T(t)$ is rising is given by $T'(t) = P(t)A'(t) + A(t)P'(t) \quad \Rightarrow$

$$\begin{aligned} T'(1999) &= P(1999)A'(1999) + A(1999)P'(1999) = (961{,}400)(\$1400/\text{yr}) + (\$30{,}593)(9200/\text{yr}) \\ &= \$1{,}345{,}960{,}000/\text{yr} + \$281{,}455{,}600/\text{yr} = \$1{,}627{,}415{,}600/\text{yr} \end{aligned}$$

So the total personal income was rising by about \$1.627 billion per year in 1999.

The term $P(t)A'(t) \approx \$1.346$ billion represents the portion of the rate of change of total income due to the existing population's increasing income. The term $A(t)P'(t) \approx \$281$ million represents the portion of the rate of change of total income due to increasing population.

94. (a) $f(20) = 10{,}000$ means that when the price of the fabric is \$20/yard, 10,000 yards will be sold.

$f'(20) = -350$ means that as the price of the fabric increases past \$20/yard, the amount of fabric which will be sold is decreasing at a rate of 350 yards per (dollar per yard).

(b) $R(p) = pf(p) \quad \Rightarrow \quad R'(p) = pf'(p) + f(p) \cdot 1 \quad \Rightarrow \quad R'(20) = 20f'(20) + f(20) \cdot 1 = 20(-350) + 10{,}000 = 3000$.

This means that as the price of the fabric increases past \$20/yard, the total revenue is increasing at \$3000/(\$/yard). Note that the Product Rule indicates that we will lose \$7000/(\$/yard) due to selling less fabric, but this loss is more than made up for by the additional revenue due to the increase in price.

95. $v = \dfrac{0.14[\text{S}]}{0.015 + [\text{S}]} \quad \Rightarrow \quad \dfrac{dv}{d[\text{S}]} = \dfrac{(0.015 + [\text{S}])(0.14) - (0.14[\text{S}])(1)}{(0.015 + [\text{S}])^2} = \dfrac{0.0021}{(0.015 + [\text{S}])^2}$.

$dv/d[\text{S}]$ represents the rate of change of the rate of an enzymatic reaction with respect to the concentration of a substrate S.

96. $B(t) = N(t)\,M(t) \quad \Rightarrow \quad B'(t) = N(t)\,M'(t) + M(t)\,N'(t)$, so

$B'(4) = N(4)\,M'(4) + M(4)\,N'(4) = 820(0.14) + 1.2(50) = 174.8$ g/week.

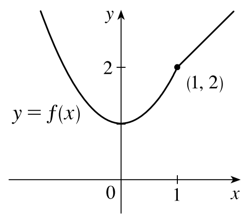
97. $f(x) = \begin{cases} x^2 + 1 & \text{if } x < 1 \\ x + 1 & \text{if } x \geq 1 \end{cases}$

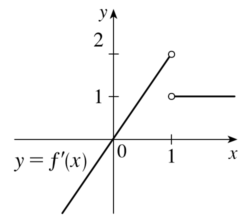
Calculate the left- and right-hand derivatives as defined in Exercise 2.2.62:

$$f'_{-}(1) = \lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^-} \frac{[(1+h)^2 + 1] - (1+1)}{h} = \lim_{h \to 0^-} \frac{h^2 + 2h}{h} = \lim_{h \to 0^-} (h+2) = 2 \text{ and}$$

$$f'_{+}(1) = \lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^+} \frac{[(1+h) + 1] - (1+1)}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} 1 = 1.$$

Since the left and right limits are different, $\lim\limits_{h \to 0} \dfrac{f(1+h) - f(1)}{h}$ does not exist, that is, $f'(1)$ does not exist. Therefore, f is not differentiable at 1.





98. $g(x) = \begin{cases} 2x & \text{if } x \leq 0 \\ 2x - x^2 & \text{if } 0 < x < 2 \\ 2 - x & \text{if } x \geq 2 \end{cases}$

Investigate the left- and right-hand derivatives at $x = 0$ and $x = 2$:

$$g'_{-}(0) = \lim_{h \to 0^-} \frac{g(0+h) - g(0)}{h} = \lim_{h \to 0^-} \frac{2h - 2(0)}{h} = 2 \text{ and}$$

$$g'_{+}(0) = \lim_{h \to 0^+} \frac{g(0+h) - g(0)}{h} = \lim_{h \to 0^+} \frac{(2h - h^2) - 2(0)}{h} = \lim_{h \to 0^+} (2-h) = 2, \text{ so } g \text{ is differentiable at } x = 0.$$

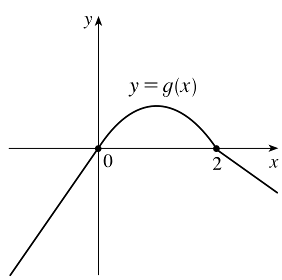
$$g'_{-}(2) = \lim_{h \to 0^-} \frac{g(2+h) - g(2)}{h} = \lim_{h \to 0^-} \frac{2(2+h) - (2+h)^2 - (2-2)}{h} = \lim_{h \to 0^-} \frac{-2h - h^2}{h} = \lim_{h \to 0^-} (-2-h) = -2$$

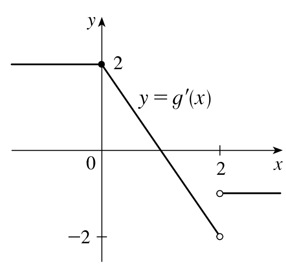
and

$$g'_{+}(2) = \lim_{h \to 0^+} \frac{g(2+h) - g(2)}{h} = \lim_{h \to 0^+} \frac{[2 - (2+h)] - (2-2)}{h} = \lim_{h \to 0^+} \frac{-h}{h} = \lim_{h \to 0^+} (-1) = -1,$$

so g is not differentiable at $x = 2$. Thus, a formula for g' is

$$g'(x) = \begin{cases} 2 & \text{if } x \leq 0 \\ 2 - 2x & \text{if } 0 < x < 2 \\ -1 & \text{if } x > 2 \end{cases}$$





99. (a) Note that $x^2 - 9 < 0$ for $x^2 < 9 \quad \Leftrightarrow \quad |x| < 3 \quad \Leftrightarrow \quad -3 < x < 3$. So

$$f(x) = \begin{cases} x^2 - 9 & \text{if } x \leq -3 \\ -x^2 + 9 & \text{if } -3 < x < 3 \\ x^2 - 9 & \text{if } x \geq 3 \end{cases} \quad \Rightarrow \quad f'(x) = \begin{cases} 2x & \text{if } x < -3 \\ -2x & \text{if } -3 < x < 3 \\ 2x & \text{if } x > 3 \end{cases} = \begin{cases} 2x & \text{if } |x| > 3 \\ -2x & \text{if } |x| < 3 \end{cases}$$

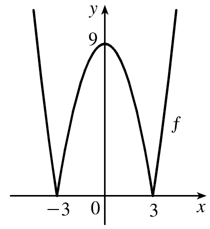
To show that $f'(3)$ does not exist we investigate $\lim_{h\to 0}\dfrac{f(3+h)-f(3)}{h}$ by computing the left- and right-hand derivatives defined in Exercise 2.2.62.

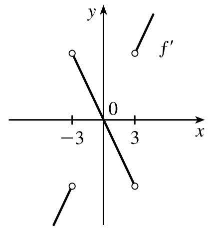
$$f'_-(3)=\lim_{h\to 0^-}\frac{f(3+h)-f(3)}{h}=\lim_{h\to 0^-}\frac{[-(3+h)^2+9]-0}{h}=\lim_{h\to 0^-}(-6-h)=-6 \quad\text{and}$$

$$f'_+(3)=\lim_{h\to 0^+}\frac{f(3+h)-f(3)}{h}=\lim_{h\to 0^+}\frac{\left[(3+h)^2-9\right]-0}{h}=\lim_{h\to 0^+}\frac{6h+h^2}{h}=\lim_{h\to 0^+}(6+h)=6.$$

Since the left and right limits are different, $\lim_{h\to 0}\dfrac{f(3+h)-f(3)}{h}$ does not exist, that is, $f'(3)$ does not exist. Similarly, $f'(-3)$ does not exist. Therefore, f is not differentiable at 3 or at -3.

(b)





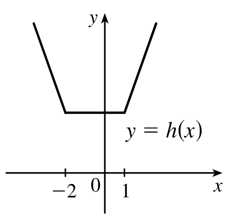
100. If $x \geq 1$, then $h(x)=|x-1|+|x+2|=x-1+x+2=2x+1$.

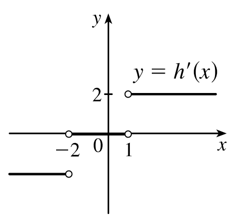
If $-2<x<1$, then $h(x)=-(x-1)+x+2=3$.

If $x\leq -2$, then $h(x)=-(x-1)-(x+2)=-2x-1$. Therefore,

$$h(x)=\begin{cases}-2x-1 & \text{if } x\leq -2\\ 3 & \text{if } -2<x<1\\ 2x+1 & \text{if } x\geq 1\end{cases}\quad\Rightarrow\quad h'(x)=\begin{cases}-2 & \text{if } x<-2\\ 0 & \text{if } -2<x<1\\ 2 & \text{if } x>1\end{cases}$$

To see that $h'(1)=\lim_{x\to 1}\dfrac{h(x)-h(1)}{x-1}$ does not exist, observe that $\lim_{x\to 1^-}\dfrac{h(x)-h(1)}{x-1}=\lim_{x\to 1^-}\dfrac{3-3}{3-1}=0$ but $\lim_{x\to 1^+}\dfrac{h(x)-h(1)}{x-1}=\lim_{x\to 1^+}\dfrac{2x-2}{x-1}=2$. Similarly, $h'(-2)$ does not exist.





101. $y=f(x)=ax^2 \quad\Rightarrow\quad f'(x)=2ax$. So the slope of the tangent to the parabola at $x=2$ is $m=2a(2)=4a$. The slope of the given line, $2x+y=b \quad\Leftrightarrow\quad y=-2x+b$, is seen to be -2, so we must have $4a=-2 \quad\Leftrightarrow\quad a=-\frac{1}{2}$. So when $x=2$, the point in question has y-coordinate $-\frac{1}{2}\cdot 2^2=-2$. Now we simply require that the given line, whose equation is $2x+y=b$, pass through the point $(2,-2)$: $2(2)+(-2)=b \quad\Leftrightarrow\quad b=2$. So we must have $a=-\frac{1}{2}$ and $b=2$.

102. (a) We use the Product Rule repeatedly: $F=fg \quad\Rightarrow\quad F'=f'g+fg' \quad\Rightarrow$

$$F''=(f''g+f'g')+(f'g'+fg'')=f''g+2f'g'+fg''.$$

(b) $F'''=f'''g+f''g'+2\,(f''g'+f'g'')+f'g''+fg'''=f'''g+3f''g'+3f'g''+fg''' \quad\Rightarrow$

$$\begin{aligned}F^{(4)}&=f^{(4)}g+f'''g'+3\,(f'''g'+f''g'')+3\,(f''g''+f'g''')+f'g'''+fg^{(4)}\\&=f^{(4)}g+4f'''g'+6f''g''+4f'g'''+fg^{(4)}\end{aligned}$$

(c) By analogy with the Binomial Theorem, we make the guess:

$$F^{(n)} = f^{(n)}g + nf^{(n-1)}g' + \binom{n}{2} f^{(n-2)}g'' + \cdots + \binom{n}{k} f^{(n-k)}g^{(k)} + \cdots + nf'g^{(n-1)} + fg^{(n)},$$

where $\binom{n}{k} = \dfrac{n!}{k!\,(n-k)!} = \dfrac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$.

103. The slope of the curve $y = c\sqrt{x}$ is $y' = \dfrac{c}{2\sqrt{x}}$ and the slope of the tangent line $y = \frac{3}{2}x + 6$ is $\frac{3}{2}$. These must be equal at the point of tangency $\left(a, c\sqrt{a}\right)$, so $\dfrac{c}{2\sqrt{a}} = \dfrac{3}{2} \Rightarrow c = 3\sqrt{a}$. The y-coordinates must be equal at $x = a$, so $c\sqrt{a} = \frac{3}{2}a + 6 \Rightarrow \left(3\sqrt{a}\right)\sqrt{a} = \frac{3}{2}a + 6 \Rightarrow 3a = \frac{3}{2}a + 6 \Rightarrow \frac{3}{2}a = 6 \Rightarrow a = 4$. Since $c = 3\sqrt{a}$, we have $c = 3\sqrt{4} = 6$.

104. f is clearly differentiable for $x < 2$ and for $x > 2$. For $x < 2$, $f'(x) = 2x$, so $f'_-(2) = 4$. For $x > 2$, $f'(x) = m$, so $f'_+(2) = m$. For f to be differentiable at $x = 2$, we need $4 = f'_-(2) = f'_+(2) = m$. So $f(x) = 4x + b$. We must also have continuity at $x = 2$, so $4 = f(2) = \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (4x + b) = 8 + b$. Hence, $b = -4$.

105. $F = f/g \Rightarrow f = Fg \Rightarrow f' = F'g + Fg' \Rightarrow F' = \dfrac{f' - Fg'}{g} = \dfrac{f' - (f/g)g'}{g} = \dfrac{f'g - fg'}{g^2}$

106. (a) $xy = c \Rightarrow y = \dfrac{c}{x}$. Let $P = \left(a, \dfrac{c}{a}\right)$. The slope of the tangent line at $x = a$ is $y'(a) = -\dfrac{c}{a^2}$. Its equation is $y - \dfrac{c}{a} = -\dfrac{c}{a^2}(x - a)$ or $y = -\dfrac{c}{a^2}x + \dfrac{2c}{a}$, so its y-intercept is $\dfrac{2c}{a}$. Setting $y = 0$ gives $x = 2a$, so the x-intercept is $2a$. The midpoint of the line segment joining $\left(0, \dfrac{2c}{a}\right)$ and $(2a, 0)$ is $\left(a, \dfrac{c}{a}\right) = P$.

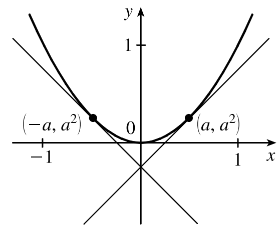
(b) We know the x- and y-intercepts of the tangent line from part (a), so the area of the triangle bounded by the axes and the tangent is $\frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}xy = \frac{1}{2}(2a)(2c/a) = 2c$, a constant.

107. *Solution 1:* Let $f(x) = x^{1000}$. Then, by the definition of a derivative, $f'(1) = \lim\limits_{x\to 1} \dfrac{f(x) - f(1)}{x - 1} = \lim\limits_{x\to 1} \dfrac{x^{1000} - 1}{x - 1}$. But this is just the limit we want to find, and we know (from the Power Rule) that $f'(x) = 1000x^{999}$, so $f'(1) = 1000(1)^{999} = 1000$. So $\lim\limits_{x\to 1} \dfrac{x^{1000} - 1}{x - 1} = 1000$.

Solution 2: Note that $(x^{1000} - 1) = (x - 1)(x^{999} + x^{998} + x^{997} + \cdots + x^2 + x + 1)$. So

$$\lim_{x\to 1} \frac{x^{1000} - 1}{x - 1} = \lim_{x\to 1} \frac{(x - 1)(x^{999} + x^{998} + x^{997} + \cdots + x^2 + x + 1)}{x - 1} = \lim_{x\to 1} (x^{999} + x^{998} + x^{997} + \cdots + x^2 + x + 1)$$

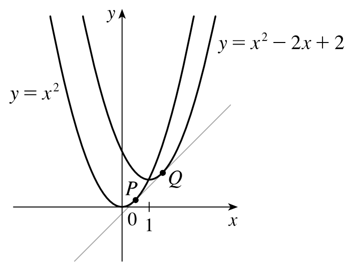
$$= \underbrace{1 + 1 + 1 + \cdots + 1 + 1 + 1}_{1000 \text{ ones}} = 1000, \text{ as above.}$$



108. In order for the two tangents to intersect on the y-axis, the points of tangency must be at equal distances from the y-axis, since the parabola $y = x^2$ is symmetric about the y-axis. Say the points of tangency are (a, a^2) and $(-a, a^2)$, for some $a > 0$. Then since the derivative of $y = x^2$ is $dy/dx = 2x$, the left-hand tangent has slope $-2a$ and equation $y - a^2 = -2a(x + a)$, or $y = -2ax - a^2$, and similarly the right-hand tangent line has equation $y - a^2 = 2a(x - a)$, or $y = 2ax - a^2$. So the two lines intersect at $(0, -a^2)$. Now if the lines are perpendicular, then the product of their slopes is -1, so $(-2a)(2a) = -1 \quad \Leftrightarrow \quad a^2 = \frac{1}{4} \quad \Leftrightarrow \quad a = \frac{1}{2}$. So the lines intersect at $\left(0, -\frac{1}{4}\right)$.

109. $y = x^2 \quad \Rightarrow \quad y' = 2x$, so the slope of a tangent line at the point (a, a^2) is $y' = 2a$ and the slope of a normal line is $-1/(2a)$, for $a \neq 0$. The slope of the normal line through the points (a, a^2) and $(0, c)$ is $\dfrac{a^2 - c}{a - 0}$, so $\dfrac{a^2 - c}{a} = -\dfrac{1}{2a} \quad \Rightarrow$ $a^2 - c = -\frac{1}{2} \quad \Rightarrow \quad a^2 = c - \frac{1}{2}$. The last equation has two solutions if $c > \frac{1}{2}$, one solution if $c = \frac{1}{2}$, and no solution if $c < \frac{1}{2}$. Since the y-axis is normal to $y = x^2$ regardless of the value of c (this is the case for $a = 0$), we have three normal lines if $c > \frac{1}{2}$ and one normal line if $c \leq \frac{1}{2}$.

110.



From the sketch, it appears that there may be a line that is tangent to both curves. The slope of the line through the points $P(a, a^2)$ and $Q(b, b^2 - 2b + 2)$ is $\dfrac{b^2 - 2b + 2 - a^2}{b - a}$. The slope of the tangent line at P is $2a$ $\quad [y' = 2x]$ and at Q is $2b - 2$ $\quad [y' = 2x - 2]$. All three slopes are equal, so $2a = 2b - 2 \quad \Leftrightarrow \quad a = b - 1$.

Also, $2b - 2 = \dfrac{b^2 - 2b + 2 - a^2}{b - a} \quad \Rightarrow \quad 2b - 2 = \dfrac{b^2 - 2b + 2 - (b - 1)^2}{b - (b - 1)} \quad \Rightarrow \quad 2b - 2 = b^2 - 2b + 2 - b^2 + 2b - 1 \quad \Rightarrow$ $2b = 3 \quad \Rightarrow \quad b = \frac{3}{2}$ and $a = \frac{3}{2} - 1 = \frac{1}{2}$. Thus, an equation of the tangent line at P is $y - \left(\frac{1}{2}\right)^2 = 2\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right)$ or $y = x - \frac{1}{4}$.

APPLIED PROJECT Building a Better Roller Coaster

1. (a) $f(x) = ax^2 + bx + c \quad \Rightarrow \quad f'(x) = 2ax + b$.

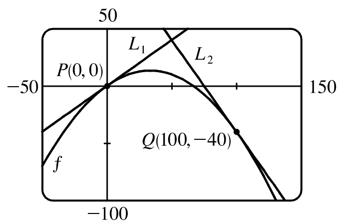
The origin is at P:	$f(0) = 0$	$\Rightarrow$	$c = 0$
The slope of the ascent is 0.8:	$f'(0) = 0.8$	$\Rightarrow$	$b = 0.8$
The slope of the drop is -1.6:	$f'(100) = -1.6$	$\Rightarrow$	$200a + b = -1.6$

(b) $b = 0.8$, so $200a + b = -1.6 \quad \Rightarrow \quad 200a + 0.8 = -1.6 \quad \Rightarrow \quad 200a = -2.4 \quad \Rightarrow \quad a = -\dfrac{2.4}{200} = -0.012$.

Thus, $f(x) = -0.012x^2 + 0.8x$.

(c) Since L_1 passes through the origin with slope 0.8, it has equation $y = 0.8x$.

The horizontal distance between P and Q is 100, so the y-coordinate at Q is $f(100) = -0.012(100)^2 + 0.8(100) = -40$. Since L_2 passes through the point $(100, -40)$ and has slope -1.6, it has equation $y + 40 = -1.6(x - 100)$ or $y = -1.6x + 120$.



(d) The difference in elevation between $P(0, 0)$ and $Q(100, -40)$ is $0 - (-40) = 40$ feet.

2. (a)

Interval	Function	First Derivative	Second Derivative
$(-\infty, 0)$	$L_1(x) = 0.8x$	$L_1'(x) = 0.8$	$L_1''(x) = 0$
$[0, 10)$	$g(x) = kx^3 + lx^2 + mx + n$	$g'(x) = 3kx^2 + 2lx + m$	$g''(x) = 6kx + 2l$
$[10, 90]$	$q(x) = ax^2 + bx + c$	$q'(x) = 2ax + b$	$q''(x) = 2a$
$(90, 100]$	$h(x) = px^3 + qx^2 + rx + s$	$h'(x) = 3px^2 + 2qx + r$	$h''(x) = 6px + 2q$
$(100, \infty)$	$L_2(x) = -1.6x + 120$	$L_2'(x) = -1.6$	$L_2''(x) = 0$

There are 4 values of x (0, 10, 90, and 100) for which we must make sure the function values are equal, the first derivative values are equal, and the second derivative values are equal. The third column in the following table contains the value of each side of the condition — these are found after solving the system in part (b).

At $x =$	Condition	Value	Resulting Equation
0	$g(0) = L_1(0)$	0	$n = 0$
	$g'(0) = L_1'(0)$	$\frac{4}{5}$	$m = 0.8$
	$g''(0) = L_1''(0)$	0	$2l = 0$
10	$g(10) = q(10)$	$\frac{68}{9}$	$1000k + 100l + 10m + n = 100a + 10b + c$
	$g'(10) = q'(10)$	$\frac{2}{3}$	$300k + 20l + m = 20a + b$
	$g''(10) = q''(10)$	$-\frac{2}{75}$	$60k + 2l = 2a$
90	$h(90) = q(90)$	$-\frac{220}{9}$	$729{,}000p + 8100q + 90r + s = 8100a + 90b + c$
	$h'(90) = q'(90)$	$-\frac{22}{15}$	$24{,}300p + 180q + r = 180a + b$
	$h''(90) = q''(90)$	$-\frac{2}{75}$	$540p + 2q = 2a$
100	$h(100) = L_2(100)$	-40	$1{,}000{,}000p + 10{,}000q + 100r + s = -40$
	$h'(100) = L_2'(100)$	$-\frac{8}{5}$	$30{,}000p + 200q + r = -1.6$
	$h''(100) = L_2''(100)$	0	$600p + 2q = 0$

(b) We can arrange our work in a 12×12 matrix as follows.

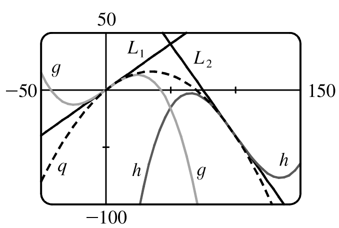
a	b	c	k	l	m	n	p	q	r	s	constant
0	0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0	0.8
0	0	0	0	2	0	0	0	0	0	0	0
−100	−10	−1	1000	100	10	1	0	0	0	0	0
−20	−1	0	300	20	1	0	0	0	0	0	0
−2	0	0	60	2	0	0	0	0	0	0	0
−8100	−90	−1	0	0	0	0	729,000	8100	90	1	0
−180	−1	0	0	0	0	0	24,300	180	1	0	0
−2	0	0	0	0	0	0	540	2	0	0	0
0	0	0	0	0	0	0	1,000,000	10,000	100	1	−40
0	0	0	0	0	0	0	30,000	200	1	0	−1.6
0	0	0	0	0	0	0	600	2	0	0	0

Solving the system gives us the formulas for q, g, and h.

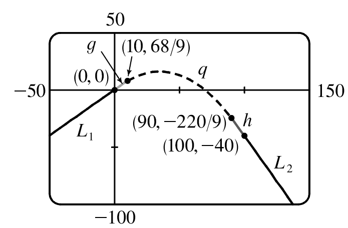
$$\left.\begin{aligned} a &= -0.01\overline{3} = -\tfrac{1}{75} \\ b &= 0.9\overline{3} = \tfrac{14}{15} \\ c &= -0.\overline{4} = -\tfrac{4}{9} \end{aligned}\right\} q(x) = -\tfrac{1}{75}x^2 + \tfrac{14}{15}x - \tfrac{4}{9} \qquad \left.\begin{aligned} k &= -0.000\overline{4} = -\tfrac{1}{2250} \\ l &= 0 \\ m &= 0.8 = \tfrac{4}{5} \\ n &= 0 \end{aligned}\right\} g(x) = -\tfrac{1}{2250}x^3 + \tfrac{4}{5}x$$

$$\left.\begin{aligned} p &= 0.000\overline{4} = \tfrac{1}{2250} \\ q &= -0.1\overline{3} = -\tfrac{2}{15} \\ r &= 11.7\overline{3} = \tfrac{176}{15} \\ s &= -324.\overline{4} = -\tfrac{2920}{9} \end{aligned}\right\} h(x) = \tfrac{1}{2250}x^3 - \tfrac{2}{15}x^2 + \tfrac{176}{15}x - \tfrac{2920}{9}$$

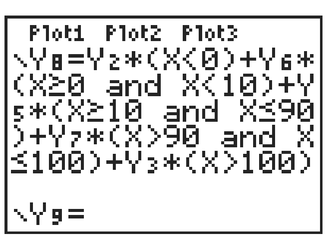
(c) Graph of L_1, q, g, h, and L_2:



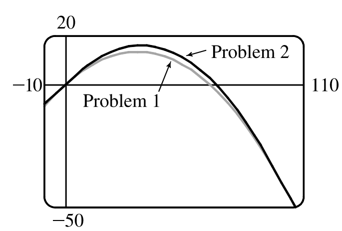
The graph of the five functions as a piecewise-defined function:



This is the piecewise-defined function assignment on a TI-83/4 Plus calculator, where $\mathrm{Y}_2 = L_1$, $\mathrm{Y}_6 = g$, $\mathrm{Y}_5 = q$, $\mathrm{Y}_7 = h$, and $\mathrm{Y}_3 = L_2$.



A comparison of the graphs in part 1(c) and part 2(c):



2.4 Derivatives of Trigonometric Functions

1. $f(x) = x^2 \sin x \overset{\text{PR}}{\Rightarrow} f'(x) = x^2 \cos x + (\sin x)(2x) = x^2 \cos x + 2x \sin x$

2. $f(x) = x \cos x + 2 \tan x \quad \Rightarrow \quad f'(x) = x(-\sin x) + (\cos x)(1) + 2 \sec^2 x = \cos x - x \sin x + 2 \sec^2 x$

3. $f(x) = 3 \cot x - 2 \cos x \quad \Rightarrow \quad f'(x) = 3(-\csc^2 x) - 2(-\sin x) = -3 \csc^2 x + 2 \sin x$

4. $y = 2 \sec x - \csc x \quad \Rightarrow \quad y' = 2(\sec x \, \tan x) - (-\csc x \, \cot x) = 2 \sec x \, \tan x + \csc x \, \cot x$

5. $y = \sec \theta \, \tan \theta \quad \Rightarrow \quad y' = \sec \theta \, (\sec^2 \theta) + \tan \theta \, (\sec \theta \tan \theta) = \sec \theta \, (\sec^2 \theta + \tan^2 \theta)$. Using the identity $1 + \tan^2 \theta = \sec^2 \theta$, we can write alternative forms of the answer as $\sec \theta \, (1 + 2 \tan^2 \theta)$ or $\sec \theta \, (2 \sec^2 \theta - 1)$.

6. $g(t) = 4 \sec t + \tan t \quad \Rightarrow \quad g'(t) = 4 \sec t \, \tan t + \sec^2 t$

7. $y = c \cos t + t^2 \sin t \quad \Rightarrow \quad y' = c(-\sin t) + t^2(\cos t) + \sin t \, (2t) = -c \sin t + t(t \cos t + 2 \sin t)$

8. $y = u(a \cos u + b \cot u) \quad \Rightarrow$

$y' = u(-a \sin u - b \csc^2 u) + (a \cos u + b \cot u) \cdot 1 = a \cos u + b \cot u - au \sin u - bu \csc^2 u$

9. $y = \dfrac{x}{2 - \tan x} \quad \Rightarrow \quad y' = \dfrac{(2 - \tan x)(1) - x(-\sec^2 x)}{(2 - \tan x)^2} = \dfrac{2 - \tan x + x \sec^2 x}{(2 - \tan x)^2}$

10. $y = \sin \theta \, \cos \theta \quad \Rightarrow \quad y' = \sin \theta(-\sin \theta) + \cos \theta(\cos \theta) = \cos^2 \theta - \sin^2 \theta \quad$ [or $\cos 2\theta$]

11. $f(\theta) = \dfrac{\sin \theta}{1 + \cos \theta} \quad \Rightarrow$

$$f'(\theta) = \frac{(1 + \cos \theta) \cos \theta - (\sin \theta)(-\sin \theta)}{(1 + \cos \theta)^2} = \frac{\cos \theta + \cos^2 \theta + \sin^2 \theta}{(1 + \cos \theta)^2} = \frac{\cos \theta + 1}{(1 + \cos \theta)^2} = \frac{1}{1 + \cos \theta}$$

12. $y = \dfrac{\cos x}{1 - \sin x} \quad \Rightarrow$

$$y' = \frac{(1 - \sin x)(-\sin x) - \cos x(-\cos x)}{(1 - \sin x)^2} = \frac{-\sin x + \sin^2 x + \cos^2 x}{(1 - \sin x)^2} = \frac{-\sin x + 1}{(1 - \sin x)^2} = \frac{1}{1 - \sin x}$$

13. $y = \dfrac{t \sin t}{1 + t} \quad \Rightarrow$

$$y' = \frac{(1 + t)(t \cos t + \sin t) - t \sin t(1)}{(1 + t)^2} = \frac{t \cos t + \sin t + t^2 \cos t + t \sin t - t \sin t}{(1 + t)^2} = \frac{(t^2 + t) \cos t + \sin t}{(1 + t)^2}$$

14. $y = \dfrac{\sin t}{1 + \tan t} \quad \Rightarrow$

$$y' = \frac{(1 + \tan t) \cos t - (\sin t) \sec^2 t}{(1 + \tan t)^2} = \frac{\cos t + \sin t - \dfrac{\sin t}{\cos^2 t}}{(1 + \tan t)^2} = \frac{\cos t + \sin t - \tan t \sec t}{(1 + \tan t)^2}$$

15. Using Exercise 2.3.87(a), $f(\theta) = \theta \cos \theta \, \sin \theta \quad \Rightarrow$

$$\begin{aligned} f'(\theta) &= 1 \cos \theta \, \sin \theta + \theta(-\sin \theta) \sin \theta + \theta \cos \theta(\cos \theta) = \cos \theta \, \sin \theta - \theta \sin^2 \theta + \theta \cos^2 \theta \\ &= \sin \theta \, \cos \theta + \theta(\cos^2 \theta - \sin^2 \theta) = \tfrac{1}{2} \sin 2\theta + \theta \cos 2\theta \quad \text{[using double-angle formulas]} \end{aligned}$$

16. Using Exercise 2.3.87(a), $f(x) = x^2 \sin x \tan x \Rightarrow$

$$f'(x) = (x^2)' \sin x \tan x + x^2(\sin x)' \tan x + x^2 \sin x\,(\tan x)' = 2x \sin x \tan x + x^2 \cos x \tan x + x^2 \sin x \sec^2 x$$
$$= 2x \sin x \tan x + x^2 \sin x + x^2 \sin x \sec^2 x = x \sin x\,(2 \tan x + x + x \sec^2 x).$$

17. $\dfrac{d}{dx}(\csc x) = \dfrac{d}{dx}\left(\dfrac{1}{\sin x}\right) = \dfrac{(\sin x)(0) - 1(\cos x)}{\sin^2 x} = \dfrac{-\cos x}{\sin^2 x} = -\dfrac{1}{\sin x} \cdot \dfrac{\cos x}{\sin x} = -\csc x \cot x$

18. $\dfrac{d}{dx}(\sec x) = \dfrac{d}{dx}\left(\dfrac{1}{\cos x}\right) = \dfrac{(\cos x)(0) - 1(-\sin x)}{\cos^2 x} = \dfrac{\sin x}{\cos^2 x} = \dfrac{1}{\cos x} \cdot \dfrac{\sin x}{\cos x} = \sec x \tan x$

19. $\dfrac{d}{dx}(\cot x) = \dfrac{d}{dx}\left(\dfrac{\cos x}{\sin x}\right) = \dfrac{(\sin x)(-\sin x) - (\cos x)(\cos x)}{\sin^2 x} = -\dfrac{\sin^2 x + \cos^2 x}{\sin^2 x} = -\dfrac{1}{\sin^2 x} = -\csc^2 x$

20. $f(x) = \cos x \Rightarrow$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} = \lim_{h\to 0} \frac{\cos x \cosh h - \sin x \sin h - \cos x}{h}$$
$$= \lim_{h\to 0}\left(\cos x\,\frac{\cos h - 1}{h} - \sin x\,\frac{\sin h}{h}\right) = \cos x \lim_{h\to 0}\frac{\cos h - 1}{h} - \sin x \lim_{h\to 0}\frac{\sin h}{h}$$
$$= (\cos x)(0) - (\sin x)(1) = -\sin x$$

21. $y = \sin x + \cos x \Rightarrow y' = \cos x - \sin x$, so $y'(0) = \cos 0 - \sin 0 = 1 - 0 = 1$. An equation of the tangent line to the curve $y = \sin x + \cos x$ at the point $(0, 1)$ is $y - 1 = 1(x - 0)$ or $y = x + 1$.

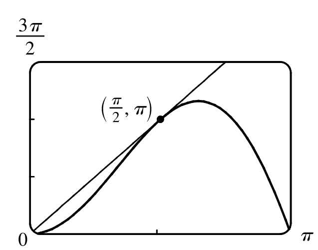
22. $y = (1 + x)\cos x \Rightarrow y' = (1 + x)(-\sin x) + \cos x \cdot 1$. At $(0, 1)$, $y' = 1$, and an equation of the tangent line is $y - 1 = 1(x - 0)$ or $y = x + 1$.

23. $y = \cos x - \sin x \Rightarrow y' = -\sin x - \cos x$, so $y'(\pi) = -\sin \pi - \cos \pi = 0 - (-1) = 1$. An equation of the tangent line to the curve $y = \cos x - \sin x$ at the point $(\pi, -1)$ is $y - (-1) = 1(x - \pi)$ or $y = x - \pi - 1$.

24. $y = x + \tan x \Rightarrow y' = 1 + \sec^2 x$, so $y'(\pi) = 1 + (-1)^2 = 2$. An equation of the tangent line to the curve $y = x + \tan x$ at the point (π, π) is $y - \pi = 2(x - \pi)$ or $y = 2x - \pi$.

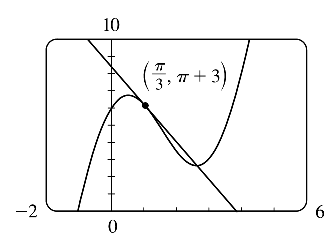
25. (a) $y = 2x \sin x \Rightarrow y' = 2(x \cos x + \sin x \cdot 1)$. At $\left(\frac{\pi}{2}, \pi\right)$, $y' = 2\left(\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\right) = 2(0 + 1) = 2$, and an equation of the tangent line is $y - \pi = 2\left(x - \frac{\pi}{2}\right)$, or $y = 2x$.

(b)



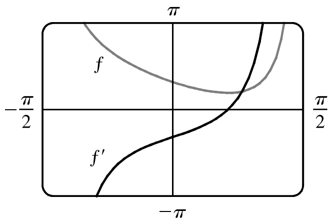
26. (a) $y = 3x + 6\cos x \Rightarrow y' = 3 - 6\sin x$. At $\left(\frac{\pi}{3}, \pi + 3\right)$, $y' = 3 - 6\sin\frac{\pi}{3} = 3 - 6\,\frac{\sqrt{3}}{2} = 3 - 3\sqrt{3}$, and an equation of the tangent line is $y - (\pi + 3) = \left(3 - 3\sqrt{3}\right)\left(x - \frac{\pi}{3}\right)$, or $y = \left(3 - 3\sqrt{3}\right)x + 3 + \pi\sqrt{3}$.

(b)



27. (a) $f(x) = \sec x - x \quad \Rightarrow \quad f'(x) = \sec x\, \tan x - 1$

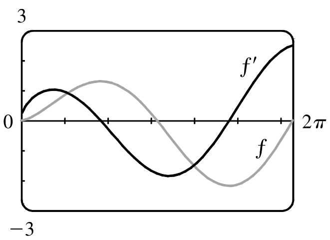
(b)



Note that $f' = 0$ where f has a minimum. Also note that f' is negative when f is decreasing and f' is positive when f is increasing.

28. (a) $f(x) = \sqrt{x}\sin x \quad \Rightarrow \quad f'(x) = \sqrt{x}\cos x + (\sin x)\left(\frac{1}{2}x^{-1/2}\right) = \sqrt{x}\cos x + \dfrac{\sin x}{2\sqrt{x}}$

(b)



Notice that $f'(x) = 0$ when f has a horizontal tangent.
f' is positive when f is increasing and f' is negative when f is decreasing.

29. $H(\theta) = \theta\sin\theta \quad \Rightarrow \quad H'(\theta) = \theta\,(\cos\theta) + (\sin\theta)\cdot 1 = \theta\cos\theta + \sin\theta \quad \Rightarrow$

$H''(\theta) = \theta\,(-\sin\theta) + (\cos\theta)\cdot 1 + \cos\theta = -\theta\sin\theta + 2\cos\theta$

30. $f(t) = \sec t \quad \Rightarrow \quad f'(t) = \sec t\, \tan t \quad \Rightarrow \quad f''(t) = (\sec t)\sec^2 t + (\tan t)\sec t\, \tan t = \sec^3 t + \sec t\, \tan^2 t$, so

$f''\left(\frac{\pi}{4}\right) = \left(\sqrt{2}\right)^3 + \sqrt{2}(1)^2 = 2\sqrt{2} + \sqrt{2} = 3\sqrt{2}$.

31. (a) $f(x) = \dfrac{\tan x - 1}{\sec x} \quad \Rightarrow$

$$f'(x) = \frac{\sec x(\sec^2 x) - (\tan x - 1)(\sec x \tan x)}{(\sec x)^2} = \frac{\sec x(\sec^2 x - \tan^2 x + \tan x)}{\sec^2 x} = \frac{1 + \tan x}{\sec x}$$

(b) $f(x) = \dfrac{\tan x - 1}{\sec x} = \dfrac{\dfrac{\sin x}{\cos x} - 1}{\dfrac{1}{\cos x}} = \dfrac{\dfrac{\sin x - \cos x}{\cos x}}{\dfrac{1}{\cos x}} = \sin x - \cos x \quad \Rightarrow \quad f'(x) = \cos x - (-\sin x) = \cos x + \sin x$

(c) From part (a), $f'(x) = \dfrac{1 + \tan x}{\sec x} = \dfrac{1}{\sec x} + \dfrac{\tan x}{\sec x} = \cos x + \sin x$, which is the expression for $f'(x)$ in part (b).

32. (a) $g(x) = f(x)\sin x \quad \Rightarrow \quad g'(x) = f(x)\cos x + \sin x \cdot f'(x)$, so

$g'\left(\frac{\pi}{3}\right) = f\left(\frac{\pi}{3}\right)\cos\frac{\pi}{3} + \sin\frac{\pi}{3}\cdot f'\left(\frac{\pi}{3}\right) = 4\cdot\frac{1}{2} + \frac{\sqrt{3}}{2}\cdot(-2) = 2 - \sqrt{3}$

(b) $h(x) = \dfrac{\cos x}{f(x)} \quad \Rightarrow \quad h'(x) = \dfrac{f(x)\cdot(-\sin x) - \cos x \cdot f'(x)}{[f(x)]^2}$, so

$$h'\left(\tfrac{\pi}{3}\right) = \frac{f\left(\frac{\pi}{3}\right)\cdot\left(-\sin\frac{\pi}{3}\right) - \cos\frac{\pi}{3}\cdot f'\left(\frac{\pi}{3}\right)}{\left[f\left(\frac{\pi}{3}\right)\right]^2} = \frac{4\left(-\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{2}\right)(-2)}{4^2} = \frac{-2\sqrt{3}+1}{16} = \frac{1-2\sqrt{3}}{16}$$

33. $f(x) = x + 2\sin x$ has a horizontal tangent when $f'(x) = 0 \quad \Leftrightarrow \quad 1 + 2\cos x = 0 \quad \Leftrightarrow \quad \cos x = -\frac{1}{2} \quad \Leftrightarrow$

$x = \frac{2\pi}{3} + 2\pi n$ or $\frac{4\pi}{3} + 2\pi n$, where n is an integer. Note that $\frac{4\pi}{3}$ and $\frac{2\pi}{3}$ are $\pm\frac{\pi}{3}$ units from π. This allows us to write the solutions in the more compact equivalent form $(2n+1)\pi \pm \frac{\pi}{3}$, n an integer.

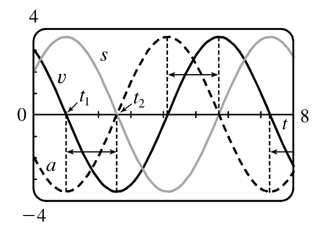
34. $y = \dfrac{\cos x}{2+\sin x} \quad\Rightarrow\quad y' = \dfrac{(2+\sin x)(-\sin x) - \cos x\,\cos x}{(2+\sin x)^2} = \dfrac{-2\sin x - \sin^2 x - \cos^2 x}{(2+\sin x)^2} = \dfrac{-2\sin x - 1}{(2+\sin x)^2} = 0$ when $-2\sin x - 1 = 0 \quad\Leftrightarrow\quad \sin x = -\frac{1}{2} \quad\Leftrightarrow\quad x = \frac{11\pi}{6} + 2\pi n$ or $x = \frac{7\pi}{6} + 2\pi n$, n an integer. So $y = \frac{1}{\sqrt{3}}$ or $y = -\frac{1}{\sqrt{3}}$ and the points on the curve with horizontal tangents are: $\left(\frac{11\pi}{6} + 2\pi n, \frac{1}{\sqrt{3}}\right)$, $\left(\frac{7\pi}{6} + 2\pi n, -\frac{1}{\sqrt{3}}\right)$, n an integer.

35. (a) $x(t) = 8\sin t \quad\Rightarrow\quad v(t) = x'(t) = 8\cos t \quad\Rightarrow\quad a(t) = x''(t) = -8\sin t$

(b) The mass at time $t = \frac{2\pi}{3}$ has position $x\left(\frac{2\pi}{3}\right) = 8\sin\frac{2\pi}{3} = 8\left(\frac{\sqrt{3}}{2}\right) = 4\sqrt{3}$, velocity $v\left(\frac{2\pi}{3}\right) = 8\cos\frac{2\pi}{3} = 8\left(-\frac{1}{2}\right) = -4$, and acceleration $a\left(\frac{2\pi}{3}\right) = -8\sin\frac{2\pi}{3} = -8\left(\frac{\sqrt{3}}{2}\right) = -4\sqrt{3}$. Since $v\left(\frac{2\pi}{3}\right) < 0$, the particle is moving to the left.

36. (a) $s(t) = 2\cos t + 3\sin t \quad\Rightarrow\quad v(t) = -2\sin t + 3\cos t \quad\Rightarrow\quad a(t) = -2\cos t - 3\sin t$

(b)

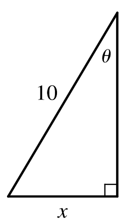


(c) $s = 0 \quad\Rightarrow\quad t_2 \approx 2.55$. So the mass passes through the equilibrium position for the first time when $t \approx 2.55$ s.

(d) $v = 0 \quad\Rightarrow\quad t_1 \approx 0.98$, $s(t_1) \approx 3.61$ cm. So the mass travels a maximum of about 3.6 cm (upward and downward) from its equilibrium position.

(e) The speed $|v|$ is greatest when $s = 0$, that is, when $t = t_2 + n\pi$, n a positive integer.

37.

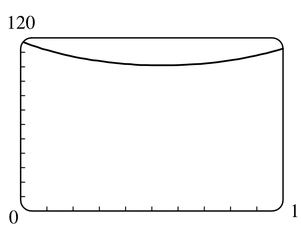


From the diagram we can see that $\sin\theta = x/10 \quad\Leftrightarrow\quad x = 10\sin\theta$. We want to find the rate of change of x with respect to θ, that is, $dx/d\theta$. Taking the derivative of $x = 10\sin\theta$, we get $dx/d\theta = 10(\cos\theta)$. So when $\theta = \frac{\pi}{3}$, $\frac{dx}{d\theta} = 10\cos\frac{\pi}{3} = 10\left(\frac{1}{2}\right) = 5$ ft/rad.

38. (a) $F = \dfrac{\mu W}{\mu\sin\theta + \cos\theta} \quad\Rightarrow\quad \dfrac{dF}{d\theta} = \dfrac{(\mu\sin\theta + \cos\theta)(0) - \mu W(\mu\cos\theta - \sin\theta)}{(\mu\sin\theta + \cos\theta)^2} = \dfrac{\mu W(\sin\theta - \mu\cos\theta)}{(\mu\sin\theta + \cos\theta)^2}$

(b) $\dfrac{dF}{d\theta} = 0 \quad\Leftrightarrow\quad \mu W(\sin\theta - \mu\cos\theta) = 0 \quad\Leftrightarrow\quad \sin\theta = \mu\cos\theta \quad\Leftrightarrow\quad \tan\theta = \mu \quad\Leftrightarrow\quad \theta = \tan^{-1}\mu$

(c)



From the graph of $F = \dfrac{0.6(50)}{0.6\sin\theta + \cos\theta}$ for $0 \le \theta \le 1$, we see that $\dfrac{dF}{d\theta} = 0 \quad\Rightarrow\quad \theta \approx 0.54$. Checking this with part (b) and $\mu = 0.6$, we calculate $\theta = \tan^{-1} 0.6 \approx 0.54$. So the value from the graph is consistent with the value in part (b).

39. $\displaystyle\lim_{x\to 0}\frac{\sin 5x}{3x} = \lim_{x\to 0}\frac{5}{3}\left(\frac{\sin 5x}{5x}\right) = \frac{5}{3}\lim_{x\to 0}\frac{\sin 5x}{5x} = \frac{5}{3}\lim_{\theta\to 0}\frac{\sin\theta}{\theta} \quad [\theta = 5x] \quad = \frac{5}{3}\cdot 1 = \frac{5}{3}$

40. $\lim\limits_{x\to 0}\dfrac{\sin x}{\sin \pi x}=\lim\limits_{x\to 0}\dfrac{\sin x}{x}\cdot\dfrac{\pi x}{\sin \pi x}\cdot\dfrac{1}{\pi}=\lim\limits_{x\to 0}\dfrac{\sin x}{x}\cdot\lim\limits_{\theta\to 0}\dfrac{\theta}{\sin\theta}\cdot\dfrac{1}{\pi}\quad[\theta=\pi x]$

$$=1\cdot\lim_{\theta\to 0}\frac{1}{\dfrac{\sin\theta}{\theta}}\cdot\frac{1}{\pi}=1\cdot 1\cdot\frac{1}{\pi}=\frac{1}{\pi}$$

41. $\lim\limits_{t\to 0}\dfrac{\tan 6t}{\sin 2t}=\lim\limits_{t\to 0}\left(\dfrac{\sin 6t}{t}\cdot\dfrac{1}{\cos 6t}\cdot\dfrac{t}{\sin 2t}\right)=\lim\limits_{t\to 0}\dfrac{6\sin 6t}{6t}\cdot\lim\limits_{t\to 0}\dfrac{1}{\cos 6t}\cdot\lim\limits_{t\to 0}\dfrac{2t}{2\sin 2t}$

$$=6\lim_{t\to 0}\frac{\sin 6t}{6t}\cdot\lim_{t\to 0}\frac{1}{\cos 6t}\cdot\frac{1}{2}\lim_{t\to 0}\frac{2t}{\sin 2t}=6(1)\cdot\frac{1}{1}\cdot\frac{1}{2}(1)=3$$

42. $\lim\limits_{\theta\to 0}\dfrac{\cos\theta-1}{\sin\theta}=\lim\limits_{\theta\to 0}\dfrac{\dfrac{\cos\theta-1}{\theta}}{\dfrac{\sin\theta}{\theta}}=\dfrac{\lim\limits_{\theta\to 0}\dfrac{\cos\theta-1}{\theta}}{\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}}=\dfrac{0}{1}=0$

43. $\lim\limits_{x\to 0}\dfrac{\sin 3x}{5x^3-4x}=\lim\limits_{x\to 0}\left(\dfrac{\sin 3x}{3x}\cdot\dfrac{3}{5x^2-4}\right)=\lim\limits_{x\to 0}\dfrac{\sin 3x}{3x}\cdot\lim\limits_{x\to 0}\dfrac{3}{5x^2-4}=1\cdot\left(\dfrac{3}{-4}\right)=-\dfrac{3}{4}$

44. $\lim\limits_{x\to 0}\dfrac{\sin 3x\sin 5x}{x^2}=\lim\limits_{x\to 0}\left(\dfrac{3\sin 3x}{3x}\cdot\dfrac{5\sin 5x}{5x}\right)=\lim\limits_{x\to 0}\dfrac{3\sin 3x}{3x}\cdot\lim\limits_{x\to 0}\dfrac{5\sin 5x}{5x}$

$$=3\lim_{x\to 0}\frac{\sin 3x}{3x}\cdot 5\lim_{x\to 0}\frac{\sin 5x}{5x}=3(1)\cdot 5(1)=15$$

45. Divide numerator and denominator by θ. ($\sin\theta$ also works.)

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta+\tan\theta}=\lim_{\theta\to 0}\frac{\dfrac{\sin\theta}{\theta}}{1+\dfrac{\sin\theta}{\theta}\cdot\dfrac{1}{\cos\theta}}=\frac{\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}}{1+\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}\lim\limits_{\theta\to 0}\dfrac{1}{\cos\theta}}=\frac{1}{1+1\cdot 1}=\frac{1}{2}$$

46. $\lim\limits_{x\to 0}\csc x\,\sin(\sin x)=\lim\limits_{x\to 0}\dfrac{\sin(\sin x)}{\sin x}=\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}\quad[\text{As } x\to 0,\ \theta=\sin x\to 0.]\quad=1$

47. $\lim\limits_{\theta\to 0}\dfrac{\cos\theta-1}{2\theta^2}=\lim\limits_{\theta\to 0}\dfrac{\cos\theta-1}{2\theta^2}\cdot\dfrac{\cos\theta+1}{\cos\theta+1}=\lim\limits_{\theta\to 0}\dfrac{\cos^2\theta-1}{2\theta^2(\cos\theta+1)}=\lim\limits_{\theta\to 0}\dfrac{-\sin^2\theta}{2\theta^2(\cos\theta+1)}$

$$=-\frac{1}{2}\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\frac{\sin\theta}{\theta}\cdot\frac{1}{\cos\theta+1}=-\frac{1}{2}\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\lim_{\theta\to 0}\frac{1}{\cos\theta+1}$$

$$=-\frac{1}{2}\cdot 1\cdot 1\cdot\frac{1}{1+1}=-\frac{1}{4}$$

48. $\lim\limits_{x\to 0}\dfrac{\sin(x^2)}{x}=\lim\limits_{x\to 0}\left[x\cdot\dfrac{\sin(x^2)}{x\cdot x}\right]=\lim\limits_{x\to 0}x\cdot\lim\limits_{x\to 0}\dfrac{\sin(x^2)}{x^2}=0\cdot\lim\limits_{y\to 0^+}\dfrac{\sin y}{y}\quad[\text{where } y=x^2]\quad=0\cdot 1=0$

49. $\lim\limits_{x\to\pi/4}\dfrac{1-\tan x}{\sin x-\cos x}=\lim\limits_{x\to\pi/4}\dfrac{\left(1-\dfrac{\sin x}{\cos x}\right)\cdot\cos x}{(\sin x-\cos x)\cdot\cos x}=\lim\limits_{x\to\pi/4}\dfrac{\cos x-\sin x}{(\sin x-\cos x)\cos x}=\lim\limits_{x\to\pi/4}\dfrac{-1}{\cos x}=\dfrac{-1}{1/\sqrt{2}}=-\sqrt{2}$

50. $\lim\limits_{x\to 1}\dfrac{\sin(x-1)}{x^2+x-2}=\lim\limits_{x\to 1}\dfrac{\sin(x-1)}{(x+2)(x-1)}=\lim\limits_{x\to 1}\dfrac{1}{x+2}\lim\limits_{x\to 1}\dfrac{\sin(x-1)}{x-1}=\frac{1}{3}\cdot 1=\frac{1}{3}$

51. $\dfrac{d}{dx}(\sin x) = \cos x \quad \Rightarrow \quad \dfrac{d^2}{dx^2}(\sin x) = -\sin x \quad \Rightarrow \quad \dfrac{d^3}{dx^3}(\sin x) = -\cos x \quad \Rightarrow \quad \dfrac{d^4}{dx^4}(\sin x) = \sin x.$

The derivatives of $\sin x$ occur in a cycle of four. Since $99 = 4(24) + 3$, we have $\dfrac{d^{99}}{dx^{99}}(\sin x) = \dfrac{d^3}{dx^3}(\sin x) = -\cos x$.

