Solution Manual for
SIGNALS AND SYSTEMS USING MATLAB

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Chapter 0

From the Ground Up

0.1 Basic Problems

0.1 Let $z = 8 + j3$ and $v = 9 - j2$,

(a) Find

$$(i)\ \mathcal{R}e(z) + \mathcal{I}m(v), \quad (ii)\ |z+v|, \quad (iii)\ |zv|, \quad (iv)\ \angle z + \angle v, \quad (v)\ |v/z|, \quad (vi)\ \angle(v/z)$$

(b) Find the trigonometric and polar forms of

$$(i)\ z+v, \quad (ii)\ zv, \quad (iii)\ z^* \quad (iv)\ zz^*, \quad (v)\ z-v$$

Answers: (a) $\mathcal{R}e(z) + \mathcal{I}m(v) = 6$; $|v/z| = \sqrt{85}/\sqrt{73}$; (b) $zz^* = |z|^2 = 73$.
Solution

(a) i. $\mathcal{R}e(z) + \mathcal{I}m(v) = 8 - 2 = 6$

ii. $|z+v| = |17 + j1| = \sqrt{17^2 + 1}$

iii. $|zv| = |72 - j16 + j27 + 6| = |78 + j11| = \sqrt{78^2 + 11^2}$

iv. $\angle z + \angle v = \tan^{-1}(3/8) - \tan^{-1}(2/9)$

v. $|v/z| = |v|/|z| = \sqrt{85}/\sqrt{73}$

vi. $\angle(v/z) = -\tan^{-1}(2/9) - \tan^{-1}(3/8)$

(b) i. $z + v = 17 + j = \sqrt{17^2+1}e^{j\tan^{-1}(1/17)}$

ii. $zv = 78 + j11 = \sqrt{78^2 + 11^2}e^{j\tan^{-1}(11/78)}$

iii. $z^* = 8 - j3 = \sqrt{64+9}(e^{-j\tan^{-1}(3/8)})^* = \sqrt{73}e^{j\tan^{-1}(3/8)}$

iv. $zz^* = |z|^2 = 73$

v. $z - v = -1 + j5 = \sqrt{1+25}e^{-j\tan^{-1}(5)}$

0.2 Use Euler's identity to

(a) show that

$$(i)\ \cos(\theta - \pi/2) = \sin(\theta),\ \ (ii)\ -\sin(\theta - \pi/2) = \cos(\theta),\ \ (iii)\ \cos(\theta) = \sin(\theta + \pi/2).$$

(b) to find

$$(i)\quad \int_0^1 \cos(2\pi t)\sin(2\pi t)dt,\ \ (ii)\quad \int_0^1 \cos^2(2\pi t)dt.$$

Answers: (b) 0 and $1/2$.

Solution

(a) We have

i. $\cos(\theta - \pi/2) = 0.5(e^{j(\theta-\pi/2)} + e^{-j(\theta-\pi/2)}) = -j0.5(e^{j\theta} - e^{-j\theta}) = \sin(\theta)$

ii. $-\sin(\theta - \pi/2) = 0.5j(e^{j(\theta-\pi/2)} - e^{-j(\theta-\pi/2)}) = 0.5j(-j)(e^{j\theta} + e^{-j\theta}) = \cos(\theta)$

iii. $\sin(\theta + \pi/2) = (je^{j\theta} + je^{-j\theta})/(2j) = \cos(\theta)$

(b) i. $\cos(2\pi t)\sin(2\pi t) = (1/4j)(e^{j4\pi t} - e^{-j4\pi t})$ so that

$$\int_0^1 \cos(2\pi t)\sin(2\pi t)dt = \frac{1}{4j}\frac{e^{j4\pi t}}{4\pi j}|_0^1 + \frac{1}{4j}\frac{e^{-j4\pi t}}{4\pi j}|_0^1 = 0 + 0 = 0$$

ii. We have

$$\cos^2(2\pi t) = \frac{1}{4}(e^{j4\pi t} + 2 + e^{-j4\pi t}) = \frac{1}{2}(1 + \cos(4\pi t))$$

so that its integral is $1/2$ since the integral of $\cos(4\pi t)$ is over two of its periods and it is zero.

0.3 Use Euler's identity to

(a) show the identities

$$
\begin{array}{ll}
(i) & \cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \\
(ii) & \sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta),
\end{array}
$$

(b) find an expression for $\cos(\alpha)\cos(\beta)$, and for $\sin(\alpha)\sin(\beta)$.

Answers: $e^{j\alpha}e^{j\beta} = \cos(\alpha+\beta) + j\sin(\alpha+\beta) = [\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)] + j[\sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)]$.

Solution
(a) Using Euler's identity the product

$$
\begin{aligned}
e^{j\alpha}e^{j\beta} &= (\cos(\alpha) + j\sin(\alpha))(\cos(\beta) + j\sin(\beta)) \\
&= [\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)] + j[\sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)]
\end{aligned}
$$

while

$$
e^{j(\alpha+\beta)} = \cos(\alpha+\beta) + j\sin(\alpha+\beta)
$$

so that equating the real and imaginary parts of the above two equations we get the desired trigonometric identities.

(b) We have

$$
\begin{aligned}
\cos(\alpha)\cos(\beta) &= 0.5(e^{j\alpha} + e^{-j\alpha})\,0.5(e^{j\beta} + e^{-j\beta}) \\
&= 0.25(e^{j(\alpha+\beta)} + e^{-j(\alpha+\beta)}) + 0.25(e^{j(\alpha-\beta)} + e^{-j(\alpha-\beta)}) \\
&= 0.5\cos(\alpha+\beta) + 0.5\cos(\alpha-\beta)
\end{aligned}
$$

Now,

$$
\begin{aligned}
\sin(\alpha)\sin(\beta) &= \cos(\alpha-\pi/2)\cos(\beta-\pi/2) \\
&= 0.5\cos(\alpha-\pi/2+\beta-\pi/2) + 0.5\cos(\alpha-\pi/2-\beta+\pi/2) \\
&= 0.5\cos(\alpha+\beta-\pi) + 0.5\cos(\alpha-\beta) \\
&= -0.5\cos(\alpha+\beta) + 0.5\cos(\alpha-\beta)
\end{aligned}
$$

0.4 Consider the calculation of roots of an equation $z^N = \alpha$ where $N \geq 1$ is an integer and $\alpha = |\alpha|e^{j\phi}$ a nonzero complex number.

(a) First verify that there are exactly N roots for this equation and that they are given by $z_k = re^{j\theta_k}$ where $r = |\alpha|^{1/N}$ and $\theta_k = (\phi + 2\pi k)/N$ for $k = 0, 1, \cdots, N-1$.

(b) Use the above result to find the roots of the following equations

$$(i)\ z^2 = 1, \quad (ii)\ z^2 = -1, \quad (iii)\ z^3 = 1, \quad (iv)\ z^3 = -1.$$

and plot them in a polar plane (i..e., indicating their magnitude and phase). Explain how the roots are distributed in the polar plane.

Answers: Roots of $z^3 = -1 = 1e^{j\pi}$ are $z_k = 1e^{j(\pi+2\pi k)/3}$, $k = 0, 1, 2$, equally spaced around circle of radius r.

Solution

(a) Replacing $z_k = |\alpha|^{1/N}e^{j(\phi+2\pi k)/N}$ in z^N we get $z_k^N = |\alpha|e^{j(\phi+2\pi k)} = |\alpha|e^{j(\phi)} = \alpha$ for any value of $k = 0, \cdots, N-1$.

(b) Applying the above result we have:

- For $z^2 = 1 = 1e^{j2\pi}$ the roots are $z_k = 1e^{j(2\pi+2\pi k)/2}$, $k = 0, 1$. When $k = 0$, $z_0 = e^{j\pi} = -1$ and $z_1 = e^{j2\pi} = 1$.
- When $z^2 = -1 = 1e^{j\pi}$ the roots are $z_k = 1e^{j(\pi+2\pi k)/2}$, $k = 0, 1$. When $k = 0$, $z_0 = e^{j\pi/2} = j$, and $z_1 = e^{j3\pi/2} = -j$.
- For $z^3 = 1 = 1e^{j2\pi}$ the roots are $z_k = 1e^{j(2\pi+2\pi k)/3}$, $k = 0, 1, 2$. When $k = 0$, $z_0 = e^{j2\pi/3}$; for $k = 1$, $z_1 = e^{j4\pi/3} = e^{-j2\pi/3} = z_0^*$; and for $k = 2$, $z_2 = 1e^{j(2\pi)} = 1$.
- When $z^3 = -1 = 1e^{j\pi}$ the roots are $z_k = 1e^{j(\pi+2\pi k)/3}$, $k = 0, 1, 2$. When $k = 0$, $z_0 = e^{j\pi/3}$; for $k = 1$, $z_1 = e^{j\pi} = -1$; and for $k = 2$, $z_2 = 1e^{j(5\pi)/3} = 1e^{j(-\pi)/3} = z_0^*$

(c) Notice that the roots are equally spaced around a circle of radius r and that the complex roots appear as pairs of complex conjugate roots.

0.5 Consider a function of $z = 1 + j1$, $w = e^z$

(a) Find (i) $\log(w)$, (ii) $\mathcal{R}e(w)$, (iii) $\mathcal{I}m(w)$
(b) What is $w + w^*$, where w^* is the complex conjugate of w?
(c) Determine $|w|$, $\angle w$ and $|\log(w)|^2$?
(d) Express $\cos(1)$ in terms of w using Euler's identity.

Answers: $\log(w) = z$; $w + w^* = 2\mathcal{R}e[w] = 2e\cos(1)$.

Solution
(a) If $w = e^z$ then

$$\log(w) = z = 1 + j1$$

given that the log and e functions are the inverse of each other.
The real and imaginary of w are

$$w = e^z = e^1 e^{j1} = \underbrace{e\cos(1)}_{\text{real part}} + j \underbrace{e\sin(1)}_{\text{imaginary part}}$$

(b) The imaginary parts are cancelled and the real parts added twice in

$$w + w^* = 2\mathcal{R}e[w] = 2e\cos(1)$$

(c) Replacing z

$$w = e^z = e^1 e^{j1}$$

so that $|w| = e$ and $\angle w = 1$.
Using the result in (a)

$$|\log(w)|^2 = |z|^2 = 2$$

(d) According to Euler's equation

$$\cos(1) = 0.5(e^j + e^{-j}) = 0.5\left(\frac{w}{e} + \frac{w^*}{e}\right)$$

which can be verified using $w + w^*$ obtained above.

0.6 A phasor can be thought of as a vector, representing a complex number, rotating around the polar plane at a certain frequency in radians/second. The projection of such a vector onto the real axis gives a cosine with a certain amplitude and phase. This problem will show the algebra of phasors which would help you with some of the trigonometric identities that are hard to remember.

(a) When you plot $y(t) = A\sin(\Omega_0 t)$ you notice that it is a cosine $x(t) = A\cos(\Omega_0 t)$ shifted in time, i.e.,

$$y(t) = A\sin(\Omega_0 t) = A\cos(\Omega_0(t - \Delta_t)) = x(t - \Delta_t)$$

how much is this shift Δ_t? Better yet, what is $\Delta_\theta = \Omega_0\Delta_t$ or the shift in phase? One thus only need to consider cosine functions with different phase shifts instead of sines and cosines.

(b) From above, the phasor that generates $x(t) = A\cos(\Omega_0 t)$ is Ae^{j0} so that $x(t) = \mathcal{R}e[Ae^{j0}e^{j\Omega_0 t}]$. The phasor corresponding to the sine $y(t)$ should then be $Ae^{-j\pi/2}$. Obtain an expression for $y(t)$ similar to the one for $x(t)$ in terms of this phasor.

(c) From the above results, give the phasors corresponding to $-x(t) = -A\cos(\Omega_0 t)$ and $-y(t) = -\sin(\Omega_0 t)$. Plot the phasors that generate $\cos$, $\sin$, $-\cos$ and $-\sin$ for a given frequency. Do you see now how these functions are connected? How many radians do you need to shift in positive or negative direction to get a sine from a cosine, etc.

(d) Suppose then you have the sum of two sinusoids, for instance $z(t) = x(t) + y(t) = A\cos(\Omega_0 t) + A\sin(\Omega_0 t)$, adding the corresponding phasors for $x(t)$ and $y(t)$ at some time, e.g., $t = 0$, which is just a sum of two vectors, you should get a vector and the corresponding phasor. For $x(t)$, $y(t)$, obtain their corresponding phasors and then obtain from them the phasor corresponding to $z(t) = x(t) + y(t)$.

(e) Find the phasors corresponding to

$$(i)\ 4\cos(2t + \pi/3),\ (ii)\ -4\sin(2t + \pi/3),\ (iii)\ 4\cos(2t + \pi/3) - 4\sin(2t + \pi/3)$$

Answers: $\sin(\Omega_0 t) = \cos(\Omega_0(t - T_0/4)) = \cos(\Omega_0 t - \pi/2)$ since $\Omega_0 = 2\pi/T_0$; $z(t) = \sqrt{2}A\cos(\Omega_0 t - \pi/4)$; (e) (i) $4e^{j\pi/3}$; (iii) $4\sqrt{2}e^{j7\pi/12}$.

Solution

(a) Shifting to the right a cosine by a fourth of its period we get a sinusoid, thus

$$\sin(\Omega_0 t) = \cos(\Omega_0(t - T_0/4)) = \cos(\Omega_0 t - \Omega_0 T_0/4) = \cos(\Omega_0 t - \pi/2)$$

since $\Omega_0 = 2\pi/T_0$ or $\Omega_0 T_0 = 2\pi$.

(b) The phasor that generates a sine is $Ae^{-j\pi/2}$ since

$$y(t) = \mathcal{R}e[Ae^{-j\pi/2}e^{j\Omega_0 t}] = \mathcal{R}e[Ae^{j(\Omega_0 t - \pi/2)}] = A\cos(\Omega_0 t - \pi/2)$$

which equals $A\sin(\Omega_0 t)$.

(c) The phasors corresponding to $-x(t) = -A\cos(\Omega_0 t) = A\cos(\Omega_0 t + \pi)$ is $Ae^{j\pi}$. For

$$-y(t) = -A\sin(\Omega_0 t) = -A\cos(\Omega_0 t - \pi/2) = A\cos(\Omega_0 t - \pi/2 + \pi) = A\cos(\Omega_0 t + \pi/2)$$

the phasor is $Ae^{j\pi/2}$. Thus, relating any sinusoid to the corresponding cosine, the magnitude and angle of this cosine gives the magnitude and phase of the phasor that generates the given sinusoid.

(d) If $z(t) = x(t) + y(t) = A\cos(\Omega_0 t) + A\sin(\Omega_0 t)$, the phasor corresponding to $z(t)$ is the sum of the phasors Ae^{j0}, corresponding to $A\cos(\Omega_0 t)$, with the phasor $Ae^{-j\pi/2}$, corresponding to $A\sin(\Omega_0 t)$, which gives $\sqrt{2}Ae^{-j\pi/4}$ (equivalently the sum of a vector with length A and angle 0 with another vector of length A and angle $-\pi/2$). We have that

$$z(t) = \mathcal{R}e\left[\sqrt{2}Ae^{-j\pi/4}e^{j\Omega_0 t}\right] = \sqrt{2}A\cos(\Omega_0 t - \pi/4)$$

(e) i. Phasor $4e^{j\pi/3}$

ii. $-4\sin(2t + \pi/3) = 4\cos(2t + \pi/3 + \pi/2)$ with phasor $4e^{j5\pi/6}$

iii. We have

$$\begin{aligned} 4\cos(2t + \pi/3) - 4\sin(2t + \pi/3) &= \mathcal{R}e[(4e^{j\pi/3} + 4e^{j(\pi/2+\pi/3)})e^{j2t}] \\ &= \mathcal{R}e[4e^{j\pi/3}\underbrace{(1 + e^{j\pi/2})}_{\sqrt{2}e^{j\pi/4}}e^{j2t}] \\ &= \mathcal{R}e[4\sqrt{2}e^{j7\pi/12}e^{j2t}] \end{aligned}$$

so that the phasor is $4\sqrt{2}e^{j7\pi/12}$

0.7 To get an idea of the number of bits generated and processed by a digital system consider the following applications:

(a) A compact disc (CD) is capable of storing 75 minutes of "CD quality" stereo (left and right channels are recorded) music. Calculate the number of bits that are stored in the CD as raw data.
Hint: find out what 'CD quality' means in the binary representation of each sample.

(b) Find out what the vocoder in your cell phone is used for. To attaining "telephone quality" voice you use a sampling rate of $10,000$ samples/sec, and that each sample is represented by 8 bits. Calculate the number of bits that your cell-phone has to process every second that you talk. Why would you then need a vocoder?

(c) Find out whether text messaging is cheaper or more expensive than voice. Explain how the text messaging works.

(d) Find out how an audio CD and an audio DVD compare. Find out why it is said that a vinyl long-play record reproduces sounds much better. Are we going backwards with digital technology in music recording? Explain.

(e) To understand why video streaming in the internet is many times of low quality, consider the amount of data that needs to be processed by a video compressor every second. Assume the size of a video frame, in pixels, is 352×240, and that an acceptable quality for the image is obtained by allocating 8 bits/pixel and to avoid jerking effects we use 60 frames/second.

- How many pixels need to be processed every second?
- How many bits would be available for transmission every second?
- The above is raw data, compression changes the whole picture (literally), find out what some of the compression methods are.

Answers: (a) About 6.4 Gbs; vocoder (short for voice encoder) reduces number of transmitted bits while keeping voice recognizable.

Solution
(a) Assuming a maximum frequency of 22.05 kHz for the acoustic signal, the numbers of bytes (8 bits per byte) for two channels (stereo) and a 75 minutes recording is greater or equal to: $2 \times 22,050$ samples/channel/second $\times$ 2 bytes/sample $\times$ 2 channels $\times$ 75 minutes $\times$ 60 seconds/minute $= 7.938 \times 10^8$ bytes. Multiplying by 8 we get the number of bits. CD quality means that the signal is sampled at 44.1 kHz and each sample is represented by 16 bits or 2 bytes.

(b) The raw data would consist of 8 (bits/sample) $\times 10,000$ (samples/sec)=$80,000$ bits/sec. The vocoder is part of a larger unit called a digital signal processor chip set. It uses various procedures to reduce the number of bits that are transmitted while still keeping your voice recognizable. When there is silence it does not transmit, letting another signal use the channel during pauses.

(c) Texting between cell phones is possible by sending short messages (160 characters) using the short message services (SMS). Whenever your cell-phone communicates with the cell phone tower there is an exchange of messages over the control channel for localization, and call setup. This channel provides a pathway for SMS messages by sending packets of data. Except for the cost of storing messages, the procedure is rather inexpensive and convenient to users.

(d) For CD audio the sampling rate is 44.1 kHz with 16 bits/sample. For DVD audio the sampling rate is 192 kHz with 24 bits/sample. The sampling process requires getting rid of high frequencies in the signal, also each sample is only approximated by the binary representation, so analog recording could sound better in some cases.

