Chapter 2 Answers

2.1. (a) We know that

$$y_1[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] \qquad \text{(S2.1-1)}$$

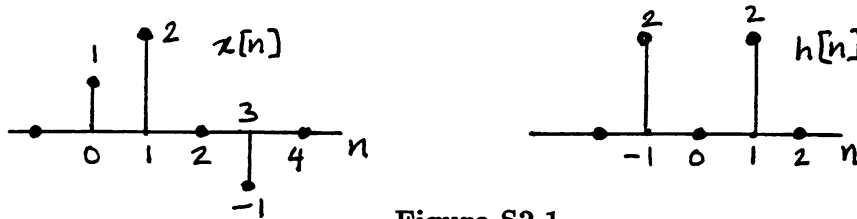The signals $x[n]$ and $h[n]$ are as shown in Figure S2.1.

Figure S2.1

From this figure, we can easily see that the above convolution sum reduces to

$$
\begin{aligned}
y_1[n] &= h[-1]x[n+1] + h[1]x[n-1] \\
&= 2x[n+1] + 2x[n-1]
\end{aligned}
$$

This gives

$$y_1[n] = 2\delta[n+1] + 4\delta[n] + 2\delta[n-1] + 2\delta[n-2] - 2\delta[n-4]$$

(b) We know that

$$y_2[n] = x[n+2] * h[n] = \sum_{k=-\infty}^{\infty} h[k]x[n+2-k]$$

Comparing with eq. (S2.1-1), we see that

$$y_2[n] = y_1[n+2]$$

(c) We may rewrite eq. (S2.1-1) as

$$y_1[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Similarly, we may write

$$y_3[n] = x[n] * h[n+2] = \sum_{k=-\infty}^{\infty} x[k]h[n+2-k]$$

Comparing this with eq. (S2.1), we see that

$$y_3[n] = y_1[n+2]$$

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2.2. Using the given definition for the signal $h[n]$, we may write

$$h[k] = \left(\frac{1}{2}\right)^{k-1} \{u[k+3] - u[k-10]\}$$

The signal $h[k]$ is non zero only in the range $-3 \le k \le 9$. From this we know that the signal $h[-k]$ is non zero only in the range $-9 \le k \le 3$. If we now shift the signal $h[-k]$ by n to the right, then the resultant signal $h[n-k]$ will be non zero in the range $(n-9) \le k \le (n+3)$. Therefore,

$$A = n - 9, \qquad B = n + 3$$

2.3. Let us define the signals

$$x_1[n] = \left(\frac{1}{2}\right)^n u[n]$$

and

$$h_1[n] = u[n].$$

We note that

$$x[n] = x_1[n-2] \qquad \text{and} \qquad h[n] = h_1[n+2]$$

Now,

$$
\begin{aligned}
y[n] &= x[n] * h[n] = x_1[n-2] * h_1[n+2] \\
&= \sum_{k=-\infty}^{\infty} x_1[k-2] h_1[n-k+2]
\end{aligned}
$$

By replacing k with $m+2$ in the abovr summation, we obtain

$$y[n] = \sum_{m=-\infty}^{\infty} x_1[m] h_1[n-m] = x_1[n] * h_1[n]$$

Using the results of Example 2.1 in the text book, we may write

$$y[n] = 2 \left[1 - \left(\frac{1}{2}\right)^{n+1}\right] u[n]$$

2.4. We know that

$$y[n] = x[n] * h[n] = \sum_{k=\infty}^{\infty} x[k] h[n-k]$$

The signals $x[n]$ and $y[n]$ are as shown in Figure S2.4. From this figure, we see that the above summation reduces to

$$y[n] = x[3]h[n-3] + x[4]h[n-4] + x[5]h[n-5] + x[6]h[n-6] + x[7]h[n-7] + x[8]h[n-8]$$

This gives

$$y[n] = \begin{cases} n-6, & 7 \le n \le 11 \\ 6, & 12 \le n \le 18 \\ 24-n, & 19 \le n \le 23 \\ 0, & \text{otherwise} \end{cases}$$
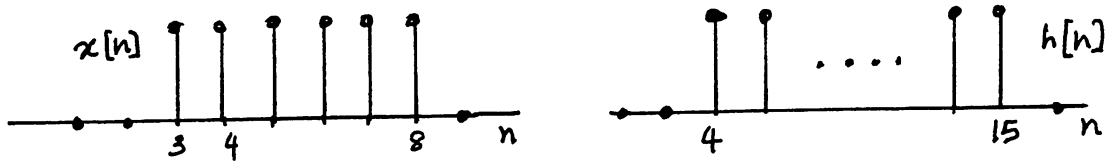
Figure S2.4

2.5. The signal $y[n]$ is

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

In this case, this summation reduces to

$$y[n] = \sum_{k=0}^{9} x[k]h[n-k] = \sum_{k=0}^{9} h[n-k]$$

From this it is clear that $y[n]$ is a summation of shifted replicas of $h[n]$. Since the last replica will begin at $n = 9$ and $h[n]$ is zero for $n > N$, $y[n]$ is zero for $n > N + 9$. Using this and the fact that $y[14] = 0$, we may conclude that N can *at most* be 4. Furthermore, since $y[4] = 5$, we can conclude that $h[n]$ has *at least* 5 non-zero points. The only value of N which satifies both these conditions is 4.

2.6. From the given information, we have:

$$
\begin{aligned}
y[n] &= x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\
&= \sum_{k=-\infty}^{\infty} (\frac{1}{3})^{-k} u[-k-1] u[n-k-1] \\
&= \sum_{k=-\infty}^{-1} (\frac{1}{3})^{-k} u[n-k-1] \\
&= \sum_{k=1}^{\infty} (\frac{1}{3})^{k} u[n+k-1]
\end{aligned}
$$

Replacing k by $p-1$,

$$y[n] = \sum_{p=0}^{\infty} (\frac{1}{3})^{p+1} u[n+p] \qquad (S2.6\text{--}1)$$

For $n \geq 0$ the above equation reduces to,

$$y[n] = \sum_{p=0}^{\infty} (\frac{1}{3})^{p+1} = \frac{1}{3} \frac{1}{1-\frac{1}{3}} = \frac{1}{2}.$$

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For $n < 0$ eq. (S2.6-1) reduces to,

$$y[n] = \sum_{p=-n}^{\infty} (\frac{1}{3})^{p+1} = (\frac{1}{3})^{-n+1} \sum_{p=0}^{\infty} (\frac{1}{3})^p$$

$$= (\frac{1}{3})^{-n+1} \frac{1}{1 - \frac{1}{3}} = (\frac{1}{3})^{-n} \frac{1}{2} = \frac{3^n}{2}$$

Therefore,

$$y[n] = \begin{cases} (3^n/2), & n < 0 \\ (1/2), & n \geq 0 \end{cases}$$

2.7. (a) Given that

$$x[n] = \delta[n - 1],$$

we see that

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]g[n - 2k] = g[n - 2] = u[n - 2] - u[n - 6]$$

(b) Given that

$$x[n] = \delta[n - 2],$$

we see that

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]g[n - 2k] = g[n - 4] = u[n - 4] - u[n - 8]$$

(c) The input to the system in part (b) is the same as the input in part (a) shifted by 1 to the right. If S is time invariant then the system output obtained in part (b) has to the be the same as the system output obtained in part (a) shifted by 1 to the right. Clealry, this is not the case. Therefore, the system is **not** LTI.

(d) If $x[n] = u[n]$, then

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]g[n - 2k]$$

$$= \sum_{k=0}^{\infty} g[n - 2k]$$

The signal $g[n - 2k]$ is plotted for $k = 0, 1, 2$ in Figure S2.7. From this figure it is clear that

$$y[n] = \begin{cases} 1, & n = 0, 1 \\ 2, & n > 1 \\ 0, & \text{otherwise} \end{cases} = 2u[n] - \delta[n] - \delta[n - 1]$$
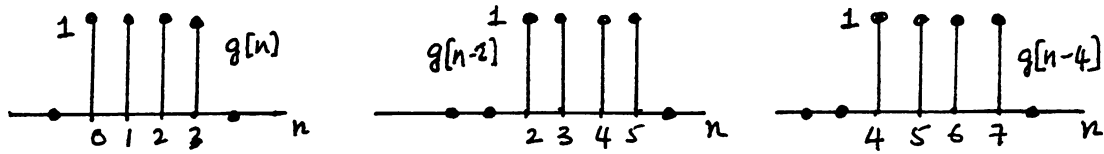
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Figure S2.7

2.8.   Using the convolution integral,

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t - \tau)d\tau.$$

Given that $h(t) = \delta(t + 2) + 2\delta(t + 1)$, the above integral reduces to

$$x(t) * y(t) = x(t + 2) + 2x(t + 1)$$

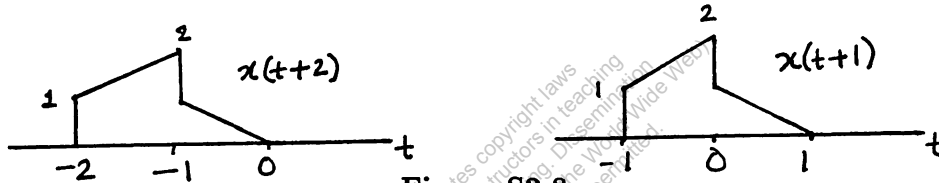The signals $x(t + 2)$ and $2x(t + 1)$ are plotted in Figure S2.8.

Figure S2.8

Using these plots, we can easily show that

$$y(t) = \begin{cases} t + 3, & -2 < t \le -1 \\ t + 4, & -1 < t \le 0 \\ 2 - 2t, & 0 < t \le 1 \\ 0, & \text{otherwise} \end{cases}$$

2.9.   Using the given definition for the signal $h(t)$, we may write

$$h(\tau) = e^{2\tau}u(-\tau + 4) + e^{-2\tau}u(\tau - 5) = \begin{cases} e^{-2\tau}, & \tau > 5 \\ e^{2\tau}, & \tau < 4 \\ 0, & 4 < \tau < 5 \end{cases}$$

Therefore,

$$h(-\tau) = \begin{cases} e^{2\tau}, & \tau < -5 \\ e^{-2\tau}, & \tau > -4 \\ 0, & -5 < \tau < -4 \end{cases}$$

If we now shift the signal $h(-\tau)$ by t to the right, then the resultant signal $h(t - \tau)$ will be

$$h(t - \tau) = \begin{cases} e^{-2(t-\tau)}, & \tau < t - 5 \\ e^{2(t-\tau)}, & \tau > t - 4 \\ 0, & (t - 5) < \tau < (t - 4) \end{cases}$$

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Therefore,
$$A = t - 5, \qquad B = t - 4.$$

2.10. From the given information, we may sketch $x(t)$ and $h(t)$ as shown in Figure S2.10.

    (a) With the aid of the plots in Figure S2.10, we can show that $y(t) = x(t) * h(t)$ is as shown in Figure S2.10.
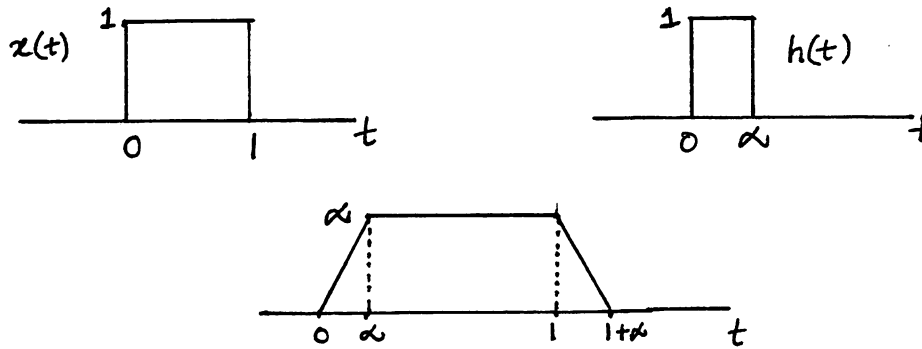

Figure S2.10

Therefore,
$$y(t) = \begin{cases} t, & 0 \leq t \leq \alpha \\ \alpha, & \alpha \leq t \leq 1 \\ 1 + \alpha - t, & 1 \leq t \leq (1 + \alpha) \\ 0, & \text{otherwise} \end{cases}$$

    (b) From the plot of $y(t)$, it is clear that $\frac{dy(t)}{dt}$ has discontinuities at 0, α, 1, and $1 + \alpha$. If we want $\frac{dy(t)}{dt}$ to have only three discontinuities, then we need to ensure that $\alpha = 1$.

2.11. **(a)** From the given information, we see that $h(t)$ is non zero only for $0 \leq t \leq \infty$. Therefore,
$$\begin{aligned} y(t) &= x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)x(t - \tau)d\tau \\ &= \int_{0}^{\infty} e^{-3\tau}(u(t - \tau - 3) - u(t - \tau - 5))d\tau \end{aligned}$$

We can easily show that $(u(t - \tau - 3) - u(t - \tau - 5))$ is non zero only in the range $(t - 5) < \tau < (t - 3)$. Therefore, for $t \leq 3$, the above integral evaluates to zero. For $3 < t \leq 5$, the above integral is
$$y(t) = \int_{0}^{t-3} e^{-3\tau}d\tau = \frac{1 - e^{-3(t-3)}}{3}$$

For $t > 5$, the integral is
$$y(t) = \int_{t-5}^{t-3} e^{-3\tau}d\tau = \frac{(1 - e^{-6})e^{-3(t-5)}}{3}$$

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Therefore, the result of this convolution may be expressed as

$$y(t) = \begin{cases} 0, & -\infty < t \le 3 \\ \frac{1-e^{-3(t-3)}}{3}, & 3 < t \le 5 \\ \frac{(1-e^{-6})e^{-3(t-5)}}{3}, & 5 < t \le \infty \end{cases}$$

(b) By differentiating $x(t)$ with respect to time we get

$$\frac{dx(t)}{dt} = \delta(t-3) - \delta(t-5)$$

Therefore,

$$g(t) = \frac{dx(t)}{dt} * h(t) = e^{-3(t-3)}u(t-3) - e^{-3(t-5)}u(t-5).$$

(c) From the result of part (a), we may compute the derivative of $y(t)$ to be

$$\frac{dy(t)}{dt} = \begin{cases} 0, & -\infty < t \le 3 \\ e^{-3(t-3)}, & 3 < t \le 5 \\ (e^{-6}-1)e^{-3(t-5)}, & 5 < t \le \infty \end{cases}$$

This is exactly equal to $g(t)$. Therefore, $g(t) = \frac{dy(t)}{dt}$.

2.12. The signal $y(t)$ may be written as

$$y(t) = \cdots + e^{-(t+6)}u(t+6) + e^{-(t+3)}u(t+3) + e^{-t}u(t) + e^{-(t-3)}u(t-3) + e^{-(t-6)}u(t-6) + \cdots$$

In the range $0 \le t < 3$, we may write $y(t)$ as

$$\begin{aligned} y(t) &= \cdots + e^{-(t+6)}u(t+6) + e^{-(t+3)}u(t+3) + e^{-t}u(t) \\ &= e^{-t} + e^{-(t+3)} + e^{-(t+6)} + \cdots \\ &= e^{-t}(1 + e^{-3} + e^{-6} + \cdots) \\ &= e^{-t}\frac{1}{1-e^{-3}} \end{aligned}$$

Therefore, $A = \frac{1}{1-e^{-3}}$.

2.13. (a) We require that

$$\left(\frac{1}{5}\right)^n u[n] - A\left(\frac{1}{5}\right)^{(n-1)} u[n-1] = \delta[n]$$

Putting $n = 1$ and solving for A gives $A = \frac{1}{5}$.

(b) From part (a), we know that

$$h[n] - \frac{1}{5}h[n-1] = \delta[n]$$

$$h[n] * \left(\delta[n] - \frac{1}{5}\delta[n-1]\right) = \delta[n]$$

From the definition of an inverse system, we may argue that

$$g[n] = \delta[n] - \frac{1}{5}\delta[n-1].$$

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2.14. (a) We first determine if $h_1(t)$ is absolutely integrable as follows

$$\int_{-\infty}^{\infty} |h_1(\tau)|d\tau = \int_0^{\infty} e^{-t}d\tau = 1$$

Therefore, $h_1(t)$ is the impulse response of a stable LTI system.

(b) We determine if $h_2(t)$ is absolutely integrable as follows

$$\int_{-\infty}^{\infty} |h_2(\tau)|d\tau = \int_0^{\infty} e^{-t}|\cos(2t)|d\tau$$

This integral is clearly finite-valued because $e^{-t}|\cos(2t)|$ is an exponentially decaying function in the range $0 \le t \le \infty$. Therefore, $h_2(t)$ is the impulse response of a stable LTI system.

2.15. (a) We determine if $h_1[n]$ is absolutely summable as follows

$$\sum_{k=-\infty}^{\infty} |h_1[k]| = \sum_{k=0}^{\infty} k|\cos(\frac{\pi}{4}k)|$$

This sum does not have a finite value because the function $k|\cos(\frac{\pi}{4}k)|$ increases as the value of k increases. Therefore, $h_1[n]$ cannot be the impulse response of a stable LTI system.

(b) We determine if $h_2[n]$ is absolutely summable as follows

$$\sum_{k=-\infty}^{\infty} |h_2[k]| = \sum_{k=-\infty}^{10} 3^k \approx 3^{11}/2$$

Therefore, $h_2[n]$ is the impulse response of a stable LTI system.