52. Let $f(x) = x \sin x$ and $h(x) = \sin x$, so $f(x) = xh(x)$. Then $f'(x) = h(x) + xh'(x)$,

$f''(x) = h'(x) + h'(x) + xh''(x) = 2h'(x) + xh''(x)$,

$f'''(x) = 2h''(x) + h''(x) + xh'''(x) = 3h''(x) + xh'''(x), \cdots, f^{(n)}(x) = nh^{(n-1)}(x) + xh^{(n)}(x)$.

Since $34 = 4(8) + 2$, we have $h^{(34)}(x) = h^{(2)}(x) = \dfrac{d^2}{dx^2}(\sin x) = -\sin x$ and $h^{(35)}(x) = -\cos x$.

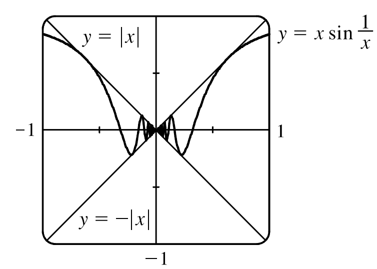
Thus, $\dfrac{d^{35}}{dx^{35}}(x \sin x) = 35h^{(34)}(x) + xh^{(35)}(x) = -35 \sin x - x \cos x$.

53. $y = A \sin x + B \cos x \quad \Rightarrow \quad y' = A \cos x - B \sin x \quad \Rightarrow \quad y'' = -A \sin x - B \cos x$. Substituting these expressions for y, y', and y'' into the given differential equation $y'' + y' - 2y = \sin x$ gives us

$(-A \sin x - B \cos x) + (A \cos x - B \sin x) - 2(A \sin x + B \cos x) = \sin x \quad \Leftrightarrow$

$-3A \sin x - B \sin x + A \cos x - 3B \cos x = \sin x \quad \Leftrightarrow \quad (-3A - B) \sin x + (A - 3B) \cos x = 1 \sin x$, so we must have $-3A - B = 1$ and $A - 3B = 0$ (since 0 is the coefficient of $\cos x$ on the right side). Solving for A and B, we add the first equation to three times the second to get $B = -\frac{1}{10}$ and $A = -\frac{3}{10}$.

54. Since $-1 \le \sin(1/x) \le 1$, we have (as illustrated in the figure) $-|x| \le x \sin(1/x) \le |x|$. We know that $\lim\limits_{x \to 0}(|x|) = 0$ and $\lim\limits_{x \to 0}(-|x|) = 0$; so by the Squeeze Theorem, $\lim\limits_{x \to 0} x \sin(1/x) = 0$.



55. (a) $\dfrac{d}{dx} \tan x = \dfrac{d}{dx} \dfrac{\sin x}{\cos x} \quad \Rightarrow \quad \sec^2 x = \dfrac{\cos x \, \cos x - \sin x \, (-\sin x)}{\cos^2 x} = \dfrac{\cos^2 x + \sin^2 x}{\cos^2 x}$. So $\sec^2 x = \dfrac{1}{\cos^2 x}$.

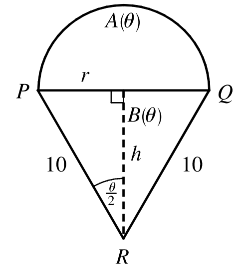
(b) $\dfrac{d}{dx} \sec x = \dfrac{d}{dx} \dfrac{1}{\cos x} \quad \Rightarrow \quad \sec x \, \tan x = \dfrac{(\cos x)(0) - 1(-\sin x)}{\cos^2 x}$. So $\sec x \, \tan x = \dfrac{\sin x}{\cos^2 x}$.

(c) $\dfrac{d}{dx}(\sin x + \cos x) = \dfrac{d}{dx} \dfrac{1 + \cot x}{\csc x} \quad \Rightarrow$

$$\cos x - \sin x = \frac{\csc x \, (-\csc^2 x) - (1 + \cot x)(-\csc x \, \cot x)}{\csc^2 x} = \frac{\csc x \, [-\csc^2 x + (1 + \cot x) \, \cot x]}{\csc^2 x}$$
$$= \frac{-\csc^2 x + \cot^2 x + \cot x}{\csc x} = \frac{-1 + \cot x}{\csc x}$$

So $\cos x - \sin x = \dfrac{\cot x - 1}{\csc x}$.

56. We get the following formulas for r and h in terms of θ:



$$\sin\frac{\theta}{2} = \frac{r}{10} \quad\Rightarrow\quad r = 10\sin\frac{\theta}{2} \quad\text{and}\quad \cos\frac{\theta}{2} = \frac{h}{10} \quad\Rightarrow\quad h = 10\cos\frac{\theta}{2}$$

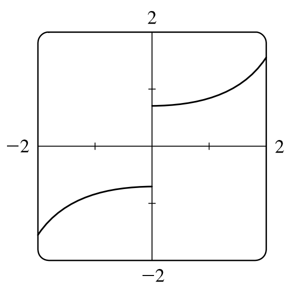
Now $A(\theta) = \frac{1}{2}\pi r^2$ and $B(\theta) = \frac{1}{2}(2r)h = rh$. So

$$\lim_{\theta\to 0^+}\frac{A(\theta)}{B(\theta)} = \lim_{\theta\to 0^+}\frac{\frac{1}{2}\pi r^2}{rh} = \frac{1}{2}\pi\lim_{\theta\to 0^+}\frac{r}{h} = \frac{1}{2}\pi\lim_{\theta\to 0^+}\frac{10\sin(\theta/2)}{10\cos(\theta/2)}$$
$$= \frac{1}{2}\pi\lim_{\theta\to 0^+}\tan(\theta/2) = 0$$

57. By the definition of radian measure, $s = r\theta$, where r is the radius of the circle. By drawing the bisector of the angle θ, we can see that $\sin\frac{\theta}{2} = \frac{d/2}{r} \quad\Rightarrow\quad d = 2r\sin\frac{\theta}{2}$. So $\lim\limits_{\theta\to 0^+}\frac{s}{d} = \lim\limits_{\theta\to 0^+}\frac{r\theta}{2r\sin(\theta/2)} = \lim\limits_{\theta\to 0^+}\frac{2\cdot(\theta/2)}{2\sin(\theta/2)} = \lim\limits_{\theta\to 0}\frac{\theta/2}{\sin(\theta/2)} = 1$.

[This is just the reciprocal of the limit $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$ combined with the fact that as $\theta \to 0$, $\frac{\theta}{2} \to 0$ also.]

58. (a)



It appears that $f(x) = \dfrac{x}{\sqrt{1-\cos 2x}}$ has a jump discontinuity at $x = 0$.

(b) Using the identity $\cos 2x = 1 - \sin^2 x$, we have $\dfrac{x}{\sqrt{1-\cos 2x}} = \dfrac{x}{\sqrt{1-(1-2\sin^2 x)}} = \dfrac{x}{\sqrt{2\sin^2 x}} = \dfrac{x}{\sqrt{2}\,|\sin x|}$.

Thus,
$$\lim_{x\to 0^-}\frac{x}{\sqrt{1-\cos 2x}} = \lim_{x\to 0^-}\frac{x}{\sqrt{2}\,|\sin x|} = \frac{1}{\sqrt{2}}\lim_{x\to 0^-}\frac{x}{-(\sin x)}$$
$$= -\frac{1}{\sqrt{2}}\lim_{x\to 0^-}\frac{1}{\sin x/x} = -\frac{1}{\sqrt{2}}\cdot\frac{1}{1} = -\frac{\sqrt{2}}{2}$$

Evaluating $\lim\limits_{x\to 0^+} f(x)$ is similar, but $|\sin x| = +\sin x$, so we get $\frac{1}{2}\sqrt{2}$. These values appear to be reasonable values for the graph, so they confirm our answer to part (a).

Another method: Multiply numerator and denominator by $\sqrt{1+\cos 2x}$.

2.5 The Chain Rule

1. Let $u = g(x) = 1 + 4x$ and $y = f(u) = \sqrt[3]{u}$. Then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = (\frac{1}{3}u^{-2/3})(4) = \dfrac{4}{3\sqrt[3]{(1+4x)^2}}$.

2. Let $u = g(x) = 2x^3 + 5$ and $y = f(u) = u^4$. Then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = (4u^3)(6x^2) = 24x^2(2x^3+5)^3$.

3. Let $u = g(x) = \pi x$ and $y = f(u) = \tan u$. Then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = (\sec^2 u)(\pi) = \pi\sec^2 \pi x$.

4. Let $u = g(x) = \cot x$ and $y = f(u) = \sin u$. Then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = (\cos u)(-\csc^2 x) = -\cos(\cot x)\csc^2 x$.

5. Let $u = g(x) = \sin x$ and $y = f(u) = \sqrt{u}$. Then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = \frac{1}{2}u^{-1/2}\cos x = \dfrac{\cos x}{2\sqrt{u}} = \dfrac{\cos x}{2\sqrt{\sin x}}$.

6. Let $u = g(x) = \sqrt{x}$ and $y = f(u) = \sin u$. Then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = (\cos u)\left(\frac{1}{2}x^{-1/2}\right) = \dfrac{\cos u}{2\sqrt{x}} = \dfrac{\cos\sqrt{x}}{2\sqrt{x}}$.

7. $F(x) = (5x^6 + 2x^3)^4 \overset{\text{CR}}{\Rightarrow} F'(x) = 4(5x^6 + 2x^3)^3 \cdot \dfrac{d}{dx}(5x^6 + 2x^3) = 4(5x^6 + 2x^3)^3(30x^5 + 6x^2)$.

We can factor as follows: $4(x^3)^3(5x^3 + 2)^3 6x^2(5x^3 + 1) = 24x^{11}(5x^3 + 2)^3(5x^3 + 1)$

8. $F(x) = (1 + x + x^2)^{99} \overset{\text{CR}}{\Rightarrow} F'(x) = 99(1 + x + x^2)^{98} \cdot \dfrac{d}{dx}\left(1 + x + x^2\right) = 99(1 + x + x^2)^{98}(1 + 2x)$

9. $f(x) = \sqrt{5x + 1} = (5x + 1)^{1/2} \overset{\text{CR}}{\Rightarrow} f'(x) = \frac{1}{2}(5x + 1)^{-1/2} \cdot \dfrac{d}{dx}(5x + 1) = \dfrac{5}{2\sqrt{5x + 1}}$

10. $g(x) = (2 - \sin x)^{3/2} \overset{\text{CR}}{\Rightarrow}$

$g'(x) = \frac{3}{2}(2 - \sin x)^{1/2} \cdot \dfrac{d}{dx}(2 - \sin x) = \frac{3}{2}(2 - \sin x)^{1/2}(-\cos x) = -\frac{3}{2}\cos x\,(2 - \sin x)^{1/2}$

11. $A(t) = \dfrac{1}{(\cos t + \tan t)^2} = (\cos t + \tan t)^{-2} \overset{\text{CR}}{\Rightarrow} A'(t) = -2(\cos t + \tan t)^{-3}(-\sin t + \sec^2 t) = \dfrac{2(\sin t - \sec^2 t)}{(\cos t + \tan t)^3}$

12. $f(x) = \dfrac{1}{\sqrt[3]{x^2 - 1}} = (x^2 - 1)^{-1/3} \Rightarrow f'(x) = -\dfrac{1}{3}(x^2 - 1)^{-4/3}(2x) = \dfrac{-2x}{3(x^2 - 1)^{4/3}}$

13. $f(\theta) = \cos(\theta^2) \Rightarrow f'(\theta) = -\sin(\theta^2) \cdot \dfrac{d}{d\theta}(\theta^2) = -\sin(\theta^2) \cdot (2\theta) = -2\theta\sin(\theta^2)$

14. $g(\theta) = \cos^2\theta = (\cos\theta)^2 \Rightarrow g'(\theta) = 2(\cos\theta)^1(-\sin\theta) = -2\sin\theta\cos\theta = -\sin 2\theta$

15. $h(v) = v\sqrt[3]{1 + v^2} = v(1 + v^2)^{1/3} \overset{\text{PR}}{\Rightarrow}$

$h'(v) = v \cdot \frac{1}{3}(1 + v^2)^{-2/3}(2v) + (1 + v^2)^{1/3} \cdot 1 = \frac{1}{3}(1 + v^2)^{-2/3}[2v^2 + 3(1 + v^2)] = \dfrac{5v^2 + 3}{3(\sqrt[3]{1 + v^2})^2}$

16. $f(t) = t\sin\pi t \Rightarrow f'(t) = t(\cos\pi t) \cdot \pi + (\sin\pi t) \cdot 1 = \pi t\cos\pi t + \sin\pi t$

17. $f(x) = (2x - 3)^4(x^2 + x + 1)^5 \Rightarrow$

$$\begin{aligned} f'(x) &= (2x - 3)^4 \cdot 5(x^2 + x + 1)^4(2x + 1) + (x^2 + x + 1)^5 \cdot 4(2x - 3)^3 \cdot 2 \\ &= (2x - 3)^3(x^2 + x + 1)^4[(2x - 3) \cdot 5(2x + 1) + (x^2 + x + 1) \cdot 8] \\ &= (2x - 3)^3(x^2 + x + 1)^4(20x^2 - 20x - 15 + 8x^2 + 8x + 8) = (2x - 3)^3(x^2 + x + 1)^4(28x^2 - 12x - 7) \end{aligned}$$

18. $g(x) = (x^2+1)^3(x^2+2)^6 \quad \Rightarrow$

$$g'(x) = (x^2+1)^3 \cdot 6(x^2+2)^5 \cdot 2x + (x^2+2)^6 \cdot 3(x^2+1)^2 \cdot 2x$$
$$= 6x(x^2+1)^2(x^2+2)^5[2(x^2+1)+(x^2+2)] = 6x(x^2+1)^2(x^2+2)^5(3x^2+4)$$

19. $h(t) = (t+1)^{2/3}(2t^2-1)^3 \quad \Rightarrow$

$$h'(t) = (t+1)^{2/3} \cdot 3(2t^2-1)^2 \cdot 4t + (2t^2-1)^3 \cdot \tfrac{2}{3}(t+1)^{-1/3} = \tfrac{2}{3}(t+1)^{-1/3}(2t^2-1)^2[18t(t+1)+(2t^2-1)]$$
$$= \tfrac{2}{3}(t+1)^{-1/3}(2t^2-1)^2(20t^2+18t-1)$$

20. $F(t) = (3t-1)^4(2t+1)^{-3} \quad \Rightarrow$

$$F'(t) = (3t-1)^4(-3)(2t+1)^{-4}(2) + (2t+1)^{-3} \cdot 4(3t-1)^3(3)$$
$$= 6(3t-1)^3(2t+1)^{-4}[-(3t-1)+2(2t+1)] = 6(3t-1)^3(2t+1)^{-4}(t+3)$$

21. $g(u) = \left(\dfrac{u^3-1}{u^3+1}\right)^8 \quad \Rightarrow$

$$g'(u) = 8\left(\frac{u^3-1}{u^3+1}\right)^7 \frac{d}{du}\frac{u^3-1}{u^3+1} = 8\,\frac{(u^3-1)^7}{(u^3+1)^7}\,\frac{(u^3+1)(3u^2)-(u^3-1)(3u^2)}{(u^3+1)^2}$$
$$= 8\,\frac{(u^3-1)^7}{(u^3+1)^7}\,\frac{3u^2[(u^3+1)-(u^3-1)]}{(u^3+1)^2} = 8\frac{(u^3-1)^7}{(u^3+1)^7}\,\frac{3u^2(2)}{(u^3+1)^2} = \frac{48u^2(u^3-1)^7}{(u^3+1)^9}$$

22. $y = \left(x+\dfrac{1}{x}\right)^5 \quad \Rightarrow \quad y' = 5\left(x+\dfrac{1}{x}\right)^4 \dfrac{d}{dx}\left(x+\dfrac{1}{x}\right) = 5\left(x+\dfrac{1}{x}\right)^4\left(1-\dfrac{1}{x^2}\right).$

Another form of the answer is $\dfrac{5(x^2+1)^4(x^2-1)}{x^6}$.

23. $y = \sqrt{\dfrac{x}{x+1}} = \left(\dfrac{x}{x+1}\right)^{1/2} \quad \Rightarrow$

$$y' = \frac{1}{2}\left(\frac{x}{x+1}\right)^{-1/2} \frac{d}{dx}\left(\frac{x}{x+1}\right) = \frac{1}{2}\,\frac{x^{-1/2}}{(x+1)^{-1/2}}\,\frac{(x+1)(1)-x(1)}{(x+1)^2}$$
$$= \frac{1}{2}\,\frac{(x+1)^{1/2}}{x^{1/2}}\,\frac{1}{(x+1)^2} = \frac{1}{2\sqrt{x}(x+1)^{3/2}}$$

24. $U(y) = \left(\dfrac{y^4+1}{y^2+1}\right)^5 \quad \Rightarrow$

$$U'(y) = 5\left(\frac{y^4+1}{y^2+1}\right)^4 \frac{(y^2+1)(4y^3)-(y^4+1)(2y)}{(y^2+1)^2} = \frac{5(y^4+1)^4 2y[2y^2(y^2+1)-(y^4+1)]}{(y^2+1)^4(y^2+1)^2}$$
$$= \frac{10y(y^4+1)^4(y^4+2y^2-1)}{(y^2+1)^6}$$

25. $h(\theta) = \tan(\theta^2 \sin\theta) \quad \overset{\text{CR}}{\Rightarrow}$

$$h'(\theta) = \sec^2(\theta^2\sin\theta) \cdot \frac{d}{d\theta}(\theta^2\sin\theta) = \sec^2(\theta^2\sin\theta)\cdot[\theta^2\cos\theta + (\sin\theta)(2\theta)] = \theta\sec^2(\theta^2\sin\theta)(\theta\cos\theta + 2\sin\theta)$$

26. $f(t)=\sqrt{\dfrac{t}{t^2+4}}=\left(\dfrac{t}{t^2+4}\right)^{1/2} \quad\Rightarrow$

$$f'(t)=\frac{1}{2}\left(\frac{t}{t^2+4}\right)^{-1/2}\cdot\frac{d}{dt}\left(\frac{t}{t^2+4}\right)=\frac{1}{2}\left(\frac{t^2+4}{t}\right)^{1/2}\cdot\frac{(t^2+4)(1)-t(2t)}{(t^2+4)^2}$$
$$=\frac{(t^2+4)^{1/2}}{2t^{1/2}}\cdot\frac{t^2+4-2t^2}{(t^2+4)^2}=\frac{4-t^2}{2t^{1/2}(t^2+4)^{3/2}}$$

27. $y=\dfrac{\cos x}{\sqrt{1+\sin x}}=(\cos x)(1+\sin x)^{-1/2} \quad\Rightarrow$

$$y'=(\cos x)\cdot\left(-\tfrac{1}{2}\right)(1+\sin x)^{-3/2}\cos x+(1+\sin x)^{-1/2}(-\sin x)$$
$$=-\tfrac{1}{2}(1+\sin x)^{-3/2}[\cos^2 x+2(1+\sin x)\,\sin x]=-\tfrac{1}{2}(1+\sin x)^{-3/2}(\cos^2 x+2\sin x+2\sin^2 x)$$
$$=-\tfrac{1}{2}(1+\sin x)^{-3/2}(1+2\sin x+\sin^2 x)=-\tfrac{1}{2}(1+\sin x)^{-3/2}(1+\sin x)^2$$
$$=-\tfrac{1}{2}(1+\sin x)^{1/2} \text{ or } -\tfrac{1}{2}\sqrt{1+\sin x}$$

28. $F(t)=\dfrac{t^2}{\sqrt{t^3+1}} \quad\Rightarrow$

$$F'(t)=\frac{(t^3+1)^{1/2}(2t)-t^2\cdot\frac{1}{2}(t^3+1)^{-1/2}(3t^2)}{\left(\sqrt{t^3+1}\right)^2}=\frac{t(t^3+1)^{-1/2}\left[2(t^3+1)-\frac{3}{2}t^3\right]}{(t^3+1)^1}$$
$$=\frac{t\left(\frac{1}{2}t^3+2\right)}{(t^3+1)^{3/2}}=\frac{t(t^3+4)}{2(t^3+1)^{3/2}}$$

29. $H(r)=\dfrac{(r^2-1)^3}{(2r+1)^5} \quad\Rightarrow$

$$H'(r)=\frac{(2r+1)^5\cdot 3(r^2-1)^2(2r)-(r^2-1)^3\cdot 5(2r+1)^4(2)}{[(2r+1)^5]^2}=\frac{2(2r+1)^4(r^2-1)^2[3r(2r+1)-5(r^2-1)]}{(2r+1)^{10}}$$
$$=\frac{2(r^2-1)^2(6r^2+3r-5r^2+5)}{(2r+1)^6}=\frac{2(r^2-1)^2(r^2+3r+5)}{(2r+1)^6}$$

30. $s(t)=\sqrt{\dfrac{1+\sin t}{1+\cos t}}=\left(\dfrac{1+\sin t}{1+\cos t}\right)^{1/2} \quad\Rightarrow$

$$s'(t)=\frac{1}{2}\left(\frac{1+\sin t}{1+\cos t}\right)^{-1/2}\frac{(1+\cos t)\cos t-(1+\sin t)(-\sin t)}{(1+\cos t)^2}$$
$$=\frac{1}{2}\frac{(1+\sin t)^{-1/2}}{(1+\cos t)^{-1/2}}\frac{\cos t+\cos^2 t+\sin t+\sin^2 t}{(1+\cos t)^2}=\frac{\cos t+\sin t+1}{2\sqrt{1+\sin t}\,(1+\cos t)^{3/2}}$$

31. $y=\cos(\sec 4x) \quad\Rightarrow$

$$y'=-\sin(\sec 4x)\,\frac{d}{dx}\,\sec 4x=-\sin(\sec 4x)\cdot\sec 4x\tan 4x\cdot 4=-4\sin(\sec 4x)\sec 4x\tan 4x$$

32. $J(\theta)=\tan^2(n\theta)=[\tan(n\theta)]^2 \quad\Rightarrow$

$$J'(\theta)=2\,[\tan(n\theta)]^1\,\frac{d}{d\theta}\tan(n\theta)=2\tan(n\theta)\,\sec^2(n\theta)\cdot n=2n\tan(n\theta)\,\sec^2(n\theta)$$

33. $y = \sin\sqrt{1+x^2} \quad\Rightarrow\quad y' = \cos\sqrt{1+x^2}\cdot\frac{1}{2}(1+x^2)^{-1/2}\cdot 2x = \left(x\,\cos\sqrt{1+x^2}\,\right)/\sqrt{1+x^2}$

34. $y = \sqrt{\sin(1+x^2)} = [\sin(1+x^2)]^{1/2} \quad\Rightarrow\quad y' = \frac{1}{2}[\sin(1+x^2)]^{-1/2}\cdot\cos(1+x^2)\cdot 2x = \dfrac{x\cos(1+x^2)}{\sqrt{\sin(1+x^2)}}$

35. $y = \left(\dfrac{1-\cos 2x}{1+\cos 2x}\right)^4 \quad\Rightarrow$

$$y' = 4\left(\frac{1-\cos 2x}{1+\cos 2x}\right)^3\cdot\frac{(1+\cos 2x)(2\sin 2x)+(1-\cos 2x)(-2\sin 2x)}{(1+\cos 2x)^2}$$
$$= 4\left(\frac{1-\cos 2x}{1+\cos 2x}\right)^3\cdot\frac{2\sin 2x\,(1+\cos 2x+1-\cos 2x)}{(1+\cos 2x)^2} = \frac{4(1-\cos 2x)^3}{(1+\cos 2x)^3}\,\frac{2\sin 2x\,(2)}{(1+\cos 2x)^2} = \frac{16\sin 2x\,(1-\cos 2x)^3}{(1+\cos 2x)^5}$$

36. $y = x\sin\dfrac{1}{x} \quad\Rightarrow\quad y' = \sin\dfrac{1}{x} + x\cos\dfrac{1}{x}\left(-\dfrac{1}{x^2}\right) = \sin\dfrac{1}{x} - \dfrac{1}{x}\cos\dfrac{1}{x}$

37. $y = \cot^2(\sin\theta) = [\cot(\sin\theta)]^2 \quad\Rightarrow$

$$y' = 2[\cot(\sin\theta)]\cdot\frac{d}{d\theta}[\cot(\sin\theta)] = 2\cot(\sin\theta)\cdot[-\csc^2(\sin\theta)\cdot\cos\theta] = -2\cos\theta\,\cot(\sin\theta)\,\csc^2(\sin\theta)$$

38. $y = \sin(t+\cos\sqrt{t}\,) \quad\Rightarrow$

$$y' = \cos(t+\cos\sqrt{t}\,)\cdot\frac{d}{dt}(t+\cos\sqrt{t}\,) = \cos(t+\cos\sqrt{t}\,)\cdot\left(1-\sin\sqrt{t}\cdot\frac{1}{2\sqrt{t}}\right) = \cos(t+\cos\sqrt{t}\,)\,\frac{2\sqrt{t}-\sin\sqrt{t}}{2\sqrt{t}}$$

39. $f(t) = \tan(\sec(\cos t)) \quad\Rightarrow$

$$f'(t) = \sec^2(\sec(\cos t))\cdot\frac{d}{dt}\,\sec(\cos t) = \sec^2(\sec(\cos t))\cdot\sec(\cos t)\tan(\cos t)\cdot\frac{d}{dt}\,\cos t$$
$$= -\sin t\,\sec^2(\sec(\cos t))\sec(\cos t)\tan(\cos t)$$

40. $g(u) = [(u^2-1)^6 - 3u]^4 \quad\Rightarrow$

$$g'(u) = 4[(u^2-1)^6-3u]^3\cdot\frac{d}{du}[(u^2-1)^6-3u] = 4[(u^2-1)^6-3u]^3\cdot[6(u^2-1)^5\cdot 2u-3]$$
$$= 12[(u^2-1)^6-3u]^3[4u(u^2-1)^5-1]$$

41. $y = \sqrt{x+\sqrt{x}} \quad\Rightarrow\quad y' = \frac{1}{2}\,(x+\sqrt{x}\,)^{-1/2}\left(1+\frac{1}{2}x^{-1/2}\right) = \dfrac{1}{2\sqrt{x+\sqrt{x}}}\left(1+\dfrac{1}{2\sqrt{x}}\right)$

42. $y = \sqrt{x+\sqrt{x+\sqrt{x}}} \quad\Rightarrow\quad y' = \frac{1}{2}\left(x+\sqrt{x+\sqrt{x}}\right)^{-1/2}\left[1+\frac{1}{2}\left(x+\sqrt{x}\right)^{-1/2}\left(1+\frac{1}{2}x^{-1/2}\right)\right]$

43. $g(x) = (2r\sin rx + n)^p \quad\Rightarrow\quad g'(x) = p(2r\sin rx+n)^{p-1}(2r\cos rx\cdot r) = p(2r\sin rx+n)^{p-1}(2r^2\cos rx)$

44. $y = \cos^4(\sin^3 x) = [\cos(\sin^3 x)]^4 \quad\Rightarrow$

$$y' = 4[\cos(\sin^3 x)]^3(-\sin(\sin^3 x))\,3\sin^2 x\,\cos x = -12\sin^2 x\,\cos x\,\cos^3(\sin^3 x)\,\sin(\sin^3 x)$$

45. $y = \cos\sqrt{\sin(\tan \pi x)} = \cos(\sin(\tan \pi x))^{1/2} \quad \Rightarrow$

$$y' = -\sin(\sin(\tan \pi x))^{1/2} \cdot \frac{d}{dx}(\sin(\tan \pi x))^{1/2} = -\sin(\sin(\tan \pi x))^{1/2} \cdot \tfrac{1}{2}(\sin(\tan \pi x))^{-1/2} \cdot \frac{d}{dx}(\sin(\tan \pi x))$$

$$= \frac{-\sin\sqrt{\sin(\tan \pi x)}}{2\sqrt{\sin(\tan \pi x)}} \cdot \cos(\tan \pi x) \cdot \frac{d}{dx}\tan \pi x = \frac{-\sin\sqrt{\sin(\tan \pi x)}}{2\sqrt{\sin(\tan \pi x)}} \cdot \cos(\tan \pi x) \cdot \sec^2(\pi x) \cdot \pi$$

$$= \frac{-\pi\cos(\tan \pi x)\sec^2(\pi x)\sin\sqrt{\sin(\tan \pi x)}}{2\sqrt{\sin(\tan \pi x)}}$$

46. $y = \left[x + (x + \sin^2 x)^3\right]^4 \quad \Rightarrow \quad y' = 4\left[x + (x + \sin^2 x)^3\right]^3 \cdot \left[1 + 3(x + \sin^2 x)^2 \cdot (1 + 2\sin x\, \cos x)\right]$

47. $y = \cos(\sin 3\theta) \quad \Rightarrow \quad y' = -\sin(\sin 3\theta) \cdot (\cos 3\theta) \cdot 3 = -3\cos 3\theta\, \sin(\sin 3\theta) \quad \Rightarrow$

$$y'' = -3\left[(\cos 3\theta)\cos(\sin 3\theta)(\cos 3\theta) \cdot 3 + \sin(\sin 3\theta)(-\sin 3\theta) \cdot 3\right] = -9\cos^2(3\theta)\cos(\sin 3\theta) + 9(\sin 3\theta)\sin(\sin 3\theta)$$

48. $y = \dfrac{1}{(1 + \tan x)^2} = (1 + \tan x)^{-2} \quad \Rightarrow \quad y' = -2(1 + \tan x)^{-3}\sec^2 x = \dfrac{-2\sec^2 x}{(1 + \tan x)^3}$.

Using the Product Rule with $y' = \left[-2(1 + \tan x)^{-3}\right](\sec x)^2$, we get

$$y'' = -2(1 + \tan x)^{-3} \cdot 2(\sec x)(\sec x\, \tan x) + (\sec x)^2 \cdot 6(1 + \tan x)^{-4}\sec^2 x$$

$$= 2\sec^2 x\,(1 + \tan x)^{-4}\left[-2(1 + \tan x)\tan x + 3\sec^2 x\right] \qquad \left[\begin{array}{c}\text{2 is the lesser exponent for } \sec x \\ \text{and } -4 \text{ for } (1 + \tan x)\end{array}\right]$$

$$= 2\sec^2 x\,(1 + \tan x)^{-4}\left[-2\tan x - 2\tan^2 x + 3(\tan^2 x + 1)\right]$$

$$= \frac{2\sec^2 x\left(\tan^2 x - 2\tan x + 3\right)}{(1 + \tan x)^4}$$

49. $y = \sqrt{1 - \sec t} \quad \Rightarrow \quad y' = \tfrac{1}{2}(1 - \sec t)^{-1/2}(-\sec t\, \tan t) = \dfrac{-\sec t\, \tan t}{2\sqrt{1 - \sec t}}$.

Using the Product Rule with $y' = \left(-\tfrac{1}{2}\sec t\, \tan t\right)(1 - \sec t)^{-1/2}$, we get

$$y'' = \left(-\tfrac{1}{2}\sec t\, \tan t\right)\left[-\tfrac{1}{2}(1 - \sec t)^{-3/2}(-\sec t\, \tan t)\right] + (1 - \sec t)^{-1/2}\left(-\tfrac{1}{2}\right)[\sec t\, \sec^2 t + \tan t\, \sec t\, \tan t].$$

Now factor out $-\tfrac{1}{2}\sec t(1 - \sec t)^{-3/2}$. Note that $-\tfrac{3}{2}$ is the lesser exponent on $(1 - \sec t)$. Continuing,

$$y'' = -\tfrac{1}{2}\sec t\,(1 - \sec t)^{-3/2}\left[\tfrac{1}{2}\sec t\, \tan^2 t + (1 - \sec t)(\sec^2 t + \tan^2 t)\right]$$

$$= -\tfrac{1}{2}\sec t\,(1 - \sec t)^{-3/2}\left(\tfrac{1}{2}\sec t\, \tan^2 t + \sec^2 t + \tan^2 t - \sec^3 t - \sec t\, \tan^2 t\right)$$

$$= -\tfrac{1}{2}\sec t\,(1 - \sec t)^{-3/2}\left[-\tfrac{1}{2}\sec t\,(\sec^2 t - 1) + \sec^2 t + (\sec^2 t - 1) - \sec^3 t\right]$$

$$= -\tfrac{1}{2}\sec t\,(1 - \sec t)^{-3/2}\left(-\tfrac{3}{2}\sec^3 t + 2\sec^2 t + \tfrac{1}{2}\sec t - 1\right)$$

$$= \sec t\,(1 - \sec t)^{-3/2}\left(\tfrac{3}{4}\sec^3 t - \sec^2 t - \tfrac{1}{4}\sec t + \tfrac{1}{2}\right)$$

$$= \frac{\sec t\,(3\sec^3 t - 4\sec^2 t - \sec t + 2)}{4(1 - \sec t)^{3/2}}$$

There are many other correct forms of y'', such as $y'' = \dfrac{\sec t\,(3\sec t + 2)\sqrt{1 - \sec t}}{4}$. We chose to find a factored form with only secants in the final form.

50. $y = \dfrac{4x}{\sqrt{x+1}} \quad \Rightarrow$

$$y' = \frac{\sqrt{x+1}\cdot 4 - 4x \cdot \frac{1}{2}(x+1)^{-1/2}}{\left(\sqrt{x+1}\right)^2} = \frac{4\sqrt{x+1} - 2x/\sqrt{x+1}}{x+1} = \frac{4(x+1) - 2x}{(x+1)^{3/2}} = \frac{2x+4}{(x+1)^{3/2}} \quad \Rightarrow$$

$$y'' = \frac{(x+1)^{3/2}\cdot 2 - (2x+4)\cdot \frac{3}{2}(x+1)^{1/2}}{[(x+1)^{3/2}]^2} = \frac{(x+1)^{1/2}[2(x+1) - 3(x+2)]}{(x+1)^3} = \frac{2x+2-3x-6}{(x+1)^{5/2}} = \frac{-x-4}{(x+1)^{5/2}}$$

51. $y = (3x-1)^{-6} \quad \Rightarrow \quad y' = -6(3x-1)^{-7}\cdot 3 = -18(3x-1)^{-7}$. At $(0,1)$, $y' = -18(-1)^{-7} = -18(-1) = 18$, and an equation of the tangent line is $y - 1 = 18(x-0)$, or $y = 18x + 1$.

52. $y = \sqrt{1+x^3} = (1+x^3)^{1/2} \quad \Rightarrow \quad y' = \frac{1}{2}(1+x^3)^{-1/2}\cdot 3x^2 = \dfrac{3x^2}{2\sqrt{1+x^3}}$. At $(2,3)$, $y' = \dfrac{3\cdot 4}{2\sqrt{9}} = 2$, and an equation of the tangent line is $y - 3 = 2(x-2)$, or $y = 2x - 1$.

53. $y = \sin(\sin x) \quad \Rightarrow \quad y' = \cos(\sin x)\cdot \cos x$. At $(\pi, 0)$, $y' = \cos(\sin\pi)\cdot\cos\pi = \cos(0)\cdot(-1) = 1(-1) = -1$, and an equation of the tangent line is $y - 0 = -1(x-\pi)$, or $y = -x + \pi$.

54. $y = \sin^2 x\cos x \quad \Rightarrow \quad y' = \sin^2 x(-\sin x) + \cos x(2\sin x\cos x)$. At $(\pi/2, 0)$, $y' = 1(-1) + 0 = -1$, and an equation of the tangent line is $y - 0 = -1\left(x - \frac{\pi}{2}\right)$, or $y = -x + \frac{\pi}{2}$.

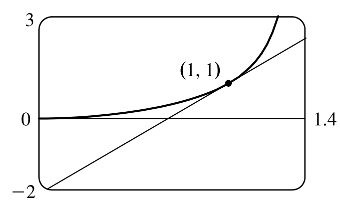
55. (a) $y = f(x) = \tan\left(\frac{\pi}{4}x^2\right) \quad \Rightarrow \quad f'(x) = \sec^2\left(\frac{\pi}{4}x^2\right)\left(2\cdot\frac{\pi}{4}x\right)$.

The slope of the tangent at $(1,1)$ is thus

$f'(1) = \sec^2\frac{\pi}{4}\left(\frac{\pi}{2}\right) = 2\cdot\frac{\pi}{2} = \pi$, and its equation

is $y - 1 = \pi(x-1)$ or $y = \pi x - \pi + 1$.

(b)



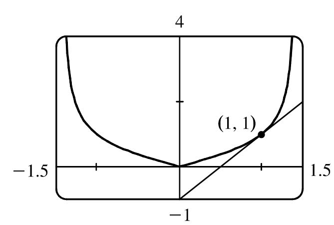
56. (a) For $x > 0$, $|x| = x$, and $y = f(x) = \dfrac{x}{\sqrt{2-x^2}} \quad \Rightarrow$

$$f'(x) = \frac{\sqrt{2-x^2}\,(1) - x\left(\frac{1}{2}\right)(2-x^2)^{-1/2}(-2x)}{\left(\sqrt{2-x^2}\right)^2}\cdot\frac{(2-x^2)^{1/2}}{(2-x^2)^{1/2}}$$

$$= \frac{(2-x^2)+x^2}{(2-x^2)^{3/2}} = \frac{2}{(2-x^2)^{3/2}}$$

So at $(1,1)$, the slope of the tangent line is $f'(1) = 2$ and its equation is $y - 1 = 2(x-1)$ or $y = 2x - 1$.

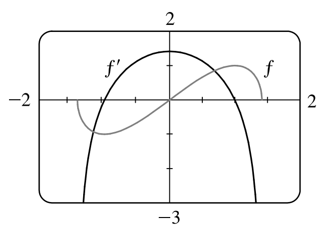
(b)



57. (a) $f(x) = x\sqrt{2-x^2} = x(2-x^2)^{1/2} \quad \Rightarrow$

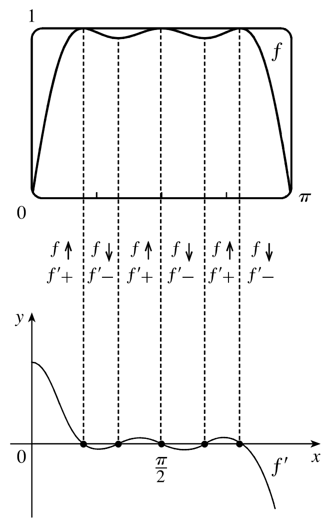
$$f'(x) = x\cdot\tfrac{1}{2}(2-x^2)^{-1/2}(-2x) + (2-x^2)^{1/2}\cdot 1 = (2-x^2)^{-1/2}\left[-x^2 + (2-x^2)\right] = \frac{2-2x^2}{\sqrt{2-x^2}}$$

(b)



$f' = 0$ when f has a horizontal tangent line, f' is negative when f is decreasing, and f' is positive when f is increasing.

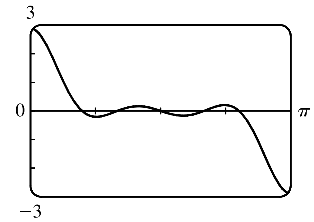
58. (a)



From the graph of f, we see that there are 5 horizontal tangents, so there must be 5 zeros on the graph of f'. From the symmetry of the graph of f, we must have the graph of f' as high at $x = 0$ as it is low at $x = \pi$. The intervals of increase and decrease as well as the signs of f' are indicated in the figure.

(b) $f(x) = \sin(x + \sin 2x) \quad \Rightarrow$

$$f'(x) = \cos(x + \sin 2x) \cdot \frac{d}{dx}(x + \sin 2x) = \cos(x + \sin 2x)(1 + 2\cos 2x)$$



59. For the tangent line to be horizontal, $f'(x) = 0$. $f(x) = 2\sin x + \sin^2 x \quad \Rightarrow \quad f'(x) = 2\cos x + 2\sin x\,\cos x = 0 \quad \Leftrightarrow$ $2\cos x(1 + \sin x) = 0 \quad \Leftrightarrow \quad \cos x = 0$ or $\sin x = -1$, so $x = \frac{\pi}{2} + 2n\pi$ or $\frac{3\pi}{2} + 2n\pi$, where n is any integer. Now $f\left(\frac{\pi}{2}\right) = 3$ and $f\left(\frac{3\pi}{2}\right) = -1$, so the points on the curve with a horizontal tangent are $\left(\frac{\pi}{2} + 2n\pi, 3\right)$ and $\left(\frac{3\pi}{2} + 2n\pi, -1\right)$, where n is any integer.

60. $y = \sqrt{1 + 2x} \quad \Rightarrow \quad y' = \frac{1}{2}(1 + 2x)^{-1/2} \cdot 2 = \dfrac{1}{\sqrt{1 + 2x}}$. The line $6x + 2y = 1$ $\left(\text{or } y = -3x + \frac{1}{2}\right)$ has slope -3, so the tangent line perpendicular to it must have slope $\frac{1}{3}$. Thus, $\dfrac{1}{3} = \dfrac{1}{\sqrt{1 + 2x}} \quad \Leftrightarrow \quad \sqrt{1 + 2x} = 3 \quad \Rightarrow \quad 1 + 2x = 9 \quad \Leftrightarrow$ $2x = 8 \quad \Leftrightarrow \quad x = 4$. When $x = 4$, $y = \sqrt{1 + 2(4)} = 3$, so the point is $(4, 3)$.

61. $F(x) = f(g(x)) \quad \Rightarrow \quad F'(x) = f'(g(x)) \cdot g'(x)$, so $F'(5) = f'(g(5)) \cdot g'(5) = f'(-2) \cdot 6 = 4 \cdot 6 = 24$.

62. $h(x) = \sqrt{4 + 3f(x)} \quad \Rightarrow \quad h'(x) = \frac{1}{2}(4 + 3f(x))^{-1/2} \cdot 3f'(x)$, so

$h'(1) = \frac{1}{2}(4 + 3f(1))^{-1/2} \cdot 3f'(1) = \frac{1}{2}(4 + 3 \cdot 7)^{-1/2} \cdot 3 \cdot 4 = \frac{6}{\sqrt{25}} = \frac{6}{5}$.

63. (a) $h(x) = f(g(x)) \quad \Rightarrow \quad h'(x) = f'(g(x)) \cdot g'(x)$, so $h'(1) = f'(g(1)) \cdot g'(1) = f'(2) \cdot 6 = 5 \cdot 6 = 30$.

(b) $H(x) = g(f(x)) \quad \Rightarrow \quad H'(x) = g'(f(x)) \cdot f'(x)$, so $H'(1) = g'(f(1)) \cdot f'(1) = g'(3) \cdot 4 = 9 \cdot 4 = 36$.

64. (a) $F(x) = f(f(x)) \quad \Rightarrow \quad F'(x) = f'(f(x)) \cdot f'(x)$, so $F'(2) = f'(f(2)) \cdot f'(2) = f'(1) \cdot 5 = 4 \cdot 5 = 20$.

(b) $G(x) = g(g(x)) \quad \Rightarrow \quad G'(x) = g'(g(x)) \cdot g'(x)$, so $G'(3) = g'(g(3)) \cdot g'(3) = g'(2) \cdot 9 = 7 \cdot 9 = 63$.

65. (a) $u(x) = f(g(x)) \;\Rightarrow\; u'(x) = f'(g(x))g'(x)$. So $u'(1) = f'(g(1))g'(1) = f'(3)g'(1)$. To find $f'(3)$, note that f is linear from $(2,4)$ to $(6,3)$, so its slope is $\dfrac{3-4}{6-2} = -\dfrac{1}{4}$. To find $g'(1)$, note that g is linear from $(0,6)$ to $(2,0)$, so its slope is $\dfrac{0-6}{2-0} = -3$. Thus, $f'(3)g'(1) = \left(-\frac{1}{4}\right)(-3) = \frac{3}{4}$.

(b) $v(x) = g(f(x)) \;\Rightarrow\; v'(x) = g'(f(x))f'(x)$. So $v'(1) = g'(f(1))f'(1) = g'(2)f'(1)$, which does not exist since $g'(2)$ does not exist.

(c) $w(x) = g(g(x)) \;\Rightarrow\; w'(x) = g'(g(x))g'(x)$. So $w'(1) = g'(g(1))g'(1) = g'(3)g'(1)$. To find $g'(3)$, note that g is linear from $(2,0)$ to $(5,2)$, so its slope is $\dfrac{2-0}{5-2} = \dfrac{2}{3}$. Thus, $g'(3)g'(1) = \left(\frac{2}{3}\right)(-3) = -2$.

66. (a) $h(x) = f(f(x)) \;\Rightarrow\; h'(x) = f'(f(x))f'(x)$. So $h'(2) = f'(f(2))f'(2) = f'(1)f'(2) \approx (-1)(-1) = 1$.

(b) $g(x) = f(x^2) \;\Rightarrow\; g'(x) = f'(x^2)\cdot \dfrac{d}{dx}\left(x^2\right) = f'(x^2)(2x)$. So $g'(2) = f'(2^2)(2\cdot 2) = 4f'(4) \approx 4(2) = 8$.

67. The point $(3,2)$ is on the graph of f, so $f(3) = 2$. The tangent line at $(3,2)$ has slope $\dfrac{\Delta y}{\Delta x} = \dfrac{-4}{6} = -\dfrac{2}{3}$.

$g(x) = \sqrt{f(x)} \;\Rightarrow\; g'(x) = \frac{1}{2}[f(x)]^{-1/2}\cdot f'(x) \;\Rightarrow$

$g'(3) = \frac{1}{2}[f(3)]^{-1/2}\cdot f'(3) = \frac{1}{2}(2)^{-1/2}(-\frac{2}{3}) = -\dfrac{1}{3\sqrt{2}}$ or $-\frac{1}{6}\sqrt{2}$.

68. (a) $F(x) = f(x^\alpha) \;\Rightarrow\; F'(x) = f'(x^\alpha)\dfrac{d}{dx}(x^\alpha) = f'(x^\alpha)\alpha x^{\alpha-1}$

(b) $G(x) = [f(x)]^\alpha \;\Rightarrow\; G'(x) = \alpha\,[f(x)]^{\alpha-1}\,f'(x)$

69. $r(x) = f(g(h(x))) \;\Rightarrow\; r'(x) = f'(g(h(x)))\cdot g'(h(x))\cdot h'(x)$, so

$r'(1) = f'(g(h(1)))\cdot g'(h(1))\cdot h'(1) = f'(g(2))\cdot g'(2)\cdot 4 = f'(3)\cdot 5\cdot 4 = 6\cdot 5\cdot 4 = 120$

70. $f(x) = xg(x^2) \;\Rightarrow\; f'(x) = xg'(x^2)\,2x + g(x^2)\cdot 1 = 2x^2g'(x^2) + g(x^2) \;\Rightarrow$

$f''(x) = 2x^2g''(x^2)\,2x + g'(x^2)\,4x + g'(x^2)\,2x = 4x^3g''(x^2) + 4xg'(x^2) + 2xg'(x^2) = 6xg'(x^2) + 4x^3g''(x^2)$

71. $F(x) = f(3f(4f(x))) \;\Rightarrow$

$$\begin{aligned} F'(x) &= f'(3f(4f(x)))\cdot\frac{d}{dx}(3f(4f(x))) = f'(3f(4f(x)))\cdot 3f'(4f(x))\cdot\frac{d}{dx}(4f(x)) \\ &= f'(3f(4f(x)))\cdot 3f'(4f(x))\cdot 4f'(x), \quad \text{so} \end{aligned}$$

$F'(0) = f'(3f(4f(0)))\cdot 3f'(4f(0))\cdot 4f'(0) = f'(3f(4\cdot 0))\cdot 3f'(4\cdot 0)\cdot 4\cdot 2 = f'(3\cdot 0)\cdot 3\cdot 2\cdot 4\cdot 2 = 2\cdot 3\cdot 2\cdot 4\cdot 2 = 96.$

72. $F(x) = f(xf(xf(x))) \;\Rightarrow$

$$\begin{aligned} F'(x) &= f'(xf(xf(x)))\cdot\frac{d}{dx}(xf(xf(x))) = f'(xf(xf(x)))\cdot\left[x\cdot f'(xf(x))\cdot\frac{d}{dx}(xf(x)) + f(xf(x))\cdot 1\right] \\ &= f'(xf(xf(x)))\cdot[xf'(xf(x))\cdot(xf'(x)+f(x)\cdot 1) + f(xf(x))], \quad \text{so} \end{aligned}$$

$$\begin{aligned} F'(1) &= f'(f(f(1)))\cdot[f'(f(1))\cdot(f'(1)+f(1)) + f(f(1))] = f'(f(2))\cdot[f'(2)\cdot(4+2)+f(2)] \\ &= f'(3)\cdot[5\cdot 6+3] = 6\cdot 33 = 198. \end{aligned}$$

73. Let $f(x) = \cos x$. Then $Df(2x) = 2f'(2x)$, $D^2 f(2x) = 2^2 f''(2x)$, $D^3 f(2x) = 2^3 f'''(2x)$, $\ldots$, $D^{(n)} f(2x) = 2^n f^{(n)}(2x)$. Since the derivatives of $\cos x$ occur in a cycle of four, and since $103 = 4(25) + 3$, we have $f^{(103)}(x) = f^{(3)}(x) = \sin x$ and $D^{103} \cos 2x = 2^{103} f^{(103)}(2x) = 2^{103} \sin 2x$.

74. Let $f(x) = x \sin \pi x$ and $h(x) = \sin \pi x$, so $f(x) = xh(x)$. Then $Df(x) = xh'(x) + h(x)$, $D^2 f(x) = xh''(x) + h'(x) + h'(x) = xh''(x) + 2h'(x)$, $D^3 f(x) = xh'''(x) + h''(x) + 2h''(x) = xh'''(x) + 3h''(x)$, $\ldots$, $D^n f(x) = xh^{(n)}(x) + nh^{(n-1)}(x)$. We now find a pattern for the derivatives of h: $h'(x) = \pi \cos \pi x$, $h''(x) = -\pi^2 \sin \pi x$, $h'''(x) = -\pi^3 \cos \pi x$, $h^4(x) = \pi^4 \sin \pi x$, and so on. Since $34 = 4(8) + 2$, we have $h^{(34)}(x) = -\pi^{34} \sin \pi x$ and $h^{(35)}(x) = -\pi^{35} \cos \pi x$. Thus,

$$D^{35} f(x) = xh^{(35)}(x) + 35h^{(34)}(x) = x(-\pi^{35} \cos \pi x) + 35(-\pi^{34} \sin \pi x) = -\pi^{35} x \cos \pi x - 35\pi^{34} \sin \pi x.$$

75. $s(t) = 10 + \frac{1}{4} \sin(10\pi t) \quad \Rightarrow \quad$ the velocity after t seconds is $v(t) = s'(t) = \frac{1}{4} \cos(10\pi t)(10\pi) = \frac{5\pi}{2} \cos(10\pi t)$ cm/s.

76. (a) $s = A\cos(\omega t + \delta) \quad \Rightarrow \quad$ velocity $= s' = -\omega A \sin(\omega t + \delta)$.

(b) If $A \neq 0$ and $\omega \neq 0$, then $s' = 0 \quad \Leftrightarrow \quad \sin(\omega t + \delta) = 0 \quad \Leftrightarrow \quad \omega t + \delta = n\pi \quad \Leftrightarrow \quad t = \dfrac{n\pi - \delta}{\omega}$, n an integer.

