Solutions Manual

Fourth Edition

SIGNALS & SYSTEMS

Continuous and Discrete

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PREFACE

This manual contains solutions to all end-of-chapter problems and all computer exercises contained in the Fourth Edition of *Signals and Systems: Continuous and Discrete*. The manual is divided into two separate parts. Part I contains solutions to the end-of-chapter problems and Part II contains solutions to the computer exercises.

All computer exercises are developed using MATLAB as are all end-of-chapter problems specifying the use of MATLAB. In several parts of this manual, especially in Chapters 8 and 9, MathCAD is used for a little variety in a few problems.

Thanks go to Carol Baker for her expert typing skills and for her help in assembling the final product.

While we have done our best to insure that the problem solutions contained herein are correct, it is inevitable that manuals such as this are never perfect. We apologize in advance for any frustration caused by such errors.

R.E.Z.
W.H.T.
D.R.F.

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PART I

SOLUTIONS TO END-OF-CHAPTER PROBLEMS

CHAPTER 1

Problem 1-1

(a) Write the acceleration as

$$a(t) = \begin{cases} \alpha t, & t \le t_0 \\ 0, & t > t_0 \end{cases}$$

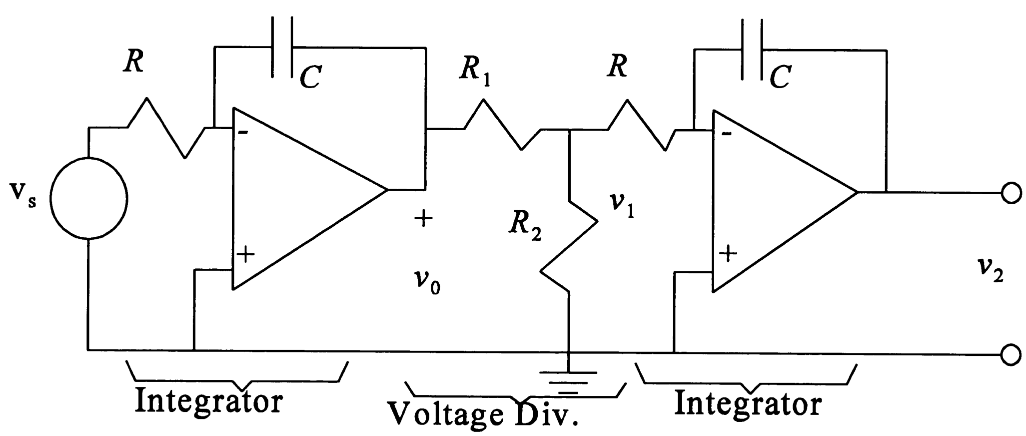
Thus the velocity and position are, respectively, given by

$$v(t) = \int_0^t a(\lambda)d\lambda = \begin{cases} \alpha t^2/2, & t \le t_0 \\ \alpha t_0^2/2, & t > t_0 \end{cases}$$

and

$$x(t) = \int_0^t v(\lambda)d\lambda = \begin{cases} \alpha t^3/6, & t \le t_0 \\ \alpha t_0^3/6 + \alpha t_0^2(t - t_0)/2, & t > t_0 \end{cases}$$

For $t_0 = 72$ s and $\alpha = 5/9$ m/s^2, we have $x(t) = (5/54)t^3$, $t \le 72$ s. At $t = t_0 = 72$ s (burnout), we have $x(t_0) = 35.56$ km.
(b) See the figure below for the integrator.



$$v_2(t) = -\frac{1}{RC}\int_0^t v_1(\lambda)d\lambda$$

Assume that $R_2 << R$. The input impedance to the op-amp integrator is therefore much larger than the output impedance of the previous stage, and

$$v_1(t) = \frac{R_2}{R_1 + R_2} v_0(t)$$

From Example 1-2,

$$v_0(t) = -\frac{1}{RC}\left(\frac{\beta t^2}{2}\right) = -\frac{\beta t^2}{2RC}$$

Therefore,

$$v_2(t) = -\frac{1}{RC}\int_0^t \frac{R_2}{R_1 + R_2}\left(-\frac{\beta}{2RC}\right)\lambda^2 d\lambda$$

Integrating and setting $t = t_0$, we obtain

$$v_2(t_0) = \frac{R_2}{R_1 + R_2}\left(\frac{\beta t_0^2}{2RC}\right)\left(\frac{t_0}{3RC}\right) = 10 \text{ V}$$

The second factor on the right is 10 V because of the maximum output limitation on the first integrator. Thus, we require that

$$\frac{R_2}{R_1 + R_2}\left(\frac{t_0}{3RC}\right) = 1$$

For example, from Example 1-2 we have $RC = 0.36$ s. With $t_0 = 72$ s and $R_1 = 10$ k ohms, we get $R_2 = 152$ ohms.

Problem 1-2

(a) Let $n = 0, 1, 2, 3, ..., N$. Then

$$\begin{aligned} v(T) &= v(0) + Ta(T) \quad \text{(a)} \\ v(2T) &= v(T) + Ta(2T) \quad \text{(b)} \\ &\cdots \\ v(NT) &= v[(N-1)T] + Ta(NT) \quad \text{(c)} \end{aligned}$$

Substitute (a) into (b) and so on until (c is reached. This gives

$$v(NT) = v(0) + T\sum_{n=1}^{N} a(nT)$$

(b) Let $n = 0, 1, 2, 3, ..., N$. Then

$$\begin{aligned} v(T) &= v(0) + (T/2)[a(0) + a(T)] \quad \text{(a)} \\ v(2T) &= v(T) + (T/2)[a(T) + a(2T)] \quad \text{(b)} \\ &\cdots \\ v(NT) &= v[(N-1)T] + (T/2)\{a[(N-1)T] + a(NT)\} \quad \text{(c)} \end{aligned}$$

Substitute (a) into (b) and so on until (c) is reached. The result is as given in the problem statement.

Problem 1-3

(a) A maximum departure of the weight from equilibrium of 1 cm requires a spring constant of

$$K = \frac{Ma_{\max}}{x_{\max}} = \frac{(0.002)(20)}{0.01} = 4 \text{ kg/s}^2$$

(b) For a minimum increment of 0.5 mm = 0.0005 m, we have

$$\Delta a_{\min} = \frac{K\Delta x_{\min}}{M} = \frac{4(0.0005)}{0.002} = 1 \text{ m/s}^2$$

(c) The velocity is given by

$$v_r(t) = \int_0^t a(\lambda)d\lambda = \int_0^t 20\,d\lambda = \begin{cases} 20t, & 0 \le 50 \text{ s} \\ 1000, & t > 50 \text{ s} \end{cases}$$

Problem 1-4

K is the same as in Example 1-1 because M, $x_{\max}$, and $a_{\max}$ are the same. Also, $\Delta a_{\min}$ is the same. The velocity profile is

$$v_r(t) = \begin{cases} \int_0^t 20\,d\lambda = 20t, \ 0 \le t < 10 \\ 200, \ 10 \le t < 20 \\ 200 + \int_{20}^t 20\,d\lambda = 200 + 20(t - 20), \ 20 \le t < 30 \\ 400, \ t > 30 \end{cases}$$

Problem 1-5

From (1-15) and using the $x(t)$ given in the problem, we have

$$\begin{aligned} s(t) &= \cos(\omega_0 t) + \alpha\beta\cos[\omega_0(t - 2\tau)] \\ &= [1 + \alpha\beta\cos(2\omega_0\tau)]\cos(\omega_0 t) + \alpha\beta\sin(2\omega_0\tau)\sin(\omega_0 t) \\ &= A(\tau)\cos[\omega_0 t - \theta(\tau)] \\ &= A(\tau)\cos\theta(\tau)\cos(\omega_0 t) + A(\tau)\sin\theta(\tau)\sin(\omega_0 t) \end{aligned}$$

Set coefficients of like sin/cos terms equal on each side of the identity to obtain

$$\begin{aligned} A(\tau)\cos\theta(\tau) &= 1 + \alpha\beta\cos(2\omega_0\tau) \\ A(\tau)\sin\theta(\tau) &= \alpha\beta\sin(2\omega_0\tau) \end{aligned}$$

Square and add to obtain

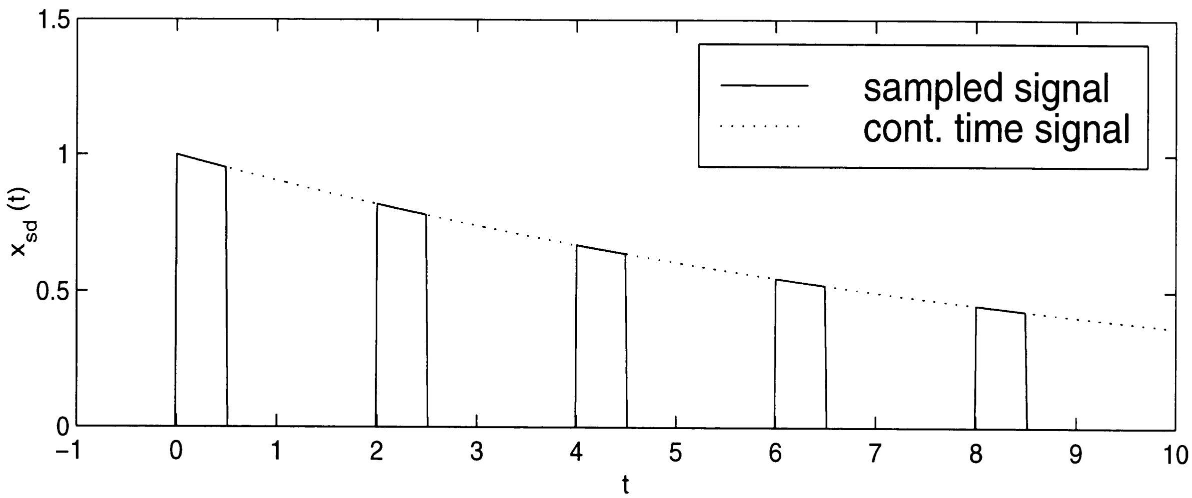
$$A(\tau) = \sqrt{1 + 2\alpha\beta\cos(2\omega_0\tau) + (\alpha\beta)^2}$$

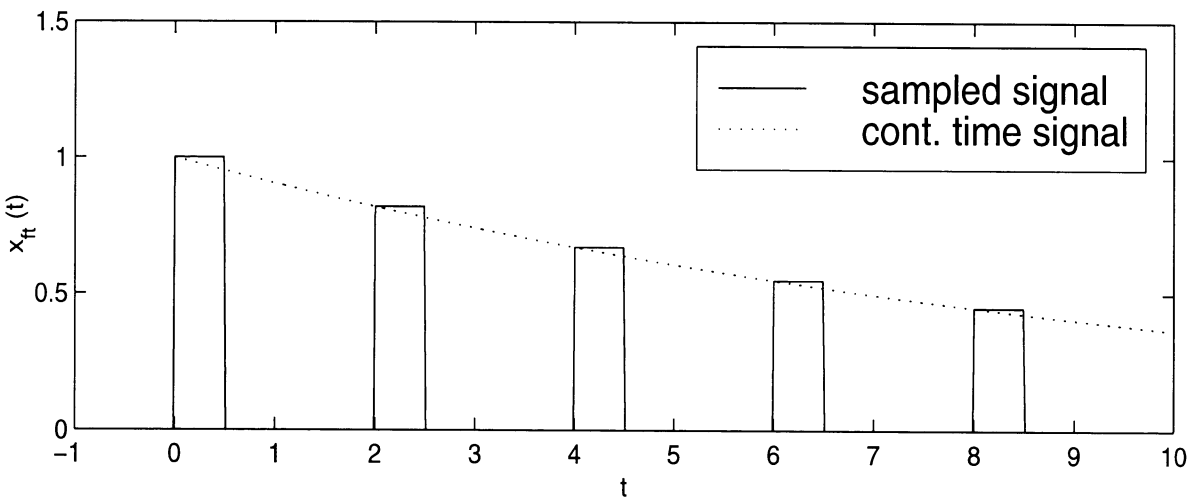
Divide the second equation by the first to obtain

$$\frac{\sin\theta(\tau)}{\cos\theta(\tau)} = \tan\theta(\tau) = \frac{\alpha\beta\sin(2\omega_0\tau)}{1 + \alpha\beta\cos(2\omega_0\tau)}$$

Problem 1-6

Sketches of the analog and sampled signals for both cases are shown below[(a) top and (b) bottom]:





Problem 1-7

(a) The impulse-sampled signal is

$$x_{\text{imp. samp}}(t) = \cos(2\pi t) \sum_{n=-\infty}^{\infty} \delta(t - 0.1n)$$
$$= \sum_{n=-\infty}^{\infty} \cos(2\pi t)\,\delta(t - 0.1n)$$
$$= \sum_{n=-\infty}^{\infty} \cos(0.2\pi n)\,\delta(t - 0.1n)$$

where property (1-59) for the unit impulse has been used to get the last result.
(b) The unit-pulse train sampled signal is

$$x_{\text{unit pulse samp}}(t) = \cos(2\pi t) \sum_{n=-\infty}^{\infty} \delta[t - 0.1n]$$
$$= \sum_{n=-\infty}^{\infty} \cos(2\pi t)\,\delta[t - 0.1n]$$
$$= \sum_{n=-\infty}^{\infty} \cos(0.2\pi n)\,\delta[t - 0.1n]$$

where the fact that the unit pulse is 1 for its argument 0 and 0 otherwise has been used.

Problem 1-8

(a) The signal can be developed in terms of equations as follows:

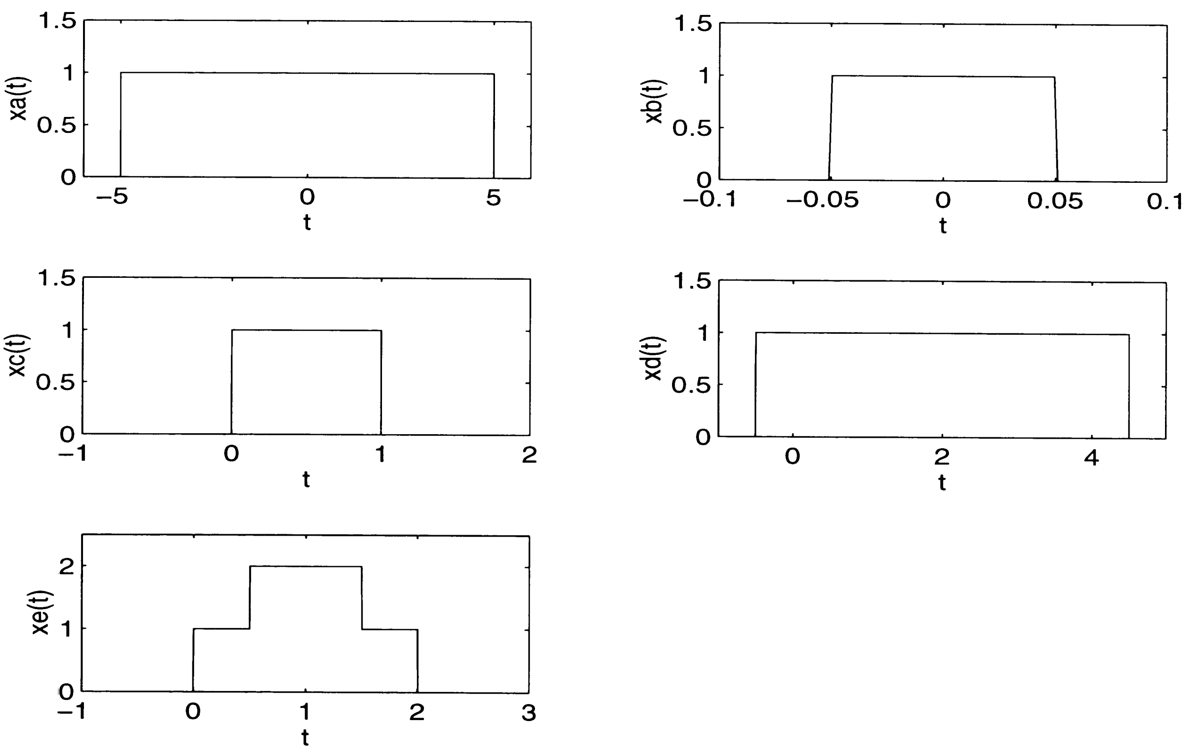
$$\Pi(0.1t) = \begin{cases} 1, & |0.1t| \le 1/2 \\ 0, & \text{otherwise} \end{cases}$$
$$= \begin{cases} 1, & |t| \le 10/2 = 5 \\ 0, & \text{otherwise} \end{cases}$$

This is a rectangular pulse of amplitude 1 between -5 and 5 and 0 otherwise. A sketch will be given at the end of the problem solution.
(b) Following a procedure similar to that of (a) one finds that this is a rectangular pulse of amplitude 1 between -0.05 and 0.05 and 0 otherwise. A sketch will be given at the end of the problem solution.
(c) This is a rectangular pulse of amplitude 1 between 0 and 1 and 0 otherwise. A sketch will be given at the end of the problem solution.
(d) This is a rectangular pulse of amplitude 1 between 0.5 and 4.5 and 0 otherwise. A sketch will be given at the end of the problem solution.

(e) The first term of this signal is a rectangular pulse of amplitude 1 between 0 and 2 and 0 otherwise. The second term is a rectangular pulse of amplitude 1 between 0.5 and 1.5 and 0 otherwise. Where both pulses are nonzero, the total amplitude is 2; where only one pulse is nonzero the amplitude is 1. A sketch is provided below.

The MATLAB program below uses the special function given in Section 1-6 (page 32) of the text to provide the plots.

```
%      Sketches for Problem 1-8
%
t = -6:0.0015:6;
xa = pls_fn(0.1*t);
xb = pls_fn(10*t);
xc = pls_fn(t - 0.5);
xd = pls_fn((t - 2)/5);
xe = pls_fn((t - 1)/2) + pls_fn(t - 1);
subplot(3,2,1),plot(t, xa,'-w'), axis([-6 6 0 1.5]),xlabel('t'),ylabel('xa(t)')
subplot(3,2,2),plot(t, xb,'-w'), axis([-.1 .1 0 1.5]),xlabel('t'),ylabel('xb(t)')
subplot(3,2,3),plot(t, xc,'-w'), axis([-1 2 0 1.5]),xlabel('t'),ylabel('xc(t)')
subplot(3,2,4),plot(t, xd,'-w'), axis([-1 5 0 1.5]),xlabel('t'),ylabel('xd(t)')
subplot(3,2,5),plot(t, xe,'-w'), axis([-1 3 0 2.5]),xlabel('t'),ylabel('xe(t)')
```



Problem 1-9

(a) $2\pi f_0 = 50\pi$, so $T_0 = 1/f_0 = 1/25 = 0.04$ s. (b) $2\pi f_0 = 60\pi$, so $T_0 = 1/f_0 = 1/30 = 0.0333$ s. (c) $2\pi f_0 = 70\pi$, so $T_0 = 1/f_0 = 1/35 = 0.0286$ s. (d) We have $50\pi = 2\pi m f_0$ and $60\pi = 2\pi n f_0$, where m and n are integers and f_0 is the largest constant that satisfies these equations. The largest f_0 is 5 Hz with $m = 5$ and $n = 6$. (e) We have $50\pi = 2\pi m f_0$ and $70\pi = 2\pi n f_0$, where m and n are integers and f_0 is the largest constant that satisfies these equations. The largest f_0 is 5 Hz with $m = 5$ and $n = 7$.

Problem 1-10

(a) $|A| = 4.2426$; angle(A) = 0.7854 radians; $B = 5.0 + j\,8.6603$, so Re(B) = 5 and Im(B) = 8.6603. (b) $A+B = 8.0 + j11.6603$. (c) $A - B = -2.0 - j5.6603$. (d) $A*B = -10.9808 + j40.9808$. (e) $A/B = 0.4098 - j0.1098$.

Problem 1-11

(a) $2\pi f_0 = 10\pi$, so $T_0 = 1/f_0 = 1/5 = 0.2$ s. (b) $2\pi f_0 = 17\pi$, so $T_0 = 1/f_0 = 1/8.5 = 0.1176$ s. (c) $2\pi f_0 = 19\pi$, so $T_0 = 1/f_0 = 1/9.5 = 0.1053$ s. (d) We have $10\pi = 2\pi m f_0$ and $17\pi = 2\pi n f_0$, where m and n are integers and f_0 is the largest constant that satisfies these equations. The largest f_0 is 0.5 Hz with $m = 10$ and $n = 17$. (e) We have $10\pi = 2\pi m f_0$ and $19\pi = 2\pi n f_0$, where m and n are integers and f_0 is the largest constant that satisfies these equations. The largest f_0 is 0.5 Hz with $m = 10$ and $n = 19$. (f) We have $17\pi = 2\pi m f_0$ and $19\pi = 2\pi n f_0$, where m and n are integers and f_0 is the largest constant that satisfies these equations. The largest f_0 is 0.5 Hz with $m = 17$ and $n = 19$.

Problem 1-12

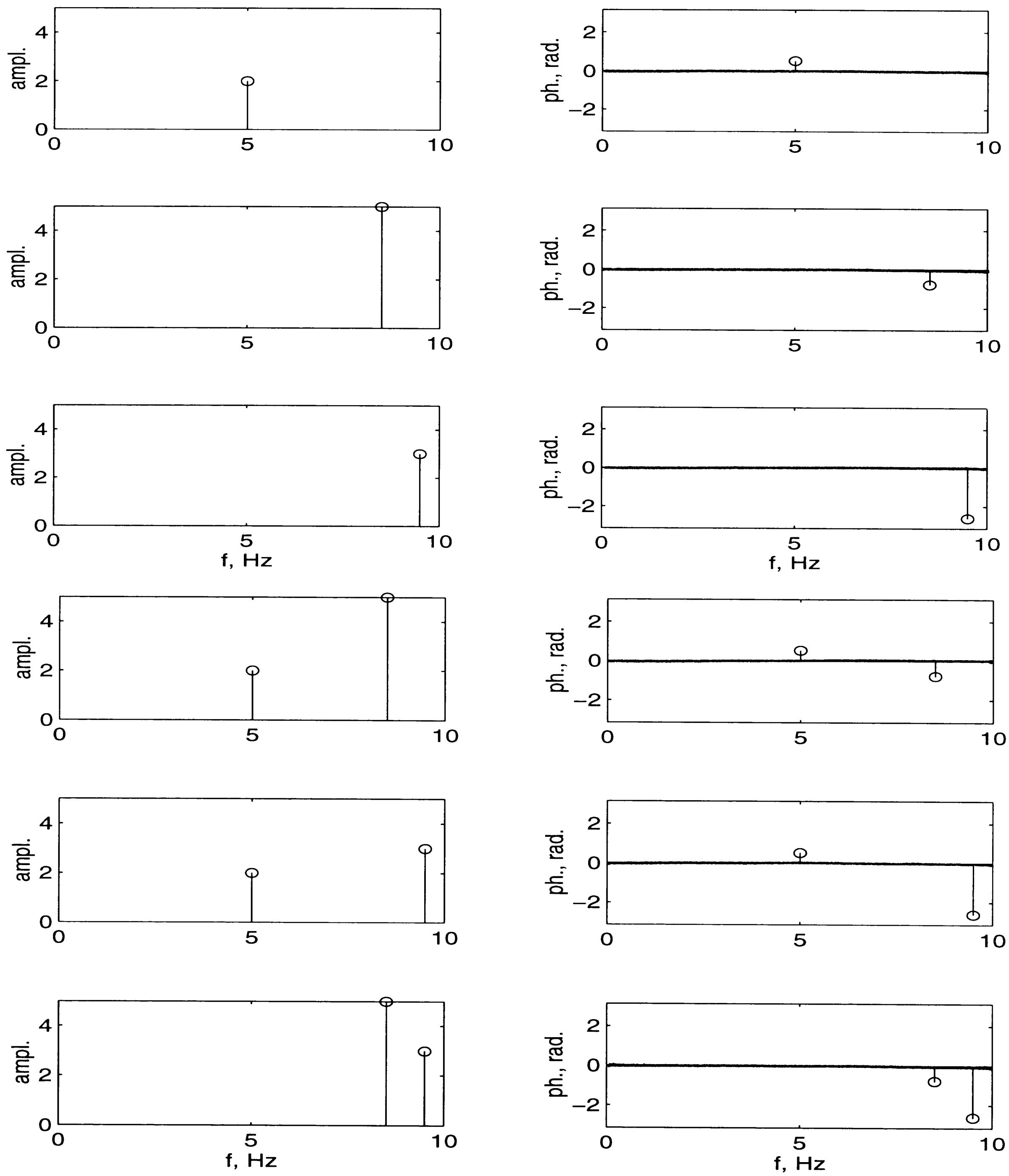
(a) Written as the real part of rotating phasors:

$$x_a(t) = \mathrm{Re}[2e^{j(10\pi t + \pi/6)}];\ x_b(t) = \mathrm{Re}[5e^{j(17\pi t - \pi/4)}]$$
$$x_c(t) = \mathrm{Re}[3e^{j(10\pi t - \pi/3 - \pi/2)}] = \mathrm{Re}[3e^{j(10\pi t - 5\pi/6)}]$$
$$x_d(t) = \mathrm{Re}[2e^{j(10\pi t + \pi/6)} + 5e^{j(17\pi t - \pi/4)}];\ x_e(t) = \mathrm{Re}[2e^{j(10\pi t + \pi/6)} + 3e^{j(10\pi t - 5\pi/6)}]$$
$$x_f(t) = \mathrm{Re}[5e^{j(17\pi t - \pi/4)} + 3e^{j(10\pi t - 5\pi/6)}]$$

(b) In terms of counter rotating phasors, the signals are:

$$x_a(t) = [e^{j(10\pi t + \pi/6)} + e^{-j(10\pi t + \pi/6)}];\ x_b(t) = [2.5e^{j(17\pi t - \pi/4)} + 2.5e^{j(17\pi t - \pi/4)}]$$
$$x_c(t) = [1.5e^{j(10\pi t - 5\pi/6)} + 1.5e^{-j(10\pi t - 5\pi/6)}]$$
$$x_d(t) = [e^{j(10\pi t + \pi/6)} + e^{-j(10\pi t + \pi/6)} + 2.5e^{j(17\pi t - \pi/4)} + 2.5e^{-j(17\pi t - \pi/4)}]$$
$$x_e(t) = [e^{j(10\pi t + \pi/6)} + e^{-j(10\pi t + \pi/6)} + 1.5e^{j(10\pi t - 5\pi/6)} + 1.5e^{-j(10\pi t - 5\pi/6)}]$$
$$x_f(t) = [2.5e^{j(17\pi t - \pi/4)} + 2.5e^{-j(17\pi t - \pi/4)} + 1.5e^{j(10\pi t - 5\pi/6)} + 1.5e^{-j(10\pi t - 5\pi/6)}]$$

(c) Single-sided spectra are plotted below. Double-sided amplitude spectra are obtained by halving the lines and taking mirror image; phase spectra are obtained by taking antisymmetric mirror image.