(e) The number of pixels processed every second is: 352×240 pixels/frame $\times 60$ frames/sec. The number of bits available for transmission every second is obtained by multiplying the above answer by 8 bits/pixel. There many compression methods JPEG, MPEG, etc.

0.8 The geometric series

$$S = \sum_{n=0}^{N-1} \alpha^n$$

will be used quite frequently in the next chapters so let us look at some of its properties:

(a) Suppose $\alpha = 1$ what is S equal to?

(b) Suppose $\alpha \neq 1$ show that

$$S = \frac{1 - \alpha^N}{1 - \alpha}$$

Verify that $(1 - \alpha)S = (1 - \alpha^N)$. Why do you need the constraint that $\alpha \neq 1$? Would this sum exist if $\alpha > 1$? Explain.

(c) Suppose now that $N = \infty$, under what conditions will S exist? if it does, what would S be equal to? Explain.

(d) Suppose again that $N = \infty$ in the definition of S. The derivative of S with respect to α is

$$S_1 = \frac{dS}{d\alpha} = \sum_{n=0}^{\infty} n\alpha^{n-1}$$

obtain a rational expression to find S_1 .

Answers: $S = N$ when $\alpha = 1$, $S = (1 - \alpha^N)/(1 - \alpha)$ when $\alpha \neq 1$.

Solution
(a) If $\alpha = 1$ then

$$S = \sum_{n=0}^{N-1} 1 = \underbrace{1 + 1 + \cdots + 1}_{N \text{ times}} = N$$

(b) The expression

$$\begin{aligned} S(1 - \alpha) &= S - \alpha S \\ &= (1 + \alpha + \cdots + \alpha^{N-1}) - (\alpha + \alpha^2 + \cdots + \alpha^{N-1} + \alpha^N) \\ &= 1 - \alpha^N \end{aligned}$$

as the intermediate terms cancel. So that

$$S = \frac{1 - \alpha^N}{1 - \alpha}, \quad \alpha \neq 1$$

Since we do not want the denominator $1 - \alpha$ to be zero, the above requires that $\alpha \neq 1$. If $\alpha = 1$ the sum was found in (a). As a finite sum, it exists for any finite values of α.
Putting (a) and (b) together we have

$$S = \begin{cases} (1 - \alpha^N)/(1 - \alpha) & \alpha \neq 1 \\ N & \alpha = 1 \end{cases}$$

(c) If N is infinite, the sum is of infinite length and we need to impose the condition that $|\alpha| < 1$ so that α^n decays as $n \to \infty$. In that case, the term $\alpha^N \to 0$ as $N \to \infty$, and the sum is

$$S = \frac{1}{1 - \alpha} \qquad |\alpha| < 1$$

If $|\alpha| \geq 1$ this sum does not exist, i.e., it becomes infinite.
(d) The derivative becomes

$$S_1 = \frac{dS}{d\alpha} = \sum_{n=0}^{\infty} n\alpha^{n-1} = \frac{1}{(1-\alpha)^2}.$$

0.2 Problems using MATLAB

0.9 Derivative and finite difference — Let $y(t) = dx(t)/dt$, where $x(t) = 4\cos(2\pi t)$, $-\infty < t < \infty$. Find $y(t)$ analytically and determine a value of T_s for which $\Delta[x(nT_s)]/T_s = y(nT_s)$ (consider as possible values $T_s = 0.01$ and $T_s = 0.1$). Use the MATLAB function *diff* or create your own to compute the finite difference. Plot the finite difference in the range $[0, 1]$ and compare it with the actual derivative $y(t)$ in that range. Explain your results for the given values of T_s.
Answers: $y(t) = -8\pi\sin(2\pi t)$ has same sampling period as $x(t)$, $T_s \leq 0.5$; $T_s = 0.01$ gives better results.

Solution
The derivative is

$$y(t) = \frac{dx(t)}{dt} = -8\pi\sin(2\pi t)$$

which has the same frequency as $x(t)$, thus the sampling period should be like in the previous problem, $T_s \leq 0.5$.

```
% Pr. 0.9
clear all
% actual derivative
Tss=0.0001;t1=0:Tss:3;
y=-8*pi*sin(2*pi*t1);
figure(2)
% forward difference
Ts=0.01;t=[0:Ts:3];N=length(t);
subplot(211)
xa=4*cos(2*pi*t);  % sampled signal
der1_x=forwardiff(xa,Ts,t,y,t1);

clear der1_x
% forward difference
Ts=0.1;t=[0:Ts:3];N=length(t);
subplot(212)
xa=4*cos(2*pi*t);  % sampled signal
der1_x=forwardiff(xa,Ts,t,y,t1);
```

The function *forwardiff* computes and plots the forward difference and the actual derivative.

```
function der=forwardiff(xa,Ts,t,y,t1)
% % forward difference
% % xa: sampled signal using Ts
% % y: actual derivative defined in t
 N=length(t);n=0:N-2;
 der=diff(xa)/Ts;
 stem(n*Ts,der,'filled');grid;xlabel('t, nT_s')
 hold on
 plot(t1,y,'r'); legend('forward difference','derivative')
 hold off
```

For $T_s = 0.1$ the finite difference looks like the actual derivative but shifted, while for $T_s = 0.01$ it does not.

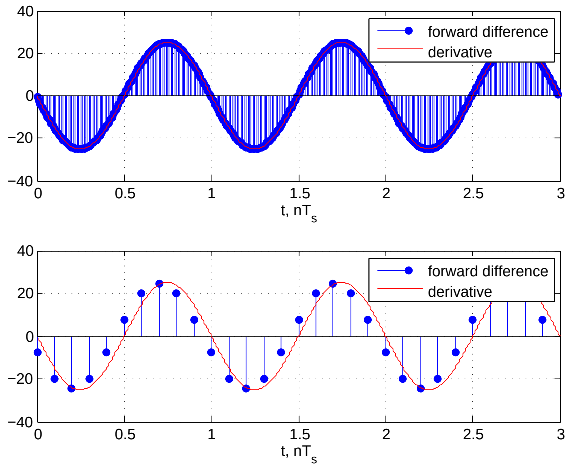


Figure 1: Problem 9: $T_s = 0.01$ sec (top) and $T_s = 0.1$ sec (bottom)

0.10 Backward difference — Another definition for the finite difference is the backward difference:

$$\Delta_1[x(nT_s)] = x(nT_s) - x((n-1)T_s)$$

($\Delta_1[x(nT_s)]/T_s$ approximates the derivative of $x(t)$).

(a) Indicate how this new definition connects with the finite difference defined earlier in this Chapter.

(b) Solve Problem 9 with MATLAB using this new finite difference and compare your results with the ones obtained there.

(c) For the value of $T_s = 0.1$, use the average of the two finite differences to approximate the derivative of the analog signal $x(t)$. Compare this result with the previous ones. Provide an expression for calculating this new finite difference directly.

Answers: $\Delta_1[x(n+1)] = x(n+1) - x(n) = \Delta[x(n)]$; $0.5\,\{\Delta_1[x(n)] + \Delta[x(n)]\} = 0.5[x(n+1) - x(n-1)]$.

Solution
(a) The backward finite difference (let $T_s = 1$ for simplicity)

$$\Delta_1[x(n)] = x(n) - x(n-1)$$

is connected with the forward finite difference $\Delta[x(n)]$ given in the chapter as follows

$$\Delta_1[x(n+1)] = x(n+1) - x(n) = \Delta[x(n)]$$

That is, $\Delta[x(n)]$ is $\Delta_1[x(n)]$ shifted one sample to the left.
(b) (c) The average of the two finite differences gives

$$0.5\,\{\Delta_1[x(n)] + \Delta[x(n)]\} = 0.5[x(n+1) - x(n-1)]$$

which gives a better approximation to the derivative than either of the given finite differences. The following script is used to compute Δ_1 and the average.

```
%  Pro 0.10
%  compares forward/backward differences
%  with new average difference
Ts=0.1;
for k=0:N-2,
   x1=4*cos(2*pi*(k-1)*Ts);
   x2=4*cos(2*pi*k*Ts);
   der_x(k+1)=x2-x1; % backward difference
end
der_x=der_x/Ts;
Tss=0.0001;t1=0:Tss:3;
y=-8*pi*sin(2*pi*t1); % actual derivative
n=0:N-2;
figure(3)
 subplot(211)
 stem(n*Ts,der_x,'k');grid
 hold on
 stem(n*Ts,der1_x,'b','filled') % derv1_x forward difference
                                % from Pr.  0.2
 hold on
 plot(t1,y,'r'); xlabel('t, nT_s')
 legend('bck diff','forwd diff', 'derivative')
```

```
hold off
subplot(212)
stem(n*Ts,0.5*(der_x+der1_x));grid;xlabel('t, nT_s') % average
hold on
plot(t1,y,'r')
hold off
legend('average diff','derivative')
```

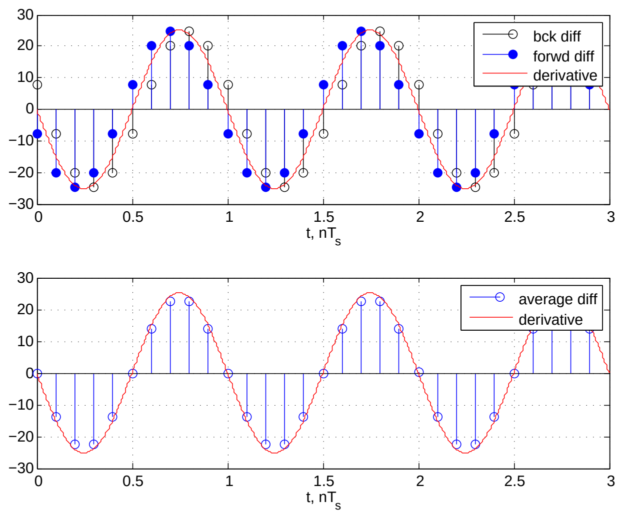


Figure 2: Problem 10: Comparison of different finite differences.

0.11 Sums and Gauss — Three laws in the computation of sums are

Distributive: $\sum_k ca_k = c\sum_k a_k$

Associative: $\sum_k (a_k + b_k) = \sum_k a_k + \sum_k b_k$

Commutative: $\sum_k a_k = \sum_{p(k)} a_{p(k)}$

for any permutation $p(k)$ of the set of integers k in the summation.

(a) Explain why the above rules make sense when computing sums. To do that consider

$$\sum_k a_k = \sum_{k=0}^{2} a_k, \text{ and } \sum_k b_k = \sum_{k=0}^{2} b_k.$$

Let c be a constant, and choose any permutation of the values $[0, 1, 2]$ for instance $[2, 1, 0]$ or $[1, 0, 2]$.

(b) The trick that Gauss played when he was a preschooler can be explained by using the above rules. Suppose you want to find the sum of the integers from 0 to $10,000$ (Gauss did it for integers between 0 and 100 but he was then just a little boy, and we can do better!). That is, we want to find S where

$$S = \sum_{k=0}^{10,000} k = 0 + 1 + 2 + \cdots + 10000$$

to do so consider

$$2S = \sum_{k=0}^{10,000} k + \sum_{k=10,000}^{0} k$$

and apply the above rules to find then S. Come up with a MATLAB function of your own to do this sum.

(c) Find the sum of an arithmetic progression

$$S_1 = \sum_{k=0}^{N} (\alpha + \beta k)$$

for constants α and β, using the given three rules.

(d) Find out if MATLAB can do these sums symbolically, i.e., without having numerical values. Use the found symbolic function to calculate the sum in the previous item when $\alpha = \beta = 1$ and $N = 100$.

Answers: $N = 10,000$, $S = N(N+1)/2$; $S_1 = \alpha(N+1) + \beta(N(N+1))/2$.

Solution

(a) The distributive and the associative laws are equivalent to the ones for integrals, indeed

$$\sum_k ca_k = c(\cdots + a_{-1} + a_0 + a_1 + \cdots) = c\sum_k a_k$$

since c does not depend on k. Likewise

$$\sum_k [a_k + b_k] = (\cdots + a_{-1} + b_{-1} + a_0 + b_0 + a_1 + b_1 \cdots) = \sum_k a_k + \sum_k b_k$$

Finally, when adding a set of numbers the order in which they are added does not change the result. For instance,

$$a_0 + a_1 + a_2 + a_3 = a_0 + a_2 + a_1 + a_3$$

(b) Gauss' trick can be shown in general as follows. Let $S = \sum_{k=0}^{N} k$ then

$$2S = \sum_{k=0}^{N} k + \sum_{k=N}^{0} k$$

letting $\ell = -k + N$ in the second summation we have

$$2S = \sum_{k=0}^{N} k + \sum_{\ell=0}^{N} (N - \ell) = \sum_{k=0}^{N} (k + N - k) = N \sum_{k=0}^{N} 1 = N(N+1)$$

where we let the dummy variables of the two sums be equal. We thus have that for $N = 10^4$

$$S = \frac{N(N+1)}{2} = \frac{10^4(10^4+1)}{2} \approx 0.5 \times 10^8$$

(c) Using the above properties of the sum,

$$\begin{aligned} S_1 &= \sum_{k=0}^{N} (\alpha + \beta k) = \alpha \sum_{k=0}^{N} 1 + \beta \sum_{k=0}^{N} k \\ &= \alpha(N+1) + \beta \frac{N(N+1)}{2} \end{aligned}$$

(d) The following script computes numerically and symbolically the various sums.

```
% Pro  0.11
clear all
% numeric
N=100;
S1=[0:1:N];
S2=[N:-1:0];
S=sum(S1+S2)/2
% symbolic
syms S1 N alpha beta k
simple(symsum(alpha+beta*k,0,N))
% computing sum for specific values of alpha, beta and N
subs(symsum(alpha+beta*k,0,N),{alpha,beta,N},{1,1,100})

S = 5050

((2*alpha + N*beta)*(N + 1))/2

5151
```

The answers shown at the bottom.

0.12 Integrals and sums — Suppose you wish to find the area under a signal $x(t)$ using sums. You will need the following result found above

$$\sum_{n=0}^{N} n = \frac{N(N+1)}{2}$$

(a) Consider first $x(t) = t$, $0 \leq t \leq 1$, and zero otherwise. The area under this signal is 0.5. The integral can be approximated from above and below as

$$\sum_{n=1}^{N-1} (nT_s)T_s < \int_0^1 t dt < \sum_{n=1}^{N} (nT_s)T_s$$

where $NT_s = 1$ (i.e., we divide the interval $[0, 1]$ into N intervals of width T_s). Graphically show for $N = 4$ that the above equation makes sense by showing the right and left bounds as approximations for the area under $x(t)$.

(b) Let $T_s = 0.001$, use the symbolic function *symsum* to compute the left and right bounds for the above integral. Find the average of these results and compare it with the actual value of the integral.

(c) Verify the symbolic results by finding the sums on the left and the right of the above inequality using the summation given at the beginning of the problem. What happens when $N \to \infty$.

(d) Write a MATLAB script to compute the area under the signal $y(t) = t^2$ from $0 \leq t \leq 1$. Let $T_s = 0.001$. Compare the average of the lower and upper bounds to the value of the integral.

Answer: For $T_s = 1/N$

$$\left[\frac{(N-1)(N-2) + 2(N-1)}{2N^2}\right] \leq \frac{1}{2} \leq \left[\frac{(N-1)(N-2) + 2(N-1)}{2N^2}\right] + \frac{1}{N}$$

Solution

(a) The following figure shows the upper and lower bounds when approximating the integral of t:

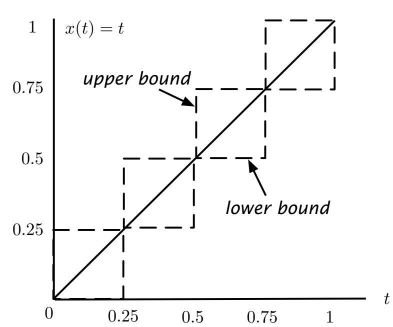


Figure 3: Problem 12: Upper and lower bounds of the integral of t when $N = 4$.

(b) (c) The lower bound for the integral is

$$\begin{aligned} S_\ell &= \sum_{n=1}^{N-1} (nT_s)T_s = T_s^2 \sum_{n=1}^{N-1} n = T_s^2 \sum_{\ell=0}^{N-2} (\ell + 1) \\ &= T_s^2 \left[\frac{(N-1)(N-2)}{2} + (N-1)\right] \end{aligned}$$

The definite integral is

$$\int_0^1 t dt = \frac{1}{2}$$

The upper bound is

$$S_u = \sum_{n=1}^{N} (nT_s)T_s = S_\ell + NT_s^2$$

Letting $NT_s = 1$, or $T_s = 1/N$ we have then that

$$\left[\frac{(N-1)(N-2)+2(N-1)}{2N^2}\right] \leq \frac{1}{2} \leq \left[\frac{(N-1)(N-2)+2(N-1)}{2N^2}\right] + \frac{1}{N}$$

for large N the upper and the lower bound tend to $1/2$.

The following script computes the lower and upper bound of the integral of t.

```
% Pr. 0.12
clear all
Ts=0.001;N=1/Ts;
% integral of t from 0 to 1 is 0.5
syms S1 n T k
% lower bound
 n=subs(N);T=subs(Ts);
 y=simple(symsum(k*T^2,1,n-1));
 yy=subs(y)
% upper bound
 z=simple(symsum(k*T^2,1,n));
 zz=subs(z)
% average
 int= 0.5*(yy+zz)
```

giving the following results (the actual integral is $1/2$).

```
yy = 0.4995
zz =  0.5005
int =  0.5000
```

(d) For $y(t) = t^2$, $0 \leq t \leq 1$, the following script computes the upper and the lower bounds and their average:

```
%% integral of t^2 from 0 to 1 is 0.333
% lower bound
 y1=simple(symsum(k^2*T^3,1,n-1));
 yy1=subs(y1)
%  upper bound
 z1=simple(symsum(k^2*T^3,1,n));
 zz1=subs(z1)
% average
 int= 0.5*(yy1+zz1)
```

giving the following results, in this case the value of the definite integral is $1/3$.

```
yy1 = 0.3328
zz1 = 0.3338
int =   0.3333
```

0.13 Exponentials — The exponential $x(t) = e^{at}$ for $t \geq 0$ and zero otherwise is a very common continuous-time signal. Likewise, $y(n) = \alpha^n$ for integers $n \geq 0$ and zero otherwise is a very common discrete-time signal. Let us see how they are related. Do the following using MATLAB:

(a) Let $a = -0.5$, plot $x(t)$

(b) Let $a = -1$, plot the corresponding signal $x(t)$. Does this signal go to zero faster than the exponential for $a = -0.5$?

(c) Suppose we sample the signal $x(t)$ using $T_s = 1$ what would be $x(nT_s)$ and how can it be related to $y(n)$, i.e., what is the value of α that would make the two equal?

(d) Suppose that a current $x(t) = e^{-0.5t}$ for $t \geq 0$ and zero otherwise is applied to a discharged capacitor of capacitance $C = 1$ F at $t = 0$. What would be the voltage in the capacitor at $t = 1$ second?