77. (a) $B(t) = 4.0 + 0.35 \sin \dfrac{2\pi t}{5.4} \quad \Rightarrow \quad \dfrac{dB}{dt} = \left(0.35 \cos \dfrac{2\pi t}{5.4}\right)\left(\dfrac{2\pi}{5.4}\right) = \dfrac{0.7\pi}{5.4} \cos \dfrac{2\pi t}{5.4} = \dfrac{7\pi}{54} \cos \dfrac{2\pi t}{5.4}$

(b) At $t = 1$, $\dfrac{dB}{dt} = \dfrac{7\pi}{54} \cos \dfrac{2\pi}{5.4} \approx 0.16$.

78. $L(t) = 12 + 2.8 \sin\left(\frac{2\pi}{365}(t - 80)\right) \quad \Rightarrow \quad L'(t) = 2.8 \cos\left(\frac{2\pi}{365}(t - 80)\right)\left(\frac{2\pi}{365}\right)$.

On March 21, $t = 80$, and $L'(80) \approx 0.0482$ hours per day. On May 21, $t = 141$, and $L'(141) \approx 0.02398$, which is approximately one-half of $L'(80)$.

79. By the Chain Rule, $a(t) = \dfrac{dv}{dt} = \dfrac{dv}{ds}\dfrac{ds}{dt} = \dfrac{dv}{ds} v(t) = v(t) \dfrac{dv}{ds}$. The derivative dv/dt is the rate of change of the velocity with respect to time (in other words, the acceleration) whereas the derivative dv/ds is the rate of change of the velocity with respect to the displacement.

80. (a) The derivative dV/dr represents the rate of change of the volume with respect to the radius and the derivative dV/dt represents the rate of change of the volume with respect to time.

(b) Since $V = \dfrac{4}{3}\pi r^3$, $\dfrac{dV}{dt} = \dfrac{dV}{dr}\dfrac{dr}{dt} = 4\pi r^2 \dfrac{dr}{dt}$.

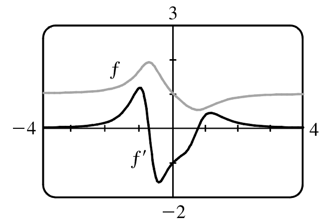
81. (a) Derive gives $g'(t) = \dfrac{45(t-2)^8}{(2t+1)^{10}}$ without simplifying. With either Maple or Mathematica, we first get

$g'(t) = 9\dfrac{(t-2)^8}{(2t+1)^9} - 18\dfrac{(t-2)^9}{(2t+1)^{10}}$, and the simplification command results in the expression given by Derive.

(b) Derive gives $y' = 2(x^3 - x + 1)^3(2x + 1)^4(17x^3 + 6x^2 - 9x + 3)$ without simplifying. With either Maple or Mathematica, we first get $y' = 10(2x + 1)^4(x^3 - x + 1)^4 + 4(2x + 1)^5(x^3 - x + 1)^3(3x^2 - 1)$. If we use

Mathematica's `Factor` or `Simplify`, or Maple's `factor`, we get the above expression, but Maple's `simplify` gives the polynomial expansion instead. For locating horizontal tangents, the factored form is the most helpful.

82. (a) $f(x) = \left(\dfrac{x^4 - x + 1}{x^4 + x + 1}\right)^{1/2}$. Derive gives $f'(x) = \dfrac{(3x^4 - 1)\sqrt{\dfrac{x^4 - x + 1}{x^4 + x + 1}}}{(x^4 + x + 1)(x^4 - x + 1)}$ whereas either Maple or Mathematica give $f'(x) = \dfrac{3x^4 - 1}{\sqrt{\dfrac{x^4 - x + 1}{x^4 + x + 1}}\,(x^4 + x + 1)^2}$ after simplification.



(b) $f'(x) = 0 \quad \Leftrightarrow \quad 3x^4 - 1 = 0 \quad \Leftrightarrow \quad x = \pm\sqrt[4]{\frac{1}{3}} \approx \pm 0.7598$.

(c) Yes. $f'(x) = 0$ where f has horizontal tangents. f' has two maxima and one minimum where f has inflection points.

83. (a) If f is even, then $f(x) = f(-x)$. Using the Chain Rule to differentiate this equation, we get $f'(x) = f'(-x)\dfrac{d}{dx}(-x) = -f'(-x)$. Thus, $f'(-x) = -f'(x)$, so f' is odd.

(b) If f is odd, then $f(x) = -f(-x)$. Differentiating this equation, we get $f'(x) = -f'(-x)(-1) = f'(-x)$, so f' is even.

84.
$$\left[\frac{f(x)}{g(x)}\right]' = \left\{f(x)\,[g(x)]^{-1}\right\}' = f'(x)\,[g(x)]^{-1} + (-1)\,[g(x)]^{-2}\,g'(x)f(x)$$
$$= \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{[g(x)]^2} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

This is an alternative derivation of the *formula* in the Quotient Rule. But part of the purpose of the Quotient Rule is to show that if f and g are differentiable, so is f/g. The proof in Section 2.3 does that; this one doesn't.

85. (a)
$$\begin{aligned}\frac{d}{dx}(\sin^n x\,\cos nx) &= n\sin^{n-1} x\,\cos x\,\cos nx + \sin^n x\,(-n\sin nx) && \text{[Product Rule]}\\ &= n\sin^{n-1} x\,(\cos nx\,\cos x - \sin nx\,\sin x) && [\text{factor out } n\sin^{n-1} x]\\ &= n\sin^{n-1} x\cos(nx + x) && \text{[Addition Formula for cosine]}\\ &= n\sin^{n-1} x\cos[(n+1)x] && [\text{factor out } x]\end{aligned}$$

(b)
$$\begin{aligned}\frac{d}{dx}(\cos^n x\,\cos nx) &= n\cos^{n-1} x\,(-\sin x)\cos nx + \cos^n x\,(-n\sin nx) && \text{[Product Rule]}\\ &= -n\cos^{n-1} x\,(\cos nx\,\sin x + \sin nx\,\cos x) && [\text{factor out } -n\cos^{n-1} x]\\ &= -n\cos^{n-1} x\sin(nx + x) && \text{[Addition Formula for sine]}\\ &= -n\cos^{n-1} x\sin[(n+1)x] && [\text{factor out } x]\end{aligned}$$

86. "The rate of change of y^5 with respect to x is eighty times the rate of change of y with respect to x" $\Leftrightarrow$ $\dfrac{d}{dx}y^5 = 80\dfrac{dy}{dx} \quad \Leftrightarrow \quad 5y^4\dfrac{dy}{dx} = 80\dfrac{dy}{dx} \quad \Leftrightarrow \quad 5y^4 = 80$ (Note that $dy/dx \neq 0$ since the curve never has a horizontal tangent) $\Leftrightarrow \quad y^4 = 16 \quad \Leftrightarrow \quad y = 2$ (since $y > 0$ for all x)

87. Since $\theta^\circ = \left(\frac{\pi}{180}\right)\theta$ rad, we have $\dfrac{d}{d\theta}(\sin\theta^\circ) = \dfrac{d}{d\theta}\left(\sin\frac{\pi}{180}\theta\right) = \frac{\pi}{180}\cos\frac{\pi}{180}\theta = \frac{\pi}{180}\cos\theta^\circ$.

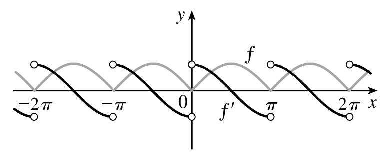
88. (a) $f(x) = |x| = \sqrt{x^2} = (x^2)^{1/2} \quad\Rightarrow\quad f'(x) = \frac{1}{2}(x^2)^{-1/2}(2x) = x/\sqrt{x^2} = x/|x|$ for $x \neq 0$.

f is not differentiable at $x = 0$.

(b) $f(x) = |\sin x| = \sqrt{\sin^2 x} \quad\Rightarrow$

$$f'(x) = \tfrac{1}{2}(\sin^2 x)^{-1/2} 2\sin x\cos x = \frac{\sin x}{|\sin x|}\cos x$$

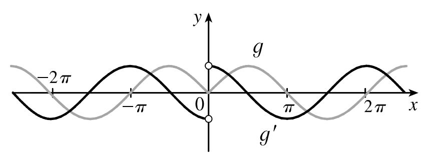
$$= \begin{cases} \cos x & \text{if } \sin x > 0 \\ -\cos x & \text{if } \sin x < 0 \end{cases}$$



f is not differentiable when $x = n\pi$, n an integer.

(c) $g(x) = \sin|x| = \sin\sqrt{x^2} \quad\Rightarrow$

$$g'(x) = \cos|x| \cdot \frac{x}{|x|} = \frac{x}{|x|}\cos x = \begin{cases} \cos x & \text{if } x > 0 \\ -\cos x & \text{if } x < 0 \end{cases}$$



g is not differentiable at 0.

89. The Chain Rule says that $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$, so

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{du}\frac{du}{dx}\right) = \left[\frac{d}{dx}\left(\frac{dy}{du}\right)\right]\frac{du}{dx} + \frac{dy}{du}\frac{d}{dx}\left(\frac{du}{dx}\right) \quad \text{[Product Rule]}$$

$$= \left[\frac{d}{du}\left(\frac{dy}{du}\right)\frac{du}{dx}\right]\frac{du}{dx} + \frac{dy}{du}\frac{d^2u}{dx^2} = \frac{d^2y}{du^2}\left(\frac{du}{dx}\right)^2 + \frac{dy}{du}\frac{d^2u}{dx^2}$$

90. From Exercise 89, $\dfrac{d^2y}{dx^2} = \dfrac{d^2y}{du^2}\left(\dfrac{du}{dx}\right)^2 + \dfrac{dy}{du}\dfrac{d^2u}{dx^2} \quad\Rightarrow$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{d^2y}{du^2}\left(\frac{du}{dx}\right)^2\right] + \frac{d}{dx}\left[\frac{dy}{du}\frac{d^2u}{dx^2}\right]$$

$$= \left[\frac{d}{dx}\left(\frac{d^2y}{du^2}\right)\right]\left(\frac{du}{dx}\right)^2 + \left[\frac{d}{dx}\left(\frac{du}{dx}\right)^2\right]\frac{d^2y}{du^2} + \left[\frac{d}{dx}\left(\frac{dy}{du}\right)\right]\frac{d^2u}{dx^2} + \left[\frac{d}{dx}\left(\frac{d^2u}{dx^2}\right)\right]\frac{dy}{du}$$

$$= \left[\frac{d}{du}\left(\frac{d^2y}{du^2}\right)\frac{du}{dx}\right]\left(\frac{du}{dx}\right)^2 + 2\frac{du}{dx}\frac{d^2u}{dx^2}\frac{d^2y}{du^2} + \left[\frac{d}{du}\left(\frac{dy}{du}\right)\frac{du}{dx}\right]\left(\frac{d^2u}{dx^2}\right) + \frac{d^3u}{dx^3}\frac{dy}{du}$$

$$= \frac{d^3y}{du^3}\left(\frac{du}{dx}\right)^3 + 3\frac{du}{dx}\frac{d^2u}{dx^2}\frac{d^2y}{du^2} + \frac{dy}{du}\frac{d^3u}{dx^3}$$

APPLIED PROJECT Where Should a Pilot Start Descent?

1. Condition (i) will hold if and only if all of the following four conditions hold:

(α) $P(0) = 0$

(β) $P'(0) = 0$ (for a smooth landing)

(γ) $P'(\ell) = 0$ (since the plane is cruising horizontally when it begins its descent)

(δ) $P(\ell) = h$.

[continued]

First of all, condition α implies that $P(0) = d = 0$, so $P(x) = ax^3 + bx^2 + cx \quad \Rightarrow \quad P'(x) = 3ax^2 + 2bx + c$. But $P'(0) = c = 0$ by condition β. So $P'(\ell) = 3a\ell^2 + 2b\ell = \ell\,(3a\ell + 2b)$. Now by condition γ, $3a\ell + 2b = 0 \quad \Rightarrow \quad a = -\dfrac{2b}{3\ell}$.

Therefore, $P(x) = -\dfrac{2b}{3\ell}x^3 + bx^2$. Setting $P(\ell) = h$ for condition δ, we get $P(\ell) = -\dfrac{2b}{3\ell}\ell^3 + b\ell^2 = h \quad \Rightarrow$

$-\dfrac{2}{3}b\ell^2 + b\ell^2 = h \quad \Rightarrow \quad \dfrac{1}{3}b\ell^2 = h \quad \Rightarrow \quad b = \dfrac{3h}{\ell^2} \quad \Rightarrow \quad a = -\dfrac{2h}{\ell^3}$. So $y = P(x) = -\dfrac{2h}{\ell^3}x^3 + \dfrac{3h}{\ell^2}x^2$.

2. By condition (ii), $\dfrac{dx}{dt} = -v$ for all t, so $x(t) = \ell - vt$. Condition (iii) states that $\left|\dfrac{d^2y}{dt^2}\right| \le k$. By the Chain Rule, we have $\dfrac{dy}{dt} = \dfrac{dy}{dx}\dfrac{dx}{dt} = -\dfrac{2h}{\ell^3}\left(3x^2\right)\dfrac{dx}{dt} + \dfrac{3h}{\ell^2}(2x)\dfrac{dx}{dt} = \dfrac{6hx^2v}{\ell^3} - \dfrac{6hxv}{\ell^2} \quad (\text{for } x \le \ell) \quad \Rightarrow$

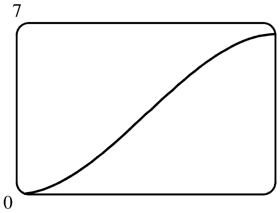
$\dfrac{d^2y}{dt^2} = \dfrac{6hv}{\ell^3}(2x)\dfrac{dx}{dt} - \dfrac{6hv}{\ell^2}\dfrac{dx}{dt} = -\dfrac{12hv^2}{\ell^3}x + \dfrac{6hv^2}{\ell^2}$. In particular, when $t = 0$, $x = \ell$ and so

$\left.\dfrac{d^2y}{dt^2}\right|_{t=0} = -\dfrac{12hv^2}{\ell^3}\ell + \dfrac{6hv^2}{\ell^2} = -\dfrac{6hv^2}{\ell^2}$. Thus, $\left|\dfrac{d^2y}{dt^2}\right|_{t=0} = \dfrac{6hv^2}{\ell^2} \le k$. (This condition also follows from taking $x = 0$.)

3. We substitute $k = 860\text{ mi/h}^2$, $h = 35{,}000\text{ ft} \times \dfrac{1\text{ mi}}{5280\text{ ft}}$, and $v = 300\text{ mi/h}$ into the result of part (b):

$\dfrac{6\left(35{,}000 \cdot \frac{1}{5280}\right)(300)^2}{\ell^2} \le 860 \quad \Rightarrow \quad \ell \ge 300\sqrt{6 \cdot \dfrac{35{,}000}{5280 \cdot 860}} \approx 64.5$ miles.

4. Substituting the values of h and ℓ in Problem 3 into $P(x) = -\dfrac{2h}{\ell^3}x^3 + \dfrac{3h}{\ell^2}x^2$ gives us $P(x) = ax^3 + bx^2$, where $a \approx -4.937 \times 10^{-5}$ and $b \approx 4.78 \times 10^{-3}$.



2.6 Implicit Differentiation

1. (a) $\dfrac{d}{dx}(9x^2 - y^2) = \dfrac{d}{dx}(1) \quad \Rightarrow \quad 18x - 2y\,y' = 0 \quad \Rightarrow \quad 2y\,y' = 18x \quad \Rightarrow \quad y' = \dfrac{9x}{y}$

(b) $9x^2 - y^2 = 1 \quad \Rightarrow \quad y^2 = 9x^2 - 1 \quad \Rightarrow \quad y = \pm\sqrt{9x^2 - 1}$, so $y' = \pm\frac{1}{2}(9x^2 - 1)^{-1/2}(18x) = \pm\dfrac{9x}{\sqrt{9x^2 - 1}}$.

(c) From part (a), $y' = \dfrac{9x}{y} = \dfrac{9x}{\pm\sqrt{9x^2 - 1}}$, which agrees with part (b).

2. (a) $\dfrac{d}{dx}(2x^2 + x + xy) = \dfrac{d}{dx}(1) \quad \Rightarrow \quad 4x + 1 + xy' + y \cdot 1 = 0 \quad \Rightarrow \quad xy' = -4x - y - 1 \quad \Rightarrow \quad y' = -\dfrac{4x + y + 1}{x}$

(b) $2x^2 + x + xy = 1 \quad \Rightarrow \quad xy = 1 - 2x^2 - x \quad \Rightarrow \quad y = \dfrac{1}{x} - 2x - 1$, so $y' = -\dfrac{1}{x^2} - 2$

(c) From part (a),

$y' = -\dfrac{4x + y + 1}{x} = -4 - \dfrac{1}{x}y - \dfrac{1}{x} = -4 - \dfrac{1}{x}\left(\dfrac{1}{x} - 2x - 1 - \dfrac{1}{x}\right) = -4 - \dfrac{1}{x^2} + 2 + \dfrac{1}{x} - \dfrac{1}{x} = -\dfrac{1}{x^2} - 2$, which agrees with part (b).

3. (a) $\frac{d}{dx}\left(\sqrt{x}+\sqrt{y}\right)=\frac{d}{dx}(1) \quad\Rightarrow\quad \frac{1}{2}x^{-1/2}+\frac{1}{2}y^{-1/2}y'=0 \quad\Rightarrow\quad \frac{1}{2\sqrt{y}}\,y'=-\frac{1}{2\sqrt{x}} \quad\Rightarrow\quad y'=-\frac{\sqrt{y}}{\sqrt{x}}$

(b) $\sqrt{x}+\sqrt{y}=1 \quad\Rightarrow\quad \sqrt{y}=1-\sqrt{x} \quad\Rightarrow\quad y=\left(1-\sqrt{x}\right)^2 \quad\Rightarrow\quad y=1-2\sqrt{x}+x$, so

$y'=-2\cdot\frac{1}{2}x^{-1/2}+1=1-\frac{1}{\sqrt{x}}$.

(c) From part (a), $y'=-\frac{\sqrt{y}}{\sqrt{x}}=-\frac{1-\sqrt{x}}{\sqrt{x}}$ [from part (b)] $=-\frac{1}{\sqrt{x}}+1$, which agrees with part (b).

4. (a) $\frac{d}{dx}\left(\frac{2}{x}-\frac{1}{y}\right)=\frac{d}{dx}(4) \quad\Rightarrow\quad -2x^{-2}+y^{-2}y'=0 \quad\Rightarrow\quad \frac{1}{y^2}\,y'=\frac{2}{x^2} \quad\Rightarrow\quad y'=\frac{2y^2}{x^2}$

(b) $\frac{2}{x}-\frac{1}{y}=4 \quad\Rightarrow\quad \frac{1}{y}=\frac{2}{x}-4 \quad\Rightarrow\quad \frac{1}{y}=\frac{2-4x}{x} \quad\Rightarrow\quad y=\frac{x}{2-4x}$, so

$y'=\frac{(2-4x)(1)-x(-4)}{(2-4x)^2}=\frac{2}{(2-4x)^2}\quad\left[\text{or }\frac{1}{2(1-2x)^2}\right]$.

(c) From part (a), $y'=\frac{2y^2}{x^2}=\frac{2\left(\frac{x}{2-4x}\right)^2}{x^2}$ [from part (b)] $=\frac{2x^2}{x^2(2-4x)^2}=\frac{2}{(2-4x)^2}$, which agrees with part (b).

5. $\frac{d}{dx}\left(x^2-4xy+y^2\right)=\frac{d}{dx}(4) \quad\Rightarrow\quad 2x-4[xy'+y(1)]+2y\,y'=0 \quad\Rightarrow\quad 2y\,y'-4xy'=4y-2x \quad\Rightarrow$

$y'(y-2x)=2y-x \quad\Rightarrow\quad y'=\frac{2y-x}{y-2x}$

6. $\frac{d}{dx}\left(2x^2+xy-y^2\right)=\frac{d}{dx}(2) \quad\Rightarrow\quad 4x+xy'+y(1)-2y\,y'=0 \quad\Rightarrow\quad xy'-2y\,y'=-4x-y \quad\Rightarrow$

$(x-2y)y'=-4x-y \quad\Rightarrow\quad y'=\frac{-4x-y}{x-2y}$

7. $\frac{d}{dx}\left(x^4+x^2y^2+y^3\right)=\frac{d}{dx}(5) \quad\Rightarrow\quad 4x^3+x^2\cdot 2y\,y'+y^2\cdot 2x+3y^2y'=0 \quad\Rightarrow\quad 2x^2y\,y'+3y^2y'=-4x^3-2xy^2 \quad\Rightarrow$

$(2x^2y+3y^2)y'=-4x^3-2xy^2 \quad\Rightarrow\quad y'=\frac{-4x^3-2xy^2}{2x^2y+3y^2}=-\frac{2x(2x^2+y^2)}{y(2x^2+3y)}$

8. $\frac{d}{dx}\left(x^3-xy^2+y^3\right)=\frac{d}{dx}(1) \quad\Rightarrow\quad 3x^2-x\cdot 2y\,y'-y^2\cdot 1+3y^2y'=0 \quad\Rightarrow\quad 3y^2y'-2x\,y\,y'=y^2-3x^2 \quad\Rightarrow$

$(3y^2-2xy)\,y'=y^2-3x^2 \quad\Rightarrow\quad y'=\frac{y^2-3x^2}{3y^2-2xy}=\frac{y^2-3x^2}{y(3y-2x)}$

9. $\frac{d}{dx}\left(\frac{x^2}{x+y}\right)=\frac{d}{dx}(y^2+1) \quad\Rightarrow\quad \frac{(x+y)(2x)-x^2(1+y')}{(x+y)^2}=2y\,y' \quad\Rightarrow$

$2x^2+2xy-x^2-x^2\,y'=2y(x+y)^2\,y' \quad\Rightarrow\quad x^2+2xy=2y(x+y)^2\,y'+x^2\,y' \quad\Rightarrow$

$x(x+2y)=\left[2y(x^2+2xy+y^2)+x^2\right]y' \quad\Rightarrow\quad y'=\frac{x(x+2y)}{2x^2y+4xy^2+2y^3+x^2}$

Or: Start by clearing fractions and then differentiate implicitly.

10. $\dfrac{d}{dx}\left(y^5+x^2y^3\right)=\dfrac{d}{dx}\left(1+x^4y\right) \quad\Rightarrow\quad 5y^4y'+x^2\cdot 3y^2y'+y^3\cdot 2x=0+x^4y'+y\cdot 4x^3 \quad\Rightarrow$

$y'\left(5y^4+3x^2y^2-x^4\right)=4x^3y-2xy^3 \quad\Rightarrow\quad y'=\dfrac{4x^3y-2xy^3}{5y^4+3x^2y^2-x^4}$

11. $\dfrac{d}{dx}(y\cos x)=\dfrac{d}{dx}(x^2+y^2) \quad\Rightarrow\quad y(-\sin x)+\cos x\cdot y'=2x+2y\,y' \quad\Rightarrow\quad \cos x\cdot y'-2y\,y'=2x+y\sin x \quad\Rightarrow$

$y'(\cos x-2y)=2x+y\sin x \quad\Rightarrow\quad y'=\dfrac{2x+y\sin x}{\cos x-2y}$

12. $\dfrac{d}{dx}\cos(xy)=\dfrac{d}{dx}(1+\sin y) \quad\Rightarrow\quad -\sin(xy)(xy'+y\cdot 1)=\cos y\cdot y' \quad\Rightarrow\quad -xy'\sin(xy)-\cos y\cdot y'=y\sin(xy) \quad\Rightarrow$

$y'[-x\sin(xy)-\cos y]=y\sin(xy) \quad\Rightarrow\quad y'=\dfrac{y\sin(xy)}{-x\sin(xy)-\cos y}=-\dfrac{y\sin(xy)}{x\sin(xy)+\cos y}$

13. $\dfrac{d}{dx}\sqrt{x+y}=\dfrac{d}{dx}\left(x^4+y^4\right) \quad\Rightarrow\quad \frac{1}{2}\left(x+y\right)^{-1/2}\left(1+y'\right)=4x^3+4y^3y' \quad\Rightarrow$

$\dfrac{1}{2\sqrt{x+y}}+\dfrac{1}{2\sqrt{x+y}}\,y'=4x^3+4y^3y' \quad\Rightarrow\quad \dfrac{1}{2\sqrt{x+y}}-4x^3=4y^3y'-\dfrac{1}{2\sqrt{x+y}}y' \quad\Rightarrow$

$\dfrac{1-8x^3\sqrt{x+y}}{2\sqrt{x+y}}=\dfrac{8y^3\sqrt{x+y}-1}{2\sqrt{x+y}}\,y' \quad\Rightarrow\quad y'=\dfrac{1-8x^3\sqrt{x+y}}{8y^3\sqrt{x+y}-1}$

14. $\dfrac{d}{dx}\left[y\sin(x^2)\right]=\dfrac{d}{dx}\left[x\sin(y^2)\right] \quad\Rightarrow\quad y\cos(x^2)\cdot 2x+\sin(x^2)\cdot y'=x\cos(y^2)\cdot 2y\,y'+\sin(y^2)\cdot 1 \quad\Rightarrow$

$y'\left[\sin(x^2)-2xy\cos(y^2)\right]=\sin(y^2)-2xy\cos(x^2) \quad\Rightarrow\quad y'=\dfrac{\sin(y^2)-2xy\cos(x^2)}{\sin(x^2)-2xy\cos(y^2)}$

15. $\dfrac{d}{dx}\tan(x/y)=\dfrac{d}{dx}(x+y) \quad\Rightarrow\quad \sec^2(x/y)\cdot\dfrac{y\cdot 1-x\cdot y'}{y^2}=1+y' \quad\Rightarrow$

$y\sec^2(x/y)-x\sec^2(x/y)\cdot y'=y^2+y^2y' \quad\Rightarrow\quad y\sec^2(x/y)-y^2=y^2y'+x\sec^2(x/y) \quad\Rightarrow$

$y\sec^2(x/y)-y^2=\left[y^2+x\sec^2(x/y)\right]\cdot y' \quad\Rightarrow\quad y'=\dfrac{y\sec^2(x/y)-y^2}{y^2+x\sec^2(x/y)}$

16. $\dfrac{d}{dx}(xy)=\dfrac{d}{dx}\sqrt{x^2+y^2} \quad\Rightarrow\quad xy'+y(1)=\frac{1}{2}\left(x^2+y^2\right)^{-1/2}(2x+2y\,y') \quad\Rightarrow$

$xy'+y=\dfrac{x}{\sqrt{x^2+y^2}}+\dfrac{y}{\sqrt{x^2+y^2}}\,y' \quad\Rightarrow\quad xy'-\dfrac{y}{\sqrt{x^2+y^2}}\,y'=\dfrac{x}{\sqrt{x^2+y^2}}-y \quad\Rightarrow$

$\dfrac{x\sqrt{x^2+y^2}-y}{\sqrt{x^2+y^2}}\,y'=\dfrac{x-y\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} \quad\Rightarrow\quad y'=\dfrac{x-y\sqrt{x^2+y^2}}{x\sqrt{x^2+y^2}-y}$

17. $\dfrac{d}{dx}\sqrt{xy}=\dfrac{d}{dx}(1+x^2y) \quad\Rightarrow\quad \frac{1}{2}(xy)^{-1/2}(xy'+y\cdot 1)=0+x^2y'+y\cdot 2x \quad\Rightarrow$

$\dfrac{x}{2\sqrt{xy}}\,y'+\dfrac{y}{2\sqrt{xy}}=x^2y'+2xy \quad\Rightarrow\quad y'\left(\dfrac{x}{2\sqrt{xy}}-x^2\right)=2xy-\dfrac{y}{2\sqrt{xy}} \quad\Rightarrow$

$y'\left(\dfrac{x-2x^2\sqrt{xy}}{2\sqrt{xy}}\right)=\dfrac{4xy\sqrt{xy}-y}{2\sqrt{xy}} \quad\Rightarrow\quad y'=\dfrac{4xy\sqrt{xy}-y}{x-2x^2\sqrt{xy}}$

18. $\frac{d}{dx}(x\sin y + y\sin x) = \frac{d}{dx}(1) \quad\Rightarrow\quad x\cos y\cdot y' + \sin y\cdot 1 + y\cos x + \sin x\cdot y' = 0 \quad\Rightarrow$

$x\cos y\cdot y' + \sin x\cdot y' = -\sin y - y\cos x \quad\Rightarrow\quad y'(x\cos y + \sin x) = -\sin y - y\cos x \quad\Rightarrow\quad y' = \frac{-\sin y - y\cos x}{x\cos y + \sin x}$

19. $\frac{d}{dx}\sin(xy) = \frac{d}{dx}\cos(x+y) \quad\Rightarrow\quad \cos(xy)\cdot(xy' + y\cdot 1) = -\sin(x+y)\cdot(1+y') \quad\Rightarrow$

$x\cos(xy)\,y' + y\cos(xy) = -\sin(x+y) - y'\sin(x+y) \quad\Rightarrow$

$x\cos(xy)\,y' + y'\sin(x+y) = -y\cos(xy) - \sin(x+y) \quad\Rightarrow$

$[x\cos(xy) + \sin(x+y)]\,y' = -1\,[y\cos(xy) + \sin(x+y)] \quad\Rightarrow\quad y' = -\frac{y\cos(xy) + \sin(x+y)}{x\cos(xy) + \sin(x+y)}$

20. $\frac{d}{dx}\tan(x-y) = \frac{d}{dx}\left(\frac{y}{1+x^2}\right) \quad\Rightarrow\quad (1+x^2)\tan(x-y) = y \quad\Rightarrow$

$(1+x^2)\sec^2(x-y)\cdot(1-y') + \tan(x-y)\cdot 2x = y' \quad\Rightarrow$

$(1+x^2)\sec^2(x-y) - (1+x^2)\sec^2(x-y)\cdot y' + 2x\tan(x-y) = y' \quad\Rightarrow$

$(1+x^2)\sec^2(x-y) + 2x\tan(x-y) = \left[1 + (1+x^2)\sec^2(x-y)\right]\cdot y' \quad\Rightarrow$

$y' = \frac{(1+x^2)\sec^2(x-y) + 2x\tan(x-y)}{1 + (1+x^2)\sec^2(x-y)}$

21. $\frac{d}{dx}\left\{f(x) + x^2[f(x)]^3\right\} = \frac{d}{dx}(10) \quad\Rightarrow\quad f'(x) + x^2\cdot 3[f(x)]^2\cdot f'(x) + [f(x)]^3\cdot 2x = 0.$ If $x = 1$, we have

$f'(1) + 1^2\cdot 3[f(1)]^2\cdot f'(1) + [f(1)]^3\cdot 2(1) = 0 \quad\Rightarrow\quad f'(1) + 1\cdot 3\cdot 2^2\cdot f'(1) + 2^3\cdot 2 = 0 \quad\Rightarrow$

$f'(1) + 12f'(1) = -16 \quad\Rightarrow\quad 13f'(1) = -16 \quad\Rightarrow\quad f'(1) = -\frac{16}{13}.$

22. $\frac{d}{dx}[g(x) + x\sin g(x)] = \frac{d}{dx}(x^2) \quad\Rightarrow\quad g'(x) + x\cos g(x)\cdot g'(x) + \sin g(x)\cdot 1 = 2x.$ If $x = 0$, we have

$g'(0) + 0 + \sin g(0) = 2(0) \quad\Rightarrow\quad g'(0) + \sin 0 = 0 \quad\Rightarrow\quad g'(0) + 0 = 0 \quad\Rightarrow\quad g'(0) = 0.$

23. $\frac{d}{dy}(x^4y^2 - x^3y + 2xy^3) = \frac{d}{dy}(0) \quad\Rightarrow\quad x^4\cdot 2y + y^2\cdot 4x^3\,x' - (x^3\cdot 1 + y\cdot 3x^2\,x') + 2(x\cdot 3y^2 + y^3\cdot x') = 0 \quad\Rightarrow$

$4x^3y^2\,x' - 3x^2y\,x' + 2y^3\,x' = -2x^4y + x^3 - 6xy^2 \quad\Rightarrow\quad (4x^3y^2 - 3x^2y + 2y^3)\,x' = -2x^4y + x^3 - 6xy^2 \quad\Rightarrow$

$x' = \frac{dx}{dy} = \frac{-2x^4y + x^3 - 6xy^2}{4x^3y^2 - 3x^2y + 2y^3}$

24. $\frac{d}{dy}(y\sec x) = \frac{d}{dy}(x\tan y) \quad\Rightarrow\quad y\cdot\sec x\,\tan x\cdot x' + \sec x\cdot 1 = x\cdot\sec^2 y + \tan y\cdot x' \quad\Rightarrow$

$y\sec x\,\tan x\cdot x' - \tan y\cdot x' = x\sec^2 y - \sec x \quad\Rightarrow\quad (y\sec x\,\tan x - \tan y)\,x' = x\sec^2 y - \sec x \quad\Rightarrow$

$x' = \frac{dx}{dy} = \frac{x\sec^2 y - \sec x}{y\sec x\,\tan x - \tan y}$

25. $y \sin 2x = x \cos 2y \quad \Rightarrow \quad y \cdot \cos 2x \cdot 2 + \sin 2x \cdot y' = x(-\sin 2y \cdot 2y') + \cos(2y) \cdot 1 \quad \Rightarrow$

$\sin 2x \cdot y' + 2x \sin 2y \cdot y' = -2y \cos 2x + \cos 2y \quad \Rightarrow \quad y'(\sin 2x + 2x \sin 2y) = -2y \cos 2x + \cos 2y \quad \Rightarrow$

$y' = \dfrac{-2y \cos 2x + \cos 2y}{\sin 2x + 2x \sin 2y}$. When $x = \frac{\pi}{2}$ and $y = \frac{\pi}{4}$, we have $y' = \dfrac{(-\pi/2)(-1) + 0}{0 + \pi \cdot 1} = \dfrac{\pi/2}{\pi} = \dfrac{1}{2}$, so an equation of the tangent line is $y - \frac{\pi}{4} = \frac{1}{2}(x - \frac{\pi}{2})$, or $y = \frac{1}{2}x$.

26. $\sin(x + y) = 2x - 2y \quad \Rightarrow \quad \cos(x + y) \cdot (1 + y') = 2 - 2y' \quad \Rightarrow \quad \cos(x + y) \cdot y' + 2y' = 2 - \cos(x + y) \quad \Rightarrow$

$y'[\cos(x + y) + 2] = 2 - \cos(x + y) \quad \Rightarrow \quad y' = \dfrac{2 - \cos(x + y)}{\cos(x + y) + 2}$. When $x = \pi$ and $y = \pi$, we have $y' = \dfrac{2 - 1}{1 + 2} = \dfrac{1}{3}$, so an equation of the tangent line is $y - \pi = \frac{1}{3}(x - \pi)$, or $y = \frac{1}{3}x + \frac{2\pi}{3}$.

27. $x^2 - xy - y^2 = 1 \quad \Rightarrow \quad 2x - (xy' + y \cdot 1) - 2y\,y' = 0 \quad \Rightarrow \quad 2x - xy' - y - 2y\,y' = 0 \quad \Rightarrow \quad 2x - y = xy' + 2y\,y' \quad \Rightarrow$

$2x - y = (x + 2y)\,y' \quad \Rightarrow \quad y' = \dfrac{2x - y}{x + 2y}$. When $x = 2$ and $y = 1$, we have $y' = \dfrac{4 - 1}{2 + 2} = \dfrac{3}{4}$, so an equation of the tangent line is $y - 1 = \frac{3}{4}(x - 2)$, or $y = \frac{3}{4}x - \frac{1}{2}$.

28. $x^2 + 2xy + 4y^2 = 12 \quad \Rightarrow \quad 2x + 2x\,y' + 2y + 8y\,y' = 0 \quad \Rightarrow \quad 2x\,y' + 8y\,y' = -2x - 2y \quad \Rightarrow$

$(x + 4y)\,y' = -x - y \quad \Rightarrow \quad y' = -\dfrac{x + y}{x + 4y}$. When $x = 2$ and $y = 1$, we have $y' = -\dfrac{2 + 1}{2 + 4} = -\dfrac{1}{2}$, so an equation of the tangent line is $y - 1 = -\frac{1}{2}(x - 2)$ or $y = -\frac{1}{2}x + 2$.

29. $x^2 + y^2 = (2x^2 + 2y^2 - x)^2 \quad \Rightarrow \quad 2x + 2y\,y' = 2(2x^2 + 2y^2 - x)(4x + 4y\,y' - 1)$. When $x = 0$ and $y = \frac{1}{2}$, we have $0 + y' = 2(\frac{1}{2})(2y' - 1) \quad \Rightarrow \quad y' = 2y' - 1 \quad \Rightarrow \quad y' = 1$, so an equation of the tangent line is $y - \frac{1}{2} = 1(x - 0)$ or $y = x + \frac{1}{2}$.

30. $x^{2/3} + y^{2/3} = 4 \quad \Rightarrow \quad \frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0 \quad \Rightarrow \quad \dfrac{1}{\sqrt[3]{x}} + \dfrac{y'}{\sqrt[3]{y}} = 0 \quad \Rightarrow \quad y' = -\dfrac{\sqrt[3]{y}}{\sqrt[3]{x}}$. When $x = -3\sqrt{3}$ and $y = 1$, we have $y' = -\dfrac{1}{\left(-3\sqrt{3}\right)^{1/3}} = -\dfrac{\left(-3\sqrt{3}\right)^{2/3}}{-3\sqrt{3}} = \dfrac{3}{3\sqrt{3}} = \dfrac{1}{\sqrt{3}}$, so an equation of the tangent line is $y - 1 = \frac{1}{\sqrt{3}}\left(x + 3\sqrt{3}\right)$ or $y = \frac{1}{\sqrt{3}}x + 4$.

31. $2(x^2 + y^2)^2 = 25(x^2 - y^2) \quad \Rightarrow \quad 4(x^2 + y^2)(2x + 2y\,y') = 25(2x - 2y\,y') \quad \Rightarrow$

$4(x + y\,y')(x^2 + y^2) = 25(x - y\,y') \quad \Rightarrow \quad 4y\,y'(x^2 + y^2) + 25yy' = 25x - 4x(x^2 + y^2) \quad \Rightarrow$

$y' = \dfrac{25x - 4x(x^2 + y^2)}{25y + 4y(x^2 + y^2)}$. When $x = 3$ and $y = 1$, we have $y' = \frac{75 - 120}{25 + 40} = -\frac{45}{65} = -\frac{9}{13}$, so an equation of the tangent line is $y - 1 = -\frac{9}{13}(x - 3)$ or $y = -\frac{9}{13}x + \frac{40}{13}$.

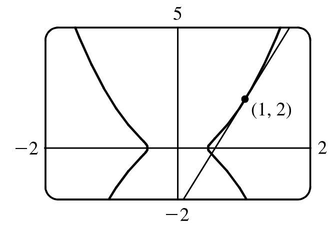
32. $y^2(y^2-4)=x^2(x^2-5) \;\Rightarrow\; y^4-4y^2=x^4-5x^2 \;\Rightarrow\; 4y^3\,y'-8y\,y'=4x^3-10x$.

When $x=0$ and $y=-2$, we have $-32y'+16y'=0 \;\Rightarrow\; -16y'=0 \;\Rightarrow\; y'=0$, so an equation of the tangent line is $y+2=0(x-0)$ or $y=-2$.

33. (a) $y^2=5x^4-x^2 \;\Rightarrow\; 2y\,y'=5(4x^3)-2x \;\Rightarrow\; y'=\dfrac{10x^3-x}{y}$.

So at the point $(1,2)$ we have $y'=\dfrac{10(1)^3-1}{2}=\dfrac{9}{2}$, and an equation of the tangent line is $y-2=\frac{9}{2}(x-1)$ or $y=\frac{9}{2}x-\frac{5}{2}$.

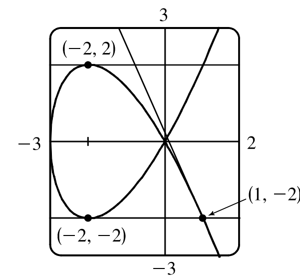
(b)



34. (a) $y^2=x^3+3x^2 \;\Rightarrow\; 2y\,y'=3x^2+3(2x) \;\Rightarrow\; y'=\dfrac{3x^2+6x}{2y}$. So at the point $(1,-2)$ we have $y'=\dfrac{3(1)^2+6(1)}{2(-2)}=-\dfrac{9}{4}$, and an equation of the tangent line is $y+2=-\frac{9}{4}(x-1)$ or $y=-\frac{9}{4}x+\frac{1}{4}$.

(b) The curve has a horizontal tangent where $y'=0 \;\Leftrightarrow$
$3x^2+6x=0 \;\Leftrightarrow\; 3x(x+2)=0 \;\Leftrightarrow\; x=0$ or $x=-2$.
But note that at $x=0$, $y=0$ also, so the derivative does not exist.
At $x=-2$, $y^2=(-2)^3+3(-2)^2=-8+12=4$, so $y=\pm 2$.
So the two points at which the curve has a horizontal tangent are $(-2,-2)$ and $(-2,2)$.

(c)



35. $x^2+4y^2=4 \;\Rightarrow\; 2x+8y\,y'=0 \;\Rightarrow\; y'=-x/(4y) \;\Rightarrow$

$$y''=-\frac{1}{4}\,\frac{y\cdot 1-x\cdot y'}{y^2}=-\frac{1}{4}\,\frac{y-x[-x/(4y)]}{y^2}=-\frac{1}{4}\,\frac{4y^2+x^2}{4y^3}=-\frac{1}{4}\,\frac{4}{4y^3}\qquad\left[\begin{array}{c}\text{since } x \text{ and } y \text{ must satisfy the}\\ \text{original equation } x^2+4y^2=4\end{array}\right]$$

Thus, $y''=-\dfrac{1}{4y^3}$.

36. $x^2+xy+y^2=3 \;\Rightarrow\; 2x+xy'+y+2y\,y'=0 \;\Rightarrow\; (x+2y)y'=-2x-y \;\Rightarrow\; y'=\dfrac{-2x-y}{x+2y}$.

Differentiating $2x+xy'+y+2y\,y'=0$ to find y'' gives $2+xy''+y'+y'+2y\,y''+2y'y'=0 \;\Rightarrow$

$$(x+2y)\,y''=-2-2y'-2(y')^2=-2\left[1-\frac{2x+y}{x+2y}+\left(\frac{2x+y}{x+2y}\right)^2\right]\;\Rightarrow$$

$$\begin{aligned}
y''&=-\frac{2}{x+2y}\left[\frac{(x+2y)^2-(2x+y)(x+2y)+(2x+y)^2}{(x+2y)^2}\right]\\
&=-\frac{2}{(x+2y)^3}\left(x^2+4xy+4y^2-2x^2-4xy-xy-2y^2+4x^2+4xy+y^2\right)\\
&=-\frac{2}{(x+2y)^3}\left(3x^2+3xy+3y^2\right)=-\frac{2}{(x+2y)^3}\,(9)\qquad\left[\begin{array}{c}\text{since } x \text{ and } y \text{ must satisfy the}\\ \text{original equation } x^2+xy+y^2=3\end{array}\right]
\end{aligned}$$

Thus, $y''=-\dfrac{18}{(x+2y)^3}$.

37. $\sin y + \cos x = 1 \quad \Rightarrow \quad \cos y \cdot y' - \sin x = 0 \quad \Rightarrow \quad y' = \dfrac{\sin x}{\cos y} \quad \Rightarrow$

$$y'' = \frac{\cos y\, \cos x - \sin x(-\sin y)\, y'}{(\cos y)^2} = \frac{\cos y\, \cos x + \sin x\, \sin y(\sin x/\cos y)}{\cos^2 y}$$
$$= \frac{\cos^2 y\, \cos x + \sin^2 x\, \sin y}{\cos^2 y\, \cos y} = \frac{\cos^2 y\, \cos x + \sin^2 x\, \sin y}{\cos^3 y}$$

Using $\sin y + \cos x = 1$, the expression for y'' can be simplified to $y'' = (\cos^2 x + \sin y)/\cos^3 y$.

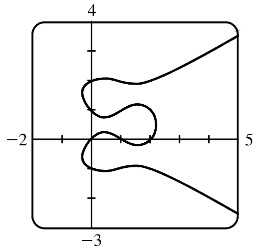
38. $x^3 - y^3 = 7 \quad \Rightarrow \quad 3x^2 - 3y^2 y' = 0 \quad \Rightarrow \quad y' = \dfrac{x^2}{y^2} \quad \Rightarrow$

$$y'' = \frac{y^2(2x) - x^2(2y\, y')}{(y^2)^2} = \frac{2xy[y - x(x^2/y^2)]}{y^4} = \frac{2x(y - x^3/y^2)}{y^3} = \frac{2x(y^3 - x^3)}{y^3 y^2} = \frac{2x(-7)}{y^5} = \frac{-14x}{y^5}$$

39. If $x = 0$ in $xy + y^3 = 1$, then we get $y^3 = 1 \quad \Rightarrow \quad y = 1$, so the point where $x = 0$ is $(0, 1)$. Differentiating implicitly with respect to x gives us $xy' + y \cdot 1 + 3y^2\, y' = 0$. Substituting 0 for x and 1 for y gives us $1 + 3y' = 0 \quad \Rightarrow \quad y' = -\frac{1}{3}$. Differentiating $xy' + y + 3y^2\, y' = 0$ implicitly with respect to x gives us $xy'' + y' + y' + 3(y^2 y'' + y' \cdot 2y\, y') = 0$. Now substitute 0 for x, 1 for y, and $-\frac{1}{3}$ for y'. $\;0 - \frac{1}{3} - \frac{1}{3} + 3\left[y'' + \left(-\frac{1}{3}\right) \cdot 2\left(-\frac{1}{3}\right)\right] = 0 \quad \Rightarrow \quad 3\left(y'' + \frac{2}{9}\right) = \frac{2}{3} \quad \Rightarrow$
$y'' + \frac{2}{9} = \frac{2}{9} \quad \Rightarrow \quad y'' = 0$.

40. If $x = 1$ in $x^2 + xy + y^3 = 1$, then we get $1 + y + y^3 = 1 \quad \Rightarrow \quad y^3 + y = 0 \quad \Rightarrow \quad y(y^2 + 1) \quad \Rightarrow \quad y = 0$, so the point where $x = 1$ is $(1, 0)$. Differentiating implicitly with respect to x gives us $2x + xy' + y \cdot 1 + 3y^2 \cdot y' = 0$. Substituting 1 for x and 0 for y gives us $2 + y' + 0 + 0 = 0 \quad \Rightarrow \quad y' = -2$. Differentiating $2x + xy' + y + 3y^2 y' = 0$ implicitly with respect to x gives us $2 + xy'' + y' \cdot 1 + y' + 3(y^2 y'' + y' \cdot 2yy') = 0$. Now substitute 1 for x, 0 for y, and -2 for y'.
$2 + y'' + (-2) + (-2) + 3(0 + 0) = 0 \quad \Rightarrow \quad y'' = 2$. Differentiating $2 + xy'' + 2y' + 3y^2 y'' + 6y(y')^2 = 0$ implicitly with respect to x gives us $xy''' + y'' \cdot 1 + 2y'' + 3(y^2 y''' + y'' \cdot 2yy') + 6[y \cdot 2y'y'' + (y')^2 y'] = 0$. Now substitute 1 for x, 0 for y, -2 for y', and 2 for y''. $\;y''' + 2 + 4 + 3(0 + 0) + 6[0 + (-8)] = 0 \quad \Rightarrow \quad y''' = -2 - 4 + 48 = 42$.

41. (a) There are eight points with horizontal tangents: four at $x \approx 1.57735$ and four at $x \approx 0.42265$.

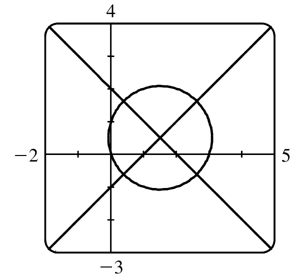


(b) $y' = \dfrac{3x^2 - 6x + 2}{2(2y^3 - 3y^2 - y + 1)} \quad \Rightarrow \quad y' = -1$ at $(0, 1)$ and $y' = \frac{1}{3}$ at $(0, 2)$.

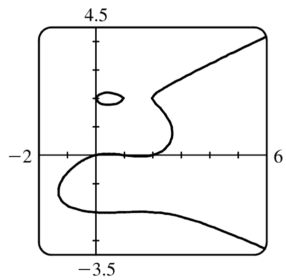
Equations of the tangent lines are $y = -x + 1$ and $y = \frac{1}{3}x + 2$.

(c) $y' = 0 \quad \Rightarrow \quad 3x^2 - 6x + 2 = 0 \quad \Rightarrow \quad x = 1 \pm \frac{1}{3}\sqrt{3}$

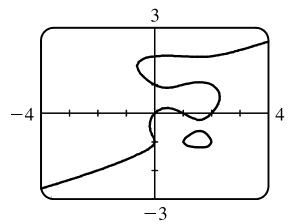
(d) By multiplying the right side of the equation by $x-3$, we obtain the first graph. By modifying the equation in other ways, we can generate the other graphs.



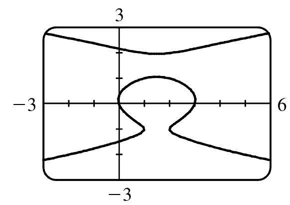
$y(y^2-1)(y-2)$
$= x(x-1)(x-2)(x-3)$



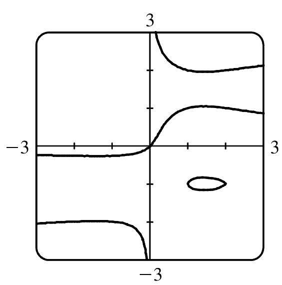
$y(y^2-4)(y-2)$
$= x(x-1)(x-2)$



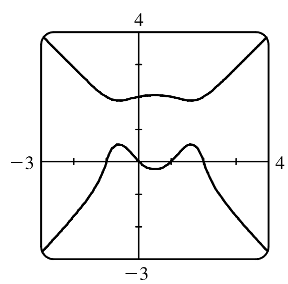
$y(y+1)(y^2-1)(y-2)$
$= x(x-1)(x-2)$



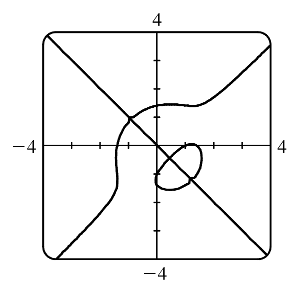
$(y+1)(y^2-1)(y-2)$
$= (x-1)(x-2)$



$x(y+1)(y^2-1)(y-2)$
$= y(x-1)(x-2)$

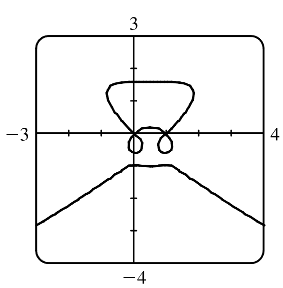


$y(y^2+1)(y-2)$
$= x(x^2-1)(x-2)$



$y(y+1)(y^2-2)$
$= x(x-1)(x^2-2)$

42. (a)



(b) $\dfrac{d}{dx}(2y^3+y^2-y^5) = \dfrac{d}{dx}(x^4-2x^3+x^2) \quad\Rightarrow$

$6y^2y' + 2y\,y' - 5y^4y' = 4x^3 - 6x^2 + 2x \quad\Rightarrow$

$y' = \dfrac{2x(2x^2-3x+1)}{6y^2+2y-5y^4} = \dfrac{2x(2x-1)(x-1)}{y(6y+2-5y^3)}$. From the graph and the values for which $y'=0$, we speculate that there are 9 points with horizontal tangents: 3 at $x=0$, 3 at $x=\frac{1}{2}$, and 3 at $x=1$. The three horizontal tangents along the top of the wagon are hard to find, but by limiting the y-range of the graph (to $[1.6, 1.7]$, for example) they are distinguishable.

43. From Exercise 31, a tangent to the lemniscate will be horizontal if $y' = 0 \quad\Rightarrow\quad 25x - 4x(x^2 + y^2) = 0 \quad\Rightarrow$ $x[25 - 4(x^2 + y^2)] = 0 \quad\Rightarrow\quad x^2 + y^2 = \frac{25}{4}$ **(1)**. (Note that when x is 0, y is also 0, and there is no horizontal tangent at the origin.) Substituting $\frac{25}{4}$ for $x^2 + y^2$ in the equation of the lemniscate, $2(x^2 + y^2)^2 = 25(x^2 - y^2)$, we get $x^2 - y^2 = \frac{25}{8}$ **(2)**. Solving **(1)** and **(2)**, we have $x^2 = \frac{75}{16}$ and $y^2 = \frac{25}{16}$, so the four points are $\left(\pm\frac{5\sqrt{3}}{4}, \pm\frac{5}{4}\right)$.

44. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \quad\Rightarrow\quad \dfrac{2x}{a^2} + \dfrac{2yy'}{b^2} = 0 \quad\Rightarrow\quad y' = -\dfrac{b^2x}{a^2y} \quad\Rightarrow$ an equation of the tangent line at (x_0, y_0) is $y - y_0 = \dfrac{-b^2x_0}{a^2y_0}(x - x_0)$. Multiplying both sides by $\dfrac{y_0}{b^2}$ gives $\dfrac{y_0y}{b^2} - \dfrac{y_0^2}{b^2} = -\dfrac{x_0x}{a^2} + \dfrac{x_0^2}{a^2}$. Since (x_0, y_0) lies on the ellipse, we have $\dfrac{x_0x}{a^2} + \dfrac{y_0y}{b^2} = \dfrac{x_0^2}{a^2} + \dfrac{y_0^2}{b^2} = 1$.

45. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1 \quad\Rightarrow\quad \dfrac{2x}{a^2} - \dfrac{2yy'}{b^2} = 0 \quad\Rightarrow\quad y' = \dfrac{b^2x}{a^2y} \quad\Rightarrow$ an equation of the tangent line at (x_0, y_0) is $y - y_0 = \dfrac{b^2x_0}{a^2y_0}(x - x_0)$. Multiplying both sides by $\dfrac{y_0}{b^2}$ gives $\dfrac{y_0y}{b^2} - \dfrac{y_0^2}{b^2} = \dfrac{x_0x}{a^2} - \dfrac{x_0^2}{a^2}$. Since (x_0, y_0) lies on the hyperbola, we have $\dfrac{x_0x}{a^2} - \dfrac{y_0y}{b^2} = \dfrac{x_0^2}{a^2} - \dfrac{y_0^2}{b^2} = 1$.

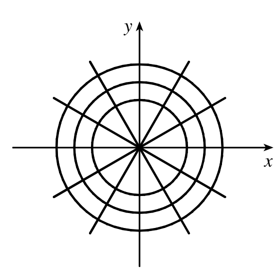
46. $\sqrt{x} + \sqrt{y} = \sqrt{c} \quad\Rightarrow\quad \dfrac{1}{2\sqrt{x}} + \dfrac{y'}{2\sqrt{y}} = 0 \quad\Rightarrow\quad y' = -\dfrac{\sqrt{y}}{\sqrt{x}} \quad\Rightarrow$ an equation of the tangent line at (x_0, y_0) is $y - y_0 = -\dfrac{\sqrt{y_0}}{\sqrt{x_0}}(x - x_0)$. Now $x = 0 \quad\Rightarrow\quad y = y_0 - \dfrac{\sqrt{y_0}}{\sqrt{x_0}}(-x_0) = y_0 + \sqrt{x_0}\sqrt{y_0}$, so the y-intercept is $y_0 + \sqrt{x_0}\sqrt{y_0}$. And $y = 0 \quad\Rightarrow\quad -y_0 = -\dfrac{\sqrt{y_0}}{\sqrt{x_0}}(x - x_0) \quad\Rightarrow\quad x - x_0 = \dfrac{y_0\sqrt{x_0}}{\sqrt{y_0}} \quad\Rightarrow$ $x = x_0 + \sqrt{x_0}\sqrt{y_0}$, so the x-intercept is $x_0 + \sqrt{x_0}\sqrt{y_0}$. The sum of the intercepts is

$$\left(y_0 + \sqrt{x_0}\sqrt{y_0}\right) + \left(x_0 + \sqrt{x_0}\sqrt{y_0}\right) = x_0 + 2\sqrt{x_0}\sqrt{y_0} + y_0 = \left(\sqrt{x_0} + \sqrt{y_0}\right)^2 = \left(\sqrt{c}\right)^2 = c.$$

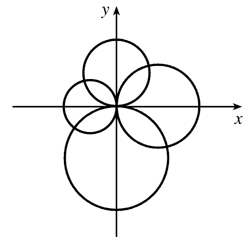
47. If the circle has radius r, its equation is $x^2 + y^2 = r^2 \quad\Rightarrow\quad 2x + 2yy' = 0 \quad\Rightarrow\quad y' = -\dfrac{x}{y}$, so the slope of the tangent line at $P(x_0, y_0)$ is $-\dfrac{x_0}{y_0}$. The negative reciprocal of that slope is $\dfrac{-1}{-x_0/y_0} = \dfrac{y_0}{x_0}$, which is the slope of OP, so the tangent line at P is perpendicular to the radius OP.

48. $y^q = x^p \quad\Rightarrow\quad qy^{q-1}y' = px^{p-1} \quad\Rightarrow\quad y' = \dfrac{px^{p-1}}{qy^{q-1}} = \dfrac{px^{p-1}y}{qy^q} = \dfrac{px^{p-1}x^{p/q}}{qx^p} = \dfrac{p}{q}x^{(p/q)-1}$

49. $x^2 + y^2 = r^2$ is a circle with center O and $ax + by = 0$ is a line through O [assume a and b are not both zero]. $x^2 + y^2 = r^2 \quad\Rightarrow\quad 2x + 2yy' = 0 \quad\Rightarrow\quad y' = -x/y$, so the slope of the tangent line at $P_0\,(x_0, y_0)$ is $-x_0/y_0$. The slope of the line OP_0 is y_0/x_0, which is the negative reciprocal of $-x_0/y_0$. Hence, the curves are orthogonal, and the families of curves are orthogonal trajectories of each other.



50. The circles $x^2 + y^2 = ax$ and $x^2 + y^2 = by$ intersect at the origin where the tangents are vertical and horizontal [assume a and b are both nonzero]. If (x_0, y_0) is the other point of intersection, then $x_0^2 + y_0^2 = ax_0$ (**1**) and $x_0^2 + y_0^2 = by_0$ (**2**).



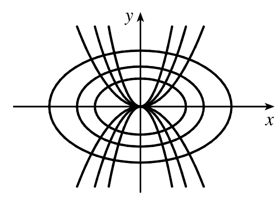
Now $x^2 + y^2 = ax \;\Rightarrow\; 2x + 2yy' = a \;\Rightarrow\; y' = \dfrac{a - 2x}{2y}$ and $x^2 + y^2 = by \;\Rightarrow$

$2x + 2yy' = by' \;\Rightarrow\; y' = \dfrac{2x}{b - 2y}$. Thus, the curves are orthogonal at (x_0, y_0) $\Leftrightarrow$

$\dfrac{a - 2x_0}{2y_0} = -\dfrac{b - 2y_0}{2x_0} \;\Leftrightarrow\; 2ax_0 - 4x_0^2 = 4y_0^2 - 2by_0 \;\Leftrightarrow\; ax_0 + by_0 = 2(x_0^2 + y_0^2)$,

which is true by (**1**) and (**2**).

51. $y = cx^2 \;\Rightarrow\; y' = 2cx$ and $x^2 + 2y^2 = k$ [assume $k > 0$] $\Rightarrow\; 2x + 4yy' = 0 \;\Rightarrow$

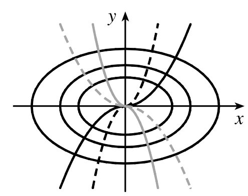


$2yy' = -x \;\Rightarrow\; y' = -\dfrac{x}{2(y)} = -\dfrac{x}{2(cx^2)} = -\dfrac{1}{2cx}$, so the curves are orthogonal if

$c \neq 0$. If $c = 0$, then the horizontal line $y = cx^2 = 0$ intersects $x^2 + 2y^2 = k$ orthogonally

at $\left(\pm\sqrt{k}, 0\right)$, since the ellipse $x^2 + 2y^2 = k$ has vertical tangents at those two points.

52. $y = ax^3 \;\Rightarrow\; y' = 3ax^2$ and $x^2 + 3y^2 = b$ [assume $b > 0$] $\Rightarrow\; 2x + 6yy' = 0 \;\Rightarrow$

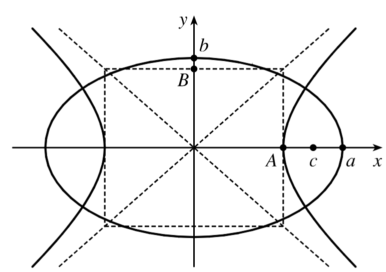


$3yy' = -x \;\Rightarrow\; y' = -\dfrac{x}{3(y)} = -\dfrac{x}{3(ax^3)} = -\dfrac{1}{3ax^2}$, so the curves are orthogonal if

$a \neq 0$. If $a = 0$, then the horizontal line $y = ax^3 = 0$ intesects $x^2 + 3y^2 = b$ orthogonally

at $\left(\pm\sqrt{b}, 0\right)$, since the ellipse $x^2 + 3y^2 = b$ has vertical tangents at those two points.

53. Since $A^2 < a^2$, we are assured that there are four points of intersection.



(1) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \;\Rightarrow\; \dfrac{2x}{a^2} + \dfrac{2yy'}{b^2} = 0 \;\Rightarrow\; \dfrac{yy'}{b^2} = -\dfrac{x}{a^2} \;\Rightarrow$

$y' = m_1 = -\dfrac{xb^2}{ya^2}$.

(2) $\dfrac{x^2}{A^2} - \dfrac{y^2}{B^2} = 1 \;\Rightarrow\; \dfrac{2x}{A^2} - \dfrac{2yy'}{B^2} = 0 \;\Rightarrow\; \dfrac{yy'}{B^2} = \dfrac{x}{A^2} \;\Rightarrow$

$y' = m_2 = \dfrac{xB^2}{yA^2}$.

Now $m_1 m_2 = -\dfrac{xb^2}{ya^2} \cdot \dfrac{xB^2}{yA^2} = -\dfrac{b^2B^2}{a^2A^2} \cdot \dfrac{x^2}{y^2}$ (**3**). Subtracting equations, (**1**) − (**2**), gives us $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{x^2}{A^2} + \dfrac{y^2}{B^2} = 0 \;\Rightarrow$

$\dfrac{y^2}{b^2} + \dfrac{y^2}{B^2} = \dfrac{x^2}{A^2} - \dfrac{x^2}{a^2} \;\Rightarrow\; \dfrac{y^2B^2 + y^2b^2}{b^2B^2} = \dfrac{x^2a^2 - x^2A^2}{A^2a^2} \;\Rightarrow\; \dfrac{y^2(b^2 + B^2)}{b^2B^2} = \dfrac{x^2(a^2 - A^2)}{a^2A^2}$ (**4**). Since

$a^2 - b^2 = A^2 + B^2$, we have $a^2 - A^2 = b^2 + B^2$. Thus, equation (**4**) becomes $\dfrac{y^2}{b^2B^2} = \dfrac{x^2}{A^2a^2} \;\Rightarrow\; \dfrac{x^2}{y^2} = \dfrac{A^2a^2}{b^2B^2}$, and

substituting for $\dfrac{x^2}{y^2}$ in equation (**3**) gives us $m_1 m_2 = -\dfrac{b^2B^2}{a^2A^2} \cdot \dfrac{a^2A^2}{b^2B^2} = -1$. Hence, the ellipse and hyperbola are orthogonal trajectories.

54. $y = (x+c)^{-1} \Rightarrow y' = -(x+c)^{-2}$ and $y = a(x+k)^{1/3} \Rightarrow y' = \frac{1}{3}a(x+k)^{-2/3}$, so the curves are othogonal if the product of the slopes is -1, that is, $\dfrac{-1}{(x+c)^2} \cdot \dfrac{a}{3(x+k)^{2/3}} = -1 \Rightarrow a = 3(x+c)^2(x+k)^{2/3} \Rightarrow$

$a = 3\left(\dfrac{1}{y}\right)^2\left(\dfrac{y}{a}\right)^2$ [since $y^2 = (x+c)^{-2}$ and $y^2 = a^2(x+k)^{2/3}$] $\Rightarrow a = 3\left(\dfrac{1}{a^2}\right) \Rightarrow a^3 = 3 \Rightarrow a = \sqrt[3]{3}$.

55. (a) $\left(P + \dfrac{n^2a}{V^2}\right)(V - nb) = nRT \Rightarrow PV - Pnb + \dfrac{n^2a}{V} - \dfrac{n^3ab}{V^2} = nRT \Rightarrow$

$$\frac{d}{dP}(PV - Pnb + n^2aV^{-1} - n^3abV^{-2}) = \frac{d}{dP}(nRT) \Rightarrow$$

$$PV' + V \cdot 1 - nb - n^2aV^{-2} \cdot V' + 2n^3abV^{-3} \cdot V' = 0 \Rightarrow V'(P - n^2aV^{-2} + 2n^3abV^{-3}) = nb - V \Rightarrow$$

$$V' = \frac{nb - V}{P - n^2aV^{-2} + 2n^3abV^{-3}} \text{ or } \frac{dV}{dP} = \frac{V^3(nb - V)}{PV^3 - n^2aV + 2n^3ab}$$

(b) Using the last expression for dV/dP from part (a), we get

$$\frac{dV}{dP} = \frac{(10\text{ L})^3[(1\text{ mole})(0.04267\text{ L/mole}) - 10\text{ L}]}{\left[\begin{array}{l}(2.5\text{ atm})(10\text{ L})^3 - (1\text{ mole})^2(3.592\text{ L}^2\text{-atm/ mole}^2)(10\text{ L}) \\ \qquad + 2(1\text{ mole})^3(3.592\text{ L}^2\text{-atm/ mole}^2)(0.04267\text{ L/ mole})\end{array}\right]}$$

$$= \frac{-9957.33\text{ L}^4}{2464.386541\text{ L}^3\text{-atm}} \approx -4.04\text{ L/ atm}.$$

56. (a) $x^2 + xy + y^2 + 1 = 0 \Rightarrow 2x + xy' + y \cdot 1 + 2yy' + 0 = 0 \Rightarrow y'(x + 2y) = -2x - y \Rightarrow y' = \dfrac{-2x - y}{x + 2y}$

(b) Plotting the curve in part (a) gives us an empty graph, that is, there are no points that satisfy the equation. If there were any points that satisfied the equation, then x and y would have opposite signs; otherwise, all the terms are positive and their sum can not equal 0. $x^2 + xy + y^2 + 1 = 0 \Rightarrow x^2 + 2xy + y^2 - xy + 1 = 0 \Rightarrow (x+y)^2 = xy - 1$. The left side of the last equation is nonnegative, but the right side is at most -1, so that proves there are no points that satisfy the equation.

Another solution: $x^2 + xy + y^2 + 1 = \frac{1}{2}x^2 + xy + \frac{1}{2}y^2 + \frac{1}{2}x^2 + \frac{1}{2}y^2 + 1 = \frac{1}{2}(x^2 + 2xy + y^2) + \frac{1}{2}(x^2 + y^2) + 1$

$$= \tfrac{1}{2}(x+y)^2 + \tfrac{1}{2}(x^2 + y^2) + 1 \geq 1$$

Another solution: Regarding $x^2 + xy + y^2 + 1 = 0$ as a quadratic in x, the discriminant is $y^2 - 4(y^2 + 1) = -3y^2 - 4$. This is negative, so there are no real solutions.

(c) The expression for y' in part (a) is meaningless; that is, since the equation in part (a) has no solution, it does not implicitly define a function y of x, and therefore it is meaningless to consider y'.

57. To find the points at which the ellipse $x^2 - xy + y^2 = 3$ crosses the x-axis, let $y = 0$ and solve for x.

$y = 0 \Rightarrow x^2 - x(0) + 0^2 = 3 \Leftrightarrow x = \pm\sqrt{3}$. So the graph of the ellipse crosses the x-axis at the points $(\pm\sqrt{3}, 0)$.

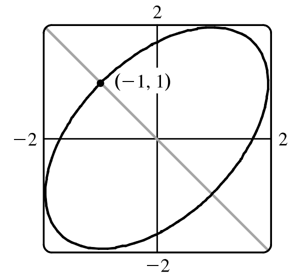
[continued]

Using implicit differentiation to find y', we get $2x - xy' - y + 2yy' = 0 \;\Rightarrow\; y'(2y - x) = y - 2x \;\Leftrightarrow\; y' = \dfrac{y - 2x}{2y - x}$.

So y' at $(\sqrt{3}, 0)$ is $\dfrac{0 - 2\sqrt{3}}{2(0) - \sqrt{3}} = 2$ and y' at $(-\sqrt{3}, 0)$ is $\dfrac{0 + 2\sqrt{3}}{2(0) + \sqrt{3}} = 2$. Thus, the tangent lines at these points are parallel.

58. (a) We use implicit differentiation to find $y' = \dfrac{y - 2x}{2y - x}$ as in Exercise 57. The slope of the tangent line at $(-1, 1)$ is $m = \dfrac{1 - 2(-1)}{2(1) - (-1)} = \dfrac{3}{3} = 1$, so the slope of the normal line is $-\dfrac{1}{m} = -1$, and its equation is $y - 1 = -1(x + 1) \;\Leftrightarrow\; y = -x$. Substituting $-x$ for y in the equation of the ellipse, we get $x^2 - x(-x) + (-x)^2 = 3 \;\Rightarrow\; 3x^2 = 3 \;\Leftrightarrow\; x = \pm 1$. So the normal line must intersect the ellipse again at $x = 1$, and since the equation of the line is $y = -x$, the other point of intersection must be $(1, -1)$.

(b)



59. $x^2y^2 + xy = 2 \;\Rightarrow\; x^2 \cdot 2yy' + y^2 \cdot 2x + x \cdot y' + y \cdot 1 = 0 \;\Leftrightarrow\; y'(2x^2y + x) = -2xy^2 - y \;\Leftrightarrow$

$y' = -\dfrac{2xy^2 + y}{2x^2y + x}$. So $-\dfrac{2xy^2 + y}{2x^2y + x} = -1 \;\Leftrightarrow\; 2xy^2 + y = 2x^2y + x \;\Leftrightarrow\; y(2xy + 1) = x(2xy + 1) \;\Leftrightarrow$

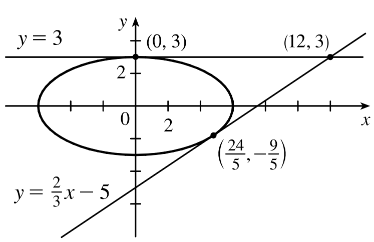
$y(2xy + 1) - x(2xy + 1) = 0 \;\Leftrightarrow\; (2xy + 1)(y - x) = 0 \;\Leftrightarrow\; xy = -\frac{1}{2}$ or $y = x$. But $xy = -\frac{1}{2} \;\Rightarrow$

$x^2y^2 + xy = \frac{1}{4} - \frac{1}{2} \neq 2$, so we must have $x = y$. Then $x^2y^2 + xy = 2 \;\Rightarrow\; x^4 + x^2 = 2 \;\Leftrightarrow\; x^4 + x^2 - 2 = 0 \;\Leftrightarrow$

$(x^2 + 2)(x^2 - 1) = 0$. So $x^2 = -2$, which is impossible, or $x^2 = 1 \;\Leftrightarrow\; x = \pm 1$. Since $x = y$, the points on the curve where the tangent line has a slope of -1 are $(-1, -1)$ and $(1, 1)$.

60. $x^2 + 4y^2 = 36 \;\Rightarrow\; 2x + 8yy' = 0 \;\Rightarrow\; y' = -\dfrac{x}{4y}$. Let (a, b) be a point on $x^2 + 4y^2 = 36$ whose tangent line passes through $(12, 3)$. The tangent line is then $y - 3 = -\dfrac{a}{4b}(x - 12)$, so $b - 3 = -\dfrac{a}{4b}(a - 12)$. Multiplying both sides by $4b$ gives $4b^2 - 12b = -a^2 + 12a$, so $4b^2 + a^2 = 12(a + b)$. But $4b^2 + a^2 = 36$, so $36 = 12(a + b) \;\Rightarrow\; a + b = 3 \;\Rightarrow$ $b = 3 - a$. Substituting $3 - a$ for b into $a^2 + 4b^2 = 36$ gives $a^2 + 4(3 - a)^2 = 36 \;\Leftrightarrow\; a^2 + 36 - 24a + 4a^2 = 36 \;\Leftrightarrow$ $5a^2 - 24a = 0 \;\Leftrightarrow\; a(5a - 24) = 0$, so $a = 0$ or $a = \frac{24}{5}$. If $a = 0$, $b = 3 - 0 = 3$, and if $a = \frac{24}{5}$, $b = 3 - \frac{24}{5} = -\frac{9}{5}$.

So the two points on the ellipse are $(0, 3)$ and $\left(\frac{24}{5}, -\frac{9}{5}\right)$. Using $y - 3 = -\dfrac{a}{4b}(x - 12)$ with $(a, b) = (0, 3)$ gives us the tangent line $y - 3 = 0$ or $y = 3$. With $(a, b) = \left(\frac{24}{5}, -\frac{9}{5}\right)$, we have $y - 3 = -\frac{24/5}{4(-9/5)}(x - 12) \;\Leftrightarrow\; y - 3 = \frac{2}{3}(x - 12) \;\Leftrightarrow\; y = \frac{2}{3}x - 5$. A graph of the ellipse and the tangent lines confirms our results.



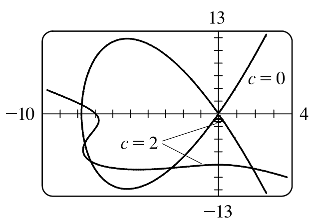
61. (a) $y = J(x)$ and $xy'' + y' + xy = 0 \;\Rightarrow\; xJ''(x) + J'(x) + xJ(x) = 0$. If $x = 0$, we have $0 + J'(0) + 0 = 0$, so $J'(0) = 0$.

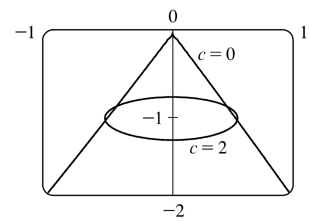
(b) Differentiating $xy'' + y' + xy = 0$ implicitly, we get $xy''' + y'' \cdot 1 + y'' + xy' + y \cdot 1 = 0 \;\Rightarrow$ $xy''' + 2y'' + xy' + y = 0$, so $xJ'''(x) + 2J''(x) + xJ'(x) + J(x) = 0$. If $x = 0$, we have $0 + 2J''(0) + 0 + 1 \quad [J(0) = 1 \text{ is given}] \quad = 0 \;\Rightarrow\; 2J''(0) = -1 \;\Rightarrow\; J''(0) = -\frac{1}{2}$.

62. $x^2 + 4y^2 = 5 \;\Rightarrow\; 2x + 4(2yy') = 0 \;\Rightarrow\; y' = -\dfrac{x}{4y}$. Now let h be the height of the lamp, and let (a, b) be the point of tangency of the line passing through the points $(3, h)$ and $(-5, 0)$. This line has slope $(h - 0)/[3 - (-5)] = \frac{1}{8}h$. But the slope of the tangent line through the point (a, b) can be expressed as $y' = -\dfrac{a}{4b}$, or as $\dfrac{b - 0}{a - (-5)} = \dfrac{b}{a + 5}$ [since the line passes through $(-5, 0)$ and (a, b)], so $-\dfrac{a}{4b} = \dfrac{b}{a + 5} \;\Leftrightarrow\; 4b^2 = -a^2 - 5a \;\Leftrightarrow\; a^2 + 4b^2 = -5a$. But $a^2 + 4b^2 = 5$ [since (a, b) is on the ellipse], so $5 = -5a \;\Leftrightarrow\; a = -1$. Then $4b^2 = -a^2 - 5a = -1 - 5(-1) = 4 \;\Rightarrow\; b = 1$, since the point is on the top half of the ellipse. So $\dfrac{h}{8} = \dfrac{b}{a + 5} = \dfrac{1}{-1 + 5} = \dfrac{1}{4} \;\Rightarrow\; h = 2$. So the lamp is located 2 units above the x-axis.

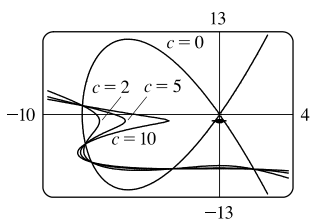
LABORATORY PROJECT Families of Implicit Curves

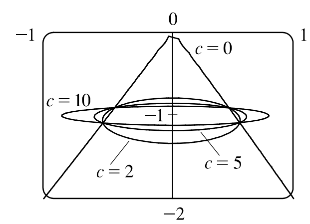
1. (a) There appear to be nine points of intersection. The "inner four" near the origin are about $(\pm 0.2, -0.9)$ and $(\pm 0.3, -1.1)$. The "outer five" are about $(2.0, -8.9)$, $(-2.8, -8.8)$, $(-7.5, -7.7)$, $(-7.8, -4.7)$, and $(-8.0, 1.5)$.





(b) We see from the graphs with $c = 5$ and $c = 10$, and for other values of c, that the curves change shape but the nine points of intersection are the same.

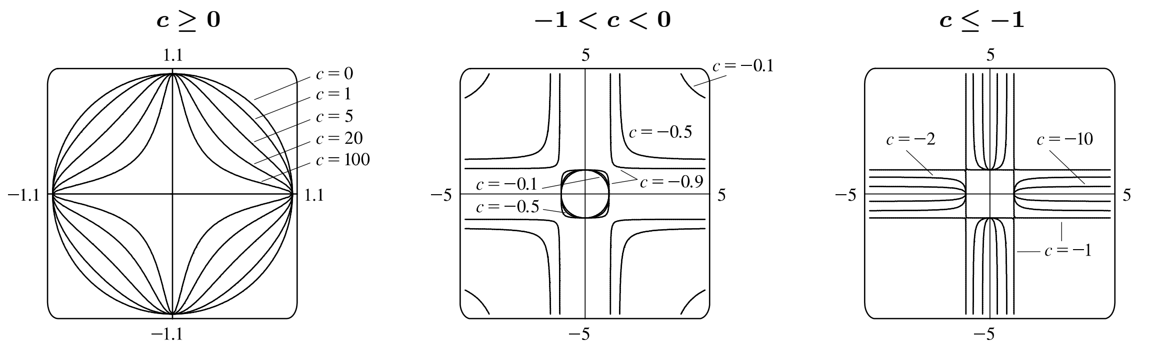




2. (a) If $c = 0$, the graph is the unit circle. As c increases, the graph looks more diamondlike and then more crosslike (see the graph for $c \geq 0$).

For $-1 < c < 0$ (see the graph), there are four hyperboliclike branches as well as an ellipticlike curve bounded by $|x| \leq 1$ and $|y| \leq 1$ for values of c close to 0. As c gets closer to -1, the branches and the curve become more rectangular, approaching the lines $|x| = 1$ and $|y| = 1$.

For $c = -1$, we get the lines $x = \pm 1$ and $y = \pm 1$. As c decreases, we get four test-tubelike curves (see the graph) that are bounded by $|x| = 1$ and $|y| = 1$, and get thinner as $|c|$ gets larger.



(b) The curve for $c = -1$ is described in part (a). When $c = -1$, we get $x^2 + y^2 - x^2y^2 = 1 \quad \Leftrightarrow$ $0 = x^2y^2 - x^2 - y^2 + 1 \quad \Leftrightarrow \quad 0 = (x^2 - 1)(y^2 - 1) \quad \Leftrightarrow \quad x = \pm 1$ or $y = \pm 1$, which algebraically proves that the graph consists of the stated lines.

(c) $\dfrac{d}{dx}(x^2 + y^2 + cx^2y^2) = \dfrac{d}{dx}(1) \quad \Rightarrow \quad 2x + 2y\,y' + c(x^2 \cdot 2y\,y' + y^2 \cdot 2x) = 0 \quad \Rightarrow$

$2y\,y' + 2cx^2y\,y' = -2x - 2cxy^2 \quad \Rightarrow \quad 2y(1 + cx^2)y' = -2x(1 + cy^2) \quad \Rightarrow \quad y' = -\dfrac{x(1 + cy^2)}{y(1 + cx^2)}$.

For $c = -1$, $y' = -\dfrac{x(1 - y^2)}{y(1 - x^2)} = -\dfrac{x(1 + y)(1 - y)}{y(1 + x)(1 - x)}$, so $y' = 0$ when $y = \pm 1$ or $x = 0$ (which leads to $y = \pm 1$) and y' is undefined when $x = \pm 1$ or $y = 0$ (which leads to $x = \pm 1$). Since the graph consists of the lines $x = \pm 1$ and $y = \pm 1$, the slope at any point on the graph is undefined or 0, which is consistent with the expression found for y'.

2.7 Rates of Change in the Natural and Social Sciences

1. (a) $s = f(t) = t^3 - 9t^2 + 24t$ (in feet) $\quad \Rightarrow \quad v(t) = f'(t) = 3t^2 - 18t + 24$ (in ft/s)

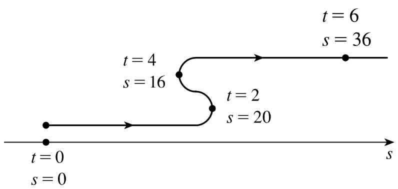
(b) $v(1) = 3(1)^2 - 18(1) + 24 = 9$ ft/s

(c) The particle is at rest when $v(t) = 0$. $3t^2 - 18t + 24 = 0 \quad \Leftrightarrow \quad 3(t^2 - 6t + 8) = 0 \quad \Leftrightarrow \quad 3(t - 2)(t - 4) = 0 \quad \Rightarrow$ $t = 2$ s or $t = 4$ s.

(d) The particle is moving in the positive direction when $v(t) > 0$. $3(t - 2)(t - 4) > 0 \quad \Leftrightarrow \quad 0 \leq t < 2$ or $t > 4$.

(e) v changes sign at $t = 2$ and 4 in the interval $[0, 6]$. The total distance traveled during the first 6 seconds is $|f(2) - f(0)| + |f(4) - f(2)| + |f(6) - f(4)| = |20 - 0| + |16 - 20| + |36 - 16| = 20 + 4 + 20 = 44$ ft.

(f)

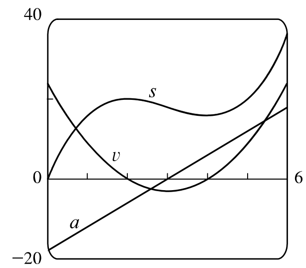


(g) $v(t) = 3t^2 - 18t + 24 \quad \Rightarrow$

$a(t) = v'(t) = 6t - 18$ [in (ft/s)/s or ft/s^2].

$a(1) = 6 - 18 = -12$ ft/s^2.

(h)



(i) $a(t) > 0 \quad \Leftrightarrow \quad 6t - 18 > 0 \quad \Leftrightarrow \quad t > 3$. The particle is speeding up when v and a have the same sign. From the figure in part (h), we see that v and a are both positive when $4 < t \leq 6$ and both negative when $2 < t < 3$. Thus, the particle is speeding up when $2 < t < 3$ and $4 < t \leq 6$. The particle is slowing down when v and a have opposite signs; that is, when $0 \leq t < 2$ and $3 < t < 4$.

2. (a) $s = f(t) = 0.01t^4 - 0.04t^3$ (in feet) $\quad \Rightarrow \quad v(t) = f'(t) = 0.04t^3 - 0.12t^2$ (in ft/s)

(b) $v(3) = 0.04(3)^3 - 0.12(3)^2 = 0$ ft/s

(c) The particle is at rest when $v(t) = 0$. $\quad 0.04t^3 - 0.12t^2 = 0 \quad \Leftrightarrow \quad 0.04t^2(t - 3) = 0 \quad \Leftrightarrow \quad t = 0$ s or 3 s.

(d) The particle is moving in the positive direction when $v(t) > 0$. $\quad 0.04t^2(t - 3) > 0 \quad \Leftrightarrow \quad t > 3$.

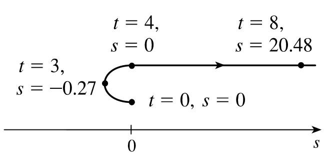
(e) See Exercise 1(e).

$|f(3) - f(0)| = |-0.27 - 0| = 0.27$.

$|f(8) - f(3)| = |20.48 - (-0.27)| = 20.75$.

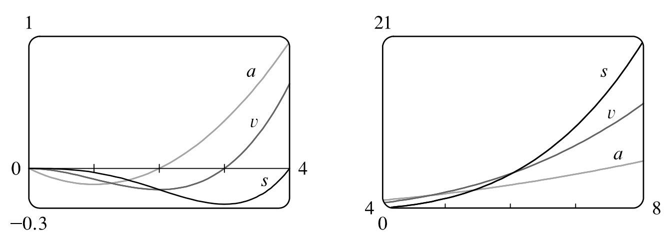
The total distance is $0.27 + 20.75 = 21.02$ ft.

(f)



(g) $v(t) = 0.04t^3 - 0.12t^2 \quad \Rightarrow \quad a(t) = v'(t) = 0.12t^2 - 0.24t$. $\quad a(3) = 0.12(3)^2 - 0.24(3) = 0.36$ (ft/s)/s or ft/s^2.

(h) Here we show the graph of s, v, and a for $0 \leq t \leq 4$ and $4 \leq t \leq 8$.



(i) The particle is speeding up when v and a have the same sign. This occurs when $0 < t < 2$ $\quad$ [v and a are both negative] and when $t > 3$ $\quad$ [v and a are both positive]. It is slowing down when v and a have opposite signs; that is, when $2 < t < 3$.

3. (a) $s = f(t) = \sin(\pi t/2)$ (in feet) $\Rightarrow$ $v(t) = f'(t) = \cos(\pi t/2) \cdot (\pi/2) = \frac{\pi}{2}\cos(\pi t/2)$ (in ft/s)

(b) $v(1) = \frac{\pi}{2}\cos\frac{\pi}{2} = \frac{\pi}{2}(0) = 0$ ft/s

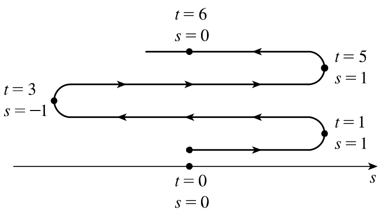
(c) The particle is at rest when $v(t) = 0$. $\frac{\pi}{2}\cos\frac{\pi}{2}t = 0 \Leftrightarrow \cos\frac{\pi}{2}t = 0 \Leftrightarrow \frac{\pi}{2}t = \frac{\pi}{2} + n\pi \Leftrightarrow t = 1 + 2n$, where n is a nonnegative integer since $t \geq 0$.

(d) The particle is moving in the positive direction when $v(t) > 0$. From part (c), we see that v changes sign at every positive odd integer. v is positive when $0 < t < 1$, $3 < t < 5$, $7 < t < 9$, and so on.

(e) v changes sign at $t = 1$, 3, and 5 in the interval $[0, 6]$. The total distance traveled during the first 6 seconds is

$$|f(1) - f(0)| + |f(3) - f(1)| + |f(5) - f(3)| + |f(6) - f(5)| = |1 - 0| + |-1 - 1| + |1 - (-1)| + |0 - 1|$$
$$= 1 + 2 + 2 + 1 = 6 \text{ ft}$$

(f)

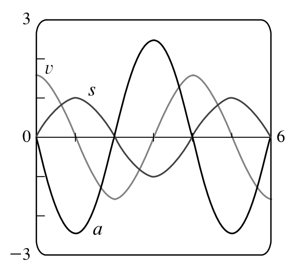


(g) $v(t) = \frac{\pi}{2}\cos(\pi t/2)$ $\Rightarrow$

$$a(t) = v'(t) = \frac{\pi}{2}\left[-\sin(\pi t/2) \cdot (\pi/2)\right]$$
$$= (-\pi^2/4)\sin(\pi t/2) \text{ ft/s}^2$$

$$a(1) = (-\pi^2/4)\sin(\pi/2) = -\pi^2/4 \text{ ft/s}^2$$

(h)



(i) The particle is speeding up when v and a have the same sign. From the figure in part (h), we see that v and a are both positive when $3 < t < 4$ and both negative when $1 < t < 2$ and $5 < t < 6$. Thus, the particle is speeding up when $1 < t < 2$, $3 < t < 4$, and $5 < t < 6$. The particle is slowing down when v and a have opposite signs; that is, when $0 < t < 1$, $2 < t < 3$, and $4 < t < 5$.

4. (a) $s = f(t) = \dfrac{9t}{t^2 + 9}$ (in feet) $\Rightarrow$ $v(t) = f'(t) = \dfrac{(t^2+9)(9) - 9t(2t)}{(t^2+9)^2} = \dfrac{-9t^2 + 81}{(t^2+9)^2} = \dfrac{-9(t^2 - 9)}{(t^2+9)^2}$ (in ft/s)

(b) $v(1) = \dfrac{-9(1 - 9)}{(1 + 9)^2} = \dfrac{72}{100} = 0.72$ ft/s

(c) The particle is at rest when $v(t) = 0$. $\dfrac{-9(t^2 - 9)}{(t^2+9)^2} = 0 \Leftrightarrow t^2 - 9 = 0 \Rightarrow t = 3$ s [since $t \geq 0$].

(d) The particle is moving in the positive direction when $v(t) > 0$.

$$\frac{-9(t^2 - 9)}{(t^2+9)^2} > 0 \Rightarrow -9(t^2 - 9) > 0 \Rightarrow t^2 - 9 < 0 \Rightarrow t^2 < 9 \Rightarrow 0 \leq t < 3.$$

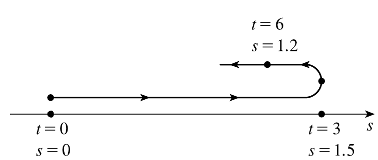
(e) Since the particle is moving in the positve direction and in the negative direction, we need to calculate the distance traveled in the intervals $[0, 3]$ and$[3, 6]$, respectively.

$|f(3) - f(0)| = \left|\frac{27}{18} - 0\right| = \frac{3}{2}$

$|f(6) - f(3)| = \left|\frac{54}{45} - \frac{27}{18}\right| = \frac{3}{10}$

The total distance is $\frac{3}{2} + \frac{3}{10} = \frac{9}{5}$ or 1.8 ft.

(f)

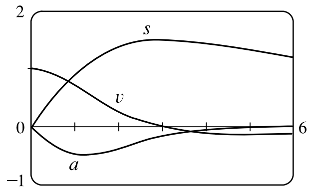


(g) $v(t) = -9\,\dfrac{t^2-9}{(t^2+9)^2} \quad\Rightarrow$

$$a(t) = v'(t) = -9\,\frac{(t^2+9)^2(2t)-(t^2-9)2(t^2+9)(2t)}{[(t^2+9)^2]^2} = -9\,\frac{2t(t^2+9)[(t^2+9)-2(t^2-9)]}{(t^2+9)^4} = \frac{18t(t^2-27)}{(t^2+9)^3}.$$

$$a(1) = \frac{18(-26)}{10^3} = -0.468\ \text{ft/s}^2$$

(h)



(i) The particle is speeding up when v and a have the same sign. a is negative for $0 < t < \sqrt{27}$ [≈ 5.2], so from the figure in part (h), we see that v and a are both negative for $3 < t < 3\sqrt{3}$. The particle is slowing down when v and a have opposite signs. This occurs when $0 < t < 3$ and when $t > 3\sqrt{3}$.

5. (a) From the figure, the velocity v is positive on the interval $(0,2)$ and negative on the interval $(2,3)$. The acceleration a is positive (negative) when the slope of the tangent line is positive (negative), so the acceleration is positive on the interval $(0,1)$, and negative on the interval $(1,3)$. The particle is speeding up when v and a have the same sign, that is, on the interval $(0,1)$ when $v > 0$ and $a > 0$, and on the interval $(2,3)$ when $v < 0$ and $a < 0$. The particle is slowing down when v and a have opposite signs, that is, on the interval $(1,2)$ when $v > 0$ and $a < 0$.

(b) $v > 0$ on $(0,3)$ and $v < 0$ on $(3,4)$. $a > 0$ on $(1,2)$ and $a < 0$ on $(0,1)$ and $(2,4)$. The particle is speeding up on $(1,2)$ $[v > 0, a > 0]$ and on $(3,4)$ $[v < 0, a < 0]$. The particle is slowing down on $(0,1)$ and $(2,3)$ $[v > 0, a < 0]$.

6. (a) The velocity v is positive when s is increasing, that is, on the intervals $(0,1)$ and $(3,4)$; and it is negative when s is decreasing, that is, on the interval $(1,3)$. The acceleration a is positive when the graph of s is concave upward (v is increasing), that is, on the interval $(2,4)$; and it is negative when the graph of s is concave downward (v is decreasing), that is, on the interval $(0,2)$. The particle is speeding up on the interval $(1,2)$ $[v < 0, a < 0]$ and on $(3,4)$ $[v > 0, a > 0]$. The particle is slowing down on the interval $(0,1)$ $[v > 0, a < 0]$ and on $(2,3)$ $[v < 0, a > 0]$.

(b) The velocity v is positive on $(3,4)$ and negative on $(0,3)$. The acceleration a is positive on $(0,1)$ and $(2,4)$ and negative on $(1,2)$. The particle is speeding up on the interval $(1,2)$ $[v < 0, a < 0]$ and on $(3,4)$ $[v > 0, a > 0]$. The particle is slowing down on the interval $(0,1)$ $[v < 0, a > 0]$ and on $(2,3)$ $[v < 0, a > 0]$.

7. (a) $h(t) = 2 + 24.5t - 4.9t^2 \quad\Rightarrow\quad v(t) = h'(t) = 24.5 - 9.8t$. The velocity after 2 s is $v(2) = 24.5 - 9.8(2) = 4.9$ m/s and after 4 s is $v(4) = 24.5 - 9.8(4) = -14.7$ m/s.

(b) The projectile reaches its maximum height when the velocity is zero. $v(t) = 0 \quad \Leftrightarrow \quad 24.5 - 9.8t = 0 \quad \Leftrightarrow$

$t = \dfrac{24.5}{9.8} = 2.5$ s.

(c) The maximum height occurs when $t = 2.5$. $h(2.5) = 2 + 24.5(2.5) - 4.9(2.5)^2 = 32.625$ m $\left[\text{or } 32\frac{5}{8}\text{ m}\right]$.

(d) The projectile hits the ground when $h = 0 \quad \Leftrightarrow \quad 2 + 24.5t - 4.9t^2 = 0 \quad \Leftrightarrow$

$t = \dfrac{-24.5 \pm \sqrt{24.5^2 - 4(-4.9)(2)}}{2(-4.9)} \quad \Rightarrow \quad t = t_f \approx 5.08$ s [since $t \geq 0$].

(e) The projectile hits the ground when $t = t_f$. Its velocity is $v(t_f) = 24.5 - 9.8t_f \approx -25.3$ m/s [downward].

8. (a) At maximum height the velocity of the ball is 0 ft/s. $v(t) = s'(t) = 80 - 32t = 0 \quad \Leftrightarrow \quad 32t = 80 \quad \Leftrightarrow \quad t = \frac{5}{2}$.

So the maximum height is $s\left(\frac{5}{2}\right) = 80\left(\frac{5}{2}\right) - 16\left(\frac{5}{2}\right)^2 = 200 - 100 = 100$ ft.

(b) $s(t) = 80t - 16t^2 = 96 \quad \Leftrightarrow \quad 16t^2 - 80t + 96 = 0 \quad \Leftrightarrow \quad 16(t^2 - 5t + 6) = 0 \quad \Leftrightarrow \quad 16(t-3)(t-2) = 0$.

So the ball has a height of 96 ft on the way up at $t = 2$ and on the way down at $t = 3$. At these times the velocities are $v(2) = 80 - 32(2) = 16$ ft/s and $v(3) = 80 - 32(3) = -16$ ft/s, respectively.

9. (a) $h(t) = 15t - 1.86t^2 \quad \Rightarrow \quad v(t) = h'(t) = 15 - 3.72t$. The velocity after 2 s is $v(2) = 15 - 3.72(2) = 7.56$ m/s.

(b) $25 = h \quad \Leftrightarrow \quad 1.86t^2 - 15t + 25 = 0 \quad \Leftrightarrow \quad t = \dfrac{15 \pm \sqrt{15^2 - 4(1.86)(25)}}{2(1.86)} \quad \Leftrightarrow \quad t = t_1 \approx 2.35$ or $t = t_2 \approx 5.71$.

The velocities are $v(t_1) = 15 - 3.72t_1 \approx 6.24$ m/s [upward] and $v(t_2) = 15 - 3.72t_2 \approx -6.24$ m/s [downward].

10. (a) $s(t) = t^4 - 4t^3 - 20t^2 + 20t \quad \Rightarrow \quad v(t) = s'(t) = 4t^3 - 12t^2 - 40t + 20$. $v = 20 \quad \Leftrightarrow$

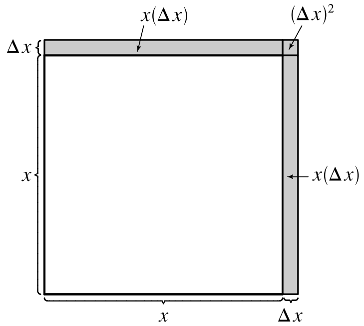
$4t^3 - 12t^2 - 40t + 20 = 20 \quad \Leftrightarrow \quad 4t^3 - 12t^2 - 40t = 0 \quad \Leftrightarrow \quad 4t(t^2 - 3t - 10) = 0 \quad \Leftrightarrow$

$4t(t-5)(t+2) = 0 \quad \Leftrightarrow \quad t = 0$ s or 5 s [for $t \geq 0$].

(b) $a(t) = v'(t) = 12t^2 - 24t - 40$. $a = 0 \quad \Leftrightarrow \quad 12t^2 - 24t - 40 = 0 \quad \Leftrightarrow \quad 4(3t^2 - 6t - 10) = 0 \quad \Leftrightarrow$

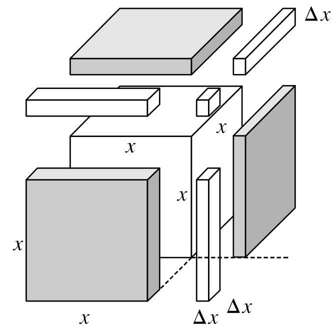
$t = \dfrac{6 \pm \sqrt{6^2 - 4(3)(-10)}}{2(3)} = 1 \pm \dfrac{1}{3}\sqrt{39} \approx 3.08$ s [for $t \geq 0$]. At this time, the acceleration changes from negative to positive and the velocity attains its minimum value.

11. (a) $A(x) = x^2 \quad \Rightarrow \quad A'(x) = 2x$. $A'(15) = 30\ \text{mm}^2/\text{mm}$ is the rate at which the area is increasing with respect to the side length as x reaches 15 mm.



(b) The perimeter is $P(x) = 4x$, so $A'(x) = 2x = \frac{1}{2}(4x) = \frac{1}{2}P(x)$. The figure suggests that if Δx is small, then the change in the area of the square is approximately half of its perimeter (2 of the 4 sides) times Δx. From the figure, $\Delta A = 2x\,(\Delta x) + (\Delta x)^2$. If Δx is small, then $\Delta A \approx 2x\,(\Delta x)$ and so $\Delta A/\Delta x \approx 2x$.

12. (a) $V(x) = x^3 \Rightarrow \dfrac{dV}{dx} = 3x^2$. $\left.\dfrac{dV}{dx}\right|_{x=3} = 3(3)^2 = 27\ \text{mm}^3/\text{mm}$ is the rate at which the volume is increasing as x increases past 3 mm.



(b) The surface area is $S(x) = 6x^2$, so $V'(x) = 3x^2 = \frac{1}{2}(6x^2) = \frac{1}{2}S(x)$. The figure suggests that if Δx is small, then the change in the volume of the cube is approximately half of its surface area (the area of 3 of the 6 faces) times Δx. From the figure, $\Delta V = 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3$. If Δx is small, then $\Delta V \approx 3x^2(\Delta x)$ and so $\Delta V/\Delta x \approx 3x^2$.

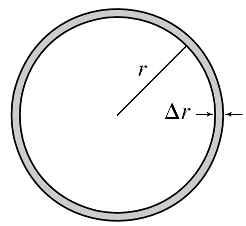
13. (a) Using $A(r) = \pi r^2$, we find that the average rate of change is:

(i) $\dfrac{A(3) - A(2)}{3 - 2} = \dfrac{9\pi - 4\pi}{1} = 5\pi$ (ii) $\dfrac{A(2.5) - A(2)}{2.5 - 2} = \dfrac{6.25\pi - 4\pi}{0.5} = 4.5\pi$

(iii) $\dfrac{A(2.1) - A(2)}{2.1 - 2} = \dfrac{4.41\pi - 4\pi}{0.1} = 4.1\pi$

(b) $A(r) = \pi r^2 \Rightarrow A'(r) = 2\pi r$, so $A'(2) = 4\pi$.

(c) The circumference is $C(r) = 2\pi r = A'(r)$. The figure suggests that if Δr is small, then the change in the area of the circle (a ring around the outside) is approximately equal to its circumference times Δr. Straightening out this ring gives us a shape that is approximately rectangular with length $2\pi r$ and width Δr, so $\Delta A \approx 2\pi r(\Delta r)$. Algebraically, $\Delta A = A(r + \Delta r) - A(r) = \pi(r + \Delta r)^2 - \pi r^2 = 2\pi r(\Delta r) + \pi(\Delta r)^2$. So we see that if Δr is small, then $\Delta A \approx 2\pi r(\Delta r)$ and therefore, $\Delta A/\Delta r \approx 2\pi r$.



14. After t seconds the radius is $r = 60t$, so the area is $A(t) = \pi(60t)^2 = 3600\pi t^2 \Rightarrow A'(t) = 7200\pi t \Rightarrow$

(a) $A'(1) = 7200\pi\ \text{cm}^2/\text{s}$ (b) $A'(3) = 21{,}600\pi\ \text{cm}^2/\text{s}$ (c) $A'(5) = 36{,}000\pi\ \text{cm}^2/\text{s}$

As time goes by, the area grows at an increasing rate. In fact, the rate of change is linear with respect to time.

15. $S(r) = 4\pi r^2 \Rightarrow S'(r) = 8\pi r \Rightarrow$

(a) $S'(1) = 8\pi\ \text{ft}^2/\text{ft}$ (b) $S'(2) = 16\pi\ \text{ft}^2/\text{ft}$ (c) $S'(3) = 24\pi\ \text{ft}^2/\text{ft}$

As the radius increases, the surface area grows at an increasing rate. In fact, the rate of change is linear with respect to the radius.

16. (a) Using $V(r) = \frac{4}{3}\pi r^3$, we find that the average rate of change is:

(i) $\dfrac{V(8) - V(5)}{8 - 5} = \dfrac{\frac{4}{3}\pi(512) - \frac{4}{3}\pi(125)}{3} = 172\pi\ \mu\text{m}^3/\mu\text{m}$

(ii) $\dfrac{V(6) - V(5)}{6 - 5} = \dfrac{\frac{4}{3}\pi(216) - \frac{4}{3}\pi(125)}{1} = 121.\overline{3}\pi\ \mu\text{m}^3/\mu\text{m}$

(iii) $\dfrac{V(5.1) - V(5)}{5.1 - 5} = \dfrac{\frac{4}{3}\pi(5.1)^3 - \frac{4}{3}\pi(5)^3}{0.1} = 102.01\overline{3}\pi\ \mu\text{m}^3/\mu\text{m}$

(b) $V'(r) = 4\pi r^2$, so $V'(5) = 100\pi\ \mu\text{m}^3/\mu\text{m}$.

(c) $V(r) = \frac{4}{3}\pi r^3 \quad \Rightarrow \quad V'(r) = 4\pi r^2 = S(r)$. By analogy with Exercise 13(c), we can say that the change in the volume of the spherical shell, ΔV, is approximately equal to its thickness, Δr, times the surface area of the inner sphere. Thus, $\Delta V \approx 4\pi r^2(\Delta r)$ and so $\Delta V/\Delta r \approx 4\pi r^2$.

17. The mass is $f(x) = 3x^2$, so the linear density at x is $\rho(x) = f'(x) = 6x$.

(a) $\rho(1) = 6$ kg/m (b) $\rho(2) = 12$ kg/m (c) $\rho(3) = 18$ kg/m

Since ρ is an increasing function, the density will be the highest at the right end of the rod and lowest at the left end.