2.16. (a) **True.** This may be easily argued by noting that convolution may be viewed as the process of carrying out the superposition of a number of echos of $h[n]$. The first such echo will occur at the location of the first non zero sample of $x[n]$. In this case, the first echo will occur at N_1. The echo of $h[n]$ which occurs at $n = N_1$ will have its first non zero sample at the time location $N_1 + N_2$. Therefore, for all values of n which are lesser that $N_1 + N_2$, the output $y[n]$ is zero.

(b) **False.** Consider

$$
\begin{aligned}
y[n] &= x[n] * h[n] \\
&= \sum_{k=-\infty}^{\infty} x[k]h[n-k]
\end{aligned}
$$

From this,

$$
\begin{aligned}
y[n-1] &= \sum_{k=-\infty}^{\infty} x[k]h[n-1-k] \\
&= x[n] * h[n-1]
\end{aligned}
$$

This shows that the given statement is false.

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(c) True. Consider

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau.$$

From this,

$$
\begin{aligned}
y(-t) &= \int_{-\infty}^{\infty} x(\tau)h(-t-\tau)d\tau \\
&= \int_{-\infty}^{\infty} x(-\tau)h(-t+\tau)d\tau \\
&= x(-t) * h(-t)
\end{aligned}
$$

This shows that the given statement is true.

(d) True. This may be argued by considering

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau.$$

In Figure S2.16, we plot $x(\tau)$ and $h(t-\tau)$ under the assumptions that (1) $x(t) = 0$ for $t > T_1$ and (2) $h(t) = 0$ for $t > T_2$. Clearly, the product $x(\tau)h(t-\tau)$ is zero if
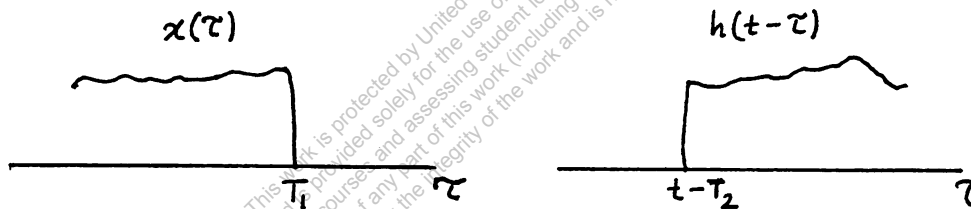

Figure S2.16

$t - T_2 > T_1$. Therefore, $y(t) = 0$ for $t > T_1 + T_2$.

2.17. (a) We know that $y(t)$ is the sum of the particular and homogeneous solutions to the given differential equation. We first determine the particular solution $y_p(t)$ by using the method specified in Example 2.14. Since we are given that the input is $x(t) = e^{(-1+3j)t}u(t)$ for $t > 0$, we hypothesize that for $t > 0$

$$y_p(t) = Ke^{(-1+3j)t}.$$

Substituting for $x(t)$ and $y(t)$ in the given differential equation,

$$(-1+3j)Ke^{(-1+3j)t} + 4Ke^{(-1+3j)t} = e^{(-1+3j)t}$$

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This gives

$$(-1+3j)K + 4K = 1, \quad \Rightarrow K = \frac{1}{3(1+j)}$$

Therefore,

$$y_p(t) = \frac{1}{3(1+j)}e^{(-1+3j)t}, \quad t > 0$$

In order to determine the homogeneous solution, we hypothesize that

$$y_h(t) = Ae^{st}$$

Since the homogeneous solution has to satisfy the following differential equation

$$\frac{dy_h(t)}{dt} + 4y_h(t) = 0,$$

we obtain

$$Ase^{st} + 4Ae^{st} = Ae^{st}(s+4) = 0.$$

This implies that $s = -4$ for any A. The overall solution to the differential equation now becomes

$$y(t) = Ae^{-4t} + \frac{1}{3(1+j)}e^{(-1+3j)t}, \quad t > 0$$

Now in order to determine the constant A, we use the fact that the system satisfies the condition of initial rest. Given that $y(0) = 0$, we may conclude that

$$A + \frac{1}{3(1+j)} = 0, \quad A = \frac{-1}{3(1+j)}$$

Therefore for $t > 0$,

$$y(t) = \frac{1}{3(1+j)}\left[-e^{-4t} + e^{(-1+3j)t}\right], \quad t > 0$$

Since the system satisfies the condition of initial rest, $y(t) = 0$ for $t < 0$. Therefore,

$$y(t) = \frac{1-j}{6}\left[-e^{-4t} + e^{(-1+3j)t}\right]u(t)$$

(b) The output will now be the real part of the answer obtained in part (a).

$$y(t) = \frac{1}{6}\left[e^{-t}\cos 3t + e^{-t}\sin 3t - e^{-4t}\right]u(t).$$

2.18. Since the system is causal, $y[n] = 0$ for $n < 1$. Now,

$$
\begin{aligned}
y[1] &= \frac{1}{4}y[0] + x[1] = 0 + 1 = 1 \\
y[2] &= \frac{1}{4}y[1] + x[2] = \frac{1}{4} + 0 = \frac{1}{4} \\
y[3] &= \frac{1}{4}y[2] + x[3] = \frac{1}{16} + 0 = \frac{1}{16} \\
&\vdots \\
y[m] &= (\frac{1}{4})^{m-1} \\
&\vdots
\end{aligned}
$$

Therefore,

$$
y[n] = (\frac{1}{4})^{n-1}u[n-1]
$$

2.19. **(a)** Consider the difference equation relating $y[n]$ and $w[n]$ for S_2:

$$
y[n] = \alpha y[n-1] + \beta w[n]
$$

From this we may write

$$
w[n] = \frac{1}{\beta}y[n] - \frac{\alpha}{\beta}y[n-1]
$$

and

$$
w[n-1] = \frac{1}{\beta}y[n-1] - \frac{\alpha}{\beta}y[n-2]
$$

Weighting the previous equation by 1/2 and subtracting from the one before, we obtain

$$
w[n] - \frac{1}{2}w[n-1] = \frac{1}{\beta}y[n] - \frac{\alpha}{\beta}y[n-1] - \frac{1}{2\beta}y[n-1] + \frac{\alpha}{2\beta}y[n-2]
$$

Substituting this in the difference equation relating $w[n]$ and $x[n]$ for S_1,

$$
\frac{1}{\beta}y[n] - \frac{\alpha}{\beta}y[n-1] - \frac{1}{2\beta}y[n-1] + \frac{\alpha}{2\beta}y[n-2] = x[n]
$$

That is,

$$
y[n] = (\alpha + \frac{1}{2})y[n-1] - \frac{\alpha}{2}y[n-2] + \beta x[n]
$$

Comparing with the given equation relating $y[n]$ and $x[n]$, we obtain

$$
\alpha = \frac{1}{4}, \qquad \beta = 1
$$

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(b) The difference equations relating the input and output of the systems S_1 and S_2 are

$$w[n] = \frac{1}{2}w[n-1] + x[n] \quad \text{and} \quad y[n] = \frac{1}{4}y[n-1] + w[n]$$

From these, we can use the method specifed in Example 2.15 to show that the impulse responses of S_1 and S_2 are

$$h_1[n] = \left(\frac{1}{2}\right)^n u[n]$$

and

$$h_2[n] = \left(\frac{1}{4}\right)^n u[n],$$

respectively. The overall impulse response of the system made up of a cascade of S_1 and S_2 will be

$$
\begin{aligned}
h[n] &= h_1[n] * h_2[n] = \sum_{k=-\infty}^{\infty} h_1[k]h_2[n-k] \\
&= \sum_{k=0}^{\infty} (\tfrac{1}{2})^k (\tfrac{1}{4})^{n-k} u[n-k] \\
&= \sum_{k=0}^{n} (\tfrac{1}{2})^k (\tfrac{1}{4})^{n-k} = \sum_{k=0}^{n} (\tfrac{1}{2})^{2(n-k)} \\
&= [2(\tfrac{1}{2})^n - (\tfrac{1}{4})^n]u[n]
\end{aligned}
$$

2.20. **(a)**

$$\int_{-\infty}^{\infty} u_0(t)\cos(t)dt = \int_{-\infty}^{\infty} \delta(t)dt = 1$$

(b)

$$\int_{0}^{5} \sin(2\pi t)\delta(t+3)dt = \sin(6\pi) = 0$$

(c) In order to evaluate the integral

$$\int_{-5}^{5} u_1(1-\tau)\cos(2\pi\tau)d\tau,$$

consider the signal

$$x(t) = \cos(2\pi t)[u(t+5) - u(t-5)].$$

We know that

$$
\begin{aligned}
\frac{dx(t)}{dt} &= u_1(t) * x(t) = \int_{-\infty}^{\infty} u_1(t-\tau)x(\tau)d\tau \\
&= \int_{-5}^{5} u_1(t-\tau)\cos(2\pi\tau)d\tau
\end{aligned}
$$

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Now,

$$\left.\frac{dx(t)}{dt}\right|_{t=1} = \int_{-5}^{5} u_1(1-\tau)\cos(2\pi\tau)d\tau$$

which is the desired integral. We now evaluate the value of the integral as

$$\left.\frac{dx(t)}{dt}\right|_{t=1} = \sin(2\pi t)|_{t=1} = 0.$$

2.21. **(a)** The desired convolution is

$$\begin{aligned}
y[n] &= x[n] * h[n] \\
&= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\
&= \beta^n \sum_{k=0}^{n} (\alpha/\beta)^k \text{ for } n \geq 0 \\
&= [\frac{\beta^{n+1} - \alpha^{n+1}}{\beta - \alpha}]u[n] \text{ for } \alpha \neq \beta.
\end{aligned}$$

(b) From (a),

$$y[n] = \alpha^n \left[\sum_{k=0}^{n} 1\right] u[n] = (n+1)\alpha^n u[n].$$

(c) For $n \leq 6$,

$$y[n] = 4^n \left\{ \sum_{k=0}^{\infty} (-\frac{1}{8})^k - \sum_{k=0}^{3} (-\frac{1}{8})^k \right\}.$$

For $n > 6$,

$$y[n] = 4^n \left\{ \sum_{k=0}^{\infty} (-\frac{1}{8})^k - \sum_{k=0}^{n-1} (-\frac{1}{8})^k \right\}.$$

Therefore,

$$y[n] = \begin{cases} (8/9)(-1/8)^4 4^n, & n \leq 6 \\ (8/9)(-1/2)^n, & n > 6 \end{cases}$$

(d) The desired convolution is

$$\begin{aligned}
y[n] &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\
&= x[0]h[n] + x[1]h[n-1] + x[2]h[n-2] + x[3]h[n-3] + x[4]h[n-4] \\
&= h[n] + h[n-1] + h[n-2] + h[n-3] + h[n-4].
\end{aligned}$$

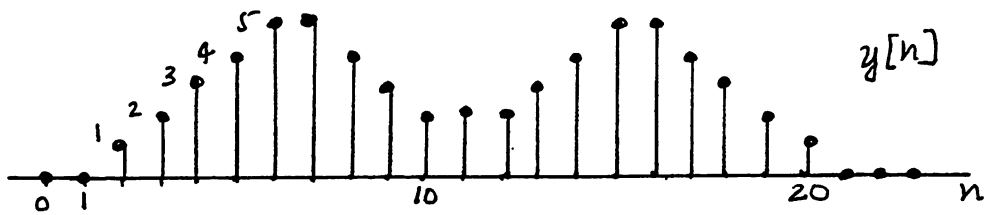This is as shown in Figure S2.21.

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Figure S2.21

2.22. **(a)** The desired convolution is

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$= \int_0^t e^{-\alpha t}e^{-\beta(t-\tau)}d\tau, \ t \geq 0$$

Then

$$y(t) = \begin{cases} \dfrac{e^{-\beta t}\{e^{-(\alpha-\beta)t}-1\}}{\beta-\alpha}u(t) & \alpha \neq \beta \\ te^{-\beta t}u(t) & \alpha = \beta \end{cases}.$$

(b) The desired convolution is

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$= \int_0^2 h(t-\tau)d\tau - \int_2^5 h(t-\tau)d\tau.$$

This may be written as

$$y(t) = \begin{cases} \displaystyle\int_0^2 e^{2(t-\tau)}d\tau - \int_2^5 e^{2(t-\tau)}d\tau, & t \leq 1 \\ \displaystyle\int_{t-1}^2 e^{2(t-\tau)}d\tau - \int_2^5 e^{2(t-\tau)}d\tau, & 1 \leq t \leq 3 \\ \displaystyle-\int_{t-1}^5 e^{2(t-\tau)}d\tau, & 3 \leq t \leq 6 \\ 0, & 6 < t \end{cases}$$

Therefore,

$$y(t) = \begin{cases} (1/2)[e^{2t} - 2e^{2(t-2)} + e^{2(t-5)}], & t \leq 1 \\ (1/2)[e^2 + e^{2(t-5)} - 2e^{2(t-2)}], & 1 \leq t \leq 3 \\ (1/2)[e^{2(t-5)} - e^2], & 3 \leq t \leq 6 \\ 0, & 6 < t \end{cases}$$

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(c) The desired convolution is

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$= \int_0^2 \sin(\pi\tau)h(t-\tau)d\tau.$$

This gives us

$$y(t) = \begin{cases} 0, & t < 1 \\ (2/\pi)[1 - \cos\{\pi(t-1)\}], & 1 < t < 3 \\ (2/\pi)[\cos\{\pi(t-3)\} - 1], & 3 < t < 5 \\ 0, & 5 < t \end{cases}$$

(d) Let

$$h(t) = h_1(t) - \frac{1}{3}\delta(t-2),$$

where

$$h_1(t) = \begin{cases} 4/3, & 0 \le t \le 1 \\ 0, & \text{otherwise} \end{cases}.$$

Now,

$$y(t) = h(t) * x(t) = [h_1(t) * x(t)] - \frac{1}{3}x(t-2).$$

We have

$$h_1(t) * x(t) = \int_{t-1}^t \frac{4}{3}(a\tau + b)d\tau = \frac{4}{3}[\frac{1}{2}at^2 - \frac{1}{2}a(t-1)^2 + bt - b(t-1)].$$

Therefore,

$$y(t) = \frac{4}{3}[\frac{1}{2}at^2 - \frac{1}{2}a(t-1)^2 + bt - b(t-1)] - \frac{1}{3}[a(t-2) + b] = at + b = x(t).$$

(e) $x(t)$ periodic implies $y(t)$ periodic. $\therefore$ determine 1 period only. We have

$$y(t) = \begin{cases} \int_{t-1}^{-\frac{1}{2}}(t-\tau-1)d\tau + \int_{-\frac{1}{2}}^t (1-t+\tau)d\tau = \frac{1}{4} + t - t^2, & -\frac{1}{2} < t < \frac{1}{2} \\ \int_{t-1}^{\frac{1}{2}}(1-t+\tau)d\tau + \int_{\frac{1}{2}}^t (t-1-\tau)d\tau = t^2 - 3t + 7/4, & \frac{1}{2} < t < \frac{3}{2} \end{cases}.$$

The period of $y(t)$ is 2.

2.23. $y(t)$ is sketched in Figure S2.23 for the different values of T.

2.24. (a) We are given that $h_2[n] = \delta[n] + \delta[n-1]$. Therefore,

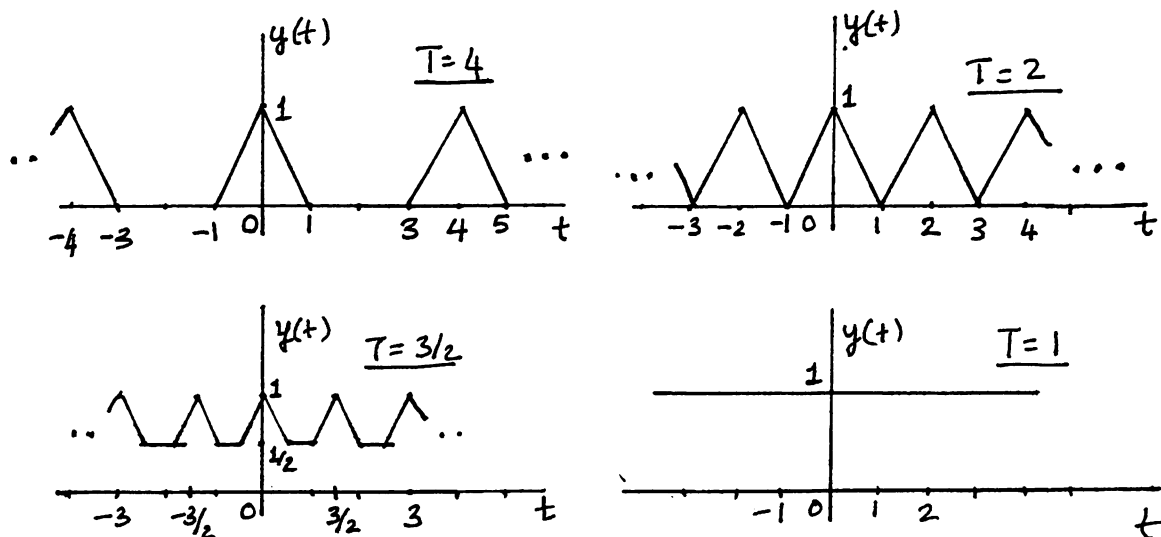$$h_2[n] * h_2[n] = \delta[n] + 2\delta[n-1] + \delta[n-2].$$

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Figure S2.23

Since

$$h[n] = h_1[n] * [h_2[n] * h_2[n]],$$

we get

$$h[n] = h_1[n] + 2h_1[n-1] + h_1[n-2].$$

Therefore,

$$
\begin{array}{lcl}
h[0] = h_1[0] & \Rightarrow & h_1[0] = 1, \\
h[1] = h_1[1] + 2h_1[0] & \Rightarrow & h_1[1] = 3, \\
h[2] = h_1[2] + 2h_1[1] + h_1[0] & \Rightarrow & h_1[2] = 3, \\
h[3] = h_1[3] + 2h_1[2] + h_1[1] & \Rightarrow & h_1[3] = 2, \\
h[4] = h_1[4] + 2h_1[3] + h_1[2] & \Rightarrow & h_1[4] = 1 \\
h[5] = h_1[5] + 2h_1[4] + h_1[3] & \Rightarrow & h_1[5] = 0.
\end{array}
$$

$h_1[n] = 0$ for $n < 0$ and $n \geq 5$.