(e) How would you obtain an approximate result to the above problem using a computer? Explain.

Answers: $0 < e^{-\alpha t} < e^{-\beta t}$ for $\alpha > \beta \geq 0$; $v_c(1) = 0.79$.

Solution
(a)(b) We have that

$$0 < e^{-\alpha t} < e^{-\beta t}$$

for $\alpha > \beta \geq 0$.

```
% Pr.  0.13
clear all
% compare two exponentials
 t=[0:0.001:10];
 x=exp(-0.5*t);
 x1=exp(-1*t);
 figure(6)
 plot(t,x,t,x1,'r');
 legend('Exponential Signal, a=-0.5','Exponential Signal, a=-1')
 grid
 axis([0 10 0 1.1 ]); xlabel('time')
```

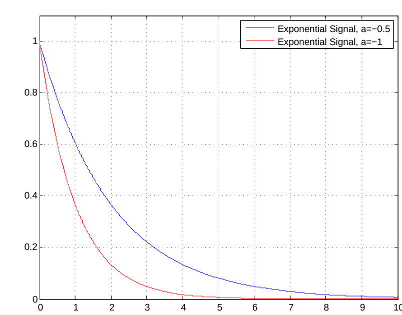


Figure 4: Problem 13: Comparison of exponentials $e^{-0.5t}$ and e^{-t} for $t \geq 0$ and 0 otherwise.

(c) Sampling $x(t) = e^{at}$ using $T_s = 1$, we get

$$x(t)|_{t=n} = e^{an} = \alpha^n$$

where $\alpha = e^a > 0$
(d) The voltage in the capacitor is given by

$$v_c(t) = \frac{1}{C}\int_0^t e^{-0.5\tau}d\tau + v_c(0)$$

with a initial voltage $v_c(0) = 0$. Letting $C = 1$, we have

$$v_c(t) = \frac{e^{-0.5\tau}}{-0.5}|_0^t = 2(1 - e^{-0.5t})$$

so that at $t = 1$ the voltage in the capacitor is $v_c(1) = 2 - 2e^{-0.5} = 0.79$.
(e) Letting $NT_s = 1$, the definite integral is approximated, from below, by

$$\sum_{n=0}^{N-1} T_s e^{-0.5(n+1)T_s}$$

if we let $\alpha = e^{-0.5T_s}$ the above sum becomes

$$T_s \sum_{n=0}^{N-1} \alpha^{n+1} = T_s\alpha\frac{1 - \alpha^N}{1 - \alpha}$$

which is computed using the following script:

```
% compute value of Int (the integral)
 N=1000;Ts=1/N;alpha=exp(- 0.5*Ts);
 Int=Ts*alpha*(1-alpha^N)/(1-alpha)

Int =  0.7867
```

approximating the analytic result found above.

0.14 **Algebra of complex numbers —** Consider complex numbers $z = 1 + j$, $w = -1 + j$, $v = -1 - j$ and $u = 1 - j$. You may use MATLAB *compass* to plot vectors corresponding to complex numbers to verify your analytic results.

(a) In the complex plane, indicate the point (x, y) that corresponds to z and then show a vector $\vec{z}$ that joins the point (x, y) to the origin. What is the magnitude and the angle corresponding to z or $\vec{z}$?

(b) Do the same for the complex numbers w, v and u. Plot the four complex numbers and find their sum $z + w + v + u$ analytically and graphically.

(c) Find the ratios z/w, w/v, and u/z. Determine the real and imaginary parts of each, as well as their magnitudes and phases. Using the ratios find u/w.

(d) The phase of a complex number is only significant when the magnitude of the complex number is significant. Consider z and $y = 10^{-16}z$, compare their magnitudes and phases. What would you say about the phase of y?

Answers: $|w| = \sqrt{2},\ \angle w = 3\pi/4$, $|v| = \sqrt{2},\ \angle v = 5\pi/4$, $|u| = \sqrt{2},\ \angle u = -\pi/4$.

Solution

(a) The point (1,1) in the two-dimensional plane corresponds to $z = 1 + j$. The magnitude and phase are

$$|z| = \sqrt{1+1} = \sqrt{2}$$
$$\angle z = \tan^{-1}(1) = \pi/4$$

(b) For the other complex numbers:

$$|w| = \sqrt{2}, \quad \angle w = \pi - \pi/4 = 3\pi/4$$
$$|v| = \sqrt{2}, \quad \angle v = \pi + \pi/4 = 5\pi/4$$
$$|u| = \sqrt{2}, \quad \angle u = -\pi/4$$

The sum of these complex numbers

$$z + w + v + u = 0$$

(c) The ratios

$$\frac{z}{w} = \frac{1+j}{-1+j} = \frac{\sqrt{2}e^{j\pi/4}}{\sqrt{2}e^{j3\pi/4}} = 1e^{-j\pi/2} = -j$$
$$\frac{w}{v} = \frac{-1+j}{-1-j} = \frac{\sqrt{2}e^{j3\pi/4}}{\sqrt{2}e^{j5\pi/4}} = 1e^{-j\pi/2} = -j$$
$$\frac{u}{z} = \frac{1-j}{1+j} = \frac{\sqrt{2}e^{-j\pi/4}}{\sqrt{2}e^{j\pi/4}} = 1e^{-j\pi/2} = -j$$

Also, multiplying numerator and denominator by the by the conjugate of the denominator we get the above results. For instance,

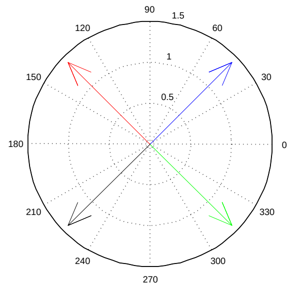
$$\frac{z}{w} = \frac{1+j}{-1+j} = \frac{(1+j)(-1-j)}{2} = \frac{-1-j-j-j^2}{2} = \frac{-2j}{2} = -j$$

and similarly for the others. Using these ratios we have

$$\frac{u}{w} = \frac{u}{z} \times \frac{z}{w} = (-j)(-j) = -1.$$

(d) $y = 10^{-6} = j10^{-6} = 10^{-6}z$ so that

$$|y| = 10^{-6}|z| = 10^{-6}$$
$$\angle y = \pi/4$$



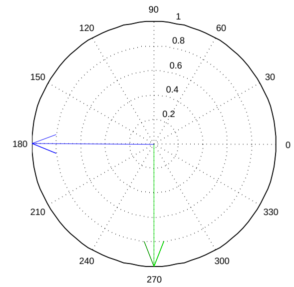


Figure 5: Problem 14: Results of complex calculations in parts (a) z, w, v, u and (b) $z/w, w/v, u/z, z/w$

Although the magnitude of y is negligible, its phase is equal to that of z.

The results are verified by the following script:

```
% Pro 0.14
z=1+j; w=-1+j; v=-1-j;u=1-j;
figure(1)
compass(1,1)
hold on
compass(-1,1,'r')
hold on
compass(-1,-1,'k')
hold on
compass(1,-1,'g')
hold off
% part (a)
abs(z)
angle(z)
% part (b)
abs(w)
angle(w)
abs(v)
angle(v)
abs(u)
angle(u)
r=z+w+v+u
%part (c)
r1=z/w
r2=w/v
r3=u/z
r4=u/z
r5=u/w
figure(2)
compass(real(r1),imag(r1))
hold on
compass(real(r2),imag(r2),'r')
hold on
```

```
compass(real(r3),imag(r3),'k')
hold on
compass(real(r4),imag(r4),'g')
hold on
compass(real(r5),imag(r5),'b')
hold off
% part (c)
z
y=z*1e-16
abs(y)
angle(y)/pi
```

Chapter 1

Continuous–time Signals

1.1 Basic Problems

1.1 Consider the following continuous-time signal

$$x(t) = \begin{cases} 1-t & 0 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Carefully plot $x(t)$ and then find and plot the following signals:

(a) $x(t+1)$, $x(t-1)$ and $x(-t)$

(b) $0.5[x(t) + x(-t)]$ and $0.5[x(t) - x(-t)]$

(c) $x(2t)$ and $x(0.5t)$

(d) $y(t) = dx(t)/dt$ and

$$z(t) = \int_{-\infty}^{t} y(\tau)/d\tau$$

Answers: $x(t+1)$ is $x(t)$ shifted left by 1; $0.5[x(t) + x(-t)]$ discontinuous at $t = 0$.

Solution

Notice that $0.5[x(t) + x(-t)]$, the even component of $x(t)$, is discontinuous at $t = 0$, it is 1 at $t = 0$ but 0.5 at $t \pm \epsilon$ for $\epsilon \to 0$. Likewise the odd component of $x(t)$, or $0.5[x(t) - x(-t)]$, must be zero at $t = 0$ so that when added to the even component one gets $x(t)$.

$z(t)$ equals $x(t)$. See Fig. 1.

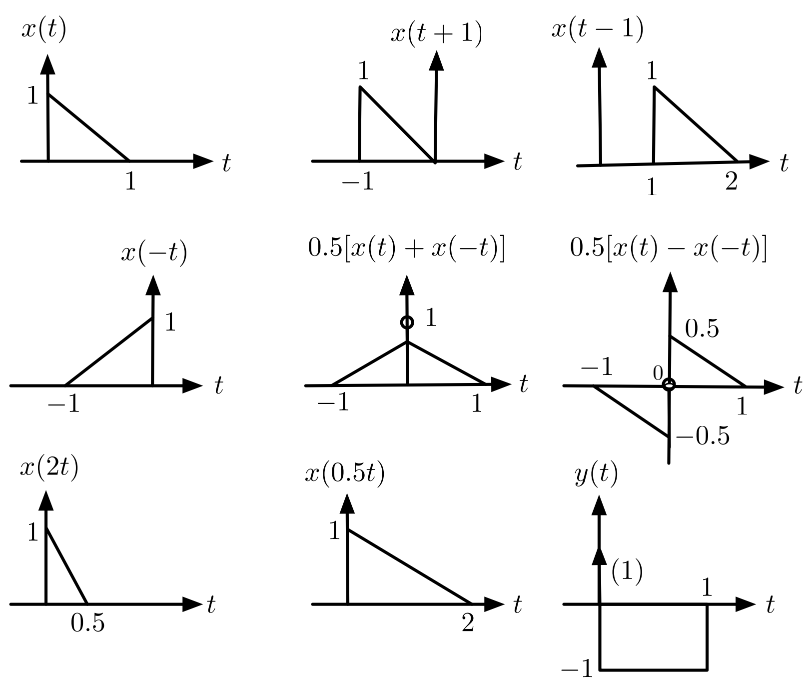


Figure 1.1: Problem 1

1.2 The following problems relate to the symmetry of the signal:

(a) Consider a causal exponential $x(t) = e^{-t}u(t)$.

i. Plot $x(t)$ and explain why it is called causal. Is $x(t)$ an even or an odd signal?

ii. Is it true that $0.5e^{-|t|}$ is the even component of $x(t)$? Explain

(b) Using Euler's identity $x(t) = e^{jt} = \cos(t) + j\sin(t)$. Find the even $x_e(t)$ and the odd $x_o(t)$ components of $x(t)$.

(c) A signal $x(t)$ is known to be even, and not exactly zero for all time, explain why

$$\int_{-\infty}^{\infty} x(t)\sin(\Omega_0 t)dt = 0.$$

(d) Is it true that

$$\int_{-\infty}^{\infty} [x(t) + x(-t)]\sin(\Omega_0 t)dt$$

for any signal $x(t)$ which is not exactly zero for all time?

Answer: (a) (ii) yes, it is true; (b) $x_e(t) = \cos(t)$; (c) integrand is odd; (d) $x(t) + x(-t)$ is even.

Solution

(a) We have that

i. $x(t)$ is causal because it is zero for $t < 0$. It is neither even nor odd.

ii. Yes, the even component of $x(t)$ is

$$\begin{aligned} x_e(t) &= 0.5[x(-t) + x(t)] \\ &= 0.5[e^t u(-t) + e^{-t}u(t)] = 0.5e^{-|t|} \end{aligned}$$

(b) $x(t) = \cos(t) + j\sin(t)$ is a complex signal, $x_e(t) = 0.5[e^{jt} + e^{-jt}] = \cos(t)$ so $x_o(t) = j\sin(t)$.

(c) The product of the even signal $x(t)$ with the sine, which is odd, gives an odd signal and because of this symmetry the integral is zero.

(d) Yes, because $x(t)+x(-t) = 2x_e(t)$, i.e., twice the even component of $x(t)$, and multiplied by the sine it is an odd function.

1.3 Do reflection and time-shifting commute? That is, do the two block diagrams in Fig. 1.2 provide identical signals, i.e., is $y(t)$ equal to $z(t)$? To provide an answer to this consider the signal $x(t)$ shown in Fig. 1.2 is

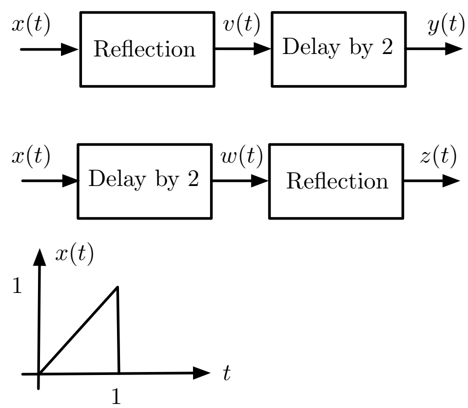


Figure 1.2: Problem 3

the input to the two block diagrams. Find $y(t)$ and $z(t)$, plot them and compare these plots. What is your conclusion? Explain.
Answers: Operations do not commute.

Solution
The signal $x(t) = t[u(t) - u(t-1)]$ so that its reflection is

$$v(t) = x(-t) = -t[u(-t) - u(-t-1)]$$

and delaying $v(t)$ by 2 is

$$\begin{aligned} y(t) &= v(t-2) = -(t-2)[u(-(t-2)) - u(-(t-2)-1)] \\ &= (-t+2)[u(-t+2) - u(-t+1)] = (2-t)[u(t-1) - u(t-2)] \end{aligned}$$

On the other hand, the delaying of $x(t)$ by 2 gives

$$w(t) = x(t-2) = (t-2)[u(t-2) - u(t-3)]$$

which when reflected gives

$$z(t) = w(-t) = (-t-2)[u(-t-2) - u(-t-3)]$$

Comparing $y(t)$ and $z(t)$ we can see that these operations do not commute, that the order in which these operations are done cannot be changed, so that $y(t) \neq z(t)$ as shown in Fig. 1.3.

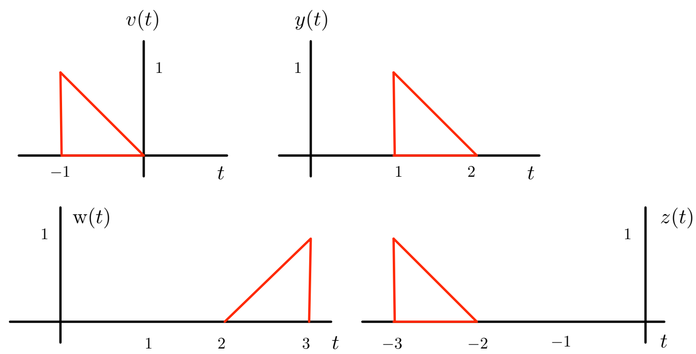


Figure 1.3: Problem 3: Reflection and delaying do not commute, $y(t) \neq z(t)$.

1.4 The following problems relate to the periodicity of signals:

(a) Determine the frequency Ω_0 in rad/sec, the corresponding frequency f_0 in Hz, and the fundamental period T_0 sec of these signals defined in $-\infty < t < \infty$,

$$(i)\ \ \cos(2\pi t), \quad (ii)\ \ \sin(t - \pi/4), \quad (iii)\ \ \tan(\pi t)$$

(b) Find the fundamental period T of $z(t) = 1 + \sin(t) + \sin(3t), \quad -\infty < t < \infty$.

(c) If $x(t)$ is periodic of fundamental period $T_0 = 1$, determine the fundamental period of the following signals

$$(i)\ \ y(t) = 2 + x(t), \quad (ii)\ \ w(t) = x(2t), \quad (iii)\ \ v(t) = 1/x(t)$$

(d) What is the fundamental frequency f_0, in Hz, of

$$(i)\ \ x(t) = 2\cos(t), \quad (ii)\ \ y(t) = 3\cos(2\pi t + \pi/4), \quad (iii)\ \ c(t) = 1/\cos(t)$$

(e) If $z(t)$ is periodic of fundamental period T_0, is $z_e(t) = 0.5[z(t)+z(-t)]$ also periodic? If so determine its fundamental period T_0. What about $z_o(t) = 0.5[z(t) - z(-t)]$?

Answers: (a) (iii) the frequency is $f_0 = 1/2$ Hz; (b) $T = 2\pi$; (c) $x(2t)$ has fundamental period $1/2$; (d) $c(t)$ has $f_0 = 1/(2\pi)$ Hz; (e) $z_e(t)$ is periodic of fundamental period T_0.

Solution

(a) Using $\Omega_0 = 2\pi f_0 = 2\pi/T_0$ for

i. $\cos(2\pi t)$: $\Omega_0 = 2\pi$ rad/sec, $f_0 = 1$ Hz and $T_0 = 1$ sec.

ii. $\sin(t - \pi/4)$: $\Omega_0 = 1$ rad/sec, $f_0 = 1/(2\pi)$ Hz and $T_0 = 2\pi$ sec.

iii. $\tan(\pi t) = \sin(\pi t)/\cos(\pi t)$: $\Omega_0 = \pi$ rad/sec, $f_0 = 1/2$ Hz and $T_0 = 2$ sec.

(b) The fundamental period of $\sin(t)$ is $T_0 = 2\pi$, and $T_1 = 2\pi/3$ is the fundamental period of $\sin(3t)$, $T_1/T_0 = 1/3$ so $3T_1 = T_0 = 2\pi$ is the fundamental period of $z(t)$.

(c) i. $y(t)$ is periodic of fundamental period $T_0 = 1$.

ii. $w(t) = x(2t)$ is $x(t)$ compressed by a factor of 2 so its fundamental period is $T_0/2 = 1/2$, the fundamental period of $z(t)$.

iii. $v(t)$ has same fundamental period as $x(t)$, $T_0 = 1$, indeed $v(t + kT_0) = 1/x(t + kT_0) = 1/x(t)$.

(d) i. $x(t) = 2\cos(t)$, $\Omega_0 = 2\pi f_0 = 1$ so $f_0 = 1/(2\pi)$

ii. $y(t) = 3\cos(2\pi t + \pi/4)$, $\Omega_0 = 2\pi f_0 = 2\pi$ so $f_0 = 1$

iii. $c(t) = 1/\cos(t)$, of fundamental period $T_0 = 2\pi$, so $f_0 = 1/(2\pi)$.