18. $V(t) = 5000\left(1 - \frac{1}{40}t\right)^2 \quad \Rightarrow \quad V'(t) = 5000 \cdot 2\left(1 - \frac{1}{40}t\right)\left(-\frac{1}{40}\right) = -250\left(1 - \frac{1}{40}t\right)$

(a) $V'(5) = -250\left(1 - \frac{5}{40}\right) = -218.75$ gal/min (b) $V'(10) = -250\left(1 - \frac{10}{40}\right) = -187.5$ gal/min

(c) $V'(20) = -250\left(1 - \frac{20}{40}\right) = -125$ gal/min (d) $V'(40) = -250\left(1 - \frac{40}{40}\right) = 0$ gal/min

The water is flowing out the fastest at the beginning—when $t = 0$, $V'(t) = -250$ gal/min. The water is flowing out the slowest at the end—when $t = 40$, $V'(t) = 0$. As the tank empties, the water flows out more slowly.

19. The quantity of charge is $Q(t) = t^3 - 2t^2 + 6t + 2$, so the current is $Q'(t) = 3t^2 - 4t + 6$.

(a) $Q'(0.5) = 3(0.5)^2 - 4(0.5) + 6 = 4.75$ A (b) $Q'(1) = 3(1)^2 - 4(1) + 6 = 5$ A

The current is lowest when Q' has a minimum. $Q''(t) = 6t - 4 < 0$ when $t < \frac{2}{3}$. So the current decreases when $t < \frac{2}{3}$ and increases when $t > \frac{2}{3}$. Thus, the current is lowest at $t = \frac{2}{3}$ s.

20. (a) $F = \dfrac{GmM}{r^2} = (GmM)r^{-2} \quad \Rightarrow \quad \dfrac{dF}{dr} = -2(GmM)r^{-3} = -\dfrac{2GmM}{r^3}$, which is the rate of change of the force with respect to the distance between the bodies. The minus sign indicates that as the distance r between the bodies increases, the magnitude of the force F exerted by the body of mass m on the body of mass M is decreasing.

(b) Given $F'(20{,}000) = -2$, find $F'(10{,}000)$. $-2 = -\dfrac{2GmM}{20{,}000^3} \quad \Rightarrow \quad GmM = 20{,}000^3$.

$$F'(10,000) = -\frac{2(20{,}000^3)}{10{,}000^3} = -2 \cdot 2^3 = -16 \text{ N/km}$$

21. With $m = m_0\left(1 - \dfrac{v^2}{c^2}\right)^{-1/2}$,

$$F = \frac{d}{dt}(mv) = m\frac{d}{dt}(v) + v\frac{d}{dt}(m) = m_0\left(1 - \frac{v^2}{c^2}\right)^{-1/2} \cdot a + v \cdot m_0\left[-\frac{1}{2}\left(1 - \frac{v^2}{c^2}\right)^{-3/2}\right]\left(-\frac{2v}{c^2}\right)\frac{d}{dt}(v)$$

$$= m_0\left(1 - \frac{v^2}{c^2}\right)^{-3/2} \cdot a\left[\left(1 - \frac{v^2}{c^2}\right) + \frac{v^2}{c^2}\right] = \frac{m_0 a}{(1 - v^2/c^2)^{3/2}}$$

Note that we factored out $(1 - v^2/c^2)^{-3/2}$ since $-3/2$ was the lesser exponent. Also note that $\dfrac{d}{dt}(v) = a$.

22. (a) $D(t) = 7 + 5\cos[0.503(t - 6.75)] \quad \Rightarrow \quad D'(t) = -5\sin[0.503(t - 6.75)](0.503) = -2.515\sin[0.503(t - 6.75)]$.

At 3:00 AM, $t = 3$, and $D'(3) = -2.515\sin[0.503(-3.75)] \approx 2.39$ m/h (rising).

(b) At 6:00 AM, $t = 6$, and $D'(6) = -2.515\sin[0.503(-0.75)] \approx 0.93$ m/h (rising).

(c) At 9:00 AM, $t = 9$, and $D'(9) = -2.515\sin[0.503(2.25)] \approx -2.28$ m/h (falling).

(d) At noon, $t = 12$, and $D'(12) = -2.515\sin[0.503(5.25)] \approx -1.21$ m/h (falling).

23. (a) To find the rate of change of volume with respect to pressure, we first solve for V in terms of P.

$$PV = C \Rightarrow V = \frac{C}{P} \Rightarrow \frac{dV}{dP} = -\frac{C}{P^2}.$$

(b) From the formula for dV/dP in part (a), we see that as P increases, the absolute value of dV/dP decreases.

Thus, the volume is decreasing more rapidly at the beginning.

(c) $\beta = -\frac{1}{V}\frac{dV}{dP} = -\frac{1}{V}\left(-\frac{C}{P^2}\right) = \frac{C}{(PV)P} = \frac{C}{CP} = \frac{1}{P}$

24. (a) $[\text{C}] = \frac{a^2kt}{akt+1} \Rightarrow$ rate of reaction $= \frac{d[\text{C}]}{dt} = \frac{(akt+1)(a^2k) - (a^2kt)(ak)}{(akt+1)^2} = \frac{a^2k(akt+1-akt)}{(akt+1)^2} = \frac{a^2k}{(akt+1)^2}$

(b) If $x = [\text{C}]$, then $a - x = a - \frac{a^2kt}{akt+1} = \frac{a^2kt + a - a^2kt}{akt+1} = \frac{a}{akt+1}$.

So $k(a-x)^2 = k\left(\frac{a}{akt+1}\right)^2 = \frac{a^2k}{(akt+1)^2} = \frac{d[\text{C}]}{dt}$ [from part (a)] $= \frac{dx}{dt}$.

25. (a) **1920:** $m_1 = \frac{1860 - 1750}{1920 - 1910} = \frac{110}{10} = 11$, $m_2 = \frac{2070 - 1860}{1930 - 1920} = \frac{210}{10} = 21$,

$(m_1 + m_2)/2 = (11 + 21)/2 = 16$ million/year

1980: $m_1 = \frac{4450 - 3710}{1980 - 1970} = \frac{740}{10} = 74$, $m_2 = \frac{5280 - 4450}{1990 - 1980} = \frac{830}{10} = 83$,

$(m_1 + m_2)/2 = (74 + 83)/2 = 78.5$ million/year

(b) $P(t) = at^3 + bt^2 + ct + d$ (in millions of people), where $a \approx -0.000\,284\,900\,3$, $b \approx 0.522\,433\,122\,43$, $c \approx -6.395\,641\,396$, and $d \approx 1720.586\,081$.

(c) $P(t) = at^3 + bt^2 + ct + d \Rightarrow P'(t) = 3at^2 + 2bt + c$ (in millions of people per year)

(d) 1920 corresponds to $t = 20$ and $P'(20) \approx 14.16$ million/year. 1980 corresponds to $t = 80$ and $P'(80) \approx 71.72$ million/year. These estimates are smaller than the estimates in part (a).

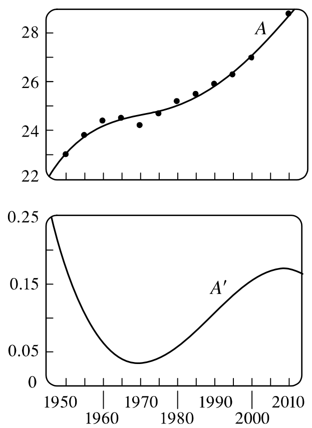
(e) $P'(85) \approx 76.24$ million/year.

26. (a) $A(t) = at^4 + bt^3 + ct^2 + dt + e$, where $a \approx -1.199\,781 \times 10^{-6}$, $b \approx 9.545\,853 \times 10^3$, $c \approx -28.478\,550$, $d \approx 37{,}757.105\,467$, and $e \approx -1.877\,031 \times 10^7$.

(b) $A(t) = at^4 + bt^3 + ct^2 + dt + e \Rightarrow A'(t) = 4at^3 + 3bt^2 + 2ct + d$.

(c) Part (b) gives $A'(1990) \approx 0.106$ years of age per year.

(d)



27. (a) Using $v = \dfrac{P}{4\eta l}(R^2 - r^2)$ with $R = 0.01$, $l = 3$, $P = 3000$, and $\eta = 0.027$, we have v as a function of r:

$v(r) = \dfrac{3000}{4(0.027)3}(0.01^2 - r^2)$. $v(0) = 0.\overline{925}$ cm/s, $v(0.005) = 0.69\overline{4}$ cm/s, $v(0.01) = 0$.

(b) $v(r) = \dfrac{P}{4\eta l}(R^2 - r^2) \Rightarrow v'(r) = \dfrac{P}{4\eta l}(-2r) = -\dfrac{Pr}{2\eta l}$. When $l = 3$, $P = 3000$, and $\eta = 0.027$, we have

$v'(r) = -\dfrac{3000r}{2(0.027)3}$. $v'(0) = 0$, $v'(0.005) = -92.\overline{592}$ (cm/s)/cm, and $v'(0.01) = -185.\overline{185}$ (cm/s)/cm.

(c) The velocity is greatest where $r = 0$ (at the center) and the velocity is changing most where $r = R = 0.01$ cm (at the edge).

28. (a) (i) $f = \dfrac{1}{2L}\sqrt{\dfrac{T}{\rho}} = \left(\dfrac{1}{2}\sqrt{\dfrac{T}{\rho}}\right)L^{-1} \Rightarrow \dfrac{df}{dL} = -\left(\dfrac{1}{2}\sqrt{\dfrac{T}{\rho}}\right)L^{-2} = -\dfrac{1}{2L^2}\sqrt{\dfrac{T}{\rho}}$

(ii) $f = \dfrac{1}{2L}\sqrt{\dfrac{T}{\rho}} = \left(\dfrac{1}{2L\sqrt{\rho}}\right)T^{1/2} \Rightarrow \dfrac{df}{dT} = \dfrac{1}{2}\left(\dfrac{1}{2L\sqrt{\rho}}\right)T^{-1/2} = \dfrac{1}{4L\sqrt{T\rho}}$

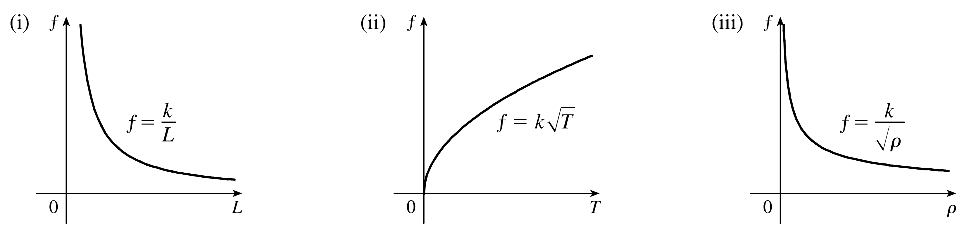
(iii) $f = \dfrac{1}{2L}\sqrt{\dfrac{T}{\rho}} = \left(\dfrac{\sqrt{T}}{2L}\right)\rho^{-1/2} \Rightarrow \dfrac{df}{d\rho} = -\dfrac{1}{2}\left(\dfrac{\sqrt{T}}{2L}\right)\rho^{-3/2} = -\dfrac{\sqrt{T}}{4L\rho^{3/2}}$

(b) *Note:* Illustrating tangent lines on the generic figures may help to explain the results.

(i) $\dfrac{df}{dL} < 0$ and L is decreasing $\Rightarrow$ f is increasing $\Rightarrow$ higher note

(ii) $\dfrac{df}{dT} > 0$ and T is increasing $\Rightarrow$ f is increasing $\Rightarrow$ higher note

(iii) $\dfrac{df}{d\rho} < 0$ and ρ is increasing $\Rightarrow$ f is decreasing $\Rightarrow$ lower note



29. (a) $C(x) = 2000 + 3x + 0.01x^2 + 0.0002x^3 \Rightarrow C'(x) = 0 + 3(1) + 0.01(2x) + 0.0002(3x^2) = 3 + 0.02x + 0.0006x^2$

(b) $C'(100) = 3 + 0.02(100) + 0.0006(100)^2 = 3 + 2 + 6 = \$11$/pair. $C'(100)$ is the rate at which the cost is increasing as the 100th pair of jeans is produced. It predicts the (approximate) cost of the 101st pair.

(c) The cost of manufacturing the 101st pair of jeans is

$C(101) - C(100) = 2611.0702 - 2600 = 11.0702 \approx \$11.07$. This is close to the marginal cost from part (b).

30. (a) $C(q) = 84 + 0.16q - 0.0006q^2 + 0.000003q^3 \Rightarrow C'(q) = 0.16 - 0.0012q + 0.000009q^2$, and

$C'(100) = 0.16 - 0.0012(100) + 0.000009(100)^2 = 0.13$. This is the rate at which the cost is increasing as the 100th item is produced.

(b) The actual cost of producing the 101st item is $C(101) - C(100) = 97.13030299 - 97 \approx \$0.13$

31. (a) $A(x) = \dfrac{p(x)}{x} \quad \Rightarrow \quad A'(x) = \dfrac{xp'(x) - p(x) \cdot 1}{x^2} = \dfrac{xp'(x) - p(x)}{x^2}$.

$A'(x) > 0 \quad \Rightarrow \quad A(x)$ is increasing; that is, the average productivity increases as the size of the workforce increases.

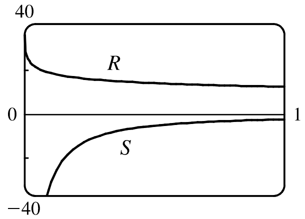
(b) $p'(x)$ is greater than the average productivity $\quad \Rightarrow \quad p'(x) > A(x) \quad \Rightarrow \quad p'(x) > \dfrac{p(x)}{x} \quad \Rightarrow \quad xp'(x) > p(x) \quad \Rightarrow$

$xp'(x) - p(x) > 0 \quad \Rightarrow \quad \dfrac{xp'(x) - p(x)}{x^2} > 0 \quad \Rightarrow \quad A'(x) > 0.$

32. (a) $R = \dfrac{40 + 24x^{0.4}}{1 + 4x^{0.4}} \quad \Rightarrow \quad S = \dfrac{dR}{dx} = \dfrac{(1 + 4x^{0.4})(9.6x^{-0.6}) - (40 + 24x^{0.4})(1.6x^{-0.6})}{(1 + 4x^{0.4})^2}$

$$= \frac{9.6x^{-0.6} + 38.4x^{-0.2} - 64x^{-0.6} - 38.4x^{-0.2}}{(1 + 4x^{0.4})^2} = -\frac{54.4x^{-0.6}}{(1 + 4x^{0.4})^2}$$

(b)



At low levels of brightness, R is quite large $[R(0) = 40]$ and is quickly decreasing, that is, S is negative with large absolute value. This is to be expected: at low levels of brightness, the eye is more sensitive to slight changes than it is at higher levels of brightness.

33. $PV = nRT \quad \Rightarrow \quad T = \dfrac{PV}{nR} = \dfrac{PV}{(10)(0.0821)} = \dfrac{1}{0.821}(PV)$. Using the Product Rule, we have

$$\frac{dT}{dt} = \frac{1}{0.821}\left[P(t)V'(t) + V(t)P'(t)\right] = \frac{1}{0.821}\left[(8)(-0.15) + (10)(0.10)\right] \approx -0.2436 \text{ K/min.}$$

34. $f(r) = 2\sqrt{Dr} \quad \Rightarrow \quad f'(r) = 2 \cdot \frac{1}{2}(Dr)^{-1/2} \cdot D = \dfrac{D}{\sqrt{Dr}} = \sqrt{\dfrac{D}{r}}$. $f'(r)$ is the rate of change of the wave speed with respect to the reproductive rate.

35. (a) If the populations are stable, then the growth rates are neither positive nor negative; that is, $\dfrac{dC}{dt} = 0$ and $\dfrac{dW}{dt} = 0$.

(b) "The caribou go extinct" means that the population is zero, or mathematically, $C = 0$.

(c) We have the equations $\dfrac{dC}{dt} = aC - bCW$ and $\dfrac{dW}{dt} = -cW + dCW$. Let $dC/dt = dW/dt = 0$, $a = 0.05$, $b = 0.001$, $c = 0.05$, and $d = 0.0001$ to obtain $0.05C - 0.001CW = 0$ **(1)** and $-0.05W + 0.0001CW = 0$ **(2)**. Adding 10 times **(2)** to **(1)** eliminates the CW-terms and gives us $0.05C - 0.5W = 0 \quad \Rightarrow \quad C = 10W$. Substituting $C = 10W$ into **(1)** results in $0.05(10W) - 0.001(10W)W = 0 \quad \Leftrightarrow \quad 0.5W - 0.01W^2 = 0 \quad \Leftrightarrow \quad 50W - W^2 = 0 \quad \Leftrightarrow$ $W(50 - W) = 0 \quad \Leftrightarrow \quad W = 0$ or 50. Since $C = 10W$, $C = 0$ or 500. Thus, the population pairs (C, W) that lead to stable populations are $(0, 0)$ and $(500, 50)$. So it is possible for the two species to live in harmony.

36. (a) If $dP/dt = 0$, the population is stable (it is constant).

(b) $\dfrac{dP}{dt} = 0 \;\Rightarrow\; \beta P = r_0\left(1 - \dfrac{P}{P_c}\right)P \;\Rightarrow\; \dfrac{\beta}{r_0} = 1 - \dfrac{P}{P_c} \;\Rightarrow\; \dfrac{P}{P_c} = 1 - \dfrac{\beta}{r_0} \;\Rightarrow\; P = P_c\left(1 - \dfrac{\beta}{r_0}\right).$

If $P_c = 10{,}000$, $r_0 = 5\% = 0.05$, and $\beta = 4\% = 0.04$, then $P = 10{,}000\left(1 - \frac{4}{5}\right) = 2000$.

(c) If $\beta = 0.05$, then $P = 10{,}000\left(1 - \frac{5}{5}\right) = 0$. There is no stable population.

2.8 Related Rates

1. $V = x^3 \;\Rightarrow\; \dfrac{dV}{dt} = \dfrac{dV}{dx}\dfrac{dx}{dt} = 3x^2\,\dfrac{dx}{dt}$

2. (a) $A = \pi r^2 \;\Rightarrow\; \dfrac{dA}{dt} = \dfrac{dA}{dr}\dfrac{dr}{dt} = 2\pi r\,\dfrac{dr}{dt}$ (b) $\dfrac{dA}{dt} = 2\pi r\,\dfrac{dr}{dt} = 2\pi(30\text{ m})(1\text{ m/s}) = 60\pi\text{ m}^2\text{/s}$

3. Let s denote the side of a square. The square's area A is given by $A = s^2$. Differentiating with respect to t gives us $\dfrac{dA}{dt} = 2s\,\dfrac{ds}{dt}$. When $A = 16$, $s = 4$. Substitution 4 for s and 6 for $\dfrac{ds}{dt}$ gives us $\dfrac{dA}{dt} = 2(4)(6) = 48\text{ cm}^2\text{/s}$.

4. $A = \ell w \;\Rightarrow\; \dfrac{dA}{dt} = \ell\cdot\dfrac{dw}{dt} + w\cdot\dfrac{d\ell}{dt} = 20(3) + 10(8) = 140\text{ cm}^2\text{/s}$.

5. $V = \pi r^2 h = \pi(5)^2 h = 25\pi h \;\Rightarrow\; \dfrac{dV}{dt} = 25\pi\,\dfrac{dh}{dt} \;\Rightarrow\; 3 = 25\pi\,\dfrac{dh}{dt} \;\Rightarrow\; \dfrac{dh}{dt} = \dfrac{3}{25\pi}\text{ m/min}$.

6. $V = \frac{4}{3}\pi r^3 \;\Rightarrow\; \dfrac{dV}{dt} = \frac{4}{3}\pi\cdot 3r^2\,\dfrac{dr}{dt} \;\Rightarrow\; \dfrac{dV}{dt} = 4\pi\left(\frac{1}{2}\cdot 80\right)^2(4) = 25{,}600\pi\text{ mm}^3\text{/s}$.

7. $S = 4\pi r^2 \;\Rightarrow\; \dfrac{dS}{dt} = 4\pi\cdot 2r\,\dfrac{dr}{dt} \;\Rightarrow\; \dfrac{dS}{dt} = 4\pi\cdot 2\cdot 8\cdot 2 = 128\pi\text{ cm}^2\text{/min}$.

8. (a) $A = \frac{1}{2}ab\sin\theta \;\Rightarrow\; \dfrac{dA}{dt} = \frac{1}{2}ab\cos\theta\,\dfrac{d\theta}{dt} = \frac{1}{2}(2)(3)\left(\cos\frac{\pi}{3}\right)(0.2) = 3\left(\frac{1}{2}\right)(0.2) = 0.3\text{ cm}^2\text{/min}$.

(b) $A = \frac{1}{2}ab\sin\theta \;\Rightarrow$

$$\frac{dA}{dt} = \frac{1}{2}a\left(b\cos\theta\,\frac{d\theta}{dt} + \sin\theta\,\frac{db}{dt}\right) = \tfrac{1}{2}(2)\left[3\left(\cos\tfrac{\pi}{3}\right)(0.2) + \left(\sin\tfrac{\pi}{3}\right)(1.5)\right]$$
$$= 3\left(\tfrac{1}{2}\right)(0.2) + \tfrac{1}{2}\sqrt{3}\left(\tfrac{3}{2}\right) = 0.3 + \tfrac{3}{4}\sqrt{3}\text{ cm}^2\text{/min}\quad[\approx 1.6]$$

(c) $A = \frac{1}{2}ab\sin\theta \;\Rightarrow$

$$\frac{dA}{dt} = \frac{1}{2}\left(\frac{da}{dt}\,b\sin\theta + a\,\frac{db}{dt}\sin\theta + ab\cos\theta\,\frac{d\theta}{dt}\right)\qquad\text{[by Exercise 2.3.87(a)]}$$
$$= \tfrac{1}{2}\left[(2.5)(3)\left(\tfrac{1}{2}\sqrt{3}\right) + (2)(1.5)\left(\tfrac{1}{2}\sqrt{3}\right) + (2)(3)\left(\tfrac{1}{2}\right)(0.2)\right]$$
$$= \left(\tfrac{15}{8}\sqrt{3} + \tfrac{3}{4}\sqrt{3} + 0.3\right) = \left(\tfrac{21}{8}\sqrt{3} + 0.3\right)\text{ cm}^2\text{/min}\quad[\approx 4.85]$$

Note how this answer relates to the answer in part (a) [θ changing] and part (b) [b and θ changing].

9. (a) $y = \sqrt{2x+1}$ and $\dfrac{dx}{dt} = 3 \quad \Rightarrow \quad \dfrac{dy}{dt} = \dfrac{dy}{dx}\dfrac{dx}{dt} = \dfrac{1}{2}(2x+1)^{-1/2} \cdot 2 \cdot 3 = \dfrac{3}{\sqrt{2x+1}}$. When $x = 4$, $\dfrac{dy}{dt} = \dfrac{3}{\sqrt{9}} = 1$.

(b) $y = \sqrt{2x+1} \quad \Rightarrow \quad y^2 = 2x+1 \quad \Rightarrow \quad 2x = y^2 - 1 \quad \Rightarrow \quad x = \frac{1}{2}y^2 - \frac{1}{2}$ and $\dfrac{dy}{dt} = 5 \quad \Rightarrow$

$\dfrac{dx}{dt} = \dfrac{dx}{dy}\dfrac{dy}{dt} = y \cdot 5 = 5y$. When $x = 12$, $y = \sqrt{25} = 5$, so $\dfrac{dx}{dt} = 5(5) = 25$.

10. (a) $\dfrac{d}{dt}(4x^2 + 9y^2) = \dfrac{d}{dt}(36) \quad \Rightarrow \quad 8x\,\dfrac{dx}{dt} + 18y\,\dfrac{dy}{dt} = 0 \quad \Rightarrow \quad 4x\,\dfrac{dx}{dt} + 9y\,\dfrac{dy}{dt} = 0 \quad \Rightarrow$

$4(2)\,\dfrac{dx}{dt} + 9\left(\dfrac{2}{3}\sqrt{5}\right)\left(\dfrac{1}{3}\right) = 0 \quad \Rightarrow \quad 8\,\dfrac{dx}{dt} = -2\sqrt{5} \quad \Rightarrow \quad \dfrac{dx}{dt} = -\dfrac{1}{4}\sqrt{5}$

(b) $4x\,\dfrac{dx}{dt} + 9y\,\dfrac{dy}{dt} = 0 \quad \Rightarrow \quad 4(-2)(3) + 9\left(\dfrac{2}{3}\sqrt{5}\right)\dfrac{dy}{dt} = 0 \quad \Rightarrow \quad 6\sqrt{5}\,\dfrac{dy}{dt} = 24 \quad \Rightarrow \quad \dfrac{dy}{dt} = \dfrac{4}{\sqrt{5}}$

11. $\dfrac{d}{dt}(x^2 + y^2 + z^2) = \dfrac{d}{dt}(9) \quad \Rightarrow \quad 2x\,\dfrac{dx}{dt} + 2y\,\dfrac{dy}{dt} + 2z\dfrac{dz}{dt} = 0 \quad \Rightarrow \quad x\,\dfrac{dx}{dt} + y\,\dfrac{dy}{dt} + z\,\dfrac{dz}{dt} = 0.$

If $\dfrac{dx}{dt} = 5$, $\dfrac{dy}{dt} = 4$ and $(x, y, z) = (2, 2, 1)$, then $2(5) + 2(4) + 1\,\dfrac{dz}{dt} = 0 \quad \Rightarrow \quad \dfrac{dz}{dt} = -18.$

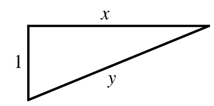
12. $\dfrac{d}{dt}(xy) = \dfrac{d}{dt}(8) \quad \Rightarrow \quad x\,\dfrac{dy}{dt} + y\,\dfrac{dx}{dt} = 0.$ If $\dfrac{dy}{dt} = -3$ cm/s and $(x, y) = (4, 2)$, then $4(-3) + 2\,\dfrac{dx}{dt} = 0 \quad \Rightarrow$

$\dfrac{dx}{dt} = 6$. Thus, the x-coordinate is increasing at a rate of 6 cm/s.

13. (a) Given: a plane flying horizontally at an altitude of 1 mi and a speed of 500 mi/h passes directly over a radar station. If we let t be time (in hours) and x be the horizontal distance traveled by the plane (in mi), then we are given that $dx/dt = 500$ mi/h.

(b) Unknown: the rate at which the distance from the plane to the station is increasing when it is 2 mi from the station. If we let y be the distance from the plane to the station, then we want to find dy/dt when $y = 2$ mi.

(c)



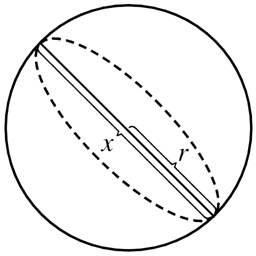
(d) By the Pythagorean Theorem, $y^2 = x^2 + 1 \quad \Rightarrow \quad 2y\,(dy/dt) = 2x\,(dx/dt)$.

(e) $\dfrac{dy}{dt} = \dfrac{x}{y}\dfrac{dx}{dt} = \dfrac{x}{y}(500)$. Since $y^2 = x^2 + 1$, when $y = 2$, $x = \sqrt{3}$, so $\dfrac{dy}{dt} = \dfrac{\sqrt{3}}{2}(500) = 250\sqrt{3} \approx 433$ mi/h.

14. (a) Given: the rate of decrease of the surface area is 1 cm^2/min. If we let t be time (in minutes) and S be the surface area (in cm^2), then we are given that $dS/dt = -1$ cm^2/s.

(b) Unknown: the rate of decrease of the diameter when the diameter is 10 cm. If we let x be the diameter, then we want to find dx/dt when $x = 10$ cm.

(c)



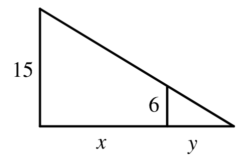
(d) If the radius is r and the diameter $x = 2r$, then $r = \frac{1}{2}x$ and $S = 4\pi r^2 = 4\pi\left(\frac{1}{2}x\right)^2 = \pi x^2 \Rightarrow$

$$\frac{dS}{dt} = \frac{dS}{dx}\frac{dx}{dt} = 2\pi x\,\frac{dx}{dt}.$$

(e) $-1 = \frac{dS}{dt} = 2\pi x\,\frac{dx}{dt} \Rightarrow \frac{dx}{dt} = -\frac{1}{2\pi x}$. When $x = 10$, $\frac{dx}{dt} = -\frac{1}{20\pi}$. So the rate of decrease is $\frac{1}{20\pi}$ cm/min.

15. (a) Given: a man 6 ft tall walks away from a street light mounted on a 15-ft-tall pole at a rate of 5 ft/s. If we let t be time (in s) and x be the distance from the pole to the man (in ft), then we are given that $dx/dt = 5$ ft/s.

(b) Unknown: the rate at which the tip of his shadow is moving when he is 40 ft from the pole. If we let y be the distance from the man to the tip of his shadow (in ft), then we want to find $\frac{d}{dt}(x + y)$ when $x = 40$ ft.

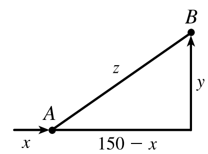
(c) 

(d) By similar triangles, $\frac{15}{6} = \frac{x+y}{y} \Rightarrow 15y = 6x + 6y \Rightarrow 9y = 6x \Rightarrow y = \frac{2}{3}x$.

(e) The tip of the shadow moves at a rate of $\frac{d}{dt}(x + y) = \frac{d}{dt}\left(x + \frac{2}{3}x\right) = \frac{5}{3}\frac{dx}{dt} = \frac{5}{3}(5) = \frac{25}{3}$ ft/s.

16. (a) Given: at noon, ship A is 150 km west of ship B; ship A is sailing east at 35 km/h, and ship B is sailing north at 25 km/h. If we let t be time (in hours), x be the distance traveled by ship A (in km), and y be the distance traveled by ship B (in km), then we are given that $dx/dt = 35$ km/h and $dy/dt = 25$ km/h.

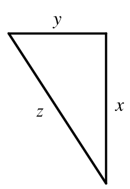
(b) Unknown: the rate at which the distance between the ships is changing at 4:00 PM. If we let z be the distance between the ships, then we want to find dz/dt when $t = 4$ h.

(c) 

(d) $z^2 = (150 - x)^2 + y^2 \Rightarrow 2z\,\frac{dz}{dt} = 2(150 - x)\left(-\frac{dx}{dt}\right) + 2y\,\frac{dy}{dt}$

(e) At 4:00 PM, $x = 4(35) = 140$ and $y = 4(25) = 100 \Rightarrow z = \sqrt{(150 - 140)^2 + 100^2} = \sqrt{10{,}100}$.

So $\frac{dz}{dt} = \frac{1}{z}\left[(x - 150)\,\frac{dx}{dt} + y\,\frac{dy}{dt}\right] = \frac{-10(35) + 100(25)}{\sqrt{10{,}100}} = \frac{215}{\sqrt{101}} \approx 21.4$ km/h.

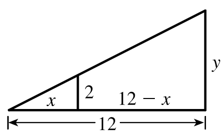
17. 

We are given that $\frac{dx}{dt} = 60$ mi/h and $\frac{dy}{dt} = 25$ mi/h. $z^2 = x^2 + y^2 \Rightarrow$

$2z\,\frac{dz}{dt} = 2x\,\frac{dx}{dt} + 2y\,\frac{dy}{dt} \Rightarrow z\,\frac{dz}{dt} = x\,\frac{dx}{dt} + y\,\frac{dy}{dt} \Rightarrow \frac{dz}{dt} = \frac{1}{z}\left(x\,\frac{dx}{dt} + y\,\frac{dy}{dt}\right)$.

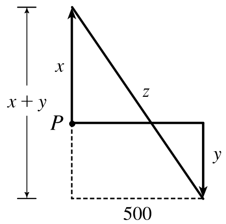
After 2 hours, $x = 2\,(60) = 120$ and $y = 2\,(25) = 50 \Rightarrow z = \sqrt{120^2 + 50^2} = 130$,

so $\frac{dz}{dt} = \frac{1}{z}\left(x\,\frac{dx}{dt} + y\,\frac{dy}{dt}\right) = \frac{120(60) + 50(25)}{130} = 65$ mi/h.

18. 

We are given that $\frac{dx}{dt} = 1.6$ m/s. By similar triangles, $\frac{y}{12} = \frac{2}{x} \Rightarrow y = \frac{24}{x} \Rightarrow$

$\frac{dy}{dt} = -\frac{24}{x^2}\frac{dx}{dt} = -\frac{24}{x^2}(1.6)$. When $x = 8$, $\frac{dy}{dt} = -\frac{24(1.6)}{64} = -0.6$ m/s, so the shadow is decreasing at a rate of 0.6 m/s.

19. 

We are given that $\dfrac{dx}{dt} = 4$ ft/s and $\dfrac{dy}{dt} = 5$ ft/s. $z^2 = (x+y)^2 + 500^2 \quad \Rightarrow$

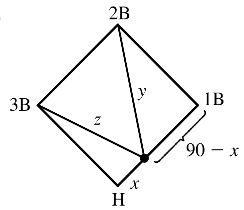
$2z\,\dfrac{dz}{dt} = 2(x+y)\left(\dfrac{dx}{dt} + \dfrac{dy}{dt}\right)$. 15 minutes after the woman starts, we have

$x = (4 \text{ ft/s})(20 \text{ min})(60 \text{ s/min}) = 4800$ ft and $y = 5 \cdot 15 \cdot 60 = 4500 \quad \Rightarrow$

$z = \sqrt{(4800 + 4500)^2 + 500^2} = \sqrt{86{,}740{,}000}$, so

$$\frac{dz}{dt} = \frac{x+y}{z}\left(\frac{dx}{dt} + \frac{dy}{dt}\right) = \frac{4800 + 4500}{\sqrt{86{,}740{,}000}}(4+5) = \frac{837}{\sqrt{8674}} \approx 8.99 \text{ ft/s}.$$

20. We are given that $\dfrac{dx}{dt} = 24$ ft/s.

(a) 

$y^2 = (90 - x)^2 + 90^2 \quad \Rightarrow \quad 2y\,\dfrac{dy}{dt} = 2(90 - x)\left(-\dfrac{dx}{dt}\right)$. When $x = 45$,

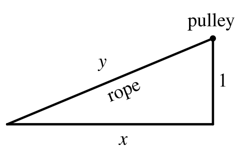
$y = \sqrt{45^2 + 90^2} = 45\sqrt{5}$, so $\dfrac{dy}{dt} = \dfrac{90 - x}{y}\left(-\dfrac{dx}{dt}\right) = \dfrac{45}{45\sqrt{5}}(-24) = -\dfrac{24}{\sqrt{5}}$,

so the distance from second base is decreasing at a rate of $\frac{24}{\sqrt{5}} \approx 10.7$ ft/s.

(b) Due to the symmetric nature of the problem in part (a), we expect to get the same answer—and we do.

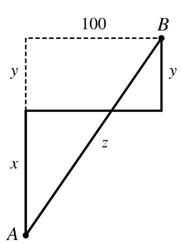
$z^2 = x^2 + 90^2 \quad \Rightarrow \quad 2z\,\dfrac{dz}{dt} = 2x\,\dfrac{dx}{dt}$. When $x = 45$, $z = 45\sqrt{5}$, so $\dfrac{dz}{dt} = \dfrac{45}{45\sqrt{5}}(24) = \dfrac{24}{\sqrt{5}} \approx 10.7$ ft/s.

21. $A = \frac{1}{2}bh$, where b is the base and h is the altitude. We are given that $\dfrac{dh}{dt} = 1$ cm/min and $\dfrac{dA}{dt} = 2$ cm^2/min. Using the Product Rule, we have $\dfrac{dA}{dt} = \dfrac{1}{2}\left(b\,\dfrac{dh}{dt} + h\,\dfrac{db}{dt}\right)$. When $h = 10$ and $A = 100$, we have $100 = \frac{1}{2}b(10) \quad \Rightarrow \quad \frac{1}{2}b = 10 \quad \Rightarrow$ $b = 20$, so $2 = \dfrac{1}{2}\left(20 \cdot 1 + 10\,\dfrac{db}{dt}\right) \quad \Rightarrow \quad 4 = 20 + 10\,\dfrac{db}{dt} \quad \Rightarrow \quad \dfrac{db}{dt} = \dfrac{4 - 20}{10} = -1.6$ cm/min.

22. 

Given $\dfrac{dy}{dt} = -1$ m/s, find $\dfrac{dx}{dt}$ when $x = 8$ m. $y^2 = x^2 + 1 \quad \Rightarrow \quad 2y\,\dfrac{dy}{dt} = 2x\,\dfrac{dx}{dt} \quad \Rightarrow$

$\dfrac{dx}{dt} = \dfrac{y}{x}\dfrac{dy}{dt} = -\dfrac{y}{x}$. When $x = 8$, $y = \sqrt{65}$, so $\dfrac{dx}{dt} = -\dfrac{\sqrt{65}}{8}$. Thus, the boat approaches the dock at $\dfrac{\sqrt{65}}{8} \approx 1.01$ m/s.

23. 

We are given that $\dfrac{dx}{dt} = 35$ km/h and $\dfrac{dy}{dt} = 25$ km/h. $z^2 = (x+y)^2 + 100^2 \quad \Rightarrow$

$2z\,\dfrac{dz}{dt} = 2(x+y)\left(\dfrac{dx}{dt} + \dfrac{dy}{dt}\right)$. At 4:00 PM, $x = 4(35) = 140$ and $y = 4(25) = 100 \quad \Rightarrow$

$z = \sqrt{(140 + 100)^2 + 100^2} = \sqrt{67{,}600} = 260$, so

$$\frac{dz}{dt} = \frac{x+y}{z}\left(\frac{dx}{dt} + \frac{dy}{dt}\right) = \frac{140 + 100}{260}(35 + 25) = \frac{720}{13} \approx 55.4 \text{ km/h}.$$

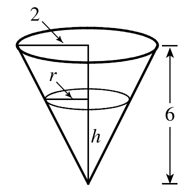
24. The distance z of the particle to the origin is given by $z = \sqrt{x^2 + y^2}$, so $z^2 = x^2 + [2\sin(\pi x/2)]^2 \quad \Rightarrow$

$2z\,\dfrac{dz}{dt} = 2x\,\dfrac{dx}{dt} + 4 \cdot 2\sin\left(\dfrac{\pi}{2}x\right)\cos\left(\dfrac{\pi}{2}x\right) \cdot \dfrac{\pi}{2}\,\dfrac{dx}{dt} \quad \Rightarrow \quad z\,\dfrac{dz}{dt} = x\,\dfrac{dx}{dt} + 2\pi\sin\left(\dfrac{\pi}{2}x\right)\cos\left(\dfrac{\pi}{2}x\right)\dfrac{dx}{dt}$. When

$(x,y)=\left(\frac{1}{3},1\right)$, $z=\sqrt{\left(\frac{1}{3}\right)^2+1^2}=\sqrt{\frac{10}{9}}=\frac{1}{3}\sqrt{10}$, so $\frac{1}{3}\sqrt{10}\,\frac{dz}{dt}=\frac{1}{3}\sqrt{10}+2\pi\,\sin\frac{\pi}{6}\,\cos\frac{\pi}{6}\cdot\sqrt{10}\;\Rightarrow$

$\frac{1}{3}\frac{dz}{dt}=\frac{1}{3}+2\pi\left(\frac{1}{2}\right)\left(\frac{1}{2}\sqrt{3}\right)\;\Rightarrow\;\frac{dz}{dt}=1+\frac{3\sqrt{3}\,\pi}{2}$ cm/s.

25. If $C=$ the rate at which water is pumped in, then $\frac{dV}{dt}=C-10{,}000$, where

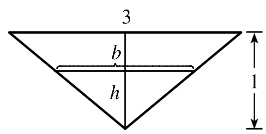


$V=\frac{1}{3}\pi r^2h$ is the volume at time t. By similar triangles, $\frac{r}{2}=\frac{h}{6}\;\Rightarrow\;r=\frac{1}{3}h\;\Rightarrow$

$V=\frac{1}{3}\pi\left(\frac{1}{3}h\right)^2h=\frac{\pi}{27}h^3\;\Rightarrow\;\frac{dV}{dt}=\frac{\pi}{9}h^2\,\frac{dh}{dt}$. When $h=200$ cm,

$\frac{dh}{dt}=20$ cm/min, so $C-10{,}000=\frac{\pi}{9}(200)^2(20)\;\Rightarrow\;C=10{,}000+\frac{800{,}000}{9}\pi\approx 289{,}253\text{ cm}^3/\text{min}$.

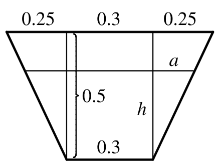
26. By similar triangles, $\frac{3}{1}=\frac{b}{h}$, so $b=3h$. The trough has volume



$V=\frac{1}{2}bh(10)=5(3h)h=15h^2\;\Rightarrow\;12=\frac{dV}{dt}=30h\,\frac{dh}{dt}\;\Rightarrow\;\frac{dh}{dt}=\frac{2}{5h}$.

When $h=\frac{1}{2}$, $\frac{dh}{dt}=\frac{2}{5\cdot\frac{1}{2}}=\frac{4}{5}$ ft/min.

27.



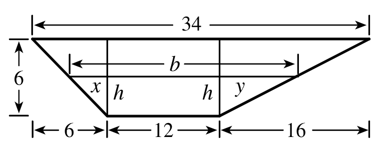
The figure is labeled in meters. The area A of a trapezoid is $\frac{1}{2}(\text{base}_1+\text{base}_2)(\text{height})$, and the volume V of the 10-meter-long trough is $10A$. Thus, the volume of the trapezoid with height h is $V=(10)\frac{1}{2}[0.3+(0.3+2a)]h$.

By similar triangles, $\frac{a}{h}=\frac{0.25}{0.5}=\frac{1}{2}$, so $2a=h\;\Rightarrow\;V=5(0.6+h)h=3h+5h^2$.

Now $\frac{dV}{dt}=\frac{dV}{dh}\frac{dh}{dt}\;\Rightarrow\;0.2=(3+10h)\frac{dh}{dt}\;\Rightarrow\;\frac{dh}{dt}=\frac{0.2}{3+10h}$. When $h=0.3$,

$\frac{dh}{dt}=\frac{0.2}{3+10(0.3)}=\frac{0.2}{6}$ m/min $=\frac{1}{30}$ m/min or $\frac{10}{3}$ cm/min.

28. The figure is drawn without the top 3 feet.



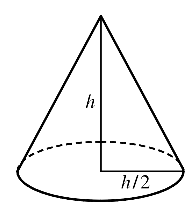
$V=\frac{1}{2}(b+12)h(20)=10(b+12)h$ and, from similar triangles,

$\frac{x}{h}=\frac{6}{6}$ and $\frac{y}{h}=\frac{16}{6}=\frac{8}{3}$, so $b=x+12+y=h+12+\frac{8h}{3}=12+\frac{11h}{3}$.

Thus, $V=10\left(24+\frac{11h}{3}\right)h=240h+\frac{110h^2}{3}$ and so $0.8=\frac{dV}{dt}=\left(240+\frac{220}{3}h\right)\frac{dh}{dt}$.

When $h=5$, $\frac{dh}{dt}=\frac{0.8}{240+5(220/3)}=\frac{3}{2275}\approx 0.00132$ ft/min.

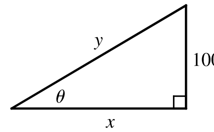
29. We are given that $\frac{dV}{dt}=30\text{ ft}^3/\text{min}$. $V=\frac{1}{3}\pi r^2h=\frac{1}{3}\pi\left(\frac{h}{2}\right)^2h=\frac{\pi h^3}{12}\;\Rightarrow$



$\frac{dV}{dt}=\frac{dV}{dh}\frac{dh}{dt}\;\Rightarrow\;30=\frac{\pi h^2}{4}\frac{dh}{dt}\;\Rightarrow\;\frac{dh}{dt}=\frac{120}{\pi h^2}$.

When $h=10$ ft, $\frac{dh}{dt}=\frac{120}{10^2\pi}=\frac{6}{5\pi}\approx 0.38$ ft/min.

30. We are given $dx/dt = 8$ ft/s. $\cot\theta = \dfrac{x}{100} \quad\Rightarrow\quad x = 100\cot\theta \quad\Rightarrow$



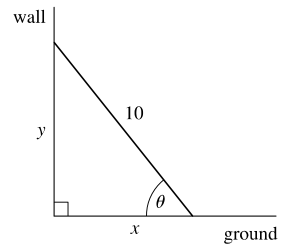
$\dfrac{dx}{dt} = -100\csc^2\theta\,\dfrac{d\theta}{dt} \quad\Rightarrow\quad \dfrac{d\theta}{dt} = -\dfrac{\sin^2\theta}{100}\cdot 8$. When $y = 200$, $\sin\theta = \dfrac{100}{200} = \dfrac{1}{2} \quad\Rightarrow$

$\dfrac{d\theta}{dt} = -\dfrac{(1/2)^2}{100}\cdot 8 = -\dfrac{1}{50}$ rad/s. The angle is decreasing at a rate of $\frac{1}{50}$ rad/s.

31. The area A of an equilateral triangle with side s is given by $A = \frac{1}{4}\sqrt{3}\,s^2$.

$\dfrac{dA}{dt} = \frac{1}{4}\sqrt{3}\cdot 2s\,\dfrac{ds}{dt} = \frac{1}{4}\sqrt{3}\cdot 2(30)(10) = 150\sqrt{3}\ \text{cm}^2/\text{min}$.

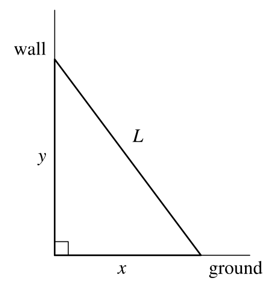
32. $\cos\theta = \dfrac{x}{10} \quad\Rightarrow\quad -\sin\theta\,\dfrac{d\theta}{dt} = \dfrac{1}{10}\,\dfrac{dx}{dt}$. From Example 2, $\dfrac{dx}{dt} = 1$ and



when $x = 6$, $y = 8$, so $\sin\theta = \dfrac{8}{10}$.

Thus, $-\dfrac{8}{10}\,\dfrac{d\theta}{dt} = \dfrac{1}{10}(1) \quad\Rightarrow\quad \dfrac{d\theta}{dt} = -\dfrac{1}{8}$ rad/s.

33. From the figure and given information, we have $x^2 + y^2 = L^2$, $\dfrac{dy}{dt} = -0.15\ \text{m/s}$, and



$\dfrac{dx}{dt} = 0.2\ \text{m/s}$ when $x = 3$ m. Differentiating implicitly with respect to t, we get

$x^2 + y^2 = L^2 \quad\Rightarrow\quad 2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 0 \quad\Rightarrow\quad y\dfrac{dy}{dt} = -x\dfrac{dx}{dt}$. Substituting the given information gives us $y(-0.15) = -3(0.2) \quad\Rightarrow\quad y = 4$ m. Thus, $3^2 + 4^2 = L^2 \quad\Rightarrow$ $L^2 = 25 \quad\Rightarrow\quad L = 5$ m.

34. According to the model in Example 2, $\dfrac{dy}{dt} = -\dfrac{x}{y}\,\dfrac{dx}{dt} \to -\infty$ as $y \to 0$, which doesn't make physical sense. For example, the model predicts that for sufficiently small y, the tip of the ladder moves at a speed greater than the speed of light. Therefore the model is not appropriate for small values of y. What actually happens is that the tip of the ladder leaves the wall at some point in its descent. For a discussion of the true situation see the article "The Falling Ladder Paradox" by Paul Scholten and Andrew Simoson in *The College Mathematics Journal,* 27, (1), January 1996, pages 49–54. Also see "On Mathematical and Physical Ladders" by M. Freeman and P. Palffy-Muhoray in the *American Journal of Physics,* 53 (3), March 1985, pages 276–277.

35. The area A of a sector of a circle with radius r and angle θ is given by $A = \frac{1}{2}r^2\theta$. Here r is constant and θ varies, so $\dfrac{dA}{dt} = \dfrac{1}{2}r^2\,\dfrac{d\theta}{dt}$. The minute hand rotates through $360^\circ = 2\pi$ radians each hour, so $\dfrac{dA}{dt} = \dfrac{1}{2}r^2(2\pi) = \pi r^2\ \text{cm}^2/\text{h}$. This answer makes sense because the minute hand sweeps through the full area of a circle, πr^2, each hour.

36. The volume of a hemisphere is $\frac{2}{3}\pi r^3$, so the volume of a hemispherical basin of radius 30 cm is $\frac{2}{3}\pi(30)^3 = 18{,}000\pi\ \text{cm}^3$. If the basin is half full, then $V = \pi\left(rh^2 - \frac{1}{3}h^3\right) \quad\Rightarrow\quad 9000\pi = \pi\left(30h^2 - \frac{1}{3}h^3\right) \quad\Rightarrow\quad \frac{1}{3}h^3 - 30h^2 + 9000 = 0 \quad\Rightarrow$

$h = H \approx 19.58$ [from a graph or numerical rootfinder; the other two solutions are less than 0 and greater than 30].

$$V = \pi\left(30h^2 - \tfrac{1}{3}h^3\right) \Rightarrow \frac{dV}{dt} = \pi\left(60h\,\frac{dh}{dt} - h^2\,\frac{dh}{dt}\right) \Rightarrow \left(2\,\frac{\text{L}}{\text{min}}\right)\left(1000\,\frac{\text{cm}^3}{\text{L}}\right) = \pi(60h - h^2)\,\frac{dh}{dt} \Rightarrow$$

$$\frac{dh}{dt} = \frac{2000}{\pi(60H - H^2)} \approx 0.804 \text{ cm/min}.$$

37. Differentiating both sides of $PV = C$ with respect to t and using the Product Rule gives us $P\,\frac{dV}{dt} + V\,\frac{dP}{dt} = 0 \Rightarrow$ $\frac{dV}{dt} = -\frac{V}{P}\,\frac{dP}{dt}$. When $V = 600$, $P = 150$ and $\frac{dP}{dt} = 20$, so we have $\frac{dV}{dt} = -\frac{600}{150}(20) = -80$. Thus, the volume is decreasing at a rate of 80 cm^3/min.

38. $PV^{1.4} = C \Rightarrow P \cdot 1.4V^{0.4}\,\frac{dV}{dt} + V^{1.4}\,\frac{dP}{dt} = 0 \Rightarrow \frac{dV}{dt} = -\frac{V^{1.4}}{P \cdot 1.4V^{0.4}}\,\frac{dP}{dt} = -\frac{V}{1.4P}\,\frac{dP}{dt}$.

When $V = 400$, $P = 80$ and $\frac{dP}{dt} = -10$, so we have $\frac{dV}{dt} = -\frac{400}{1.4(80)}(-10) = \frac{250}{7}$. Thus, the volume is increasing at a rate of $\frac{250}{7} \approx 36$ cm^3/min.

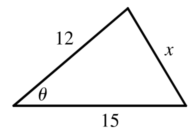
39. With $R_1 = 80$ and $R_2 = 100$, $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{80} + \frac{1}{100} = \frac{180}{8000} = \frac{9}{400}$, so $R = \frac{400}{9}$. Differentiating $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$ with respect to t, we have $-\frac{1}{R^2}\,\frac{dR}{dt} = -\frac{1}{R_1^2}\,\frac{dR_1}{dt} - \frac{1}{R_2^2}\,\frac{dR_2}{dt} \Rightarrow \frac{dR}{dt} = R^2\left(\frac{1}{R_1^2}\,\frac{dR_1}{dt} + \frac{1}{R_2^2}\,\frac{dR_2}{dt}\right)$. When $R_1 = 80$ and $R_2 = 100$, $\frac{dR}{dt} = \frac{400^2}{9^2}\left[\frac{1}{80^2}(0.3) + \frac{1}{100^2}(0.2)\right] = \frac{107}{810} \approx 0.132\ \Omega/\text{s}$.

