Even Numbered Textbook Exercise Solutions

Chapter 1

1.2. The difference is as follows:

$$\begin{bmatrix} 9 & 4 & 9 \\ 6 & 4 & 8 \\ 5 & 2 & 9 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 6 \\ 2 & 1 & 6 \\ 5 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 4 & 4 & 3 \\ 4 & 3 & 2 \\ 0 & 2 & 0 \end{bmatrix}$$

D **E** **F**

1.4. Matrix **C**, the product of **A** and **B** is found as follows:

$$\begin{bmatrix} 7 & 4 & 9 \\ 6 & 4 & 12 \\ 3 & 2 & 17 \end{bmatrix}\begin{bmatrix} 7 & 6 \\ 5 & 1 \\ 9 & 12 \end{bmatrix} = \begin{bmatrix} (7\cdot 7)+(4\cdot 5)+(9\cdot 9) & (7\cdot 6)+(4\cdot 1)+(9\cdot 12) \\ (6\cdot 7)+(4\cdot 5)+(12\cdot 9) & (6\cdot 6)+(4\cdot 1)+(12\cdot 12) \\ (3\cdot 7)+(2\cdot 5)+(17\cdot 9) & (3\cdot 6)+(2\cdot 1)+(17\cdot 12) \end{bmatrix}$$

A × **B** = **C**

$$\begin{bmatrix} 7 & 4 & 9 \\ 6 & 4 & 12 \\ 3 & 2 & 17 \end{bmatrix}\begin{bmatrix} 7 & 6 \\ 5 & 1 \\ 9 & 12 \end{bmatrix} = \begin{bmatrix} 150 & 154 \\ 170 & 184 \\ 184 & 224 \end{bmatrix}$$

A × **B** = **C**

Notice that the number of columns (3) in Matrix **A** equals the number of rows in Matrix **B**. Also note that the number of rows in Matrix **C** equals the number of rows in Matrix **A**; the number of columns in **C** equals the number of columns in Matrix **B**.

1.6.a. 1/8 = .125

b. The inverse of the identity matrix is the identity matrix:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

c. The inverse of a diagonal matrix is found by inverting each of the principle diagonal elements:

$$\begin{bmatrix} .25 & 0 \\ 0 & 2 \end{bmatrix}$$

d. First, augment the matrix with the Identity Matrix:

$$\begin{matrix} row\ 1 \\ row\ 2 \end{matrix}\begin{bmatrix} 1 & 2 & \vdots & 1 & 0 \\ 3 & 4 & \vdots & 0 & 1 \end{bmatrix}$$

Now use the Gauss-Jordan Method to transform the original matrix to an identity matrix; the resulting right-hand side will be the inverse of the original matrix:

$$\begin{matrix} 1a \\ 2a \end{matrix}\begin{bmatrix} 1 & 2 & | & 1 & 0 \\ 0 & -\frac{2}{3} & | & -1 & \frac{1}{3} \end{bmatrix}\begin{matrix} row\ 1\ \times 1 \\ row\ 2\ \times \frac{1}{3} - (1a) \end{matrix}$$

$$\begin{matrix} 1b \\ 2b \end{matrix}\begin{bmatrix} 1 & 0 & | & -2 & 1 \\ 0 & 1 & | & 1.5 & -.5 \end{bmatrix}\begin{matrix} (1a) - 2\ \times (2b) \\ (2a) \times -3/.2 \end{matrix}$$

Thus, the inverse matrix is:

$$\begin{bmatrix} -2 & 1 \\ 1.5 & -.5 \end{bmatrix}$$

e. $$\begin{bmatrix} .02 & .04 & | & 1 & 0 \\ .06 & .08 & | & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & | & 50 & 0 \\ 0 & \frac{2}{3} & | & 50 & -16\frac{2}{3} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & -100 & 50 \\ 0 & 1 & | & 75 & -25 \end{bmatrix}$$