(e) $z_e(t)$ is periodic of fundamental period T_0, indeed

$$\begin{aligned} z_e(t + T_0) &= 0.5[z(t + T_0) + z(-t - T_0))] \\ &= 0.5[z(t) + z(-t)] \end{aligned}$$

Same for $z_o(t)$ since $z_o(t) = z(t) - z_e(t)$.

1.5 In the following problems find the fundamental period of signals and determine periodicity.

(a) Find the fundamental period of the following signals, and verify it

$$(i)\ x(t) = \cos(t + \pi/4),\ (ii)\ y(t) = 2 + \sin(2\pi t),\ (iii)\ z(t) = 1 + (\cos(t)/\sin(3t))$$

(b) The signal $x_1(t)$ is periodic of fundamental period T_0, and the signal $y_1(t)$ is also periodic of fundamental period $10T_0$. Determine if the following signals are periodic, and if so give their fundamental periods

$$(i)\ \ z_1(t) = x_1(t) + 2y_1(t) \quad (ii)\ \ v_1(t) = x_1(t)/y_1(t) \quad (iii)\ \ w_1(t) = x(t) + y_1(10t).$$

Answers: (a) Fundamental period of of $y(t)$ is 1; (b) $v_1(t)$ periodic of fundamental period $10T_0$.

Solution

(a) i. $x(t) = \cos(t + \pi/4)$, $\Omega_0 = 1 = 2\pi/T_0$ so $T_0 = 2\pi$,
$x(t + kT_0) = \cos(t + k2\pi + \pi/4) = x(t)$

ii. $y(t) = 2 + \sin(2\pi t)$, $\Omega_0 = 2\pi$, $T_0 = 1$
$y(t + kT_0) = 2 + \sin(2\pi t + 2\pi k) = y(t)$

iii. $z(t) = 1 + (\cos(t)/\sin(3t))$, $T_0 = 2\pi$ fundamental period of cosine, $T_1 = 2\pi/3$ fundamental period of the sine, then $T_0/T_1 = 3$ or $T_0 = 3T_1 = 2\pi$ is the fundamental period of $z(t)$,

$$z(t + 2\pi k) = 1 + \frac{\cos(t + 2\pi k)}{\sin(3t + 6\pi k)} = z(t)$$

(b) i. $z_1(t)$ is periodic of period $10T_0$, indeed

$$\begin{aligned} z_1(t + 10T_0) &= x_1(t + 10T_0) + 2y_1(t + 10T_0) \\ &= x_1(t) + 2y_1(t) \end{aligned}$$

ii. $v_1(t)$ is periodic of fundamental period $10T_0$ as

$$v_1(t + 10T_0) = \frac{x_1(t + 10T_0)}{y_1(t + 10T_0)} = \frac{x_1(t)}{y_1(t)}$$

iii. $w_1(t)$ is periodic of fundamental period T_0, since $y_1(10T_0)$ is compressed by a factor of 10 so its fundamental period is T_0 the same as $x_1(t)$.

1.6 The following problems are about energy and power of signals.

(a) Plot the signal $x(t) = e^{-t}u(t)$ and determine its energy. What is the power of $x(t)$?

(b) How does the energy of $z(t) = e^{-|t|}$, $-\infty < t < \infty$, compare to the energy of $z_1(t) = e^{-t}u(t)$? Carefully plot the two signals.

(c) Consider the signal

$$y(t) = \text{sign}[x_i(t)] = \begin{cases} 1 & x_i(t) \geq 0 \\ -1 & x_i(t) < 0 \end{cases}$$

for $-\infty < t < \infty$, $i = 1, 2$. Find the energy an the power of $y(t)$ when

$$(a)\;\; x_1(t) = \cos(2\pi t) \qquad (b)\;\; x_2(t) = \sin(2\pi t)$$

Plot $y(t)$ in each case.

(d) Given $v(t) = \cos(t) + \cos(2t)$.

i. Compute the power of $v(t)$.

ii. Determine the power of each of the components of $v(t)$, add them and compare the result to the power of $v(t)$.

(e) Find the power of $s(t) = \cos(2\pi t)$ and of $f(t) = s(t)u(t)$. How do they compare?

Answer: (a) $E_x = 0.5$; (b) $E_z = 2E_{z_1}$; (c) $P_y = 1$; (d) $P_v = 1$.

Solution

(a) $x(t)$ is a causal decaying exponential with energy

$$E_x = \int_0^\infty e^{-2t}dt = \frac{1}{2}$$

and zero power as

$$P_x = \lim_{T\to\infty} \frac{E_x}{2T} = 0$$

(b)

$$E_z = \int_{-\infty}^\infty e^{-2|t|}dt = 2\underbrace{\int_0^\infty e^{-2t}dt}_{E_{z_1}}$$

(c) i. If $y(t) = \text{sign}[x_1(t)]$, it has the same fundamental period as $x_1(t)$, i.e., $T_0 = 1$ and $y(t)$ is a train of pulses so its energy is infinite, while

$$P_y = \int_0^1 1\, dt = 1$$

ii. Since $x_2(t) = \cos(2\pi t - \pi/2) = \cos(2\pi(t - 1/4)) = x_1(t - 1/4)$, the energy and power of $x_2(t)$ coincide with those of $x_1(t)$.

(d) $v(t) = x_1(t) + x_2(t)$ is periodic of fundamental period $T_0 = 2\pi$, and its power is

$$P_v = \frac{1}{2\pi}\int_0^{2\pi} (\cos(t) + \cos(2t))^2 dt = \frac{1}{2\pi}\int_0^{2\pi} (\cos^2(t) + \cos^2(2t) + 2\cos(t)\cos(2t))dt$$

Using

$$\cos^2(\theta) = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$$
$$\cos(\theta)\cos(\phi) = \frac{1}{2}(\cos(\theta+\phi) + \cos(\theta-\phi))$$

we have

$$\begin{aligned} P_v &= \underbrace{\frac{1}{2\pi}\int_0^{2\pi}\cos^2(t)dt}_{P_{x_1}} + \underbrace{\frac{1}{2\pi}\int_0^{2\pi}\cos^2(2t)dt}_{P_{x_2}} + \underbrace{\frac{1}{2\pi}\int_0^{2\pi} 2\cos(t)\cos(2t))dt}_{0} \\ &= \frac{1}{2} + \frac{1}{2} + 0 = 1 \end{aligned}$$

(e) Power of $x(t)$

$$\begin{aligned} P_x &= \frac{1}{T_0}\int_0^{T_0} x^2(t)dt \\ &= \int_0^1 \cos^2(2\pi t)dt \\ &= \int_0^1 (1/2 + \cos^2(4\pi t)dt = 0.5 + 0 = 0.5 \end{aligned}$$

Power of $f(t)$

$$\begin{aligned} P_f &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} y^2(t)dt \\ &= \lim_{N\to\infty}\frac{1}{2(NT_0)}\int_0^{NT_0} y^2(t)dt \\ &= \frac{1}{2T_0}\int_0^{T_0} y^2(t)dt = 0.5P_s \end{aligned}$$

1.7 Consider a circuit consisting of a sinusoidal source $v_s(t) = \cos(t)u(t)$ volts. connected in series to a resistor R and an inductor L and assume they have been connected for a very long time.

(a) Let $R = 0$, $L = 1$ H, compute the instantaneous and the average powers delivered to the inductor.

(b) Let $R = 1\ \Omega$ and $L = 1$ H, compute the instantaneous and the average powers delivered to the resistor and the inductor.

(c) Let $R = 1\ \Omega$ and $L = 0$ H compute the instantaneous and the average powers delivered to the resistor.

(d) The complex power supplied to the circuit is defined as $P = \frac{1}{2}V_s I^*$ where V_s and I are the phasors corresponding to the source and the current in the circuit, and I^* is the complex conjugate of I. Consider the values of the resistor and the inductor given above, and compute the complex power and relate it to the average power computed in each case.

Answers: (a) $P_a = 0$; (b) $P_a = 0.25$; (c) $P_a = 0.5$.

Solution

This problem can be done in the time domain or in the phasor domain. The series connection of the source $v_s(t) = \cos(t)$, the resistor R and the inductor L is equivalent to the connection of a phasor source $V_s = 1e^{j0}$, and impedances R and $j\Omega L = jL$ (the frequency of the source is $\Omega = 1$). The corresponding to the current across the resistor and the inductor, in steady state, is

$$I = \frac{V_s}{R + jL}$$

(a) $L = 1$, $R = 0$ —intuitively, the power used by the inductor is zero since only the resistor uses power.

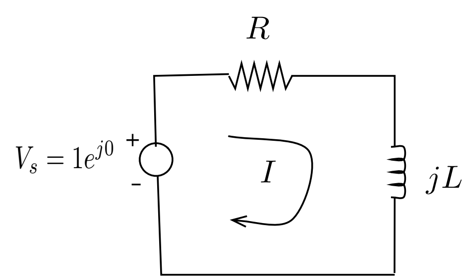


Figure 1.4: Problem 7: Phasor circuit.

In this case, the current $i(t)$ has a phasor

$$I = \frac{1}{j} = -j = 1e^{-j\pi/2}$$

so that the current across the inductor in steady state is given by

$$i(t) = \cos(t - \pi/2)$$

We can compute the average power P_a in time by finding the instantaneous power as

$$p(t) = i(t)v_s(t) = \cos(t - \pi/2)\cos(t) = \frac{1}{2}(\cos(\pi/2) + \cos(2t - \pi/2))$$

so that

$$P_a = \frac{1}{T_0}\int_0^{T_0} p(t)dt = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}[\cos(\pi/2) + \cos(2t - \pi/2)]dt = 0$$

since $\cos(\pi/2) = 0$ and the area under $\cos(2t - \pi/2)$ in a period is zero.

You probably remember from Circuits that the average power is computed using the equivalent expression

$$P_a = \frac{V_{sm} I_m}{2}\cos(\theta)$$

where V_{sm} and I_m are the peak-to-peak values of the phasors corresponding to V_s and I, and θ is the angle in the impedance of the inductor, i.e, $j1 = e^{j\pi/2}$ or $\theta = \pi/2$, and the average power is then

$$P_a = 0.5\cos(\pi/2) = 0$$

Confirming our intuition!

(b) For $L = 1$, $R = 1$, the phasor

$$I = \frac{V_s}{1+j} = \frac{\sqrt{2}}{2}e^{-j\pi/4}$$

and so in the phasor domain,

$$P_a = \frac{V_{sm} I_m}{2}\cos(\pi/4) = \frac{\sqrt{2}/2}{2}\sqrt{2}/2 = \frac{1}{4}$$

(c) $L = 0$, $R = 1$, in this case the power used by the resistor will be the power provided by the source. in this case the phasor for the current across the resistor is

$$I = V_s = 1e^{j0} \text{ so that } i(t) = \cos(t)$$

in the steady state. Thus,

$$P_a = \frac{1}{T_0}\int_0^{T_0} p(t)dt = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}[\cos(0) + \cos(2t)]dt = 0.5$$

In the phasor domain, the average power is

$$P_a = \frac{V_{sm}^2}{2}\cos(0) = \frac{1}{2}$$

(d) The complex power supplied to the circuit is given by

$$P = \frac{1}{2}V_s I^* = \frac{1}{2}(IZ)I^* = \frac{|I|^2|Z|}{2}e^{j\theta}$$

where $Z = |Z|e^{j\theta} = R + j\Omega L$ is the input impedance.

Since $\Omega = 1$, then for

- $R = 0$, $L = 1$, $Z = j$, $I = -j$ so $P = \frac{1}{2}e^{j\pi/2} = 0 + j0.5$ and $P_a = \mathcal{R}e[P] = 0$.
- $R = 1$, $L = 1$, $Z = 1+j$, $I = 1/(1+j)$ so $|I|^2 = 1/2$, $Z = \sqrt{2}$, $\theta = \pi/4$ so that $P = 0.5(0.5)\sqrt{2}e^{j\pi/4} = 0.25\sqrt{2}(\cos(\pi/4) + j\sin(\pi/4))$ and $P_a = \mathcal{R}e[P] = 0.25$.
- $R = 1$, $L = 0$, $Z = 1$, $I = 1$ so $P = \frac{1}{2}e^{j0} = 0.5 + j0$ and $P_a = \mathcal{R}e[P] = 0.5$.

The real part of the complex power corresponds to the average power used by the resistors, while the imaginary part corresponds to the reactive power which is due to inductor and capacitors only.

1.8 Consider the periodic signal $x(t) = \cos(2\Omega_0 t) + 2\cos(\Omega_0 t)$, $-\infty < t < \infty$, and $\Omega_0 = \pi$. The frequencies of the two sinusoids are said to be harmonically related.

(a) Determine the period T_0 of $x(t)$. Compute the power P_x of $x(t)$ and verify that the power P_x is the sum of the power P_1 of $x_1(t) = \cos(2\pi t)$ and the power P_2 of $x_2(t) = 2\cos(\pi t)$.

(b) Suppose that $y(t) = \cos(t) + \cos(\pi t)$, where the frequencies are not harmonically related. Find out whether $y(t)$ is periodic or not. Indicate how you would find the power P_y of $y(t)$. Would $P_y = P_1 + P_2$ where P_1 is the power of $\cos(t)$ and P_2 that of $\cos(\pi t)$? Explain what is the difference with respect to the case of harmonic frequencies.

Answers: (a) $T_0 = 2$; $P_x = 2.5$; (b) $y(t)$ is not periodic, but $P_y = P_1 + P_2$.

Solution

(a) Let $x(t) = x_1(t) + x_2(t) = \cos(2\pi t) + 2\cos(\pi t)$, so that $x_1(t)$ is a cosine of frequency $\Omega_1 = 2\pi$ or period $T_1 = 1$, and $x_2(t)$ is a cosine of frequency $\Omega_2 = \pi$ or period $T_2 = 2$. The ratio of these periods $T_2/T_1 = 2/1$ is a rational number so $x(t)$ is periodic of fundamental period $T_0 = 2T_1 = T_2 = 2$.
The average power of $x(t)$ is given by

$$P_x = \frac{1}{T_0}\int_0^{T_0} x^2(t)dt = \frac{1}{2}\int_0^2 [x_1^2(t) + x_2^2(t) + 2x_1(t)x_2(t)]dt$$

Using the trigonometric identity $\cos(\alpha)\cos(\beta) = \cos(\alpha-\beta) + \cos(\alpha+\beta)$ we have that the integral

$$\begin{aligned} \frac{1}{2}\int_0^2 2x_1(t)x_2(t)dt &= \frac{1}{2}\int_0^2 4\cos(2\pi t)\cos(\pi t)dt \\ &= \int_0^2 [\cos(\pi t) + \cos(3\pi t)]dt = 0 \end{aligned}$$

since $\cos(\pi t) + \cos(3\pi t)$ is periodic of period 2 and so its area under a period is zero. Thus,

$$\begin{aligned} P_x &= \frac{1}{2}\int_0^2 [x_1^2(t) + x_2^2(t)]dt \\ &= \frac{1}{2}\int_0^2 x_1^2(t)dt + \frac{1}{2}2\int_0^1 x_2^2(t)]dt \\ &= P_{x_1} + P_{x_2} \end{aligned}$$

so that the power of $x(t)$ equals the sum of the powers of $x_1(t)$ and $x_2(t)$ which are sinusoids of different frequencies, and thus orthogonal as we will see later.
Finally,

$$\begin{aligned} P_x &= \frac{1}{2}\int_0^2 \cos^2(2\pi t)dt + \int_0^1 4\cos^2(\pi t)dt \\ &= \frac{1}{2}\int_0^2 [0.5 + 0.5\cos(4\pi t)]dt + \int_0^1 4[0.5 + 0.5\cos(2\pi t)]dt \\ &= 0.5 + 2 = 2.5 \end{aligned}$$

remembering that the integrals of the cosines are zero (they are periodic of period 0.5 and 1 and the integrals compute their areas under one or more periods, so they are zero).
(b) The components of $y(t)$ have as periods $T_1 = 2\pi$ and $T_2 = 2$ so that $T_1/T_2 = \pi$ which is not rational so $y(t)$ is not periodic. In this case we need to find the power of $y(t)$ by finding the integral over an infinite support of $y^2(t)$ which will as before give

$$P_y = P_{y_1} + P_{y_2}$$

In the case of harmonically related signals we can use the periodicity and compute one integral. However, in either case the power superposition holds.

1.9 A signal $x(t)$ is defined as $x(t) = r(t+1) - r(t) - 2u(t) + u(t-1)$.

(a) Plot $x(t)$ and indicate where it has discontinuities. Compute $y(t) = dx(t)/dt$ and plot it. How does it indicate the discontinuities? Explain.

(b) Find the integral

$$\int_{-\infty}^{t} y(\tau)d\tau$$

and give the values of the integral when $t = -1$, 0, 0.99, 1.01, 1.99 and 2.01. Is there any problem with calculating the integral at exactly $t = 1$ and $t = 2$? Explain.

Answers: $x(t)$ has discontinuities at $t = 0$ and at $t = 1$, indicated by delta functions in $dx(t)/dt$.

Solution

(a) The signal $x(t)$ is

$$x(t) = \begin{cases} 0 & t < -1 \\ t+1 & -1 \leq t \leq 0 \\ -1 & 0 < t \leq 1 \\ 0 & t > 1 \end{cases}$$

there are discontinuities at $t = 0$ and at $t = 1$. The derivative

$$\begin{aligned} y(t) &= \frac{dx(t)}{dt} \\ &= u(t+1) - u(t) - 2\delta(t) + \delta(t-1) \end{aligned}$$

indicating the discontinuities at $t = 0$, a decrease from 1 to -1, and at $t = 1$ an increase from -1 to 0.

(b) The integral

$$\begin{aligned} \int_{-\infty}^{t} y(\tau)d\tau &= \int_{-\infty}^{t} [u(\tau+1) - u(\tau) \\ &\quad -2\delta(\tau) + \delta(\tau - 1)]d\tau = x(t) \end{aligned}$$

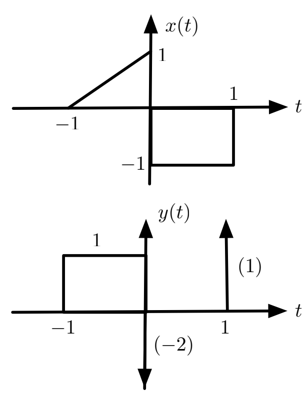


Figure 1.5: Problem 9

1.10 One of the advantages of defining the $\delta(t)$ functions is that we are now able to find the derivative of discontinuous signals. Consider a periodic sinusoid defined for all times

$$x(t) = \cos(\Omega_0 t) \quad -\infty < t < \infty$$

and a causal sinusoid defined as $x_1(t) = \cos(\Omega_0 t)u(t)$, where the unit-step function indicates that the function has a discontinuity at zero, since for $t = 0+$ the function is close to 1 and for $t = 0-$ the function is zero.

(a) Find the derivative $y(t) = dx(t)/dt$ and plot it.

(b) Find the derivative $z(t) = dx_1(t)/dt$ (treat $x_1(t)$ as the product of two functions $\cos(\Omega_0 t)$ and $u(t)$) and plot it. Express $z(t)$ in terms of $y(t)$.