40. We want to find $\frac{dB}{dt}$ when $L = 18$ using $B = 0.007W^{2/3}$ and $W = 0.12L^{2.53}$.

$$\frac{dB}{dt} = \frac{dB}{dW}\,\frac{dW}{dL}\,\frac{dL}{dt} = \left(0.007 \cdot \tfrac{2}{3}W^{-1/3}\right)(0.12 \cdot 2.53 \cdot L^{1.53})\left(\frac{20 - 15}{10{,}000{,}000}\right)$$

$$= \left[0.007 \cdot \tfrac{2}{3}(0.12 \cdot 18^{2.53})^{-1/3}\right](0.12 \cdot 2.53 \cdot 18^{1.53})\left(\frac{5}{10^7}\right) \approx 1.045 \times 10^{-8} \text{ g/yr}$$

41. We are given $d\theta/dt = 2^\circ/\text{min} = \frac{\pi}{90}$ rad/min. By the Law of Cosines,

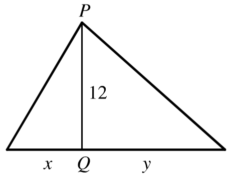


$x^2 = 12^2 + 15^2 - 2(12)(15)\cos\theta = 369 - 360\cos\theta \Rightarrow$

$2x\,\frac{dx}{dt} = 360\sin\theta\,\frac{d\theta}{dt} \Rightarrow \frac{dx}{dt} = \frac{180\sin\theta}{x}\,\frac{d\theta}{dt}$. When $\theta = 60^\circ$,

$x = \sqrt{369 - 360\cos 60^\circ} = \sqrt{189} = 3\sqrt{21}$, so $\frac{dx}{dt} = \frac{180\sin 60^\circ}{3\sqrt{21}}\,\frac{\pi}{90} = \frac{\pi\sqrt{3}}{3\sqrt{21}} = \frac{\sqrt{7}\,\pi}{21} \approx 0.396$ m/min.

42. Using Q for the origin, we are given $\frac{dx}{dt} = -2$ ft/s and need to find $\frac{dy}{dt}$ when $x = -5$.



Using the Pythagorean Theorem twice, we have $\sqrt{x^2 + 12^2} + \sqrt{y^2 + 12^2} = 39$, the total length of the rope. Differentiating with respect to t, we get

$$\frac{x}{\sqrt{x^2 + 12^2}}\,\frac{dx}{dt} + \frac{y}{\sqrt{y^2 + 12^2}}\,\frac{dy}{dt} = 0, \text{ so } \frac{dy}{dt} = -\frac{x\sqrt{y^2 + 12^2}}{y\sqrt{x^2 + 12^2}}\,\frac{dx}{dt}.$$

Now when $x = -5$, $39 = \sqrt{(-5)^2 + 12^2} + \sqrt{y^2 + 12^2} = 13 + \sqrt{y^2 + 12^2} \quad \Leftrightarrow \quad \sqrt{y^2 + 12^2} = 26$, and

$y = \sqrt{26^2 - 12^2} = \sqrt{532}$. So when $x = -5$, $\dfrac{dy}{dt} = -\dfrac{(-5)(26)}{\sqrt{532}\,(13)}(-2) = -\dfrac{10}{\sqrt{133}} \approx -0.87$ ft/s.

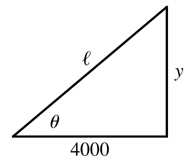
So cart B is moving towards Q at about 0.87 ft/s.

43. (a) By the Pythagorean Theorem, $4000^2 + y^2 = \ell^2$. Differentiating with respect to t,

we obtain $2y\,\dfrac{dy}{dt} = 2\ell\,\dfrac{d\ell}{dt}$. We know that $\dfrac{dy}{dt} = 600$ ft/s, so when $y = 3000$ ft,

$\ell = \sqrt{4000^2 + 3000^2} = \sqrt{25{,}000{,}000} = 5000$ ft

and $\dfrac{d\ell}{dt} = \dfrac{y}{\ell}\dfrac{dy}{dt} = \dfrac{3000}{5000}(600) = \dfrac{1800}{5} = 360$ ft/s.



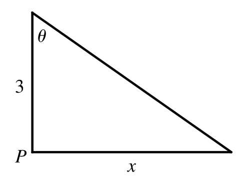
(b) Here $\tan\theta = \dfrac{y}{4000} \quad \Rightarrow \quad \dfrac{d}{dt}(\tan\theta) = \dfrac{d}{dt}\left(\dfrac{y}{4000}\right) \quad \Rightarrow \quad \sec^2\theta\,\dfrac{d\theta}{dt} = \dfrac{1}{4000}\dfrac{dy}{dt} \quad \Rightarrow \quad \dfrac{d\theta}{dt} = \dfrac{\cos^2\theta}{4000}\dfrac{dy}{dt}$. When

$y = 3000$ ft, $\dfrac{dy}{dt} = 600$ ft/s, $\ell = 5000$ and $\cos\theta = \dfrac{4000}{\ell} = \dfrac{4000}{5000} = \dfrac{4}{5}$, so $\dfrac{d\theta}{dt} = \dfrac{(4/5)^2}{4000}(600) = 0.096$ rad/s.

44. We are given that $\dfrac{d\theta}{dt} = 4(2\pi) = 8\pi$ rad/min. $x = 3\tan\theta \quad \Rightarrow$

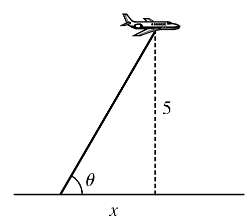
$\dfrac{dx}{dt} = 3\sec^2\theta\,\dfrac{d\theta}{dt}$. When $x = 1$, $\tan\theta = \frac{1}{3}$, so $\sec^2\theta = 1 + \left(\frac{1}{3}\right)^2 = \frac{10}{9}$

and $\dfrac{dx}{dt} = 3\left(\frac{10}{9}\right)(8\pi) = \frac{80}{3}\pi \approx 83.8$ km/min.



45. $\cot\theta = \dfrac{x}{5} \quad \Rightarrow \quad -\csc^2\theta\,\dfrac{d\theta}{dt} = \dfrac{1}{5}\dfrac{dx}{dt} \quad \Rightarrow \quad -\left(\csc\dfrac{\pi}{3}\right)^2\left(-\dfrac{\pi}{6}\right) = \dfrac{1}{5}\dfrac{dx}{dt} \quad \Rightarrow$

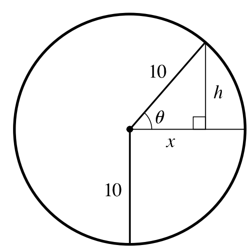
$\dfrac{dx}{dt} = \dfrac{5\pi}{6}\left(\dfrac{2}{\sqrt{3}}\right)^2 = \dfrac{10}{9}\pi$ km/min $[\approx 130$ mi/h$]$



46. We are given that $\dfrac{d\theta}{dt} = \dfrac{2\pi \text{ rad}}{2 \text{ min}} = \pi$ rad/min. By the Pythagorean Theorem, when

$h = 6$, $x = 8$, so $\sin\theta = \frac{6}{10}$ and $\cos\theta = \frac{8}{10}$. From the figure, $\sin\theta = \dfrac{h}{10} \quad \Rightarrow$

$h = 10\sin\theta$, so $\dfrac{dh}{dt} = 10\cos\theta\,\dfrac{d\theta}{dt} = 10\left(\dfrac{8}{10}\right)\pi = 8\pi$ m/min.

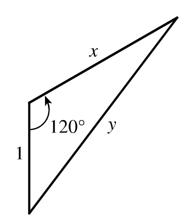


47. We are given that $\dfrac{dx}{dt} = 300$ km/h. By the Law of Cosines,

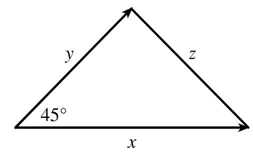
$y^2 = x^2 + 1^2 - 2(1)(x)\cos 120^\circ = x^2 + 1 - 2x\left(-\frac{1}{2}\right) = x^2 + x + 1$, so

$2y\,\dfrac{dy}{dt} = 2x\,\dfrac{dx}{dt} + \dfrac{dx}{dt} \quad \Rightarrow \quad \dfrac{dy}{dt} = \dfrac{2x+1}{2y}\dfrac{dx}{dt}$. After 1 minute, $x = \frac{300}{60} = 5$ km $\quad \Rightarrow$

$y = \sqrt{5^2 + 5 + 1} = \sqrt{31}$ km $\quad \Rightarrow \quad \dfrac{dy}{dt} = \dfrac{2(5)+1}{2\sqrt{31}}(300) = \dfrac{1650}{\sqrt{31}} \approx 296$ km/h.



48. We are given that $\dfrac{dx}{dt} = 3$ mi/h and $\dfrac{dy}{dt} = 2$ mi/h. By the Law of Cosines,



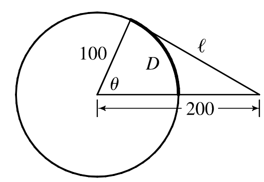
$z^2 = x^2 + y^2 - 2xy\cos 45° = x^2 + y^2 - \sqrt{2}\,xy \quad \Rightarrow$

$2z\dfrac{dz}{dt} = 2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} - \sqrt{2}\,x\dfrac{dy}{dt} - \sqrt{2}\,y\dfrac{dx}{dt}$. After 15 minutes $\left[= \frac{1}{4}\text{ h}\right]$,

we have $x = \frac{3}{4}$ and $y = \frac{2}{4} = \frac{1}{2} \quad \Rightarrow \quad z^2 = \left(\frac{3}{4}\right)^2 + \left(\frac{2}{4}\right)^2 - \sqrt{2}\left(\frac{3}{4}\right)\left(\frac{2}{4}\right) \quad \Rightarrow \quad z = \dfrac{\sqrt{13 - 6\sqrt{2}}}{4}$ and

$$\frac{dz}{dt} = \frac{2}{\sqrt{13-6\sqrt{2}}}\left[2\left(\tfrac{3}{4}\right)3 + 2\left(\tfrac{1}{2}\right)2 - \sqrt{2}\left(\tfrac{3}{4}\right)2 - \sqrt{2}\left(\tfrac{1}{2}\right)3\right] = \frac{2}{\sqrt{13-6\sqrt{2}}}\,\frac{13-6\sqrt{2}}{2} = \sqrt{13-6\sqrt{2}} \approx 2.125 \text{ mi/h}.$$

49. Let the distance between the runner and the friend be ℓ. Then by the Law of Cosines,

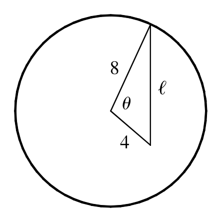


$\ell^2 = 200^2 + 100^2 - 2\cdot 200\cdot 100\cdot\cos\theta = 50{,}000 - 40{,}000\cos\theta$ $(\star)$. Differentiating implicitly with respect to t, we obtain $2\ell\dfrac{d\ell}{dt} = -40{,}000(-\sin\theta)\dfrac{d\theta}{dt}$. Now if D is the distance run when the angle is θ radians, then by the formula for the length of an arc on a circle, $s = r\theta$, we have $D = 100\theta$, so $\theta = \dfrac{1}{100}D \quad \Rightarrow \quad \dfrac{d\theta}{dt} = \dfrac{1}{100}\dfrac{dD}{dt} = \dfrac{7}{100}$. To substitute into the expression for $\dfrac{d\ell}{dt}$, we must know $\sin\theta$ at the time when $\ell = 200$, which we find from $(\star)$: $200^2 = 50{,}000 - 40{,}000\cos\theta \quad \Leftrightarrow$

$\cos\theta = \frac{1}{4} \quad \Rightarrow \quad \sin\theta = \sqrt{1 - \left(\frac{1}{4}\right)^2} = \frac{\sqrt{15}}{4}$. Substituting, we get $2(200)\dfrac{d\ell}{dt} = 40{,}000\frac{\sqrt{15}}{4}\left(\frac{7}{100}\right) \quad \Rightarrow$

$d\ell/dt = \frac{7\sqrt{15}}{4} \approx 6.78$ m/s. Whether the distance between them is increasing or decreasing depends on the direction in which the runner is running.

50. The hour hand of a clock goes around once every 12 hours or, in radians per hour, $\frac{2\pi}{12} = \frac{\pi}{6}$ rad/h. The minute hand goes around once an hour, or at the rate of 2π rad/h. So the angle θ between them (measuring clockwise from the minute hand to the hour hand) is changing at the rate of $d\theta/dt = \frac{\pi}{6} - 2\pi = -\frac{11\pi}{6}$ rad/h. Now, to relate θ to ℓ, we use the Law of Cosines: $\ell^2 = 4^2 + 8^2 - 2\cdot 4\cdot 8\cdot\cos\theta = 80 - 64\cos\theta$ $(\star)$.



Differentiating implicitly with respect to t, we get $2\ell\dfrac{d\ell}{dt} = -64(-\sin\theta)\dfrac{d\theta}{dt}$. At 1:00, the angle between the two hands is one-twelfth of the circle, that is, $\frac{2\pi}{12} = \frac{\pi}{6}$ radians. We use $(\star)$ to find ℓ at 1:00: $\ell = \sqrt{80 - 64\cos\frac{\pi}{6}} = \sqrt{80 - 32\sqrt{3}}$.

Substituting, we get $2\ell\dfrac{d\ell}{dt} = 64\sin\dfrac{\pi}{6}\left(-\dfrac{11\pi}{6}\right) \quad \Rightarrow \quad \dfrac{d\ell}{dt} = \dfrac{64\left(\frac{1}{2}\right)\left(-\frac{11\pi}{6}\right)}{2\sqrt{80-32\sqrt{3}}} = -\dfrac{88\pi}{3\sqrt{80-32\sqrt{3}}} \approx -18.6.$

So at 1:00, the distance between the tips of the hands is decreasing at a rate of 18.6 mm/h ≈ 0.005 mm/s.

2.9 Linear Approximations and Differentials

1. $f(x) = x^3 - x^2 + 3 \;\Rightarrow\; f'(x) = 3x^2 - 2x$, so $f(-2) = -9$ and $f'(-2) = 16$. Thus,

$L(x) = f(-2) + f'(-2)(x - (-2)) = -9 + 16(x + 2) = 16x + 23.$

2. $f(x) = \sin x \;\Rightarrow\; f'(x) = \cos x$, so $f\left(\frac{\pi}{6}\right) = \frac{1}{2}$ and $f'\left(\frac{\pi}{6}\right) = \frac{1}{2}\sqrt{3}$. Thus,

$L(x) = f\left(\frac{\pi}{6}\right) + f'\left(\frac{\pi}{6}\right)\left(x - \frac{\pi}{6}\right) = \frac{1}{2} + \frac{1}{2}\sqrt{3}\left(x - \frac{\pi}{6}\right) = \frac{1}{2}\sqrt{3}\,x + \frac{1}{2} - \frac{1}{12}\sqrt{3}\,\pi.$

3. $f(x) = \sqrt{x} \;\Rightarrow\; f'(x) = \frac{1}{2}x^{-1/2} = 1/(2\sqrt{x})$, so $f(4) = 2$ and $f'(4) = \frac{1}{4}$. Thus,

$L(x) = f(4) + f'(4)(x - 4) = 2 + \frac{1}{4}(x - 4) = 2 + \frac{1}{4}x - 1 = \frac{1}{4}x + 1.$

4. $f(x) = 2/\sqrt{x^2 - 5} = 2(x^2 - 5)^{-1/2} \;\Rightarrow\; f'(x) = 2\left(-\frac{1}{2}\right)(x^2 - 5)^{-3/2}(2x) = -\dfrac{2x}{(x^2 - 5)^{3/2}}$, so $f(3) = 1$ and

$f'(3) = -\frac{3}{4}$. Thus, $L(x) = f(3) + f'(3)(x - 3) = 1 - \frac{3}{4}(x - 3) = -\frac{3}{4}x + \frac{13}{4}$.

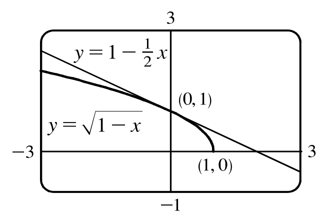
5. $f(x) = \sqrt{1 - x} \;\Rightarrow\; f'(x) = \dfrac{-1}{2\sqrt{1 - x}}$, so $f(0) = 1$ and $f'(0) = -\frac{1}{2}$.

Therefore,

$\sqrt{1 - x} = f(x) \approx f(0) + f'(0)(x - 0) = 1 + \left(-\frac{1}{2}\right)(x - 0) = 1 - \frac{1}{2}x.$

So $\sqrt{0.9} = \sqrt{1 - 0.1} \approx 1 - \frac{1}{2}(0.1) = 0.95$

and $\sqrt{0.99} = \sqrt{1 - 0.01} \approx 1 - \frac{1}{2}(0.01) = 0.995$.

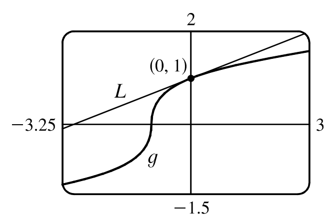


6. $g(x) = \sqrt[3]{1 + x} = (1 + x)^{1/3} \;\Rightarrow\; g'(x) = \frac{1}{3}(1 + x)^{-2/3}$, so $g(0) = 1$ and

$g'(0) = \frac{1}{3}$. Therefore, $\sqrt[3]{1 + x} = g(x) \approx g(0) + g'(0)(x - 0) = 1 + \frac{1}{3}x$.

So $\sqrt[3]{0.95} = \sqrt[3]{1 + (-0.05)} \approx 1 + \frac{1}{3}(-0.05) = 0.98\overline{3}$,

and $\sqrt[3]{1.1} = \sqrt[3]{1 + 0.1} \approx 1 + \frac{1}{3}(0.1) = 1.0\overline{3}$.

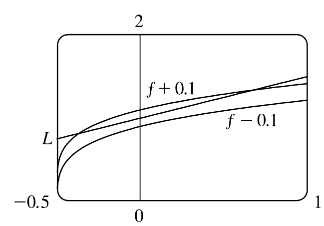


7. $f(x) = \sqrt[4]{1 + 2x} \;\Rightarrow\; f'(x) = \frac{1}{4}(1 + 2x)^{-3/4}(2) = \frac{1}{2}(1 + 2x)^{-3/4}$, so

$f(0) = 1$ and $f'(0) = \frac{1}{2}$. Thus, $f(x) \approx f(0) + f'(0)(x - 0) = 1 + \frac{1}{2}x$.

We need $\sqrt[4]{1 + 2x} - 0.1 < 1 + \frac{1}{2}x < \sqrt[4]{1 + 2x} + 0.1$, which is true when

$-0.368 < x < 0.677$.

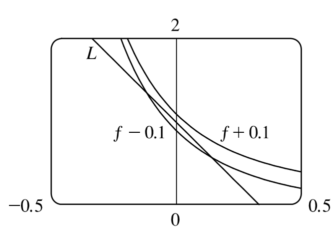


8. $f(x) = (1 + x)^{-3} \;\Rightarrow\; f'(x) = -3(1 + x)^{-4}$, so $f(0) = 1$ and

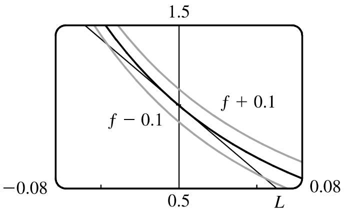
$f'(0) = -3$. Thus, $f(x) \approx f(0) + f'(0)(x - 0) = 1 - 3x$. We need

$(1 + x)^{-3} - 0.1 < 1 - 3x < (1 + x)^{-3} + 0.1$, which is true when

$-0.116 < x < 0.144$.



9. $f(x) = \dfrac{1}{(1+2x)^4} = (1+2x)^{-4} \quad \Rightarrow$

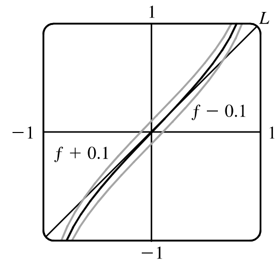


$f'(x) = -4(1+2x)^{-5}(2) = \dfrac{-8}{(1+2x)^5}$, so $f(0) = 1$ and $f'(0) = -8$.

Thus, $f(x) \approx f(0) + f'(0)(x-0) = 1 + (-8)(x-0) = 1 - 8x$.

We need $\dfrac{1}{(1+2x)^4} - 0.1 < 1 - 8x < \dfrac{1}{(1+2x)^4} + 0.1$, which is true when $-0.045 < x < 0.055$.

10. $f(x) = \tan x \quad \Rightarrow \quad f'(x) = \sec^2 x$, so $f(0) = 0$ and $f'(0) = 1$.



Thus, $f(x) \approx f(0) + f'(0)(x-0) = 0 + 1(x-0) = x$.

We need $\tan x - 0.1 < x < \tan x + 0.1$, which is true when $-0.63 < x < 0.63$.

11. (a) The differential dy is defined in terms of dx by the equation $dy = f'(x)\,dx$. For $y = f(x) = (x^2-3)^{-2}$,

$f'(x) = -2(x^2-3)^{-3}(2x) = -\dfrac{4x}{(x^2-3)^3}$, so $dy = -\dfrac{4x}{(x^2-3)^3}\,dx$.

(b) For $y = f(t) = \sqrt{1-t^4}$, $f'(t) = \frac{1}{2}(1-t^4)^{-1/2}(-4t^3) = -\dfrac{2t^3}{\sqrt{1-t^4}}$, so $dy = -\dfrac{2t^3}{\sqrt{1-t^4}}\,dt$.

12. (a) For $y = f(u) = \dfrac{1+2u}{1+3u}$, $f'(u) = \dfrac{(1+3u)(2)-(1+2u)(3)}{(1+3u)^2} = \dfrac{-1}{(1+3u)^2}$, so $dy = \dfrac{-1}{(1+3u)^2}\,du$.

(b) For $y = f(\theta) = \theta^2 \sin 2\theta$, $f'(\theta) = \theta^2(\cos 2\theta)(2) + (\sin 2\theta)(2\theta)$, so $dy = 2\theta(\theta\cos 2\theta + \sin 2\theta)\,d\theta$.

13. (a) For $y = f(t) = \tan\sqrt{t}$, $f'(t) = \sec^2\sqrt{t} \cdot \dfrac{1}{2}t^{-1/2} = \dfrac{\sec^2\sqrt{t}}{2\sqrt{t}}$, so $dy = \dfrac{\sec^2\sqrt{t}}{2\sqrt{t}}\,dt$.

(b) For $y = f(v) = \dfrac{1-v^2}{1+v^2}$,

$$f'(v) = \frac{(1+v^2)(-2v)-(1-v^2)(2v)}{(1+v^2)^2} = \frac{-2v[(1+v^2)+(1-v^2)]}{(1+v^2)^2} = \frac{-2v(2)}{(1+v^2)^2} = \frac{-4v}{(1+v^2)^2},$$

so $dy = \dfrac{-4v}{(1+v^2)^2}\,dv$.

14. (a) For $y = f(t) = \sqrt{t - \cos t}$, $f'(t) = \frac{1}{2}(t-\cos t)^{-1/2}(1+\sin t) = \dfrac{1+\sin t}{2\sqrt{t-\cos t}}$, so $dy = \dfrac{1+\sin t}{2\sqrt{t-\cos t}}\,dt$.

(b) For $y = f(x) = \dfrac{1}{x}\sin x$, $f'(x) = \dfrac{1}{x}\cos x - \dfrac{1}{x^2}\sin x = \dfrac{x\cos x - \sin x}{x^2}$, so $dy = \dfrac{x\cos x - \sin x}{x^2}\,dx$.

15. (a) $y = \tan x \quad \Rightarrow \quad dy = \sec^2 x\,dx$

(b) When $x = \pi/4$ and $dx = -0.1$, $dy = [\sec(\pi/4)]^2(-0.1) = \left(\sqrt{2}\right)^2(-0.1) = -0.2$.

16. (a) $y = \cos \pi x \quad \Rightarrow \quad dy = -\sin \pi x \cdot \pi\, dx = -\pi \sin \pi x\, dx$

(b) $x = \frac{1}{3}$ and $dx = -0.02 \quad \Rightarrow \quad dy = -\pi \sin \frac{\pi}{3}(-0.02) = \pi\left(\sqrt{3}/2\right)(0.02) = 0.01\pi\sqrt{3} \approx 0.054.$

17. (a) $y = \sqrt{3 + x^2} \quad \Rightarrow \quad dy = \dfrac{1}{2}(3 + x^2)^{-1/2}(2x)\, dx = \dfrac{x}{\sqrt{3 + x^2}}\, dx$

(b) $x = 1$ and $dx = -0.1 \quad \Rightarrow \quad dy = \dfrac{1}{\sqrt{3 + 1^2}}(-0.1) = \dfrac{1}{2}(-0.1) = -0.05.$

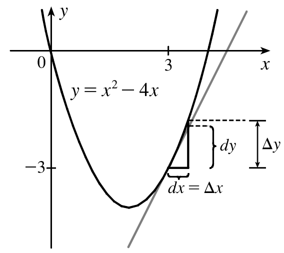
18. (a) $y = \dfrac{x + 1}{x - 1} \quad \Rightarrow \quad dy = \dfrac{(x - 1)(1) - (x + 1)(1)}{(x - 1)^2}\, dx = \dfrac{-2}{(x - 1)^2}\, dx$

(b) $x = 2$ and $dx = 0.05 \quad \Rightarrow \quad dy = \dfrac{-2}{(2 - 1)^2}(0.05) = -2(0.05) = -0.1.$

19. $y = f(x) = x^2 - 4x,\ x = 3,\ \Delta x = 0.5 \quad \Rightarrow$

$\Delta y = f(3.5) - f(3) = -1.75 - (-3) = 1.25$

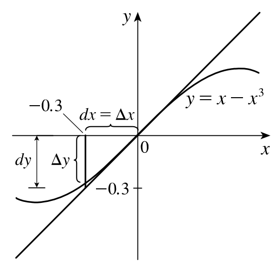
$dy = f'(x)\, dx = (2x - 4)\, dx = (6 - 4)(0.5) = 1$



20. $y = f(x) = x - x^3,\ x = 0,\ \Delta x = -0.3 \quad \Rightarrow$

$\Delta y = f(-0.3) - f(0) = -0.273 - 0 = -0.273$

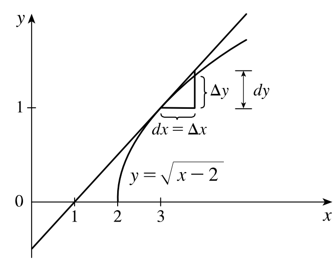
$dy = f'(x)\, dx = (1 - 3x^2)\, dx = (1 - 0)(-0.3) = -0.3$



21. $y = f(x) = \sqrt{x - 2},\ x = 3,\ \Delta x = 0.8 \quad \Rightarrow$

$\Delta y = f(3.8) - f(3) = \sqrt{1.8} - 1 \approx 0.34$

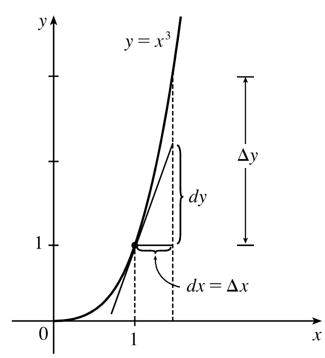
$dy = f'(x)\, dx = \dfrac{1}{2\sqrt{x - 2}}\, dx = \dfrac{1}{2(1)}(0.8) = 0.4$



22. $y = x^3,\ x = 1,\ \Delta x = 0.5 \quad \Rightarrow$

$\Delta y = (1.5)^3 - 1^3 = 3.375 - 1 = 2.375.$

$dy = 3x^2\, dx = 3(1)^2(0.5) = 1.5$



23. To estimate $(1.999)^4$, we'll find the linearization of $f(x) = x^4$ at $a = 2$. Since $f'(x) = 4x^3$, $f(2) = 16$, and $f'(2) = 32$, we have $L(x) = 16 + 32(x - 2)$. Thus, $x^4 \approx 16 + 32(x - 2)$ when x is near 2, so $(1.999)^4 \approx 16 + 32(1.999 - 2) = 16 - 0.032 = 15.968$.

24. $y = f(x) = 1/x \quad\Rightarrow\quad dy = -1/x^2\,dx$. When $x = 4$ and $dx = 0.002$, $dy = -\frac{1}{16}(0.002) = -\frac{1}{8000}$, so $\frac{1}{4.002} \approx f(4) + dy = \frac{1}{4} - \frac{1}{8000} = \frac{1999}{8000} = 0.249875$.

25. $y = f(x) = \sqrt[3]{x} \quad\Rightarrow\quad dy = \frac{1}{3}x^{-2/3}\,dx$. When $x = 1000$ and $dx = 1$, $dy = \frac{1}{3}(1000)^{-2/3}(1) = \frac{1}{300}$, so $\sqrt[3]{1001} = f(1001) \approx f(1000) + dy = 10 + \frac{1}{300} = 10.00\overline{3} \approx 10.003$.

26. $y = f(x) = \sqrt{x} \quad\Rightarrow\quad dy = \frac{1}{2}x^{-1/2}\,dx$. When $x = 100$ and $dx = 0.5$, $dy = \frac{1}{2}(100)^{-1/2}\left(\frac{1}{2}\right) = \frac{1}{40}$, so $\sqrt{100.5} = f(100.5) \approx f(100) + dy = 10 + \frac{1}{40} = 10.025$.

27. $y = f(x) = \tan x \quad\Rightarrow\quad dy = \sec^2 x\,dx$. When $x = 0°$ [i.e., 0 radians] and $dx = 2°$ [i.e., $\frac{\pi}{90}$ radians], $dy = (\sec^2 0)\left(\frac{\pi}{90}\right) = 1^2\left(\frac{\pi}{90}\right) = \frac{\pi}{90}$, so $\tan 2° = f(2°) \approx f(0°) + dy = 0 + \frac{\pi}{90} = \frac{\pi}{90} \approx 0.0349$.

28. $y = f(x) = \cos x \quad\Rightarrow\quad dy = -\sin x\,dx$. When $x = 30°$ $[\pi/6]$ and $dx = -1°$ $[-\pi/180]$, $dy = \left(-\sin\frac{\pi}{6}\right)\left(-\frac{\pi}{180}\right) = -\frac{1}{2}\left(-\frac{\pi}{180}\right) = \frac{\pi}{360}$, so $\cos 29° = f(29°) \approx f(30°) + dy = \frac{1}{2}\sqrt{3} + \frac{\pi}{360} \approx 0.875$.

29. $y = f(x) = \sec x \quad\Rightarrow\quad f'(x) = \sec x\,\tan x$, so $f(0) = 1$ and $f'(0) = 1 \cdot 0 = 0$. The linear approximation of f at 0 is $f(0) + f'(0)(x - 0) = 1 + 0(x) = 1$. Since 0.08 is close to 0, approximating $\sec 0.08$ with 1 is reasonable.

30. $y = f(x) = \sqrt{x} \quad\Rightarrow\quad f'(x) = 1/(2\sqrt{x})$, so $f(4) = 2$ and $f'(4) = \frac{1}{4}$. The linear approximation of f at 4 is $f(4) + f'(4)(x - 4) = 2 + \frac{1}{4}(x - 4)$. Now $f(4.02) = \sqrt{4.02} \approx 2 + \frac{1}{4}(0.02) = 2 + 0.005 = 2.005$, so the approximation is reasonable.

31. (a) If x is the edge length, then $V = x^3 \quad\Rightarrow\quad dV = 3x^2\,dx$. When $x = 30$ and $dx = 0.1$, $dV = 3(30)^2(0.1) = 270$, so the maximum possible error in computing the volume of the cube is about 270 cm^3. The relative error is calculated by dividing the change in V, ΔV, by V. We approximate ΔV with dV.

Relative error $= \dfrac{\Delta V}{V} \approx \dfrac{dV}{V} = \dfrac{3x^2\,dx}{x^3} = 3\,\dfrac{dx}{x} = 3\left(\dfrac{0.1}{30}\right) = 0.01$.

Percentage error = relative error $\times$ 100% $= 0.01 \times 100\% = 1\%$.

(b) $S = 6x^2 \quad\Rightarrow\quad dS = 12x\,dx$. When $x = 30$ and $dx = 0.1$, $dS = 12(30)(0.1) = 36$, so the maximum possible error in computing the surface area of the cube is about 36 cm^2.

Relative error $= \dfrac{\Delta S}{S} \approx \dfrac{dS}{S} = \dfrac{12x\,dx}{6x^2} = 2\,\dfrac{dx}{x} = 2\left(\dfrac{0.1}{30}\right) = 0.00\overline{6}$.

Percentage error = relative error $\times$ 100% $= 0.00\overline{6} \times 100\% = 0.\overline{6}\%$.

32. (a) $A = \pi r^2 \;\Rightarrow\; dA = 2\pi r\,dr$. When $r = 24$ and $dr = 0.2$, $dA = 2\pi(24)(0.2) = 9.6\pi$, so the maximum possible error in the calculated area of the disk is about $9.6\pi \approx 30\text{ cm}^2$.

(b) Relative error $= \dfrac{\Delta A}{A} \approx \dfrac{dA}{A} = \dfrac{2\pi r\,dr}{\pi r^2} = \dfrac{2\,dr}{r} = \dfrac{2(0.2)}{24} = \dfrac{0.2}{12} = \dfrac{1}{60} = 0.01\overline{6}$.

Percentage error $=$ relative error $\times 100\% = 0.01\overline{6} \times 100\% = 1.\overline{6}\%$.

33. (a) For a sphere of radius r, the circumference is $C = 2\pi r$ and the surface area is $S = 4\pi r^2$, so

$r = \dfrac{C}{2\pi} \;\Rightarrow\; S = 4\pi\left(\dfrac{C}{2\pi}\right)^2 = \dfrac{C^2}{\pi} \;\Rightarrow\; dS = \dfrac{2}{\pi}C\,dC$. When $C = 84$ and $dC = 0.5$, $dS = \dfrac{2}{\pi}(84)(0.5) = \dfrac{84}{\pi}$,

so the maximum error is about $\dfrac{84}{\pi} \approx 27\text{ cm}^2$. Relative error $\approx \dfrac{dS}{S} = \dfrac{84/\pi}{84^2/\pi} = \dfrac{1}{84} \approx 0.012 = 1.2\%$

(b) $V = \dfrac{4}{3}\pi r^3 = \dfrac{4}{3}\pi\left(\dfrac{C}{2\pi}\right)^3 = \dfrac{C^3}{6\pi^2} \;\Rightarrow\; dV = \dfrac{1}{2\pi^2}C^2\,dC$. When $C = 84$ and $dC = 0.5$,

$dV = \dfrac{1}{2\pi^2}(84)^2(0.5) = \dfrac{1764}{\pi^2}$, so the maximum error is about $\dfrac{1764}{\pi^2} \approx 179\text{ cm}^3$.

The relative error is approximately $\dfrac{dV}{V} = \dfrac{1764/\pi^2}{(84)^3/(6\pi^2)} = \dfrac{1}{56} \approx 0.018 = 1.8\%$.

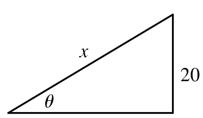
34. For a hemispherical dome, $V = \frac{2}{3}\pi r^3 \;\Rightarrow\; dV = 2\pi r^2\,dr$. When $r = \frac{1}{2}(50) = 25$ m and $dr = 0.05\text{ cm} = 0.0005$ m, $dV = 2\pi(25)^2(0.0005) = \frac{5\pi}{8}$, so the amount of paint needed is about $\frac{5\pi}{8} \approx 2\text{ m}^3$.

35. (a) $V = \pi r^2 h \;\Rightarrow\; \Delta V \approx dV = 2\pi rh\,dr = 2\pi rh\,\Delta r$

(b) The error is

$\Delta V - dV = [\pi(r+\Delta r)^2 h - \pi r^2 h] - 2\pi rh\,\Delta r = \pi r^2 h + 2\pi rh\,\Delta r + \pi(\Delta r)^2 h - \pi r^2 h - 2\pi rh\,\Delta r = \pi(\Delta r)^2 h$.

36. (a) $\sin\theta = \dfrac{20}{x} \;\Rightarrow\; x = 20\csc\theta \;\Rightarrow$



$dx = 20(-\csc\theta\,\cot\theta)\,d\theta = -20\csc 30^\circ\,\cot 30^\circ\,(\pm 1^\circ)$

$= -20(2)(\sqrt{3})\left(\pm\dfrac{\pi}{180}\right) = \pm\dfrac{2\sqrt{3}}{9}\pi$

So the maximum error is about $\pm\frac{2}{9}\sqrt{3}\,\pi \approx \pm 1.21$ cm.

(b) The relative error is $\dfrac{\Delta x}{x} \approx \dfrac{dx}{x} = \dfrac{\pm\frac{2}{9}\sqrt{3}\,\pi}{20(2)} = \pm\dfrac{\sqrt{3}}{180}\pi \approx \pm 0.03$, so the percentage error is approximately $\pm 3\%$.

37. $V = RI \;\Rightarrow\; I = \dfrac{V}{R} \;\Rightarrow\; dI = -\dfrac{V}{R^2}\,dR$. The relative error in calculating I is $\dfrac{\Delta I}{I} \approx \dfrac{dI}{I} = \dfrac{-(V/R^2)\,dR}{V/R} = -\dfrac{dR}{R}$.

Hence, the relative error in calculating I is approximately the same (in magnitude) as the relative error in R.

38. $F = kR^4 \;\Rightarrow\; dF = 4kR^3\,dR \;\Rightarrow\; \dfrac{dF}{F} = \dfrac{4kR^3\,dR}{kR^4} = 4\left(\dfrac{dR}{R}\right)$. Thus, the relative change in F is about 4 times the relative change in R. So a 5% increase in the radius corresponds to a 20% increase in blood flow.

39. (a) $dc = \dfrac{dc}{dx}\,dx = 0\,dx = 0$ (b) $d(cu) = \dfrac{d}{dx}(cu)\,dx = c\,\dfrac{du}{dx}\,dx = c\,du$

(c) $d(u+v) = \dfrac{d}{dx}(u+v)\,dx = \left(\dfrac{du}{dx} + \dfrac{dv}{dx}\right)dx = \dfrac{du}{dx}\,dx + \dfrac{dv}{dx}\,dx = du + dv$

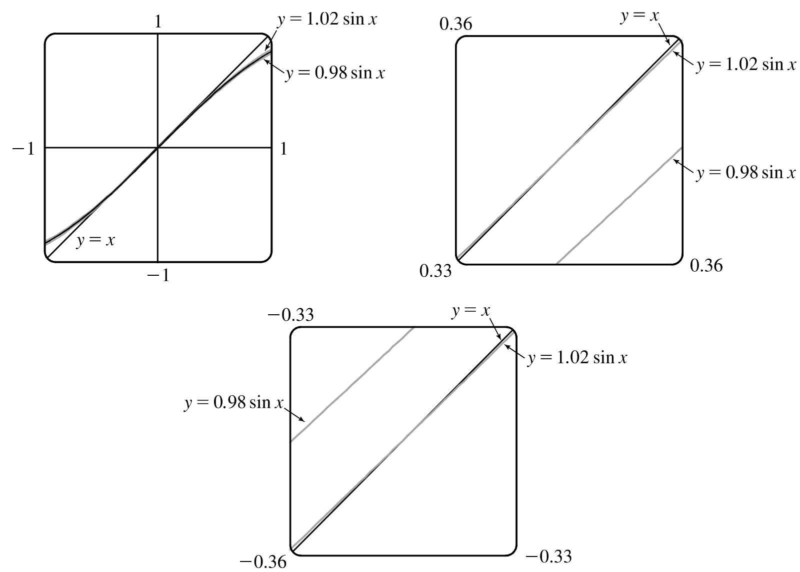
(d) $d(uv) = \dfrac{d}{dx}(uv)\,dx = \left(u\,\dfrac{dv}{dx} + v\,\dfrac{du}{dx}\right)dx = u\,\dfrac{dv}{dx}\,dx + v\,\dfrac{du}{dx}\,dx = u\,dv + v\,du$

(e) $d\left(\dfrac{u}{v}\right) = \dfrac{d}{dx}\left(\dfrac{u}{v}\right)\,dx = \dfrac{v\,\dfrac{du}{dx} - u\,\dfrac{dv}{dx}}{v^2}\,dx = \dfrac{v\,\dfrac{du}{dx}\,dx - u\,\dfrac{dv}{dx}\,dx}{v^2} = \dfrac{v\,du - u\,dv}{v^2}$

(f) $d\,(x^n) = \dfrac{d}{dx}\,(x^n)\,dx = nx^{n-1}\,dx$

40. (a) $f(x) = \sin x \;\;\Rightarrow\;\; f'(x) = \cos x$, so $f(0) = 0$ and $f'(0) = 1$. Thus, $f(x) \approx f(0) + f'(0)(x-0) = 0 + 1(x-0) = x$.

(b)



We want to know the values of x for which $y = x$ approximates $y = \sin x$ with less than a 2% difference; that is, the values of x for which

$$\left|\frac{x - \sin x}{\sin x}\right| < 0.02 \quad\Leftrightarrow\quad -0.02 < \frac{x - \sin x}{\sin x} < 0.02 \quad\Leftrightarrow$$

$$\begin{cases} -0.02\sin x < x - \sin x < 0.02\sin x & \text{if } \sin x > 0 \\ -0.02\sin x > x - \sin x > 0.02\sin x & \text{if } \sin x < 0 \end{cases} \quad\Leftrightarrow\quad \begin{cases} 0.98\sin x < x < 1.02\sin x & \text{if } \sin x > 0 \\ 1.02\sin x < x < 0.98\sin x & \text{if } \sin x < 0 \end{cases}$$

In the first figure, we see that the graphs are very close to each other near $x = 0$. Changing the viewing rectangle and using an intersect feature (see the second figure) we find that $y = x$ intersects $y = 1.02\sin x$ at $x \approx 0.344$. By symmetry, they also intersect at $x \approx -0.344$ (see the third figure). Converting 0.344 radians to degrees, we get $0.344\left(\frac{180^\circ}{\pi}\right) \approx 19.7^\circ \approx 20^\circ$, which verifies the statement.

41. (a) The graph shows that $f'(1) = 2$, so $L(x) = f(1) + f'(1)(x-1) = 5 + 2(x-1) = 2x + 3$.

$f(0.9) \approx L(0.9) = 4.8$ and $f(1.1) \approx L(1.1) = 5.2$.

(b) From the graph, we see that $f'(x)$ is positive and decreasing. This means that the slopes of the tangent lines are positive, but the tangents are becoming less steep. So the tangent lines lie *above* the curve. Thus, the estimates in part (a) are too large.

42. (a) $g'(x) = \sqrt{x^2 + 5} \Rightarrow g'(2) = \sqrt{9} = 3$. $g(1.95) \approx g(2) + g'(2)(1.95 - 2) = -4 + 3(-0.05) = -4.15$.
$g(2.05) \approx g(2) + g'(2)(2.05 - 2) = -4 + 3(0.05) = -3.85$.

(b) The formula $g'(x) = \sqrt{x^2 + 5}$ shows that $g'(x)$ is positive and increasing. This means that the slopes of the tangent lines are positive and the tangents are getting steeper. So the tangent lines lie *below* the graph of g. Hence, the estimates in part (a) are too small.

LABORATORY PROJECT Taylor Polynomials

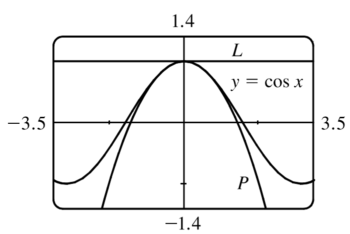
1. We first write the functions described in conditions (i), (ii), and (iii):

$$P(x) = A + Bx + Cx^2 \qquad f(x) = \cos x$$
$$P'(x) = B + 2Cx \qquad f'(x) = -\sin x$$
$$P''(x) = 2C \qquad f''(x) = -\cos x$$

So, taking $a = 0$, our three conditions become

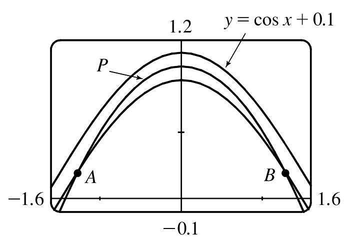
$$P(0) = f(0): \quad A = \cos 0 = 1$$
$$P'(0) = f'(0): \quad B = -\sin 0 = 0$$
$$P''(0) = f''(0): \quad 2C = -\cos 0 = -1 \Rightarrow C = -\tfrac{1}{2}$$

The desired quadratic function is $P(x) = 1 - \frac{1}{2}x^2$, so the quadratic approximation is $\cos x \approx 1 - \frac{1}{2}x^2$.



The figure shows a graph of the cosine function together with its linear approximation $L(x) = 1$ and quadratic approximation $P(x) = 1 - \frac{1}{2}x^2$ near 0. You can see that the quadratic approximation is much better than the linear one.

2. Accuracy to within 0.1 means that $\left|\cos x - \left(1 - \frac{1}{2}x^2\right)\right| < 0.1 \Leftrightarrow -0.1 < \cos x - \left(1 - \frac{1}{2}x^2\right) < 0.1 \Leftrightarrow$
$0.1 > \left(1 - \frac{1}{2}x^2\right) - \cos x > -0.1 \Leftrightarrow \cos x + 0.1 > 1 - \frac{1}{2}x^2 > \cos x - 0.1 \Leftrightarrow \cos x - 0.1 < 1 - \frac{1}{2}x^2 < \cos x + 0.1$.



From the figure we see that this is true between A and B. Zooming in or using an intersect feature, we find that the x-coordinates of B and A are about ± 1.26. Thus, the approximation $\cos x \approx 1 - \frac{1}{2}x^2$ is accurate to within 0.1 when $-1.26 < x < 1.26$.

3. If $P(x) = A + B(x-a) + C(x-a)^2$, then $P'(x) = B + 2C(x-a)$ and $P''(x) = 2C$. Applying the conditions (i), (ii), and (iii), we get

$$P(a) = f(a): \quad A = f(a)$$
$$P'(a) = f'(a): \quad B = f'(a)$$
$$P''(a) = f''(a): \quad 2C = f''(a) \quad \Rightarrow \quad C = \tfrac{1}{2}f''(a)$$

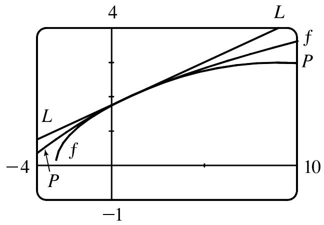
Thus, $P(x) = A + B(x-a) + C(x-a)^2$ can be written in the form $P(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$.

4. From Example 2.9.1, we have $f(1) = 2$, $f'(1) = \frac{1}{4}$, and $f'(x) = \frac{1}{2}(x+3)^{-1/2}$.

So $f''(x) = -\frac{1}{4}(x+3)^{-3/2} \quad \Rightarrow \quad f''(1) = -\frac{1}{32}$.

From Problem 3, the quadratic approximation $P(x)$ is

$\sqrt{x+3} \approx f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2 = 2 + \frac{1}{4}(x-1) - \frac{1}{64}(x-1)^2$.



The figure shows the function $f(x) = \sqrt{x+3}$ together with its linear approximation $L(x) = \frac{1}{4}x + \frac{7}{4}$ and its quadratic approximation $P(x)$. You can see that $P(x)$ is a better approximation than $L(x)$ and this is borne out by the numerical values in the following chart.

	from $L(x)$	actual value	from $P(x)$
$\sqrt{3.98}$	1.9950	1.99499373 . . .	1.99499375
$\sqrt{4.05}$	2.0125	2.01246118 . . .	2.01246094
$\sqrt{4.2}$	2.0500	2.04939015 . . .	2.04937500

5. $T_n(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots + c_n(x-a)^n$. If we put $x = a$ in this equation, then all terms after the first are 0 and we get $T_n(a) = c_0$. Now we differentiate $T_n(x)$ and obtain $T_n'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots + nc_n(x-a)^{n-1}$. Substituting $x = a$ gives $T_n'(a) = c_1$. Differentiating again, we have $T_n''(x) = 2c_2 + 2\cdot 3c_3(x-a) + 3\cdot 4c_4(x-a^2) + \cdots + (n-1)nc_n(x-a)^{n-2}$ and so $T_n''(a) = 2c_2$. Continuing in this manner, we get $T_n'''(x) = 2\cdot 3c_3 + 2\cdot 3\cdot 4c_4(x-a) + \cdots + (n-2)(n-1)nc_n(x-a)^{n-3}$ and $T_n'''(a) = 2\cdot 3c_3$. By now we see the pattern. If we continue to differentiate and substitute $x = a$, we obtain $T_n^{(4)}(a) = 2\cdot 3\cdot 4c_4$ and in general, for any integer k between 1 and n, $T_n^{(k)}(a) = 2\cdot 3\cdot 4\cdot 5\cdot \cdots \cdot kc_k = k!\,c_k \quad \Rightarrow$ $c_k = \dfrac{T_n^{(k)}(a)}{k!}$. Because we want T_n and f to have the same derivatives at a, we require that $c_k = \dfrac{f^{(k)}(a)}{k!}$ for $k = 1, 2, \ldots, n$.

6. $T_n(x) = f(a) + f'(a)(x-a) + \dfrac{f''(a)}{2!}(x-a)^2 + \cdots + \dfrac{f^{(n)}(a)}{n!}(x-a)^n$. To compute the coefficients in this equation we need to calculate the derivatives of f at 0:

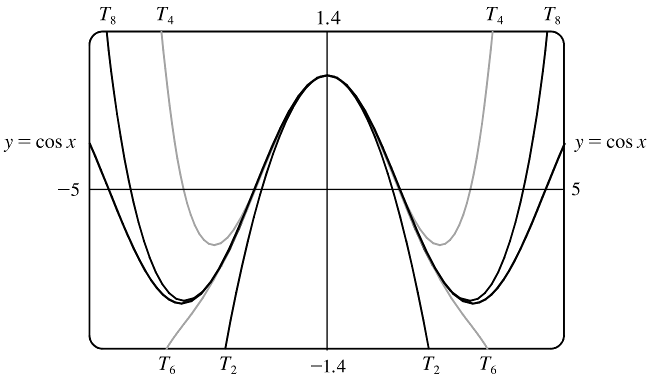
$$f(x) = \cos x \qquad f(0) = \cos 0 = 1$$
$$f'(x) = -\sin x \qquad f'(0) = -\sin 0 = 0$$
$$f''(x) = -\cos x \qquad f''(0) = -1$$
$$f'''(x) = \sin x \qquad f'''(0) = 0$$
$$f^{(4)}(x) = \cos x \qquad f^{(4)}(0) = 1$$

We see that the derivatives repeat in a cycle of length 4, so $f^{(5)}(0) = 0$, $f^{(6)}(0) = -1$, $f^{(7)}(0) = 0$, and $f^{(8)}(0) = 1$. From the original expression for $T_n(x)$, with $n = 8$ and $a = 0$, we have

$$T_8(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \cdots + \frac{f^{(8)}(0)}{8!}(x-0)^8$$

$$= 1 + 0 \cdot x + \frac{-1}{2!}x^2 + 0 \cdot x^3 + \frac{1}{4!}x^4 + 0 \cdot x^5 + \frac{-1}{6!}x^6 + 0 \cdot x^7 + \frac{1}{8!}x^8 = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}$$

and the desired approximation is $\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}$. The Taylor polynomials T_2, T_4, and T_6 consist of the initial terms of T_8 up through degree 2, 4, and 6, respectively. Therefore, $T_2(x) = 1 - \frac{x^2}{2!}$, $T_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$, and $T_6(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$. We graph T_2, T_4, T_6, T_8, and f:



Notice that $T_2(x)$ is a good approximation to $\cos x$ near 0, $T_4(x)$ is a good approximation on a larger interval, $T_6(x)$ is a better approximation, and $T_8(x)$ is better still. Each successive Taylor polynomial is a good approximation on a larger interval than the previous one.

2 Review

TRUE-FALSE QUIZ

1. False. See the note after Theorem 2.2.4.

2. True. This is the Sum Rule.

3. False. See the warning before the Product Rule.

4. True. This is the Chain Rule.

5. True. $\frac{d}{dx}\sqrt{f(x)} = \frac{d}{dx}[f(x)]^{1/2} = \frac{1}{2}[f(x)]^{-1/2}\,f'(x) = \frac{f'(x)}{2\sqrt{f(x)}}$

6. False. $\frac{d}{dx}\,f(\sqrt{x}\,) = f'(\sqrt{x}\,)\cdot \tfrac{1}{2}x^{-1/2} = \frac{f'(\sqrt{x}\,)}{2\sqrt{x}}$, which is not $\frac{f'(x)}{2\sqrt{x}}$.

7. False. $f(x) = \left|x^2 + x\right| = x^2 + x$ for $x \ge 0$ or $x \le -1$ and $\left|x^2 + x\right| = -(x^2 + x)$ for $-1 < x < 0$. So $f'(x) = 2x + 1$ for $x > 0$ or $x < -1$ and $f'(x) = -(2x + 1)$ for $-1 < x < 0$. But $|2x + 1| = 2x + 1$ for $x \ge -\frac{1}{2}$ and $|2x + 1| = -2x - 1$ for $x < -\frac{1}{2}$.