Problem 1-13

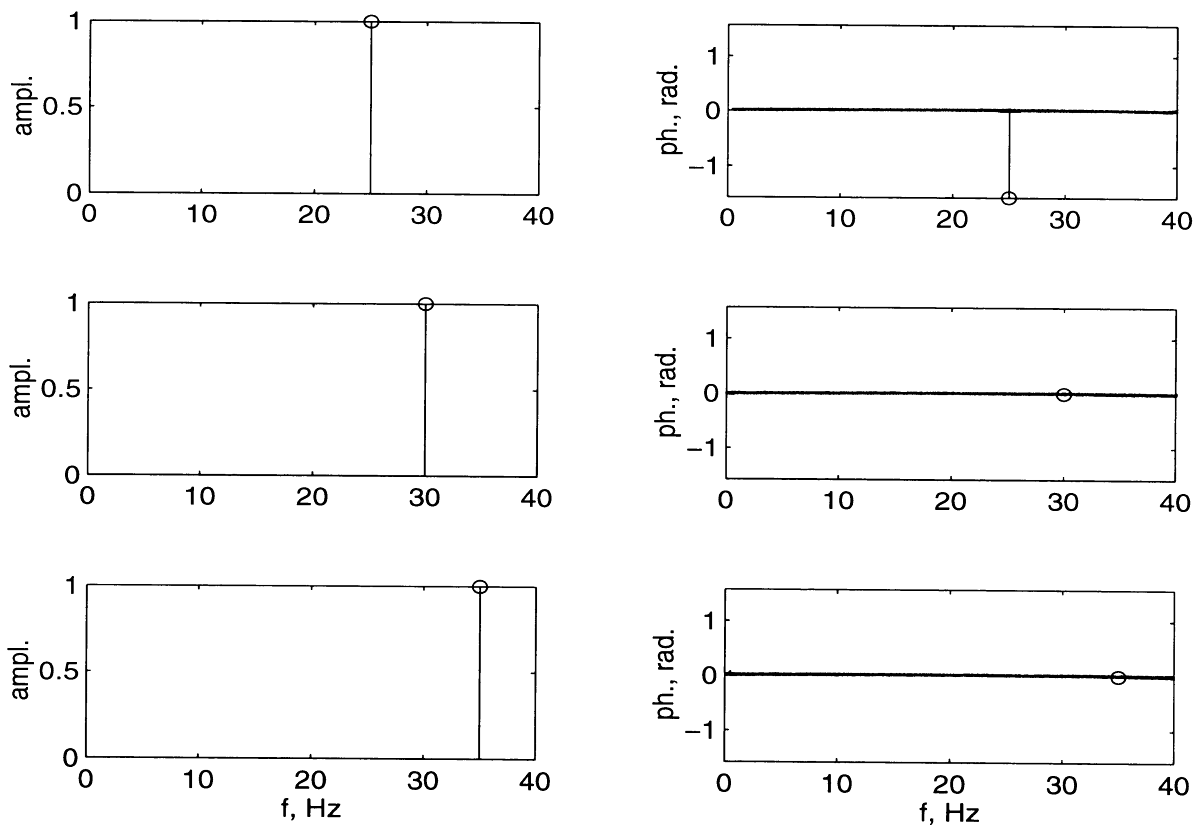
(a) Written as the real part of rotating phasors:

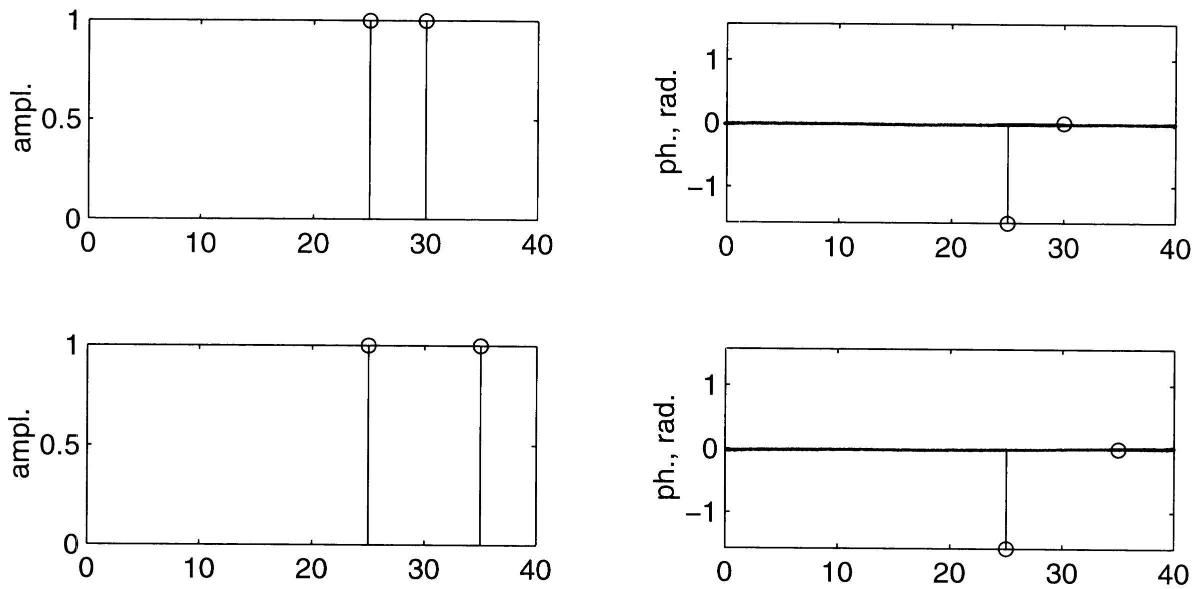
$$x_a(t) = \mathrm{Re}[e^{j(50\pi t - \pi/2)}];\ x_b(t) = \mathrm{Re}[e^{j60\pi t}];\ x_c(t) = \mathrm{Re}[e^{j70\pi t}]$$
$$x_d(t) = \mathrm{Re}[e^{j(50\pi t - \pi/2)} + e^{j60\pi t}];\ x_e(t) = \mathrm{Re}[e^{j(50\pi t - \pi/2)} + e^{j70\pi t}]$$

(b) In terms of counter rotating phasors, the signals are:

$$x_a(t) = \mathrm{Re}[0.5e^{j(50\pi t - \pi/2)} + 0.5e^{-j(50\pi t - \pi/2)}];\ x_b(t) = \mathrm{Re}[0.5e^{j60\pi t} + 0.5e^{-j60\pi t}]$$
$$x_c(t) = \mathrm{Re}[0.5e^{j70\pi t} + 0.5e^{-j70\pi t}]$$
$$x_d(t) = \mathrm{Re}[0.5e^{j(50\pi t - \pi/2)} + 0.5e^{-j(50\pi t - \pi/2)} + 0.5e^{j60\pi t} + 0.5e^{-j60\pi t}]$$
$$x_e(t) = \mathrm{Re}[0.5e^{j(50\pi t - \pi/2)} + 0.5e^{-j(50\pi t - \pi/2)} + 0.5e^{j70\pi t} + \ + 0.5e^{-j70\pi t}]$$

(c) The single-sided amplitude and phase spectra are shown below. See Prob. 1-12c for comments on obtaining double-sided spectra from single-sided spectra.

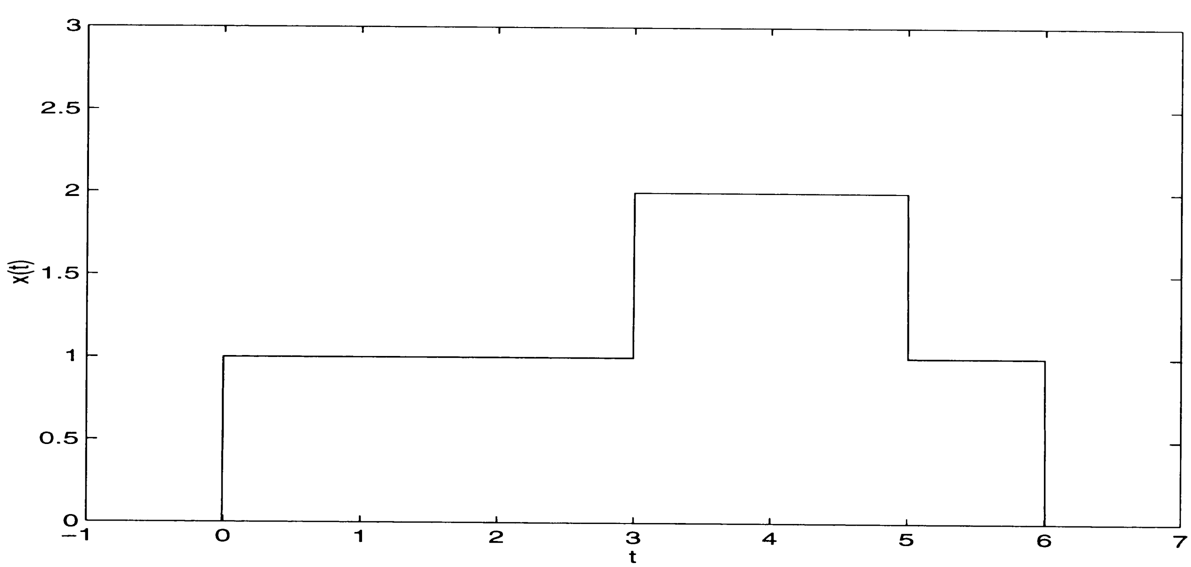




Problem 1-14

(a) A sketch is given below:

From the figure, it is evident that $x(t) = u(t) + u(t - 3) - u(t - 5) - u(t - 6)$.



(b) The derivative of $x(t)$ is $dx(t)/dt = \delta(t) + \delta(t - 3) - \delta(t - 5) - \delta(t - 6)$

Problem 1-15

Note that

$$\sin(\omega_0 t + \theta) = \frac{1}{2j}e^{j\theta}e^{j\omega_0 t} - \frac{1}{2j}e^{-j\theta}e^{-j\omega_0 t}$$
$$= \frac{1}{2}e^{j(\theta - \pi/2)}e^{j\omega_0 t} + \frac{1}{2}e^{-j(\theta - \pi/2)}e^{-j\omega_0 t}$$

Thus we conclude the following:

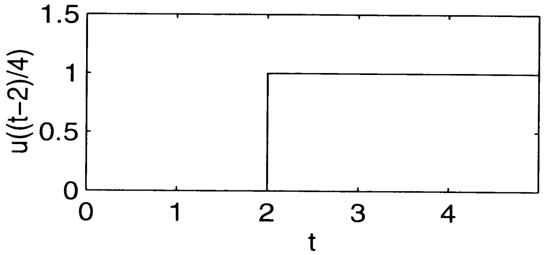
(1) The amplitude spectrum does not change;

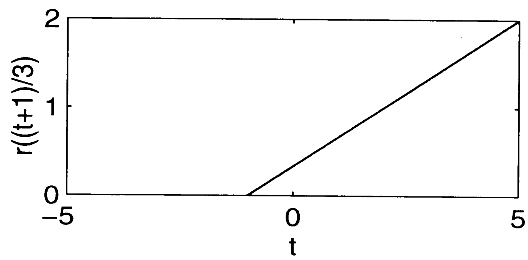
(2) The phase spectrum has a $-\pi/2$ radian phase shift with respect to the cosine-convention phase spectrum. This destroys the odd symmetry present in the phase spectrum using the real part convention.

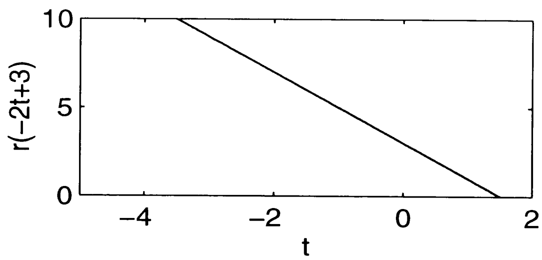
Problem 1-16

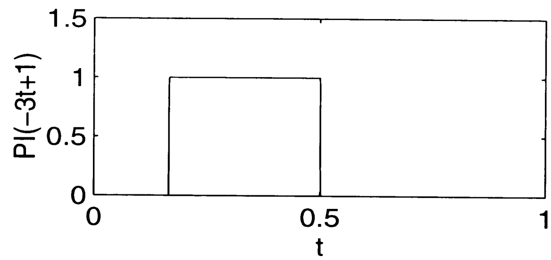
A MATLAB script is provided below to show all the plots:

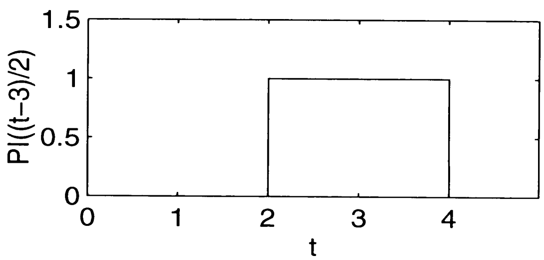
```
%       Plots for Problem 1-16
%
t = -5:.001:5;
xa = stp_fn((t-2)/4);
xb = rmp_fn((t+1)/3);
xc = rmp_fn(-2*t+3);
xd = pls_fn(-3*t+1);
xe = pls_fn((t-3)/2);
subplot(3,2,1),plot(t,xa,'-w'),xlabel('t'),ylabel('u((t-2)/4)'),...
        axis([0 5 0 1.5])
subplot(3,2,2),plot(t,xb,'-w'),xlabel('t'),ylabel('r((t+1)/3)')
subplot(3,2,3),plot(t,xc,'-w'),xlabel('t'),ylabel('r(-2t+3)'),...
        axis([-5 2 0 10])
subplot(3,2,4),plot(t,xd,'-w'),xlabel('t'),ylabel('PI(-3t+1)'),...
        axis([0 1 0 1.5])
subplot(3,2,5),plot(t,xe,'-w'),xlabel('t'),ylabel('PI((t-3)/2)'),...
        axis([0 5 0 1.5])
```











Problem 1-17

From (1-37)

$$u_{-3}(t) = \begin{cases} t^2/2, & t \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Therefore, from (1-35a) with $i = -3$,

$$u_{-4}(t) = \int_{-\infty}^{t} \frac{1}{2}\lambda^2 u(\lambda) d\lambda = \begin{cases} t^3/(2\times 3), & t \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

In general,

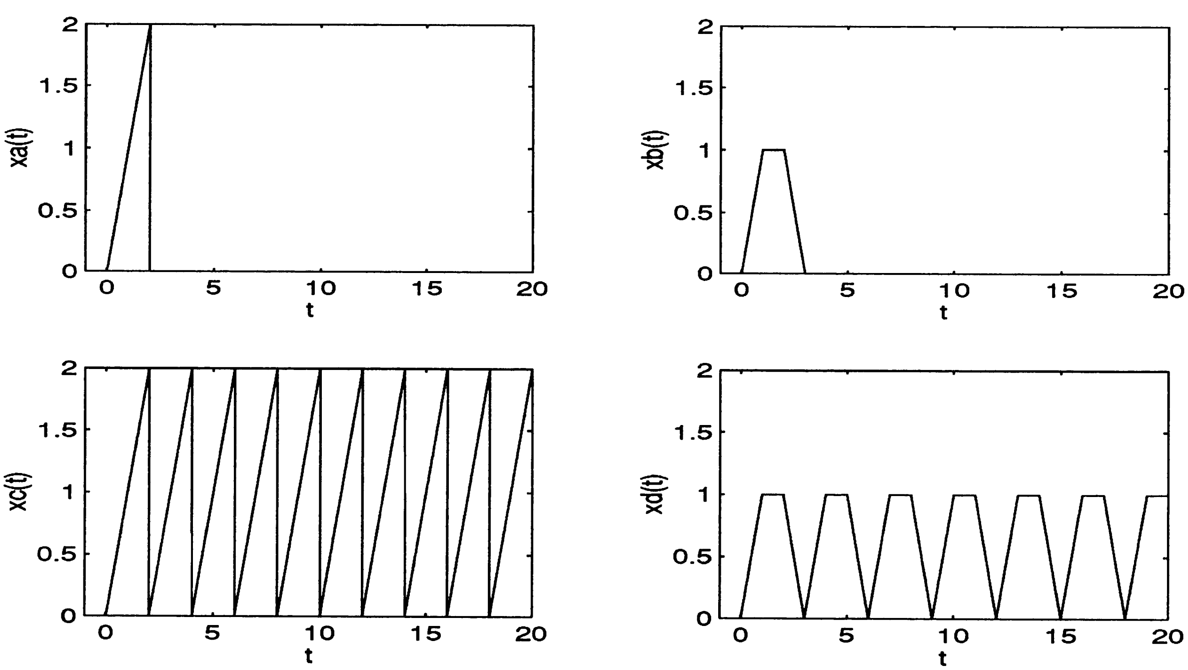
$$u_{-n}(t) = \begin{cases} t^{n-1}/(n-1)!, & t \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Problem 1-18

A MATLAB script is given below for making the plots. Note that use was made of functions to plot the repetitive signals in (c) and (d).

```
%       Plots for Problem 1-18
%
clf
t = -1:.005:20;
xa = xa_fn(t);
xb = xb_fn(t);
xc = xa;
xd = xb;
for n = 1:10
        xc = xc + xa_fn(t - 2*n);
        xd = xd + xb_fn(t - 3*n);
end
subplot(2,2,1),plot(t,xa),xlabel('t'),ylabel('xa(t)'),...
        axis([-1 20 0 2])
subplot(2,2,2),plot(t,xb),xlabel('t'),ylabel('xb(t)'),...
        axis([-1 20 0 2])
subplot(2,2,3),plot(t,xc),xlabel('t'),ylabel('xc(t)'),...
        axis([-1 20 0 2])
subplot(2,2,4),plot(t,xd),xlabel('t'),ylabel('xd(t)'),...
        axis([-1 20 0 2])
```

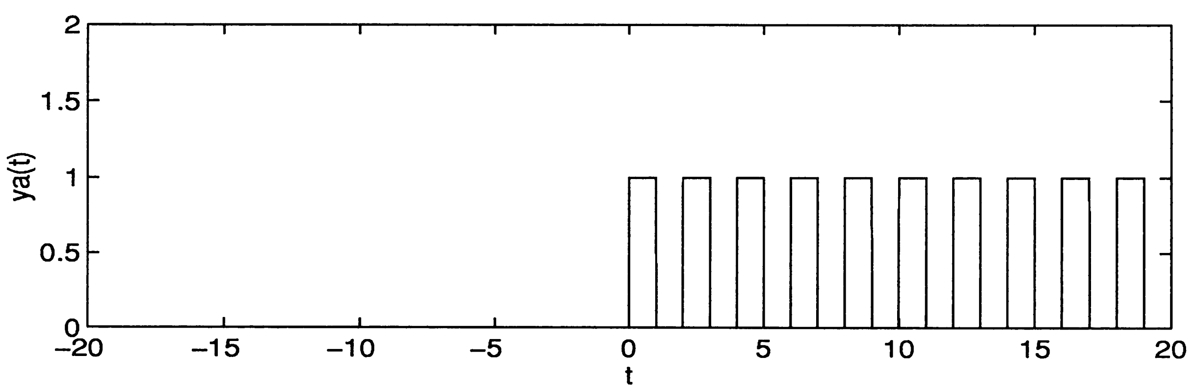
```
%       Function to compute xa for problem 1-18
%

function x = xa_fn(t)
x = rmp_fn(t).*stp_fn(2 - t);
%       Function to compute xb for problem 1-18
%

function x = xb_fn(t)
x = rmp_fn(t)-rmp_fn(t-1)-rmp_fn(t-2)+rmp_fn(t-3);
```

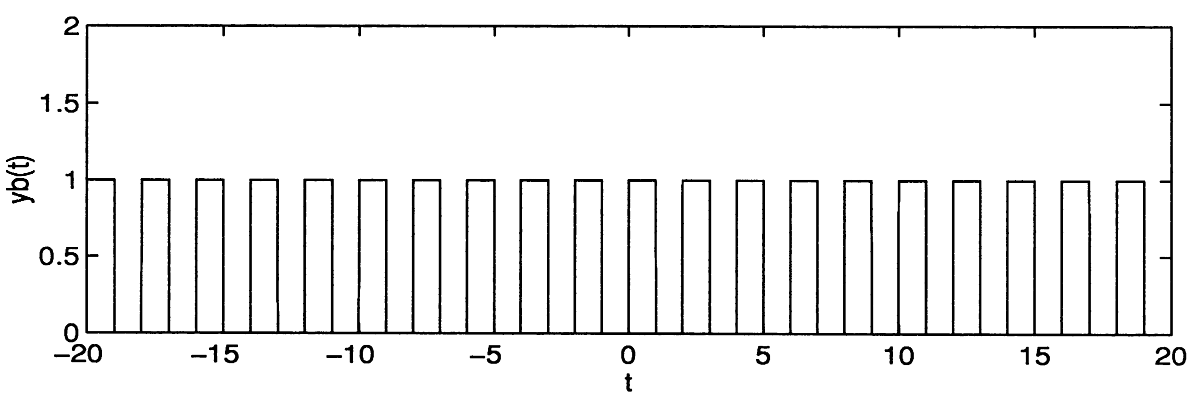


Problem 1-19

(a) Note that for $n = 0$ the summand can be written as $\Pi(t - 1/2)$. The MATLAB script below provides the plots for parts (a) and (c). The signal in part (a) is not periodic because it starts at $t = 0$. The signal of part (c) is periodic because it starts at $t = -\infty$.

```
%	Plots for Problem 1-19
%
clg
t = -20:.005:20;
ya = pls_fn(t - 0.5);
yb = pls_fn(t - 0.5);
for n = 1:10
	ya = ya + pls_fn(t-.5-2*n);
	yb = yb + pls_fn(t-.5-2*n)+ pls_fn(t-.5+2*n);
end
subplot(2,1,1),plot(t, ya, '-w'),xlabel('t'),ylabel('ya(t)'),...
	axis([-20 20 0 2])
subplot(2,1,2),plot(t, yb, '-w'),xlabel('t'),ylabel('ya(t)'),...
	axis([-20 20 0 2])
```





Problem 1-20

Representations for the signals are given below (others may be possible):

$$x_a(t) = \sum_{n=0}^{\infty} r(t - 3n)u(2 - t - 3n)$$
$$x_b(t) = \sum_{n=0}^{\infty} u(t - 4n)u(2 - t - 4n)$$
$$x_c(t) = \sum_{n=0}^{\infty} 2\delta(t - 2.5n)$$
$$x_d(t) = \sum_{n=0}^{\infty} \frac{2}{3}u(t - 3n)r(3 - t - 3n)$$

Problem 1-21

One possible representation for each (these follow from the results of Prob. 1-17) is:

$$x_1(t) = 2u_{-3}(t - 1)u(2 - t) + u(t - 2)u(4 - t) + 2u_{-3}(5 - t)u(t - 4)$$
$$x_2(t) = \frac{3}{2}u_{-4}(t)u(2 - t) + \frac{4}{9}u_{-3}(5 - t)u(t - 2)$$

Problem 1-22

One possible representation for each is

$$x_1(t) = u(t) + r(t - 1) - 2r(t - 2) + r(t - 3) - u(t - 4)$$
$$x_2(t) = u(t) - 2u(t - 1) + 2u(t - 2) - u(t - 3)$$
$$x_3(t) = r(t) - r(t - 1) - r(t - 3) + r(t - 4)$$
$$x_4(t) = r(t) - 2u(t - 1) - r(t - 2)$$

Problem 1-23

(a) First note that the integral of the function is 1, no matter what the value for ϵ:

$$I = \int_{-\infty}^{\infty} \delta_\epsilon(t)dt = \int_0^{\infty} \frac{1}{\epsilon}e^{-t/\epsilon}dt = -e^{-t/\epsilon}\Big|_0^{\infty} = 1$$

Second, note that the pulse becomes infinitely narrow and infinitely high as $\epsilon \to \infty$.

(b) Use the integral

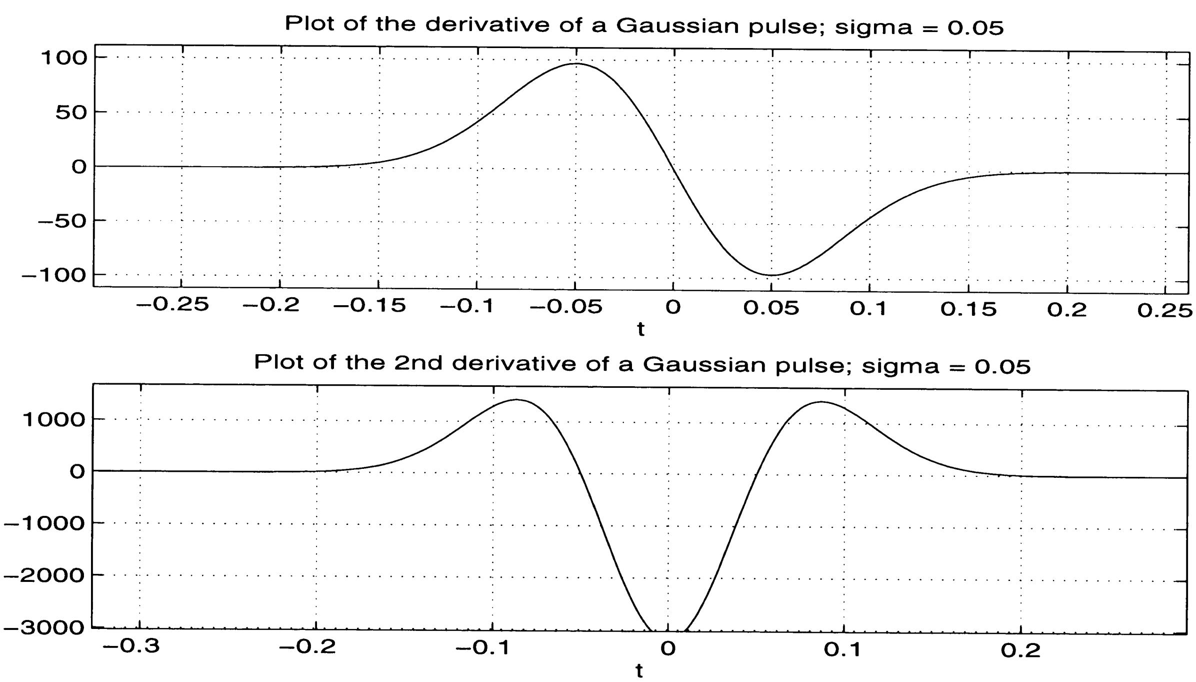
$$\int_{-\infty}^{\infty} e^{-\alpha t^2} dt = \sqrt{\frac{\pi}{\alpha}}$$

to show that the area under the given function is 1. Then note that as $\sigma \to 0$, the function becomes infinitely narrow and infinitely high. Thus the properties of a delta function are satisfied.