(c) Verify that the integral $\int_{-\infty}^{t} z(\tau)d\tau$ gives back $x_1(t)$.

Answers: (a) $y(t) = -\Omega_0 \sin(\Omega_0 t)$; (b) $z(t) = y(t)u(t) + \delta(t)$.

Solution

(a) $x(t)$, $-\infty < t < \infty$, is a continuous signal and its derivative exists and it is

$$y(t) = \frac{d\cos(\Omega_0 t)}{dt} = -\Omega_0 \sin(\Omega_0 t)$$

(b) $x_1(t)$ has a discontinuity at $t = 0$, and so its derivative will have a $\delta(t)$ function. Indeed, its derivative is

$$\begin{aligned} z(t) &= \frac{d\cos(\Omega_0 t)u(t)}{dt} \\ &= \frac{d\cos(\Omega_0 t)}{dt}u(t) + \cos(\Omega_0 t)\frac{du(t)}{dt} \\ &= -\Omega_0 \sin(\Omega_0 t)u(t) + \cos(\Omega_0 t)\delta(t) \\ &= -\Omega_0 \sin(\Omega_0 t)u(t) + \cos(0)\delta(t) \\ &= -\Omega_0 \sin(\Omega_0 t)u(t) + \delta(t) \end{aligned}$$

(c) The integral of $z(t)$ is zero for $t < 0$, and

$$\begin{aligned} \int_{-\infty}^{t} z(t')dt' &= \int_{0}^{t} -\Omega_0 \sin(\Omega_0 t')dt' + \int_{0-}^{t} \delta(t')dt' \\ &= [\cos(\Omega_0 t) - 1] + 1 = \cos(\Omega_0 t) \qquad t > 0 \end{aligned}$$

or $\cos(\Omega_0 t)u(t)$.

1.11 Let $x(t) = t[u(t) - u(t-1)]$, we would like to consider its expanded and compressed versions.

(a) Plot $x(2t)$ and determine if it is a compressed or expanded version of $x(t)$.

(b) Plot $x(t/2)$ and determine if it is a compressed or expanded version of $x(t)$.

(c) Suppose $x(t)$ is an acoustic signal, e.g., a music signal recorded in a magnetic tape, what would be a possible application of the expanding and compression operations? Explain.

Answers: (a) $x(2t) = 2t[u(t) - u(t-0.5)]$, compressed.

Solution

(a) The signal $x(t) = t$ for $0 \leq t \leq 1$, zero otherwise. Then

$$x(2t) = \begin{cases} 2t & 0 \leq 2t \leq 1 \text{ or } 0 \leq t \leq 1/2 \\ 0 & \text{otherwise} \end{cases}$$

that is, the signal has been compressed — instead of being between 0 and 1, it is now between 0 and 0.5.

(b) Likewise, the signal

$$x(t/2) = \begin{cases} t/2 & 0 \leq t/2 \leq 1 \text{ or } 0 \leq t \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

i.e., the signal has been expanded, its support has doubled.

The following figure illustrates the compressed and expanded signals $x(2t)$ and $x(t/2)$.

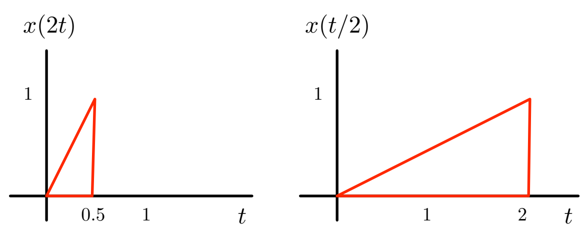


Figure 1.6: Problem 11: Compressed $x(2t)$, expanded $x(t/2)$ signals.

(c) If the acoustic signal is recorded in a tape, we can play it faster (contraction) or slower (expansion) than the speed at which it was recorded. Thus the signal can be made to last a desired amount of time, which might be helpful whenever an allocated time is reserved for broadcasting it.

1.12 Consider the signal $x(t)$ in Fig. 1.7.

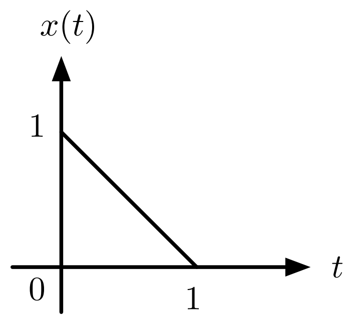


Figure 1.7: Problem 12

(a) Plot the even-odd decomposition of $x(t)$, i.e., find and plot the even $x_e(t)$ and the odd $x_o(t)$ components of $x(t)$.

(b) Show that the energy of the signal $x(t)$ can be expressed as the sum of the energies of its even and odd components, i.e. that

$$\int_{-\infty}^{\infty} x^2(t)dt = \int_{-\infty}^{\infty} x_e^2(t)dt + \int_{-\infty}^{\infty} x_o^2(t)dt$$

(c) Verify that the energy of $x(t)$ is equal to the sum of the energies of $x_e(t)$ and $x_o(t)$.

Answers: $x_o(t) = -0.5(1+t)[u(t+1) - u(t)] + 0.5(1-t)[u(t) - u(t-1)]$.

Solution

(a) Because of the discontinuity of $x(t)$ at $t = 0$ the even component of $x(t)$ is a triangle with $x_e(0) = 1$, i.e.,

$$x_e(t) = \begin{cases} 0.5(1-t) & 0 < t \leq 1 \\ 0.5(1+t) & -1 \leq t < 0 \\ 1 & t = 0 \end{cases}$$

while the odd component is

$$x_o(t) = \begin{cases} 0.5(1-t) & 0 < t \leq 1 \\ -0.5(1+t) & -1 \leq t < 0 \\ 0 & t = 0 \end{cases}$$

(b) The energy of $x(t)$ is

$$\begin{aligned} \int_{-\infty}^{\infty} x^2(t)dt &= \int_{-\infty}^{\infty} [x_e(t) + x_o(t)]^2 dt \\ &= \int_{-\infty}^{\infty} x_e^2(t)dt + \int_{-\infty}^{\infty} x_o^2(t)dt + 2\int_{-\infty}^{\infty} x_e(t)x_o(t)dt \end{aligned}$$

where the last equation on the right is zero, given that the integrand is odd.

(c) The energy of $x(t) = 1 - t$, $0 \leq t \leq 1$ and zero otherwise, is given by

$$\int_{-\infty}^{\infty} x^2(t)dt = \int_0^1 (1-t)^2 dt = t - t^2 + \frac{t^3}{3}\Big|_0^1 = \frac{1}{3}$$

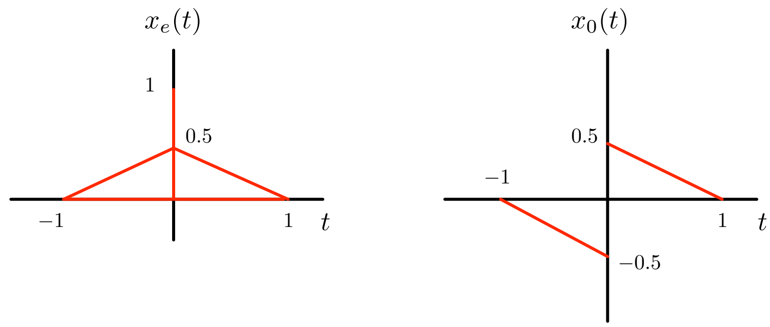


Figure 1.8: Problem 12: Even and odd decomposition of $x(t)$.

The energy of the even component is

$$\int_{-\infty}^{\infty} x_e^2(t)dt = 0.25\int_{-1}^{0}(1+t)^2dt + 0.25\int_{0}^{1}(1-t)^2dt = 0.5\int_{0}^{1}(1-t)^2dt$$

where the discontinuity at $t = 0$ does not change the above result. The energy of the odd component is

$$\int_{-\infty}^{\infty} x_o^2(t)dt = 0.25\int_{-1}^{0}(1+t)^2dt + 0.25\int_{0}^{1}(1-t)^2dt = 0.5\int_{0}^{1}(1-t)^2dt$$

so that

$$E_x = E_{x_e} + E_{x_o}$$

1.13 A periodic signal can be generated by repeating a period.

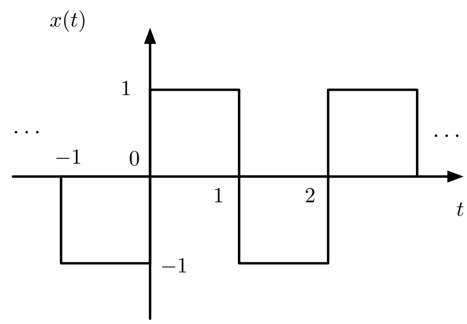


Figure 1.9: Problem 13

(a) Find the function $g(t)$, defined in $0 \leq t \leq 2$ only, in terms of basic signals and such that when repeated using a period of 2 generates the periodic signal $x(t)$ shown in Fig. 1.9.

(b) Obtain an expression for $x(t)$ in terms of $g(t)$ and shifted versions of it.

(c) Suppose we shift and multiply by a constant the periodic signal $x(t)$ to get new signals $y(t) = 2x(t-2)$, $z(t) = x(t+2)$ and $v(t) = 3x(t)$ are these signals periodic?

(d) Let then $w(t) = dx(t)/dt$, and plot it. Is $w(t)$ periodic? If so, determine its period.

Answers: (a) $g(t) = u(t) - 2u(t-1) + u(t-2)$; (c) Signals $y(t)$, $v(t)$ are periodic.

Solution
(a) The function $g(t)$ corresponding to the first period of $x(t)$ is given by

$$g(t) = u(t) - 2u(t-1) + u(t-2)$$

(b) The periodic signal $x(t)$ is

$$\begin{aligned} x(t) = & \quad g(t) + g(t-2) + g(t-4) + \cdots \\ & + \quad g(t+2) + g(t+4) + \cdots = \sum_{k=-\infty}^{\infty} g(t+2k) \end{aligned}$$

(c) Yes, the signals $y(t)$, $z(t)$ and $v(t)$ are periodic of period $T_0 = 2$ as can be easily verified.
(d) The derivative of $x(t)$ is

$$\begin{aligned} w(t) = 2\delta(t) & - 2\delta(t-1) + 2\delta(t-2) + \cdots \\ & - 2\delta(t+1) + 2\delta(t+2) + \cdots \end{aligned}$$

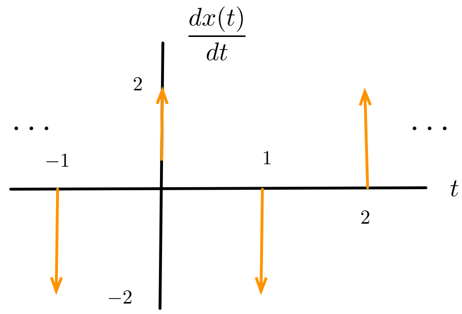


Figure 1.10: Problem 13: Derivative of $x(t)$.

which can be seen to be periodic of period $T_0 = 2$.

1.14 For a complex exponential signal $x(t) = 2e^{j2\pi t}$

(a) Suppose $y(t) = e^{j\pi t}$, would the sum of these signals $z(t) = x(t) + y(t)$ be also periodic? If so, what is the fundamental period of $z(t)$?

(b) Suppose we then generate a signal $v(t) = x(t)y(t)$, with the $x(t)$ and $y(t)$ signals given before, is $v(t)$ periodic? If so, what is its fundamental period?

Answers: (a) $z(t)$ is periodic of period $T_1 = 2$; (b) $v(t)$ is periodic of period $T_3 = 2/3$.

Solution
(a) $\Omega_0 = 2\pi = 2\pi f_0$ (rad/sec), so $f_0 = 1/T_0 = 1$ (Hz) and $T_0 = 1$ sec.
The sum

$$\begin{aligned} z(t) &= x(t) + y(t) \\ &= (2\cos(2\pi t) + \cos(\pi t)) + j(2\sin(2\pi t) + \sin(\pi t)) \end{aligned}$$

is also periodic of period $T_1 = 2$.
(b) $v(t) = x(t)y(t) = 2e^{j3\pi t}$ with frequency $\Omega_3 = 3\pi$ so that

$$T_3 = 2\pi/\Omega_3 = 2/3$$

1.15 Consider the train of triangular pulses $x(t)$ in Fig. 1.11.

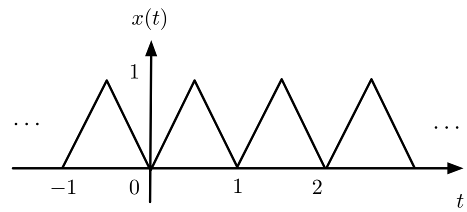


Figure 1.11: Problem 15

(a) Carefully plot the the derivative of $x(t)$, $y(t) = dx(t)/dt$.

(b) Can you compute

$$z(t) = \int_{-\infty}^{\infty} [x(t) - 0.5]dt?$$

If so, what is it equal to? If not, explain why not.

(c) Is $x(t)$ a finite energy signal? how about $y(t)$?

Answers: (a) $y(t) = \sum_k [u(t-k) - 2u(t-0.5-k) + u(t-1-k)]$; (c) $x(t)$, $y(t)$ have infinite energy.

Solution

(a) The derivative signal $y(t) = dx(t)/dt$ is a train of rectangular pulses. Indeed, if $x_1(t) = r(t) - 2r(t - 0.5) + r(t-1)$ is the first period of $x(t)$ then

$$x(t) = \sum_{k=-\infty}^{\infty} x_1(t-k)$$

its derivative is

$$y(t) = \frac{dx(t)}{dt} = \sum_{k=-\infty}^{\infty} \frac{dx_1(t-k)}{dt}$$

where

$$\frac{dx_1(t-k)}{dt} = u(t-k) - 2u(t-0.5-k) + u(t-1-k)$$

(b) The signal $x(t) - 0.5$ has an average of zero, so its integral

$$z(t) = \lim_{N\to\infty} N \int_0^1 (x(t) - 0.5)dt = 0$$

(c) Neither is a finite energy signal.

1.2 Problems using MATLAB

1.16 Signal energy and RC circuit — The signal $x(t) = e^{-|t|}$ is defined for all values of t.

(a) Plot the signal $x(t)$ and determine if this signal is finite energy.

(b) If you determine that $x(t)$ is absolutely integrable, or that the following integral

$$\int_{-\infty}^{\infty} |x(t)|dt$$

is finite, could you say that $x(t)$ has finite energy? Explain why or why not. HINT: Plot $|x(t)|$ and $|x(t)|^2$ as functions of time.

(c) From your results above, is it true the energy E_y of the signal

$$y(t) = e^{-t}\cos(2\pi t)u(t)$$

is less than half the energy of $x(t)$? Explain. To verify your result, use symbolic MATLAB to plot $y(t)$ and to compute its energy.

(d) To discharge a capacitor of 1 mF charged with a voltage of 1 volt we connect it, at time $t = 0$, with a resistor of $R\ \Omega$. When we measure the voltage in the resistor we find it to be $v_R(t) = e^{-t}u(t)$. Determine the resistance R. If the capacitor has a capacitance of 1 μF, what would be R? In general, how are R and C related?

Answers: (a) $E_x = 1$; (c) $E_y = E_x/2$; (d) $R = 1/C$.

Solution

The given signal $x(t) = e^{-|t|}$ is even, positive and decays to zero as $t \to \pm\infty$

(a) The signal is finite energy as

$$E_x = \int_{-\infty}^{\infty} x^2(t)dt \quad = \quad 2\int_0^{\infty} e^{-2t}dt = 2\frac{e^{-2t}}{-2}\Big|_0^{\infty} = 1$$

(b) The signal $x(t)$ is absolutely integrable as

$$\int_{-\infty}^{\infty} |x(t)|dt \quad = \quad 2\int_0^{\infty} e^{-t}dt = 2\frac{e^{-2t}}{-1}\Big|_0^{\infty} = 2$$

Notice that $0 < x^2(t) < x(t)$ and so the knowledge that $x(t)$ is absolutely integrable (i.e., that the above integral is finite) would imply that $x(t)$ has finite energy (i.e., the integral calculated in (b) is finite).

(c) The energy of $y(t)$ is

$$E_y = \int_0^{\infty} e^{-2t}\cos^2(2\pi t)dt < \int_0^{\infty} e^{-2t}dt = E_x/2 = 1/2$$

since $\cos^2(2\pi t) \leq 1$ (the decaying sinusoid is bounded by the envelope $e^{-2t}u(t)$).

```
% Pro 1.16
clear all; clf
syms x y t z
x=exp(-abs(t));
% computation of integrals
% for increasing values of time
for k=1:100,
    zi=2*int(x,t,0,k/10); yi=2*int(x^2,t,0,k/10); vi=int((exp(-t)*cos(2*pi*t))^2,0,k/10);
```

```
    zz(k)=subs(zi);  yy(k)=subs(yi);  vv(k)=subs(vi);
end
t1=[1:100]/10;
figure(1)
 subplot(221)
 ezplot(x,[-10,10]);grid
 axis([-10 10 0 1]);title('x(t)=e^{-|t|}')
 subplot(222)
 plot(t1,zz);grid;title('integral of |x(t)|');xlabel('t')
 subplot(223)
 plot(t1,yy);grid;title('integral of |x(t)|^2');xlabel('t')
 subplot(224)
 plot(t1,vv);grid;title('integral of |e^{-t}cos(2\pi t)|^2');xlabel('t')
figure(2)
 ezplot((exp(-t)*cos(2*pi*t))^2,[0,5]);grid
 axis([0 5 0 1])
 hold on
 ezplot((exp(-t))^2,[0,5])
 axis([0 5 0 1]);title('envelope of |y(t)|^2')
 hold off
```

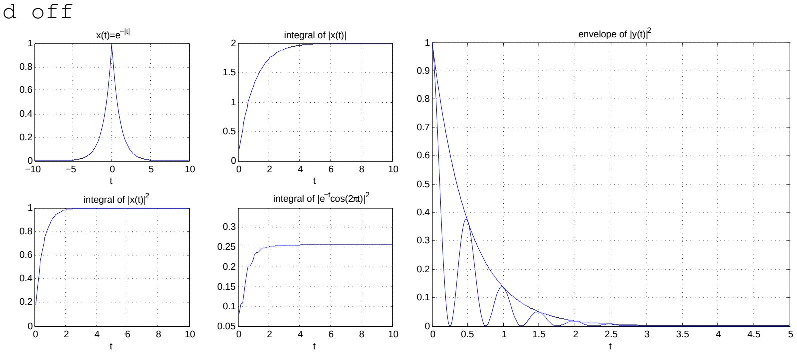


Figure 1.12: Problem 16: signal $x(t)$, and the integrals of $|x(t)|$, $|x(t)|^2$ and $|y(t)|^2$ (left). Right: envelope of $|y(t)|^2$.