(b) In this case,

$$y[n] = x[n] * h[n] = h[n] - h[n-1].$$

2.25. **(a)** We may write $x[n]$ as

$$x[n] = \left(\frac{1}{3}\right)^{|n|}.$$

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Now, the desired convolution is

$$
\begin{aligned}
y[n] &= h[n] * x[n] \\
&= \sum_{k=-\infty}^{-1} (1/3)^{-k}(1/4)^{n-k}u[n-k+3] + \sum_{k=0}^{\infty}(1/3)^{k}(1/4)^{n-k}u[n-k+3] \\
&= (1/12)\sum_{k=0}^{\infty}(1/3)^{k}(1/4)^{n+k}u[n+k+4] + \sum_{k=0}^{\infty}(1/3)^{k}(1/4)^{n-k}u[n-k+3]
\end{aligned}
$$

By consider each summation in the above equation separately, we may show that

$$
y[n] = \begin{cases} (12^4/11)3^n, & n < -4 \\ (1/11)4^4, & n = -4 \\ (1/4)^n(1/11) + -3(1/4)^n + 3(256)(1/3)^n, & n \geq -3 \end{cases}.
$$

(b) Now consider the convolution

$$
y_1[n] = [(1/3)^n u[n]] * [(1/4)^n u[n+3]].
$$

We may show that

$$
y_1[n] = \begin{cases} 0, & n < -3 \\ -3(1/4)^n + 3(256)(1/3)^n, & n \geq -3 \end{cases}.
$$

Also, consider the convolution

$$
y_2[n] = [(3)^n u[-n-1]] * [(1/4)^n u[n+3]].
$$

We may show that

$$
y_2[n] = \begin{cases} (12^4/11)3^n, & n < -4 \\ (1/4)^n(1/11), & n \geq -3 \end{cases}.
$$

Clearly, $y_1[n] + y_2[n] = y[n]$ obtained in the previous part.

2.26. (a) We have

$$
\begin{aligned}
y_1[n] = x_1[n] * x_2[n] &= \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k] \\
&= \sum_{k=0}^{\infty}(0.5)^k u[n+3-k].
\end{aligned}
$$

This evaluates to

$$
y_1[n] = x_1[n] * x_2[n] = \begin{cases} 2\left\{1 - (1/2)^{n+4}\right\}, & n \geq -3 \\ 0, & \text{otherwise} \end{cases}.
$$

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(b) Now,
$$y[n] = x_3[n] * y_1[n] = y_1[n] - y_1[n-1].$$

Therefore,

$$y[n] = \begin{cases} 2\left\{1 - (1/2)^{n+3}\right\} + 2\left\{1 - (1/2)^{n+4}\right\} = (1/2)^{n+3}, & n \geq -2 \\ 1, & n = -3 \\ 0, & \text{otherwise} \end{cases}.$$

Therefore, $y[n] = (1/2)^{n+3}u[n+3]$.

(c) We have
$$y_2[n] = x_2[n] * x_3[n] = u[n+3] - u[n+2] = \delta[n+3].$$

(d) From the result of part (c), we get

$$y[n] = y_2[n] * x_1[n] = x_1[n+3] = (1/2)^{n+3}u[n+3].$$

2.27. The proof is as follows.

$$\begin{aligned} A_y &= \int_{-\infty}^{\infty} y(t)dt \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau dt \\ &= \int_{-\infty}^{\infty} x(\tau)\int_{-\infty}^{\infty} h(t-\tau)dt d\tau \\ &= \int_{-\infty}^{\infty} x(\tau)A_h d\tau \\ &= A_x A_h \end{aligned}$$

2.28. **(a)** Causal because $h[n] = 0$ for $n < 0$. Stable because $\sum_{n=0}^{\infty}(\frac{1}{5})^n = 5/4 < \infty$.

(b) Not causal because $h[n] \neq 0$ for $n < 0$. Stable because $\sum_{n=-2}^{\infty}(0.8)^n = 5 < \infty$.

(c) Anti-causal because $h[n] = 0$ for $n > 0$. Unstable because $\sum_{n=-\infty}^{0}(1/2)^n = \infty$.

(d) Not causal because $h[n] \neq 0$ for $n < 0$. Stable because $\sum_{n=-\infty}^{3} 5^n = \frac{625}{4} < \infty$

(e) Causal because $h[n] = 0$ for $n < 0$. Unstable because the second term becomes infinite as $n \to \infty$.

(f) Not causal because $h[n] \neq 0$ for $n < 0$. Stable because $\sum_{n-\infty}^{\infty}|h[n]| = 305/3 < \infty$

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(g) Causal because $h[n] = 0$ for $n < 0$. Stable because $\displaystyle\sum_{n=-\infty}^{\infty} |h[n]| = 1 < \infty$.

2.29. **(a)** Causal because $h(t) = 0$ for $t < 0$. Stable because $\displaystyle\int_{-\infty}^{\infty} |h(t)|dt = e^{-8}/4 < \infty$.

(b) Not causal because $h(t) \neq 0$ for $t < 0$. Unstable because $\displaystyle\int_{-\infty}^{\infty} |h(t)| = \infty$.

(c) Not causal because $h(t) \neq 0$ for $t < 0$. a Stable because $\displaystyle\int_{-\infty}^{\infty} |h(t)|dt = e^{100}/2 < \infty$.

(d) Not causal because $h(t) \neq 0$ for $t < 0$. Stable because $\displaystyle\int_{-\infty}^{\infty} |h(t)|dt = e^{-2}/2 < \infty$.

(e) Not causal because $h(t) \neq 0$ for $t < 0$. Stable because $\displaystyle\int_{-\infty}^{\infty} |h(t)|dt = 1/3 < \infty$.

(f) Causal because $h(t) = 0$ for $t < 0$. Stable because $\displaystyle\int_{-\infty}^{\infty} |h(t)|dt = 1 < \infty$.

(g) Causal because $h(t) = 0$ for $t < 0$. Unstable because $\displaystyle\int_{-\infty}^{\infty} |h(t)|dt = \infty$.

2.30. We need to find the output of the system when the input is $x[n] = \delta[n]$. Since we are asked to assume initial rest, we may conclude that $y[n] = 0$ for $n < 0$. Now,

$$y[n] = x[n] - 2y[n-1].$$

Therefore,

$$y[0] = x[0] - 2y[-1] = 1, \quad y[1] = x[1] - 2y[0] = -2, \quad y[2] = x[2] + 2y[2] = -4$$

and so on. In closed form,

$$y[n] = (-2)^n u[n].$$

This is the impulse response of the system.

2.31. Initial rest implies that $y[n] = 0$ for $n < -2$. Now

$$y[n] = x[n] + 2x[n-2] - 2y[n-1].$$

Therefore,

$$y[-2] = 1, \quad y[-1] = 0, \quad y[0] = 5, \quad y[1] = -$$

$y[4] = 56, y[5] = -110, \quad y[n] = -110(-2)^{n-5} \quad \text{for } n \geq 5$.

2.32. **(a)** If $y_h[n] = A(1/2)^n$, then we need to verify

$$A\left(\frac{1}{2}\right)^n - \frac{1}{2}A\left(\frac{1}{2}\right)^{n-1} = 0.$$

Clearly this is true.

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(b) We now require that for $n \geq 0$

$$B \left(\frac{1}{3}\right)^n - \frac{1}{2} B \left(\frac{1}{3}\right)^{n-1} = \left(\frac{1}{3}\right)^n.$$

Therefore, $B = -2$.

(c) From eq. (P2.32-1), we know that $y[0] = x[0] + (1/2)y[-1] = x[0] = 1$. Now we also have

$$y[0] = A + B \quad \Rightarrow \quad A = 1 - B = 3.$$

2.33. (a) (i) From Example 2.14, we know that

$$y_1(t) = \left[\frac{1}{5} e^{3t} - \frac{1}{5} e^{-2t}\right] u(t).$$

(ii) We solve this along the lines of Example 2.14. First assume that $y_p(t)$ is of the form Ke^{2t} for $t > 0$. Then using eq. (P2.33-1), we get for $t > 0$

$$2Ke^{2t} + 2Ke^{2t} = e^{2t} \quad \Rightarrow \quad K = \frac{1}{4}.$$

We now know that $y_p(t) = \frac{1}{4} e^{2t}$ for $t > 0$. We may hypothesize the homogeneous solution to be of the form

$$y_h(t) = Ae^{-2t}.$$

Therefore,

$$y_2(t) = Ae^{-2t} + \frac{1}{4} e^{2t}, \quad \text{for } t > 0.$$

Assuming initial rest, we can conclude that $y_2(t) = 0$ for $t \leq 0$. Therefore,

$$y_2(0) = 0 = A + \frac{1}{4} \quad \Rightarrow \quad A = -\frac{1}{4}.$$

Then,

$$y_2(t) = \left[-\frac{1}{4} e^{2t} + \frac{1}{4} e^{-2t}\right] u(t).$$

(iii) Let the input be $x_3(t) = \alpha e^{3t} u(t) + \beta e^{2t} u(t)$. Assume that the particular solution $y_p(t)$ is of the form

$$y_p(t) = K_1 \alpha e^{3t} + K_2 \beta e^{2t}$$

for $t > 0$. Using eq. (P2.33-1), we get

$$3K_1 \alpha e^{3t} + 2K_2 \beta e^{2t} + 2K_1 \alpha e^{3t} + 2K_2 \beta e^{2t} = \alpha^{3t} + \beta e^{2t}.$$

Equating the coefficients of e^{3t} and e^{2t} on both sides, we get

$$K_1 = \frac{1}{5} \quad \text{and} \quad K_2 = \frac{1}{4}.$$

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Now hypothesizing that $y_h(t) = Ae^{-2t}$, we get

$$y_3(t) = \frac{1}{5}\alpha e^{3t} + \frac{1}{4}\beta e^{2t} + Ae^{-2t}$$

for $t > 0$. Assuming initial rest,

$$y_3(0) = 0 = A + \alpha/5 + \beta/4 \quad \Rightarrow \quad A = -\left(\frac{\alpha}{5} + \frac{\beta}{4}\right).$$

Therefore,

$$y_3(t) = \left\{\frac{1}{5}\alpha e^{3t} + \frac{1}{4}\beta e^{2t} - \left(\frac{\alpha}{5} + \frac{\beta}{4}\right)e^{-2t}\right\}u(t).$$

Clearly, $y_3(t) = \alpha y_1(t) + \beta y_2(t)$.

(iv) For the input-output pair $x_1(t)$ and $y_1(t)$, we may use eq. (P2.33-1) and the initial rest condition to write

$$\frac{dy_1(t)}{dt} + 2y_1(t) = x_1(t), \qquad y_1(t) = 0 \text{ for } t < t_1.$$
$$(S2.33-1)$$

For the input-output pair $x_2(t)$ and $y_2(t)$, we may use eq. (P2.33-1) and the initial rest condition to write

$$\frac{dy_2(t)}{dt} + 2y_2(t) = x_2(t), \qquad y_2(t) = 0 \text{ for } t < t_2.$$
$$(S2.33-2)$$

Scaling eq. (S2.33-1) by α and eq. (S2.33-2) by β and summing, we get

$$\frac{d}{dt}\{\alpha y_1(t) + \beta y_2(t)\} + 2\{\alpha y_1(t) + \beta y_2(t)\} = \alpha x_1(t) + \beta x_2(t),$$

and

$$y_1(t) + y_2(t) = 0 \text{ for } t < \min(t_1, t_2).$$

By inspection, it is clear that the output is $y_3(t) = \alpha y_1(t) + \beta y_2(t)$ when the input is $x_3(t) = \alpha x_1(t) + \beta x_2(t)$. Furthermore, $y_3(t) = 0$ for $t < t_3$, where t_3 denotes the time until which $x_3(t) = 0$.

(b) (i) Using the result of (a-ii), we may write

$$y_1(t) = \frac{K}{4}\left[e^{2t} - e^{-2t}\right]u(t).$$

(ii) We solve this along the lines of Example 2.14. First assume that $y_p(t)$ is of the form $KYe^{2(t-T)}$ for $t > T$. Then using eq. (P2.33-1), we get for $t > T$

$$2Ke^{2(t-T)} + 2Ke^{2(t-T)} = e^{2t} \quad \Rightarrow \quad K = \frac{1}{4}.$$

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We now know that $y_p(t) = \frac{K}{4}e^{2(t-T)}$ for $t > T$. We may hypothesize the homogeneous solution to be of the form

$$y_h(t) = Ae^{-2t}.$$

Therefore,

$$y_2(t) = Ae^{-2t} + \frac{K}{4}e^{2(t-T)}, \quad \text{for } t > T.$$

Assuming initial rest, we can conclude that $y_2(t) = 0$ for $t \leq T$. Therefore,

$$y_2(T) = 0 = Ae^{-2T} + \frac{K}{4} \quad \Rightarrow \quad A = -\frac{K}{4}e^{2T}.$$

Then,

$$y_2(t) = \left[-\frac{K}{4}e^{-2(t-T)} + \frac{K}{4}e^{2(t-T)} \right] u(t - T).$$

Clearly, $y_2(t) = y_1(t - T)$.

(iii) Consider the input-output pair $x_1(t) \rightarrow y_1(t)$ where $x_1(t) = 0$ for $t < t_0$. Note that

$$\frac{dy_1(t)}{dt} + 2y_1(t) = x_1(t), \quad y_1(t) = 0, \text{ for } t < t_0.$$

Since the derivative is a time-invariant operation, we may now write

$$\frac{dy_1(t - T)}{dt} + 2y_1(t - T) = x_1(t - T), \quad y_1(t) = 0, \text{ for } t < t_0.$$

This suggests that if the input is a signal of the form $x_2(t) = x_1(t - T)$, then the output is a signal of the form $y_2(t) = y_1(t - T)$. Also, note that the new output $y_2(t)$ will be zero for $t < t_0 + T$. This supports time-invariance since $x_2(t)$ is zero for $t < t_0 + T$. Therefore, we may conclude that the system is time-invariant.

2.34. (a) Consider $x_1(t) \xrightarrow{S} y_1(t)$ and $x_2(t) \xrightarrow{S} y_2(t)$. We know that $y_1(1) = y_2(1) = 1$. Now consider a third input to the system which is $x_3(t) = x_1(t) + x_2(t)$. Let the corresponding output be $y_3(t)$. Now, note that $y_3(1) = 1 \neq y_1(1) + y_2(1)$. Therefore, the system is not linear. A specific example follows.

Consider an input signal $x_1(t) = e^{2t}u(t)$. From Problem 2.33(a-ii), we know that the corresponding output for $t > 0$ is

$$y_1(t) = \frac{1}{4}e^{2t} + Ae^{-2t}.$$

Using the fact that $y_1(1) = 1$, we get for $t > 0$

$$y_1(t) = \frac{1}{4}e^{2t} + \left(1 - \frac{e}{4}\right)e^{-2(t-1)}.$$

Now, consider a second signal $x_2(t) = 0$. Then, the corresponding output is

$$y_2(t) = Be^{-2t}$$

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for $t > 0$. Using the fact that $y_2(1) = 1$, we get for $t > 0$

$$y_2(t) = e^{-2(t-1)}.$$

Now consider a third signal $x_3(t) = x_1(t) + x_2(t) = x_1(t)$. Note that the output will still be $y_3(t) = y_1(t)$ for $t > 0$. Clearly, $y_3(t) \neq y_1(t) + y_2(t)$ for $t > 0$. Therefore, the system is not linear.

(b) Again consider an input signal $x_1(t) = e^{2t}u(t)$. From part (a), we know that the corresponding output for $t > 0$ with $y_1(1) = 1$ is

$$y_1(t) = \frac{1}{4}e^{2t} + \left(1 - \frac{e}{4}\right)e^{-2(t-1)}.$$

Now, consider an input signal of the form $x_2(t) = x_1(t - T) = e^{2(t-T)}u(t - T)$. Then for $t > T$,

$$y_2(t) = \frac{1}{4}e^{2(t-T)} + Ae^{-2t}.$$

Using the fact that $y_2(1) = 1$ and also assuming that $T < 1$, , we get for $t > T$

$$y_2(t) = \frac{1}{4}e^{2(t-T)} + \left(1 - \frac{1}{4}e^{2(1-T)}\right)e^{-2(t-1)}.$$

Now note that $y_2(t) \neq y_1(t - T)$ for $t > T$. Therefore, the system is not time invariant.

(c) In order to show that the system is incrementally linear with the auxiliary condition specified as $y(1) = 1$, we need to first show that the system is linear with the auxiliary condition specified as $y(1) = 0$.