Problem 1-24

A MATLAB script using symbolic operations is given below for plotting the desired functions:

```
%       Plots for Problem 1-24
%
format short
sigma = 0.05;
y = 'exp(-t^2/(2*0.05^2))/sqrt(2*pi*0.05^2)';
y_prime = diff(y)
y_dbl_prime = diff(y_prime)
subplot(2,1,1),ezplot(y_prime),...
        title(['Plot of the derivative of a Gaussian pulse; sigma = ',num2str(sigma)])
subplot(2,1,2),ezplot(y_dbl_prime),...
        title(['Plot of the 2nd derivative of a Gaussian pulse; sigma = ',num2str(sigma)])
```



Problem 1-25

Using the stated rules, the first derivative is

$$\frac{dh(t)}{dt} = e^{-\alpha t}\frac{du(t)}{dt} - \alpha e^{-\alpha t}u(t) = 1\times\delta(t) - \alpha e^{-\alpha t}u(t)$$

The second derivative is

$$\frac{dh^2(t)}{dt^2} = \frac{d\delta(t)}{dt} + \alpha^2 e^{-\alpha t}u(t) - \alpha e^{-\alpha t}\frac{du(t)}{dt} = \frac{d\delta(t)}{dt} + \alpha^2 e^{-\alpha t}u(t) - \alpha\,\delta(t)$$

Problem 1-26

(a) The integral is zero because the delta function is outside the range of integration.
(b) The integral evaluates as follows:

$$\int_0^5 \cos(2\pi t)\,\delta(t-2)\,dt = \cos(4\pi) = 1$$

(c) This integral can be evaluated as

$$\int_0^5 \cos(2\pi t)\,\delta(t-0.5)\,dt = \cos(\pi) = -1$$

(d) The value of this integral is 0:

$$\int_{-\infty}^{\infty} (t-2)^2\,\delta(t-2)\,dt = (2-2)^2 = 0$$

(e) This integral evaluates to

$$\int_{-\infty}^{\infty} t^2\,\delta(t-2)\,dt = 2^2 = 4$$

Problem 1-27

(a) Using (1-66), this integral becomes

$$\int_{-\infty}^{\infty} e^{3t}\ddot{\delta}(t-2)\,dt = (-1)^2\frac{d^2}{dt^2}e^{3t}\bigg|_{t=2} = 9e^6$$

(b) Again applying (1-66), we have

$$\int_{0}^{10} \cos(2\pi t)\,\dddot{\delta}(t-0.5)\,dt = (-1)^3\frac{d^3}{dt^3}\cos(2\pi t)\bigg|_{t=0.5} = -(2\pi)^3\sin(2\pi t)\big|_{t=0.5} = 0$$

(c) Using (1-66) we get

$$\int_{-\infty}^{\infty}\left[e^{-3t}+\cos(2\pi t)\right]\dot{\delta}(t)\,dt = (-1)\frac{d}{dt}\left[e^{-3t}+\cos(2\pi t)\right]_{t=0} = -\left[-3e^{-3t}-2\pi\sin(2\pi t)\right]_{t=0} = 3$$

Problem 1-28

Match coefficients of like derivatives of $\delta(t)$ on either side of the given equations:
(a) In this case, we obtain

$$10 = 3 + C_3 \text{ or } C_3 = 7;\; C_1 = 5;\; 2 + C_2 = 6 \text{ or } C_2 = 4$$

(b) The resulting equations are

$$3 + C_1 = 0 \text{ or } C_1 = -3;\; C_2 = 0;\; C_3 = 0;\; C_4 = 0;\; C_5 = 0$$

Problem 1-29

(a) (1) This plots to a triangle 2 units high, centered on $t = 0$, and going from $t = -2$ to 2.
(2) This is a rectangle of unit height starting at $t = 0$ and ending at $t = 10$.
(3) This is a step of height 2 starting at $t = 0$ with an impulse of unit area at $t = 2$ superimposed.
(4) This an impulse of area 2 at $t = 2$.

(b) One possible representation is

$$x(t) = r(t+4) - r(t+2) + u(t) - 3r(t-4) + 3r(t-5)$$

Problem 1-30

A possible representation is

$$x(t) = 2u(t) - u(t-2) + u(t-4) - r(t-6) + r(t-8)$$

Problem 1-31

(a) Using the sifting property of the delta function, we get

$$\int_{-\infty}^{\infty} t^3 \delta(t-3)\,dt = t^3\big|_{t=3} = 27$$

(b) Using (1-66), we get

$$\int_{-\infty}^{\infty} [3t + \cos(2\pi t)]\,\dot{\delta}(t-5)\,dt = (-1)\frac{d}{dt}[3t + \cos(2\pi t)]_{t=5} = -[3 - 2\pi\sin(2\pi t)]_{t=5} = -3$$

(c) From (1-66) we have

$$\int_{-\infty}^{\infty} (1+t^2)\,\dot{\delta}(t-1.5)\,dt = (-1)\frac{d}{dt}(1+t^2)\big|_{t=1.5} = -(2t)\big|_{t=1.5} = -3$$

Problem 1-32

(a) A possible representation is

$$x_a(t) = A[2u(t) - 2u(t-T) + u(t-2T) - u(t-3T)]$$

(b) One representation of this signal is

$$x_b(t) = r(t) - 2r(t-1) + 2r(t-3) - r(t-4)]$$

(c) One way of writing this signal is

$$x_c(t) = r(t-1) - 2r(t-2) + r(t-3) + 0.5[u(t-1.5) - u(t-2.5)]$$

Problem 1-33

(a) This is a decaying exponential starting at $t = 0$. Its energy is

$$E = \int_0^\infty e^{-20t}\,dt = \left.\frac{e^{-20t}}{-20}\right|_0^\infty = \frac{1}{20}\ \text{J}$$

(b) This is a rectangular pulse starting at $t = 0$ and ending at $t = 15$. Its energy is

$$E = \int_{-\infty}^{\infty} [u(t) - u(t-15)]^2\,dt = \int_0^{15} 1^2\,dt = 15\ \text{J}$$

(c) This is a cosine burst starting at $t = 0$ and ending at $t = 2$. It contains 10 cycles. Its energy is calculated as

$$E = \int_{-\infty}^{\infty} \cos^2(10\pi t)\,[u(t)u(2-t)]^2\,dt = \int_0^2 \cos^2(10\pi t)\,dt = \int_0^2 \left[\frac{1}{2} + \frac{1}{2}\cos(10\pi t)\right]dt = 1\ \text{J}$$

(d) This is a triangle going from $t = 0$ and ending at $t = 2$ of unit height. For $0 \le t \le 1$ its equation is just t. The integral of t^2 from 0 to 1 can be doubled to yield the total energy with the result

$$E = 2\int_0^1 t^2\,dt = 2\left.\frac{t^3}{3}\right|_0^t = \frac{2}{3}\ \text{J}$$

Problem 1-34

(a) Note that $x_1(t)$ is symmetrical about $t = 2$. Therefore

$$E_1 = 2\left[\int_0^2 1^2 dt + \int_1^2 t^2 dt\right] = 2\left[t\,|_0^2 + \frac{t^3}{3}\Big|_1^2\right] = \frac{20}{3}\ \text{J}$$

(b) Note that $x_2^2(t) = 1$ for t between 0 and 3, and is 0 otherwise. Therefore

$$E_2 = \int_0^3 1^2 dt = 3\ \text{J}$$

(c) Note that $x_3(t)$ is symmetrical about $t = 2$ which allows the energy to be calculated as

$$E_3 = 2\left[\int_0^1 t^2 dt + \int_1^2 1^2 dt\right] = 2\left[\frac{t^3}{3}\Big|_0^1 + t\,|_1^2\right] = \frac{8}{3}\ \text{J}$$

(d) Note that $x_4^2(t)$ is symmetrical about $t = 1$ which allows the energy of $x_4(t)$ to be calculated as

$$E_4 = 2\int_0^1 t^2 dt = 2\frac{t^3}{3}\Big|_0^1 = \frac{2}{3}\ \text{J}$$

Problem 1-35

Only (a) and (b) are energy signals. For (a)

$$E_a = \int_0^2 t^2 dt = \frac{t^3}{3}\Big|_0^2 = \frac{8}{3}\ \text{J}$$

For (b), we note that it is symmetric about $t = 1.5$. It is a ramp from 0 to 1 and constant from 1 to 1.5, which yields

$$E_b = 2\left[\int_0^1 t^2 dt + \int_1^{1.5} 1^2 dt\right] = 2\left[\frac{t^3}{3}\Big|_0^1 + t\,|_1^{1.5}\right] = 2[1/3 + 1.5 - 1] = 5/3\ \text{J}$$

The other functions are semi-infinite in extent, so their squares will integrate to infinity.

Problem 1-36

The average powers of (a) - (c) are ½ W; for (d) and (e), the powers are 1 W. These are obtained by squaring the amplitudes of the separate frequency components, dividing by 2 to get power, and adding. This is permissible since the sinusoids have frequencies that are integer multiples of a fundamental frequency.

Problem 1-37

(a) $P = 2^2/2 = 2$ W; (b) $P = 5^2/2 = 12.5$ W; (c) $P = 3^2/2 = 4.5$ W; (d) $P = 2^2/2 + 5^2/2 = 14.5$ W; (e) $P = 2^2/2 + 3^2/2 = 6.5$ W; (f) $P = 5^2/2 + 3^2/2 = 17$ W.

Problem 1-38

(a) Power:

$$P_a = \lim_{T \to \infty} \frac{1}{2T}\left[\int_0^1 1^2\,dt + \int_1^2 6^2\,dt + \int_2^T 4^2\,dt\right] = 0 + 0 + \lim_{T \to \infty} \frac{16(T-2)}{2T} = 8 \text{ W}$$

(b) Energy:

$$E_b = \int_0^1 1^2\,dt + \int_1^2 6^2\,dt = 37 \text{ J}$$

(c) Energy:

$$E_c = \int_0^\infty e^{-10t}\,dt = -\frac{e^{-10t}}{10}\Bigg|_0^\infty = \frac{1}{10} \text{ J}$$

(d) Power:

$$P_d = \lim_{T \to \infty} \frac{1}{2T}\int_0^T \left[e^{-5t} + 1\right]^2 dt = \lim_{T \to \infty} \frac{1}{2T}\int_0^T \left[e^{-10t} + 2e^{-5t} + 1\right] dt$$

$$= \lim_{T \to \infty} \frac{1}{2T}\left[-\frac{e^{-10t}}{10} - \frac{2e^{-5t}}{5} + t\right]_0^T = \frac{1}{2} \text{ W}$$

(e) Power: similarly to (d), it can be shown that $P_e = ½$ W. (f) Neither: it can be shown that both the power and energy are infinite. (g) Power: $P_g = ½$ W. (h) Neither: $E_h = \infty$ and $P_h = 0$.

Problem 1-39

(a) Yes. The frequencies of its separate components are commensurable: $f_1 = 3\times1$ Hz and $f_2 = 5\times1$ Hz. Therefore, the fundamental frequency is 1 Hz and the period is 1 s.
(b) Its amplitude spectrum consists of a line of height 2 at 3 Hz and a line of height 4 at 5 Hz. Its phase spectrum consists of a line of height $-\pi/3$ at 3 Hz and a line of height $-\pi/2$ at 5 Hz.
(c) Written as the sum of counter-rotating phasors, the signal is

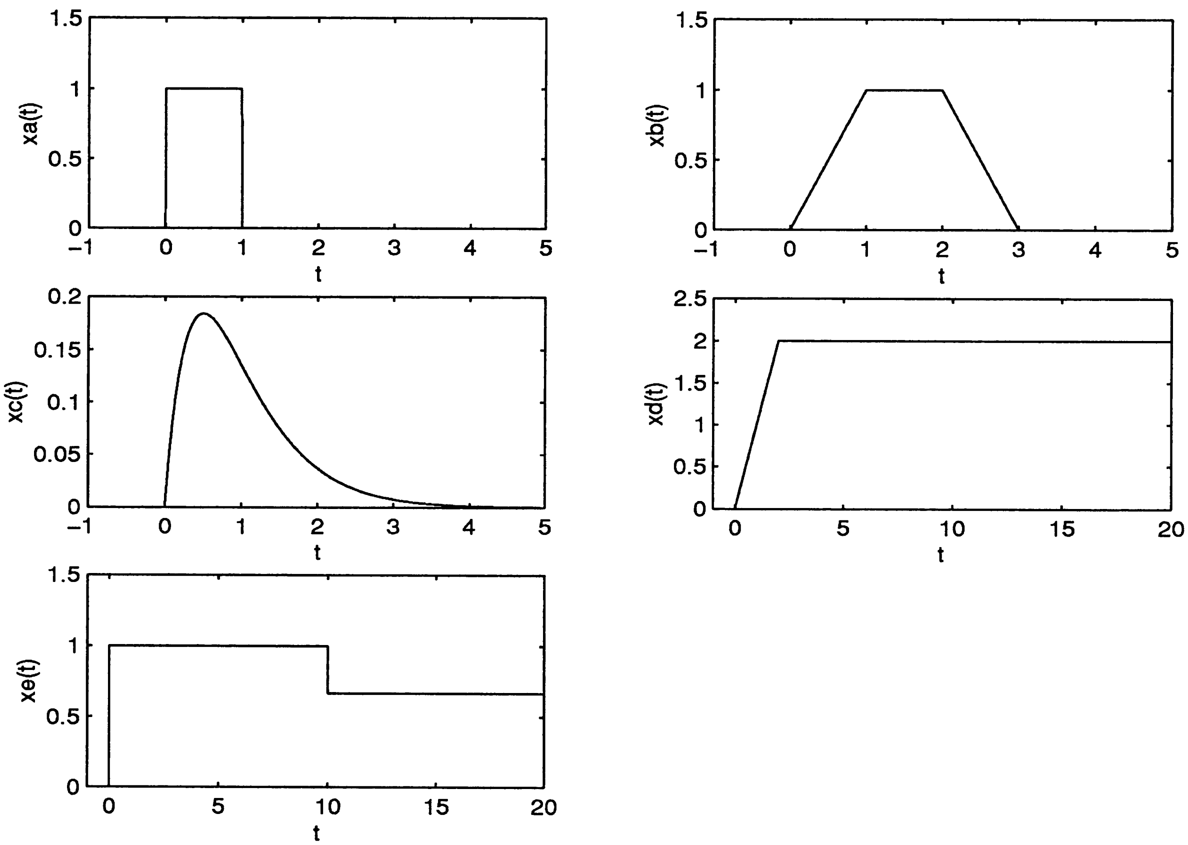
$$x(t) = e^{-j\pi/3}e^{j6\pi t} + e^{j\pi/3}e^{-j6\pi t} + 2e^{-j\pi/2}e^{j10\pi t} + 2e^{j\pi/2}e^{-j10\pi t}$$

(d) See (b): for the amplitude spectrum, halve the lines and take the mirror image about $f = 0$; for the phase spectrum, take the antisymmetric image about $f = 0$.
(e) It is clear that it is a power signal because it is the sum of sinusoids whose frequencies are harmonics of a fundamental frequency. The total power is $2^2/2 + 4^2/2 = 10$ W.

Problem 1-40

(a) through (c) are energy signals; (d) and (e) are power signals. By applying the definitions of energy and power, (1-75) and (1-76), respectively, the energies are $E_a = 1$ J, $E_b = 5/3$ J, $E_c = 1/32$ J, $P_d = 2$ W, and $P_e = 2/9$ W. The MATLAB script given below plots these signals:

```
%       Plots for Problem 1-40
%
t = -1:.005:20;
xa = stp_fn(t) - stp_fn(t-1);
xb = rmp_fn(t) - rmp_fn(t-1) - rmp_fn(t-2) + rmp_fn(t-3);
xc = t.*exp(-2*t).*stp_fn(t);
xd = rmp_fn(t) - rmp_fn(t-2);
xe = stp_fn(t) - (1/3)*stp_fn(t-10);
subplot(3,2,1),plot(t,xa,'-w'),xlabel('t'),ylabel('xa(t)'),...
        axis([-1 5 0 1.5])
subplot(3,2,2),plot(t,xb),xlabel('t'),ylabel('xb(t)'),...
        axis([-1 5 0 1.5])
subplot(3,2,3),plot(t,xc),xlabel('t'),ylabel('xc(t)'),...
        axis([-1 5 0 .2])
subplot(3,2,4),plot(t,xd),xlabel('t'),ylabel('xd(t)'),...
        axis([-1 20 0 2.5])
subplot(3,2,5),plot(t,xe),xlabel('t'),ylabel('xe(t)'),...
        axis([-1 20 0 1.5])
```



Problem 1-41

(a) Only (1) is periodic; $f_1 = 2.5$ Hz $= 0.5m$ and $f_2 = 3$ Hz $= 0.5n$ where the integers m and n are 5 and 6, respectively. The fundamental frequency is 0.5 Hz and the period is 2 s.
(b) Signals (1) and (2) are power signals. Their powers are both 1 W.
(c) Only signal (3) is an energy signal; its energy is 1/20 J. Signal (4) is neither energy nor power.

Problem 1-42

By definition, the average power is

$$P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

For a periodic signal $x(t) = x(t + T_0)$, and the integral can be broken into segments one period long plus the end pieces that are less than a period. Because of periodicity, these integrals are equal with the exception of the end pieces. Thus, we can write the integral as

$$\int_{-T}^{T} |x(t)|^2 dt = 2N \int_{t_0}^{t_0 + T_0} |x(t)|^2 dt + \epsilon_{-N} + \epsilon_N$$

where the latter two terms represent the integrals over the end intervals. Also, $2T = 2NT_0 + \Delta t_1 + \Delta t_2$. The latter two time segments are the lengths of the end intervals which are less than a period. Thus, the power expression becomes

$$P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt = \lim_{T \to \infty} \frac{1}{2T} \left[\frac{2N \int_{t_0}^{t_0 + T_0} |x(t)|^2 dt + \epsilon_{-N} + \epsilon_N}{2NT_0 + \Delta t_1 + \Delta t_2} \right] = \frac{1}{T_0} \int_{t_0}^{t_0 + T_0} |x(t)|^2 dt$$

Problem 1-43

Use the trigonometric identity for $\sin^2 (x) = ½ - ½ \cos(2x)$ to write the signal as

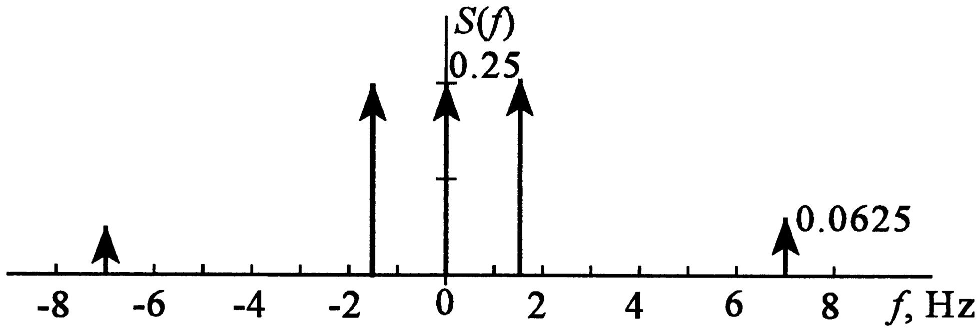
$$x(t) = \frac{1}{2} - \frac{1}{2}\cos(14\pi t - \pi/3) + \cos(3\pi t - \pi/3) = \frac{1}{2} + \frac{1}{2}\cos(14\pi t - \pi/3 + \pi) + \cos(3\pi t - \pi/3)$$

$$= \frac{1}{2} + \frac{1}{2}\cos(14\pi t + 2\pi/3) + \cos(3\pi t - \pi/3)$$

(a) Its single-sided amplitude spectrum consists of a line of height ½ at $f = 0$, a line of height 1 at $f = 1.5$ Hz, and a line of height ½ at $f = 7$ Hz. Its single-sided phase spectrum consists of no line at $f = 0$, a line of height $-\pi/3$ at $f = 1.5$ Hz, and a line of height $2\pi/3$ at $f = 7$ Hz.
(b) To get the double-sided amplitude spectrum, halve the lines in the single-sided spectrum and take its mirror image about $f = 0$. To get the double-sided phase spectrum, take the antisymmetric image of the single-sided spectrum about $f = 0$.

Problem 1-44

The signal has frequency components at 0, 1.5, and 7 Hz of amplitudes ½, 1, and ½, respectively. The power at dc is $(½)^2 = 0.25$ W which is placed at the single frequency $f = 0$ Hz. The power at the other frequencies is split between the positive and corresponding negative frequency. Thus, at $f = 1.5$ Hz we have $(1)^2/4 = 0.25$ W (one 2 in the denominator is from computing power in a sinusoid and the other 2 is from splitting it between positive and negative frequencies) and similarly at $f = -1.5$ Hz. At $f = 7$ Hz, we have a power of $(1/2)^2/4 = 0.0625$ W with a similar power at $f = -7$ Hz. All these are represented by impulses of the appropriate weights, so the plot is as shown below:



Problem 1-45

(a) Following the solution to Problem 1-44, we have spectral components at $f = 10$, 15, and 20 Hz of amplitudes 16, 6, and 4, respectively. The power in these components gets split between positive and negative frequencies. Thus, and $f = 10$ Hz we have a power of $(16)^2/4 = 64$ W with a corresponding power at $f = -10$ Hz. At $f = 15$ Hz we have a power of $(6)^2/4 = 9$ W with a corresponding power at $f = -15$ Hz. Finally, at $f = 20$ Hz we have a power of $(4)^2/4 = 4$ W with a corresponding power at $f = -20$ Hz. Mathematically, this can be expressed as

$$S_x(f) = 64[\delta(f - 10) + \delta(f + 10)] + 9[\delta(f - 15) + \delta(f + 15)] + 4[\delta(f - 20) + \delta(f + 20)]$$

(b) The power contained between 12 and 22 Hz is

$$P[12 \le |f| \le 22 \text{ Hz}] = \int_{-22}^{-12} S_x(f)\,df + \int_{12}^{22} S_x(f)\,df = 2\int_{12}^{22} S_x(f)\,df = 26 \text{ W}$$

CHAPTER 2

Problem 2-1

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Problem 2-2

(a) First order; (b) first order (differentiate once to get rid of the integral on y); (c) zero order; (d) first order; (e) second order.

Problem 2-3

(a), (b), (c), and (e) are fixed; (d) is not because of the time-varying coefficient, t^2.

Problem 2-4

Only (c) and (d) are nonlinear. Superposition will not hold in (e) because of the term +10. As an example to show linearity, consider (d):

$$\frac{dy_1(t)}{dt} + t^2 y_1(t) = \int_{-\infty}^{t} x_1(\lambda)\,d\lambda$$

$$\frac{dy_2(t)}{dt} + t^2 y_2(t) = \int_{-\infty}^{t} x_2(\lambda)\,d\lambda$$

Multiply the first equation by a constant, say a, and the second equation by another constant, say b; add to obtain:

$$\frac{d[ay_1(t) + by_2(t)]}{dt} + t^2[ay_1(t) + by_2(t)] = \int_{-\infty}^{t} [ax_1(\lambda) + bx_2(\lambda)]\,d\lambda$$

This is of the same form as the original equation.

Problem 2-5

Noncausal. Consider $t = 0.25$, which gives $y(0.25) = x(0.5)$; i.e., the output depends on a future value of the input.

Problem 2-6

(a) Nonlinear. The proof is similar to Example 2-4 in the text. (b) Noncausal because of the +2 in the argument of x. Consider $t = 0$; the output at time 0 depends on the value of the input at time 2, or a future value.

Problem 2-7

(a) Linear. Consider the responses to two arbitrary inputs:

$$y_1(t) = x_1(t^2)$$
$$y_2(t) = x_2(t^2)$$

Multiply first by a and the second by b and add to get

$$ay_1(t) + by_2(t) = ax_1(t^2) + bx_2(t^2)$$

That is, for the input $ax_1(t) + bx_2(t)$, we replace t by t^2 to get the new output which is the right-hand side of the above equation.
(b) Time varying. Consider the response to the delayed input:

$$y_a(t) = x(t^2 - \tau)$$

Now consider the delayed output due to the undelayed input:

$$y(t - \tau) = x[(t - \tau)^2]$$

Clearly the two are not the same.
(c) Noncausal. Consider $t = 2$ which gives $y(2) = x(4)$; i.e., the output depends on a future value of the input.
(d) Not zero memory. This follows from (c) where it was found that the output does not depend only on values of the input at the present time only.