(d) For a value C for the capacitor, considering the initial condition the source for the RC circuit the KVL equation for $t \geq 0$ is:

$$v_R(t) + \frac{1}{C}\int_0^t i(\tau)d\tau = 1, \quad \text{or}$$
$$e^{-t} + \frac{1}{CR}\int_0^t e^{-\tau}d\tau = 1$$

after replacing the voltage and current in the resistor. Solving the integral we obtain

$$e^{-t} + \frac{1}{RC}(1 - e^{-t}) = 1$$

so that for $t = 0$ we get an identity indicating the initial condition is satisfied by the solution. For $t \to \infty$ we get $1/RC = 1$. So that $R = 1/C$ in general, for $C = 1$ mF then $R = 1$ KΩ and for $C = 1\mu = 10^{-6}$F, then $R = 10^6\Omega$ or 1 MΩ.

1.17 Periodicity of sum of sinusoids —

(a) Consider the periodic signals $x_1(t) = 4\cos(\pi t)$ and $x_2(t) = -\sin(3\pi t + \pi/2)$. Find the periods T_1 of $x_1(t)$ and T_2 of $x_2(t)$ and determine if $x(t) = x_1(t) + x_2(t)$ is periodic. If so, what is its period T_0?

(b) Two periodic signals $x_1(t)$ and $x_2(t)$ have periods T_1 and T_2 such that their ratio $T_1/T_2 = 3/12$, determine the period of $x(t) = x_1(t) + x_2(t)$.

(c) Determine whether $x_1(t) + x_2(t)$, $x_3(t) + x_4(t)$ are periodic when

- $x_1(t) = 4\cos(2\pi t)$ and $x_2(t) = -\sin(3\pi t + \pi/2)$,
- $x_3(t) = 4\cos(2t)$ and $x_4(t) = -\sin(3\pi t + \pi/2)$

Use symbolic MATLAB to plot $x_1(t) + x_2(t)$, $x_3(t) + x_4(t)$ and confirm your analytic result about their periodicity or lack of periodicity.

Answers: (b) $T_0 = 4T_1 = T_2$; (c) $x_1(t) + x_2(t)$ is periodic, $x_3(t) + x_4(t)$ is non–periodic.

Solution
(a) The signal $x_1(t) = 4\cos(\pi t)$ has frequency $\Omega_1 = 2\pi/2$ so that the period of $x_1(t)$ is $T_1 = 2$. Likewise the signal $x_2(t) = -\sin(3\pi t + \pi/2)$ has frequency $\Omega_2 = 3\pi = 2\pi/(2/3)$ so that it is periodic of period $T_2 = 2/3$. The signal $x(t)$ is periodic of fundamental period $T_0 = 2$ as the ratio $T_1/T_2 = 2/(2/3) = 3$ so that $T_0 = 3T_2 = T_1 = 2$.
(b) The ratio of the two periods is

$$\frac{T_1}{T_2} = \frac{3}{3 \times 4} = \frac{1}{4}$$

so that

$$T_0 = 4T_1 = T_2$$

is the period of $x(t) = x_1(t) + x_2(t)$.
(c) In general, if the ratio of the periods of two periodic signals is

$$\frac{T_1}{T_2} = \frac{M}{K}$$

for integers M and K, not divisible by each other, then $T_0 = KT_1 = MT_2$ is the period of the sum of the periodic signals. If the ratio is not rational (i.e., M and/or K are not integers) then the sum of the two periodic signals is not periodic.
The following script is used to show that $x_1(t) + x_2(t)$ is periodic, while $x_3(t) + x_4(t)$ is not.

```
% Pro 1.17
clear all; clf
syms x1 x2 x3 x4 t
x1=4*cos(2*pi*t); x2=-sin(3*pi*t+pi/2);
x3=4*cos(2*t);x4=x2;
figure(3)
  subplot(211)
  ezplot(x1+x2,[0 10]);grid
  subplot(212)
  ezplot(x3+x4,[0 10]);grid
```

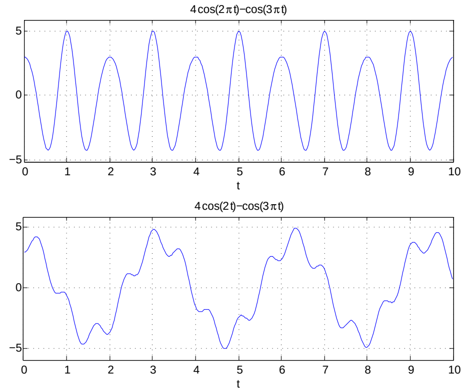


Figure 1.13: Problem 17: periodic $x_1(t) + x_2(t)$ (top), non–periodic $x_3(t) + x_4(t)$ (bottom).

1.18 Impulse signal generation — When defining the impulse or $\delta(t)$ signal the shape of the signal used to do so is not important. Whether we use the rectangular pulse we considered in this Chapter or another pulse, or even a signal that is not a pulse, in the limit we obtain the same impulse signal. Consider the following cases:

(a) The triangular pulse

$$\Lambda_\Delta(t) = \frac{1}{\Delta}\left(1 - \left|\frac{t}{\Delta}\right|\right)(u(t+\Delta) - u(t-\Delta))$$

Carefully plot it, compute its area, and find its limit as $\Delta \to 0$. What do you obtain in the limit? Explain.

(b) Consider the signal

$$S_\Delta(t) = \frac{\sin(\pi t/\Delta)}{\pi t}$$

Use the properties of the sinc signal $S(t) = \sin(\pi t)/(\pi t)$ to express $S_\Delta(t)$ in terms of $S(t)$. Then find its area, and the limit as $\Delta \to 0$. Use symbolic MATLAB to show that for decreasing values of Δ the $S_\Delta(t)$ becomes like the impulse signal.

Answers: $S_\Delta(0) = 1/\Delta$, $S_\Delta(t) = 0$ at $t = \pm k\Delta$.

Solution

(a) The triangular pulse has a width of 2Δ and a height of $1/\Delta$, its area is 1. The following MATLAB script can be used to see the limit as $\Delta \to 0$

```
% Pr. 1.18
clear all; clf
% part (a)
delta=0.1;
t=[-delta:0.05:delta];N=length(t);
lambda=zeros(1,N);
figure(5)
for k=1:6,
    lambda=(1-abs(t/delta))/delta;
    delta=delta/2;
    plot(t,lambda);xlabel('t')
    axis([-0.1 0.1 0 330]);grid
    hold on
    pause(0.5)
end
grid
hold off
```

(b) The signal $S_\Delta(t) = 1/\Delta s(t/\Delta)$ so that

$$S_\Delta(t) = \frac{1}{\Delta}\;\frac{\sin(\pi t/\Delta)}{\pi t/\Delta} = \frac{\sin(\pi t/\Delta)}{\pi t}$$

and so

$$S_\Delta(0) = \lim_{t\to\ 0}(\pi/\Delta)\;\frac{\cos(\pi t/\Delta)}{\pi} = 1/\Delta$$

and $S_\Delta(t)$ is zero at

$$\pi t/\Delta = \pm k\pi \qquad k \neq 0 \text{ integer}$$

or $t = \pm k\Delta$ and finally the integral

$$\int_{-\infty}^{\infty} S_\Delta(t)dt = \int_{-\infty}^{\infty} \frac{\sin(\tau\pi)}{\pi\Delta\tau}\Delta d\tau = 1$$

where we used $\tau = t/\Delta$. The following script illustrates the limit as $\Delta \to 0$.

```
% part (b)
 syms S t
 delta=1;
 figure(6)
 for k=1:4,
    delta=delta/k;
    S=(1/delta)*sinc(t/delta);
    ezplot(S,[-2 2])
    axis([-2 2  -8 30])
    hold on
    I=subs(int(S,t,-100*delta, 100*delta)) % area under sinc
    pause(0.5)
 end
 grid;xlabel('t')
 hold off
```

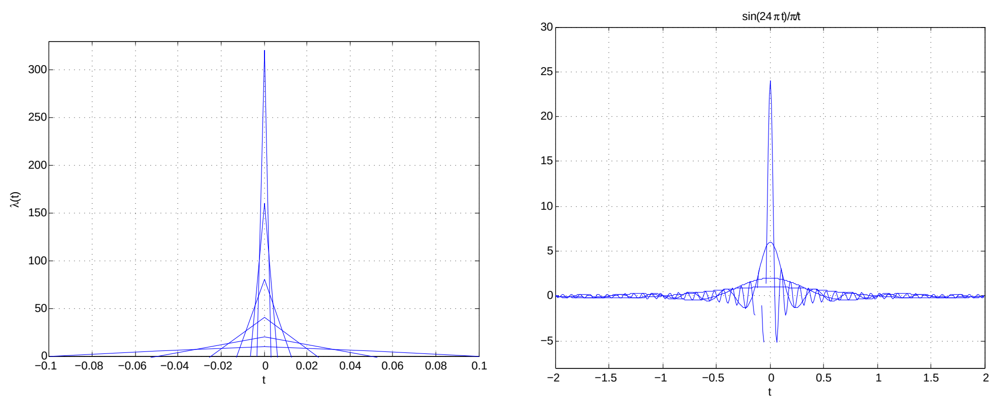


Figure 1.14: Problem 18: approximation of $\delta(t)$ using triangular (left) or sinc (right) functions

1.19 Contraction and expansion and periodicity — Consider the periodic signal $x(t) = \cos(\pi t)$ of fundamental period $T_0 = 2$ sec.

(a) Is the expanded signal $x(t/2)$ periodic? if periodic indicate its period.

(b) Is the compressed signal $x(2t)$ periodic? if periodic indicate its period.

(c) Use MATLAB to plot the above two signals and verify your analytic results.

Answers: (a) $x(t/2)$ is periodic of fundamentsl period 4.

Solution

(a) The expanded signal $x(t/2)$ is periodic. The first period of $x(t)$ is $x_1(t)$ for $0 \leq t \leq 2$, and so the period of $x(t/2)$ is $x_1(t/2)$ which is supported in $0 \leq t/2 \leq 2$ or $0 \leq t \leq 4$, so the period of $x(t/2)$ is 4.

(b) The compressed signal $x(2t)$ is periodic. The first period of $x(t)$, $x_1(t)$ for $0 \leq t \leq 2$, becomes $x_1(2t)$ for $0 \leq 2t \leq 2$ or $0 \leq t \leq 1$, its support is halved. So the period of $x(2t)$ is 1.

```
% Pr. 1.19 part(b)
clear all; clf
t=0:0.002:8; t1=0:0.001:8; t2=0:0.004:8;
x=cos(pi*t);  x1=cos(pi*t1/2); x2=cos(pi*2*t2);
figure(1)
 subplot(211)
 plot(t1,x1)
 hold on
 plot(t,x,'r')
 xlabel('t (sec)')
 ylabel('x(t/2), x(t)')
 legend('expanded signal', 'original signal')
 subplot(212)
 plot(t2,x2)
 hold on
 plot(t,x,'r')
 xlabel('t (sec) ')
 ylabel('x(2t), x(t)')
 hold off
 legend('compressed signal', 'original signal')
```

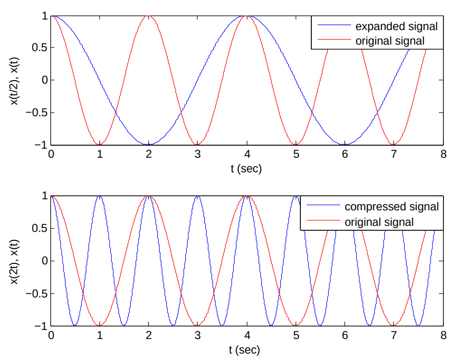


Figure 1.15: Problem 19: expanded and compressed sinusoids vs original sinusoid.

1.20 Full-wave rectified signal — Consider the full-wave rectified signal

$$y(t) = |\sin(\pi t)| \qquad -\infty < t < \infty.$$

(a) As a periodic signal $y(t)$ does not have finite energy, but it has a finite power P_y. Find it.

(b) It is always useful to get a quick estimate of the power of a periodic signal by finding a bound for the signal squared. Find a bound for $|y(t)|^2$ and show that $P_y < 1$.

(c) Use symbolic MATLAB to check if the full-wave rectified signal has finite power and if that value coincides with the P_y you found above. Plot the signal and provide the script for the computation of the power. How does it coincide with the analytical result?

Answers: (a) $P_y = 0.5$

Solution

(a) The power of the full-wave rectified signal is

$$P_y = \int_0^1 |\sin(\pi t)|^2 dt$$

because the period of $y(t)$ is $T = 1$. A simpler expression for $\sin^2(\pi t)$ can be computed using Euler's equation

$$\begin{aligned}
\sin^2(\pi t) &= \left[\frac{e^{j\pi t} - e^{-j\pi t}}{2j}\right]^2 \\
&= \frac{-1}{4}\left[e^{j2\pi t} - 2 + e^{-j2\pi t}\right] \\
&= 0.5(1 - \cos(2\pi t))
\end{aligned}$$

Since $\cos(2\pi t)$ has a period 1 its integral over a period is zero, thus

$$P_y = 0.5$$

(b) A pulse $\rho(t) = u(t) - u(t-1)$ covers one of the periods of $y(t)$ and thus the area under the full-wave rectified signal is $P_y < 1$ the area of the pulse squared.

(c) The following script is used to calculate the power which is found to be 1/2

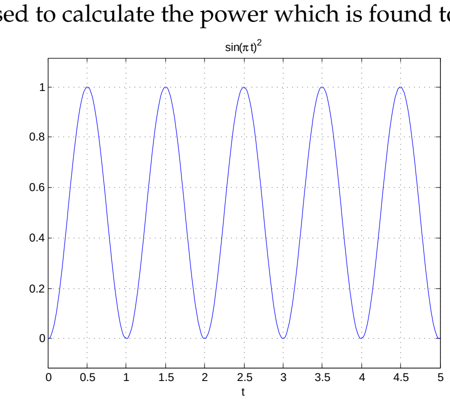


Figure 1.16: Problem 20: magnitude squared signal used to compute power.

```
%  Pro 1.20, part (c)
clear all;clf
syms x t
```

```
x=sin(pi*t); T=1;
figure(8)
ezplot(x^2,[0,5*T]);grid
P=int(x^2,t,0,T)/T
```

1.21 Shifting and scaling a discretized analog signal— The discretized approximation of a pulse is given by

$$w(nTs) = \begin{cases} 1 & -N/4 \leq n \leq -1 \\ -1 & 1 \leq n \leq (N/4)+1 \\ 0 & \text{otherwise} \end{cases}$$

where $N = 10000$ and $T_s = 0.001$ seconds.

(a) Obtain this signal and let the plotted signal using *plot* be the analog signal. Determine the duration of the analog signal.

(b) There are two possible ways to visualize the shifting of an analog signal. Since when advancing or delaying a signal the values of the signal remain the same, it is only the time values that change we could visualize a shift by changing the time scale to be a shifted version of the original scale. Using this approach plot the shifted signal $w(t-2)$.

(c) The other way to visualize the time shifting is to obtain the values of the shifted signal and plot it against the original time support. This way we could continue processing the signal while with the previous approach we can only visualize it. Using this approach obtain $w(t-2)$ and then plot it.

(d) Obtain the scaled and shifted approximations to $w(1.5t)$ and $w(1.5t-2)$ using our function *scale_shift* and comment on your results.

Answers: The duration of the pulse is 5.001 sec.

Solution
The duration of the pulse is

$$(N/4 + 1 + N/4)T_s = (N/2 + 1)T_s = 5.001 sec.$$

The following script is used to find the shifted signal by the two approaches.

```
% Pro 1.21
clear all; clf
Ts=0.001; T=5;N=2*T/Ts; t=-T:Ts:T;
w= [zeros(1,N/4) ones(1,N/4) -ones(1,N/4+1) zeros(1,N/4)];
delay=2;M=delay/Ts;
figure(1)
subplot(131)
plot(t,w); axis([-2*T 2*T 1.1*min(w) 1.1*max(w)]);grid
xlabel('t');ylabel('w(t)')
% part b
t2=t+delay;
subplot(132)
plot(t2,w); axis([-2*T 2*T 1.1*min(w) 1.1*max(w)]);grid
xlabel('t');ylabel('w(t-2)')
% part c
w2=[zeros(1,M) w(1:length(w)-M)];
subplot(133)
plot(t,w2,'r');axis([-2*T 2*T 1.1*min(w) 1.1*max(w)]);grid
xlabel('t');ylabel('w(t-2)')
% scaling and shifting
% scaling and shifting  of window
[w1,t2,t3]=scale_shift(w,1.5,delay,T,Ts);
figure(2)
 subplot(131)
 plot(t,w); axis([-2*T 2*T 1.1*min(w) 1.1*max(w)]);grid
```

```
xlabel('t');ylabel('w(t)')
subplot(132)
plot(t2,w1);axis([-2*T 2*T 1.1*min(w1) 1.1*max(w1)]);grid
xlabel('t');ylabel('w(1.5t)')
subplot(133)
plot(t3,w1); axis([-2*T 2*T 1.1*min(w1) 1.1*max(w1)]);grid
xlabel('t');ylabel('w(1.5t-2)')
%%%%%
   function [z3,t1,t2]=scale_shift (z,gamma,delay,T,Ts)
   % perfoms scale and shift of digitized signal
   % gamma positive real with two decimal
   % shf positive real
   % [-T T] range of signal
   % Ts sampling period
       beta1=100;alpha1=round(gamma,2)*beta1;
       g=gcd(beta1,alpha1);beta=beta1/g;alpha=alpha1/g;
       z1=interp(z,beta);z2=decimate(z1,alpha);
       t1=-T/gamma:Ts:T/gamma;
       M=length(t1);
       z3=z2(1:M);
       t2=t1+delay;
```

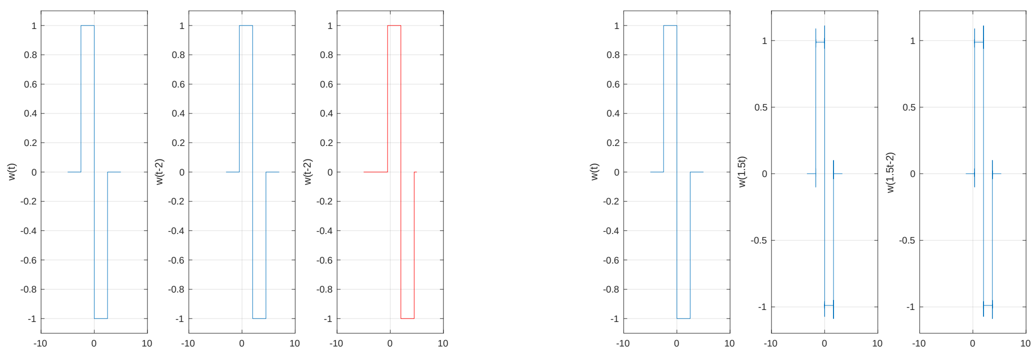


Figure 1.17: Problem 21: Shifting and scaling of a pulse.

The sharp values at the edges of the pulse are due to the discontinuities in the pulse.