For an input-output pair $x_1(t)$ and $y_1(t)$, we may use eq. (P2.33-1) and the fact that $y_1(1) = 0$ to write

$$\frac{dy_1(t)}{dt} + 2y_1(t) = x_1(t), \qquad y_1(1) = 0. \tag{S2.34-1}$$

For an input-output pair $x_2(t)$ and $y_2(t)$, we may use eq. (P2.33-1) and the initial rest condition to write

$$\frac{dy_2(t)}{dt} + 2y_2(t) = x_2(t), \qquad y_2(1) = 0. \tag{S2.34-2}$$

Scaling eq. (S2.34-1) by α and eq. (S2.34-2) by β and summing, we get

$$\frac{d}{dt}\{\alpha y_1(t) + \beta y_2(t)\} + 2\{\alpha y_1(t) + \beta y_2(t)\} = \alpha x_1(t) + \beta x_2(t)$$

and

$$y_3(1) = y_1(1) + y_2(1) = 0.$$

By inspection, it is clear that the output is $y_3(t) = \alpha y_1(t) + \beta y_2(t)$ when the input is $x_3(t) = \alpha x_1(t) + \beta x_2(t)$. Furthermore, $y_3(1) = 0 = y_1(1) + y_2(1)$. Therefore, the system is linear.

Therefore, the overall system may be treated as the cascade of a linear system with an adder which adds the response of the system to the auxiliary conditions alone.

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(d) In the previous part, we showed that the system is linear when $y(1) = 0$. In order to show that the system is not time-invariant, consider an input of the form $x_1(t) = e^{2t}u(t)$. From part (a), we know that the corresponding output will be

$$y_1(t) = \frac{1}{4}e^{2t} + Ae^{-2t}.$$

Using the fact that $y_1(1) = 0$, we get for $t > 0$

$$y_1(t) = \frac{1}{4}e^{2t} - \frac{1}{4}e^{-2(t-2)}.$$

Now consider an input of the form $x_2(t) = x_1(t - 1/2)$. Note that $y_2(1) = 0$. Clearly, $y_2(1) \neq y_1(1 - 1/2) = (1/4)(e - e^3)$. Therefore, $y_2(t) \neq y_1(t - 1/2)$ for all t. This implies that the system is not time invariant.

(e) A proof which is very similar to the proof for linearity used in part (c) may be used here. We may show that the system is not time invariant by using the method outlined in part (d).

2.35. (a) Since the system is linear, the response $y_1(t) = 0$ for all t.

(b) Now let us find the output $y_2(t)$ when the input is $x_2(t)$. The particular solution is of the form

$$y_p(t) = Y, \quad t > -1.$$

Substituting in eq. (P2.33-1), we get

$$2Y = 1.$$

Now, including the homogeneous solution which is of the form $y_h(t) = Ae^{-2t}$, we get the overall solution:

$$y_2(t) = Ae^{-2t} + \frac{1}{2}, \quad t > -1.$$

Since $y(0) = 0$, we get

$$y_2(t) = -\frac{1}{2}e^{-2t} + \frac{1}{2}, \quad t > -1. \tag{S2.35-1}$$

For $t < -1$, we note that $x_2(t) = 0$. Thus the particular solution is zero in this range and

$$y_2(t) = Be^{-2t}, \quad t < -1. \tag{S2.35-2}$$

Since the two pieces of the solution for $y_2(t)$ in eqs. (S2.35-1) and (S2.35-2) must match at $t = -1$, we can determine B from the equation

$$\frac{1}{2} - \frac{1}{2}e^2 = Be^2$$

which yields

$$y_2(t) = \left(\frac{1}{2} - \frac{1}{2}e^2\right)e^{-2(t+1)}, \quad t < -1.$$

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Now note that since $x_1(t) = x_2(t)$ for $t < -1$, it must be true that for a causal system $y_1(t) = y_2(t)$ for $t < -1$. However the results of parts (a) and (b) show that this is not true. Therefore, the system is not causal.

2.36. (a) Consider an input $x_1[n]$ such that $x_1[n] = 0$ for $n < n_1$. The corresponding output will be

$$y_1[n] = \frac{1}{2}y_1[n-1] + x_1[n], \qquad y_1[n] = 0 \text{ for } n < n_1.$$
$$\text{(S2.36-1)}$$

Also, consider another input $x_2[n]$ such that $x_2[n] = 0$ for $n < n_2$. The corresponding output will be

$$y_2[n] = \frac{1}{2}y_2[n-1] + x_2[n], \qquad y_2[n] = 0 \text{ for } n < n_2.$$
$$\text{(S2.36-2)}$$

Scaling eq. (S2.36-1) by α and eq. (S2.36-2) by β and summing, we get

$$\alpha y_1[n] + \beta y_2[n] = \frac{\alpha}{2}y_1[n-1] + \frac{\beta}{2}y_2[n-1] + \alpha x_1[n] + \beta x_2[n].$$

By inspection, it is clear that the output is $y_3[n] = \alpha y_1[n] + \beta y_2[n]$ when the input is $x_3[n] = \alpha x_1[n] + \beta x_2[n]$. Furthermore, $y_3(1) = 0 = y_1(1) + y_2(1)$. Therefore, the system is linear.

(b) Let us consider two inputs

$$x_1[n] = 0, \qquad \text{for all } n,$$

and

$$x_2[n] = \begin{cases} 0, & n < -1 \\ 1, & n \geq -1 \end{cases}.$$

Since the system is linear, the response to $x_1[n]$ is $y_1[n] = 0$ for all n. Now let us find the output $y_2[n]$ when the input is $x_2[n]$. Since $y_2[0] = 0$,

$$y_2[1] = (1/2)0 + 0 = 0, \qquad y_2[2] = (1/2)0 + 0 = 0, \qquad \cdots.$$

Therefore, $y_2[n] = 0$ for $n \geq 0$. Now, for $n < 0$, note that

$$y_2[0] = (1/2)y_2[-1] + x[0].$$

Therefore, $y_2[-1] = -2$. Proceeding similarly, we get $y_2[-2] = -4$, $y_2[-3] = -8$, and so on. Therefore, $y_2[n] = -(1/2)^n u[-n-1]$.

Now note that since $x_1[n] = x_2[n]$ for $n < 0$, it must be true that for a causal system $y_1[n] = y_2[n]$ for $n < 0$. However, the results obtained above show that this is not true. Therefore, the system is not causal.

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2.37. Let us consider two inputs
$$x_1(t) = 0, \quad \text{for all } t$$
and
$$x_2(t) = e^t[u(t) - u(t-1)].$$
Since the system is linear, the response $y_1(t) = 0$ for all t.

Now let us find the output $y_2(t)$ when the input is $x_2(t)$. The particular solution is of the form
$$y_p(t) = Ye^t, \quad 0 < t < 1.$$
Substituting in eq. (P2.33-1), we get
$$3Y = 1.$$

Now, including the homogeneous solution which is of the form $y_h(t) = Ae^{-2t}$, we get the overall solution:
$$y_2(t) = Ae^{-2t} + \frac{1}{3}e^t, \quad 0 < t < 1.$$
Assuming final rest, we have $y(1) = 0$. Using this we get $A = -e^3/3$. Therefore,
$$y_2(t) = -\frac{1}{3}e^{-2t+3} + \frac{1}{3}e^t, \quad 0 < t < 1. \tag{S2.37-1}$$

For $t < 0$, we note that $x_2(t) = 0$. Thus the particular solution is zero in this range and
$$y_2(t) = Be^{-2t}, \quad t < 0. \tag{S2.37-2}$$

Since the two pieces of the solution for $y_2(t)$ in eqs. (S2.37-1) and (S2.37-2) must match at $t = 0$, we can determine B from the equation
$$\frac{1}{3} - \frac{1}{3}e^3 = B$$
which yields
$$y_2(t) = \left(\frac{1}{3} - \frac{1}{3}e^3\right)e^{-2t}, \quad t < 0.$$

Now note that since $x_1(t) = x_2(t)$ for $t < 0$, it must be true that for a causal system $y_1(t) = y_2(t)$ for $t < 0$. However, the results of obtained above show that this is not true. Therefore, the system is not causal.

2.38. The block diagrams are as shown in Figure S2.38.

2.39. The block diagrams are as shown in Figure S2.39.

2.40. (a) Note that
$$y(t) = \int_{-\infty}^{t} e^{-(t-\tau)}x(\tau-2)d\tau = \int_{-\infty}^{t-2} e^{-(t-2-\tau')}x(\tau')d\tau'.$$
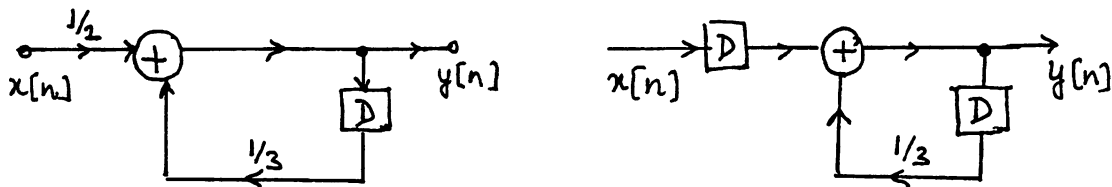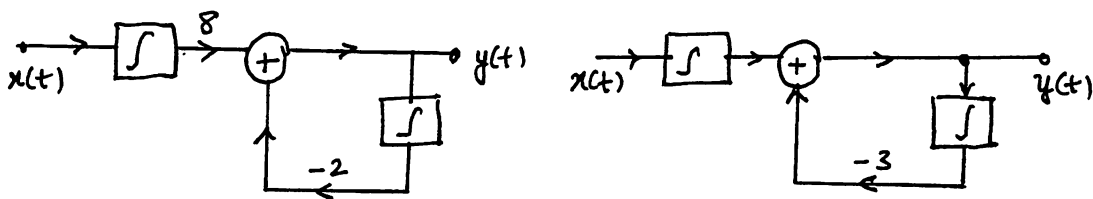Therefore,
$$h(t) = e^{-(t-2)}u(t-2).$$

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Figure S2.38

Figure S2.39

(b) We have

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$
$$= \int_{2}^{\infty} e^{-(\tau-2)}[u(t-\tau+1) - u(t-\tau-2)]$$

$h(\tau)$ and $x(t-\tau)$ are as shown in the figure below.

Using this figure, we may write

$$y(t) = \begin{cases} 0, & t < 1 \\ \int_{2}^{t+1} e^{-(\tau-2)}d\tau = 1 - e^{-(t-1)}, & 1 < t < 4 \\ \int_{t-2}^{t+1} e^{-(\tau-2)}d\tau = e^{-(t-4)}[1-e^{-3}], & t > 4 \end{cases}$$

2.41. **(a)** We may write

$$\begin{aligned} g[n] &= x[n] - \alpha x[n-1] \\ &= \alpha^n u[n] - \alpha^n u[n-1] \\ &= \delta[n]. \end{aligned}$$
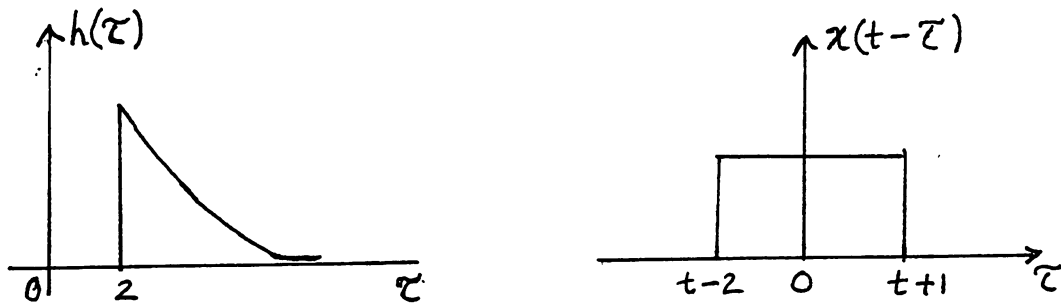
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Figure S2.40

(b) Note that $g[n] = x[n] * \{\delta[n] - \alpha\delta[n-1]\}$. Therefore, from part (a), we know that $x[n] * \{\delta[n] - \alpha\delta[n-1]\} = \delta[n]$. Using this we may write

$$x[n] * \{\delta[n-1] - \alpha\delta[n-2]\} = \delta[n-1],$$
$$x[n] * \{\delta[n+1] - \alpha\delta[n]\} = \delta[n+1],$$
$$x[n] * \{\delta[n+2] - \alpha\delta[n+1]\} = \delta[n+2].$$

Now note that

$$x[n] * h[n] = 4\delta[n+2] + 2\delta[n+1] + \delta[n] + \frac{1}{2}\delta[n-1].$$

Therefore,

$$
\begin{aligned}
x[n] * h[n] = \ & 4x[n] * \{\delta[n+2] - \alpha\delta[n+1]\} \\
& + \ 2x[n] * \{\delta[n+1] - \alpha\delta[n]\} \\
& + \ x[n] * \{\delta[n] - \alpha\delta[n-1]\} \\
& + \ (1/2)x[n] * \{\delta[n-1] - \alpha\delta[n-2]\}
\end{aligned}
$$

This may be written as

$$
\begin{aligned}
x[n] * h[n] = x[n] \ * \ & \{4\delta[n+2] - 4\alpha\delta[n+1] + 2\delta[n+1] \\
& - \ 2\alpha\delta[n] + \delta[n] - \alpha\delta[n-1] \\
& + \ (1/2)\delta[n-1] - (1/2)\delta[n-2]
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
h[n] = \ & 4\delta[n+2] + (2-4\alpha)\delta[n+1] + (1-2\alpha)\delta[n] \\
& + \ (1/2 - \alpha)\delta[n-1] - (1/2)\delta[n-2]
\end{aligned}
$$

2.42. We have

$$y(t) = x(t) * h(t) = \int_{-0.5}^{0.5} e^{j\omega_0(t-\tau)}d\tau.$$

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Therefore,

$$y(0) = \int_{-0.5}^{0.5} e^{-j\omega_0 \tau} d\tau = \frac{2}{\omega_0} \sin(\omega_0/2).$$

(a) If $\omega_0 = 2\pi$, then $y(0) = 0$.

(b) Clearly, our answer to part (a) is not unique. Any $\omega_0 = 2k\pi$, $k \in \mathcal{I}$ and $k \neq 0$ will suffice.

2.43. (a) We first have

$$
\begin{aligned}
[x(t) * h(t)] * g(t) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau) h(\sigma' - \tau) g(t - \sigma') d\tau d\sigma' \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau) h(\sigma) g(t - \sigma - \tau) d\tau d\sigma
\end{aligned}
$$

Also,

$$
\begin{aligned}
x(t) * [h(t) * g(t)] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(t - \sigma') h(\tau) g(\sigma' - \tau) d\sigma' d\tau \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\sigma) h(\tau) g(t - \tau - \sigma) d\tau d\sigma \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau) h(\sigma) g(t - \sigma - \tau) d\tau d\sigma
\end{aligned}
$$

The equality is proved.

(b) (i) We first have

$$w[n] = u[n] * h_1[n] = \sum_{k=0}^{n} \left(-\frac{1}{2}\right)^k = \frac{2}{3}\left[1 - (-\frac{1}{2})^{n+1}\right] u[n].$$

Now,

$$y[n] = w[n] * h_2[n] = (n+1)u[n].$$

(ii) We first have

$$g[n] = h_1[n] * h_2[n] = \sum_{k=0}^{n} \left(-\frac{1}{2}\right)^k + \frac{1}{2}\sum_{k=0}^{n-1}(-\frac{1}{2})^k = u[n]$$

Now,

$$y[n] = u[n] * g[n] = u[n] * u[n] = (n+1)u[n].$$

The same result was obtained in both parts (i) and (ii).

(c) Note that

$$x[n] * (h_2[n] * h_1[n]) = (x[n] * h_2[n]) * h_1[n].$$

Also note that

$$x[n] * h_2[n] = \alpha^n u[n] - \alpha^n u[n-1] = \delta[n].$$

Therefore,

$$x[n] * h_1[n] * h_2[n] = \delta[n] * \sin 8n = \sin 8n.$$

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2.44. **(a)** We have

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-T_1}^{T_1} x(\tau)h(t-\tau)d\tau.$$

Note that $h(-\tau) = 0$ for $|\tau| > T_2$. Therefore, $h(t-\tau) = 0$ for $\tau > t+T_2$ and $\tau < -T_2+t$. Therefore, the above integral evaluates to zero either if $T_1 < -T_2 + t$ or $T_2 + t < -T_1$. This implies that the convolution integral is zero if $t > |T_1 + T_2|$.

(b) **(i)** We have

$$y[n] = h[n] * x[n] = \sum_{k=N_0}^{N_1} h[k]x[n-k].$$

Note that $x[-k] \neq 0$ for $-N_3 \leq n \leq -N_2$. Therefore, $x[-k+n] \neq 0$ for $-N_3 + n \leq k \leq -N_2 + n$. Clearly, the convolution sum is not zero if $-N_3 + n \leq N_1$ **and** $-N_2 + n \geq N_0$. Therefore, $y[n]$ is nonzero for $n \leq N_1 + N_3$ and $n \geq N_0 + N_2$.

(ii) We can easily show that $M_y = M_h + M_x - 1$.

(c) $h[n] = 0$ for $n > 5$.

(d) From the figure it is clear that

$$y(t) = h(t) * x(t) = \int_{-2}^{-1} x(t-\tau)d\tau + x(t-6).$$

Therefore,

$$y(0) = \int_{-2}^{-1} x(\tau)d\tau + x(-6).$$

This implies that $x(t)$ must be known for $1 \leq t \leq 2$ and for $t = -6$.