Problem 2-8

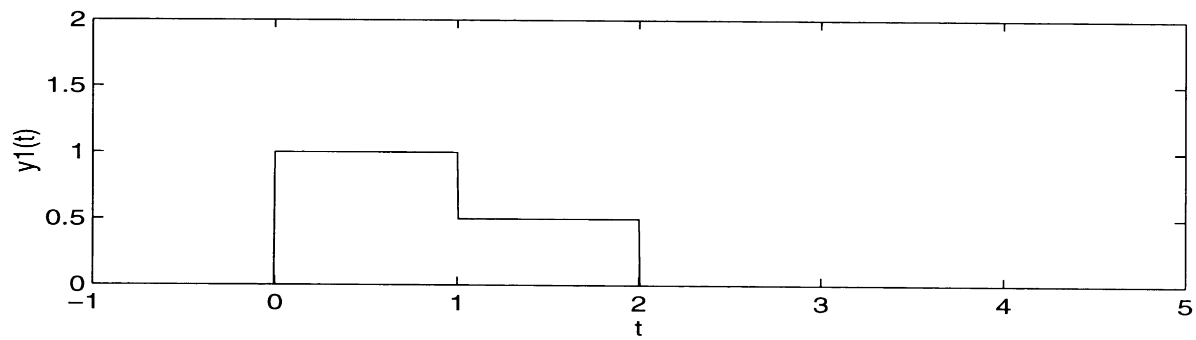
(a) Consider two inputs and the corresponding outputs:

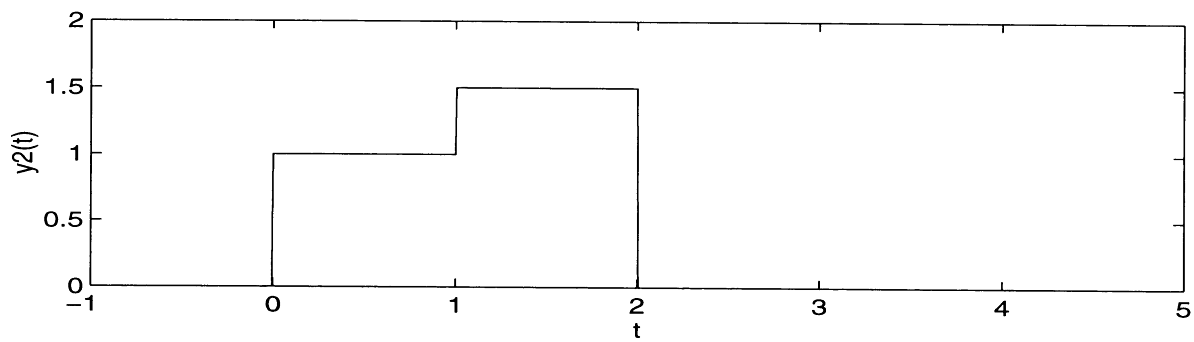
$$y_1(t) = x_1(t) + \alpha x_1(t - \tau_0)$$
$$y_2(t) = x_2(t) + \alpha x_2(t - \tau_0)$$

Multiply the top equation by a and the bottom by b (two constants); add to get

$$ay_1(t) + by_2(t) = ax_1(t) + bx_2(t) + \alpha[ax_2(t - \tau_0) + bx_1(t - \tau_0)]$$

This is of the same form as the original input/output relationship, so linarity is proved.
(b) The only way for the system to be zero memory is for τ_0 to be 0.
(c) It is causal only if $\tau_0 \geq 0$, for in that case the system doesn't respond before the input is applied.
(d) See the MATLAB plots below (α = 0.5 and 1.5 in that order):





Problem 2-9

(a) Consider two different inputs:

$$y_1(t) = \frac{1}{T_1 + T_2} \int_{t-T_1}^{t+T_2} x_1(\lambda)\, d\lambda$$

$$y_2(t) = \frac{1}{T_1 + T_2} \int_{t-T_1}^{t+T_2} x_2(\lambda)\, d\lambda$$

Multiply the first by a and the second by b (two arbitrary constants); add and rearrange to obtain

$$ay_1(t) + by_2(t) = \frac{1}{T_1 + T_2} \int_{t-T_1}^{t+T_2} [ax_1(\lambda) + bx_2(\lambda)]\, d\lambda$$

This is of the same form as the defining equation, so the system is linear.
(b) For causality, the output can't depend on future values of the input. This requires that $T_1 \geq 0$, $T_2 \leq 0$, and $T_1 > -T_2$.

Problem 2-10

Using Kirchoff's voltage equation and Ohm's law, the appropriate equations are

$$x(t) = L\frac{di(t)}{dt} + y(t)$$
$$y(t) = Ri(t)$$
$$\frac{di(t)}{dt} = \frac{1}{R}\frac{dy(t)}{dt}$$

Substitute the last equation in the first and rearrange to obtain

$$\frac{dy(t)}{dt} + \frac{R}{L}y(t) = \frac{R}{L}x(t)$$

(b) The proof is similar to those of Problems 2-8 and 2-9.
(c) Consider

$$\frac{dy(t-\tau)}{dt} = \frac{dy(t')}{dt'}\frac{dt'}{dt} \text{ where } t' = t - \tau$$

Thus

$$\frac{dy(t-\tau)}{dt} + \frac{R}{L}y(t-\tau) = \frac{R}{L}x(t-\tau)$$

which shows that the system is fixed.
(d) Note that the solution to the homogeneous equation is

$$y_H(t) = Ae^{-Rt/L},\ t > 0$$

Assume a complete solution of this form where A is time varying. Substitute into the differential equation of part (a) to obtain

$$A(t) = \int_0^t \frac{R}{L}x(\lambda)e^{R\lambda/L}d\lambda + A_0$$

Since the inductor current is assumed 0 at $t = 0$, this gives $A_0 = 0$, so the solution to the differential equation is

$$y(t) = \int_0^t \frac{R}{L}x(\lambda)\exp\left[-\frac{R}{L}(t-\lambda)\right]d\lambda$$

Problem 2-11

Property	a	b	c	d	e	f
Linear	X		X		X	
Causal	X	X	X	X		
Fixed	X	X				
Dynamic	X	X	X	X	X	X
Order	2	3	2	2	0	2

Problem 2-12

(a) Linear; first order; causal; time invariant. (b) Linear; first order; causal; time varying.
(c) Nonlinear; second order; causal; time invariant. (d) Nonlinear; zero order; causal; time invariant.

Problem 2-13

(a) Write the system equations for two arbitrary inputs:

$$y_1(t) = x_1(t)\cos(100\pi t)$$
$$y_2(t) = x_2(t)\cos(100\pi t)$$

Multiply the first equation by an arbitrary constant, a, and the second by another constant, b. Add to obtain

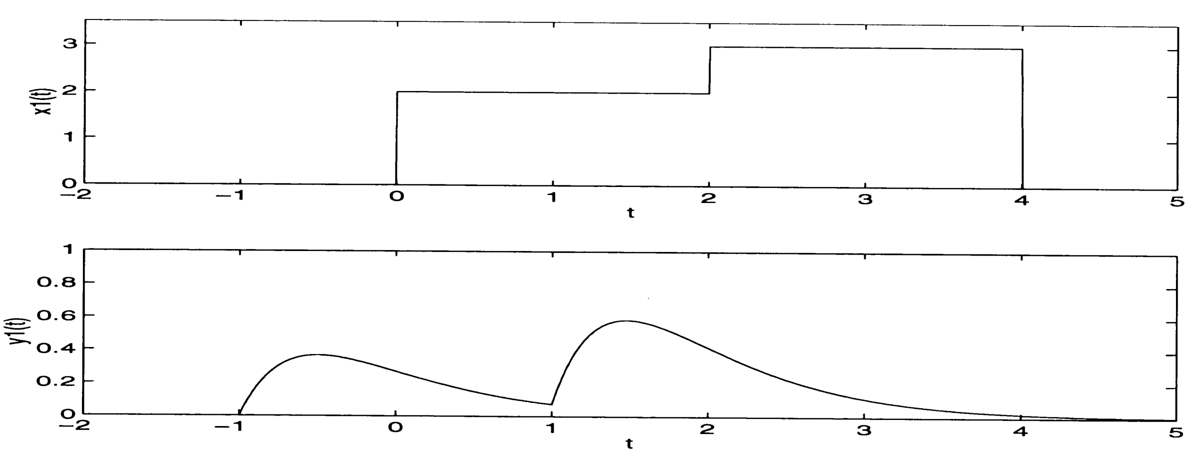
$$ay_1(t) + by_2(t) = [ax_1(t) + bx_2(t)]\cos(100\pi t)$$

This is of the same form as the system equation, so the system is linear. (b) The system is time-varying because of the $\cos(2\pi t)$ multiplying $x(t)$. (c) It is causal because the output does not depend on the future input. (d) It is instantaneous because the output depends only on the input at the present time.

Problem 2-14

(a) The system can't be causal because the output exists before the input is applied.
(b) Since the system is linear and fixed, superposition can be applied to get

$$y_{new}(t) = 2(t + 1)\exp[2(t + 1)]u(t + 1) + 3(t - 1)\exp[2(t - 1)]u(t - 1)$$



Problem 2-15

Given that

$$x_1(t) \rightarrow y_1(t) \text{ and } x_2(t) \rightarrow y_2(t)$$

where the arrow is read "produces the output", then

$$a_1 x_1(t) + a_2 x_2(t) \rightarrow a_1 y_1(t) + a_2 y_2(t)$$

Now let

$$x_2(t) = b_1 x_{21}(t) + b_2 x_{22}(t)$$

By superposition, the response is

$$y_2(t) = b_1 y_{21}(t) + b_2 y_{22}(t)$$

where x_{21} produces the output y_{21}, etc. Thus

$$\begin{aligned} a_1 y_1 + a_2 y_2 &= a_1 y_1 + a_2 [b_1 y_{21} + b_2 y_{22}] \\ &= a_1 y_1 + a_2' y_2' + a_2' y_3' \end{aligned}$$

where

$$a_2' = a_2 b_1;\ a_3' = a_2 b_2;\ y_2' = y_{21};\ y_3' = y_{22}$$

Thus, superposition can be extended to three or more inputs superimposed. Induction can be used to show it for the general case of N inputs.

Problem 2-16

(a) To show that

$$h(t)*[x_1(t) + x_2(t)] = h(t)*x_1(t) + h(t)*x_2(t)$$

The left-hand side can be written as

$$\begin{aligned}\text{LHS} &= \int_{-\infty}^{\infty} h(\lambda)[x_1(t-\lambda) + x_2(t-\lambda)]\,d\lambda \\ &= \int_{-\infty}^{\infty} h(\lambda)x_1(t-\lambda)\,d\lambda + \int_{-\infty}^{\infty} h(\lambda)x_2(t-\lambda)]\,d\lambda \\ &= h(t)*x_1(t) + h(t)*x_2(t)\end{aligned}$$

which proves the relationship.
(b) To show

$$h(t)*[x_1(t)*x_2(t)] = [h(t)*x_1(t)]*x_2(t)$$

The left-hand side can be written as

$$\text{LHS} = \int_{-\infty}^{\infty} h(\lambda)y(t-\lambda)\,d\lambda \text{ where } y(t) = \int_{-\infty}^{\infty} x_1(\eta)x_2(t-\eta)\,d\eta$$

Therefore

$$y(t-\lambda) = \int_{-\infty}^{\infty} x_1(\eta)x_2(t-\lambda-\eta)\,d\eta$$

and the left-hand side can be written as

$$\begin{aligned}\text{LHS} &= \int_{-\infty}^{\infty} h(\lambda)\left\{\int_{-\infty}^{\infty} x_1(\eta)x_2(t-\lambda-\eta)\,d\eta\right\}d\lambda = \int_{-\infty}^{\infty} h(\tau-\eta)\left\{\int_{-\infty}^{\infty} x_1(\eta)x_2(t-\tau)\,d\tau\right\}d\eta \\ &= \int_{-\infty}^{\infty} x_2(t-\tau)\left\{\int_{-\infty}^{\infty} x_1(\eta)h(\tau-\eta)\,d\eta\right\} = [h(t)*x_1(t)]*x_2(t)\end{aligned}$$

which proves the relationship.
(c) Simply note that a constant can be taken outside of an integral.
(d) Let

$$y(t) = \int_{-\infty}^{\infty} h(t - \tau)x(\tau)\,d\tau$$

Look at the integrand components as functions of τ; $x(\tau)$ is 0 for $\tau < c$ and $\tau > d$, and $h(t - \tau)$ is 0 for $t - \tau < a$ and for $t - \tau > b$. There will be no overlap of the two signals in the integrand of a convolution if

$$t - a < c \text{ or } t < a + c$$
$$\text{and for } t - b > d \text{ or } t > b + d$$

That is, the result of the convolution will be zero for either of these two conditons.
(e) Integrate the convolution over all time, interchange the area and convolution integrals, change variables, and the result follows.

Problem 2-17

(a) There are three cases: (1) no overlap; (2) partial overlap; (3) full overlap. For case (1) the result is zero, and this holds for $t < -1$. For partial overlap, we have

$$y(t) = \int_0^{t+1} 2e^{-10\lambda}\,d\lambda = \frac{1}{5}\left[1 - e^{-10(t+1)}\right],\quad -1 \le t \le 1$$

For full overlap, the convolution integral is

$$y(t) = \int_{t-1}^{t+1} 2e^{-10\lambda}\,d\lambda = \frac{1}{5}\left[e^{-10(t-1)} - e^{-10(t+1)}\right],\quad t > 1$$

A sketch of the integrand for these cases will help in establishing the limits of integration.
(b) The convolution integral for this case is

$$y(t) = \int_{-\infty}^{\infty} \Pi\left[\frac{\lambda - 1}{2}\right] u(t - \lambda - 10)\,d\lambda = \int_0^2 u(t - \lambda - 10)\,d\lambda = \begin{cases} \int_0^{t-10} d\lambda = t - 10, & 10 \le t < 12 \\ \int_0^2 d\lambda = 2, & t \ge 12 \end{cases}$$

(c) The convolution integral for this case is

$$y(t) = \int_{-\infty}^{\infty} 2e^{-10\lambda}u(\lambda)u(t-\lambda-2)\,d\lambda = 2\int_{0}^{\infty} e^{-10\lambda}u(t-\lambda-2)\,d\lambda = \frac{1}{5}\left[1-e^{-10(t-2)}\right]u(t-2)$$

(d) The convolution integral for this case is

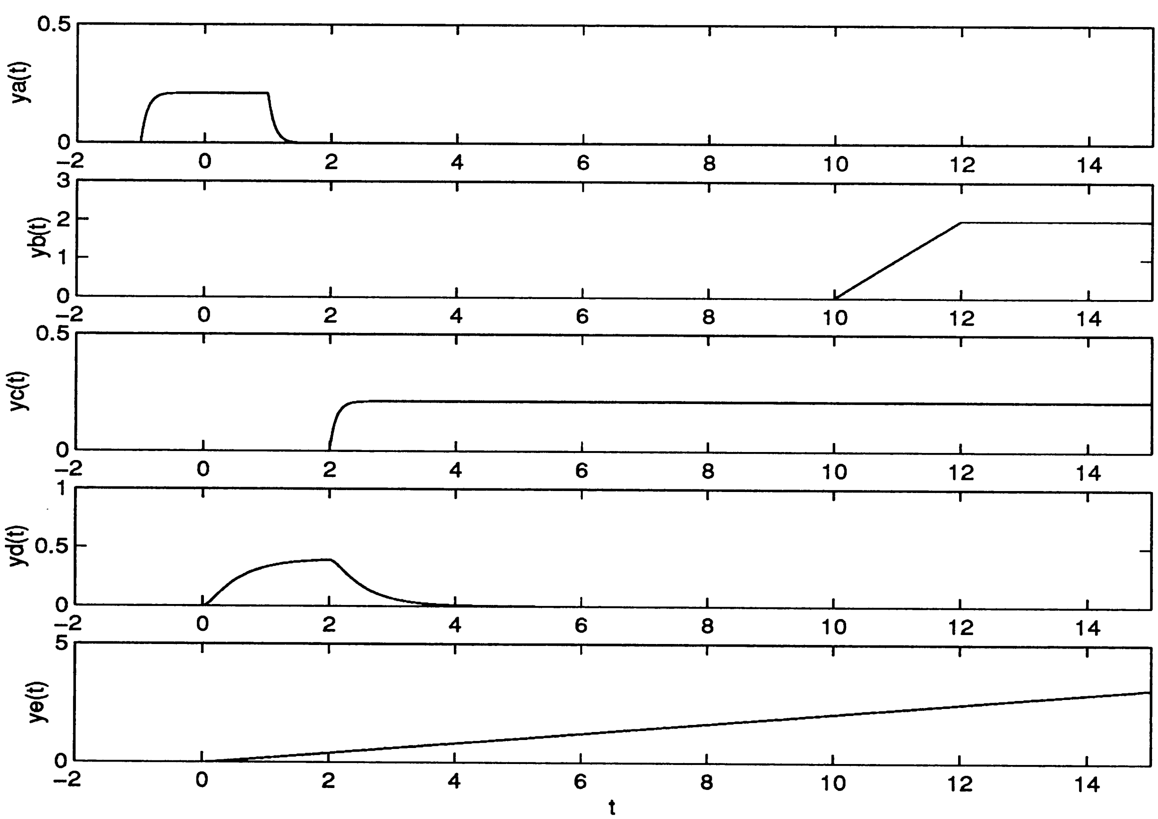
$$y(t) = \int_{-\infty}^{\infty}\left[e^{-2\lambda}-e^{-10\lambda}\right]u(\lambda)\left[u(t-\lambda)-u(t-\lambda-2)\right]d\lambda$$

$$= \left[0.5(1-e^{-2t})-0.1(1-e^{-10t})\right]u(t)-\left[0.5(1-e^{-2(t-2)})-0.1(1-e^{-10(t-2)})\right]u(t-2)$$

which follows by noting that $h(t) = u(t) - u(t-2)$.
(e) In this case, the convolution integral becomes

$$y(t) = \int_{-\infty}^{\infty} 2\lambda e^{-2(t-\lambda)}u(\lambda)u(t-\lambda)\,d\lambda = \left[t-0.5(1-e^{-2t})\right]u(t)$$

Sketches for all three cases are shown below.



Problem 2-18

The convolution is

$$y(t) = \Pi(t-0.5) * \sum_{n=-\infty}^{\infty} \delta(t-2n) = \sum_{n=-\infty}^{\infty} \Pi(t-0.5) * \delta(t-2n) = \sum_{n=-\infty}^{\infty} \Pi(t-0.5-2n)$$

This is a doubly infinite train of square pulses of unit width and height spaced by 2 units. The one at $t = 0$ starts at 0 and ends at 1.

Problem 2-19

(a) By KVL and Ohm's law:

$$\frac{L}{R}\frac{dh(t)}{dt} + h(t) = \delta(t)$$

where the forcing function being a delta function means that the response is the impulse response. For $t < 0$, the impulse response is zero because the input is 0 and the initial conditions are assumed 0. For $t > 0$, we solve the homogeneous differential equation to get the solution

$$h(t) = Ae^{-Rt/L},\ t > 0$$

To find the initial condition required to fix A, integrate the differential equation (with impulse forcing function) through $t = 0$:

$$\int_{0-}^{0+} \frac{L}{R}\frac{dh(t)}{dt}dt + \int_{0-}^{0+} h(t)\,dt = \int_{0-}^{0+} \delta(t)\,dt = 1$$

From the form of $h(t)$, we see that it has a step discontinuity at $t = 0$ and therefore its derivative has an impulse at $t = 0$. Thus the second term on the left-hand side integrates to 0 (it only has a step discontinuity). The first term on the left-hand side integrates to $(L/R)[h(0+) - h(0-)]$. Thus, the above equation becomes $h(0+) = R/L = A$, and the impulse response becomes

$$h(t) = \frac{R}{L}e^{-Rt/L}u(t)$$

(b) Let the voltage across the resistor be $v_R(t)$ and the voltage across the inductor be $v_L(t)$. Thus

$$v_L(t) + v_R(t) = \delta(t)$$

But the voltage across the resistor cannot be proportional to an impulse because then the current around the loop would be proportional to an impulse and this means the inductor voltage (L times the derivative of the current) would be proportional to the derivative of an impulse. Since there is no derivative of an impulse on the right-hand side to balance it, this cannot be the case. Therefore, it must be true at time 0 that $v_L(t) = \delta(t)$ and the current possesses a step at time 0. In particular,

$$i(0+) = \frac{1}{L}\int_{0-}^{0+} \delta(t)\,dt = \frac{1}{L}$$

For $t > 0$, the current around the resistor-inductor loop must satisfy

$$L\frac{di(t)}{dt} + Ri(t) = 0 \text{ or } i(t) = Ae^{-Rt/L},\ t > 0$$

The constant A can be fixed by setting $i(0) = i(0+) = 1/L$. Thus, the same result is obtained for the impulse response as obtained in part (a).

Problem 2-20

Using voltage division with an impulse forcing function, the impulse response is

$$h(t) = \frac{1}{1+1+1}\delta(t) = \frac{1}{3}\delta(t)$$

Problem 2-21

(a) Use KCL at the junction of the three elements after expressing the currents in terms of input and output voltages. The currents through the inductor, top resistor, and output resistor are, respectively, given by

$$i_1(t) = \frac{1}{L}\int_{-\infty}^{t}[x(\lambda) - y(\lambda)]\,d\lambda;\ i_2(t) = \frac{x(t) - y(t)}{R_1};\ i_3(t) = \frac{y(t)}{R_2}$$

Using KCL, we obtain

$$\frac{1}{L}\int_{-\infty}^{t}[x(\lambda) - y(\lambda)]\,d\lambda + \frac{x(t) - y(t)}{R_1} = \frac{y(t)}{R_2}$$

When rearranged, this gives the result given in the statement of the problem.
(b) Let the input be a delta function. Then the output is the impulse response, and it obeys the

differential equation

$$\frac{dh(t)}{dt} + a h(t) = b\frac{d\delta(t)}{dt} + a\delta(t)$$

where

$$a = \frac{R_1 R_2}{(R_1 + R_2)L};\ b = \frac{R_2}{R_1 + R_2}$$

Try a solution of the form

$$h(t) = Ae^{-at}u(t) + B\delta(t)$$

The delta function is necessary so that there is a derivative of a delta function on the left-hand side of the differential equation to match the one on the right-hand side. Substitute the assumed solution into the differential equation and match coefficients of like derivatives of delta functions to obtain

$$A = a(1 - b) = \frac{R_1^2 R_2}{(R_1 + R_2)^2 L};\ B = b = \frac{R_2}{R_1 + R_2}$$

Problem 2-22

Solution 1: Consider an impulse forcing function. KVL must be satisfied around the loop consisting of the impulse source, resistor, and inductor. The inductor must have an impulse of voltage across it since

$$\delta(t) = v_R(t) + v_L(t)$$

Otherwise, if an impulse appears across the resistor, the current is an impulse, and the voltage across the inductor, being proportional to the derivative of the current, would be the derivative of an impulse which isn't present on the forcing function side of the equation. Therefore, at $t = 0$,

$$v_L(t) = L\frac{di(t)}{dt} \text{ which gives } i(0+) = \frac{1}{L}\int_{0-}^{0+}\delta(t)dt = \frac{1}{L}$$

The system differential equation is

$$x(t) = R\,i(t) + L\frac{di(t)}{dt}$$

With $x(t) = \delta(t)$, we get for $t > 0$

$$i(t) = Ae^{-Rt/L}, \; t > 0$$

Using the initial condition for the current found above, we find that $A = 1/L$. Since the current is 0 for $t < 0$, the current around the loop for all time is

$$i(t) = \frac{1}{L}e^{-Rt/L}u(t)$$

The impulse response is the voltage across the inductor (with impulse input). This can be found as

$$h(t) = L\frac{d}{dt}\left[\frac{1}{L}e^{-Rt/L}u(t)\right] = -\frac{R}{L}e^{-Rt/L} + \delta(t)$$

Solution 2: By KVL

$$x(t) = Ri(t) + y(t)$$

But

$$L\frac{di(t)}{dt} = y(t) \text{ or } i(t) = \frac{1}{L}\int_{-\infty}^{t} y(\lambda)\,d\lambda$$

Therefore

$$x(t) = \frac{R}{L}\int_{-\infty}^{t} y(\lambda)\,d\lambda + y(t) \text{ or } \frac{dy(t)}{dt} + \frac{R}{L}y(t) = \frac{dx(t)}{dt}$$

With the input a unit impulse, we have

$$\frac{dh(t)}{dt} + \frac{R}{L}h(t) = \frac{d\delta(t)}{dt}$$

$h(t)$ must have a unit impulse term because there is a derivative of a unit impulse on the right-hand side. Taking the derivative of the impulse response, therefore, gives a term proportional to the derivative of a unit impulse to match the right-hand side. Thus we assume that

$$h(t) = Ae^{-Rt/L}u(t) + B\delta(t)$$

Substitute into the differential equation and match coefficients of like derivatives of impulses on both sides to obtain $B = 1$ and $A = -R/L$.