1.22 Windowing, scaling and shifting a discretized analog signal— We wish to obtain a discrete approximation to a sinusoid $x(t) = \sin(3\pi t)$ from 0 to 2.5 seconds. To do so a discretized signal $x(nT_s)$, with $T_s = 0.001$, is multiplied it by a causal window $w(nT_s)$ of duration 2.5, i.e., $w(nT_s) = 1$ for $0 \leq n \leq 2500$ and zero otherwise. Use our *scale_shift* function to find $x(2t)$ and $x(2t-5)$ for $-1 \leq t \leq 10$ and plot them.

Solution

The following script is used to find the scaled and shifted versions of a windowed signal.

```
% Pro 1.22
clear all; clf
Ts=0.001; T=5; N=2*T/Ts; t=-T:Ts:T;
w0=[zeros(1,N/2) ones(1,N/4+1) zeros(1,N/4)];
delay=2; M=delay/Ts;
% scaling and shifting of windowed signal x
 [w1,t2,t3]=scale_shift(w0,1,delay,T,Ts);
 x=sin(3*pi*t).*w1;
 gamma=2; shf=5;
 [z,t2,t3]=scale_shift (x,gamma,shf,T,Ts);
 figure(1)
 subplot(311)
 plot(t,x); axis([-1 2*T 1.1*min(x) 1.1*max(x)]);grid
 xlabel('t');ylabel('x(t)')
 subplot(312)
 plot(t2,z)
 axis([-1 2*T 1.1*min(z) 1.1*max(z)]);grid
 xlabel('t');ylabel('x(2t)')
 subplot(313)
 plot(t3,z); axis([-1 2*T 1.1*min(z) 1.1*max(z)]);grid
 xlabel('t');ylabel('x(2t-5)')
```

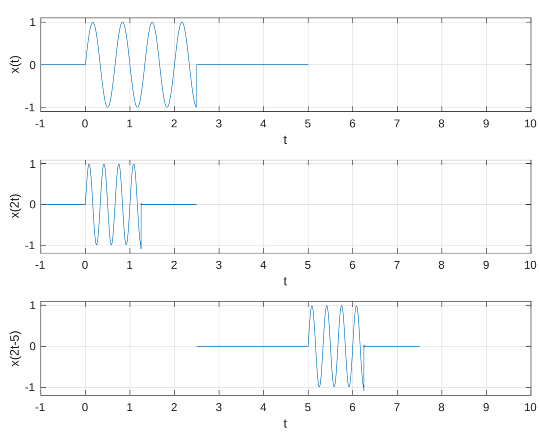


Figure 1.18: Problem 22: Windowing, scaling and shifting of a sinusoid.

1.23 Multipath effects — In wireless communications, the effects of *multi-path* significantly affect the quality of the received signal. Due to the presence of buildings, cars, etc. between the transmitter and the receiver the sent signal does not typically go from the transmitter to the receiver in a straight path (called *line of sight*). Several copies of the signal, shifted in time and frequency as well as attenuated, are received—i.e., the transmission is done over multiple paths each attenuating and shifting the sent signal. The sum of these versions of the signal appears quite different from the original signal given that constructive as well as destructive effects may occur. In this problem we consider the time shift of an actual signal to illustrate the effects of attenuation and time-shift. In the next problem we consider the effects of time and frequency shifting, and attenuation.

Assume that the MATLAB *handel.mat* signal is an analog signal $x(t)$ that it is transmitted over three paths, so that the received signal is

$$y(t) = x(t) + 0.8x(t - \tau) + 0.5x(t - 2\tau)$$

and let $\tau = 0.5$ seconds. Determine the number of samples corresponding to a delay of τ seconds by using the sampling rate Fs (samples per second) given when the file *handel* is loaded.

To simplify matters just work with a signal of duration 1 second; that is, generate a signal from *handel* with the appropriate number of samples. Plot the segment of the original *handel* signal $x(t)$ and the signal $y(t)$ to see the effect of multi-path. Use the MATLAB function *sound* to listen to the original and the received signals.

Solution

The sampling rate F_s in sample/second is given with the discretized signal. To get one second of the signal we need to take $N = F_s$ samples from the given signal. The corresponding number of samples NN for $\tau = 0.5$ sec. is then calculated and the signal $y(t)$ computed and displayed as function of time as shown in the following script. For $F_s = 8,192$ samples/sec, $NN = 4,096$ samples

```
% Pro 1.23
clear all; clf
load handel; Fs % test signal and sampling freq
N=Fs; y=y(1:N)'; % one second of handel
NN=fix(0.5*Fs)  % delay in samples
% delaying signals
t=0:1/Fs:(N-1)/Fs;
tt=0:1/Fs:(N-1)/Fs+2*NN/Fs;
 y1=[y zeros(1,2*NN)];
 y2=0.8*[zeros(1,NN) y zeros(1,NN)];
 y3=0.5*[zeros(1,2*NN) y];
yy=y1+y2+y3;
figure(9)
  subplot(211)
  plot(t,y); title('original signal');grid
  subplot(212)
  plot(tt,yy); title('multipath signal');grid
  xlabel('t (sec)')
sound(yy,Fs)
```

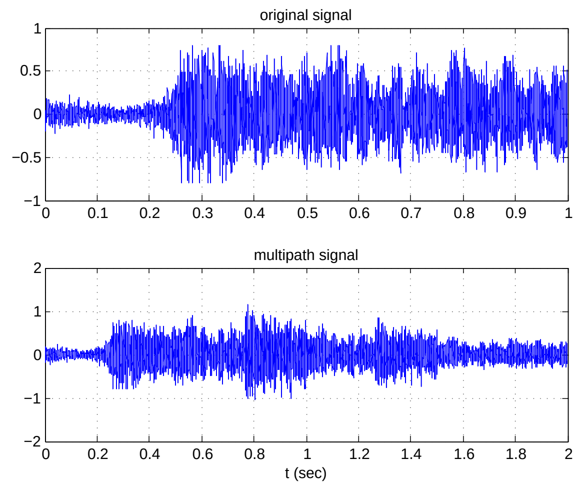


Figure 1.19: Problem 23: original 'handel' signal (top); two-path affected signal (bottom).

1.24 Multipath effects, Part 2 — Consider now the Doppler effect in wireless communications. The difference in velocity between the transmitter and the receiver causes a shift in frequency in the signal, which is called the Doppler effect. Just like the acoustic effect of a train whistle as the train goes by. To illustrate the frequency-shift effect, consider a complex exponential $x(t) = e^{j\Omega_0 t}$, assume two paths one which does not change the signal while the other causes the frequency-shift and attenuation, resulting in the signal

$$y(t) = e^{j\Omega_0 t} + \alpha e^{j\Omega_0 t} e^{j\phi t} = e^{j\Omega_0 t}\left[1 + \alpha e^{j\phi t}\right]$$

where α is the attenuation and ϕ is the Doppler frequency shift which is typically much smaller than the signal frequency. Let $\Omega_0 = \pi$, $\phi = \pi/100$, and $\alpha = 0.7$. This is analogous to the case where the received signal is the sum of the line of sight signal and an attenuated signal affected by Doppler.

(a) Consider the term $\alpha e^{j\phi t}$ a phasor with frequency $\phi = \pi/100$ to which we add 1. Use the MATLAB plotting function *compass* to plot the addition $1 + 0.7e^{j\phi t}$ for times from 0 to 256 sec changing in increments of $T = 0.5$ sec.

(b) If we write $y(t) = A(t)e^{j(\Omega_0 t + \theta(t))}$ give analytical expressions for $A(t)$ and $\theta(t)$, and compute and plot them using MATLAB for the times indicated above.

(c) Compute the real part of the signal

$$y_1(t) = x(t) + 0.7x(t-100)e^{j\phi(t-100)}$$

i.e., the effects of time and frequency delays, put together with attenuation, for the times indicated in part (a). Use the function *sound* (let $F_s = 2000$ in this function) to listen to the different signals.

Answers: $A(t) = \sqrt{1.49 + 1.4\cos(\phi t)}$, $\theta(t) = \tan^{-1}(0.7\sin(\phi t)/(1 + 0.7\cos(\phi t)))$.

Solution
(a) (b) Adding 1 to the phasor $0.7e^{j\phi t}$ gives a phasor of continuously varying magnitude and phase. Part (a) of the script below shows it.
We have

$$1 + 0.7e^{j\phi t} = 1 + 0.7\cos(\phi t) + j0.7\sin(\phi t) = A(t)e^{j\theta(t)}$$

where

$$A(t) = \sqrt{(1 + 0.7\cos(\phi t))^2 + (0.7\sin(\phi t))^2} = \sqrt{1.49 + 1.4\cos(\phi t)}$$

and

$$\theta(t) = \tan^{-1}\left[\frac{0.7\sin(\phi t)}{1 + 0.7\cos(\phi t)}\right]$$

which are computed as indicated in the script below.
(c) In this case we consider the effects of having two paths, the attenuation and the delays in time and in frequency.

```
% Pro 1.24
clear all; clf
 % part (a)
 t1=0;T=0.5; m=1;
 figure(1)
 for k=1:512,
     B=0.7*exp(j*pi*t1/100);
     A=1+B;
     A1(k)=abs(A);
     Theta(k)=angle(A)*180/pi;
     if k==20*m,
       compass(real(A),imag(A),'r')
       hold on
```

```
        compass(real(B),imag(B))
        hold on
        compass(1,0,'k')
        legend('A=B+1','B','1')
        m=m+1;
        pause(0.1)
      else
        t1=t1+T;
        hold off
      end
end
t=0:T:511*T;
% part (b)
figure(2)
  subplot(211)
  plot(t,A1);title('Magnitude of 1+e^{j\phi t}');grid
  axis([0 max(t) 0 1.1*max(A1)])
  subplot(212)
  plot(t,Theta);title('Phase (degrees) of 1+e^{j\phi t}');grid
  axis([0 max(t) 1.1*min(Theta) 1.1*max(Theta)]);xlabel('t')
% part (c)
y0=0.7*exp(j*(pi+pi/100)*t);
y1=real(exp(j*pi*t)+[zeros(1,100) y0(1:length(y0)-100)]);
t1=0:T:(length(y1)-1)*T;
figure(3)
  plot(t1,y1);title('Multi-path effects');grid
  axis([0 max(t1) 1.1*min(y1) 1.1*max(y1)]); ylabel('y_1(t)');xlabel('t')
```

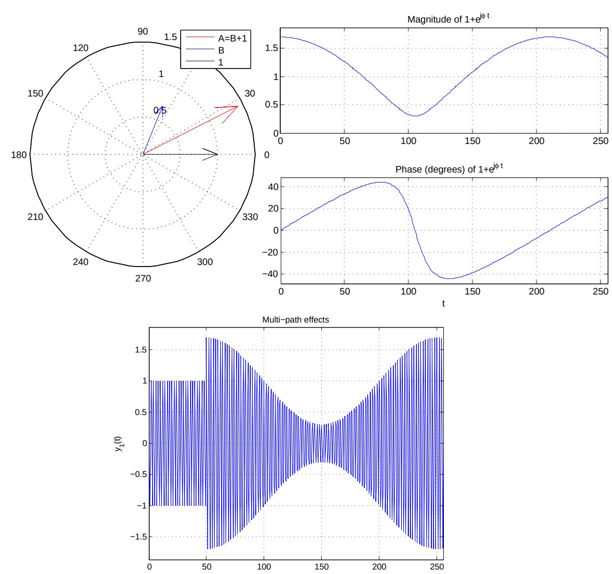


Figure 1.20: Problem 24: phasor plot (top–left); magnitude and phase of $1+e^{j\phi t}$ (top–right); resulting signal due to multipath (bottom).

1.25 Beating or pulsation — An interesting phenomenon in the generation of musical sounds is beating or pulsation. Suppose NP different players try to play a pure tone, a sinusoid of frequency 160 Hz, and that the signal recorded is the sum of these sinusoids. Assume the NP players while trying to play the pure tone end up playing tones separated by Δ Hz, so that the recorded signal is

$$y(t) = \sum_{i=1}^{NP} 10\cos(2\pi f_i t)$$

where the f_i are frequencies from 159 to 161 separated by Δ Hz. Each player playing a different frequency.

(a) Generate the signal $y(t)$ $0 \leq t \leq 200$ (sec) in MATLAB. Let each musician play a unique frequency. Consider an increasing number of players, letting NP to go from 51 players, with $\Delta = 0.04$ Hz, to 101 players with $\Delta = 0.02$ Hz. Plot $y(t)$ for each of the different number of players.

(b) Explain how this is related with multi-path and the Doppler effect discussed in the previous problems.

Solution

(a) The following script generates the signal $y(t)$ for $NP = 101$ players, and $\Delta = 0.02$ Hz (changing the NP to 51 we obtain the corresponding signal).

```
% Pro 1.25
clear all; clf
NP=101 % number of players
% NP=51
A=10; delta=2/(NP-1);
F=160-(NP-1)/2*delta:delta:160+(NP-1)/2*delta;
t=0:0.1:200;
y=zeros(1,length(t));
figure(13)
for k=1:NP,
   y=y+A*cos(2*pi*F(k)*t);
   plot(t,y);grid
   pause(0.1)
end
ylabel('y(t)'); xlabel('t')
```

The final signal looks like a sequence of very narrow pulses.

(b) In this part, one can think of a multipath with NP paths, with no attenuation but a different Doppler shift, ranging from -1 Hz to 1 Hz, in increments of 0.02 Hz.

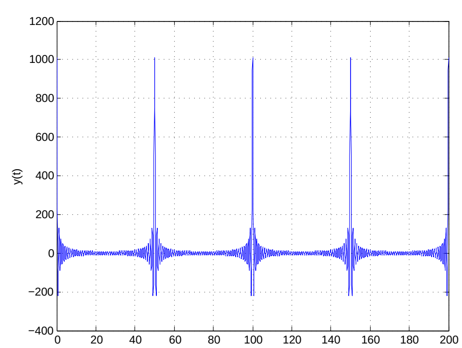


Figure 1.21: Problem 25: pulsation effect when $NP = 101$ and $\Delta = 0.02$ Hz.

1.26 **Chirps —** Pure tones or sinusoids are not very interesting to listen to. Modulation and other techniques are used to generate more interesting sounds. Chirps, which are sinusoids with time-varying frequency, are some of those more interesting sounds. For instance, the following is a chirp signal

$$y(t) = A\cos(\Omega_c t + s(t))$$

(a) Let $A = 1$, $\Omega_c = 2$, and $s(t) = t^2/4$. Use MATLAB to plot this signal for $0 \leq t \leq 40$ sec in steps of 0.05 sec. Use *sound* to listen to the signal.

(b) Let $A = 1$, $\Omega_c = 2$, and $s(t) = -2\sin(t)$ use MATLAB to plot this signal for $0 \leq t \leq 40$ sec in steps of 0.05 sec. Use *sound* to listen to the signal.

(c) What the frequency of these chirps are is not clear. The instantaneous frequency $IF(t)$ is the derivative with respect to t of the argument of the cosine. For instance for a cosine $\cos(\Omega_0 t)$ the $IF(t) = d\Omega_0 t/dt = \Omega_0$, so that the instantaneous frequency coincides with the conventional frequency. Determine the instantaneous frequencies of the two chirps and plot them. Do they make sense as frequencies? Explain.

Solution

(a)(b) The following script generates the chirps

```
% Pro 1.26
clear all;clf
t=0:0.05:40;
% chirps
y=cos(2*t+t.^2/4);
y1=cos(2*t- 2*sin(t));
figure(14)
 subplot(211)
 plot(t,y); title('linear chirp')
 axis([0 20 1.1*min(y) 1.1*max(y)]);grid
 subplot(212)
 plot(t,y1);title('sinusoidal chirp');xlabel('t')
 axis([0 20 1.1*min(y1) 1.1*max(y1)]);grid
% instantaneous frequencies
IF=2+2*t/4;
IF1=2-2*cos(2*t);
figure(15)
 subplot(211)
 plot(t,IF);title('IF of linear chirp')
 ylabel('frequency'); xlabel('t');grid
 subplot(212
 plot(t,IF1);title('IF of sinusoidal chirp')
 ylabel('frequency');xlabel('t');grid
```

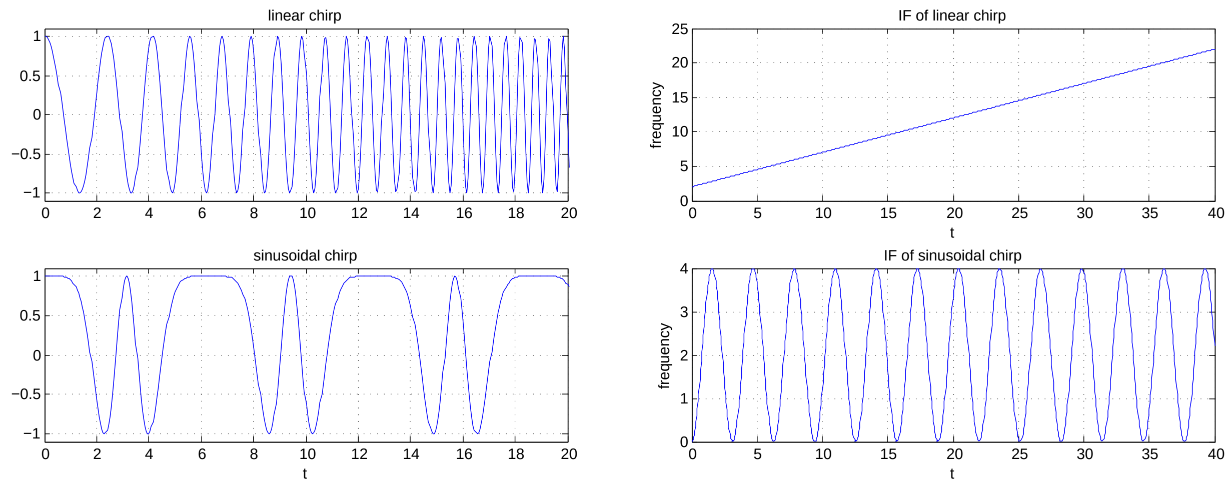


Figure 1.22: Problem 26: linear and sinusoidal chirps (left) and their corresponding instantaneous frequencies (right).

Chapter 2

Continuous–time Systems

2.1 Basic Problems

2.1 The input-output equation characterizing an amplifier that saturates once the input reaches certain values is

$$y(t) = \begin{cases} 100x(t) & -10 \leq x(t) \leq 10 \\ 1000 & x(t) > 10 \\ -1000 & x(t) < -10 \end{cases}$$

where $x(t)$ is the input and $y(t)$ the output.

(a) Plot the relation between the input $x(t)$ and the output $y(t)$. Is this a linear system? For what range of input values is the system linear, if any?

(b) Suppose the input is a sinusoid $x(t) = 20\cos(2\pi t)u(t)$, carefully plot $x(t)$ and $y(t)$ for $t = -2$ to 4.

(c) Let the input be delayed by 2 units of time, i.e, the input is $x_1(t) = x(t-2)$ find the corresponding output $y_1(t)$, and indicate how it relates to the output $y(t)$ due to $x(t)$ found above. Is the system time-invariant?

Answers: If input is always in $[-10 \;\; 10]$ system behaves linearly; system is time–invariant.