The inverse matrix is:

$$\begin{bmatrix} -100 & 50 \\ 75 & -25 \end{bmatrix}$$

f. The inverse matrix is: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

g. The inverse matrix is: $\begin{bmatrix} .04 & .04 \\ .04 & .16 \end{bmatrix}$

h. $$\begin{bmatrix} 2 & 0 & 0 & | & 1 & 0 & 0 \\ 2 & 4 & 0 & | & 0 & 1 & 0 \\ 4 & 8 & 20 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & .5 & 0 & 0 \\ 0 & 4 & 0 & | & -1 & 1 & 0 \\ 0 & 8 & 20 & | & -2 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & .5 & 0 & 0 \\ 0 & 1 & 0 & | & -.25 & .25 & 0 \\ 0 & 0 & 2 & | & 0 & -2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & .5 & 0 & 0 \\ 0 & 1 & 0 & | & -.25 & .25 & 0 \\ 0 & 0 & 1 & | & 0 & -.1 & .05 \end{bmatrix}$$

The inverse matrix is:

$$\begin{bmatrix} .5 & 0 & 0 \\ -.25 & .25 & 0 \\ 0 & -.1 & .05 \end{bmatrix}$$

1.8.a. See 6.g above for the inverse of **C**:

$$C^{-1} = \begin{bmatrix} .04 & .04 \\ .04 & .16 \end{bmatrix}$$

$$\begin{bmatrix} .04 & .04 \\ .04 & .16 \end{bmatrix} \cdot \begin{bmatrix} .01 \\ .11 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} .0048 \\ .01764 \end{bmatrix}$$

$$\boldsymbol{C^{-1}} \cdot \mathbf{s} = \mathbf{x} = \mathbf{x}$$

b. Our original system of equations is represented:

$$\begin{bmatrix} .08 & .08 & .1 & 1 \\ .08 & .32 & .2 & 1 \\ .1 & .2 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} .1 \\ .1 \\ .1 \\ .1 \end{bmatrix}$$

$$\boldsymbol{C} \cdot \mathbf{x} = \mathbf{s}$$

The elements of **C** and **s** are known; our problem is to find the weights in vector **x**. Thus we will rearrange the system from **Cx** = **s** to **x** = $\mathbf{C}^{-1}\mathbf{s}$, where $\mathbf{C}^{-1}$ is the inverse of matrix **C**. So, the time-consuming part of our problem is to find $\mathbf{C}^{-1}$. We will begin by augmenting Matrix **C** with the Identity Matrix **I**:

$$\begin{array}{r} Row\ 1 \\ Row\ 2 \\ Row\ 3 \\ Row\ 4 \end{array} \left[\begin{array}{cccc|cccc} .08 & .08 & .1 & 1 & 1 & 0 & 0 & 0 \\ .08 & .32 & .2 & 1 & 0 & 1 & 0 & 0 \\ .1 & .2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} Original \\ System \end{array}$$

$$\begin{array}{r} 1a \\ 2a \\ 3a \\ 4a \end{array} \left[\begin{array}{cccc|cccc} 1 & 1 & 1.25 & 12.5 & 12.5 & 0 & 0 & 0 \\ 0 & 3 & 1.25 & 0 & -12.5 & 12.5 & 0 & 0 \\ 0 & 1 & -1.25 & -12.5 & -12.5 & 0 & 10 & 0 \\ 0 & 0 & -1.25 & -12.5 & -12.5 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} (row1)\cdot 12.5 \\ (row2)\cdot 12.5-(1a) \\ (row3)\cdot 10(1a) \\ (row4)\cdot 1-(1a) \end{array}$$

$$\begin{array}{r} 1b \\ 2b \\ 3b \\ 4b \end{array} \left[\begin{array}{cccc|cccc} 1 & 0 & .83\bar{3} & 12.5 & 16.\bar{6} & -4.1\bar{6} & 0 & 0 \\ 0 & 1 & .41\bar{6} & 0 & -4.1\bar{6} & 4.1\bar{6} & 0 & 0 \\ 0 & 0 & -1.6\bar{6} & -12.5 & -8.3\bar{3} & -4.1\bar{6} & 10 & 0 \\ 0 & 0 & -1.25 & -12.5 & -12.5 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} (row1a)\cdot 2b \\ (row2a)\cdot 1/3 \\ (row3a)-(2b) \\ (row4a) \end{array}$$