Problem 2-23

By KVL around the loop,

$$x(t) = R_1 i(t) + \frac{1}{C}\int_{-\infty}^{t} i(\lambda)\,d\lambda + R_2 i(t)$$

But $i(t) = y(t)/R_2$. Substitute this into the integro-differential equation and differentiate once to get

$$\frac{R_1 + R_2}{R_2}\frac{dy(t)}{dt} + \frac{y(t)}{R_2 C} = \frac{dx(t)}{dt}$$

One way to find the impulse response is to find the step response and differentiate it. The solution to the homogeneous equation is

$$a(t) = Ae^{-t/(R_1 + R_2)C},\ t > 0$$

With a step input, the right-hand side of the differential equation is an impulse. To get the required initial condition, we integrate the differential equation through $t = 0$:

$$\frac{R_1 + R_2}{R_2}\int_{0-}^{0+}\frac{da(t)}{dt}dt + \frac{1}{R_2 C}\int_{0-}^{0+} a(t)\,dt = \int_{0-}^{0+}\delta(t)\,dt = 1$$

To match the right-hand side, the integrand of the first term on the left-hand side must contain a unit impulse and, therefore, the second term on the left-hand side is proportional to a unit step. Hence the integral on the second term through $t = 0$ is 0 (a step discontinuity). The first term is a perfect differential. Thus, we obtain $a(0+) = R_2/(R_1 + R_2)$ as the required initial condition, and the step response is

$$a(t) = \frac{R_2}{R_1 + R_2}e^{-t/(R_1 + R_2)C}u(t) \text{ and } h(t) = \frac{da(t)}{dt} = \frac{R_2}{R_1 + R_2}\left[\delta(t) - \frac{1}{(R_1 + R_2)C}e^{-t/(R_1 + R_2)C}u(t)\right]$$

Problem 2-24

(a) From the analysis of this circuit in Example 1-2, we have

$$y(t) = -\frac{1}{RC}\int_{-\infty}^{t} x(\lambda)\,d\lambda$$

If the input is a unit impulse, we obtain

$$h(t) = -\frac{1}{RC}\int_{-\infty}^{t} \delta(\lambda)\,d\lambda = -\frac{1}{RC}u(t)$$

(b) See the first equation above, or evaluate the superposition integral with the impulse response:

$$y(t) = \int_{-\infty}^{\infty} h(t-\lambda)x(\lambda)\,d\lambda = \int_{-\infty}^{\infty}\left[-\frac{1}{RC}u(t-\lambda)\right]x(\lambda)\,d\lambda = -\frac{1}{RC}\int_{-\infty}^{t} x(\lambda)\,d\lambda$$

Problem 2-25

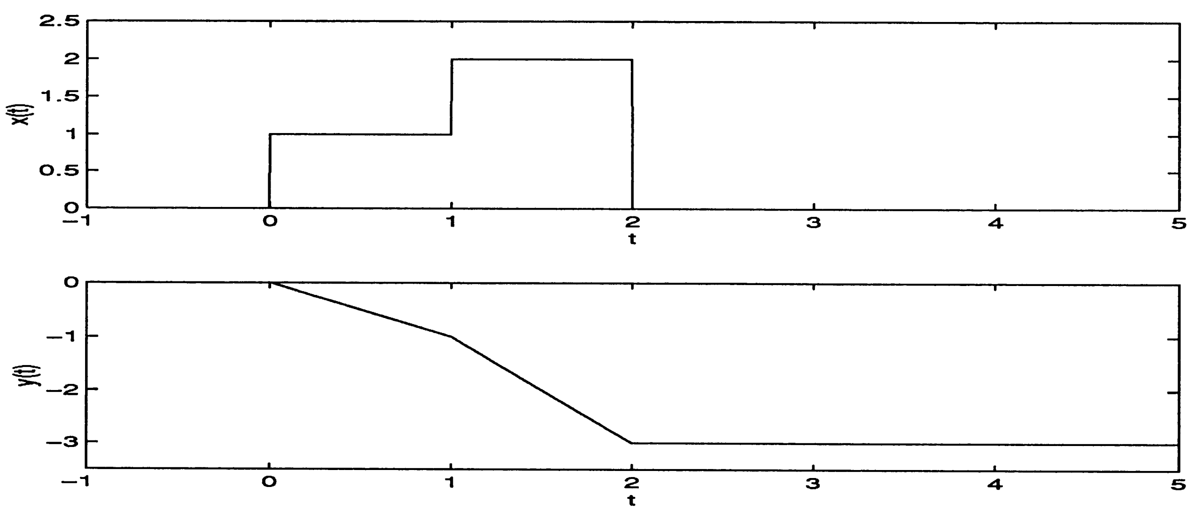
(a) To obtain the step response, integrate the impulse response, which yields

$$a(t) = -r(t)/RC$$

(b) Use superposition to obtain

$$y(t) = -[r(t) + r(t-1) - 2r(t-2)]/RC$$

The input and output are plotted below for $RC = 1$:

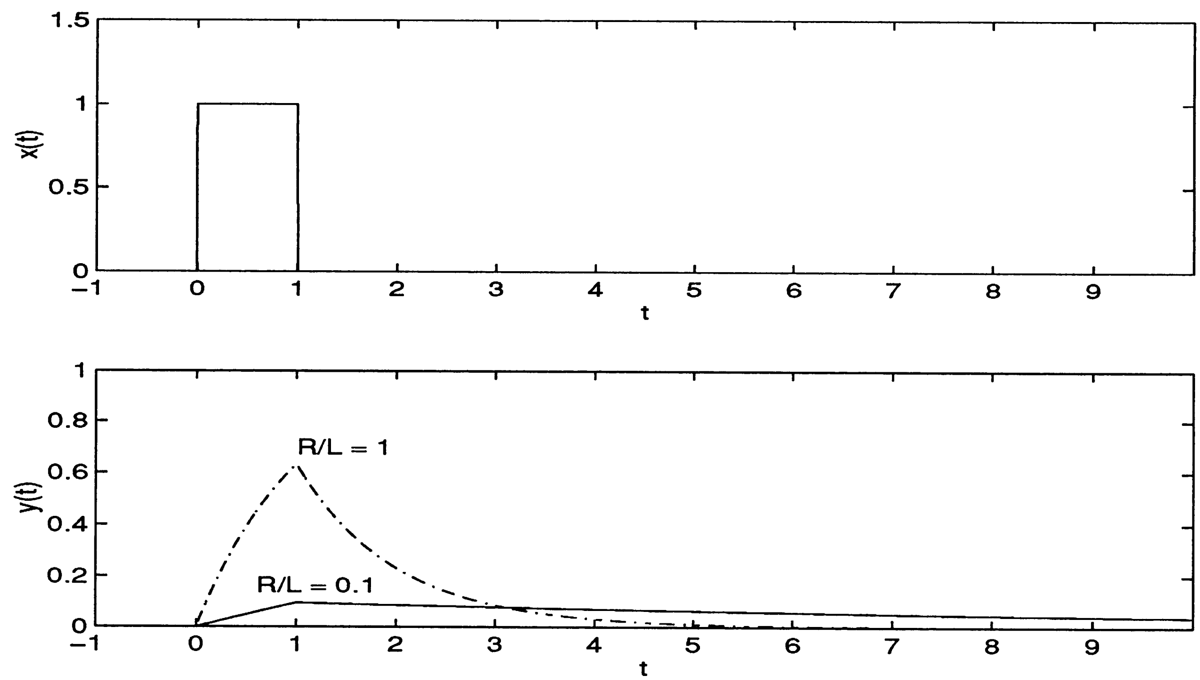


Problem 2-26

We use the impulse response found in Problem 2-19 to evaluate

$$y(t) = \int_{-\infty}^{\infty} \Pi(\lambda - 0.5)\frac{R}{L}e^{-R(t-\lambda)/L}u(t-\lambda)d\lambda = \begin{cases} 0, & t < 0 \\ 1 - e^{-Rt/L}, & 0 < t < 1 \\ \left[1 - e^{-R/L}\right]e^{-R(t-1)/L}, & t > 1 \end{cases}$$

An alternative solution is to write the input as the difference between two steps and use superpositon after finding the step response by integration of the impulse response. A sketch is provided below.



Problem 2-27

From Problem 2-22 and the given input, the superposition integral becomes

$$y(t) = \int_{-\infty}^{\infty} \Pi(\lambda - 0.5)\left[\delta(t-\lambda) - \frac{R}{L}e^{-R(t-\lambda)/L}u(t-\lambda)\right]d\lambda = \Pi(t-0.5) - \Pi(t-0.5) * \left[\frac{R}{L}e^{-Rt/L}u(t)\right]$$

The last term evaluates to the negative of the result given in the solution to Problem 2-26. A sketch is obtained by taking the plot for Problem 2-26, inverting it, and adding square pulse starting at $t = 0$ and ending at $t = 1$.

Problem 2-28

The derivative of the input is

$$\frac{dx(t)}{dt} = \delta(t) - \delta(t - 1)$$

The step response is found by integrating the impulse response found in Problem 2-19:

$$a(t) = \int_{-\infty}^{t} \frac{R}{L} e^{-R\lambda/L} u(\lambda) d\lambda = [1 - \exp(-R\,t/L)]\,u(t)$$

The output is

$$\begin{aligned} y(t) &= [\delta(t) - \delta(t - 1)] * [1 - \exp(-R\,t/L)]\,u(t) \\ &= [1 - \exp(-R\,t/L)]\,u(t) - \{[1 - \exp[-R\,(t - 1)/L]\}\,u(t - 1) \end{aligned}$$

See the solution to Problem 2-26 for a plot.

Problem 2-29

(a) The step response is

$$a_s(t) = \int_{-\infty}^{t} h(\lambda)\,d\lambda = \exp(-R\,t/L)\,u(t)$$

(b) The ramp response is

$$a_r(t) = \int_{-\infty}^{t} a_s(\lambda)\,d\lambda = \frac{L}{R}[1 - \exp(-R\,t/L)]\,u(t)$$

Problem 2-30

Write the input as

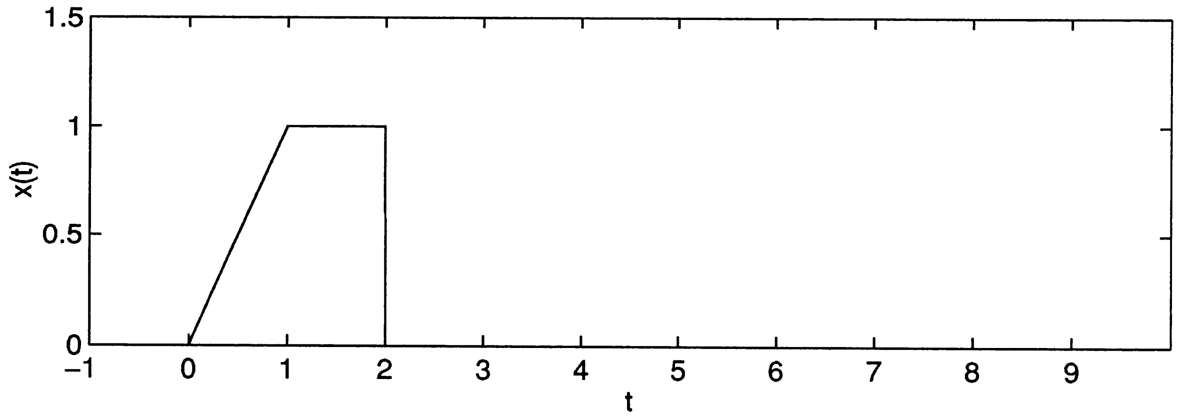
$$x(t) = r(t) - r(t-1) - u(t-2)$$

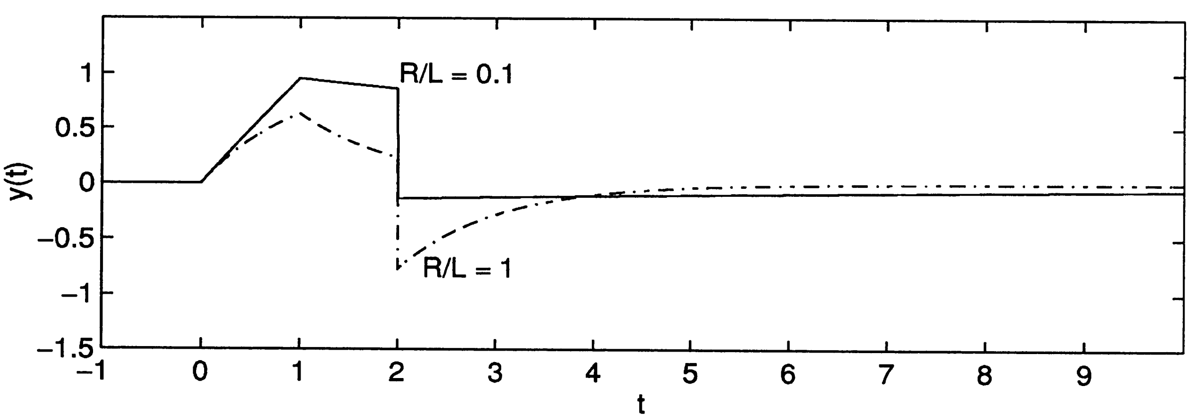
Thus, by superposition, the output is

$$y(t) = a_r(t) - a_r(t-1) - a_s(t-2)$$

$$= \frac{L}{R}[1 - \exp(-Rt/L)]u(t) - \frac{L}{R}\{1 - \exp[-R(t-1)/L)]u(t-1) - \exp[-R(t-2)/L]u(t-2)$$

Sketches of the input and output are given below:





Problem 2-31

(a) The impulse response is

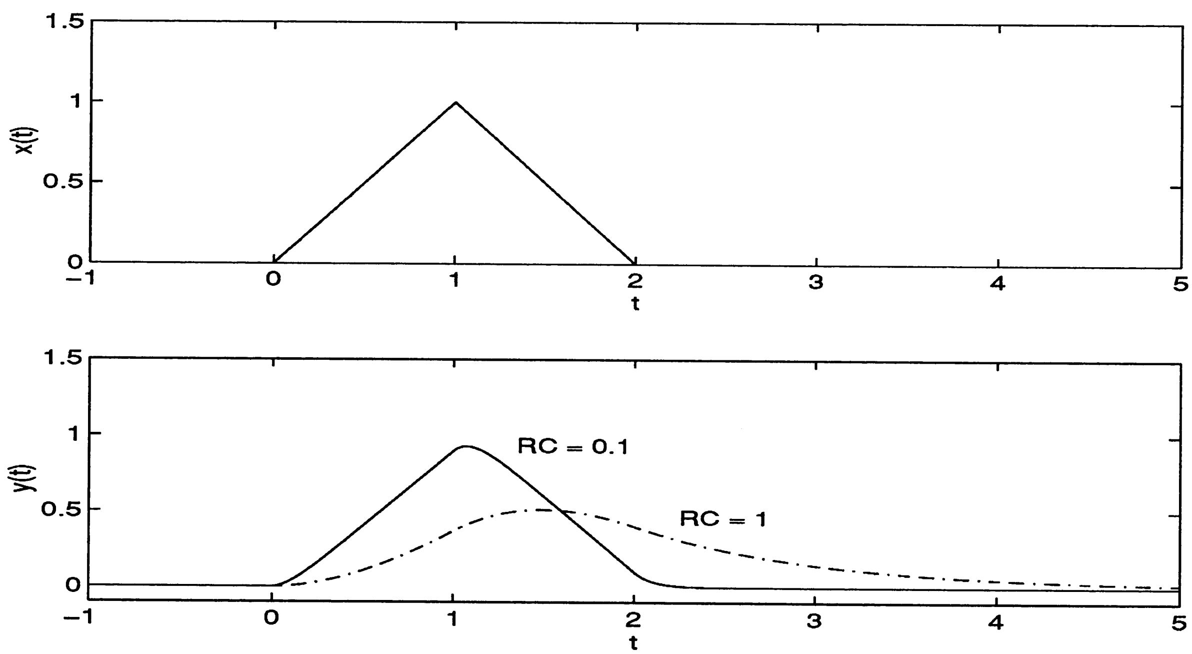
$$h(t) = \frac{1}{RC}\exp(-t/RC)\,u(t)$$

and, by integration, the step and ramp responses are

$$a_s(t) = [1 - \exp(-t/RC)]\,u(t) \text{ and } a_r(t) = r(t) - RC[1 - \exp(-t/RC)]\,u(t)$$

respectively. Using superposition, the output is

$$\begin{aligned} y(t) &= a_r(t) - 2a_r(t-1) + a_r(t-2) \\ &= r(t) - RC[1 - \exp(-t/RC)]\,u(t) \\ &\quad - 2\{r(t-1) - RC[1 - \exp(-(t-1)/RC)]\,u(t-1)\} \\ &\quad + r(t-2) - RC[1 - \exp(-(t-2)/RC)]\,u(t-2) \end{aligned}$$



(b) The response to $dx(t)/dt$ is the derivative of the response given in (a), which is

$$\begin{aligned} y(t) &= a_s(t) - 2a_s(t-1) + a_s(t-2) = [1 - \exp(-t/RC)]\,u(t) \\ &\quad - 2\{[1 - \exp(-(t-1)/RC)]\,u(t-1)\} + [1 - \exp(-(t-2)/RC)]\,u(t-2) \end{aligned}$$

Problem 2-32

(a) Use KCL at the output node to obtain

$$\frac{x(t) - y(t)}{R_1} = \frac{y(t)}{R_2} + C\frac{dy(t)}{dt}$$

When rearranged, this gives the answer given in the problem statement.
(b) The homogeneous equation for the impulse response is

$$R_1 C\frac{dh(t)}{dt} + \left(1 + \frac{R_1}{R_2}\right) y(t) = 0$$

Assume a solution of the form

$$h(t) = Ae^{pt},\ t > 0$$

Substitute the assumed solution into the homogeneous differential equation to get the characteristic equation

$$R_1 Cp + \left(1 + \frac{R_1}{R_2}\right) = 0 \text{ or } p = -\frac{R_1 R_2 C}{R_1 + R_2}$$

To get the required initial condition, integrate the differential equation, with impulse forcing function, through $t = 0$:

$$R_1 C\int_{0-}^{0+} \frac{dh(t)}{dt} dt + \left(1 + \frac{R_1}{R_2}\right)\int_{0-}^{0+} h(t)dt = \int_{0-}^{0+} \delta(t) = 1$$

The second term on the left-hand side is discontinuous at $t = 0$, but contains no impulse function; the first term on the left-hand side must contain an impulse function to balance the impulse function on the right-hand side. The first integral has an integrand that is a perfect differential, so

$$R_1 C[h(0+) - h(0-)] = 1 \text{ or } h(0+) = \frac{1}{R_1 C}$$

Substituting for $h(0+) = A$ and p, we obtain the result for the impulse response given in the problem statement.

(c) Integrate the impulse response to obtain the given answer.
(d) Note that the input can be written as

$$x(t) = u(t) - u(t-1)$$

and use superposition to obtain the given answer.
(e) Duhamel's integral simply tells us to integrate the step response. This gives

$$y_r(t) = \int_0^t \frac{R_2}{R_1 + R_2}[1 - \exp(-\lambda/\tau)]d\lambda, \; t > 0$$

Integrating and putting in the limits gives the result in the problem statement.
(f) Using superposition,

$$y(t) = y_r(t) - 2y_r(t-1) + y_r(t-2)$$

Problem 2-33

(a) The frequency response function is given in terms of the impulse response by

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt$$

From the impulse response given in Problem 2-29, this gives

$$H(j\omega) = \int_{-\infty}^{\infty}\left[\delta(t) - \frac{R}{L}\exp(-Rt/L)\,u(t)\right]\exp(-j\omega t)\,dt = \frac{j\omega}{R/L + j\omega}$$

(b) In terms of $f = \omega/2\pi$, the frequency response function is

$$H(f) = \frac{jf/f_3}{1 + jf/f_3} \text{ where } f_3 = \frac{R}{2\pi L}$$

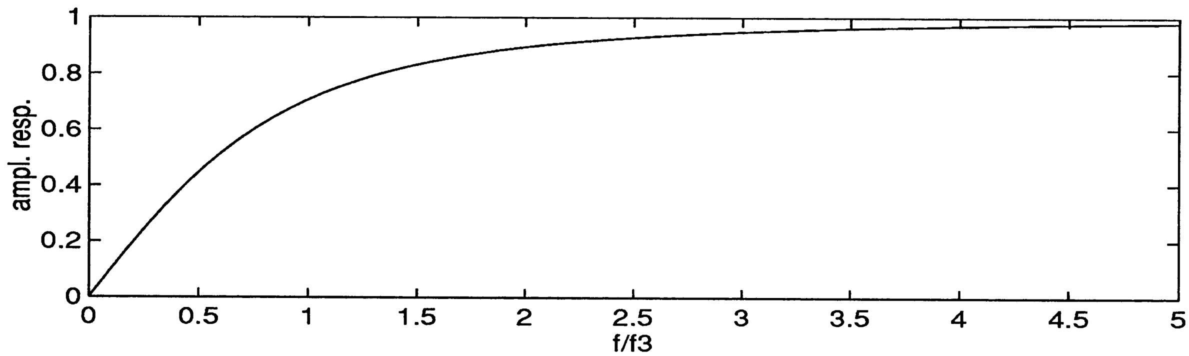
Taking the magnitude, the amplitude response function is

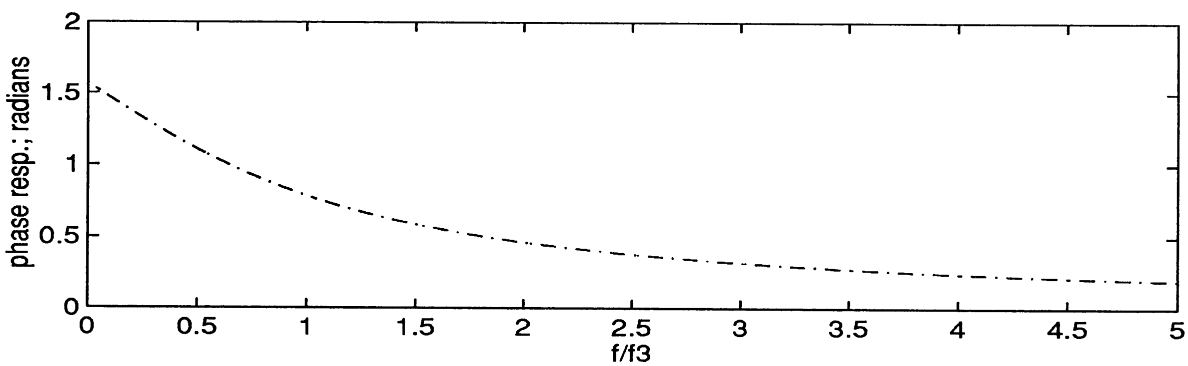
$$A(f) = \frac{|f|/f_3}{\sqrt{1 + (f/f_3)^2}}$$

(c) The phase response function is the argument of the frequency response function. It is given by

$$\theta(f) = \frac{\pi}{2} - \tan^{-1}(f/f_3)$$

Plots of the amplitude and phase response functions are given below:





Problem 2-34

(a) The frequency response is

$$H(j\omega) = \int_0^\infty \frac{1}{R_1 C} e^{-(1/\tau + j\omega)t} dt = \frac{1}{R_1 C} \frac{1}{\sqrt{1/\tau + j\omega}}, \quad \tau = \frac{R_1 R_2 C}{R_1 + R_2}$$

(b) Let $f = \omega/(2\pi)$ and $f_3 = 1/(2\pi\tau)$. The amplitude response function is

$$A(f) = \frac{\tau}{R_1 C} \frac{1}{1 + (f/f_3)^2}$$

and the phase response function is

$$\theta(f) = -\tan^{-1}(f/f_3)$$

(c) The steadystate response is

$$y(t) = A(1)\cos[2\pi t + \theta(1)] + A(2.5)\cos[5\pi t + \theta(2.5)]$$

Using the given parameter values, we obtain $\tau = 0.5$, $R_1C = 1$, and $f_3 = 1/\pi$. The amplitude and phase responses are

$$A(1) = \frac{0.5}{\sqrt{1 + \pi^2}} = 0.1517; \; \theta(1) = -\tan^{-1}(\pi) = -72.35°$$

$$A(2.5) = \frac{0.5}{\sqrt{1 + (2.5\pi)^2}} = 0.0632; \; \theta(2.5) = -\tan^{-1}(2.5\pi) = -87.7°$$

Thus, the steadystate output is

$$y(t) = 0.1517\cos[2\pi t - 72.35°] + 0.0632\cos[5\pi t - 87.7°]$$

Problem 2-35

From Problem 2-19 and the condition for BIBO stability, we obtain

$$\int_{-\infty}^{\infty} |h(t)|\, dt = \int_0^{\infty} \frac{R}{L} e^{-Rt/L} dt = 1 < \infty$$

so the system is BIBO stable.

Problem 2-36

From Problem 2-22 and the condition for BIBO stability, we have

$$\int_{-\infty}^{\infty} |h(t)|\, dt = \int_{-\infty}^{\infty} \left|\delta(t) - \frac{R}{L} e^{-Rt/L} u(t)\right| dt \le \int_{-\infty}^{\infty} \delta(t) dt + \frac{R}{L}\int_0^{\infty} e^{-Rt/L} dt = 1 + 1 = 2 < \infty$$

Therefore, the system is BIBO stable.

Problem 2-37

The condition for BIBO stability is

$$\int_{-\infty}^{\infty}\left|-\frac{1}{RC}u(t)\right|dt = \frac{1}{RC}r(t)\Big|_{-\infty}^{\infty} \rightarrow \infty$$

so the system is not BIBO stable.

Problem 2-38

The condition for BIBO stability is

$$\int_{-\infty}^{\infty}\left|\frac{1}{3}\delta(t)\right|dt = \frac{1}{3} < \infty$$

so the system is BIBO stable.