8. True. $f'(r)$ exists $\Rightarrow$ f is differentiable at r $\Rightarrow$ f is continuous at r $\Rightarrow$ $\lim_{x\to r} f(x) = f(r)$.

9. True. $g(x) = x^5 \Rightarrow g'(x) = 5x^4 \Rightarrow g'(2) = 5(2)^4 = 80$, and by the definition of the derivative,
$\lim_{x\to 2} \dfrac{g(x) - g(2)}{x - 2} = g'(2) = 5(2)^4 = 80$.

10. False. $\dfrac{d^2y}{dx^2}$ is the second derivative while $\left(\dfrac{dy}{dx}\right)^2$ is the first derivative squared. For example, if $y = x$, then $\dfrac{d^2y}{dx^2} = 0$, but $\left(\dfrac{dy}{dx}\right)^2 = 1$.

11. False. A tangent line to the parabola $y = x^2$ has slope $dy/dx = 2x$, so at $(-2, 4)$ the slope of the tangent is $2(-2) = -4$ and an equation of the tangent line is $y - 4 = -4(x + 2)$. [The given equation, $y - 4 = 2x(x + 2)$, is not even linear!]

12. True. $\dfrac{d}{dx}(\tan^2 x) = 2\tan x \sec^2 x$, and $\dfrac{d}{dx}(\sec^2 x) = 2\sec x(\sec x \tan x) = 2\tan x \sec^2 x$.
Or: $\dfrac{d}{dx}(\sec^2 x) = \dfrac{d}{dx}(1 + \tan^2 x) = \dfrac{d}{dx}(\tan^2 x)$.

13. True. If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, then $p'(x) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1$, which is a polynomial.

14. True. If $r(x) = \dfrac{p(x)}{q(x)}$, then $r'(x) = \dfrac{q(x)p'(x) - p(x)q'(x)}{[q(x)]^2}$, which is a quotient of polynomials, that is, a rational function.

15. True. $f(x) = (x^6 - x^4)^5$ is a polynomial of degree 30, so its 31st derivative, $f^{(31)}(x)$, is 0.

EXERCISES

1. (a) $s = s(t) = 1 + 2t + t^2/4$. The average velocity over the time interval $[1, 1+h]$ is

$$v_{\text{ave}} = \frac{s(1+h) - s(1)}{(1+h) - 1} = \frac{1 + 2(1+h) + (1+h)^2/4 - 13/4}{h} = \frac{10h + h^2}{4h} = \frac{10 + h}{4}$$

So for the following intervals the average velocities are:

(i) $[1, 3]$: $h = 2$, $v_{\text{ave}} = (10 + 2)/4 = 3$ m/s

(ii) $[1, 2]$: $h = 1$, $v_{\text{ave}} = (10 + 1)/4 = 2.75$ m/s

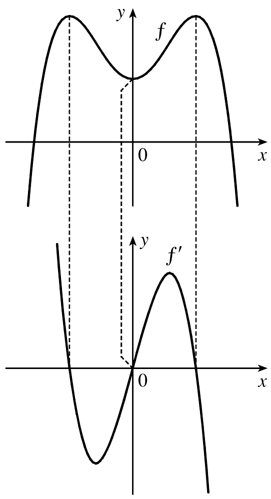
(iii) $[1, 1.5]$: $h = 0.5$, $v_{\text{ave}} = (10 + 0.5)/4 = 2.625$ m/s

(iv) $[1, 1.1]$: $h = 0.1$, $v_{\text{ave}} = (10 + 0.1)/4 = 2.525$ m/s

(b) When $t = 1$, the instantaneous velocity is $\lim_{h\to 0} \dfrac{s(1+h) - s(1)}{h} = \lim_{h\to 0} \dfrac{10 + h}{4} = \dfrac{10}{4} = 2.5$ m/s.

2. f is not differentiable: at $x = -4$ because f is not continuous, at $x = -1$ because f has a corner, at $x = 2$ because f is not continuous, and at $x = 5$ because f has a vertical tangent.

3.



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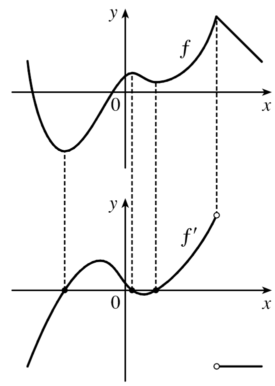


5. The graph of a has tangent lines with positive slope for $x < 0$ and negative slope for $x > 0$, and the values of c fit this pattern, so c must be the graph of the derivative of the function for a. The graph of c has horizontal tangent lines to the left and right of the x-axis and b has zeros at these points. Hence, b is the graph of the derivative of the function for c. Therefore, a is the graph of f, c is the graph of f', and b is the graph of f''.

6. $2^6 = 64$, so $f(x) = x^6$ and $a = 2$.

7. (a) $f'(r)$ is the rate at which the total cost changes with respect to the interest rate. Its units are dollars/(percent per year).

(b) The total cost of paying off the loan is increasing by \$1200/(percent per year) as the interest rate reaches 10%. So if the interest rate goes up from 10% to 11%, the cost goes up approximately \$1200.

(c) As r increases, C increases. So $f'(r)$ will always be positive.

8. (a) Drawing slope triangles, we obtain the following estimates: $F'(1950) \approx \frac{1.1}{10} = 0.11$, $F'(1965) \approx \frac{-1.6}{10} = -0.16$, and $F'(1987) \approx \frac{0.2}{10} = 0.02$.

(b) The rate of change of the average number of children born to each woman was increasing by 0.11 in 1950, decreasing by 0.16 in 1965, and increasing by 0.02 in 1987.

(c) There are many possible reasons:

- In the baby-boom era (post-WWII), there was optimism about the economy and family size was rising.
- In the baby-bust era, there was less economic optimism, and it was considered less socially responsible to have a large family.
- In the baby-boomlet era, there was increased economic optimism and a return to more conservative attitudes.

9. (a) $P'(t)$ is the rate at which the percentage of Americans under the age of 18 is changing with respect to time. Its units are percent per year (%/yr).

(b) To find $P'(t)$, we use $\lim_{h\to 0} \dfrac{P(t+h)-P(t)}{h} \approx \dfrac{P(t+h)-P(t)}{h}$ for small values of h.

For 1950: $P'(1950) \approx \dfrac{P(1960)-P(1950)}{1960-1950} = \dfrac{35.7-31.1}{10} = 0.46$

For 1960: We estimate $P'(1960)$ by using $h=-10$ and $h=10$, and then average the two results to obtain a final estimate.

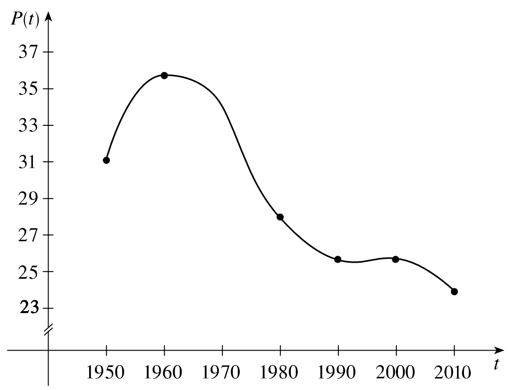
$$h=-10 \quad\Rightarrow\quad P'(1960) \approx \frac{P(1950)-P(1960)}{1950-1960} = \frac{31.1-35.7}{-10} = 0.46$$

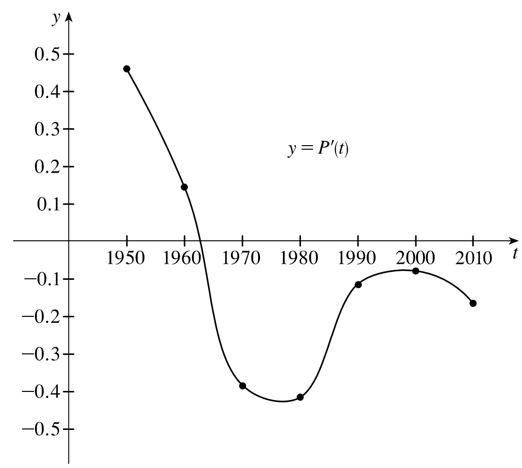
$$h=10 \quad\Rightarrow\quad P'(1960) \approx \frac{P(1970)-P(1960)}{1970-1960} = \frac{34.0-35.7}{10} = -0.17$$

So we estimate that $P'(1960) \approx \frac{1}{2}[0.46+(-0.17)] = 0.145$.

t	1950	1960	1970	1980	1990	2000	2010
$P'(t)$	0.460	0.145	−0.385	−0.415	−0.115	−0.085	−0.170

(c)





(d) We could get more accurate values for $P'(t)$ by obtaining data for the mid-decade years 1955, 1965, 1975, 1985, 1995, and 2005.

10. $f(x) = \dfrac{4-x}{3+x} \quad\Rightarrow$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\dfrac{4-(x+h)}{3+(x+h)} - \dfrac{4-x}{3+x}}{h} = \lim_{h\to 0}\frac{(4-x-h)(3+x)-(4-x)(3+x+h)}{h(3+x+h)(3+x)}$$

$$= \lim_{h\to 0}\frac{-7h}{h(3+x+h)(3+x)} = \lim_{h\to 0}\frac{-7}{(3+x+h)(3+x)} = -\frac{7}{(3+x)^2}$$

11. $f(x) = x^3+5x+4 \quad\Rightarrow$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^3+5(x+h)+4-(x^3+5x+4)}{h}$$

$$= \lim_{h\to 0}\frac{3x^2h+3xh^2+h^3+5h}{h} = \lim_{h\to 0}(3x^2+3xh+h^2+5) = 3x^2+5$$

12. (a) $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{\sqrt{3-5(x+h)}-\sqrt{3-5x}}{h} \dfrac{\sqrt{3-5(x+h)}+\sqrt{3-5x}}{\sqrt{3-5(x+h)}+\sqrt{3-5x}}$

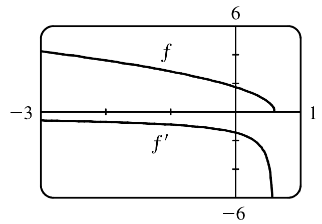
$$= \lim_{h\to 0} \frac{[3-5(x+h)]-(3-5x)}{h\left(\sqrt{3-5(x+h)}+\sqrt{3-5x}\right)} = \lim_{h\to 0} \frac{-5}{\sqrt{3-5(x+h)}+\sqrt{3-5x}} = \frac{-5}{2\sqrt{3-5x}}$$

(b) Domain of f: (the radicand must be nonnegative) $3-5x \geq 0 \;\Rightarrow$

$5x \leq 3 \;\Rightarrow\; x \in \left(-\infty, \frac{3}{5}\right]$

Domain of f': exclude $\frac{3}{5}$ because it makes the denominator zero;

$x \in \left(-\infty, \frac{3}{5}\right)$



(c) Our answer to part (a) is reasonable because $f'(x)$ is always negative and f is always decreasing.

13. $y = (x^2+x^3)^4 \;\Rightarrow\; y' = 4(x^2+x^3)^3(2x+3x^2) = 4(x^2)^3(1+x)^3x(2+3x) = 4x^7(x+1)^3(3x+2)$

14. $y = \dfrac{1}{\sqrt{x}} - \dfrac{1}{\sqrt[5]{x^3}} = x^{-1/2} - x^{-3/5} \;\Rightarrow\; y' = -\dfrac{1}{2}x^{-3/2} + \dfrac{3}{5}x^{-8/5}$ or $\dfrac{3}{5x\sqrt[5]{x^3}} - \dfrac{1}{2x\sqrt{x}}$ or $\dfrac{1}{10}x^{-8/5}(-5x^{1/10}+6)$

15. $y = \dfrac{x^2-x+2}{\sqrt{x}} = x^{3/2} - x^{1/2} + 2x^{-1/2} \;\Rightarrow\; y' = \dfrac{3}{2}x^{1/2} - \dfrac{1}{2}x^{-1/2} - x^{-3/2} = \dfrac{3}{2}\sqrt{x} - \dfrac{1}{2\sqrt{x}} - \dfrac{1}{\sqrt{x^3}}$

16. $y = \dfrac{\tan x}{1+\cos x} \;\Rightarrow\; y' = \dfrac{(1+\cos x)\sec^2 x - \tan x(-\sin x)}{(1+\cos x)^2} = \dfrac{(1+\cos x)\sec^2 x + \tan x\,\sin x}{(1+\cos x)^2}$

17. $y = x^2 \sin \pi x \;\Rightarrow\; y' = x^2(\cos \pi x)\pi + (\sin \pi x)(2x) = x(\pi x \cos \pi x + 2\sin \pi x)$

18. $y = \left(x + \dfrac{1}{x^2}\right)^{\sqrt{7}} \;\Rightarrow\; y' = \sqrt{7}\left(x + \dfrac{1}{x^2}\right)^{\sqrt{7}-1}\left(1 - \dfrac{2}{x^3}\right)$

19. $y = \dfrac{t^4-1}{t^4+1} \;\Rightarrow\; y' = \dfrac{(t^4+1)4t^3-(t^4-1)4t^3}{(t^4+1)^2} = \dfrac{4t^3[(t^4+1)-(t^4-1)]}{(t^4+1)^2} = \dfrac{8t^3}{(t^4+1)^2}$

20. $y = \sin(\cos x) \;\Rightarrow\; y' = \cos(\cos x)(-\sin x) = -\sin x\,\cos(\cos x)$

21. $y = \tan\sqrt{1-x} \;\Rightarrow\; y' = \left(\sec^2\sqrt{1-x}\right)\left(\dfrac{1}{2\sqrt{1-x}}\right)(-1) = -\dfrac{\sec^2\sqrt{1-x}}{2\sqrt{1-x}}$

22. Using the Reciprocal Rule, $g(x) = \dfrac{1}{f(x)} \;\Rightarrow\; g'(x) = -\dfrac{f'(x)}{[f(x)]^2}$, we have $y = \dfrac{1}{\sin(x-\sin x)} \;\Rightarrow$

$y' = -\dfrac{\cos(x-\sin x)(1-\cos x)}{\sin^2(x-\sin x)}$.

23. $\dfrac{d}{dx}(xy^4 + x^2y) = \dfrac{d}{dx}(x+3y) \;\Rightarrow\; x\cdot 4y^3y' + y^4\cdot 1 + x^2\cdot y' + y\cdot 2x = 1 + 3y' \;\Rightarrow$

$y'(4xy^3 + x^2 - 3) = 1 - y^4 - 2xy \;\Rightarrow\; y' = \dfrac{1-y^4-2xy}{4xy^3+x^2-3}$

24. $y = \sec(1+x^2) \;\Rightarrow\; y' = 2x\sec(1+x^2)\,\tan(1+x^2)$

25. $y = \dfrac{\sec 2\theta}{1+\tan 2\theta} \quad \Rightarrow$

$$y' = \frac{(1+\tan 2\theta)(\sec 2\theta\,\tan 2\theta \cdot 2) - (\sec 2\theta)(\sec^2 2\theta \cdot 2)}{(1+\tan 2\theta)^2} = \frac{2\sec 2\theta\,[(1+\tan 2\theta)\tan 2\theta - \sec^2 2\theta]}{(1+\tan 2\theta)^2}$$
$$= \frac{2\sec 2\theta\,(\tan 2\theta + \tan^2 2\theta - \sec^2 2\theta)}{(1+\tan 2\theta)^2} = \frac{2\sec 2\theta\,(\tan 2\theta - 1)}{(1+\tan 2\theta)^2} \qquad [1+\tan^2 x = \sec^2 x]$$

26. $\dfrac{d}{dx}(x^2\cos y + \sin 2y) = \dfrac{d}{dx}(xy) \quad \Rightarrow \quad x^2(-\sin y \cdot y') + (\cos y)(2x) + \cos 2y \cdot 2y' = x \cdot y' + y \cdot 1 \quad \Rightarrow$

$y'(-x^2 \sin y + 2\cos 2y - x) = y - 2x\cos y \quad \Rightarrow \quad y' = \dfrac{y - 2x\cos y}{2\cos 2y - x^2 \sin y - x}$

27. $y = (1 - x^{-1})^{-1} \quad \Rightarrow$

$y' = -1(1-x^{-1})^{-2}[-(-1x^{-2})] = -(1-1/x)^{-2}x^{-2} = -((x-1)/x)^{-2}x^{-2} = -(x-1)^{-2}$

28. $y = \dfrac{1}{\sqrt[3]{x+\sqrt{x}}} = \left(x+\sqrt{x}\right)^{-1/3} \quad \Rightarrow \quad y' = -\frac{1}{3}\left(x+\sqrt{x}\right)^{-4/3}\left(1+\dfrac{1}{2\sqrt{x}}\right)$

29. $\sin(xy) = x^2 - y \quad \Rightarrow \quad \cos(xy)(xy' + y \cdot 1) = 2x - y' \quad \Rightarrow \quad x\cos(xy)y' + y' = 2x - y\cos(xy) \quad \Rightarrow$

$y'[x\cos(xy) + 1] = 2x - y\cos(xy) \quad \Rightarrow \quad y' = \dfrac{2x - y\cos(xy)}{x\cos(xy)+1}$

30. $y = \sqrt{\sin\sqrt{x}} \quad \Rightarrow \quad y' = \frac{1}{2}\left(\sin\sqrt{x}\right)^{-1/2}\left(\cos\sqrt{x}\right)\left(\dfrac{1}{2\sqrt{x}}\right) = \dfrac{\cos\sqrt{x}}{4\sqrt{x\sin\sqrt{x}}}$

31. $y = \cot(3x^2+5) \quad \Rightarrow \quad y' = -\csc^2(3x^2+5)(6x) = -6x\csc^2(3x^2+5)$

32. $y = \dfrac{(x+\lambda)^4}{x^4+\lambda^4} \quad \Rightarrow \quad y' = \dfrac{(x^4+\lambda^4)(4)(x+\lambda)^3 - (x+\lambda)^4(4x^3)}{(x^4+\lambda^4)^2} = \dfrac{4(x+\lambda)^3(\lambda^4 - \lambda x^3)}{(x^4+\lambda^4)^2}$

33. $y = \sqrt{x}\cos\sqrt{x} \quad \Rightarrow$

$$y' = \sqrt{x}\,\left(\cos\sqrt{x}\right)' + \cos\sqrt{x}\,\left(\sqrt{x}\right)' = \sqrt{x}\left[-\sin\sqrt{x}\,\left(\tfrac{1}{2}x^{-1/2}\right)\right] + \cos\sqrt{x}\,\left(\tfrac{1}{2}x^{-1/2}\right)$$
$$= \tfrac{1}{2}x^{-1/2}\left(-\sqrt{x}\,\sin\sqrt{x} + \cos\sqrt{x}\right) = \frac{\cos\sqrt{x} - \sqrt{x}\,\sin\sqrt{x}}{2\sqrt{x}}$$

34. $y = (\sin mx)/x \quad \Rightarrow \quad y' = (mx\cos mx - \sin mx)/x^2$

35. $y = \tan^2(\sin\theta) = [\tan(\sin\theta)]^2 \quad \Rightarrow \quad y' = 2[\tan(\sin\theta)] \cdot \sec^2(\sin\theta) \cdot \cos\theta$

36. $x\tan y = y - 1 \quad \Rightarrow \quad \tan y + (x\sec^2 y)\,y' = y' \quad \Rightarrow \quad y' = \dfrac{\tan y}{1 - x\sec^2 y}$

37. $y = (x\tan x)^{1/5} \quad \Rightarrow \quad y' = \frac{1}{5}(x\tan x)^{-4/5}(\tan x + x\sec^2 x)$

38. $y = \dfrac{(x-1)(x-4)}{(x-2)(x-3)} = \dfrac{x^2-5x+4}{x^2-5x+6} \quad \Rightarrow \quad y' = \dfrac{(x^2-5x+6)(2x-5) - (x^2-5x+4)(2x-5)}{(x^2-5x+6)^2} = \dfrac{2(2x-5)}{(x-2)^2(x-3)^2}$

39. $y = \sin\left(\tan\sqrt{1+x^3}\right) \quad \Rightarrow \quad y' = \cos\left(\tan\sqrt{1+x^3}\right)\left(\sec^2\sqrt{1+x^3}\right)\left[3x^2/\left(2\sqrt{1+x^3}\right)\right]$

40. $y = \sin^2\left(\cos\sqrt{\sin \pi x}\right) = \left[\sin\left(\cos\sqrt{\sin \pi x}\right)\right]^2 \quad \Rightarrow$

$$\begin{aligned} y' &= 2\left[\sin\left(\cos\sqrt{\sin \pi x}\right)\right]\left[\sin\left(\cos\sqrt{\sin \pi x}\right)\right]' = 2\sin\left(\cos\sqrt{\sin \pi x}\right)\cos\left(\cos\sqrt{\sin \pi x}\right)\left(\cos\sqrt{\sin \pi x}\right)' \\ &= 2\sin\left(\cos\sqrt{\sin \pi x}\right)\cos\left(\cos\sqrt{\sin \pi x}\right)\left(-\sin\sqrt{\sin \pi x}\right)\left(\sqrt{\sin \pi x}\right)' \\ &= -2\sin\left(\cos\sqrt{\sin \pi x}\right)\cos\left(\cos\sqrt{\sin \pi x}\right)\sin\sqrt{\sin \pi x}\cdot\tfrac{1}{2}(\sin \pi x)^{-1/2}(\sin \pi x)' \\ &= \frac{-\sin\left(\cos\sqrt{\sin \pi x}\right)\cos\left(\cos\sqrt{\sin \pi x}\right)\sin\sqrt{\sin \pi x}}{\sqrt{\sin \pi x}}\cdot\cos \pi x\cdot\pi \\ &= \frac{-\pi\sin\left(\cos\sqrt{\sin \pi x}\right)\cos\left(\cos\sqrt{\sin \pi x}\right)\sin\sqrt{\sin \pi x}\,\cos \pi x}{\sqrt{\sin \pi x}} \end{aligned}$$

41. $f(t) = \sqrt{4t+1} \quad \Rightarrow \quad f'(t) = \frac{1}{2}(4t+1)^{-1/2}\cdot 4 = 2(4t+1)^{-1/2} \quad \Rightarrow$

$f''(t) = 2(-\frac{1}{2})(4t+1)^{-3/2}\cdot 4 = -4/(4t+1)^{3/2}$, so $f''(2) = -4/9^{3/2} = -\frac{4}{27}$.

42. $g(\theta) = \theta\sin\theta \quad \Rightarrow \quad g'(\theta) = \theta\cos\theta + \sin\theta\cdot 1 \quad \Rightarrow \quad g''(\theta) = \theta(-\sin\theta) + \cos\theta\cdot 1 + \cos\theta = 2\cos\theta - \theta\sin\theta$,

so $g''(\pi/6) = 2\cos(\pi/6) - (\pi/6)\sin(\pi/6) = 2\left(\sqrt{3}/2\right) - (\pi/6)(1/2) = \sqrt{3} - \pi/12$.

43. $x^6 + y^6 = 1 \quad \Rightarrow \quad 6x^5 + 6y^5y' = 0 \quad \Rightarrow \quad y' = -x^5/y^5 \quad \Rightarrow$

$$y'' = -\frac{y^5(5x^4) - x^5(5y^4y')}{(y^5)^2} = -\frac{5x^4y^4\left[y - x(-x^5/y^5)\right]}{y^{10}} = -\frac{5x^4\left[(y^6+x^6)/y^5\right]}{y^6} = -\frac{5x^4}{y^{11}}$$

44. $f(x) = (2-x)^{-1} \quad \Rightarrow \quad f'(x) = (2-x)^{-2} \quad \Rightarrow \quad f''(x) = 2(2-x)^{-3} \quad \Rightarrow \quad f'''(x) = 2\cdot 3(2-x)^{-4} \quad \Rightarrow$

$f^{(4)}(x) = 2\cdot 3\cdot 4(2-x)^{-5}$. In general, $f^{(n)}(x) = 2\cdot 3\cdot 4\cdot\cdots\cdot n(2-x)^{-(n+1)} = \dfrac{n!}{(2-x)^{(n+1)}}$.

45. $\displaystyle\lim_{x\to 0}\frac{\sec x}{1-\sin x} = \frac{\sec 0}{1-\sin 0} = \frac{1}{1-0} = 1$

46. $\displaystyle\lim_{t\to 0}\frac{t^3}{\tan^3 2t} = \lim_{t\to 0}\frac{t^3\cos^3 2t}{\sin^3 2t} = \lim_{t\to 0}\cos^3 2t\cdot\frac{1}{8\dfrac{\sin^3 2t}{(2t)^3}} = \lim_{t\to 0}\frac{\cos^3 2t}{8\left(\lim\limits_{t\to 0}\dfrac{\sin 2t}{2t}\right)^3} = \frac{1}{8\cdot 1^3} = \frac{1}{8}$

47. $y = 4\sin^2 x \quad \Rightarrow \quad y' = 4\cdot 2\sin x\cos x$. At $\left(\frac{\pi}{6}, 1\right)$, $y' = 8\cdot\frac{1}{2}\cdot\frac{\sqrt{3}}{2} = 2\sqrt{3}$, so an equation of the tangent line is $y - 1 = 2\sqrt{3}\left(x - \frac{\pi}{6}\right)$, or $y = 2\sqrt{3}\,x + 1 - \pi\sqrt{3}/3$.

48. $y = \dfrac{x^2-1}{x^2+1} \quad \Rightarrow \quad y' = \dfrac{(x^2+1)(2x) - (x^2-1)(2x)}{(x^2+1)^2} = \dfrac{4x}{(x^2+1)^2}$.

At $(0,-1)$, $y' = 0$, so an equation of the tangent line is $y + 1 = 0(x-0)$, or $y = -1$.

49. $y = \sqrt{1+4\sin x} \quad \Rightarrow \quad y' = \frac{1}{2}(1+4\sin x)^{-1/2}\cdot 4\cos x = \dfrac{2\cos x}{\sqrt{1+4\sin x}}$.

At $(0,1)$, $y' = \dfrac{2}{\sqrt{1}} = 2$, so an equation of the tangent line is $y - 1 = 2(x-0)$, or $y = 2x+1$.

The slope of the normal line is $-\frac{1}{2}$, so an equation of the normal line is $y - 1 = -\frac{1}{2}(x-0)$, or $y = -\frac{1}{2}x + 1$.

50. $x^2 + 4xy + y^2 = 13 \quad\Rightarrow\quad 2x + 4(xy' + y \cdot 1) + 2yy' = 0 \quad\Rightarrow\quad x + 2xy' + 2y + yy' = 0 \quad\Rightarrow$

$2xy' + yy' = -x - 2y \quad\Rightarrow\quad y'(2x + y) = -x - 2y \quad\Rightarrow\quad y' = \dfrac{-x - 2y}{2x + y}$.

At $(2, 1)$, $y' = \dfrac{-2 - 2}{4 + 1} = -\dfrac{4}{5}$, so an equation of the tangent line is $y - 1 = -\frac{4}{5}(x - 2)$, or $y = -\frac{4}{5}x + \frac{13}{5}$.

The slope of the normal line is $\frac{5}{4}$, so an equation of the normal line is $y - 1 = \frac{5}{4}(x - 2)$, or $y = \frac{5}{4}x - \frac{3}{2}$.

51. (a) $f(x) = x\sqrt{5 - x} \quad\Rightarrow$

$$f'(x) = x\left[\frac{1}{2}(5 - x)^{-1/2}(-1)\right] + \sqrt{5 - x} = \frac{-x}{2\sqrt{5 - x}} + \sqrt{5 - x} \cdot \frac{2\sqrt{5 - x}}{2\sqrt{5 - x}} = \frac{-x}{2\sqrt{5 - x}} + \frac{2(5 - x)}{2\sqrt{5 - x}}$$
$$= \frac{-x + 10 - 2x}{2\sqrt{5 - x}} = \frac{10 - 3x}{2\sqrt{5 - x}}$$

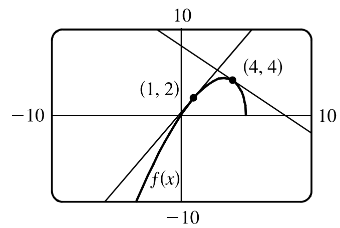
(b) At $(1, 2)$: $f'(1) = \frac{7}{4}$.

So an equation of the tangent line is $y - 2 = \frac{7}{4}(x - 1)$ or $y = \frac{7}{4}x + \frac{1}{4}$.

At $(4, 4)$: $f'(4) = -\frac{2}{2} = -1$.

So an equation of the tangent line is $y - 4 = -1(x - 4)$ or $y = -x + 8$.

(c)



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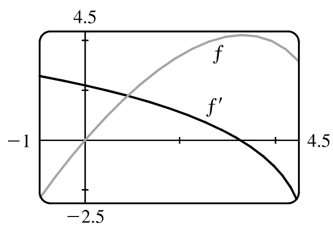
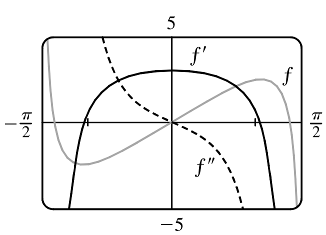


The graphs look reasonable, since f' is positive where f has tangents with positive slope, and f' is negative where f has tangents with negative slope.

52. (a) $f(x) = 4x - \tan x \quad\Rightarrow\quad f'(x) = 4 - \sec^2 x \quad\Rightarrow\quad f''(x) = -2\sec x\,(\sec x\,\tan x) = -2\sec^2 x\,\tan x$.

(b)



We can see that our answers are reasonable, since the graph of f' is 0 where f has a horizontal tangent, and the graph of f' is positive where f has tangents with positive slope and negative where f has tangents with negative slope. The same correspondence holds between the graphs of f' and f''.

53. $y = \sin x + \cos x \quad\Rightarrow\quad y' = \cos x - \sin x = 0 \quad\Leftrightarrow\quad \cos x = \sin x$ and $0 \le x \le 2\pi \quad\Leftrightarrow\quad x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$, so the points are $\left(\frac{\pi}{4}, \sqrt{2}\right)$ and $\left(\frac{5\pi}{4}, -\sqrt{2}\right)$.

54. $x^2 + 2y^2 = 1 \quad\Rightarrow\quad 2x + 4yy' = 0 \quad\Rightarrow\quad y' = -x/(2y) = 1 \quad\Leftrightarrow\quad x = -2y$. Since the points lie on the ellipse, we have $(-2y)^2 + 2y^2 = 1 \quad\Rightarrow\quad 6y^2 = 1 \quad\Rightarrow\quad y = \pm\frac{1}{\sqrt{6}}$. The points are $\left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$ and $\left(\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)$.

55. $y = f(x) = ax^2 + bx + c \quad\Rightarrow\quad f'(x) = 2ax + b$. We know that $f'(-1) = 6$ and $f'(5) = -2$, so $-2a + b = 6$ and $10a + b = -2$. Subtracting the first equation from the second gives $12a = -8 \quad\Rightarrow\quad a = -\frac{2}{3}$. Substituting $-\frac{2}{3}$ for a in the first equation gives $b = \frac{14}{3}$. Now $f(1) = 4 \quad\Rightarrow\quad 4 = a + b + c$, so $c = 4 + \frac{2}{3} - \frac{14}{3} = 0$ and hence, $f(x) = -\frac{2}{3}x^2 + \frac{14}{3}x$.

56. If $y = f(x) = \dfrac{x}{x+1}$, then $f'(x) = \dfrac{(x+1)(1) - x(1)}{(x+1)^2} = \dfrac{1}{(x+1)^2}$. When $x = a$, the equation of the tangent line is

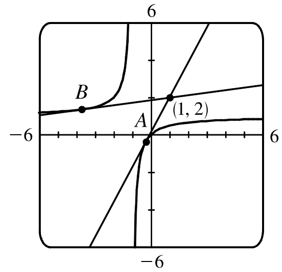
$y - \dfrac{a}{a+1} = \dfrac{1}{(a+1)^2}(x - a)$. This line passes through $(1, 2)$ when $2 - \dfrac{a}{a+1} = \dfrac{1}{(a+1)^2}(1 - a) \quad \Leftrightarrow$

$2(a+1)^2 - a(a+1) = 1 - a \quad \Leftrightarrow \quad 2a^2 + 4a + 2 - a^2 - a - 1 + a = 0 \quad \Leftrightarrow \quad a^2 + 4a + 1 = 0.$

The quadratic formula gives the roots of this equation as $a = \dfrac{-4 \pm \sqrt{4^2 - 4(1)(1)}}{2(1)} = \dfrac{-4 \pm \sqrt{12}}{2} = -2 \pm \sqrt{3}$,

so there are two such tangent lines. Since

$$f\left(-2 \pm \sqrt{3}\right) = \frac{-2 \pm \sqrt{3}}{-2 \pm \sqrt{3} + 1} = \frac{-2 \pm \sqrt{3}}{-1 \pm \sqrt{3}} \cdot \frac{-1 \mp \sqrt{3}}{-1 \mp \sqrt{3}}$$
$$= \frac{2 \pm 2\sqrt{3} \mp \sqrt{3} - 3}{1 - 3} = \frac{-1 \pm \sqrt{3}}{-2} = \frac{1 \mp \sqrt{3}}{2},$$



the lines touch the curve at $A\left(-2 + \sqrt{3}, \frac{1 - \sqrt{3}}{2}\right) \approx (-0.27, -0.37)$

and $B\left(-2 - \sqrt{3}, \frac{1 + \sqrt{3}}{2}\right) \approx (-3.73, 1.37)$.

57. $f(x) = (x - a)(x - b)(x - c) \quad \Rightarrow \quad f'(x) = (x - b)(x - c) + (x - a)(x - c) + (x - a)(x - b)$.

So $\dfrac{f'(x)}{f(x)} = \dfrac{(x-b)(x-c) + (x-a)(x-c) + (x-a)(x-b)}{(x-a)(x-b)(x-c)} = \dfrac{1}{x-a} + \dfrac{1}{x-b} + \dfrac{1}{x-c}$.

58. (a) $\cos 2x = \cos^2 x - \sin^2 x \quad \Rightarrow \quad -2\sin 2x = -2\cos x \sin x - 2\sin x \cos x \quad \Leftrightarrow \quad \sin 2x = 2\sin x \cos x$

(b) $\sin(x + a) = \sin x\, \cos a + \cos x\, \sin a \quad \Rightarrow \quad \cos(x + a) = \cos x\, \cos a - \sin x\, \sin a$.

59. (a) $S(x) = f(x) + g(x) \quad \Rightarrow \quad S'(x) = f'(x) + g'(x) \quad \Rightarrow \quad S'(1) = f'(1) + g'(1) = 3 + 1 = 4$

(b) $P(x) = f(x)\, g(x) \quad \Rightarrow \quad P'(x) = f(x)\, g'(x) + g(x)\, f'(x) \quad \Rightarrow$

$P'(2) = f(2)\, g'(2) + g(2) f'(2) = 1(4) + 1(2) = 4 + 2 = 6$

(c) $Q(x) = \dfrac{f(x)}{g(x)} \quad \Rightarrow \quad Q'(x) = \dfrac{g(x)\, f'(x) - f(x)\, g'(x)}{[g(x)]^2} \quad \Rightarrow$

$Q'(1) = \dfrac{g(1)\, f'(1) - f(1)\, g'(1)}{[g(1)]^2} = \dfrac{3(3) - 2(1)}{3^2} = \dfrac{9 - 2}{9} = \dfrac{7}{9}$

(d) $C(x) = f(g(x)) \quad \Rightarrow \quad C'(x) = f'(g(x))\, g'(x) \quad \Rightarrow \quad C'(2) = f'(g(2))\, g'(2) = f'(1) \cdot 4 = 3 \cdot 4 = 12$

60. (a) $P(x) = f(x)\, g(x) \quad \Rightarrow \quad P'(x) = f(x)\, g'(x) + g(x)\, f'(x) \quad \Rightarrow$

$P'(2) = f(2)\, g'(2) + g(2)\, f'(2) = (1)\left(\frac{6-0}{3-0}\right) + (4)\left(\frac{0-3}{3-0}\right) = (1)(2) + (4)(-1) = 2 - 4 = -2$

(b) $Q(x) = \dfrac{f(x)}{g(x)} \quad \Rightarrow \quad Q'(x) = \dfrac{g(x)\, f'(x) - f(x)\, g'(x)}{[g(x)]^2} \quad \Rightarrow$

$Q'(2) = \dfrac{g(2)\, f'(2) - f(2)\, g'(2)}{[g(2)]^2} = \dfrac{(4)(-1) - (1)(2)}{4^2} = \dfrac{-6}{16} = -\dfrac{3}{8}$

(c) $C(x) = f(g(x)) \quad \Rightarrow \quad C'(x) = f'(g(x))g'(x) \quad \Rightarrow$

$C'(2) = f'(g(2))g'(2) = f'(4)g'(2) = \left(\frac{6-0}{5-3}\right)(2) = (3)(2) = 6$

61. $f(x) = x^2 g(x) \quad \Rightarrow \quad f'(x) = x^2 g'(x) + g(x)(2x) = x[xg'(x) + 2g(x)]$

62. $f(x) = g(x^2) \quad \Rightarrow \quad f'(x) = g'(x^2)(2x) = 2xg'(x^2)$

63. $f(x) = [g(x)]^2 \quad \Rightarrow \quad f'(x) = 2[g(x)] \cdot g'(x) = 2g(x)\,g'(x)$

64. $f(x) = x^a g(x^b) \quad \Rightarrow \quad f'(x) = ax^{a-1}g(x^b) + x^a g'(x^b)(bx^{b-1}) = ax^{a-1}g(x^b) + bx^{a+b-1}g'(x^b)$

65. $f(x) = g(g(x)) \quad \Rightarrow \quad f'(x) = g'(g(x))\,g'(x)$

66. $f(x) = \sin(g(x)) \quad \Rightarrow \quad f'(x) = \cos(g(x)) \cdot g'(x)$

67. $f(x) = g(\sin x) \quad \Rightarrow \quad f'(x) = g'(\sin x) \cdot \cos x$

68. $f(x) = g\left(\tan\sqrt{x}\right) \quad \Rightarrow$

$$f'(x) = g'\left(\tan\sqrt{x}\right) \cdot \frac{d}{dx}\left(\tan\sqrt{x}\right) = g'\left(\tan\sqrt{x}\right) \cdot \sec^2\sqrt{x} \cdot \frac{d}{dx}\left(\sqrt{x}\right) = \frac{g'\left(\tan\sqrt{x}\right)\sec^2\sqrt{x}}{2\sqrt{x}}$$

69. $h(x) = \dfrac{f(x)\,g(x)}{f(x) + g(x)} \quad \Rightarrow$

$$\begin{aligned} h'(x) &= \frac{[f(x) + g(x)]\,[f(x)\,g'(x) + g(x)\,f'(x)] - f(x)\,g(x)\,[f'(x) + g'(x)]}{[f(x) + g(x)]^2} \\ &= \frac{[f(x)]^2\,g'(x) + f(x)\,g(x)\,f'(x) + f(x)\,g(x)\,g'(x) + [g(x)]^2\,f'(x) - f(x)\,g(x)\,f'(x) - f(x)\,g(x)\,g'(x)}{[f(x) + g(x)]^2} \\ &= \frac{f'(x)\,[g(x)]^2 + g'(x)\,[f(x)]^2}{[f(x) + g(x)]^2} \end{aligned}$$

70. $h(x) = \sqrt{\dfrac{f(x)}{g(x)}} \quad \Rightarrow \quad h'(x) = \dfrac{f'(x)\,g(x) - f(x)\,g'(x)}{2\sqrt{f(x)/g(x)}\,[g(x)]^2} = \dfrac{f'(x)\,g(x) - f(x)\,g'(x)}{2[g(x)]^{3/2}\sqrt{f(x)}}$

71. Using the Chain Rule repeatedly, $h(x) = f(g(\sin 4x)) \quad \Rightarrow$

$$h'(x) = f'(g(\sin 4x)) \cdot \frac{d}{dx}(g(\sin 4x)) = f'(g(\sin 4x)) \cdot g'(\sin 4x) \cdot \frac{d}{dx}(\sin 4x) = f'(g(\sin 4x))g'(\sin 4x)(\cos 4x)(4).$$

72. (a) $x = \sqrt{b^2 + c^2 t^2} \quad \Rightarrow \quad v(t) = x' = \left[1/\left(2\sqrt{b^2 + c^2 t^2}\right)\right] 2c^2 t = c^2 t/\sqrt{b^2 + c^2 t^2} \quad \Rightarrow$

$$a(t) = v'(t) = \frac{c^2\sqrt{b^2 + c^2 t^2} - c^2 t\left(c^2 t/\sqrt{b^2 + c^2 t^2}\right)}{b^2 + c^2 t^2} = \frac{b^2 c^2}{(b^2 + c^2 t^2)^{3/2}}$$

(b) $v(t) > 0$ for $t > 0$, so the particle always moves in the positive direction.

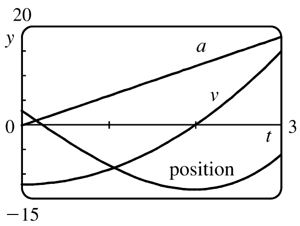
73. (a) $y = t^3 - 12t + 3 \quad \Rightarrow \quad v(t) = y' = 3t^2 - 12 \quad \Rightarrow \quad a(t) = v'(t) = 6t$

(b) $v(t) = 3(t^2 - 4) > 0$ when $t > 2$, so it moves upward when $t > 2$ and downward when $0 \le t < 2$.

(c) Distance upward $= y(3) - y(2) = -6 - (-13) = 7$,

Distance downward $= y(0) - y(2) = 3 - (-13) = 16$. Total distance $= 7 + 16 = 23$.

(d)



(e) The particle is speeding up when v and a have the same sign, that is, when $t > 2$. The particle is slowing down when v and a have opposite signs; that is, when $0 < t < 2$.

74. (a) $V = \frac{1}{3}\pi r^2 h \quad \Rightarrow \quad dV/dh = \frac{1}{3}\pi r^2 \quad [r \text{ constant}]$

(b) $V = \frac{1}{3}\pi r^2 h \quad \Rightarrow \quad dV/dr = \frac{2}{3}\pi r h \quad [h \text{ constant}]$

75. The linear density ρ is the rate of change of mass m with respect to length x.

$m = x\left(1 + \sqrt{x}\right) = x + x^{3/2} \quad \Rightarrow \quad \rho = dm/dx = 1 + \frac{3}{2}\sqrt{x}$, so the linear density when $x = 4$ is $1 + \frac{3}{2}\sqrt{4} = 4$ kg/m.

76. (a) $C(x) = 920 + 2x - 0.02x^2 + 0.00007x^3 \quad \Rightarrow \quad C'(x) = 2 - 0.04x + 0.00021x^2$

(b) $C'(100) = 2 - 4 + 2.1 = \$0.10$/unit. This value represents the rate at which costs are increasing as the hundredth unit is produced, and is the approximate cost of producing the 101st unit.

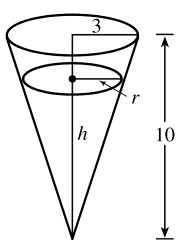
(c) The cost of producing the 101st item is $C(101) - C(100) = 990.10107 - 990 = \$0.10107$, slightly larger than $C'(100)$.

77. If $x =$ edge length, then $V = x^3 \quad \Rightarrow \quad dV/dt = 3x^2\, dx/dt = 10 \quad \Rightarrow \quad dx/dt = 10/(3x^2)$ and $S = 6x^2 \quad \Rightarrow$ $dS/dt = (12x)\, dx/dt = 12x[10/(3x^2)] = 40/x$. When $x = 30$, $dS/dt = \frac{40}{30} = \frac{4}{3}\ \text{cm}^2/\text{min}$.

78. Given $dV/dt = 2$, find dh/dt when $h = 5$. $V = \frac{1}{3}\pi r^2 h$ and, from similar triangles, $\frac{r}{h} = \frac{3}{10} \quad \Rightarrow \quad V = \frac{\pi}{3}\left(\frac{3h}{10}\right)^2 h = \frac{3\pi}{100}h^3$, so

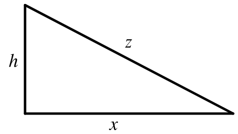
$$2 = \frac{dV}{dt} = \frac{9\pi}{100}h^2\,\frac{dh}{dt} \quad \Rightarrow \quad \frac{dh}{dt} = \frac{200}{9\pi h^2} = \frac{200}{9\pi\,(5)^2} = \frac{8}{9\pi}\ \text{cm/s}$$

when $h = 5$.



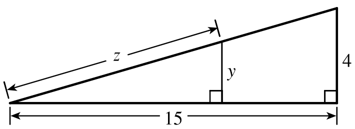
79. Given $dh/dt = 5$ and $dx/dt = 15$, find dz/dt. $z^2 = x^2 + h^2 \quad \Rightarrow$

$$2z\,\frac{dz}{dt} = 2x\,\frac{dx}{dt} + 2h\,\frac{dh}{dt} \quad \Rightarrow \quad \frac{dz}{dt} = \frac{1}{z}(15x + 5h).$$ When $t = 3$, $h = 45 + 3(5) = 60$ and $x = 15(3) = 45 \quad \Rightarrow \quad z = \sqrt{45^2 + 60^2} = 75$, so $\frac{dz}{dt} = \frac{1}{75}[15(45) + 5(60)] = 13$ ft/s.

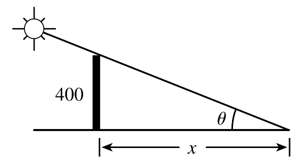


80. We are given $dz/dt = 30$ ft/s. By similar triangles, $\frac{y}{z} = \frac{4}{\sqrt{241}} \quad \Rightarrow$

$$y = \frac{4}{\sqrt{241}}z, \text{ so } \frac{dy}{dt} = \frac{4}{\sqrt{241}}\frac{dz}{dt} = \frac{120}{\sqrt{241}} \approx 7.7 \text{ ft/s.}$$



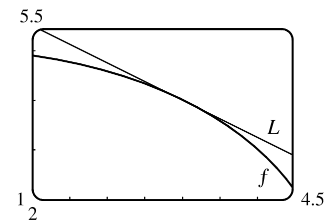
81. We are given $d\theta/dt = -0.25$ rad/h. $\tan\theta = 400/x \Rightarrow$ $x = 400\cot\theta \Rightarrow \dfrac{dx}{dt} = -400\csc^2\theta\,\dfrac{d\theta}{dt}$. When $\theta = \frac{\pi}{6}$, $\dfrac{dx}{dt} = -400(2)^2(-0.25) = 400$ ft/h.



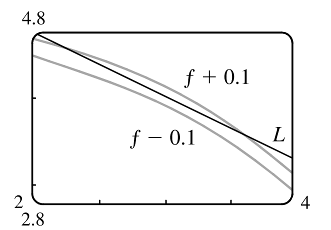
82. (a) $f(x) = \sqrt{25 - x^2} \Rightarrow f'(x) = \dfrac{-2x}{2\sqrt{25 - x^2}} = -x(25 - x^2)^{-1/2}$.

So the linear approximation to $f(x)$ near 3 is $f(x) \approx f(3) + f'(3)(x - 3) = 4 - \frac{3}{4}(x - 3)$.

(b)

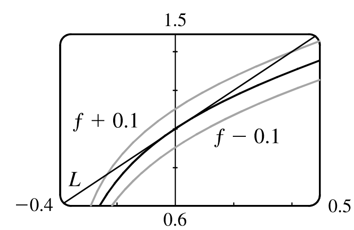


(c) For the required accuracy, we want $\sqrt{25 - x^2} - 0.1 < 4 - \frac{3}{4}(x - 3)$ and $4 - \frac{3}{4}(x - 3) < \sqrt{25 - x^2} + 0.1$. From the graph, it appears that these both hold for $2.24 < x < 3.66$.



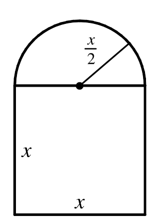
83. (a) $f(x) = \sqrt[3]{1 + 3x} = (1 + 3x)^{1/3} \Rightarrow f'(x) = (1 + 3x)^{-2/3}$, so the linearization of f at $a = 0$ is $L(x) = f(0) + f'(0)(x - 0) = 1^{1/3} + 1^{-2/3}x = 1 + x$. Thus, $\sqrt[3]{1 + 3x} \approx 1 + x \Rightarrow$ $\sqrt[3]{1.03} = \sqrt[3]{1 + 3(0.01)} \approx 1 + (0.01) = 1.01$.

(b) The linear approximation is $\sqrt[3]{1 + 3x} \approx 1 + x$, so for the required accuracy we want $\sqrt[3]{1 + 3x} - 0.1 < 1 + x < \sqrt[3]{1 + 3x} + 0.1$. From the graph, it appears that this is true when $-0.235 < x < 0.401$.



84. $y = x^3 - 2x^2 + 1 \Rightarrow dy = (3x^2 - 4x)\,dx$. When $x = 2$ and $dx = 0.2$, $dy = \left[3(2)^2 - 4(2)\right](0.2) = 0.8$.

85. $A = x^2 + \frac{1}{2}\pi\left(\frac{1}{2}x\right)^2 = \left(1 + \frac{\pi}{8}\right)x^2 \Rightarrow dA = \left(2 + \frac{\pi}{4}\right)x\,dx$. When $x = 60$ and $dx = 0.1$, $dA = \left(2 + \frac{\pi}{4}\right)60(0.1) = 12 + \frac{3\pi}{2}$, so the maximum error is approximately $12 + \frac{3\pi}{2} \approx 16.7\ \text{cm}^2$.



86. $\displaystyle\lim_{x\to 1}\frac{x^{17} - 1}{x - 1} = \left[\frac{d}{dx}x^{17}\right]_{x=1} = 17(1)^{16} = 17$

87. $\displaystyle\lim_{h\to 0}\frac{\sqrt[4]{16 + h} - 2}{h} = \left[\frac{d}{dx}\sqrt[4]{x}\right]_{x=16} = \frac{1}{4}x^{-3/4}\Big|_{x=16} = \frac{1}{4\left(\sqrt[4]{16}\right)^3} = \frac{1}{32}$

88. $\lim\limits_{\theta\to\pi/3}\dfrac{\cos\theta-0.5}{\theta-\pi/3}=\left[\dfrac{d}{d\theta}\cos\theta\right]_{\theta=\pi/3}=-\sin\dfrac{\pi}{3}=-\dfrac{\sqrt{3}}{2}$

89.
$$\begin{aligned}\lim_{x\to0}\frac{\sqrt{1+\tan x}-\sqrt{1+\sin x}}{x^3}&=\lim_{x\to0}\frac{\left(\sqrt{1+\tan x}-\sqrt{1+\sin x}\right)\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)}{x^3\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)}\\&=\lim_{x\to0}\frac{(1+\tan x)-(1+\sin x)}{x^3\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)}=\lim_{x\to0}\frac{\sin x\,(1/\cos x-1)}{x^3\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)}\cdot\frac{\cos x}{\cos x}\\&=\lim_{x\to0}\frac{\sin x\,(1-\cos x)}{x^3\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)\cos x}\cdot\frac{1+\cos x}{1+\cos x}\\&=\lim_{x\to0}\frac{\sin x\cdot\sin^2 x}{x^3\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)\cos x\,(1+\cos x)}\\&=\left(\lim_{x\to0}\frac{\sin x}{x}\right)^3\lim_{x\to0}\frac{1}{\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)\cos x\,(1+\cos x)}\\&=1^3\cdot\frac{1}{\left(\sqrt{1}+\sqrt{1}\right)\cdot1\cdot(1+1)}=\frac{1}{4}\end{aligned}$$

90. Differentiating the first given equation implicitly with respect to x and using the Chain Rule, we obtain $f(g(x))=x \;\Rightarrow$ $f'(g(x))\,g'(x)=1 \;\Rightarrow\; g'(x)=\dfrac{1}{f'(g(x))}$. Using the second given equation to expand the denominator of this expression gives $g'(x)=\dfrac{1}{1+[f(g(x))]^2}$. But the first given equation states that $f(g(x))=x$, so $g'(x)=\dfrac{1}{1+x^2}$.

91. $\dfrac{d}{dx}\,[f(2x)]=x^2 \;\Rightarrow\; f'(2x)\cdot2=x^2 \;\Rightarrow\; f'(2x)=\frac{1}{2}x^2$. Let $t=2x$. Then $f'(t)=\frac{1}{2}\left(\frac{1}{2}t\right)^2=\frac{1}{8}t^2$, so $f'(x)=\frac{1}{8}x^2$.