2.45. **(a)** **(i)** We have

$$\frac{x(t) - x(t-h)}{h} \overset{LTI}{\rightarrow} \frac{y(t) - y(t-h)}{h}.$$

Taking limit as $h \rightarrow 0$ on both sides of the above equation:

$$x'(t) \overset{LTI}{\rightarrow} y'(t)$$

(ii) Differentiating the convolution integral, we get

$$
\begin{aligned}
y'(t) &= \frac{d}{dt}\left[\int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau\right] \\
&= \int_{-\infty}^{\infty} \frac{d}{dt}[x(t-\tau)]h(\tau)d\tau \\
&= \int_{-\infty}^{\infty} x'(t-\tau)h(\tau)d\tau \\
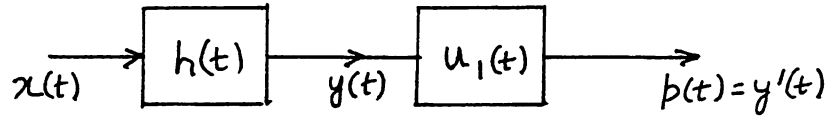&= x'(t) * h(t).
\end{aligned}
$$

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Figure S2.45

(iii) Let us name the output of the system with impulse response $u_1(t)$ as $w(t)$. Then, $w(t) = x(t) * u_1(t) = x'(t)$ and $z(t) = x'(t) * h(t)$.

Since both systems in the cascade are LTI, we may interchange their order as shown in Figure S2.45.

Then, $y(t) = x(t) * h(t)$ and $p(t) = y'(t)$. Since $z(t)$ and $p(t)$ have to be the same, we may conclude that $x'(t) * h(t) = y'(t)$.

(b) (i) We have already proved that $y'(t) = x'(t) * h(t)$. Now we may interchange $x(t)$ and $h(t)$ in the earlier proofs and they would all still hold. Therefore, we may argue that $y'(t) = x(t) * h'(t)$.

(ii) Consider

$$
\begin{aligned}
y(t) &= [x(t) * u(t)] * h'(t) \\
&= x(t) * [u(t) * u_1(t)] * h(t) \\
&= x(t) * h(t).
\end{aligned}
$$

This shows that $[x(t) * u(t)]h'(t)$ is equivalent to $x(t) * h(t)$. Now the same thing may be written as:

$$
\begin{aligned}
y(t) &= [x(t) * u(t)] * h'(t) \\
&= [[x(t) * u_1(t)] * h(t)] * u(t) \\
&= \int_{-\infty}^{t} x'(\tau) h(t - \tau) d\tau \\
&= x'(t) * [h(t) * u(t)] \\
&= x'(t) * \int_{-\infty}^{t} h(\tau) d\tau
\end{aligned}
$$

(c) Note that $x'(t) = \delta(t) - 5e^{-5t}u(t)$. Therefore, the output of the LTI system to $x'(t)$ will be $h(t) - 5\sin(\omega_0 t)$. Since this has to be equal to $y'(t) = \omega_0 \cos(\omega_0 t)$, we have

$$
h(t) = \omega_0 \cos(\omega_0 t) + 5\sin(\omega_0 t).
$$

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(d) (i) We have

$$
\begin{aligned}
y(t) &= x(t) * [u_1(t) * u(t)] * h(t) \\
&= [x(t) * u_1(t)] * [u(t) * h(t)] \\
&= x'(t) * s(t) \\
&= \int_{-\infty}^{\infty} x'(\tau)s(t-\tau)d\tau
\end{aligned}
$$

(ii) Also,

$$
\begin{aligned}
x(t) &= x(t) * \delta(t) \\
&= [x(t) * u_1(t)] * u(t) \\
&= \int_{-\infty}^{\infty} x'(\tau)u(t-\tau)d\tau
\end{aligned}
$$

(e) In this case

$$
x'(t) = e^t u(t) + \delta(t).
$$

Therefore,

$$
y(t) = s(t) + e^t u(t) * s(t).
$$

This may be written as

$$
\begin{aligned}
y(t) &= [e^{-3t} - 2e^{-2t} + 1]u(t) \\
&+ [\frac{1}{4}(e^t - e^{-3t}) \\
&- \frac{2}{3}(e^t - e^{-2t}) - e^t - 1]u(t).
\end{aligned}
$$

(f) Using the fact that $[\delta[n] - \delta[n-1]] * u[n] = \delta[n]$ gives:

$$
y[n] = [x[n] - x[n-1]] * s[n] = \sum_k [x[k] - x[k-1]]s[n-k]
$$

and

$$
x[n] = [x[n] - x[n-1]] * u[n] = \sum_{k=-\infty}^{\infty} [x[k] - x[k-1]]u[n-k].
$$

2.46. Note that

$$
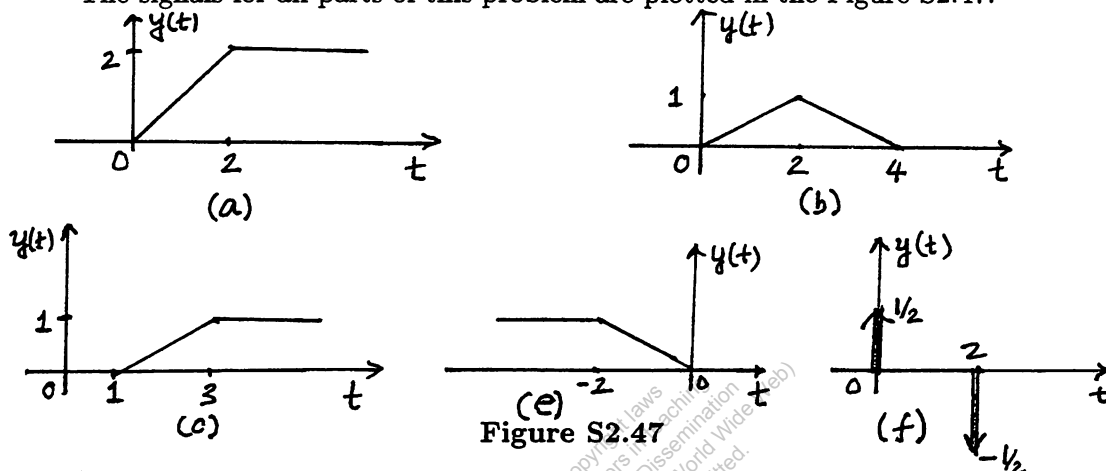\frac{dx(t)}{dt} = -6e^{-3t}u(t-1) + 2\delta(t-1) = -3x(t) + 2\delta(t-1).
$$

Given that

$$
x(t) = 2e^{-3t}u(t-1) \longrightarrow y(t)
$$

we know that $\frac{dx(t)}{dt} = -3x(t) + 2\delta(t-1)$ must yield $-3y(t) + 2h(t-1)$ at the output. From the given information, we may conclude that $2h(t-1) = e^{-2t}u(t)$. Therefore,

$$
h(t) = \frac{1}{2}e^{-2(t+1)}u(t+1).
$$

2.47. **(a)** $y(t) = 2y_0(t)$.

(b) $y(t) = y_0(t) - y_0(t-2)$.

(c) $y(t) = y_0(t-1)$.

(d) Not enough information.

(e) $y(t) = y_0(-t)$.

(f) $y(t) = y_0''(t)$.

The signals for all parts of this problem are plotted in the Figure S2.47.

Figure S2.47

2.48. **(a)** True. If $h(t)$ periodic and nonzero, then

$$\int_{-\infty}^{\infty} |h(t)|\,dt = \infty.$$

Therefore, $h(t)$ is unstable.

(b) False. For example, inverse of $h[n] = \delta[n-k]$ is $g[n] = \delta[n+k]$ which is noncausal.

(c) False. For example $h[n] = u[n]$ implies that

$$\sum_{n=-\infty}^{\infty} |h[n]| = \infty.$$

This is an unstable system.

(d) True. Assuming that $h[n]$ is bounded and nonzero in the range $n_1 \le n \le n_2$,

$$\sum_{k=n_1} n_2|h[k]| < \infty.$$

This implies that the system is stable.

(e) False. For example, $h(t) = e^t u(t)$ is causal but not stable.

(f) False. For example, the cascade of a causal system with impulse response $h_1[n] = \delta[n-1]$ and a non-causal system with impulse response $h_2[n] = \delta[n+1]$ leads to a system with overall impulse response given by $h[n] = h_1[n] * h_2[n] = \delta[n]$.

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(g) False. For example, if $h(t) = e^{-t}u(t)$, then $s(t) = (1 - e^{-t})u(t)$ and

$$\int_0^\infty |1 - e^{-t}|dt = t + e^{-t}\big|_0^\infty = \infty.$$

Although the system is stable, the step response is not absolutely integrable.

(h) True. We may write $u[n] = \sum_{k=0}^\infty \delta[n-k]$. Therefore,

$$s[n] = \sum_{k=0}^\infty h[n-k].$$

If $s[n] = 0$ for $n < 0$, then $h[n] = 0$ for $n < 0$ and the system is causal.

2.49. **(a)** It is a bounded input. $|x[n]| \le 1 = B_x$ for all n.

(b) Consider

$$
\begin{aligned}
y[0] &= \sum_{k=-\infty}^\infty x[-k]h[k] \\
&= \sum_{k=-\infty}^\infty \frac{h^2[k]}{|h[k]|} \\
&= \sum_{k=-\infty}^\infty |h[k]| \to \infty
\end{aligned}
$$

Therefore, the output is not bounded. Thus, the system is not stable and absolute summability is necessary.

(c) Let

$$x(t) = \begin{cases} 0, & \text{if } h(-t) = 0 \\ \frac{h(-t)}{|h(-t)|}, & \text{if } h(-t) \ne 0 \end{cases}.$$

Now, $|x(t)| \le 1$ for all t. Therefore, $x(t)$ is a bounded input Now,

$$
\begin{aligned}
y(0) &= \int_{-\infty}^\infty x(-\tau)h(\tau)d\tau \\
&= \int_{-\infty}^\infty \frac{h^2(\tau)}{|h(\tau)|}d\tau \\
&= \int_{-\infty}^\infty |h(t)|dt = \infty
\end{aligned}
$$

Therefore, the system is unstable if the impulse response is not absolutely integrable.

2.50. **(a)** The output will be $ax_1(t) + bx_2(t)$.

(b) The output will be $x_1(t - \tau)$.

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2.51. **(a)** For the system of Figure P2.51(a) the response to an unit impulse is

$$y_1[n] = n(\frac{1}{2})^n u[n].$$

For the system of Figure P2.51(b) the response to an unit impulse is

$$y_2[n] = 0.$$

Clearly, $y_1[n] \neq y_2[n]$.

(b) For the system of Figure P2.51(a) the response to an unit impulse is

$$y_[n] = (\frac{1}{2})^n u[n] + 2.$$

For the system of Figure P2.51(b) the response to an unit impulse is

$$y[n] = (\frac{1}{2})^n u[n] + 4.$$

Clearly, $y_1[n] \neq y_2[n]$.

2.52. We get

$$s[n] = h[n] * u[n] = \begin{cases} \sum_{k=0}^{n}(k+1)\alpha^k, & n \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Noting that

$$\sum_{k=0}^{n}(k+1)\alpha^k = \frac{d}{d\alpha}\sum_{k=0}^{n+1}\alpha^k = \frac{d}{d\alpha}\left[\frac{1-\alpha^{n+2}}{1-\alpha}\right],$$

we get

$$
\begin{aligned}
s[n] &= \left[\frac{1-(n+2)\alpha^{n+1}}{1-\alpha} + \frac{1-\alpha^{n+2}}{(1-\alpha^2)}\right]u[n] \\
&= \left[\frac{1}{(1-\alpha)^2} - \frac{\alpha}{(1-\alpha)^2}\alpha^n + \frac{\alpha}{1-\alpha}(n+1)\alpha^n\right]u[n].
\end{aligned}
$$

2.53. **(a)** Let us assume that

$$\sum_{k=0}^{N}a_k s_0^k = 0.$$

Then,

$$\sum_{k=0}^{N}a_k\frac{d^k}{dt^k}(Ae^{s_0 t}) = \sum_{k=0}^{N}Aa_k e^{s_0 t}s_0^k = 0.$$

Therefore, $Ae^{s_0^t}$ is a solution of eq. (P2.53-1).

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(b) Consider

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k}(Ate^{st}) = \sum_{k=0}^{N} Aa_k ts^k e^{st} + \sum_{k=0}^{N} Aa_k ke^{st}s^{k-1}$$

$$= Ate^{st}\sum_{k=0}^{N} a_k s^k + Ae^{st}\sum_{k=0}^{N} a_k \frac{d}{ds}(s^k)$$

$$= Ate^{st}\sum_{k=0}^{N} a_k s^k + Ae^{st}\frac{d}{ds}\sum_{k=0}^{N} a_k s^k.$$

If s_i is a solution, then $\sum_{k=0}^{N} a_k s_i^k = 0$. This implies that $te^{s_i t}$ is a solution.

(c) (i) Here,

$$s^2 + 3s + 2 = 0, \quad \Rightarrow \quad s = -2, s = -1.$$

Therefore,

$$y_h(t) = Ae^{-2t} + Be^{-t}.$$

Since $y_h(0) = 0$, $y'_h(0) = 2$, $A + B = 0$ and $2A + B = 2$. Therefore, $A = -2$, $B = 2$.

$$y(t) = 2e^{-t} - 2e^{-2t}.$$

(ii) Here,

$$s^2 + 3s + 2 = 0 \quad \Rightarrow \quad y(t) = Ae^{-2t} + Be^{-t}.$$

Since $y(0) = 1$, $y'(0) = -1$, we have $y(t) = e^{-t}$.

(iii) $y(t) = 0$ because of initial rest condition.

(iv) Here,

$$s^2 + 2s + 1 = 0 = (s+1)^2 \quad \Rightarrow \quad s = -1, \sigma = 2.$$

and

$$y(t) = Ae^{-t} + Bte^{-t}.$$

Since $y(0) = 1$, $y'(0) = 1$, $A = 1$, $B = 2$. Therefore,

$$y(t) = e^{-t} + 2te^{-t}.$$

(v) Here,

$$s^3 + s^2 - s - 1 = 0 = (s-1)(s+1)^2 \quad \Rightarrow \quad y(t) = Ae^t + Be^{-t} + Cte^{-t}.$$

Since $y(0) = 1$, $y'(0) = 1$, and $y''(0) = -2$, we get $A = 1/2$, $B = 3/4$, $C = 3/2$.
Therefore,

$$y(t) = \frac{1}{2}e^t + \frac{3}{4}e^{-t} + \frac{3}{2}te^{-t}.$$

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(vi) Here, $s = -1 \pm 2j$ and

$$y(t) = Ae^{-t}e^{2jt} + Be^{-t}e^{-2jt}.$$

Since $y(0) = 1$, $y'(0) = 1$,

$$A = \frac{1}{2}(1 - j) = B^*.$$

Therefore,

$$y(t) = e^{-t}[\cos 2t + \sin 2t].$$

2.54. **(a)** Let us assume that

$$\sum_{k=0}^{N} a_k z_0^k = 0.$$

Then, if $y[n] = Az_0^n$,

$$\sum_{k=0}^{N} a_k y[n - k] = \sum_{k=0}^{N} a_k (Az_0^{n-k}) = Az_0^n \sum_{k=0}^{N} a_k z_0^{-k} = 0.$$

Therefore, Az_0^n is a solution of eq. (P2.54-1).

(b) If $y[n] = nz^{n-1}$, then

$$\sum_{k=0}^{N} a_k y[n - k] = \sum_{k=0}^{N} a_k (n - k)z^{n-k-1}. \qquad (S2.54-1)$$

Taking the right-hand side of the equation that we want to prove,

$$R.H.S = z^{n-N} \sum_{k=0}^{N} a_k (N - k)z^{N-k-1} + (n - N)\sum_{k=0}^{N} a_k$$

$$= \sum_{k=0}^{N} a_k (n - k)z^{n-k-1} \qquad (S2.54-2)$$

Comparing eqs. (S2.54-1) and (S2.54-2), we conclude that the equation is proved.