Problem 2-39

The impulse response, for time greater than 0, satisfies

$$\frac{d^2h(t)}{dt^2} + \omega_0^2 h(t) = 0$$

Assume a solution of the form $A \exp(pt)$, substitute, and solve the resulting characteristic equation to get the roots

$$p_1 = j\omega_0 \text{ and } p_2 = -j\omega_0$$

Thus, for $t > 0$, the solution is

$$h(t) = A_1 e^{j\omega_0 t} + A_1 e^{-j\omega_0 t}$$

We can obtain the initial conditions by integrating the differential equation through 0. They are

$$\left.\frac{dh(t)}{dt}\right|_{t=0+} = 1 \text{ and } h(0+) = 0$$

This gives the following impulse response:

$$h(t) = \frac{1}{\omega_0}\sin(\omega_0 t)\,u(t)$$

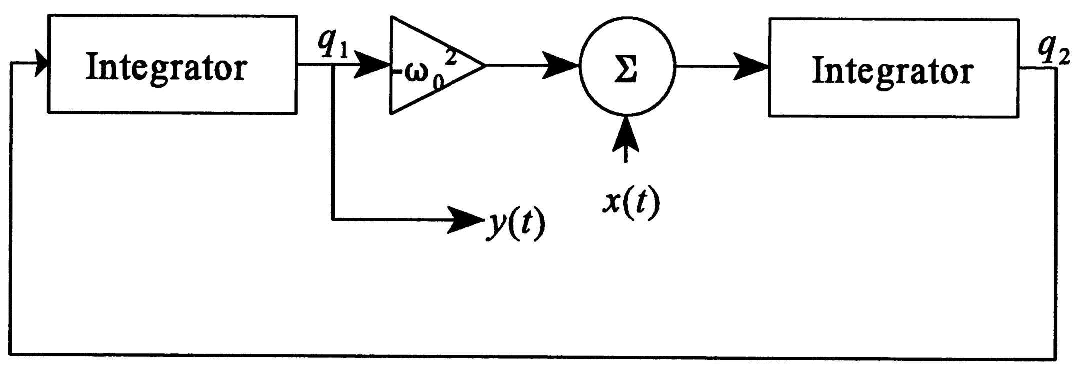
(b) Take the given equations and eliminate $q_2(t)$. Differentiate the first equation to get

$$\frac{d^2 q_1(t)}{dt^2} = \frac{dq_2(t)}{dt}$$

Substitue into the second equation to get

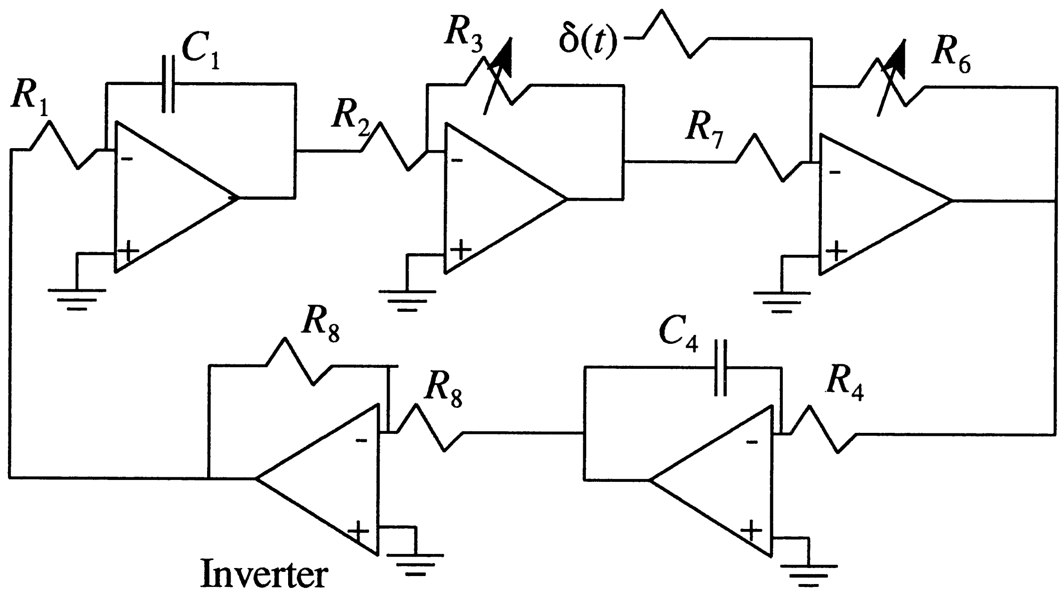
$$\frac{dq_2(t)}{dt} = -\omega_0^2 q_1(t) + x(t)$$

Let $q_1(t) = y(t)$ and the original equation results.
(c) A block diagram is given below:



(d) Possible values for the parameters given in the block diagram on the next page are

$$\frac{1}{R_1 C_1} = \frac{1}{R_4 C_4} = 1 \text{ and } 2\pi\times 10^2 \le \frac{R_3}{R_2}, \frac{R_6}{R_7} \le 2\pi\times 10^4$$



Problem 2-40

Differentiate the first equation to get

$$\frac{d^2q_1}{dt^2} = -\frac{1}{RC}\frac{dq_1}{dt} - \frac{dq_2}{dt} + \frac{1}{C}\frac{dx}{dt}$$

Substitute from the second equation to get

$$\frac{d^2q_1}{dt^2} = -\frac{1}{RC}\frac{dq_1}{dt} - \frac{1}{LC}q_1 + \frac{1}{C}\frac{dx}{dt}$$

Replace q_1 with v and rearrange to get (2-123).

Problem 2-41

(a) Consider the equation

$$(M_0 - kt)\frac{dh(t,\tau)}{dt} + (\alpha - k)h(t,\tau) = \delta(t-\tau)$$

where we will use initial conditions appropriate for the impulse response. To get the initial condition at $t = \tau$ we integrate the differential equation from $t = \tau^-$ to $t = \tau^+$. The first term on the left-hand side must be proportional to an impulse at $t = \tau$; the second term may have a jump discontinuity, but its integral through $t = \tau$ is 0. Thus, we have

$$(M_0 - k\tau)[h(\tau+,\tau) - h(\tau-,\tau)] + 0 = 1$$

By definition of the impulse response for a time-varying system, $h(\tau^-, \tau) = 0$. Thus the initial condition for the impulse response is

$$h(t = \tau+, \tau) = \frac{1}{M_0 - k\tau}$$

Now set the right-hand side of the differential equation for the impulse response to 0, separate variables, and integrate from $\lambda = \tau^+$ to $\lambda = t$ and on h from the above initial condition to $h(t, \tau)$:

$$\ln[(M_0 - k\tau)h(t,\tau)] = \ln\left[\frac{M_0 - kt}{M_0 - k\tau}\right]^{\frac{\alpha - k}{k}}, \quad t > \tau > 0$$

Exponentiating both sides and dividing through by $M_0 - k\tau$, we obtain

$$h(t,\tau) = \frac{(M_0 - kt)^{\frac{\alpha - k}{k}}}{(M_0 - k\tau)^{\frac{\alpha}{k}}}, \quad t > \tau > 0$$

(b) Since the step is applied at time zero, we evaluate the convolution integral from $\lambda = 0$ to t to get the step response. After substituting the given parameter values, the result is

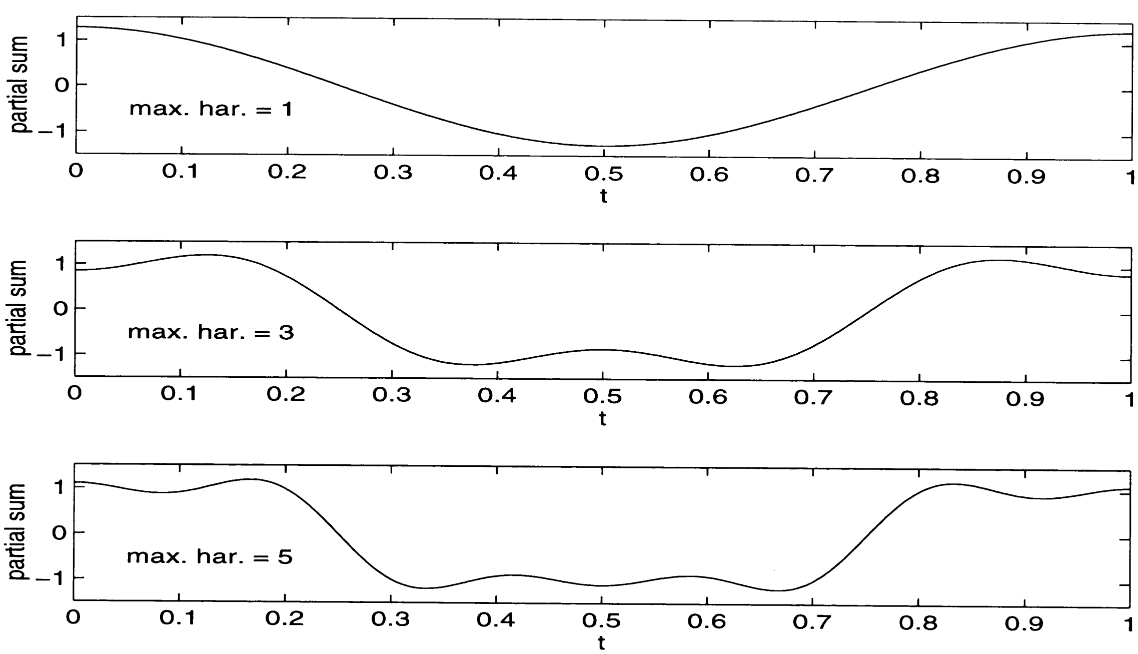
$$a_s(t,0) = \int_0^t h(t,\lambda)d\lambda = \int_0^t \frac{10-t}{10-\lambda}d\lambda = \left.\frac{10-t}{10-\lambda}\right|_0^t = \frac{t}{10}, \quad t > 0$$

CHAPTER 3

Problem 3-1

A MATLAB program for computing the partial sums is given below with plots of the partial sums through harmonics 1, 3, and 5 following:

```
%       Program to give partial Fourier sums of an
%       even square wave of unit amplitude
%
n_max = input('Enter vector of highest harmonic values desired (odd) ');
N = length(n_max);
t = 0:.002:1;
omega_0 = 2*pi;
for k = 1:N
        n = [];
        nn = [];
        n = [1:2:n_max(k)]
        nn = [1:(n_max(k)+1)/2]
        a_n = (-1).^(nn+1)*4./(pi*n);          % Form vector of Fourier cosine-coefficients
        x = a_n*cos(omega_0*n'*t);  % Rows of cosine matrix are versus time; columns versus n,
                                    % so matrix multiply sums over n
        subplot(N,1,k),plot(t, x,'-w'), xlabel('t'), ylabel('partial sum'),...
        axis([0 1 -1.5 1.5]), text(.05,-.5, ['max. har. = ', num2str(n_max(k))])
end
```



Problem 3-2

To show that

$$I_2 = \int_0^{T_0} \cos(m\omega_0 t)\cos(n\omega_0 t)\,dt = \begin{cases} 0, & m \neq n \\ T_0/2, & m = n \end{cases}$$

Use the trigonometric identity

$$\cos(u)\cos(v) = \frac{1}{2}\cos(u+v) + \frac{1}{2}\cos(u-v)$$

Then I_2 becomes

$$I_2 = \frac{1}{2}\int_0^{T_0} \cos[(m-n)\omega_0 t]\,dt + \frac{1}{2}\int_0^{T_0} \cos[(m+n)\omega_0 t]\,dt$$

If $m \neq n$, $I_2 = 0$ because both integrals are over an integer number of cycles of cosine waveforms. If $m = n$, the second integral is still 0, but the first integral evaluates to T_0 (the argument of the cosine is 0 making the integrand 1). Thus the desired result follows.

For (3-13), use the identity $\sin(m\omega_0 t)\cos(n\omega_0 t) = ½ \sin[(m+n)\omega_0 t] + ½ \sin[(m-n)\omega_0 t]$. Now both integrals will be 0 no matter if $m \neq n$ or if $m = n$ because $\sin(0) = 0$.

In deriving (3-16) the steps are identical to those used in deriving (3-15) except that the identities (3-11) and (3-13) are used.

Problem 3-3

(a) Using a trigonometric identity (see Appendix F), this may be expanded as

$$x_1(t) = \frac{1}{2} + \frac{1}{2}\cos(2\omega_0 t)$$

The right-hand side is the trigonometric Fourier series for this signal.

(b) Using Euler's theorem, we have

$$x_2(t) = \cos(200\pi t) + j\sin(200\pi t)$$

which is the trogonometric Fourier series for this signal.

(c) Using appropriate trigonometric identities

$$x_3(t) = \sin(2\pi t)\left[\frac{1}{2} + \frac{1}{2}\cos(20\pi t)\right] = \frac{1}{2}\sin(2\pi t) + \frac{1}{4}\sin(18\pi t) + \frac{1}{4}\sin(22\pi t)$$

The right-hand side is the trigonometric Fourier series for this signal.

(d) Using appropriate trigonometric identities

$$\begin{aligned} x_4(t) &= \cos(20\pi t)\left[\frac{1}{2} + \frac{1}{2}\cos(40\pi t)\right]^2 \\ &= \frac{1}{4}\cos(20\pi t)\left[1 + 2\cos(20\pi t) + \cos^2(20\pi t)\right] \\ &= \frac{1}{4}\cos(20\pi t)\left[1 + 2\cos(20\pi t) + \frac{1}{2} + \frac{1}{2}\cos(40\pi t)\right] \\ &= \frac{5}{8}\cos(20\pi t) + \frac{5}{16}\cos(60\pi t) + \frac{1}{16}\cos(100\pi t) \end{aligned}$$

The right-hand side is the trigonometric Fourier series for this signal.

Problem 3-4

This an even square wave with zero average value. Because of the zero average value, $a_0 = 0$. Because of the eveness, $b_n = 0$ for all n. To find a_n, evaluate (3-15):

$$a_n = \frac{2}{T_0}\left[\int_{-T_0/2}^{-T_0/4} -A\cos(n\omega_0 t)\,dt + \int_{-T_0/4}^{T_0/4} A\cos(n\omega_0 t)\,dt + \int_{T_0/4}^{T_0/2} -A\cos(n\omega_0 t)\,dt\right]$$

$$= \frac{2A}{T_0}\left[-\left.\frac{\sin(n\omega_0 t)}{n\omega_0}\right|_{-T_0/2}^{-T_0/4} + \left.\frac{\sin(n\omega_0 t)}{n\omega_0}\right|_{-T_0/4}^{T_0/4} - \left.\frac{\sin(n\omega_0 t)}{n\omega_0}\right|_{T_0/4}^{T_0/2}\right] = \frac{4A}{n\pi}\sin\left(\frac{n\pi}{2}\right), \text{ using } \omega_0 = \frac{2\pi}{T_0}$$

$$= \begin{cases} 0, & n \text{ even} \\ (-1)^{(n-1)/2}\dfrac{4A}{n\pi}, & n \text{ odd} \end{cases}$$

Problem 3-5

(a) Because the string shape is assumed even (origin centered at the peak location), we can use a cosine series. The dc level is 0; hence $a_0 = 0$. Since the string shape is even, only the a_m's are nonzero. They are given by

$$a_m = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} x(\lambda)\cos(m\omega_0\lambda)\,d\lambda = \frac{4}{T_0}\int_{0}^{T_0/2} x(\lambda)\cos(m\omega_0\lambda)\,d\lambda, \quad \omega_0 = \frac{2\pi}{T_0}$$

where λ is a spatial variable. The equation describing the string between 0 and $T_0/2$ is

$$f(\lambda) = -4A\lambda/T_0 + A$$

so that the integral for the Fourier coefficients becomes

$$a_m = \frac{4}{T_0}\int_0^{T_0/4}\left(-\frac{4A}{T_0}\lambda + A\right)\cos(2\pi m\lambda/T_0)\,d\lambda$$

$$= -\frac{16A}{(2\pi m)^2}\int_0^{m\pi} u\cos(u)\,du + \frac{4A}{2\pi m}\int_0^{m\pi}\cos(u)\,du = \begin{cases} 0, & m \text{ even} \\ \dfrac{8A}{(m\pi)^2}, & m \text{ odd} \end{cases}$$

For $T_0 = 9$ inches and $A = 0.5$ inches as in Example 3-3, we obtain

$$a_m = \frac{4}{(m\pi)^2}, \; m \text{ odd}, \; \omega_0 = \frac{\pi}{18}$$

The Fourier series for the triangular string shape is

$$y(x) = \frac{4}{\pi^2}\left[\cos\left(\frac{\pi x}{18}\right) + \frac{1}{9}\cos\left(\frac{3\pi x}{18}\right) + \frac{1}{25}\cos\left(\frac{5\pi x}{18}\right) + \ldots\right]$$

This differs from the result found in Example 3-3 in the following ways:

1. The new series has no dc term;
2. The fundamental frequency of the new series is half that of the old;
3. The amplitudes of the harmonics in the new seriew are half those of the old series.

(b) The error at the ends of the string ($x = \pm T_0/4$) is 0 because the triangular-wave approximation crosses the x-axis at these points.

Problem 3-6

(a) For the half-rectified sine wave

$$X_n = \frac{1}{T_0}\int_0^{T_0/2} A\sin(\omega_0 t)\,e^{-jn\omega_0 t}\,dt + \frac{1}{T_0}\int_0^{T_0/2} (0)\,e^{-jn\omega_0 t}\,dt$$

$$= \frac{1}{T_0}\int_0^{T_0/2} A\,\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}\,e^{-jn\omega_0 t}\,dt$$

$$= \frac{A}{j2T_0}\left[\frac{e^{j(1-n)\omega_0 t}}{j(1-n)\omega_0}\Bigg|_0^{T_0/2} + \frac{e^{j(1+n)\omega_0 t}}{j(1+n)\omega_0}\Bigg|_0^{T_0/2}\right] = \begin{cases} 0, & n \text{ odd} \\ \dfrac{A}{\pi(1-n^2)}, & n \text{ even and } \neq \pm 1 \\ \dfrac{A}{4j}, & n = \pm 1 \end{cases}$$

(b) For a full-rectified sine wave, the period is really $T_0/2$. Furthermore, it is an even signal, so

$$X_n = \frac{2A}{T_0}\int_0^{T_0/2} \sin(\omega_0 t)\cos(n\omega_0 t)\,dt$$

$$= \frac{A}{T_0}\int_0^{T_0/2} \{\sin[(1-n)\omega_0 t] + \sin[(1+n)\omega_0 t]\}\,dt$$

$$= \frac{A}{\omega_0 T_0}\left[-\frac{\cos[(1-n)\pi] - 1}{1-n} - \frac{\cos[(1+n)\pi] - 1}{1+n}\right],\quad n \neq \pm 1$$

$$= \begin{cases} 0, & n \text{ odd} \\ \dfrac{2A}{\pi(1-n^2)}, & n \text{ odd} \end{cases}$$

The results for $n = \pm 1$ have to be done by direct evaluation.

(c) For an even square wave

$$X_n = \frac{2A}{T_0}\left[\int_0^{T_0/4} \cos(n\omega_0 t)\,dt - \int_{T_0/2}^{T_0/2} \cos(n\omega_0 t)\,dt\right]$$

$$= \frac{2A}{T_0}\left[\frac{\sin(n\omega_0 t)}{n\omega_0}\Bigg|_0^{T_0/4} - \frac{\sin(n\omega_0 t)}{n\omega_0}\Bigg|_{T_0/4}^{T_0/2}\right] = \frac{2A\sin(n\pi/2)}{n\pi}$$

$$= \begin{cases} \frac{2A}{\pi|n|}, & n = \pm 1, \pm 5, \ldots \\ \frac{2A}{\pi|n|}, & n = \pm 3, \pm 7, \ldots \\ 0, & n \text{ even} \end{cases}$$

(d) For an even triangle signal,

$$X_n = \frac{2A}{T_0}\left[\int_0^{T_0/2}\left(1 - \frac{4t}{T_0}\right)\cos(n\omega_0 t)\,dt\right]$$

$$= \frac{2A}{T_0}\left[\frac{\sin(n\omega_0 t)}{n\omega_0}\Bigg|_0^{T_0/2} - \frac{4}{T_0}\int_0^{T_0/2} t\cos(n\omega_0 t)\,dt\right]$$

The first term in the brackets is zero upon substitution of limits. The second term must be integrated by parts or looked up in a tables. We get

$$X_n = \frac{2A}{T_0}\left(-\frac{4}{T_0}\right)\left[\frac{t\sin(n\omega_0 t)}{n\omega_0}\Bigg|_0^{T_0/2} - \int_0^{T_0/2}\frac{\sin(n\omega_0 t)}{n\omega_0}\,dt\right]$$

The first term in the brackets again evaluates to 0. The remaining integral evaluates to

$$X_n = -\frac{4A}{n\pi T_0}\frac{\cos(n\omega_0 t)}{n\omega_0}\Bigg|_0^{T_0/2} = \begin{cases} \frac{4A}{(n\pi)^2}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

Problem 3-7

Let

$$y(t) = \frac{dx(t)}{dt},\; y(t) = \sum_{n=-\infty}^{\infty} Y_n e^{jn\omega_0 t} \text{ and } x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$$

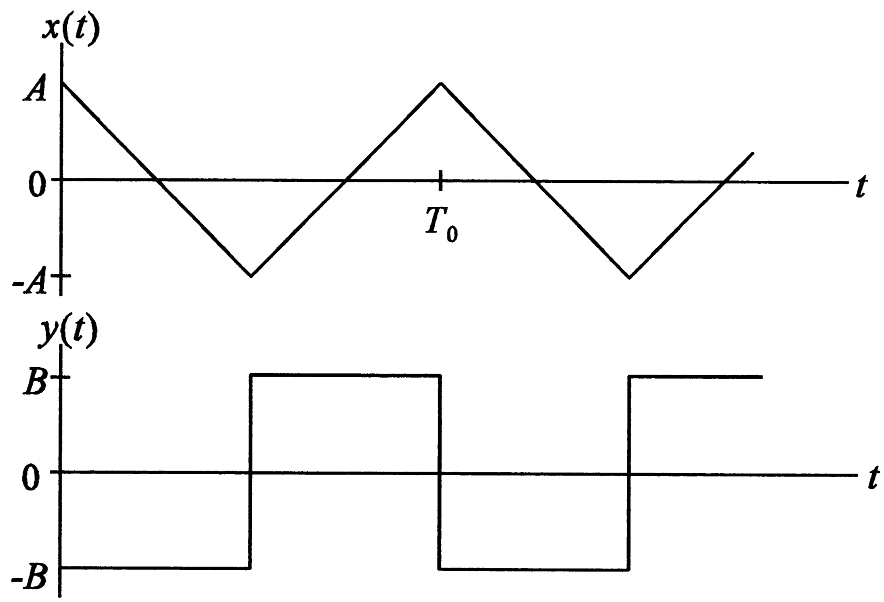
We calculate Y_n in terms of X_n as follows:

$$Y_n = \frac{1}{T_0}\int_{T_0} y(t)\, e^{-jn\omega_0 t}\, dt = \frac{1}{T_0}\int_{T_0} \frac{dx(t)}{dt}\, e^{-jn\omega_0 t}\, dt$$

Integrate the right-most integral by parts to obtain

$$Y_n = \frac{1}{T_0}\left[x(t)\, e^{-jn\omega_0 t}\Big|_0^{T_0} + jn\omega_0 \int_0^{T_0} x(t) e^{-jn\omega_0 t} dt \right] = \frac{1}{T_0}\left[x(T_0) - x(0) + jn\omega_0 \int_0^{T_0} x(t) e^{-jn\omega_0 t} dt \right] = jn\omega_0 X_n$$

Note that $\exp(jn\omega_0 T_0) = 1$ by using $\omega_0 = 2\pi/T_0$, and that $x(T_0) = x(0)$ by periodicity of $x(t)$.
(b) Differentiate a triangular wave to get a square wave as illustrated below:



The Fourier coeffients for the triangular wave are

$$X_n = \frac{4A}{(n\pi)^2},\ n \text{ odd}$$

By graphical differentiation, we see that

$$B = \frac{2A}{T_0/2} = \frac{4A}{T_0}$$

From part (a) we have

$$Y_n = jn\omega_0 X_n = jn\omega_0\left(\frac{4A}{(n\pi)^2}\right) = j\frac{8A}{\pi n T_0},\ n \text{ odd}$$

Using the value found for B in terms of A, we obtain the Fourier coeffients of a square wave knowing the Fourier coefficients of a triangular wave (or vice versa):

$$Y_n = j\frac{2B}{\pi n} = j\frac{8A}{n\pi T_0},\ n \text{ odd } (B \text{ is the square wave amplitude})$$

Problem 3-8

(a) By trigonometric identities (see Appendix F), we may write $x(t)$ as

$$x(t) = [1 - \cos(5000\pi t)]\cos(20000\pi t) = \cos(20000\pi t) - 1/2\cos(15000\pi t) - 1/2\cos(25000\pi t)$$

Using the fact that the power in a sinusoid is ½ the square of its amplitude, and the fact that we can add powers of the separate harmonics of a harmonic sum of sinusoids, we find that

$$P_{\text{ave}, x(t)} = \frac{1}{2}(1)^2 + \frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{2}\left(\frac{1}{2}\right)^2 = \frac{3}{4}\ \text{W}$$

(b) From (a) it is seen that the signal $x(t)$ consists of components with frequencies 7500, 10000, and 12500 Hz. Only the first two are passed by the telephone system, so the output power is

$$P_{\text{ave}, y(t)} = \frac{1}{2}(1)^2 + \frac{1}{2}\left(\frac{1}{2}\right)^2 = \frac{5}{8}\ \text{W};\ \text{Ratio} = \frac{5}{6}$$

Problem 3-9

For a pulse of width τ, amplitude A, and period T_0, the complex exponential Fourier series coefficients are given by

$$X_n = \frac{A\tau}{T_0}\operatorname{sinc}(nf_0\tau),\ f_0 = T_0^{-1}$$

By Parseval's theorem, the power contained in the frequency band $-\tau^{-1} \le f \le \tau^{-1}$ is

$$P[|nf_0| \le \tau^{-1}] = \sum_{n=-N}^{N} |X_n|^2$$

where $N = \lfloor 1/f_0\tau \rfloor$ where the notation means "the integer part less than". The total power in the signal is $A^2\tau/T_0$ so that the fraction of total power is

$$\frac{P[|nf_0| \le \tau^{-1}]}{P_{\text{total}}} = \frac{\tau}{T_0}\sum_{n=-N}^{N} \operatorname{sinc}^2(nf_0\tau)$$

(a) For $T_0/\tau = 2$, the fraction of total power contained within the main lobe is

$$\begin{aligned}\frac{P[|nf_0| \le \tau^{-1}]}{P_{\text{total}}} &= \frac{1}{2}\sum_{n=-2}^{2} \operatorname{sinc}^2(0.5n)\\ &= \frac{1}{2}[\operatorname{sinc}^2(-1) + \operatorname{sinc}^2(-1/2) + \operatorname{sinc}^2(0) + \operatorname{sinc}^2(1/2) + \operatorname{sinc}^2(1)]\\ &= \frac{1}{2}[1 + 2(0.4053)] = 0.9053\end{aligned}$$

(b) For $T_0/\tau = 4$, the fraction of total power contained within the main lobe is

$$\begin{aligned}\frac{P[|nf_0| \le \tau^{-1}]}{P_{\text{total}}} &= \frac{1}{4}\sum_{n=-4}^{4} \operatorname{sinc}^2(0.25n)\\ &= \frac{1}{4}[1 + 2\operatorname{sinc}^2(0.25) + 2\operatorname{sinc}^2(0.5) + 2\operatorname{sinc}^2(0.75)]\\ &= 0.9030\end{aligned}$$

(c) For $T_0/\tau = 10$, the fraction of total power contained within the main lobe is

$$\frac{P[|nf_0| \le \tau^{-1}]}{P_{\text{total}}} = \frac{1}{10}\sum_{n=-10}^{10} \operatorname{sinc}^2(0.1n)$$

$$= \frac{1}{10}[1 + 2\operatorname{sinc}^2(0.1) + 2\operatorname{sinc}^2(0.2) + 2\operatorname{sinc}^2(0.3) + 2\operatorname{sinc}^2(0.4) + 2\operatorname{sinc}^2(0.5)$$

$$+ 2\operatorname{sinc}^2(0.6) + 2\operatorname{sinc}^2(0.7) + 2\operatorname{sinc}^2(0.8) + 2\operatorname{sinc}^2(0.9)] = 0.8067$$

Problem 3-10

The waveform is odd, so

$$X_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\,[\cos(n\omega_0 t) - j\sin(n\omega_0 t)]\,dt$$

$$= -\frac{j}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\sin(n\omega_0 t)\,dt = -\frac{2j}{T_0}\int_{-0}^{T_0/2} x(t)\sin(n\omega_0 t)\,dt$$

Substitute the equation for the waveform and use integral tables to evaluate (or integrate by parts):

$$X_n = -\frac{j4A}{T_0^2}\int_0^{T_0/2} t\sin(n\omega_0 t)\,dt = -\frac{j4A}{T_0^2}\left[\sin(n\omega_0 t)\big|_0^{T_0/2} - (n\omega_0 t)\cos(n\omega_0 t)\big|_0^{T_0/2}\right] = \frac{jA}{n\pi}(-1)^n$$

Thus, the exponential Fourier series is

$$x(t) = \frac{jA}{\pi}\left[\dots + \frac{1}{3}e^{-j3\omega_0 t} - \frac{1}{2}e^{-j2\omega_0 t} + e^{-j\omega_0 t} - e^{j\omega_0 t} + \frac{1}{2}e^{j2\omega_0 t} - \frac{1}{3}e^{j3\omega_0 t} + \dots\right]$$

$$= \frac{2A}{\pi}\left[\sin(\omega_0 t) - \frac{1}{2}\sin(2\omega_0 t) + \frac{1}{3}\sin(3\omega_0 t) + \dots\right]$$

This is the same as (3-4) except for the constant out in front.