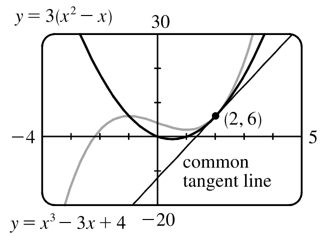
92. Let (b,c) be on the curve, that is, $b^{2/3}+c^{2/3}=a^{2/3}$. Now $x^{2/3}+y^{2/3}=a^{2/3} \;\Rightarrow\; \frac{2}{3}x^{-1/3}+\frac{2}{3}y^{-1/3}\dfrac{dy}{dx}=0$, so $\dfrac{dy}{dx}=-\dfrac{y^{1/3}}{x^{1/3}}=-\left(\dfrac{y}{x}\right)^{1/3}$, so at (b,c) the slope of the tangent line is $-(c/b)^{1/3}$ and an equation of the tangent line is $y-c=-(c/b)^{1/3}(x-b)$ or $y=-(c/b)^{1/3}x+(c+b^{2/3}c^{1/3})$. Setting $y=0$, we find that the x-intercept is $b^{1/3}c^{2/3}+b=b^{1/3}(c^{2/3}+b^{2/3})=b^{1/3}a^{2/3}$ and setting $x=0$ we find that the y-intercept is $c+b^{2/3}c^{1/3}=c^{1/3}(c^{2/3}+b^{2/3})=c^{1/3}a^{2/3}$. So the length of the tangent line between these two points is

$$\begin{aligned}\sqrt{(b^{1/3}a^{2/3})^2+(c^{1/3}a^{2/3})^2}&=\sqrt{b^{2/3}a^{4/3}+c^{2/3}a^{4/3}}=\sqrt{(b^{2/3}+c^{2/3})a^{4/3}}\\&=\sqrt{a^{2/3}a^{4/3}}=\sqrt{a^2}=a=\text{constant}\end{aligned}$$

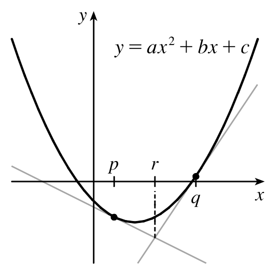
PROBLEMS PLUS

1. Let a be the x-coordinate of Q. Since the derivative of $y = 1 - x^2$ is $y' = -2x$, the slope at Q is $-2a$. But since the triangle is equilateral, $\overline{AO}/\overline{OC} = \sqrt{3}/1$, so the slope at Q is $-\sqrt{3}$. Therefore, we must have that $-2a = -\sqrt{3} \;\Rightarrow\; a = \frac{\sqrt{3}}{2}$. Thus, the point Q has coordinates $\left(\frac{\sqrt{3}}{2}, 1 - \left(\frac{\sqrt{3}}{2}\right)^2\right) = \left(\frac{\sqrt{3}}{2}, \frac{1}{4}\right)$ and by symmetry, P has coordinates $\left(-\frac{\sqrt{3}}{2}, \frac{1}{4}\right)$.

2. $y = x^3 - 3x + 4 \;\Rightarrow\; y' = 3x^2 - 3$, and $y = 3(x^2 - x) \;\Rightarrow\; y' = 6x - 3$. The slopes of the tangents of the two curves are equal when $3x^2 - 3 = 6x - 3$; that is, when $x = 0$ or 2. At $x = 0$, both tangents have slope -3, but the curves do not intersect. At $x = 2$, both tangents have slope 9 and the curves intersect at $(2, 6)$. So there is a common tangent line at $(2, 6)$, $y = 9x - 12$.



3.



We must show that r (in the figure) is halfway between p and q, that is, $r = (p + q)/2$. For the parabola $y = ax^2 + bx + c$, the slope of the tangent line is given by $y' = 2ax + b$. An equation of the tangent line at $x = p$ is $y - (ap^2 + bp + c) = (2ap + b)(x - p)$. Solving for y gives us

$$y = (2ap + b)x - 2ap^2 - bp + (ap^2 + bp + c)$$

or

$$y = (2ap + b)x + c - ap^2 \qquad \textbf{(1)}$$

Similarly, an equation of the tangent line at $x = q$ is

$$y = (2aq + b)x + c - aq^2 \qquad \textbf{(2)}$$

We can eliminate y and solve for x by subtracting equation **(1)** from equation **(2)**.

$$[(2aq + b) - (2ap + b)]x - aq^2 + ap^2 = 0$$
$$(2aq - 2ap)x = aq^2 - ap^2$$
$$2a(q - p)x = a(q^2 - p^2)$$
$$x = \frac{a(q + p)(q - p)}{2a(q - p)} = \frac{p + q}{2}$$

Thus, the x-coordinate of the point of intersection of the two tangent lines, namely r, is $(p + q)/2$.

4. We could differentiate and then simplify or we can simplify and then differentiate. The latter seems to be the simpler method.

$$\frac{\sin^2 x}{1 + \cot x} + \frac{\cos^2 x}{1 + \tan x} = \frac{\sin^2 x}{1 + \dfrac{\cos x}{\sin x}} \cdot \frac{\sin x}{\sin x} + \frac{\cos^2 x}{1 + \dfrac{\sin x}{\cos x}} \cdot \frac{\cos x}{\cos x} = \frac{\sin^3 x}{\sin x + \cos x} + \frac{\cos^3 x}{\cos x + \sin x}$$

$$= \frac{\sin^3 x + \cos^3 x}{\sin x + \cos x} \quad \text{[factor sum of cubes]} \quad = \frac{(\sin x + \cos x)(\sin^2 x - \sin x\,\cos x + \cos^2 x)}{\sin x + \cos x}$$

$$= \sin^2 x - \sin x\,\cos x + \cos^2 x = 1 - \sin x\,\cos x = 1 - \tfrac{1}{2}(2\sin x\,\cos x) = 1 - \tfrac{1}{2}\sin 2x$$

Thus, $\dfrac{d}{dx}\left(\dfrac{\sin^2 x}{1 + \cot x} + \dfrac{\cos^2 x}{1 + \tan x}\right) = \dfrac{d}{dx}\left(1 - \tfrac{1}{2}\sin 2x\right) = -\tfrac{1}{2}\cos 2x \cdot 2 = -\cos 2x$.

5. Using $f'(a) = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$, we recognize the given expression, $f(x) = \lim\limits_{t \to x} \dfrac{\sec t - \sec x}{t - x}$, as $g'(x)$ with $g(x) = \sec x$. Now $f'(\frac{\pi}{4}) = g''(\frac{\pi}{4})$, so we will find $g''(x)$. $g'(x) = \sec x \tan x \quad \Rightarrow$ $g''(x) = \sec x \sec^2 x + \tan x \sec x \tan x = \sec x(\sec^2 x + \tan^2 x)$, so $g''(\frac{\pi}{4}) = \sqrt{2}(\sqrt{2}^2 + 1^2) = \sqrt{2}(2+1) = 3\sqrt{2}$.

6. Using $f'(0) = \lim\limits_{x \to 0} \dfrac{f(x) - f(0)}{x - 0}$, we see that for the given equation, $\lim\limits_{x \to 0} \dfrac{\sqrt[3]{ax+b} - 2}{x} = \dfrac{5}{12}$, we have $f(x) = \sqrt[3]{ax+b}$, $f(0) = 2$, and $f'(0) = \frac{5}{12}$. Now $f(0) = 2 \quad \Leftrightarrow \quad \sqrt[3]{b} = 2 \quad \Leftrightarrow \quad b = 8$. Also $f'(x) = \frac{1}{3}(ax+b)^{-2/3} \cdot a$, so $f'(0) = \frac{5}{12} \quad \Leftrightarrow \quad \frac{1}{3}(8)^{-2/3} \cdot a = \frac{5}{12} \quad \Leftrightarrow \quad \frac{1}{3}(\frac{1}{4})a = \frac{5}{12} \quad \Leftrightarrow \quad a = 5$.

7. We use mathematical induction. Let S_n be the statement that $\dfrac{d^n}{dx^n}(\sin^4 x + \cos^4 x) = 4^{n-1}\cos(4x + n\pi/2)$.

S_1 is true because

$$\begin{aligned}\frac{d}{dx}(\sin^4 x + \cos^4 x) &= 4\sin^3 x\, \cos x - 4\cos^3 x\, \sin x = 4\sin x\, \cos x\left(\sin^2 x - \cos^2 x\right) x \\ &= -4\sin x\, \cos x\, \cos 2x = -2\sin 2x\, \cos 2 = -\sin 4x = \sin(-4x) \\ &= \cos\left(\tfrac{\pi}{2} - (-4x)\right) = \cos\left(\tfrac{\pi}{2} + 4x\right) = 4^{n-1}\cos\left(4x + n\tfrac{\pi}{2}\right) \text{ when } n = 1\end{aligned}$$

Now assume S_k is true, that is, $\dfrac{d^k}{dx^k}\left(\sin^4 x + \cos^4 x\right) = 4^{k-1}\cos\left(4x + k\frac{\pi}{2}\right)$. Then

$$\begin{aligned}\frac{d^{k+1}}{dx^{k+1}}(\sin^4 x + \cos^4 x) &= \frac{d}{dx}\left[\frac{d^k}{dx^k}(\sin^4 x + \cos^4 x)\right] = \frac{d}{dx}\left[4^{k-1}\cos\left(4x + k\tfrac{\pi}{2}\right)\right] \\ &= -4^{k-1}\sin\left(4x + k\tfrac{\pi}{2}\right) \cdot \frac{d}{dx}\left(4x + k\tfrac{\pi}{2}\right) = -4^k\sin\left(4x + k\tfrac{\pi}{2}\right) \\ &= 4^k\sin\left(-4x - k\tfrac{\pi}{2}\right) = 4^k\cos\left(\tfrac{\pi}{2} - \left(-4x - k\tfrac{\pi}{2}\right)\right) = 4^k\cos\left(4x + (k+1)\tfrac{\pi}{2}\right)\end{aligned}$$

which shows that S_{k+1} is true.

Therefore, $\dfrac{d^n}{dx^n}\left(\sin^4 x + \cos^4 x\right) = 4^{n-1}\cos\left(4x + n\frac{\pi}{2}\right)$ for every positive integer n, by mathematical induction.

Another proof: First write

$$\sin^4 x + \cos^4 x = (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\, \cos^2 x = 1 - \tfrac{1}{2}\sin^2 2x = 1 - \tfrac{1}{4}(1 - \cos 4x) = \tfrac{3}{4} + \tfrac{1}{4}\cos 4x$$

Then we have $\dfrac{d^n}{dx^n}\left(\sin^4 x + \cos^4 x\right) = \dfrac{d^n}{dx^n}\left(\dfrac{3}{4} + \dfrac{1}{4}\cos 4x\right) = \dfrac{1}{4} \cdot 4^n\cos\left(4x + n\dfrac{\pi}{2}\right) = 4^{n-1}\cos\left(4x + n\dfrac{\pi}{2}\right)$.

8.
$$\begin{aligned}\lim_{x \to a}\frac{f(x) - f(a)}{\sqrt{x} - \sqrt{a}} &= \lim_{x \to a}\left[\frac{f(x) - f(a)}{\sqrt{x} - \sqrt{a}} \cdot \frac{\sqrt{x} + \sqrt{a}}{\sqrt{x} + \sqrt{a}}\right] = \lim_{x \to a}\left[\frac{f(x) - f(a)}{x - a} \cdot \left(\sqrt{x} + \sqrt{a}\right)\right] \\ &= \lim_{x \to a}\frac{f(x) - f(a)}{x - a} \cdot \lim_{x \to a}\left(\sqrt{x} + \sqrt{a}\right) = f'(a) \cdot \left(\sqrt{a} + \sqrt{a}\right) = 2\sqrt{a}\, f'(a)\end{aligned}$$

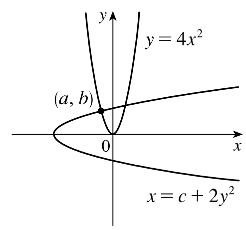
9. We must find a value x_0 such that the normal lines to the parabola $y = x^2$ at $x = \pm x_0$ intersect at a point one unit from the points $\left(\pm x_0, x_0^2\right)$. The normals to $y = x^2$ at $x = \pm x_0$ have slopes $-\dfrac{1}{\pm 2x_0}$ and pass through $\left(\pm x_0, x_0^2\right)$ respectively, so the normals have the equations $y - x_0^2 = -\dfrac{1}{2x_0}(x - x_0)$ and $y - x_0^2 = \dfrac{1}{2x_0}(x + x_0)$. The common y-intercept is $x_0^2 + \dfrac{1}{2}$.

We want to find the value of x_0 for which the distance from $\left(0, x_0^2 + \frac{1}{2}\right)$ to $\left(x_0, x_0^2\right)$ equals 1. The square of the distance is

$(x_0 - 0)^2 + \left[x_0^2 - \left(x_0^2 + \frac{1}{2}\right)\right]^2 = x_0^2 + \frac{1}{4} = 1 \quad\Leftrightarrow\quad x_0 = \pm\frac{\sqrt{3}}{2}$. For these values of x_0, the y-intercept is $x_0^2 + \frac{1}{2} = \frac{5}{4}$, so the center of the circle is at $\left(0, \frac{5}{4}\right)$.

Another solution: Let the center of the circle be $(0, a)$. Then the equation of the circle is $x^2 + (y - a)^2 = 1$. Solving with the equation of the parabola, $y = x^2$, we get $x^2 + (x^2 - a)^2 = 1 \quad\Leftrightarrow\quad x^2 + x^4 - 2ax^2 + a^2 = 1 \quad\Leftrightarrow$ $x^4 + (1 - 2a)x^2 + a^2 - 1 = 0$. The parabola and the circle will be tangent to each other when this quadratic equation in x^2 has equal roots; that is, when the discriminant is 0. Thus, $(1 - 2a)^2 - 4(a^2 - 1) = 0 \quad\Leftrightarrow$ $1 - 4a + 4a^2 - 4a^2 + 4 = 0 \quad\Leftrightarrow\quad 4a = 5$, so $a = \frac{5}{4}$. The center of the circle is $\left(0, \frac{5}{4}\right)$.

10. See the figure. The parabolas $y = 4x^2$ and $x = c + 2y^2$ intersect each other at right angles at the point (a, b) if and only if (a, b) satisfies both equations and the tangent lines at (a, b) are perpendicular. $y = 4x^2 \quad\Rightarrow\quad y' = 8x$ and $x = c + 2y^2 \quad\Rightarrow\quad 1 = 4y\,y' \quad\Rightarrow\quad y' = \dfrac{1}{4y}$, so at (a, b) we must

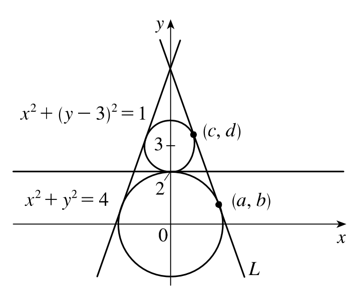


have $8a = -\dfrac{1}{1/(4b)} \quad\Rightarrow\quad 8a = -4b \quad\Rightarrow\quad b = -2a$. Since (a, b) is on both parabolas, we have **(1)** $b = 4a^2$ and **(2)** $a = c + 2b^2$. Substituting $-2a$ for b in **(1)** gives us $-2a = 4a^2 \quad\Rightarrow\quad 4a^2 + 2a = 0 \quad\Rightarrow\quad 2a(2a + 1) = 0 \quad\Rightarrow\quad a = 0$ or $a = -\frac{1}{2}$.

If $a = 0$, then $b = 0$ and $c = 0$, and the tangent lines at $(0, 0)$ are $y = 0$ and $x = 0$.

If $a = -\frac{1}{2}$, then $b = -2\left(-\frac{1}{2}\right) = 1$ and $-\frac{1}{2} = c + 2(1)^2 \quad\Rightarrow\quad c = -\frac{5}{2}$, and the tangent lines at $\left(-\frac{1}{2}, 1\right)$ are $y - 1 = -4\left(x + \frac{1}{2}\right)$ [or $y = -4x - 1$] and $y - 1 = \frac{1}{4}\left(x + \frac{1}{2}\right)$ [or $y = \frac{1}{4}x + \frac{9}{8}$].

11. See the figure. Clearly, the line $y = 2$ is tangent to both circles at the point $(0, 2)$. We'll look for a tangent line L through the points (a, b) and (c, d), and if such a line exists, then its reflection through the y-axis is another such line. The slope of L is the same at (a, b) and (c, d). Find those slopes: $x^2 + y^2 = 4 \quad\Rightarrow$ $2x + 2y\,y' = 0 \quad\Rightarrow\quad y' = -\dfrac{x}{y} \quad \left[= -\dfrac{a}{b}\right]$ and $x^2 + (y - 3)^2 = 1 \quad\Rightarrow$



$2x + 2(y - 3)y' = 0 \quad\Rightarrow\quad y' = -\dfrac{x}{y - 3} \quad \left[= -\dfrac{c}{d - 3}\right]$.

Now an equation for L can be written using either point-slope pair, so we get $y - b = -\dfrac{a}{b}(x - a) \quad \left[\text{or } y = -\dfrac{a}{b}x + \dfrac{a^2}{b} + b\right]$ and $y - d = -\dfrac{c}{d - 3}(x - c) \quad \left[\text{or } y = -\dfrac{c}{d - 3}\,x + \dfrac{c^2}{d - 3} + d\right]$. The slopes are equal, so $-\dfrac{a}{b} = -\dfrac{c}{d - 3} \quad\Leftrightarrow$

$d-3=\dfrac{bc}{a}$. Since (c,d) is a solution of $x^2+(y-3)^2=1$, we have $c^2+(d-3)^2=1$, so $c^2+\left(\dfrac{bc}{a}\right)^2=1 \;\Rightarrow$

$a^2c^2+b^2c^2=a^2 \;\Rightarrow\; c^2(a^2+b^2)=a^2 \;\Rightarrow\; 4c^2=a^2$ [since (a,b) is a solution of $x^2+y^2=4$] $\Rightarrow\; a=2c$.

Now $d-3=\dfrac{bc}{a} \;\Rightarrow\; d=3+\dfrac{bc}{2c}$, so $d=3+\dfrac{b}{2}$. The y-intercepts are equal, so $\dfrac{a^2}{b}+b=\dfrac{c^2}{d-3}+d \;\Leftrightarrow$

$\dfrac{a^2}{b}+b=\dfrac{(a/2)^2}{b/2}+\left(3+\dfrac{b}{2}\right) \;\Leftrightarrow\; \left[\dfrac{a^2}{b}+b=\dfrac{a^2}{2b}+3+\dfrac{b}{2}\right](2b) \;\Leftrightarrow\; 2a^2+2b^2=a^2+6b+b^2 \;\Leftrightarrow$

$a^2+b^2=6b \;\Leftrightarrow\; 4=6b \;\Leftrightarrow\; b=\frac{2}{3}$. It follows that $d=3+\dfrac{b}{2}=\dfrac{10}{3}$, $a^2=4-b^2=4-\frac{4}{9}=\frac{32}{9} \;\Rightarrow\; a=\frac{4}{3}\sqrt{2}$,

and $c^2=1-(d-3)^2=1-\left(\frac{1}{3}\right)^2=\frac{8}{9} \;\Rightarrow\; c=\frac{2}{3}\sqrt{2}$. Thus, L has equation $y-\dfrac{2}{3}=-\dfrac{(4/3)\sqrt{2}}{2/3}\left(x-\dfrac{4}{3}\sqrt{2}\right) \;\Leftrightarrow$

$y-\frac{2}{3}=-2\sqrt{2}\left(x-\frac{4}{3}\sqrt{2}\right) \;\Leftrightarrow\; y=-2\sqrt{2}\,x+6$. Its reflection has equation $y=2\sqrt{2}\,x+6$.

In summary, there are three lines tangent to both circles: $y=2$ touches at $(0,2)$, L touches at $\left(\frac{4}{3}\sqrt{2},\frac{2}{3}\right)$ and $\left(\frac{2}{3}\sqrt{2},\frac{10}{3}\right)$, and its reflection through the y-axis touches at $\left(-\frac{4}{3}\sqrt{2},\frac{2}{3}\right)$ and $\left(-\frac{2}{3}\sqrt{2},\frac{10}{3}\right)$.

12. $f(x)=\dfrac{x^{46}+x^{45}+2}{1+x}=\dfrac{x^{45}(x+1)+2}{x+1}=\dfrac{x^{45}(x+1)}{x+1}+\dfrac{2}{x+1}=x^{45}+2(x+1)^{-1}$, so

$f^{(46)}(x)=(x^{45})^{(46)}+2\left[(x+1)^{-1}\right]^{(46)}$. The forty-sixth derivative of any forty-fifth degree polynomial is 0, so

$(x^{45})^{46}=0$. Thus, $f^{(46)}(x)=2\left[(-1)(-2)(-3)\cdots(-46)(x+1)^{-47}\right]=2(46!)(x+1)^{-47}$ and $f^{(46)}(3)=2(46!)(4)^{-47}$

or $(46!)2^{-93}$.

13. We can assume without loss of generality that $\theta=0$ at time $t=0$, so that $\theta=12\pi t$ rad. [The angular velocity of the wheel is 360 rpm $=360\cdot(2\pi\text{ rad})/(60\text{ s})=12\pi$ rad/s.] Then the position of A as a function of time is

$A=(40\cos\theta,40\sin\theta)=(40\cos 12\pi t,40\sin 12\pi t)$, so $\sin\alpha=\dfrac{y}{1.2\text{ m}}=\dfrac{40\sin\theta}{120}=\dfrac{\sin\theta}{3}=\dfrac{1}{3}\sin 12\pi t$.

(a) Differentiating the expression for $\sin\alpha$, we get $\cos\alpha\cdot\dfrac{d\alpha}{dt}=\dfrac{1}{3}\cdot 12\pi\cdot\cos 12\pi t=4\pi\cos\theta$. When $\theta=\dfrac{\pi}{3}$, we have

$\sin\alpha=\dfrac{1}{3}\sin\theta=\dfrac{\sqrt{3}}{6}$, so $\cos\alpha=\sqrt{1-\left(\dfrac{\sqrt{3}}{6}\right)^2}=\sqrt{\dfrac{11}{12}}$ and $\dfrac{d\alpha}{dt}=\dfrac{4\pi\cos\frac{\pi}{3}}{\cos\alpha}=\dfrac{2\pi}{\sqrt{11/12}}=\dfrac{4\pi\sqrt{3}}{\sqrt{11}}\approx 6.56$ rad/s.

(b) By the Law of Cosines, $|AP|^2=|OA|^2+|OP|^2-2\,|OA|\,|OP|\cos\theta \;\Rightarrow$

$120^2=40^2+|OP|^2-2\cdot 40\,|OP|\cos\theta \;\Rightarrow\; |OP|^2-(80\cos\theta)\,|OP|-12{,}800=0 \;\Rightarrow$

$|OP|=\frac{1}{2}\left(80\cos\theta\pm\sqrt{6400\cos^2\theta+51{,}200}\right)=40\cos\theta\pm 40\sqrt{\cos^2\theta+8}=40\left(\cos\theta+\sqrt{8+\cos^2\theta}\right)$ cm

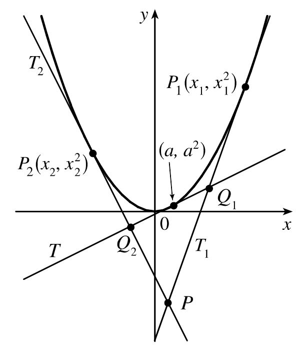
[since $|OP|>0$]. As a check, note that $|OP|=160$ cm when $\theta=0$ and $|OP|=80\sqrt{2}$ cm when $\theta=\frac{\pi}{2}$.

(c) By part (b), the x-coordinate of P is given by $x=40\left(\cos\theta+\sqrt{8+\cos^2\theta}\right)$, so

$\dfrac{dx}{dt}=\dfrac{dx}{d\theta}\dfrac{d\theta}{dt}=40\left(-\sin\theta-\dfrac{2\cos\theta\sin\theta}{2\sqrt{8+\cos^2\theta}}\right)\cdot 12\pi=-480\pi\sin\theta\left(1+\dfrac{\cos\theta}{\sqrt{8+\cos^2\theta}}\right)$ cm/s.

In particular, $dx/dt=0$ cm/s when $\theta=0$ and $dx/dt=-480\pi$ cm/s when $\theta=\frac{\pi}{2}$.

14. The equation of T_1 is $y - x_1^2 = 2x_1(x - x_1) = 2x_1x - 2x_1^2$ or $y = 2x_1x - x_1^2$.



The equation of T_2 is $y = 2x_2x - x_2^2$. Solving for the point of intersection, we get $2x(x_1 - x_2) = x_1^2 - x_2^2 \;\Rightarrow\; x = \frac{1}{2}(x_1 + x_2)$. Therefore, the coordinates of P are $\left(\frac{1}{2}(x_1 + x_2), x_1x_2\right)$. So if the point of contact of T is (a, a^2), then Q_1 is $\left(\frac{1}{2}(a + x_1), ax_1\right)$ and Q_2 is $\left(\frac{1}{2}(a + x_2), ax_2\right)$. Therefore,

$|PQ_1|^2 = \frac{1}{4}(a - x_2)^2 + x_1^2(a - x_2)^2 = (a - x_2)^2\left(\frac{1}{4} + x_1^2\right)$ and

$|PP_1|^2 = \frac{1}{4}(x_1 - x_2)^2 + x_1^2(x_1 - x_2)^2 = (x_1 - x_2)^2\left(\frac{1}{4} + x_1^2\right)$.

So $\dfrac{|PQ_1|^2}{|PP_1|^2} = \dfrac{(a - x_2)^2}{(x_1 - x_2)^2}$, and similarly $\dfrac{|PQ_2|^2}{|PP_2|^2} = \dfrac{(x_1 - a)^2}{(x_1 - x_2)^2}$. Finally, $\dfrac{|PQ_1|}{|PP_1|} + \dfrac{|PQ_2|}{|PP_2|} = \dfrac{a - x_2}{x_1 - x_2} + \dfrac{x_1 - a}{x_1 - x_2} = 1$.

15. It seems from the figure that as P approaches the point $(0, 2)$ from the right, $x_T \to \infty$ and $y_T \to 2^+$. As P approaches the point $(3, 0)$ from the left, it appears that $x_T \to 3^+$ and $y_T \to \infty$. So we guess that $x_T \in (3, \infty)$ and $y_T \in (2, \infty)$. It is more difficult to estimate the range of values for x_N and y_N. We might perhaps guess that $x_N \in (0, 3)$, and $y_N \in (-\infty, 0)$ or $(-2, 0)$.

In order to actually solve the problem, we implicitly differentiate the equation of the ellipse to find the equation of the tangent line: $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1 \;\Rightarrow\; \dfrac{2x}{9} + \dfrac{2y}{4}y' = 0$, so $y' = -\dfrac{4}{9}\dfrac{x}{y}$. So at the point (x_0, y_0) on the ellipse, an equation of the tangent line is $y - y_0 = -\dfrac{4}{9}\dfrac{x_0}{y_0}(x - x_0)$ or $4x_0x + 9y_0y = 4x_0^2 + 9y_0^2$. This can be written as $\dfrac{x_0x}{9} + \dfrac{y_0y}{4} = \dfrac{x_0^2}{9} + \dfrac{y_0^2}{4} = 1$, because (x_0, y_0) lies on the ellipse. So an equation of the tangent line is $\dfrac{x_0x}{9} + \dfrac{y_0y}{4} = 1$.

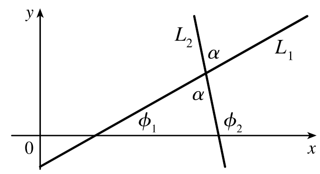
Therefore, the x-intercept x_T for the tangent line is given by $\dfrac{x_0x_T}{9} = 1 \;\Leftrightarrow\; x_T = \dfrac{9}{x_0}$, and the y-intercept y_T is given by $\dfrac{y_0y_T}{4} = 1 \;\Leftrightarrow\; y_T = \dfrac{4}{y_0}$.

So as x_0 takes on all values in $(0, 3)$, x_T takes on all values in $(3, \infty)$, and as y_0 takes on all values in $(0, 2)$, y_T takes on all values in $(2, \infty)$. At the point (x_0, y_0) on the ellipse, the slope of the normal line is $-\dfrac{1}{y'(x_0, y_0)} = \dfrac{9}{4}\dfrac{y_0}{x_0}$, and its equation is $y - y_0 = \dfrac{9}{4}\dfrac{y_0}{x_0}(x - x_0)$. So the x-intercept x_N for the normal line is given by $0 - y_0 = \dfrac{9}{4}\dfrac{y_0}{x_0}(x_N - x_0) \;\Rightarrow\; x_N = -\dfrac{4x_0}{9} + x_0 = \dfrac{5x_0}{9}$, and the y-intercept y_N is given by $y_N - y_0 = \dfrac{9}{4}\dfrac{y_0}{x_0}(0 - x_0) \;\Rightarrow\; y_N = -\dfrac{9y_0}{4} + y_0 = -\dfrac{5y_0}{4}$.

So as x_0 takes on all values in $(0, 3)$, x_N takes on all values in $\left(0, \frac{5}{3}\right)$, and as y_0 takes on all values in $(0, 2)$, y_N takes on all values in $\left(-\frac{5}{2}, 0\right)$.

16. $\displaystyle\lim_{x \to 0} \frac{\sin(3 + x)^2 - \sin 9}{x} = f'(3)$ where $f(x) = \sin x^2$. Now $f'(x) = (\cos x^2)(2x)$, so $f'(3) = 6\cos 9$.

17. (a) If the two lines L_1 and L_2 have slopes m_1 and m_2 and angles of inclination ϕ_1 and ϕ_2, then $m_1 = \tan\,\phi_1$ and $m_2 = \tan\,\phi_2$. The triangle in the figure shows that $\phi_1 + \alpha + (180° - \phi_2) = 180°$ and so $\alpha = \phi_2 - \phi_1$. Therefore, using the identity for $\tan(x-y)$, we have $\tan\alpha = \tan(\phi_2 - \phi_1) = \dfrac{\tan\phi_2 - \tan\phi_1}{1 + \tan\phi_2\tan\phi_1}$ and so $\tan\,\alpha = \dfrac{m_2 - m_1}{1 + m_1 m_2}$.



(b) (i) The parabolas intersect when $x^2 = (x-2)^2 \;\Rightarrow\; x = 1$. If $y = x^2$, then $y' = 2x$, so the slope of the tangent to $y = x^2$ at $(1,1)$ is $m_1 = 2(1) = 2$. If $y = (x-2)^2$, then $y' = 2(x-2)$, so the slope of the tangent to $y = (x-2)^2$ at $(1,1)$ is $m_2 = 2(1-2) = -2$. Therefore, $\tan\alpha = \dfrac{m_2 - m_1}{1 + m_1 m_2} = \dfrac{-2-2}{1+2(-2)} = \dfrac{4}{3}$ and so $\alpha = \tan^{-1}\left(\frac{4}{3}\right) \approx 53°$ [or 127°].

(ii) $x^2 - y^2 = 3$ and $x^2 - 4x + y^2 + 3 = 0$ intersect when $x^2 - 4x + (x^2 - 3) + 3 = 0 \;\Leftrightarrow\; 2x(x-2) = 0 \;\Rightarrow\; x = 0$ or 2, but 0 is extraneous. If $x = 2$, then $y = \pm 1$. If $x^2 - y^2 = 3$ then $2x - 2yy' = 0 \;\Rightarrow\; y' = x/y$ and $x^2 - 4x + y^2 + 3 = 0 \;\Rightarrow\; 2x - 4 + 2yy' = 0 \;\Rightarrow\; y' = \dfrac{2-x}{y}$. At $(2,1)$ the slopes are $m_1 = 2$ and $m_2 = 0$, so $\tan\alpha = \frac{0-2}{1+2\cdot 0} = -2 \;\Rightarrow\; \alpha \approx 117°$. At $(2,-1)$ the slopes are $m_1 = -2$ and $m_2 = 0$, so $\tan\alpha = \dfrac{0-(-2)}{1+(-2)(0)} = 2 \;\Rightarrow\; \alpha \approx 63°$ [or 117°].

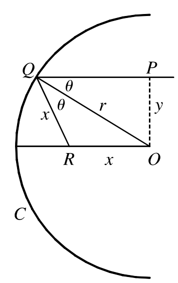
18. $y^2 = 4px \;\Rightarrow\; 2yy' = 4p \;\Rightarrow\; y' = 2p/y \;\Rightarrow\;$ slope of tangent at $P(x_1, y_1)$ is $m_1 = 2p/y_1$. The slope of FP is $m_2 = \dfrac{y_1}{x_1 - p}$, so by the formula from Problem 17(a),

$$\tan\alpha = \frac{\dfrac{y_1}{x_1 - p} - \dfrac{2p}{y_1}}{1 + \left(\dfrac{2p}{y_1}\right)\left(\dfrac{y_1}{x_1 - p}\right)} \cdot \frac{y_1\,(x_1 - p)}{y_1\,(x_1 - p)} = \frac{y_1^2 - 2p(x_1 - p)}{y_1(x_1 - p) + 2py_1} = \frac{4px_1 - 2px_1 + 2p^2}{x_1y_1 - py_1 + 2py_1}$$

$$= \frac{2p(p + x_1)}{y_1(p + x_1)} = \frac{2p}{y_1} = \text{slope of tangent at } P = \tan\beta$$

Since $0 \le \alpha, \beta \le \frac{\pi}{2}$, this proves that $\alpha = \beta$.

19. Since $\angle ROQ = \angle OQP = \theta$, the triangle QOR is isosceles, so $|QR| = |RO| = x$. By the Law of Cosines, $x^2 = x^2 + r^2 - 2rx\cos\theta$. Hence, $2rx\cos\theta = r^2$, so $x = \dfrac{r^2}{2r\cos\theta} = \dfrac{r}{2\cos\theta}$. Note that as $y \to 0^+$, $\theta \to 0^+$ (since $\sin\theta = y/r$), and hence $x \to \dfrac{r}{2\cos 0} = \dfrac{r}{2}$. Thus, as P is taken closer and closer to the x-axis, the point R approaches the midpoint of the radius AO.



20. $$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{f(x) - 0}{g(x) - 0} = \lim_{x\to 0}\frac{f(x) - f(0)}{g(x) - g(0)} = \lim_{x\to 0}\frac{\dfrac{f(x) - f(0)}{x - 0}}{\dfrac{g(x) - g(0)}{x - 0}} = \frac{\lim\limits_{x\to 0}\dfrac{f(x) - f(0)}{x - 0}}{\lim\limits_{x\to 0}\dfrac{g(x) - g(0)}{x - 0}} = \frac{f'(0)}{g'(0)}$$

21. $\lim\limits_{x\to 0}\dfrac{\sin(a+2x)-2\sin(a+x)+\sin a}{x^2}$

$$=\lim_{x\to 0}\frac{\sin a\,\cos 2x+\cos a\,\sin 2x-2\sin a\,\cos x-2\cos a\,\sin x+\sin a}{x^2}$$

$$=\lim_{x\to 0}\frac{\sin a\,(\cos 2x-2\cos x+1)+\cos a\,(\sin 2x-2\sin x)}{x^2}$$

$$=\lim_{x\to 0}\frac{\sin a\,(2\cos^2 x-1-2\cos x+1)+\cos a\,(2\sin x\,\cos x-2\sin x)}{x^2}$$

$$=\lim_{x\to 0}\frac{\sin a\,(2\cos x)(\cos x-1)+\cos a\,(2\sin x)(\cos x-1)}{x^2}$$

$$=\lim_{x\to 0}\frac{2(\cos x-1)[\sin a\,\cos x+\cos a\,\sin x](\cos x+1)}{x^2(\cos x+1)}$$

$$=\lim_{x\to 0}\frac{-2\sin^2 x\,[\sin(a+x)]}{x^2(\cos x+1)}=-2\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2\cdot\frac{\sin(a+x)}{\cos x+1}=-2(1)^2\,\frac{\sin(a+0)}{\cos 0+1}=-\sin a$$

22. Suppose that $y=mx+c$ is a tangent line to the ellipse. Then it intersects the ellipse at only one point, so the discriminant of the equation $\dfrac{x^2}{a^2}+\dfrac{(mx+c)^2}{b^2}=1 \quad\Leftrightarrow\quad (b^2+a^2m^2)x^2+2mca^2x+a^2c^2-a^2b^2=0$ must be 0; that is,

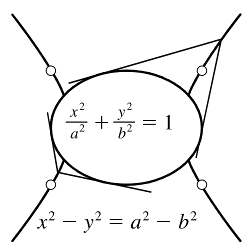
$$0=(2mca^2)^2-4(b^2+a^2m^2)(a^2c^2-a^2b^2)=4a^4c^2m^2-4a^2b^2c^2+4a^2b^4-4a^4c^2m^2+4a^4b^2m^2$$
$$=4a^2b^2(a^2m^2+b^2-c^2)$$

Therefore, $a^2m^2+b^2-c^2=0$.

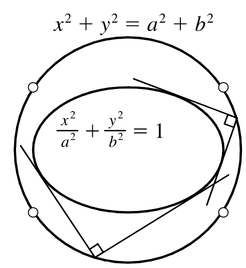
Now if a point (α,β) lies on the line $y=mx+c$, then $c=\beta-m\alpha$, so from above,

$$0=a^2m^2+b^2-(\beta-m\alpha)^2=(a^2-\alpha^2)m^2+2\alpha\beta m+b^2-\beta^2 \quad\Leftrightarrow\quad m^2+\frac{2\alpha\beta}{a^2-\alpha^2}m+\frac{b^2-\beta^2}{a^2-\alpha^2}=0.$$

(a) Suppose that the two tangent lines from the point (α,β) to the ellipse have slopes m and $\dfrac{1}{m}$. Then m and $\dfrac{1}{m}$ are roots of the equation $z^2+\dfrac{2\alpha\beta}{a^2-\alpha^2}z+\dfrac{b^2-\beta^2}{a^2-\alpha^2}=0$. This implies that $(z-m)\left(z-\dfrac{1}{m}\right)=0 \quad\Leftrightarrow\quad z^2-\left(m+\dfrac{1}{m}\right)z+m\left(\dfrac{1}{m}\right)=0$, so equating the constant terms in the two quadratic equations, we get $\dfrac{b^2-\beta^2}{a^2-\alpha^2}=m\left(\dfrac{1}{m}\right)=1$, and hence $b^2-\beta^2=a^2-\alpha^2$. So (α,β) lies on the hyperbola $x^2-y^2=a^2-b^2$.



(b) If the two tangent lines from the point (α,β) to the ellipse have slopes m and $-\dfrac{1}{m}$, then m and $-\dfrac{1}{m}$ are roots of the quadratic equation, and so $(z-m)\left(z+\dfrac{1}{m}\right)=0$, and equating the constant terms as in part (a), we get $\dfrac{b^2-\beta^2}{a^2-\alpha^2}=-1$, and hence $b^2-\beta^2=\alpha^2-a^2$. So the point (α,β) lies on the circle $x^2+y^2=a^2+b^2$.



23. $y = x^4 - 2x^2 - x \quad \Rightarrow \quad y' = 4x^3 - 4x - 1$. The equation of the tangent line at $x = a$ is $y - (a^4 - 2a^2 - a) = (4a^3 - 4a - 1)(x - a)$ or $y = (4a^3 - 4a - 1)x + (-3a^4 + 2a^2)$ and similarly for $x = b$. So if at $x = a$ and $x = b$ we have the same tangent line, then $4a^3 - 4a - 1 = 4b^3 - 4b - 1$ and $-3a^4 + 2a^2 = -3b^4 + 2b^2$. The first equation gives $a^3 - b^3 = a - b \quad \Rightarrow \quad (a - b)(a^2 + ab + b^2) = (a - b)$. Assuming $a \neq b$, we have $1 = a^2 + ab + b^2$. The second equation gives $3(a^4 - b^4) = 2(a^2 - b^2) \quad \Rightarrow \quad 3(a^2 - b^2)(a^2 + b^2) = 2(a^2 - b^2)$ which is true if $a = -b$. Substituting into $1 = a^2 + ab + b^2$ gives $1 = a^2 - a^2 + a^2 \quad \Rightarrow \quad a = \pm 1$ so that $a = 1$ and $b = -1$ or vice versa. Thus, the points $(1, -2)$ and $(-1, 0)$ have a common tangent line.

As long as there are only two such points, we are done. So we show that these are in fact the only two such points. Suppose that $a^2 - b^2 \neq 0$. Then $3(a^2 - b^2)(a^2 + b^2) = 2(a^2 - b^2)$ gives $3(a^2 + b^2) = 2$ or $a^2 + b^2 = \frac{2}{3}$. Thus, $ab = (a^2 + ab + b^2) - (a^2 + b^2) = 1 - \dfrac{2}{3} = \dfrac{1}{3}$, so $b = \dfrac{1}{3a}$. Hence, $a^2 + \dfrac{1}{9a^2} = \dfrac{2}{3}$, so $9a^4 + 1 = 6a^2 \quad \Rightarrow$ $0 = 9a^4 - 6a^2 + 1 = (3a^2 - 1)^2$. So $3a^2 - 1 = 0 \quad \Rightarrow \quad a^2 = \dfrac{1}{3} \quad \Rightarrow \quad b^2 = \dfrac{1}{9a^2} = \dfrac{1}{3} = a^2$, contradicting our assumption that $a^2 \neq b^2$.

24. Suppose that the normal lines at the three points (a_1, a_1^2), (a_2, a_2^2), and (a_3, a_3^2) intersect at a common point. Now if one of the a_i is 0 (suppose $a_1 = 0$) then by symmetry $a_2 = -a_3$, so $a_1 + a_2 + a_3 = 0$. So we can assume that none of the a_i is 0.

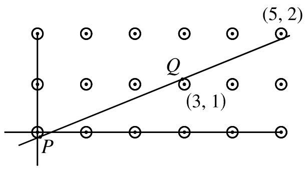
The slope of the tangent line at (a_i, a_i^2) is $2a_i$, so the slope of the normal line is $-\dfrac{1}{2a_i}$ and its equation is $y - a_i^2 = -\dfrac{1}{2a_i}(x - a_i)$. We solve for the x-coordinate of the intersection of the normal lines from (a_1, a_1^2) and (a_2, a_2^2):

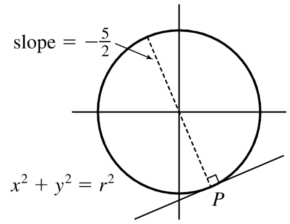
$$y = a_1^2 - \frac{1}{2a_1}(x - a_1) = a_2^2 - \frac{1}{2a_2}(x - a_2) \quad \Rightarrow \quad x\left(\frac{1}{2a_2} - \frac{1}{2a_1}\right) = a_2^2 - a_1^2 \quad \Rightarrow$$

$$x\left(\frac{a_1 - a_2}{2a_1a_2}\right) = (-a_1 - a_2)(a_1 + a_2) \quad \Leftrightarrow \quad x = -2a_1a_2(a_1 + a_2) \quad \textbf{(1)}.$$ Similarly, solving for the x-coordinate of the intersections of the normal lines from (a_1, a_1^2) and (a_3, a_3^2) gives $x = -2a_1a_3(a_1 + a_3)$ **(2)**.

Equating **(1)** and **(2)** gives $a_2(a_1 + a_2) = a_3(a_1 + a_3) \quad \Leftrightarrow \quad a_1(a_2 - a_3) = a_3^2 - a_2^2 = -(a_2 + a_3)(a_2 - a_3) \quad \Leftrightarrow$ $a_1 = -(a_2 + a_3) \quad \Leftrightarrow \quad a_1 + a_2 + a_3 = 0$.

25. Because of the periodic nature of the lattice points, it suffices to consider the points in the 5×2 grid shown. We can see that the minimum value of r occurs when there is a line with slope $\frac{2}{5}$ which touches the circle centered at $(3, 1)$ and the circles centered at $(0, 0)$ and $(5, 2)$.

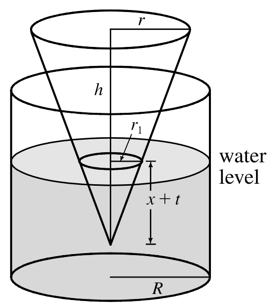




To find P, the point at which the line is tangent to the circle at $(0, 0)$, we simultaneously solve $x^2 + y^2 = r^2$ and

$y = -\frac{5}{2}x \Rightarrow x^2 + \frac{25}{4}x^2 = r^2 \Rightarrow x^2 = \frac{4}{29}r^2 \Rightarrow x = \frac{2}{\sqrt{29}}r$, $y = -\frac{5}{\sqrt{29}}r$. To find Q, we either use symmetry or solve $(x-3)^2 + (y-1)^2 = r^2$ and $y - 1 = -\frac{5}{2}(x-3)$. As above, we get $x = 3 - \frac{2}{\sqrt{29}}r$, $y = 1 + \frac{5}{\sqrt{29}}r$. Now the slope of the line PQ is $\frac{2}{5}$, so $m_{PQ} = \dfrac{1 + \frac{5}{\sqrt{29}}r - \left(-\frac{5}{\sqrt{29}}r\right)}{3 - \frac{2}{\sqrt{29}}r - \frac{2}{\sqrt{29}}r} = \dfrac{1 + \frac{10}{\sqrt{29}}r}{3 - \frac{4}{\sqrt{29}}r} = \dfrac{\sqrt{29} + 10r}{3\sqrt{29} - 4r} = \dfrac{2}{5} \Rightarrow$ $5\sqrt{29} + 50r = 6\sqrt{29} - 8r \Leftrightarrow 58r = \sqrt{29} \Leftrightarrow r = \frac{\sqrt{29}}{58}$. So the minimum value of r for which any line with slope $\frac{2}{5}$ intersects circles with radius r centered at the lattice points on the plane is $r = \frac{\sqrt{29}}{58} \approx 0.093$.

26.

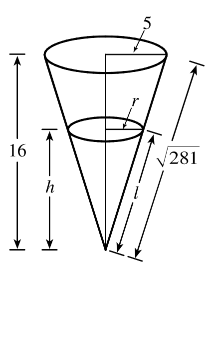


Assume the axes of the cone and the cylinder are parallel. Let H denote the initial height of the water. When the cone has been dropping for t seconds, the water level has risen x centimeters, so the tip of the cone is $x + 1t$ centimeters below the water line. We want to find dx/dt when $x + t = h$ (when the cone is completely submerged). Using similar triangles, $\dfrac{r_1}{x+t} = \dfrac{r}{h} \Rightarrow r_1 = \dfrac{r}{h}(x+t)$.

$$\begin{array}{cccccc} \text{volume of water and cone at time } t & = & \text{original volume of water} & + & \text{volume of submerged part of cone} \\ \pi R^2(H+x) & = & \pi R^2 H & + & \frac{1}{3}\pi r_1^2(x+t) \\ \pi R^2 H + \pi R^2 x & = & \pi R^2 H & + & \frac{1}{3}\pi \dfrac{r^2}{h^2}(x+t)^3 \\ 3h^2R^2x & = & r^2(x+t)^3 & & \end{array}$$

Differentiating implicitly with respect to t gives us $3h^2R^2\dfrac{dx}{dt} = r^2\left[3(x+t)^2\dfrac{dx}{dt} + 3(x+t)^2\dfrac{dt}{dt}\right] \Rightarrow$ $\dfrac{dx}{dt} = \dfrac{r^2(x+t)^2}{h^2R^2 - r^2(x+t)^2} \Rightarrow \left.\dfrac{dx}{dt}\right|_{x+t=h} = \dfrac{r^2h^2}{h^2R^2 - r^2h^2} = \dfrac{r^2}{R^2 - r^2}$. Thus, the water level is rising at a rate of $\dfrac{r^2}{R^2 - r^2}$ cm/s at the instant the cone is completely submerged.

27.



By similar triangles, $\dfrac{r}{5} = \dfrac{h}{16} \Rightarrow r = \dfrac{5h}{16}$. The volume of the cone is $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\dfrac{5h}{16}\right)^2 h = \dfrac{25\pi}{768}h^3$, so $\dfrac{dV}{dt} = \dfrac{25\pi}{256}h^2\dfrac{dh}{dt}$. Now the rate of change of the volume is also equal to the difference of what is being added (2 cm^3/min) and what is oozing out ($k\pi rl$, where πrl is the area of the cone and k is a proportionality constant). Thus, $\dfrac{dV}{dt} = 2 - k\pi rl$.

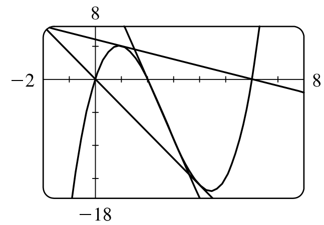
Equating the two expressions for $\dfrac{dV}{dt}$ and substituting $h = 10$, $\dfrac{dh}{dt} = -0.3$, $r = \dfrac{5(10)}{16} = \dfrac{25}{8}$, and $\dfrac{l}{\sqrt{281}} = \dfrac{10}{16} \Leftrightarrow$ $l = \dfrac{5}{8}\sqrt{281}$, we get $\dfrac{25\pi}{256}(10)^2(-0.3) = 2 - k\pi\dfrac{25}{8}\cdot\dfrac{5}{8}\sqrt{281} \Leftrightarrow \dfrac{125k\pi\sqrt{281}}{64} = 2 + \dfrac{750\pi}{256}$. Solving for k gives us

$k = \dfrac{256 + 375\pi}{250\pi\sqrt{281}}$. To maintain a certain height, the rate of oozing, $k\pi rl$, must equal the rate of the liquid being poured in; that is, $\dfrac{dV}{dt} = 0$. Thus, the rate at which we should pour the liquid into the container is

$$k\pi rl = \frac{256 + 375\pi}{250\pi\sqrt{281}} \cdot \pi \cdot \frac{25}{8} \cdot \frac{5\sqrt{281}}{8} = \frac{256 + 375\pi}{128} \approx 11.204 \text{ cm}^3/\text{min}$$

28. (a) $f(x) = x(x-2)(x-6) = x^3 - 8x^2 + 12x \quad \Rightarrow$

$f'(x) = 3x^2 - 16x + 12$. The average of the first pair of zeros is $(0+2)/2 = 1$. At $x = 1$, the slope of the tangent line is $f'(1) = -1$, so an equation of the tangent line has the form $y = -1x + b$. Since $f(1) = 5$, we have $5 = -1 + b \quad \Rightarrow \quad b = 6$ and the tangent has equation $y = -x + 6$.



Similarly, at $x = \dfrac{0+6}{2} = 3$, $y = -9x + 18$; at $x = \dfrac{2+6}{2} = 4$, $y = -4x$. From the graph, we see that each tangent line drawn at the average of two zeros intersects the graph of f at the third zero.

(b) A CAS gives $f'(x) = (x-b)(x-c) + (x-a)(x-c) + (x-a)(x-b)$ or

$f'(x) = 3x^2 - 2(a+b+c)x + ab + ac + bc$. Using the `Simplify` command, we get

$f'\left(\dfrac{a+b}{2}\right) = -\dfrac{(a-b)^2}{4}$ and $f\left(\dfrac{a+b}{2}\right) = -\dfrac{(a-b)^2}{8}(a+b-2c)$, so an equation of the tangent line at $x = \dfrac{a+b}{2}$

is $y = -\dfrac{(a-b)^2}{4}\left(x - \dfrac{a+b}{2}\right) - \dfrac{(a-b)^2}{8}(a+b-2c)$. To find the x-intercept, let $y = 0$ and use the `Solve` command. The result is $x = c$.

Using Derive, we can begin by authoring the expression $(x-a)(x-b)(x-c)$. Now load the utility file `DifferentiationApplications`. Next we author tangent $(\#1, x, (a+b)/2)$—this is the command to find an equation of the tangent line of the function in #1 whose independent variable is x at the x-value $(a+b)/2$. We then simplify that expression and obtain the equation $y = \#4$. The form in expression #4 makes it easy to see that the x-intercept is the third zero, namely c. In a similar fashion we see that b is the x-intercept for the tangent line at $(a+c)/2$ and a is the x-intercept for the tangent line at $(b+c)/2$.

```
#1:  (x - a)·(x - b)·(x - c)
#2:  LOAD(C:\Program Files\TI Education\Derive 6\Math\DifferentiationApplications.mth

                                            a + b
#3:  TANGENT ((x - a)·(x - b)·(x - c), x, -------)
                                              2

                                             2             2
                                          (a   - 2·a·b + b )·(c-x)
#4:                                       -----------------------
                                                     4
```