(c) (i) Here,

$$1 + \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2} = 0 \quad \Rightarrow \quad z = -\frac{1}{2}, \ z = -\frac{1}{4}.$$

Therefore,

$$y[n] = A(-\frac{1}{2})^n + B(-\frac{1}{4})^n.$$

Since $y[0] = 1$, $y[-1] = -6$, we get $A = -1$, $B = 2$, and

$$y[n] = 2(-\frac{1}{4})^n - (-\frac{1}{2})^n.$$

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(ii) Here,
$$z^2 - 2z + 1 = 0.$$

Therefore,
$$y[n] = A(1)^n + Bn(1)^n = A + Bn.$$

Since, $y[0] = 1$, $y[1] = 0$ we get $A = 1$, $B = -1$, and

$$y[n] = 1 - n.$$

(iii) Only difference from previous part is initial conditions. Since $y[0] = 1$, $y[10] = 21$, we get $A = 1$, $B = 2$, and
$$y[n] = 1 + 2n.$$

(iv) Here,
$$z = \frac{1}{2\sqrt{2}}(1 \pm j).$$

Therefore,
$$y[n] = A[\frac{1}{2\sqrt{2}}(1+j)]^n + B[\frac{1}{2\sqrt{2}}(1-j)]^n.$$

Since $y[0] = 0$, $y[-1] = 1$, we get $A = \frac{j}{2\sqrt{2}}$, $B = \frac{-j}{2\sqrt{2}}$, and

$$y[n] = -\frac{1}{\sqrt{2}}(\frac{1}{2})^n \sin(n\pi/4).$$

2.55. (a) $y[0] = x[0] = 1$. $h[n]$ satisfies the equation

$$h[n] = \frac{1}{2}h[n-1], \qquad n \geq 1.$$

The auxiliary condition is $h[0] = 1$. Using the method introduced in the previous problem, we have $z = 1/2$. Therefore, $h[n] = A(1/2)^n$. Using the auxiliary condition,

$$h[n] = \left(\frac{1}{2}\right)^n u[n].$$

(b) From Figure P2.55(b), we know that if $x[n] = \delta[n]$, then

$$w[n] = h_a[n] = (\frac{1}{2})^n u[n].$$

This implies that

$$y[n] = h[n] = \left(\frac{1}{2}\right)^n u[n] + 2(\frac{1}{2})^{n-1}u[n-1].$$

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(c) Plugging eq. (P2.55-3) into eq. (P2.55-1) gives:

$$\sum_m h[n-m]x[m] - \frac{1}{2}\sum_m h[n-m-1]x[m] \;=\; \sum_{m=-\infty}^{n}(\tfrac{1}{2})^{n-m}x[m] - \sum_{m=-\infty}^{n-1}(\tfrac{1}{2})^{n-m}x[m]$$

$$= (\tfrac{1}{2})^{n-n}x[n]$$

$$= x[n].$$

This implies that eq. (P2.55-3) satisfies eq. P(2.55-1).

(d) (i) Given that $a_0 \neq 0$ and that the system obeys initial rest, we get

$$a_0 y[0] = 1 \quad \Rightarrow \quad y[0] = \frac{1}{a_0}.$$

The homogeneous equation is

$$\sum_{k=0}^{N} a_k h[n-k] = 0$$

with the initial conditions

$$h[0] = 1/a_0, \qquad h[-1] = \cdots = h[-N+1] = 0.$$

(ii) We have

$$h[n] = \sum_{k=0}^{M} b_k h_1[n-k] = 0,$$

where $h_1[n]$ is as above.

(e) For $n > M$,

$$\sum_{k=0}^{N} a_k h[n-k] = 0$$

with

$$h[0] = y[0], \cdots, h[M] = y[M].$$

(f) (i) We get

$$h[n] = \begin{cases} 1, & n \text{ even}, \ n \geq 0 \\ 0, & n \text{ odd or } n < 0 \end{cases}.$$

(ii) We get

$$h[n] = \begin{cases} 1, & n \text{ even and } n \geq 0 \\ 2, & n \text{ odd and } n > 0 \\ 0, & n < 0 \end{cases}.$$

(iii) We get

$$h[n] = \begin{cases} 2, & n = 0, 2 \\ -1, & n \text{ even } n \geq 4 \\ 0, & else \end{cases}.$$

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(iv) We get

$$h[n] = \frac{1}{2}[\cos\frac{\pi n}{6} + \sqrt{3}\sin\frac{\pi n}{6}].$$

2.56. **(a)** In this case, $s + 2 = 0$ which implies that

$$y(t) = h(t) = Ae^{-2t}.$$

Since $y(0+) = 1$, $A = 1$ and

$$h(t) = e^{-2t}u(t).$$

Now consider eq. (P2.56-1).

$$
\begin{aligned}
L.H.S. &= \frac{d}{dt}\int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau + 2\int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau \\
&= \int_{-\infty}^{\infty} e^{-2(t-\tau)}\delta(t-\tau)x(\tau)d\tau \\
&= x(t) = R.H.S.
\end{aligned}
$$

This implies that $y(t)$ does solve the differential equation.

(b) Take

$$y(t) = \sum_l \alpha_l u_l(t).$$

Then

$$\sum_{k=0}^{N} a_k \sum_l \alpha_l u_{k+l}(t) = \delta(t).$$

Integrating between $t = 0^-$ and $t = 0^+$ and matching coefficients, we get $\alpha_l = 0$ except $\alpha_{-N} = 1/a_N$. This implies that for $0^- \le t \le 0^+$

$$y(t) = \frac{1}{a_N}u_{-N}(t)$$

and

$$y(0^+) = y'(0^+) = \cdots = y^{N-2'}(0^+) = 0$$

and

$$\left.\frac{d^{N-1}y(t)}{dt^{N-1}}\right|_{0+} = \frac{1}{a_N}.$$

(c) The impulse response is

$$h(t) = \sum_{k=0}^{M} b_k \frac{d^k h_b(t)}{dt^k}.$$

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(d) (i) Taking

$$y(t) = \sum_r \alpha u_r(t)$$

we get

$$\sum_r [\alpha_r u_{r+2}(t) + 3\alpha_r u_{r+1}(t) + 2\alpha_r u_r] = \delta(t)$$

This implies that $r_{max} = -2$ and $\alpha_{-2} = 1$. Therefore, $h(0+) = 0$ and $h'(0^+) = 1$ constitute the initial conditions. Now,

$$s^3 + 3s + 2 = 0 \quad \Rightarrow \quad s = -2, s = -1.$$

Therefore,

$$h(t) = Ae^{-2}t + Be^{-t}, \ t \geq 0.$$

Applying initial conditions, we get $A = -1$, $B = 1$. Therefore,

$$h(t) = (e^{-t} - e^{-2t})u_{-1}(t).$$

(ii) The initial conditions are $h(0^+) = 0$ and $h'(0^+)1$, Also, $s = -1 \pm j$. Therefore

$$h(t) = [e^{-t}\sin t]u_{-1}(t).$$

(e) From part (c), if $M \geq N$, then $\sum_{k=0}^{M} b_k \frac{d^k h_b(t)}{dt^k}$ will contain singularity terms at $t = 0$ This implies that

$$h(t) = \sum_r \alpha_r u_r(t) + \cdots.$$

(f) (i) Now,

$$\sum_r \alpha_r u_{r+1}(t) + 2\sum_r \alpha_r u_r = 3u_1(t) + u_0(t).$$

Therefore, $r_{max} = 0$. Also

$$\alpha_0 u_1(t) + \alpha_{-1}u_0(t) + 2\alpha_0 u_0(t) = 3u_1(t) + u_0(t).$$

This gives $\alpha_0 = 3$ and $\alpha_{-1} = -5$. The initial condition is $h(0^+) = -5$ and

$$h(t) = 3u_0(t) - 5e^{-2t}u_{-1}(t) = 3\delta(t) - 5e^{-2t}u(t).$$

(ii) Here, $\alpha_1 = 1$, $\alpha_0 = -3$, $\alpha_{-1} = 13$, $\alpha_{-2} = -44$. Therefore $h(0^+) = 13$ and $h'(0^+) = -44$ and

$$h(t) = u_1(t) - 3u_0(t) + 18e^{-3t}u_{-1}(t) - 5e^{-2t}u_{-1}(t).$$

2.57. (a) Realizing that $x_2[n] = y_1[n]$, we may eliminate these from the two given difference equations. This would give us

$$y_2[n] = -ay_2[n-1] + b_0 x_1[n] + b_1 x_1[n-1].$$
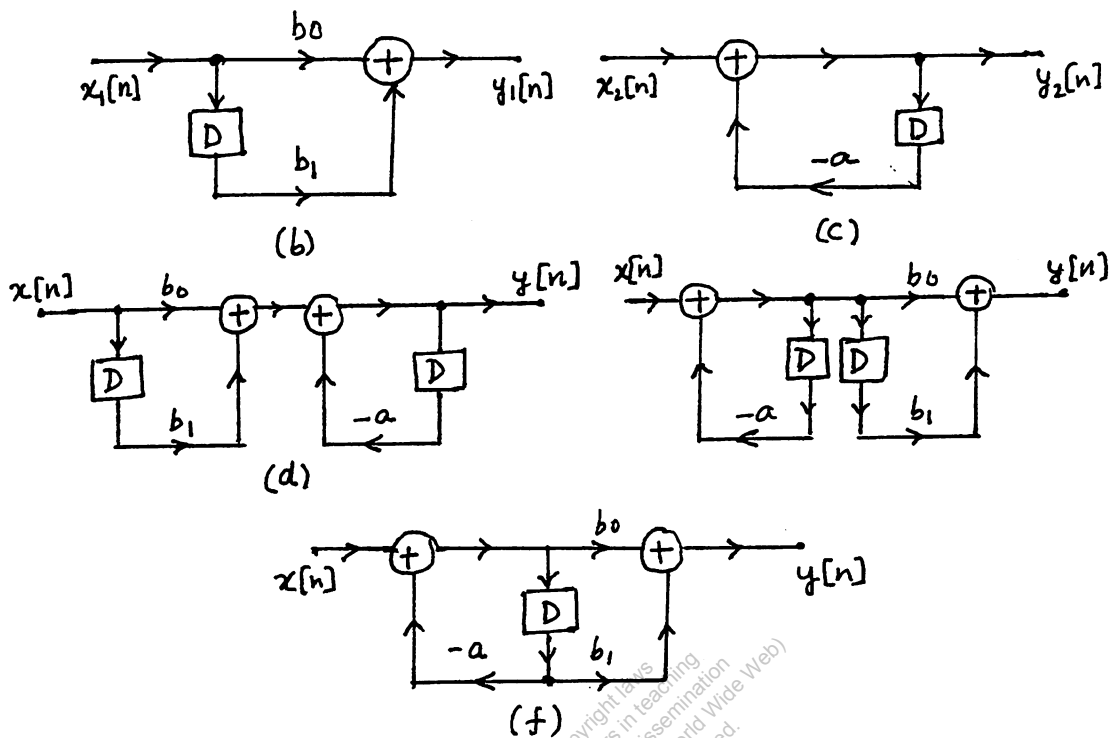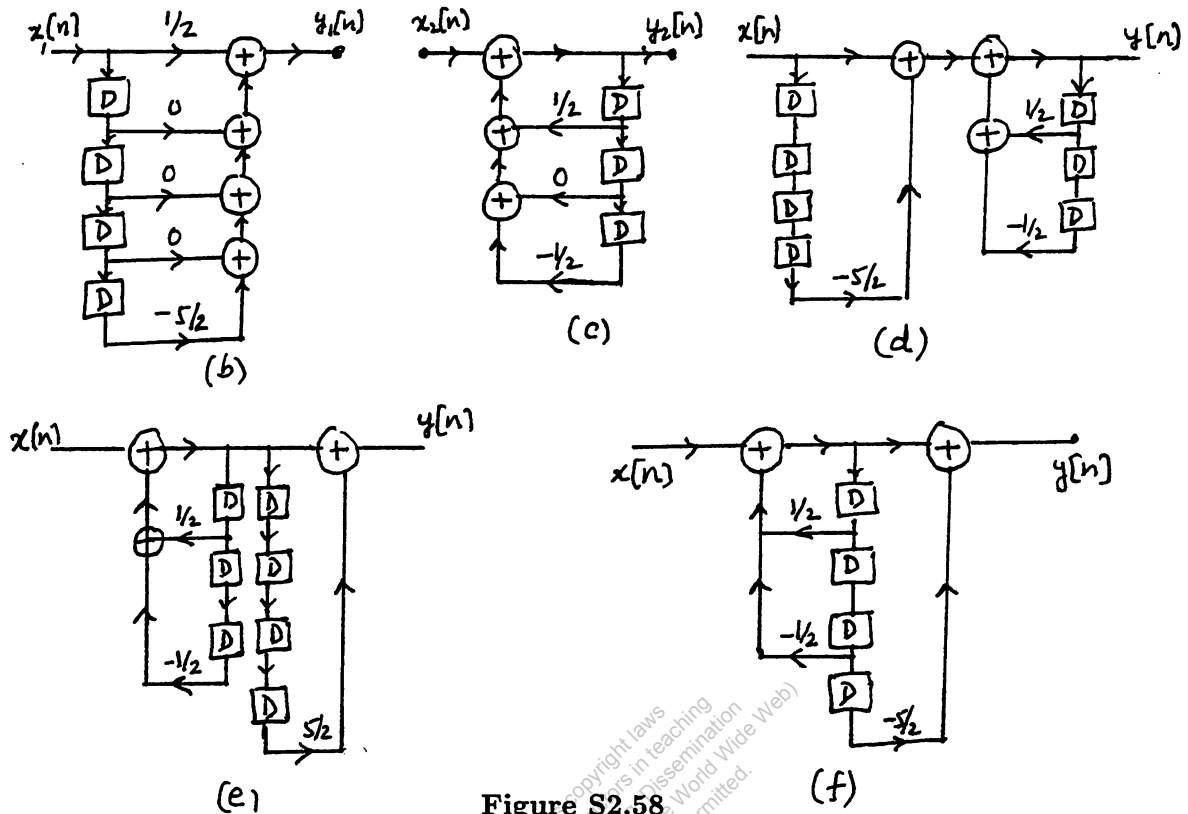
This is the same as the overall difference equation.

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Figure S2.57

(b) The figures corresponding to the remaining parts of this problem are shown in the Figure S2.57.

2.58. (a) Realizing that $x_2[n] = y_1[n]$, we may eliminate these from the two given difference equations. This would give us

$$2y_2[n] - y_2[n-1] + y_2[n-3] = x_1[n] - 5x_1[n-4].$$

This is the same as the overall difference equation.

(b) The figures corresponding to the remaining parts of this problem are shown in Figure S2.58.

2.59. (a) Integrating the given differential equation once and simplifying, we get

$$y(t) = -\frac{a_0}{a_1}\int_{-\infty}^{t} y(\tau)d\tau + \frac{b_0}{a_1}\int_{-\infty}^{t} x(\tau)d\tau + \frac{b_1}{a_1}x(t).$$

Therefore, $A = -a_0/a_1$, $B = b_1/a_1$, $C = b_0/a_1$.

(b) Realizing that $x_2(t) = y_1(t)$, we may eliminate these from the two given integral equations. This would give us

$$y_2(t) = A\int_{-\infty}^{t} y_2(\tau)d\tau + B\int_{-\infty}^{t} x_1(\tau)d\tau + Cx_1(t).$$

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Figure S2.58

(c) The figures corresponding to the remaining parts of this problem are shown in Figure S2.59.

2.60. (a) Integrating the given differential equation once and simplifying, we get

$$y(t) = \frac{-a_1}{a_2}\int_{-\infty}^{t}y(\tau)d\tau - \frac{a_0}{a_2}\int_{-\infty}^{t}\int_{-\infty}^{\tau}y(\sigma)d\sigma d\tau$$

$$+ \frac{b_0}{a_2}\int_{-\infty}^{t}\int_{-\infty}^{\tau}x(\sigma)d\sigma d\tau + \frac{b_1}{a_2}\int_{-\infty}^{t}x(\tau)d\tau + \frac{b_2}{a_1}x(t).$$

Therefore, $A = -a_1/a_2$, $B = -a_0/a_2$, $C = b_2/a_1$, $D = b_1/a_2$, $E = b_0/a_2$.

(b) Realizing that $x_2(t) = y_1(t)$, we may eliminate these from the two given integral equations.

(c) The figures corresponding to the remaining parts of this problem are shown in Figure S2.60.

2.61. (a) (i) From Kirchoff's voltage law, we know that the input voltage must equal the sum of the voltages across the inductor and capacitor. Therefore,
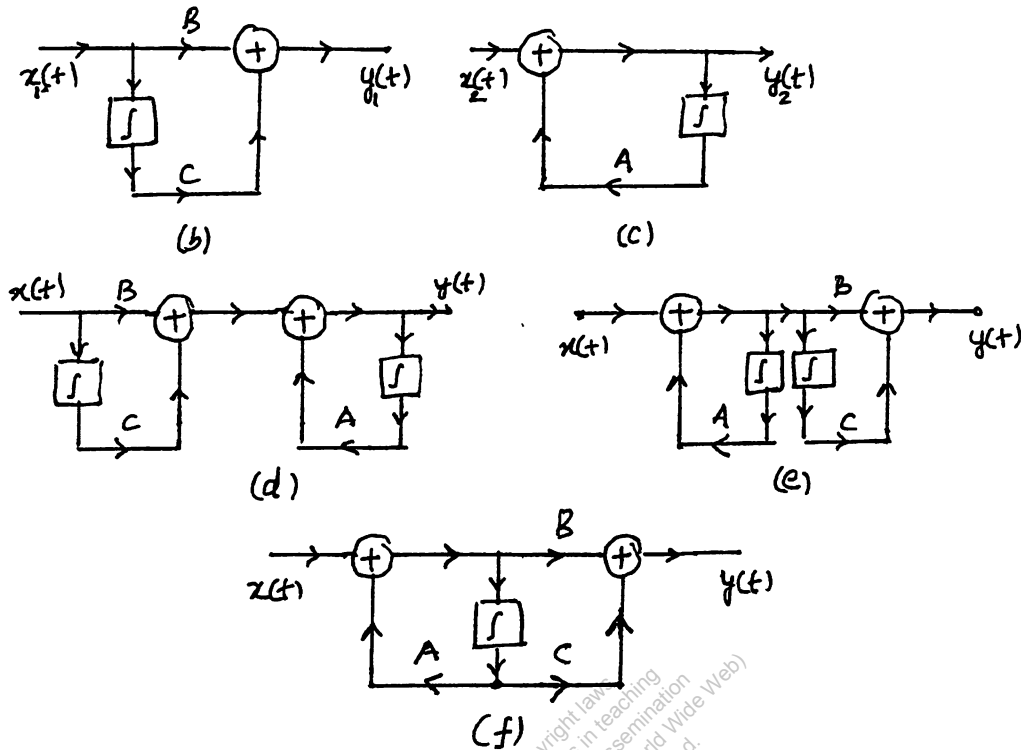
$$x(t) = LC\frac{d^2y(t)}{dt^2} + y(t).$$

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Figure S2.59

Using the values of L and C we get

$$\frac{d^2y(t)}{dt^2} + y(t) = x(t).$$

(ii) Using the results of Problem 2.53, we know that the homogeneous solution of the differential equation

$$\frac{d^2y(t)}{dt^2} + a_1\frac{dy(t)}{dt} + a_2y(t) = bx(t).$$

will have terms of the form $K_1e^{s_0t} + K_2e^{s_1t}$ where s_0 and s_1 are roots of the equation

$$s^2 + a_1s + a_2 = 0.$$

(It is assumed here that $s_0 \neq s_1$.) In this problem, $a_1 = 0$ and $a_2 = 1$. Therefore, the root of the equation are $s_0 = j$ and $s_1 = -j$.. The homogeneous solution is

$$y_h(t) = K_1e^{jt} + K_2e^{-jt}.$$

And, $\omega_1 = 1 = \omega_2$.