Problem 3-11

(a) Take the fundamental term in the series. It is

$$\frac{1}{1 + j\pi} e^{j3\pi t/2} = X_1 e^{j2\pi f_0 t}$$

By matching exponents on both sides and solving for f_0, we find that $f_0 = 3/4$ Hz; $T_0 = 1.333$ s.
(b) The average value of the waveform is given by $X_0 = 1$.
(c) The third harmonic term is given by

$$X_{-3} e^{-j9\pi t/2} + X_3 e^{j9\pi t/2} = 2|X_3|\cos(9\pi t/2 + \underline{/X_3})$$

Thus, the amplitude of the 3rd harmonic term is $2/(1 + 9\pi^2)^{1/2}$.
(d) From (c), the phase is $-\tan^{-1}(3\pi)$ radians.
(e) Substitute the values for the amplitude and phase found in parts (c) and (d) into the expression found in part (c).

Problem 3-12

(a) The exponential Fourier series coefficients are given by

$$X_n = \frac{1}{2}\int_{-1}^{1} e^{-|t|} e^{-j2\pi n t/2} dt = \frac{1}{2}\int_{-1}^{1} e^{-|t|}[\cos(\pi n t) - j\sin(\pi n t)]\,dt = \int_{0}^{1} e^{-t}\cos(\pi n t)\,dt = \frac{1 - (-1)^n e^{-1}}{1 + (n\pi)^2}$$

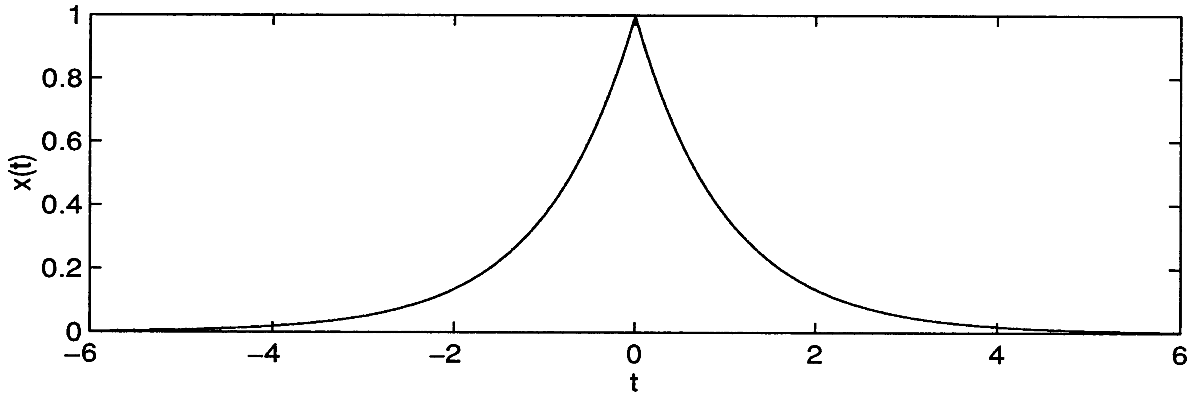
The second integral follows by the eveness of $\exp(-|t|)\cos(\pi n t)$ and oddness of $\exp(-|t|)\sin(\pi n t)$. Evaluation of the coefficients and substitution into the exponential Fourier series results in

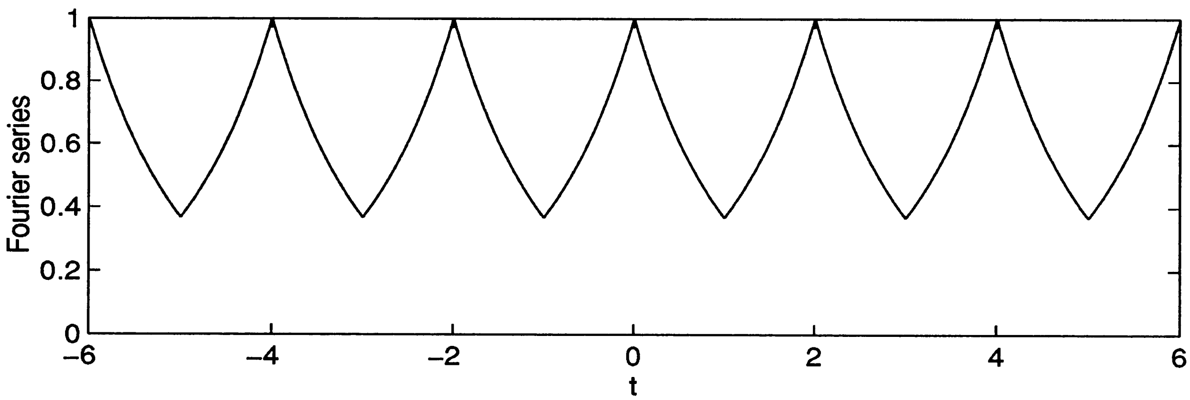
$$x(t) = \cdots + \frac{1 - e^{-1}}{1 + 4\pi^2} e^{-j2\pi t} + \frac{1 + e^{-1}}{1 + \pi^2} e^{-j\pi t} + (1 - e^{-1}) + \frac{1 + e^{-1}}{1 + \pi^2} e^{j\pi t} + \frac{1 - e^{-1}}{1 + 4\pi^2} e^{j2\pi t} + \cdots$$

(b) The trigonometric Fourier series is

$$x(t) = (1 - e^{-1}) + 2\frac{1 + e^{-1}}{1 + \pi^2}\cos(\pi t) + 2\frac{1 - e^{-1}}{1 + 4\pi^2}\cos(2\pi t) + \cdots$$

A plot of the function and the sum of the Fourier series is given on the next page.





Problem 3-13

(a) Using Euler's theorem, the exponential Fourier series is

$$
\begin{aligned}
x_a(t) &= \left[\frac{e^{j20\pi t} + e^{-j20\pi t}}{2}\right]^2 \frac{e^{j10\pi t} - e^{-j10\pi t}}{2j} \\
&= \frac{1}{8j}\left[e^{j40\pi t} + 2 + e^{-j40\pi t}\right]\left[e^{j10\pi t} - e^{-j10\pi t}\right] \\
&= -\frac{1}{8j}e^{-j50\pi t} + \frac{1}{8j}e^{-j30\pi t} - \frac{1}{4j}e^{-j10\pi t} + \frac{1}{4j}e^{j10\pi t} - \frac{1}{8j}e^{j30\pi t} + \frac{1}{8j}e^{j50\pi t}
\end{aligned}
$$

(b) A series of steps similar to those used in part (a) results in

$$\begin{aligned} x_b(t) &= \left[\frac{e^{j30\pi t} - e^{-j30\pi t}}{2j}\right]^3 + 2\frac{e^{j25\pi t} + e^{-j25\pi t}}{2} \\ &= \frac{j}{8}\left[-e^{-j90\pi t} + 3e^{-j30\pi t} - 3e^{j30\pi t} + e^{j90\pi t}\right] + e^{j25\pi t} + e^{-j25\pi t} \\ &= -\frac{j}{8}e^{-j90\pi t} + j\frac{3}{8}e^{-j30\pi t} + e^{-j25\pi t} + e^{j25\pi t} - j\frac{3}{8}e^{j30\pi t} + \frac{j}{8}e^{j90\pi t} \end{aligned}$$

(c) A series of steps used to those above gives

$$\begin{aligned} x_c(t) &= \left[\frac{e^{j40\pi t} - e^{-j40\pi t}}{2j}\right]^2\left[\frac{e^{j20\pi t} + e^{-j20\pi t}}{2}\right]^2 + \frac{e^{j10\pi t} - e^{-j10\pi t}}{2j}\,\frac{e^{j5\pi t} - e^{-j5\pi t}}{2j} \\ &= -\frac{1}{16}e^{-j120\pi t} - \frac{1}{8}e^{-j80\pi t} + \frac{1}{16}e^{-j40\pi t} - \frac{1}{4j}e^{-j15\pi t} - \frac{1}{4j}e^{-j5\pi t} \\ &\quad + \frac{1}{4} + \frac{1}{4j}e^{j5\pi t} + \frac{1}{4j}e^{j15\pi t} + \frac{1}{16}e^{j40\pi t} - \frac{1}{8}e^{j80\pi t} - \frac{1}{16}e^{j120\pi t} \end{aligned}$$

Problem 3-14

(a) Write $x(t)$ in terms of its Fourier series and replace t by $t - \tau$:

$$y(t) = x(t-\tau) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0(t-\tau)} = \sum_{n=-\infty}^{\infty} X_n e^{-jn\omega_0\tau} e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} Y_n e^{jn\omega_0 t}$$

Comparing the two sums on the right, we see that $Y_n = X_n e^{-jn\omega_0\tau}$

(b) Using a series of steps similar to those used in part (a), we have

$$y(t) = x(t)e^{j2\pi f_0 t} = \sum_{n=-\infty}^{\infty} X_n e^{j2\pi(n+1)f_0 t}$$

In the last sum, let $m = n + 1$:

$$y(t) = \sum_{m=-\infty}^{\infty} X_{m-1} e^{j2\pi m f_0 t} = \sum_{m=-\infty}^{\infty} Y_m e^{j2\pi m f_0 t}$$

comparing the two series, we find that $Y_m = X_{m-1}$.

Problem 3-15

Given

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t + \theta_n) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$$

Let

$$\cos(n\omega_0 t + \theta_n) = \cos\theta_n \cos(n\omega_0 t) - \sin\theta_n \sin(n\omega_0 t)$$

Thus, the trigonometric Fourier series can be written as

$$\begin{aligned}
x(t) &= a_0 + \sum_{n=1}^{\infty} a_n \cos\theta_n \cos(n\omega_0 t) - \sum_{n=1}^{\infty} a_n \sin\theta_n \sin(n\omega_0 t) \\
&= a_0 + \sum_{n=1}^{\infty} a_n \cos\theta_n \left(\frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2} \right) - \sum_{n=1}^{\infty} a_n \sin\theta_n \left(\frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2j} \right) \\
&= a_0 + \sum_{n=1}^{\infty} \frac{a_n}{2} (\cos\theta_n + j\sin\theta_n) e^{jn\omega_0 t} + \sum_{n=1}^{\infty} \frac{a_n}{2} (\cos\theta_n - j\sin\theta_n) e^{-jn\omega_0 t} \\
&= \equiv X_0 + \sum_{n=1}^{\infty} X_n e^{jn\omega_0 t} + \sum_{n=-\infty}^{-1} X_n e^{jn\omega_0 t} \\
&= X_0 + \sum_{n=1}^{\infty} X_n e^{jn\omega_0 t} + \sum_{n=1}^{\infty} X_{-n} e^{-jn\omega_0 t}
\end{aligned}$$

Matching coeffients in the series in the third and last lines, we obtain

$$a_0 = X_0;\ X_n = \frac{a_n}{2}(\cos\theta_n + j\sin\theta_n);\ X_{-n} = \frac{a_n}{2}(\cos\theta_n - j\sin\theta_n),\ n > 0$$

Solve these for a_n and θ_n to get

$$a_n = 2|X_n|;\ \theta_n = \tan^{-1}\left(\frac{\operatorname{Im} X_n}{\operatorname{Re} X_n} \right)$$

(b) This is obvious from the fact that cosine is an even function and sine is an odd function.

Problem 3-16

(a) The integral for the Fourier coefficients can be written as

$$X_n = \frac{1}{T_0}\int_{T_0} x(t)\,e^{-jn\omega_0 t}\,dt = \frac{1}{T_0}\int_{T_0} x(t)\cos(n\omega_0 t)\,dt - \frac{j}{T_0}\int_{T_0} x(t)\sin(n\omega_0 t)\,dt$$

Assume $x(t)$ is real and even. In the first integral on the right-hand side above, the integrand is even because the product of two even functions is even. Thus this integral is not 0 in general. In the second integral on the right-hand side above, the integrand is odd because the product of an even and an odd function is odd. Thus this integral is 0 if integrated over symmetric limits about $t = 0$ and, indeed, over any T_0-second interval by periodicity of the integrand. Hence, we conclude that the X_n's are real and, since the dependence on n is through the cosine which is an even function, X_n is an even function of n.

(b) In the expression given above, let $x(t)$ be real and odd. Now the first integral on the right-hand side is 0 because an odd times an even function is odd. The second integral on the right-hand side is not 0 in general because the product of two odd functions is even. Thus the X_n's are imaginary and odd since the dependence on n is through the sine which is an odd function.

(c) In the Fourier series for $x(t)$

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$$

replace t by $t \pm T_0/2$ to get

$$x(t \pm T_0/2) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0(t \pm T_0/2)} = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t} e^{\pm jn(2\pi/T_0)(T_0/2)}$$

$$= \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t} e^{\pm jn\pi} = -\sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t} \quad \text{(by hypothesis)}$$

The last two sums, when placed on the same side of the equation, produce

$$\sum_{n=-\infty}^{\infty} X_n\left[1 + e^{\pm jn\omega_0 t}\right] \equiv 0$$

For n odd, the bracketed term is 0, but for n even it equals 2. For the sum above to be identically 0 requires that the even-indexed X_n's be 0. For n even, the bracketed term makes these terms 0.

Problem 3-17

Property	a	b	c	d	e	f
Real coefficients		X			X	
Imaginary coefficients	X			X		
Complex coefficients			X			X
Even-indexed coefficient = 0	X	X	X	X	X	X
$X_0 = 0$	X	X	X	X	X	X

Problem 3-18

(a) $a_0 = 0$ because the average value of the waveform is 0;
(b) $b_n = 0$ for all n because the waveform is even;
(c) the X_n's are real because $x(t)$ is even;
(d) yes, so the even-indexed Fourier coefficients are 0.

Problem 3-19

Expand the second term of the given expression using a trigonometric identity to get

$$\begin{aligned} x(t) &= \cos(\omega_0 t) + \cos\Delta\omega t\cos(\omega_0 t) - \sin\Delta\omega t\sin(\omega_0 t) \\ &= (1 + \cos\Delta\omega t)\cos(\omega_0 t) - \sin\Delta\omega t\sin(\omega_0 t) \end{aligned}$$

Note that

$$A\cos(\omega_0 t + \theta) = A\cos\theta\cos\omega_0 t - A\sin\theta\sin\omega_0 t$$

Set this equal to the last equation of the first set and match multipliers of sine and cosine to get

$$A\cos\theta = 1 + \cos\Delta\omega t;\ A\sin\theta = \sin\Delta\omega t$$

Square and add these equations to get

$$A(t) = \sqrt{[1 + \cos\Delta\omega t]^2 + \sin^2\Delta\omega t} = \sqrt{2 + 2\cos\Delta\omega t} = 2|\cos(\Delta\omega t/2)|$$

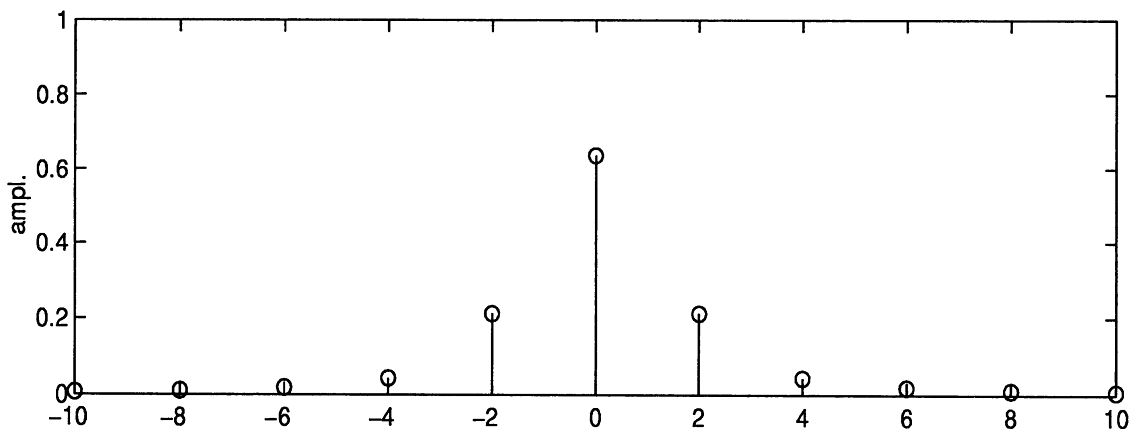
Divide the second equation by the first to get

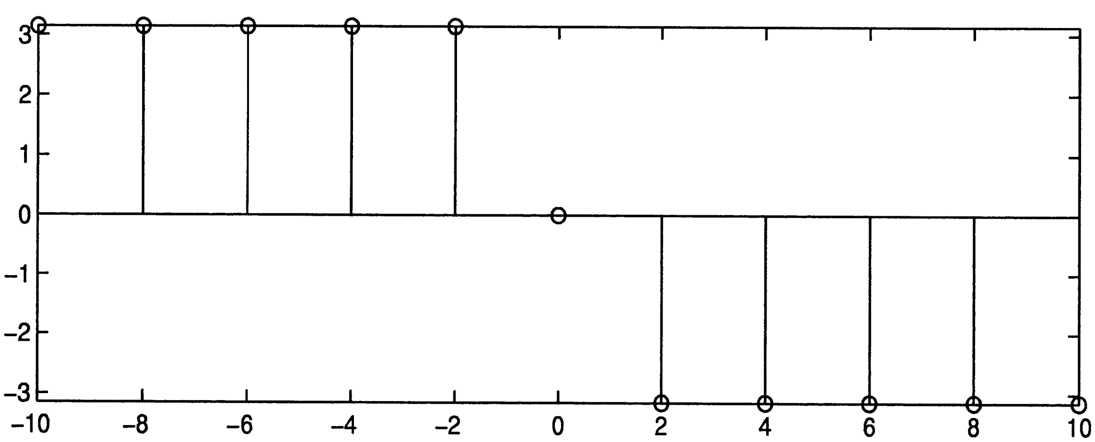
$$\theta(t) = \tan^{-1}\left[\frac{\sin\Delta\omega t}{1 + \cos\Delta\omega t}\right]$$

Thus, the sum of the two cosines becomes

$$x(t) = 2|\cos(\Delta\omega t/2)|\cos\left[\omega_0 t + \tan^{-1}\left(\frac{\sin\Delta\omega t}{1 + \cos\Delta\omega t}\right)\right]$$

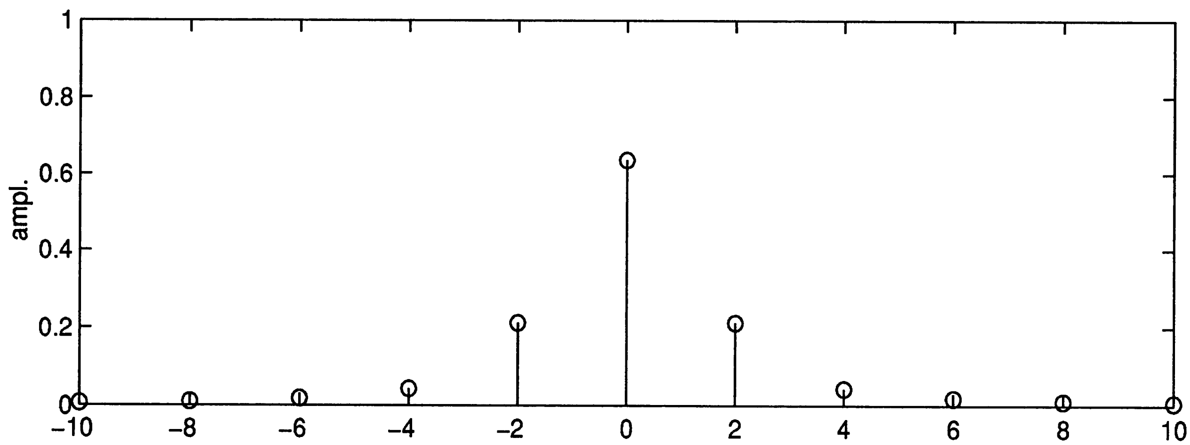
A plot is shown below for two different time scales. The ear will hear alternate reinforcements and cancellations.

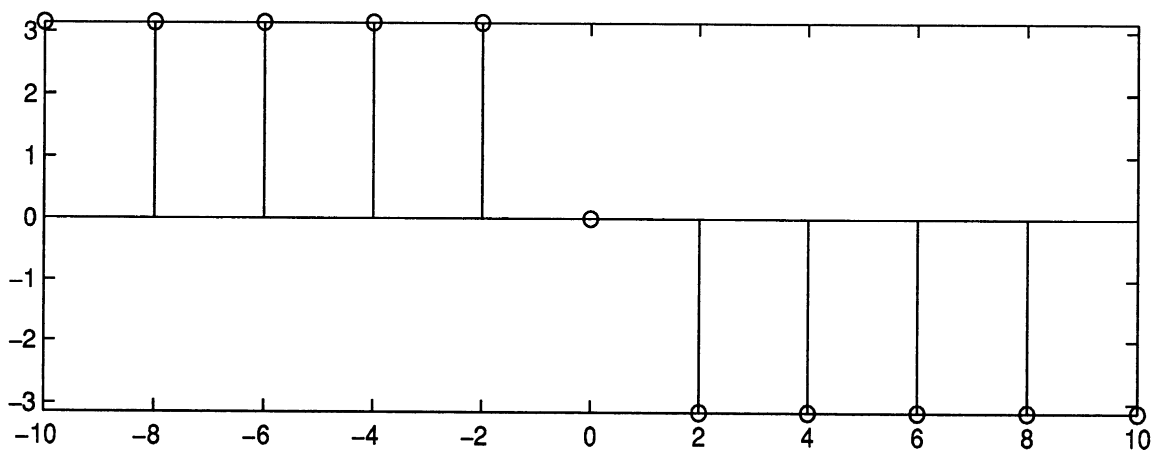




Problem 3-20

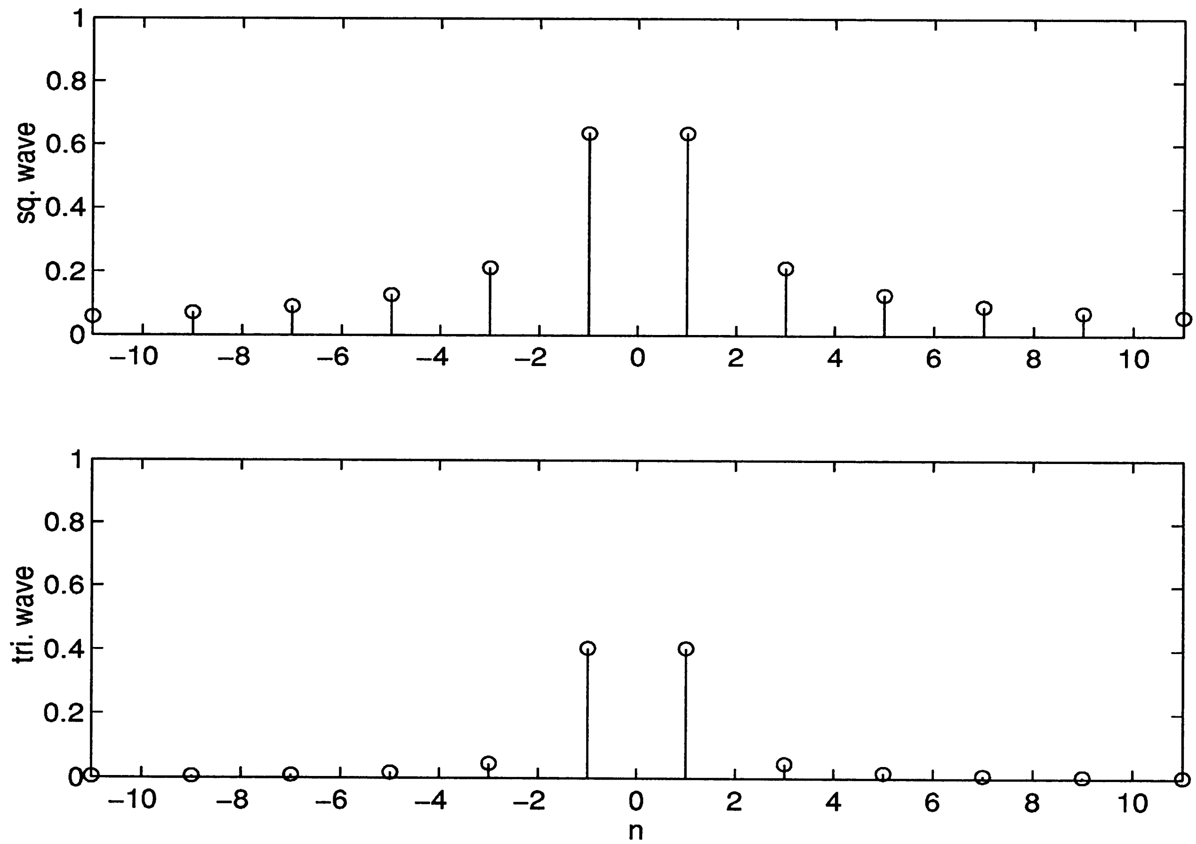
The double-sided amplitude spectra are shown below. To get the single-sided amplitude spectrum, take the portion of the double-sided spectrum for $f \geq 0$, leaving the line at $f = 0$ alone and doubling lines for $f > 0$. For the single-sided phase spectrum, simply take the positive frequency portion of the double-sided spectrum.