Solution

(a) The $y(t)$-$x(t)$ relation is a line through the origin between -10 to 10 and a constant before and after that. The system is non-linear, for instance if $x(t) = 7$ the output is $y(t) = 700$ but if we double the input, the output is not $2y(t) = 1400$ but 1000.

(b) If the inputs is always between -10 and 10 the system behaves like a linear system. In this case the output is chopped whenever $x(t)$ is above 10 or below -10. Se Fig. 2.1.

(c) Whenever the input goes below -10 or above 10 the output is -1000 and 1000, otherwise the output is $2000\cos(2\pi t)u(t)$.

(d) If the input is delayed by 2 the clipping will still occur, simply at a later time. So the system is time invariant.

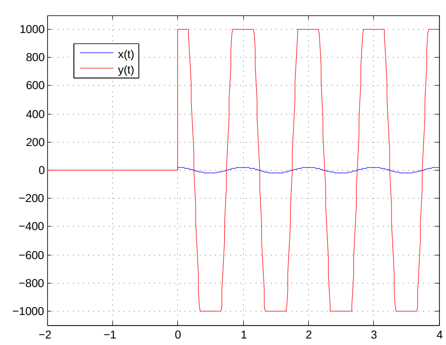


Figure 2.1: Problem 1: input and output of amplifier.

2.2 Consider an averager represented by the input/output equation

$$y(t) = \int_{t-1}^{t} x(\tau)d\tau + 2$$

where $x(t)$ is the input and $y(t)$ the output.

(a) Let the input be $x_1(t) = \delta(t)$, find graphically the corresponding output $y_1(t)$ for $-\infty < t < \infty$. Let then the input be $x_2(t) = 2x_1(t)$, find graphically the corresponding output $y_2(t)$ for $-\infty < t < \infty$. Is $y_2(t) = 2y_1(t)$? Is the system linear?

(b) Suppose the input is $x_3(t) = u(t) - u(t-1)$, graphically compute the corresponding output $y_3(t)$ for $-\infty < t < \infty$. If a new input is $x_4(t) = x_3(t-1) = u(t-1) - u(t-2)$, find graphically the corresponding output $y_4(t)$ for $-\infty < t < \infty$, and indicate if $y_4(t) = y_3(t-1)$. Accordingly, would this averager be time-invariant?

(c) Is this averager a causal system? Explain.

(d) If the input to the averager is bounded, would its output be bounded? Is the averager BIBO stable?

Answers: $y_1(t) = 2 + [u(t) - u(t-1)]$; system is non–linear, non–causal, and BIBO stable.

Solution

(a) Input $x_1(t) = \delta(t)$ gives

$$y_1(t) = \int_{t-1}^{t} \delta(\tau)d\tau + 2 = \begin{cases} 2 & t < 0 \\ 3 & 0 \leq t \leq 1 \\ 2 & t > 1 \end{cases}$$

$x_2(t) = 2x_1(t)$ gives

$$y_2(t) = 2\int_{t-1}^{t} \delta(\tau)d\tau + 2 = \begin{cases} 2 & t < 0 \\ 4 & 0 \leq t \leq 1 \\ 2 & t > 1 \end{cases}$$

Since $y_2(t) \neq 2y(t)$ system is non-linear.

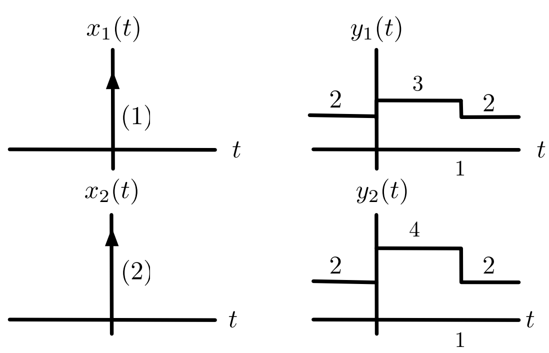


Figure 2.2: Problem 2

(b) If $x_3(t) = u(t) - u(t-1)$ then $y_3(t) = 2 + r(t) - 2r(t-1) + r(t-2)$. If $x_4(t) = x_3(t-1)$ then the corresponding output is $y_3(t-1)$, so the system is time-invariant.

(c) Non-causal, although $y(t)$ depends on present and past inputs, it is not zero when $x(t) = 0$, due to the bias of 2.

(d) If $|x(t)| < M$ we have

$$|y(t)| \leq \int_{t-1}^{t} |x(\tau)| d\tau + 2 < M + 2 < \infty$$

The system is BIBO stable.

2.3 An RC circuit in series with a voltage source $x(t)$ is represented by a ordinary differential equation

$$\frac{dy(t)}{dt} + 2y(t) = 2x(t)$$

where $y(t)$ is the voltage across the capacitor. Assume $y(0)$ is the initial voltage across the capacitor.

(a) If it is known that the resistor has a resistance R, and the capacitor $C = 1$F. Draw the circuit that corresponds to the given ordinary differential equation.

(b) For zero initial condition, and $x(t) = u(t)$, is the output of the system

$$y(t) = e^{-2t} \int_0^t e^{2\tau} d\tau?$$

If so, find and plot $y(t)$.

Answers: $R = 0.5$; $y(t) = 0.5(1 - e^{-2t})u(t)$

Solution

(a) See Fig. 3. The circuit is a series connection of a voltage source $x(t)$ with a resistor $R = 1/2\ \Omega$, and capacitor $C = 1$F. Indeed, the mesh current is $i(t) = dy(t)/dt$ so

$$x(t) = Ri(t) + y(t) = Rdy(t)/dt + y(t)$$

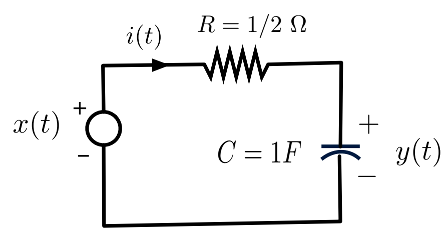


Figure 2.3: Problem 3

(b) The output is

$$y(t) = e^{-2t}\, 0.5\, e^{2\tau}|_0^t = 0.5(1 - e^{-2t})u(t)$$

and

$$\begin{aligned} \frac{dy(t)}{dt} &= e^{-2t}u(t) + 0.5(1 - e^{-2t})\delta(t) \\ &= e^{-2t}u(t) \\ \frac{dy(t)}{dt} + 2y(t) &= e^{-2t}u(t) + u(t) - e^{-2t}u(t) \\ &= u(t) \end{aligned}$$

2.4 A time-varying capacitor is characterized by the charge-voltage equation $q(t) = C(t)v(t)$. That is, the capacitance is not a constant but a function of time.

(a) Given that $i(t) = dq(t)/dt$, find the voltage-current relation for this time-varying capacitor.

(b) Let $C(t) = 1 + \cos(2\pi t)$ and $v(t) = \cos(2\pi t)$ determine the current $i_1(t)$ in the capacitor for all t.

(c) Let $C(t)$ be as above, but delay $v(t)$ by 0.25 sec., determine $i_2(t)$ for all time. Is the system TI?

Answer: (b) $i_1(t) = -2\pi \sin(2\pi t)[1 + 2\cos(2\pi t)]$

Solution: (a) The charge is

$$q(t) = C(t)v(t)$$

so that

$$i(t) = \frac{dq(t)}{dt} = C(t)\frac{dv(t)}{dt} + v(t)\frac{dC(t)}{dt}$$

(b) If $C(t) = 1 + \cos(2\pi t)$ and $v(t) = \cos(2\pi t)$, the current is

$$\begin{aligned} i_1(t) &= C(t)\frac{dv(t)}{dt} + v(t)\frac{dC(t)}{dt} \\ &= (1 + \cos(2\pi t))(-2\pi \sin(2\pi t)) - \cos(2\pi t)(2\pi \sin(2\pi t)) \\ &= -2\pi \sin(2\pi t)[1 + 2\cos(2\pi t)] \end{aligned}$$

(c) When the input is

$$v(t - 0.25) = \cos(2\pi(t - 1/4)) = \sin(2\pi t)$$

the output current is

$$\begin{aligned} i_2(t) &= C(t)\frac{dv(t - 0.25)}{dt} + v(t - 0.25)\frac{dC(t)}{dt} \\ &= (1 + \cos(2\pi t))(2\pi \cos(2\pi t)) - 2\pi \sin^2(2\pi t) \\ &= 2\pi \cos(2\pi t) + 2\pi[\cos^2(2\pi t) - \sin^2(2\pi t)] \end{aligned}$$

which is not

$$i_1(t - 0.25) = 2\pi \cos(2\pi t)[1 + \sin(2\pi t)]$$

so the system is time varying.

2.5 An analog system has the following input-output relation,

$$y(t) = \int_0^t e^{-(t-\tau)}x(\tau)d\tau \qquad t \geq 0$$

and zero otherwise. The input is $x(t)$ and $y(t)$ is the output.

(a) Is this system LTI? If so, can you determine without any computation the impulse response of the system? Explain.

(b) Is this system causal? Explain.

(c) Find the unit step response $s(t)$ and from it find the impulse response $h(t)$. Is this a BIBO stable system? Explain.

(d) Find the response due to a pulse $x(t) = u(t) - u(t-1)$.

Answers: Yes, LTI with $h(t) = e^{-t}u(t)$; causal and BIBO stable.

Solution

(a) The system is LTI since the input $x(t)$ and the output $y(t)$ are related by a convolution integral with $h(t-\tau) = e^{-(t-\tau)}u(t-\tau)$ or $h(t) = e^{-t}u(t)$.

Another way: to show that the system is linear let the input be $x_1(t) + x_2(t)$, and $x_1(t)$ and $x_2(t)$ have as outputs

$$y_i(t) = \int_0^t e^{-(t-\tau)}x_i(\tau)d\tau \qquad i = 1, 2$$

The output for $x_1(t) + x_2(t)$ is

$$\int_0^t e^{-(t-\tau)}(x_1(\tau) + x_2(\tau))d\tau = y_1(t) + y_2(t)$$

To show the time invariance let the input be $x(t-t_0)$, its output will be

$$\begin{aligned}\int_0^t e^{-(t-\tau)}x(\tau - t_0)d\tau &= \int_{-t_0}^0 e^{-((t-t_0)-\mu)}x(\mu)d\mu + \int_0^{t-t_0} e^{-((t-t_0)-\mu)}x(\mu)d\mu \\ &= \int_0^{t-t_0} e^{-((t-t_0)-\mu)}x(\mu)d\mu = y(t-t_0)\end{aligned}$$

by letting $\mu = \tau - t_0$ and using the causality of the input. The system is then TI.

Finally the impulse response is found by letting $x(t) = \delta(t)$ so that the output is

$$h(t) = \int_0^t e^{-(t-\tau)}\delta(\tau)d\tau = \int_0^t e^{-(t-0)}\delta(\tau)d\tau = \begin{cases} e^{-t} \times 1 = e^{-t} & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

(b) Yes, this system is causal as the output $y(t)$ depends on present and past values of the input.

(c) Letting $x(t) = u(t)$, the unit-step response is

$$s(t) = \int_0^t e^{-t+\tau}u(\tau)d\tau = e^{-t}\int_0^t e^{\tau}d\tau = 1 - e^{-t}$$

for $t \geq 0$ and zero otherwise. The impulse response as indicated before is $h(t) = ds(t)/dt = e^{-t}u(t)$. The BIBO stability of the system is then determined by checking whether the impulse response is absolutely integrable or not,

$$\int_{-\infty}^{\infty} |h(t)|dt = \int_0^{\infty} e^{-t}dt = -e^{-t}\,|_0^{\infty} = 1$$

so yes it is BIBO stable.

(d) Using superposition, the response to the pulse $x_1(t) = u(t) - u(t-1)$ would be

$$y_1(t) = y(t) - y(t-1) = (1 - e^{-t})u(t) - (1 - e^{-(t-1)})u(t-1)$$

which starts at zero, grows to a maximum of $1 - e^{-1}$ at $t = 1$ and goes down to zero as $t \to \infty$.

2.6 A fundamental property of linear time-invariant systems is that whenever the input of the system is a sinusoid of a certain frequency the output will also be a sinusoid of the same frequency but with an amplitude and phase determined by the system. For the following systems let the input be $x(t) = \cos(t)$, $-\infty < t < \infty$, find the output $y(t)$ and determine if the system is LTI,

$$\begin{array}{ll} (a)\;\; y(t) = |x(t)|^2, & (b)\;\; y(t) = 0.5[x(t) + x(t-1)], \\ (c)\;\; y(t) = x(t)u(t), & (d)\;\; y(t) = \frac{1}{2}\int_{t-2}^{t} x(\tau)d\tau \end{array}$$

Answers: (a) $y(t) = 0.5(1 + \cos(2t))$; (c) system is not LTI.

Solution
The input to all the systems is $x(t) = \cos(t)$, $-\infty < t < \infty$
(a) The system is non-linear, as the output

$$y(t) = \cos^2(t) = 0.5(1 + \cos(2t))$$

has frequency components of frequencies 0 and 2 (rad/sec) which are not in the input.
(b) The output is

$$y(t) = 0.5\cos(t) + 0.5\cos(t-1)$$

having the same frequencies as the input so it is LTI.
(c) The output

$$y(t) = \cos(t)u(t)$$

is not LTI. This is not a periodic signal, and it has frequencies different from the one at the input due to the multiplication by the $u(t)$.
(d) The output is

$$y(t) = 0.5\sin(\tau)|_{t-2}^{t} = 0.5\sin(t) - 0.5\sin(t-2)$$

having the same frequency as the input so it is LTI.

2.7 Consider the system where for an input $x(t)$ the output is $y(t) = x(t)f(t)$.

(a) Let $f(t) = u(t) - u(t - 10)$, determine whether the system with input $x(t)$ and output $y(t)$ is linear, time-invariant, causal and BIBO stable.

(b) Suppose $x(t) = 4\cos(\pi t/2)$, and $f(t) = \cos(6\pi t/7)$ and both are periodic, is the output $y(t)$ also periodic? what frequencies are present in the output? is this system linear? is it time-invariant? Explain.

(c) Let $f(t) = u(t) - u(t - 2)$ and the input $x(t) = u(t)$, find the corresponding output $y(t)$. Suppose you shift the input so that it is $x_1(t) = x(t - 3)$ what is the corresponding output $y_1(t)$. Is the system time-invariant? Explain.

Answer: (a) System is time–varying, BIBO stable.

Solution

(a) Since $f(t)$ is not a constant, the system is a modulator thus linear but time varying. Linearity is clearly satisfied. If $x(t) \neq 0$ is the input and we shift it to get as input $x(t - 11)$ the corresponding output is zero different from $y(t - 11)$. Thus the system is time varying. Since $y(t)$ depends on $x(t)$ the system is causal. For $x(t)$ bounded, i.e., $|x(t)| < M < \infty$, the output is also bounded, $|y(t)| < M|f(t)| < \infty$ so the system is BIBO stable.

(b) The modulated signal is

$$x(t)f(t) = 2[\cos((\pi/2 + 6\pi/7)t) + \cos((6\pi/7 - \pi/2)t) = 2(\cos(19\pi t/14) + \cos(5\pi t/14)$$

with periods of $T_0 = 28/19$ and $T_1 = 28/5$ for the two components. The ratio

$$\frac{T_0}{T_1} = \frac{5}{9}$$

i.e., it is rational so the modulated signal is periodic of period $5T_1 = 19T_0 = 28$, which is easily verified. The frequencies at the output are not present at the input so the system is linear but not time–invariant ($f(t)$ is a function of t).

(c) If $x(t) = u(t)$, the modulated signal is $y(t) = u(t) - u(t - 2)$, and if we shift the input so that it is $x(t - 3) = u(t - 3)$ the corresponding output is $u(t - 3)[u(t) - u(t - 2)] = 0$ different from the previous output shifted by 3, therefore the system is time-varying.

2.8 The response of a first-order system is for $t \geq 0$

$$y(t) = y(0)e^{-t} + \int_0^t e^{-(t-\tau)}x(\tau)d\tau$$

and zero otherwise.

(a) Consider $y(0) = 0$ is the system linear? If $y(0) \neq 0$, is the system linear? Explain.

(b) If $x(t) = 0$, what is the response of the system called? If $y(0) = 0$ what is the response of the system to any input $x(t)$ called?

(c) Let $y(0) = 0$, find the response due to $\delta(t)$. What is this response called?

(d) When $y(0) = 0$, and $x(t) = u(t)$ call the corresponding output $s(t)$. Find $s(t)$ and calculate $ds(t)/dt$, what does this correspond to from the above results.

Answers: If $y(0) = 0$, $y(t) = [x * h](t)$ is zero–state response with $h(t) = e^{-t}u(t)$.

Solution

(a) If $y(0) = 0$ the system is linear, indeed for an input $\alpha x_1(t) + \beta x_2(t)$ with $y_1(t)$ the response due to $x_1(t)$ and $y_2(t)$ the response due to $x_2(t)$ we have

$$\int_0^t e^{-(t-\tau)}[\alpha a x_1(\tau) + \beta x_2(\tau)]d\tau = \alpha y_1(t) + \beta y_2(t)$$

If $y(0) \neq 0$, the output for input $\alpha x_1(t)$ is

$$y(0)e^{-t} + \int_0^t e^{-(t-\tau)}\alpha x_1(\tau)d\tau = y(0)e^{-t} + \alpha y_1(t)$$

which is not $\alpha y_1(t)$ thus it is not linear.

(b) If the input is $x(t) = 0$, then $y(t) = y(0)e^{-t}u(t)$ is the zero-input response, due completely to the initial condition. If $y(0) = 0$ the response

$$y(t) = \int_0^t e^{-(t-\tau)}x(\tau)d\tau$$

(which is the convolution integral of the impulse response $h(t) = e^{-t}u(t)$ with $x(t)$) is the zero-state response.

(c) The impulse response, obtained when $y(0) = 0$, $x(t) = \delta(t)$, and $y(t) = h(t)$ is

$$h(t) = \int_{0-}^t e^{-(t-\tau)}\delta(\tau)d\tau = e^{-t}\int_{0-}^t e^0\delta(\tau)d\tau = \begin{cases} e^{-t} & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

(d) If $x(t) = u(t)$ and $y(0) = 0$, then $y(t) = s(t)$ given by

$$s(t) = \int_0^t e^{-(t-\tau)}d\tau = (1 - e^{-t})u(t)$$

which is shown in FIg. 12

Notice the relation between the unit-step and the impulse response:

$$\begin{aligned} \frac{ds(t)}{dt} &= \delta(t) - e^{-t}\delta(t) + e^{-t}u(t) \\ &= e^{-t}u(t) = h(t) \end{aligned}$$

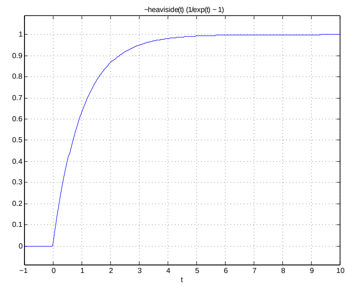


Figure 2.4: Problem 8: Unit-step response