(iii) If the voltage and current are restricted to be real, then $K_1 = K_2 = K$. Therefore,
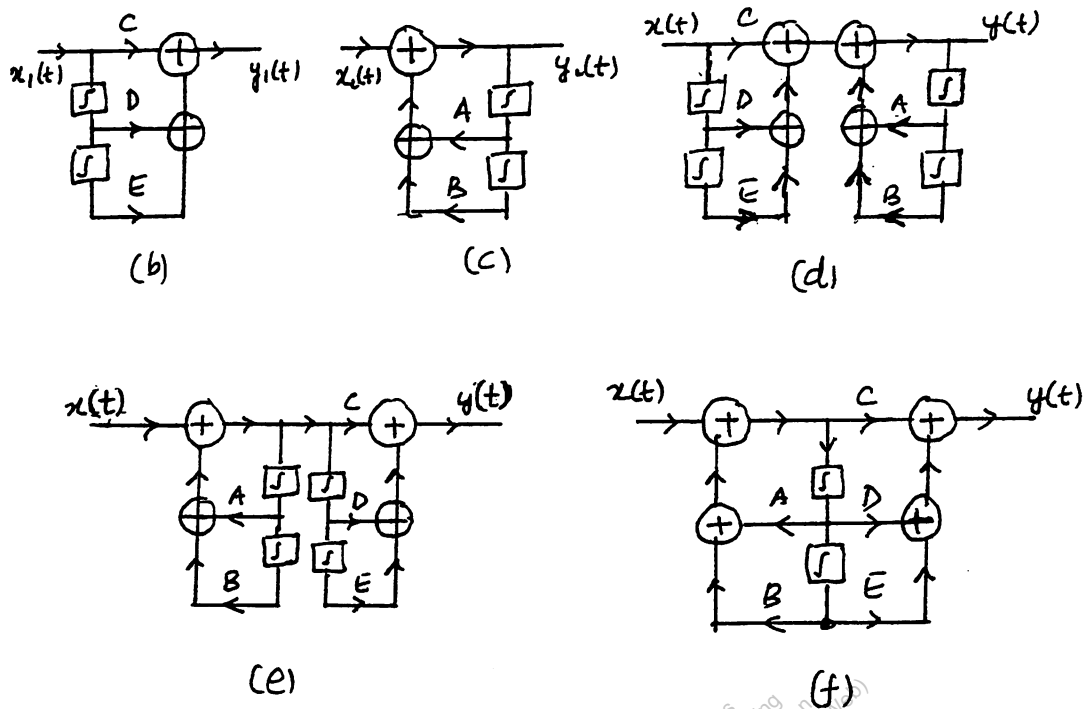
$$y_h(t) = 2K\cos(t) = 2K\sin(t + \pi/2).$$

Figure S2.60

(b) (i) From Kirchoff's voltage law, we know that the input voltage must equal the sum of the voltages across the resistor and capacitor. Therefore,

$$x(t) = RC\frac{dy(t)}{dt} + y(t).$$

Using the values of R, L, and C we get

$$\frac{dy(t)}{dt} + y(t) = x(t).$$

(ii) The natural response of the system is the homogeneous solution of the above differential equation. Using the results of Problem 2.53, we know that the homogeneous solution of the differential equation

$$\frac{dy(t)}{dt} + a_1 y(t) = bx(t).$$

will have terms of the form $Ae^{s_0 t}$ where s_0 is the root of the equation

$$s + a_1 = 0.$$

In this problem, $a_1 = 1$. Therefore, the root of the equation are $s_0 = -1$. The homogeneous solution is

$$y_h(t) = Ke^{-t}.$$

And, $a = 1$.

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(c) (i) From Kirchoff's voltage law, we know that the input voltage must equal the sum of the voltages across the resistor, inductor, and capacitor. Therefore,

$$x(t) = LC\frac{d^2y(t)}{dt^2} + RC\frac{dy(t)}{dt} + y(t).$$

Using the values of R, L, and C we get

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 5y(t) = 5x(t).$$

(ii) Using the results of Problem 2.53, we know that the homogeneous solution of the differential equation

$$\frac{d^2y(t)}{dt^2} + a_1\frac{dy(t)}{dt} + a_2y(t) = bx(t).$$

will have terms of the form $K_1e^{s_0t} + K_2e^{s_1t}$ where s_0 and s_1 are roots of the equation

$$s^2 + a_1s + a_2 = 0.$$

(It is assumed here that $s_0 \neq s_1$.) In this problem, $a_1 = 2$ and $a_2 = 5$. Therefore, the root of the equation are $s_0 = -1 + 2j$ and $s_1 = -1 - 2j$. The homogeneous solution is

$$y_h(t) = K_1e^{-t}e^{2jt} + K_2e^{-t}e^{-2jt}.$$

And, $a = 1$.

(iii) If the voltage and current are restricted to be real, then $K_1 = K_2 = K$. Therefore,

$$y_h(t) = 2Ke^{-t}\cos(2t) = 2Ke^{-t}\sin(2t + \pi/2).$$

2.62. (a) The force $x(t)$ must equal the sum of the force required to displace the mass and the force required to stretch the spring. Therefore,

$$x(t) = m\frac{d^2y(t)}{dt^2} + Ky(t) = x(t).$$

Substituting the values of m and K, we get

$$\frac{d^2y(t)}{dt^2} + 4y(t) = 2x(t).$$

Using the results of Problem 2.53, we know that the homogeneous solution of the differential equation

$$\frac{d^2y(t)}{dt^2} + a_1\frac{dy(t)}{dt} + a_2y(t) = bx(t).$$

will have terms of the form $K_1e^{s_0t} + K_2e^{s_1t}$ where s_0 and s_1 are roots of the equation

$$s^2 + a_1s + a_2 = 0.$$

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(It is assumed here that $s_0 \neq s_1$.) In this problem, $a_1 = 0$ and $a_2 = 4$. Therefore, the root of the equation are $s_0 = +2j$ and $s_1 = -2j$. The homogeneous solution is

$$y_h(t) = K_1 e^{2jt} + K_2 e^{-2jt}.$$

Assuming that $y(t)$ is real, we have $K_1 = K_2 = K$. Therefore,

$$y_h(t) = 2K\cos(2t).$$

Clearly, $y_h(t)$ is periodic.

(b) The force $x(t)$ must equal the sum of the force required to displace the mass and the force required to stretch the spring. Therefore,

$$x(t) = m\frac{dy(t)}{dt} + by(t).$$

Substituting the values of m and b, we get

$$\frac{dy(t)}{dt} + \frac{y(t)}{10000} = \frac{x(t)}{1000}.$$

Using the results of Problem 2.53, we know that the homogeneous solution of the differential equation

$$\frac{dy(t)}{dt} + a_1 y(t) = bx(t).$$

will have terms of the form $Ae^{s_0 t}$ where s_0 is the root of the equation

$$s + a_1 = 0.$$

In this problem, $a_1 = 1/10000$. Therefore, the root of the equation are $s_0 = -10^{-4}$. The homogeneous solution is

$$y_h(t) = Ke^{-10^{-4}t}.$$

Clearly, $y_h(t)$ decreases with increasing t.

(c) (i) We know that the input force $x(t) = $ (Force required to displace mass by $y(t)$) + (Force required to displace dashpot by $y(t)$) + (Force required to displace spring by $y(t)$). Therefore,

$$x(t) = m\frac{d^2 y(t)}{dt^2} + b\frac{dy(t)}{dt} + Ky(t).$$

Using the values of m, b, and K we get

$$\frac{d^2 y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 2y(t) = x(t).$$

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(ii) Using the results of Problem 2.53, we know that the homogeneous solution of the differential equation

$$\frac{d^2y(t)}{dt^2} + a_1\frac{dy(t)}{dt} + a_2 y(t) = b_1 x(t).$$

will have terms of the form $K_1 e^{s_0 t} + K_2 e^{s_1 t}$ where s_0 and s_1 are roots of the equation

$$s^2 + a_1 s + a_2 = 0.$$

(It is assumed here that $s_0 \neq s_1$.) In this problem, $a_1 = 2$ and $a_2 = 2$. Therefore, the root of the equation are $s_0 = -1 + j$ and $s_1 = -1 - j$. The homogeneous solution is

$$y_h(t) = K_1 e^{-t} e^{jt} + K_2 e^{-t} e^{-jt}.$$

And, $a = 1$.

(iii) If the force is restricted to be real, then $K_1 = K_2 = K$. Therefore,

$$y_h(t) = 2K e^{-t} \cos(t) = 2K e^{-t} \sin(t + \pi/2).$$

2.63. **(a)** We have

$$
\begin{aligned}
y[n] &= \text{Amt. borrowed} - \text{Amt. paid} + \text{Compounded Amt from prev. month} \\
&= 100,000\delta[n] + 1.01y[n-1] - Du[n-1].
\end{aligned}
$$

Therefore,

$$y[n] = 1.01y[n-1] - D, \qquad n > 0$$

and $y[0] = 100,000$ and $\gamma = 1.01$.

(b) We have

$$y_p[n] = 1.01 y_p[n-1] - D.$$

This implies that $y_p[n] = 100D$. Also the homogeneous solution is of the form

$$y_h[n] = A(1.01)^n.$$

Therefore,

$$y[n] = y_h[n] + y_p[n] = A(1.01)^n + 100D$$

Using the initial condition $y[0] = 100000$, we have

$$A = 100000 - 100D.$$

Therefore,

$$y[n] = (100000 - 100D)(1.01)^n + 100D.$$

(c) We have

$$y[360] = 0 = (P - 100D)(1.01)^{360} + 100D.$$

Therefore, $D = \$1028.60$.

(d) Total payment $= \$370,296$.

(e) The toughest question in this book!!

2.64. **(a)** We have $y(t) = x(t) * h(t)$ and $x(t) = y(t) * g(t)$. Therefore, $g(t) * h(t) = \delta(t)$. Now,

$$h(t) * g(t)|_{t=nT} = \sum_{k=0}^{n} h_k g_{n-k} \delta(t - nT).$$

Therefore we want

$$\sum_{k=0}^{n} h_k g_{n-k} = \begin{cases} 1, & n = 0 \\ 0, & n = 1, 2, 3, \cdots \end{cases}.$$

Therefore,

$$g_0 = \frac{1}{h_0}, \qquad g_1 = -\frac{h_1}{h_0^2}, \qquad g_2 = -\frac{1}{h_0}\left[\frac{-h_1^2}{h_0^2} + \frac{h_2}{h_0}\right], \cdots.$$

(b) In this case, $g_0 = 1$, $g_1 = -1/2$, $g_2 = (-1/2)^2$, $g_3 = (-1/2)^3$, and so on. This implies that

$$g(t) = \delta(t) + \sum_{k=1}^{\infty} \left(-\frac{1}{2}\right)^k \delta(t - kT).$$

(c) (i) Here, $h(t) = \sum_{k=0}^{\infty} \alpha^k \delta(t - T)$.

(ii) If $0 < \alpha < 1$, then $\alpha^k < 1$. Therefore, $h(t)$ is bounded and absolutely integrable and corresponds to a stable system. If $\alpha > 1$, then $h(t)$ is not absolutely integrable making the system unstable.

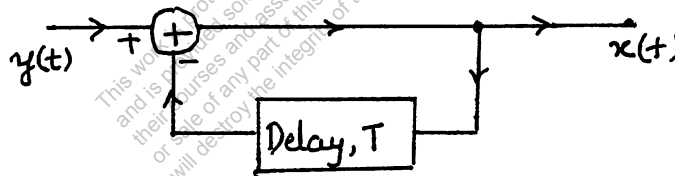**(iii)** Here, $g(t) = 1 - \alpha\delta(t - T)$. The inverse system is as shown in the figure below.

Figure S2.64

(d) If $x_1[n] = \delta[n]$, $y[n] = h[n]$. If $x_2[n] = \frac{1}{2}\delta[n] + \frac{1}{2}\delta[n - N]$, $y[n] = h[n]$.

2.65. **(a)** The autocorrelation sequences are as shown in Figure S2.65.

(b) The autocorrelation sequences are as shown in Figure S2.65.

(c) We get

$$\phi_{xy}[n] = \sum_{k=-\infty}^{\infty} h[-k]\phi_{xx}[n - k].$$

Therefore, $\phi_{xy}[n]$ may be viewed as

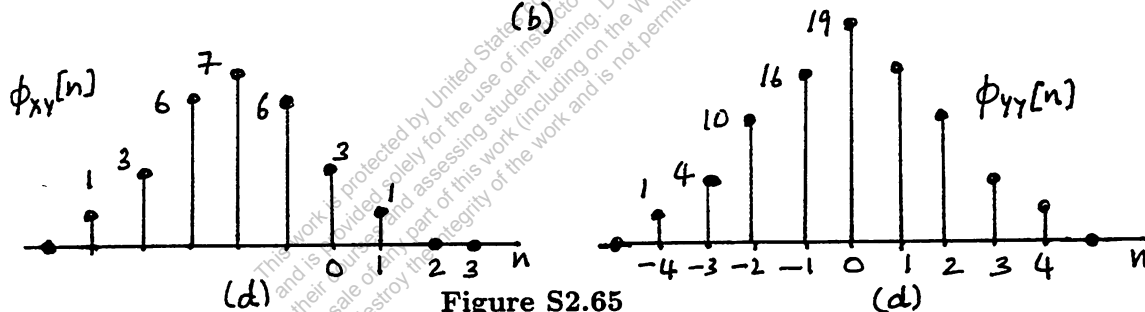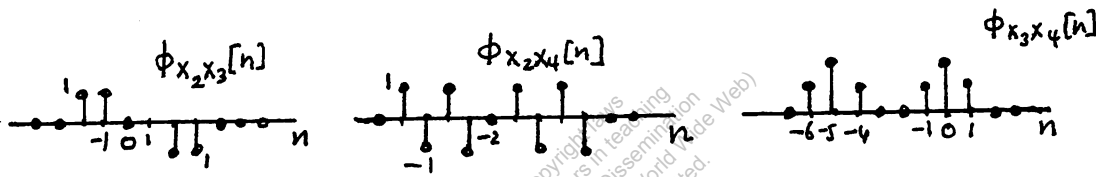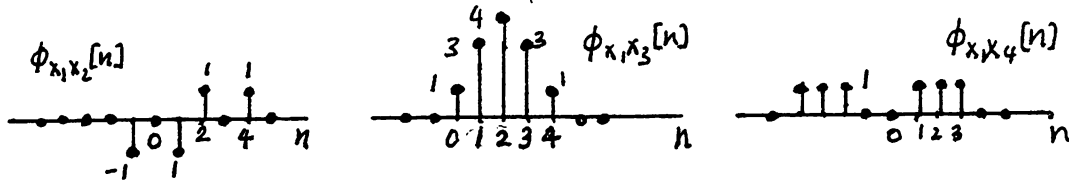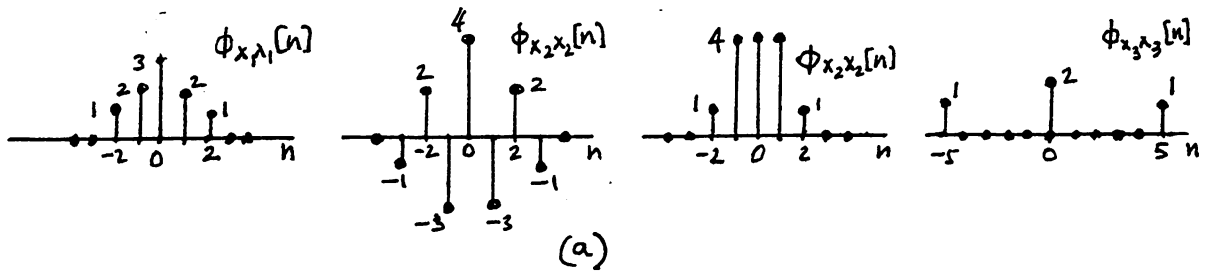$$\phi_{xx}[n] \rightarrow \boxed{h[-n]} \rightarrow \phi_{xy}[n].$$

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Figure S2.65
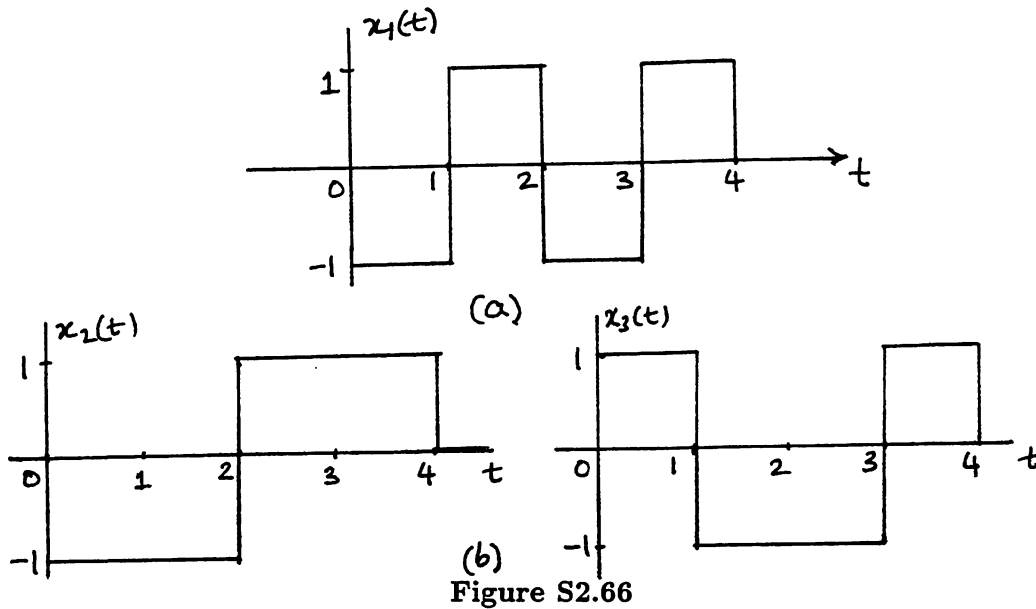
Also,

$$\phi_{yy}[n] = \sum_k \phi_{xx}[n-k]\phi_{hh}[k].$$

Therefore, $\phi_{yy}[n]$ may be viewed as

$$\phi_{xx}[n] \rightarrow \boxed{h[n] * h[-n]} \rightarrow \phi_{yy}[n].$$

(d) $\phi_{xy}[n]$ and ϕ_{yy} are as shown in Figure S2.65.