Problem 3-21

The two amplitude spectra are compared below. The spectrum for the triangular wave goes to 0 with n fastest, and therfore requires the least bandwidth.



Problem 3-22

(a) See Fig. 3-11(a), except that the nulls of the sinc function are at multiples of 500 Hz and the lines are spaced by 125 Hz.
(b) See Fig. 3-11(b), except that the nulls of the sinc function are at multiples of 1000 Hz and the lines are spaced by 125 Hz.
(c) See Fig. 3-11(c), except that the nulls of the sinc function are at multiples of 500 Hz and the lines are spaced by 62.5 Hz.

Problem 3-23

Start with (3-69) after expressing the transfer function in terms of amplitude and phase:

$$y(t) = \sum_{n=-\infty}^{\infty} |X_n| e^{j\underline{/X_n}} |H(n\omega_0)| e^{j\underline{/H(n\omega_0)}} e^{jn\omega_0 t}$$

Recall that for $x(t)$ and $h(t)$, the magnitude functions are even and the phase functions are odd. Thus, we have the series of steps given below:

$$\begin{aligned} y(t) &= \sum_{n=-\infty}^{-1} |X_n|\,|H(n\omega_0)|\, e^{j[n\omega_0 t + \underline{/X_n} + \underline{/H(n\omega_0)}]} + X_0 + \sum_{n=1}^{\infty} |X_n|\,|H(n\omega_0)|\, e^{j[n\omega_0 t + \underline{/X_n} + \underline{/H(n\omega_0)}]} \\ &= \sum_{m=1}^{\infty} |X_{-m}|\,|H(-m\omega_0)|\, e^{j[-m\omega_0 t + \underline{/X_{-m}} + \underline{/H(-m\omega_0)}]} + X_0 + \sum_{n=1}^{\infty} |X_n|\,|H(n\omega_0)|\, e^{j[n\omega_0 t + \underline{/X_n} + \underline{/H(n\omega_0)}]} \\ &= X_0 + \sum_{n=1}^{\infty} 2|X_n|\,|H(n\omega_0)| \frac{e^{j[n\omega_0 t + \underline{/X_n} + \underline{/H(n\omega_0)}]}}{2} \\ &= X_0 + \sum_{n=1}^{\infty} 2|X_n|\,|H(n\omega_0)| \cos[n\omega_0 t + \underline{/X_n} + \underline{/H(n\omega_0)}] \end{aligned}$$

Problem 3-24

Apply (3-73). The transfer function of the system is

$$H(j\omega) = \frac{1}{1 + j\omega L/R} = \frac{1}{\sqrt{1 + (\omega L/R)^2}} e^{-j\tan^{-1}(\omega L/R)}$$

From Table 3-1, the exponential Fourier series coefficients for a triangular signal are

$$X_n = \frac{4A}{(n\pi)^2}, \quad n \text{ odd}$$

Thus, (3-73) becomes

$$y(t) = \sum_{n=-\infty,\ n \text{ odd}}^{\infty} \left(\frac{8A}{(n\pi)^2 \sqrt{1 + n^2}} \right) \cos[n\omega_0 t - \tan^{-1}(n)]$$

Problem 3-25

(a) Rewrite the output as

$$y(t) = \cos[10\pi(t - 1/40)] + 5\cos[20\pi(t - 1/40)]$$

The delays in both components are the same, so there is no phase or delay distortion. However, the amplitudes are in a different ratio in the output than the input, so there is amplitude distortion.
(b) Rewrite the output as

$$y(t) = \cos[10\pi(t - 1/40)] + 2\cos[20\pi(t - 1/80)]$$

There is phase or delay distortion, but there is amplitude distortion.
(c) Rewrite the output as

$$y(t) = \cos[10\pi(t - 1/40)] + 2\cos[20\pi(t - 1/40)]$$

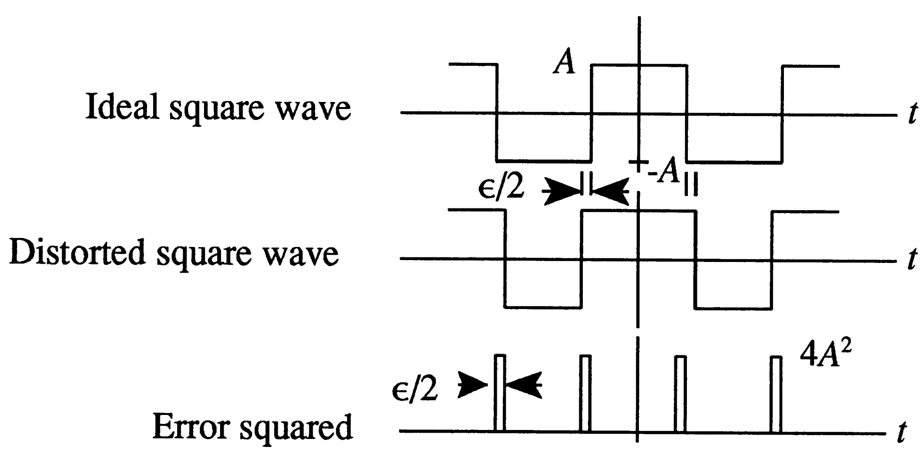
Comparing this with the input, it is clear that there is no distortion.
(d) Rewrite the output as

$$y(t) = 2\cos[10\pi(t - 1/40)] + 2\cos[20\pi(t - 1/160)]$$

This shows that there is amplitude distortion, but no phase distortion.

Problem 3-26

The ideal square wave, distorted square wave, and error-squared function are shown below:



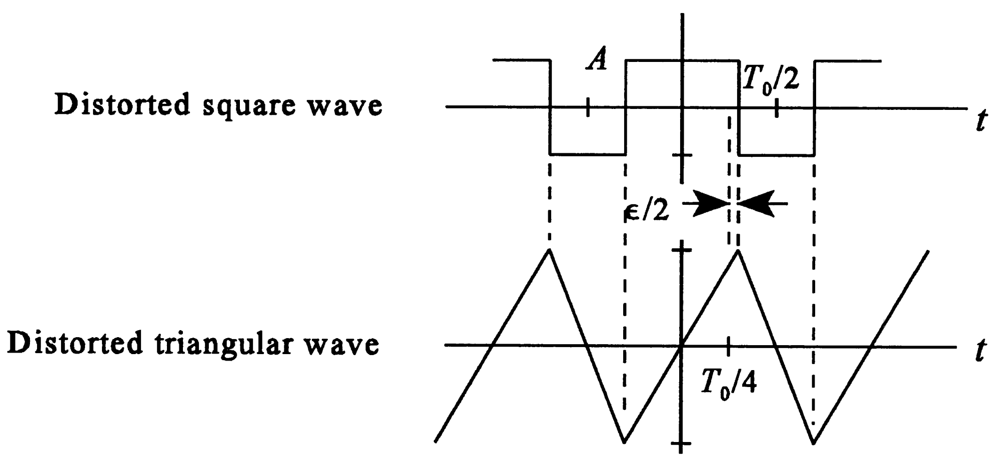
From the figure, the integral-squared error over one period is

$$\mathrm{ISE} = (4A^2)(\epsilon/2)(2) = 4A^2\epsilon$$

The mean-squared error is

$$\text{Mean-squared error} = \frac{\mathrm{ISE}}{T_0} = 4A^2\frac{\epsilon}{T_0}$$

(b) Assume ac coupling between the multivibrator and the RC integrator. The integrator will have a long positive-slope ramp and a short negative-slope ramp from integrating the unequal distorted square wave half cycles. The figure below illustrates this.



(Note that strictly speaking the triangular wave should be inverted due to the operational amplifier.)
(c) For the sinusoidal mode, find the Fourier series of the square wave. Since the waveform is even (see the distorted square wave above), we use a Fourier cosine series. The dc component is

$$a_0 = \frac{1}{T_0}\int_{T_0} x(t)dt = A\epsilon/T_0$$

For $n > 0$,

$$
\begin{aligned}
a_n &= \frac{4}{T_0}\int_0^{T_0/2} x(t)\cos(n\omega_0 t)\,dt \\
&= \frac{4}{T_0}\left[\int_0^{T_0/4+\epsilon/2} A\cos(n\omega_0 t)\,dt + \int_{T_0/4+\epsilon/2}^{T_0/2} -A\cos(n\omega_0 t)\,dt\right] \\
&= \frac{4A}{n\pi}\sin[n\pi(\epsilon/T_0 + 0.5)]
\end{aligned}
$$

The output power for the nth harmonic is

$$P_{n,\ \mathrm{out}} = \frac{1}{2}|H(n\omega_0)|^2 a_n^2$$

where the transfer function of the filter is given in the problem statement.

(I) The ratio of 2nd harmonic power to fundamental power in the output is

$$\frac{P_{2,\ \mathrm{out}}}{P_{1,\ \mathrm{out}}} = \mathrm{HC}_2 = \left[\frac{\sin[2\pi(\epsilon/T_0 + 0.5)]}{2\sin[\pi(\epsilon/T_0 + 0.5)]}\right]^2 \frac{1}{1 + 9Q^2/4}$$

(II) The ratio of 3rd harmonic power to fundamental power in the output is

$$\frac{P_{3,\ \mathrm{out}}}{P_{1,\ \mathrm{out}}} = \mathrm{HC}_3 = \left[\frac{\sin[3\pi(\epsilon/T_0 + 0.5)]}{3\sin[\pi(\epsilon/T_0 + 0.5)]}\right]^2 \frac{1}{1 + 64Q^2/9}$$

If one solves $\mathrm{HC}_2 = 0.001$ for Q, the result is 3.23, while the solution of $\mathrm{HC}_3 = 0.0005$ is $Q = 5.029$. Thus, the latter is the most stringent condition.

Problem 3-27

For (c), one derivative produces impulses so the spectral lines approach 0 as $1/n$ as $n \to \infty$. For (d), two derivatives produce impulses, so the spectral lines approach 0 as $1/n^2$ as $n \to \infty$.

Problem 3-28

Two derivatives produce impulses, so the spectral lines approach 0 as $1/n^2$ as $n \to \infty$.

Problem 3-29

Expand the last term of (3-92) to get

$$\sum_{n=1}^{\infty}\left|d_n - \int_T x(t)\phi_n^*(t)\,dt\right|^2 = \sum_{n=1}^{\infty}\left[|d_n|^2 - d_n\int_T x^*(t)\phi_n(t)\,dt - d_n^*\int_T x(t)\phi_n^*(t)\,dt + \left|\int_T x(t)\phi_n^*(t)\,dt\right|^2\right]$$

Sum each term individually and substitute into (3-92):

$$\epsilon_N = \int_T |x(t)|^2 dt - \sum_{n=1}^{N}\left|\int_T x(t)\phi_n^*(t)\,dt\right|^2 + \sum_{n=1}^{N}|d_n|^2 - \sum_{n=1}^{N} d_n^*\int_T x(t)\phi_n^*(t)\,dt$$
$$- \sum_{n=1}^{N} d_n\int_T x^*(t)\phi_n(t)\,dt + \sum_{n=1}^{N}\left|\int_T x(t)\phi_n^*(t)\,dt\right|^2$$

It is seen that the 2nd and last terms on the right hand side cancel, giving (3-90).

Problem 3-30

(a) Normalize $f_1(t)$ to get $\phi_1(t)$:

$$\int_0^{\infty}(A_1 e^{-t})^2 dt = A_1^2\int_0^{\infty} e^{-2t}dt = A_1^2/2 = 1 \text{ or } A_1 = \sqrt{2}$$

Thus

$$\phi_1(t) = \sqrt{2}e^{-t}u(t)$$

(b) Let

$$\overline{\phi_2}(t) = f_2(t) - B\phi_1(t) = A_2 e^{-2t} - B\sqrt{2}e^{-t},\ t \geq 0$$

We require that

$$\int_0^{\infty}\overline{\phi_2}(t)\,\phi_1(t)\,dt = 0 \text{ or } \int_0^{\infty}\left(A_2 e^{-3t} - B\sqrt{2}e^{-2t}\right)dt = 0 \text{ or } \frac{\sqrt{2}}{3}A_2 - B = 0 \text{ or } B = \frac{\sqrt{2}}{3}A_2$$

Thus

$$\overline{\phi_2}(t) = A_2\left[e^{-2t} - \frac{2}{3}e^{-t}\right]u(t)$$

To find A_2, evaluate the normalization integral:

$$\int_0^\infty \overline{\phi_2^2}(t)\,dt = 1 \text{ which gives } A_2 = 6$$

The second orthonormal function is, therefore,

$$\phi_2(t) = \left(6e^{-2t} - 4e^{-t}\right)u(t)$$

(c) Carrying out the steps of orthogonalization on the third function as suggested in the problem statement results in

$$\phi_3(t) = \frac{10\sqrt{6}}{17}\left[e^{-3t} - \frac{6}{5}e^{-2t} + \frac{3}{10}e^{-t}\right]u(t)$$

There is no obvious generalization for general n.

Problem 3-31

(a) From (3-94) and Table 3-1,

$$\text{ISE}_1 = \left[A^2 - 2\left(\frac{2A}{\pi}\right)^2\right]T_0 = A^2T_0\left[1 - \frac{8}{\pi^2}\right]$$

(b) Again using (3-94) and Table 3-1, we obtain the result

$$\text{ISE}_2 = \left[A^2 - 2\left(\frac{2A}{\pi}\right)^2 - 2\left(\frac{2A}{3\pi}\right)^2\right]T_0 = A^2T_0\left(1 - \frac{8}{\pi^2} - \frac{8}{9\pi^2}\right) = A^2T_0\left(1 - \frac{80}{9\pi^2}\right)$$

CHAPTER 4

Problem 4-1

(a) The Fourier transform of this signal is

$$X_a(f) = \int_{-\infty}^{\infty} Ae^{-\alpha t}u(t)e^{-j2\pi ft}dt = \int_{0}^{\infty} Ae^{-(\alpha + j2\pi f)t}dt = A\frac{e^{-(\alpha + j2\pi f)t}}{-(\alpha + j2\pi f)}\Bigg|_0^{\infty} = \frac{A}{\alpha + j2\pi f}$$

where the evaluation of the upper limit gives 0 because the problem statement says that $\alpha > 0$.
(b) The evaluation of this Fourier transform gives

$$X_b(f) = \int_{-\infty}^{\infty} Ae^{\alpha t}u(-t)e^{-j2\pi ft}dt = \int_{-\infty}^{0} Ae^{(\alpha - j2\pi f)t}dt = A\frac{e^{(\alpha - j2\pi f)t}}{(\alpha - j2\pi f)}\Bigg|_{-\infty}^{0} = \frac{A}{\alpha - j2\pi f}$$

(c) By direct evaluation, or using the results of the previous two parts noting that $x_c(t) = x_a(t) + x_b(t)$, gives

$$X_c(f) = A\left[\frac{1}{\alpha + j2\pi f} + \frac{1}{\alpha - j2\pi f}\right] = \frac{2A\alpha}{\alpha^2 + (2\pi f)^2}$$

(d) This signal is the difference between the signals of parts (a) and (b), and it follows that its Fourier transform is

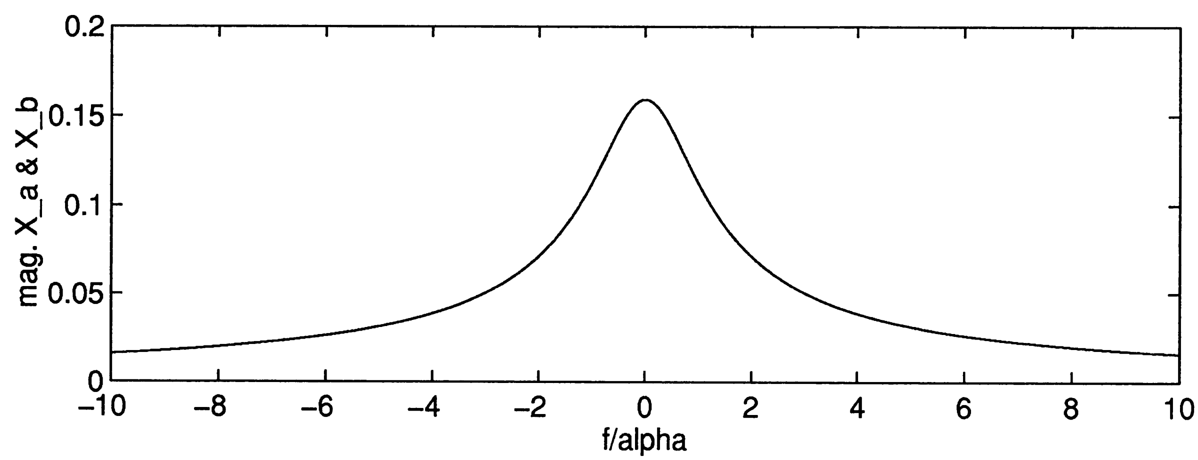
$$X_d(f) = A\left[\frac{1}{\alpha + j2\pi f} - \frac{1}{\alpha - j2\pi f}\right] = \frac{-j4A\pi f}{\alpha^2 + (2\pi f)^2}$$

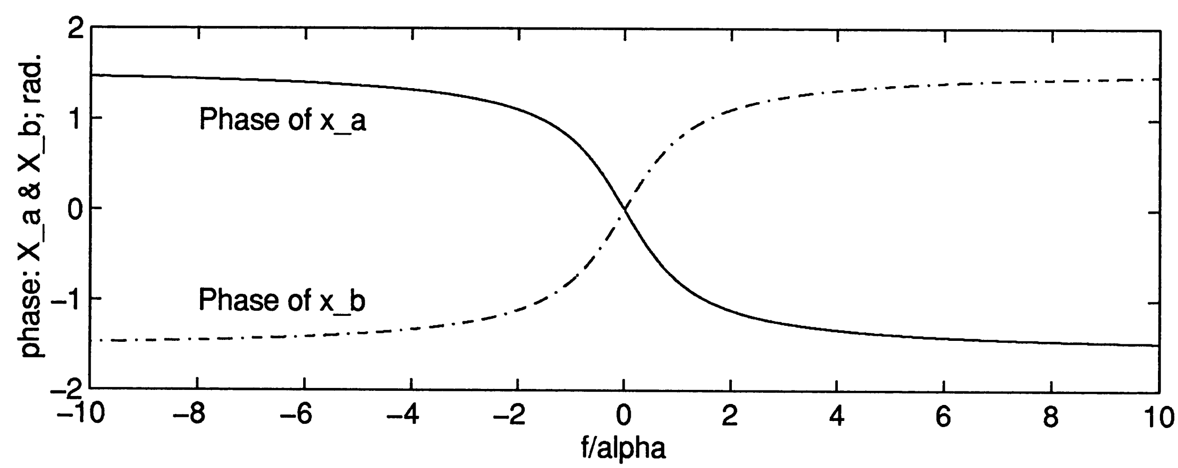
Problem 4-2

The Fourier transforms in terms of magnitude and phase are

$$X_a(f) = \frac{A}{\sqrt{\alpha^2 + (2\pi f)^2}} e^{-j\tan^{-1}(2\pi f/\alpha)} \text{ and } X_b(f) = \frac{A}{\sqrt{\alpha^2 + (2\pi f)^2}} e^{j\tan^{-1}(2\pi f/\alpha)}$$

Thus, both magnitude functions are the same and the phase functions are negations of each other. Plots are shown below:



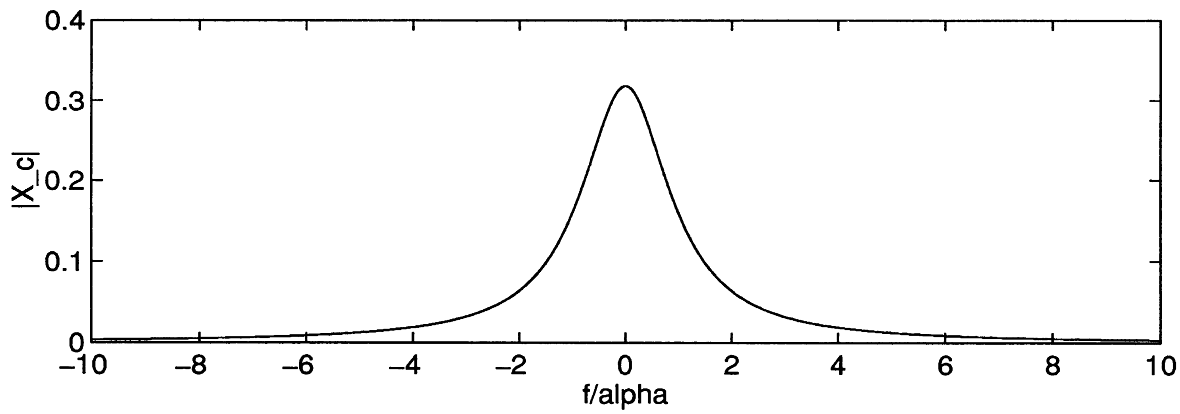


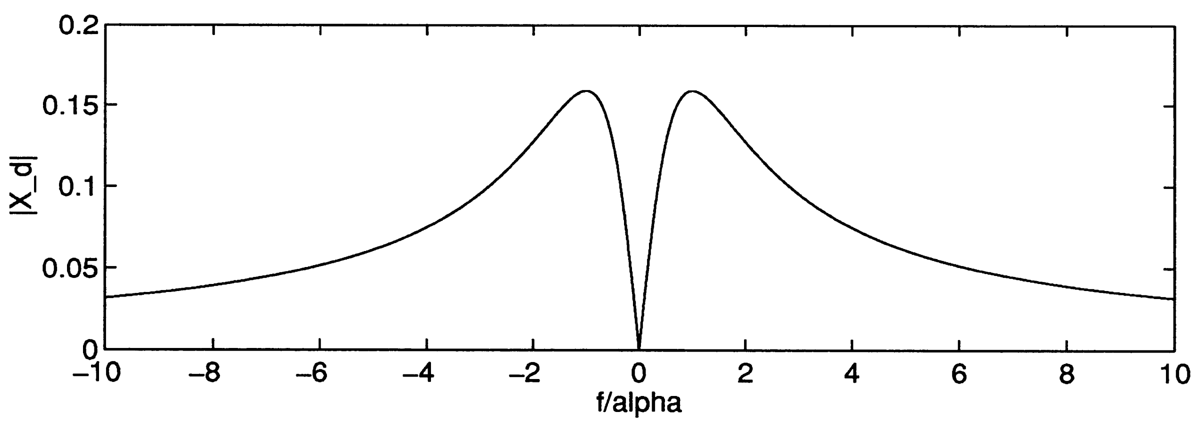
Problem 4-3

The phase function for $x_c(f)$ is identically 0. The phase function for $x_d(f)$ is $\pi/2$ for $f < 0$ and $-\pi/2$ for $f > 0$ or vice versa. The magnitude functions are given by

$$X_c(f) = \frac{2A\alpha}{\alpha^2 + (2\pi f)^2} \quad \text{and} \quad X_d(f) = \frac{4A\pi|f|}{\alpha^2 + (2\pi f)^2}$$

Plots are given below ($A = 1$):





Problem 4-4

(a) Write the integral in terms of its real and imaginary parts, where $x(t)$ is assumed real and even here:

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\, dt = \int_{-\infty}^{\infty} x(t)\cos(2\pi ft)\, dt - j\int_{-\infty}^{\infty} x(t)\sin(2\pi ft)\, dt$$

The second integral is 0 because its integrand is odd (an even times an odd function) and is integrated over symmetric limits about $f = 0$. The first integral is not 0 in general because its integrand is even, being the product of two even functions. Thus, the Fourier integral for $x(t)$ is purely real. Furthermore it is an even function of f, since this dependence comes through the cosine which is an even function. Since the integrand of the first integral is even, the integral can be evaluated by integrating from 0 to ∞ and doubling the result.

(b) Again write the integral in terms of its real and imaginary parts, where $x(t)$ is assumed real and odd. Now the first integral is 0 because its integrand is odd (an even times an odd function) and is integrated over symmetric limits about $f = 0$. The second integral is not 0 in general because its integrand is even, being the product of two odd functions. Thus, the Fourier integral for $x(t)$ is purely imaginary. Furthermore it is an odd function of f, since this dependence comes through the sine which is an odd function. Since the integrand of the second integral is even, the integral can be evaluated by integrating from 0 to ∞ and doubling the result.

Problem 4-5

(a) The integrand of the Fourier integral can be written in this case as

$$\begin{aligned} x(t)\, e^{-j2\pi ft} &= [x_R(t) + jx_I(t)]\,[\cos(2\pi ft) + j\sin(2\pi ft)] \\ &= x_R(t)\cos(2\pi ft) + x_I(t)\sin(2\pi ft) + j[x_I(t)\cos(2\pi ft) - x_R(t)\sin(2\pi ft)] \end{aligned}$$

Substitution of this into the Fourier transform integral and separating the integral into a sum of integrals gives the result of the problem statement.

(b) Under the conditions given, the second and last integrals of the problem statement are 0 since their integrands are odd and the integrals are over intervals symmetric about $t = 0$. Hence, only the first and third integrals are left. When put together as one integral, the result is that given in the problem statement.

(c) In this case, the first and third integrals are 0 since their integrands are the products of an odd function and cosine, which is even thus giving an odd integrand. When the second and fourth integrals are put together as one, the given result follows.