2.66. (a) The plot of $x_1(t)$ is as shown in Figure S2.66.

(b) The plots of $x_2(t)$ and $x_2(t)$ are as shown in Figure S2.66.

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Figure S2.66

(c) $x_1(t) * h_2(t) = x_2(t) * h_3(t) = x_1(t) * h_3(t) = 0$ for $t = 4$.

2.67. (a) The autocorrelation functions are:

$$\phi_{x_1 x_1}(t) = \begin{cases} \frac{1}{24}t^3 - \frac{1}{2}t + \frac{2}{3}, & 0 \le t \le 2 \\ 0, & t > 2 \end{cases} \quad \text{and} \quad \phi_{x_1 x_1}(t) = \phi_{x_1 x_1}(-t).$$

and

$$\phi_{x_2 x_2}(t) = \begin{cases} 7(1-t), & 0 \le t \le 1 \\ 1-t, & 1 \le t \le 2 \\ t-3, & 2 \le t \le 3 \\ 3-t, & 3 \le t \le 4 \\ t-5, & 4 \le t \le 5 \\ 5-t, & 5 \le t \le 6 \\ t-7, & 6 \le t \le 7 \\ 0, & t > 7 \end{cases} \quad \text{and} \quad \phi_{x_2 x_2}(t) = \phi_{x_2 x_2}(-t).$$

(b) If the impulse response is $h(t) = x(T - t)$, then $y(t) = \phi_{xx}(t - T)$.

(c) We have

$$
\begin{aligned}
y(T) &= \int_0^T x(\tau)h(T - \tau)d\tau \\
&\le M^{1/2}\left[\int_0^T x^2(t)dt\right]^{1/2}.
\end{aligned}
$$

Therefore, $y(t)$ is at most $M^{1/2}\left[\int_0^T x^2(t)dt\right]^{1/2}$.

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If we now choose

$$h(t) = \sqrt{\dfrac{M}{\displaystyle\int_0^T x^2(t)dt}}\, x(T-t).$$

then

$$y(T) = M^{1/2}[\int_0^T x^2(t)dt]^{1/2}.$$

Clearly, $y(T)$ is maximized for the above choice of $h(t)$.

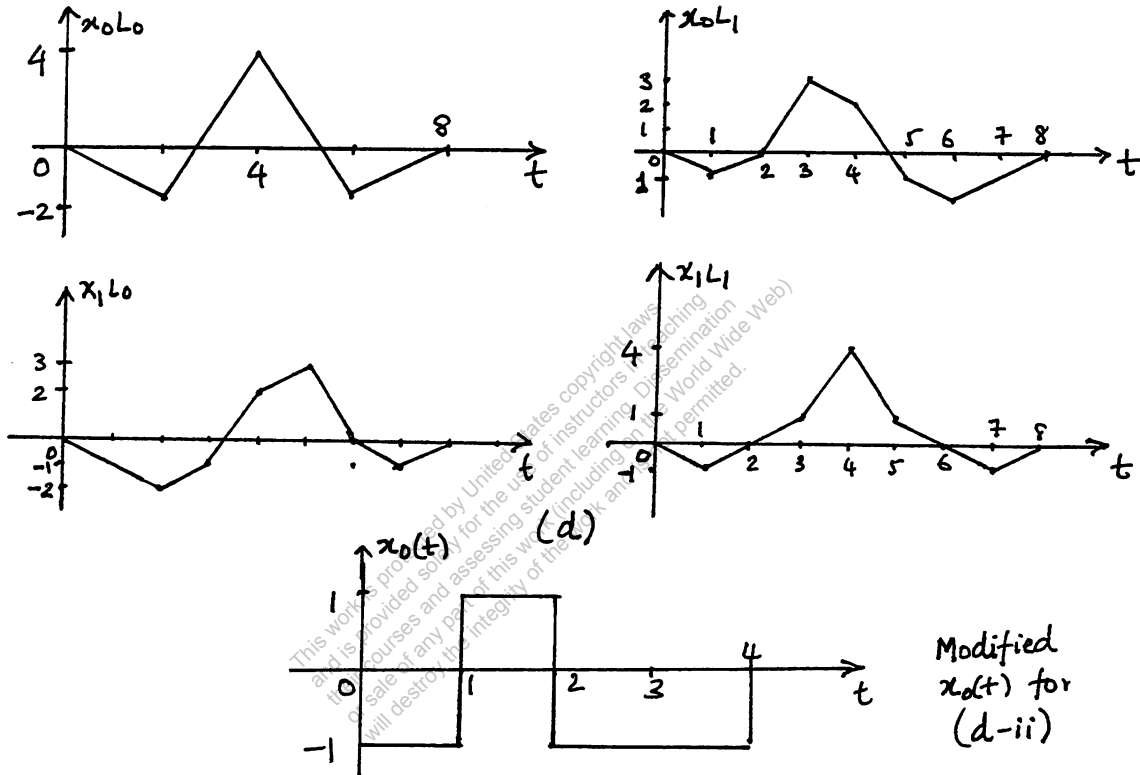**(d)** (i) The responses are as sketched in Figure S2.67.

Figure S2.67

(ii) Let the impulse responses of L_0 and L_1 be $h_{L_0}(t)$ and $h_{L_1}(t)$. Then,

$$x_0(t) * h_{L_0}(t)|_{t=4} = 4$$
$$x_0(t) * h_{L_1}(t)|_{t=4} = 2$$
$$x_1(t) * h_{L_0}(t)|_{t=4} = 2$$
$$x_1(t) * h_{L_1}(t)|_{t=4} = 4$$

To make the job of the receiver easier, modify $x_0(t)$ as shown in the figure below.

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2.68. We have

$$\phi_{pp}(\tau) = \int p(\tau)p(t+\tau)dt$$

$$\leq \left[\int p^2(\tau)d\tau\right]^{1/2}\left[\int p^2(t+\tau)d\tau\right]^{1/2}$$

$$\leq \int p^2(\tau)d\tau$$

Therefore,

$$\phi_{pp}(\tau) \leq \phi_{pp}(0) \quad \Rightarrow \quad \phi_{pp}(0) = \max_t \phi_{pp}(t).$$

Also,

$$\phi_{xp}(t) = \phi_{pp}(t-t_0) \quad \Rightarrow \quad \phi_{xp}(t_0) = \phi_{pp}(0) = \max_t \phi_{xp}(t).$$

2.69. **(a)** Let $g(\tau) = x(t-\tau)$. Then

$$\int_{-\infty}^{\infty} g(\tau)u_1(\tau)d\tau = -g'(0) = -x'(t).$$

(b) Consider $r(t) = g(t)f(t)$. Then,

$$\int_{-\infty}^{\infty} r(t)u_1(t)dt = -r'(0) = -g'(0)f(0) - g(0)f'(0).$$

Also,

$$\int_{-\infty}^{\infty} g(t)f(0)u_1(t)dt - \int_{-\infty}^{\infty} g(t)f'(0)u_0(t)dt = -g'(0)f(0) - g(0)f'(0)$$

which is the same as above.

(c) $\int_{\infty}^{\infty} g(\tau)u_2(\tau)d\tau = g''(0).$

(d) We have

$$\int_{-\infty} g(\tau)f(\tau)u_2(\tau)d\tau = \frac{d^2}{dt^2}[g(-t)f(-t)]|_{t=0}$$

$$= -\frac{d}{dt}[g'(-t)f(-t) + g(-t)f'(-t)]_{t=0}$$

$$= g''(0)f(0) - 2g'(0)f'(0) + g(0)f''(0)$$

Therefore,

$$f(t)u_2(t) = f(0)u_2(t) - 2f'(0)u_1(t) + f''(0)u_0(t).$$

2.70. **(a)** We have

$$\sum_{m=-\infty}^{\infty} x[m]u_1[m] = \sum_m x[m]\{\delta[m] - \delta[m-1]\}$$

$$= x[0] - x[1].$$

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(b) We have

$$
\begin{aligned}
x[n]u_1[n] &= x[0]\delta[n] - x[1]\delta[n-1] + [x[0]\delta[n-1] - x[0]\delta[n-1]] \\
&= x[0]u_1[n] - \{x[1] - x[0]\}\delta[n-1] \\
&= x[0]\delta[n] - x[1]\delta[n-1] + x[1]\delta[n] - x[1]\delta[n] \\
&= x[1]u_1[n] - \{x[1] - x[0]\}\delta[n]
\end{aligned}
$$

(c) We have

$$
u_2[n] = u_1[n] * u_1[n] = \delta[n] - 2\delta[n-1] + \delta[n-2]
$$

and

$$
u_3[n] = \delta[n] - 3\delta[n-1] + 3\delta[n-2] - \delta[n-3].
$$

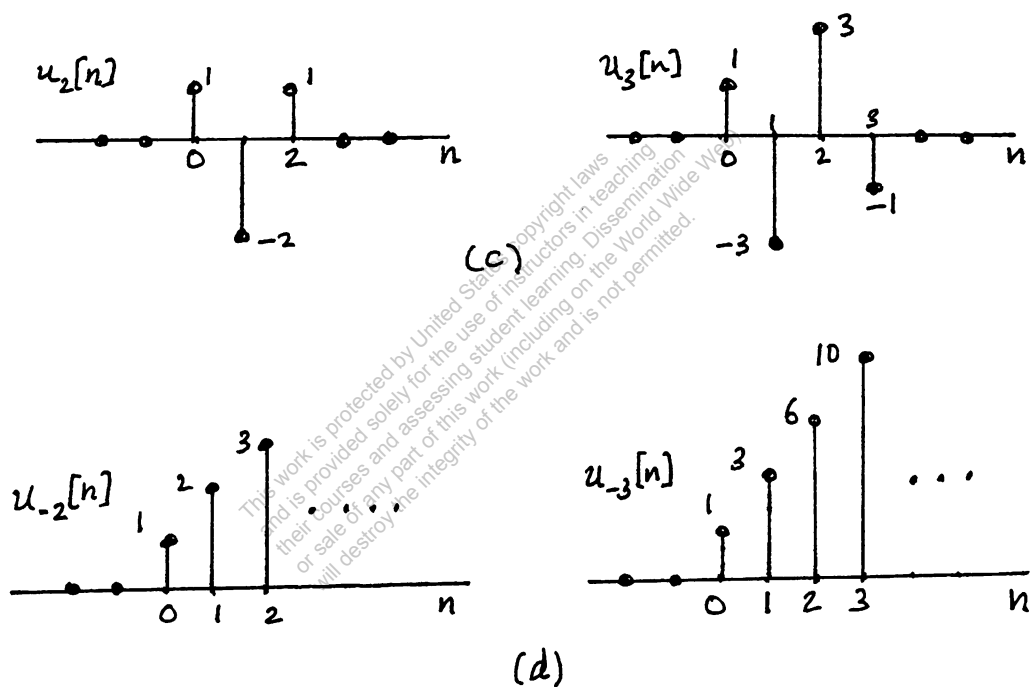The plots for these signals are as shown in Figure S2.70.

Figure S2.70

(d) We have

$$
u_{-2}[n] = (n+1), \qquad n \geq 0
$$

and

$$
u_{-3}[n] = \frac{(n+1)(n+2)}{2}, \qquad n \geq 0.
$$

The plots for these signals are as shown in the Figure S2.70.

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(e) The statement is true for $k = 1, 2, 3$. Assume it is true for k. Then, for $k > 0$

$$u_{k+1}[n] = u_1[n] * u_k[n] = u_k[n] - u_k[n-1].$$

By induction, we may now claim that the statement is true for all $k > 0$.

(f) For $k = 1$, $u_{-1}[n] = u[n]$ which shows that the statement is true. For $k = 2$,

$$u_{-2}[n] = \frac{(n+1)!}{n!}u[n] = (n+1)u[n]$$

which again shows that the statement is true. Assume that it is true for $k - 1 > 0$. Then,

$$u_{-(k-1)}[n] = u_{-k}[n] - u_{-k}[n-1]. \tag{S2.70-1}$$

Also,

$$\begin{aligned} u_{-(k-1)}[n] &= \frac{(n+k-2)!}{n!(k-2)!}u[n] \\ &= \frac{(n+k-1)!}{n!(k-1)!}u[n] - \frac{(n+k-2)!}{(n-1)!(k-2)!}u[n-2]. \end{aligned}$$

Using the above equation with eq. (S2.70-1), we get

$$u_{-k}[n] = \frac{(n+k-1)!}{n!(k-1)!}u[n].$$

By induction, we may now claim that the statement is true for all $k > 0$.

2.71. (a) We have

$$x(t) * [u_1(t) * u(t)] = x(t) = 1, \qquad \text{for all } t,$$
$$[x(t) * u_1(t)] * u(t) = 0 * u(t) = 0 \qquad \text{for all } t,$$

and

$$[x(t) * u(t)] * u_1(t) = \infty * u_1(t) = \text{undefined}.$$

(b) We have $x(t) = e^{-t}$, $h(t) = e^{-t}u(t)$, and $g(t) = u_1(t) + \delta(t)$. Therefore,

$$x(t) * [h(t) * g(t)] = x(t) = e^{-t},$$

$$[x(t) * g(t)] * h(t) = 0,$$

and

$$g(t) * [x(t) * h(t)] = g(t) * e^{-t}\int_0^\infty 1d\tau = \text{undefined}.$$

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(c) We have

$$x[n] * [h[n] * g[n]] = \left(\frac{1}{2}\right)^n * \delta[n] = \frac{1}{2}^n,$$

$$(x[n] * g[n]) * h[n] = 0 * h[n] = 0,$$

and

$$(x[n] * h[n]) * g[n] = \{(\frac{1}{2})^n \sum_{k=0}^{\infty} 1\} * g[n] = \infty.$$

(d) Let $h(t) = u_1(t)$. Then if the input is $x_1(t) = 0$, the output will be $y_1(t) = 0$. Now if $x_2(t) =$ constant, then $y_2(t) = 0$. Therefore, the system is not invertible.

Now note that

$$\left| \int_{-\infty}^{t} x_2(\tau)d\tau \right| = \begin{cases} 0 & \text{if } x_2(t) = 0 \forall t \\ \infty & \text{if } x_2(t) \neq 0 \end{cases}.$$

Therefore, if $\left|\int_{-\infty}^{t} cdt\right|_{t\to\infty} \neq \infty$, then only $x_2(t) = 0$ will yield $y_2(t) = 0$. Therefore, the system is invertible.

2.72. We have

$$\delta_\Delta(t) = \frac{1}{\Delta} u(t) * [\delta(t) - \delta(t - T)].$$

Differentiating both sides we get

$$\frac{d}{dt} \delta_\Delta t = \frac{1}{\Delta} u'(t) * [\delta(t) - \delta(t - T)]$$
$$= \frac{1}{\Delta} \delta(t) * [\delta(t) - \delta(t - T)]$$
$$= \frac{1}{\Delta} [\delta(t) - \delta(t - T)]$$

2.73. For $k = 1$, $u_{-1}(t) = u(t)$. Therefore, the given statement is true for $k = 1$. Now assume that it is true for some $k > 1$. Then,

$$u_{-(k+1)}(t) = u(t) * u_{-k}(t)$$
$$= \int_{-\infty}^{t} u_{-k}(t) = \int_{0}^{t} u_{-k}(\tau)d\tau$$
$$= \int_{0}^{t} \frac{\tau^{k-1}}{(k-1)!}, \quad t \geq 0$$
$$= \frac{\tau^k}{k(k-1)!}\bigg|_{\tau=t\geq0} = \frac{t^k}{k!} u(t).$$

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