$$\begin{array}{r} 1c \\ 2c \\ 3c \\ 4c \end{array} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 6.25 & 12.5 & -6.25 & 5 & 0 \\ 0 & 1 & 0 & -3.125 & -6.25 & 3.125 & 2.5 & 0 \\ 0 & 0 & 1 & 7.5 & 5 & 2.5 & -6 & 0 \\ 0 & 0 & 0 & -3.125 & -6.25 & 3.125 & 7.5 & 1 \end{array}\right] \begin{array}{c} (1b)-(3c)\cdot\ .83 \\ (2b)-(3c)\cdot\ .416 \\ (3b)\cdot -1/1.6 \\ (4b)-(3c)\cdot -1.25 \end{array}$$

$$\begin{array}{r} 1d \\ 2d \\ 3d \\ 4d \end{array} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & -10 & 2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 10 & -1 \\ 0 & 0 & 1 & 0 & -10 & 10 & -24 & 2.4 \\ 0 & 0 & 0 & 1 & 2 & -1 & 2.4 & -.32 \end{array}\right] \begin{array}{c} (1c)-(4d)\cdot\ 6.2 \\ (2c)-(4d)\cdot -3.125 \\ (3c)-(4d)\cdot\ 7.5 \\ (4c)\cdot -1/(3.125) \end{array}$$

$$I \qquad\qquad\qquad C^{-1}$$

$$\begin{bmatrix} 0 & 0 & -10 & 2 \\ 0 & 0 & 10 & -1 \\ -10 & 10 & -24 & 2.4 \\ 2 & -1 & 2.4 & -.32 \end{bmatrix} \begin{bmatrix} .1 \\ .1 \\ .1 \\ .1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -.8 \\ .9 \\ -2.16 \\ .308 \end{bmatrix}$$

$$\boldsymbol{C}^{-1} \cdot \boldsymbol{s} = \boldsymbol{x} = \boldsymbol{x}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -10 & 2 \\ 0 & 0 & 10 & -1 \\ -10 & 10 & -24 & 2.4 \\ 2 & -1 & 2.4 & -.32 \end{bmatrix} \begin{bmatrix} .1 \\ .1 \\ .1 \\ .1 \end{bmatrix} = \begin{bmatrix} -.8 \\ .9 \\ -2.16 \\ .308 \end{bmatrix}$$

$$\mathbf{x} = \mathbf{C}^{-1} \cdot \mathbf{s} = \mathbf{x}$$

Now it is clear that:

$x_1 = (0\times.1) + (0\times.1) + (-10\times.1) + (2\times.1) = -.8$
$x_2 = (0\times.1) + (0\times.1) + (10\times.1) + (-1\times.1) = .9$
$x_3 = (-10\times.1) + (10\times.1) + (-24\times.1) + (2.4 \times .1) = -2.16$
$x_4 = (2\times.1) + (-1\times.1) + (2.4\times.1) + (-.32\times.1) = .308$

1.10. We will use the Power Rule for all three parts of this problem.

a. When there is one coefficient, one variable and one exponent, the polynomial function is written as follows: $y = c \cdot x^n$, where c = 7 and n = 4. The derivative of y with respect to x is $\frac{dy}{dx} = c \cdot n \cdot x^{n-1} = 7 \times 4 \times x^{4-1} = 28x^3$.

b. When there is one variable x, m coefficients (c_j) and m exponents (n_j) as in $y = \sum_{j=1}^{m} c_j \cdot x^{n_j}$, the derivative y with respect to x is $\frac{dy}{dx} = \sum_{j=1}^{m} c_j \cdot n_j \cdot x^{n_j - 1}$. Thus, when $y = 5x^2 - 3x + 2$, its derivative with respect to x is: $dy/dx = 10x - 3$.

c. When $y = -7x^2 + 4x + 5$, its first derivative with respect to x is: $dy/dx = -14x + 4$.

1.12.a. $s'(t) = 3t^2 - 6t + 1$; $s'(1) = -2$ per year

The annual rate of change of stumpage value at the exact instant time t=1 is -2. This means that at that moment (t=1), the stumpage value tends to decline at an annual rate of 2 over a very short interval. In reality, the value will not normally decline exactly $2, because the rate of decline is not linear (in this example), and will curve away from the tangent line at the point on the graph where t=1. The smaller the time change from time t=1, the closer that the derivative will estimate the change in stumpage value per unit time. For example, from time t=1 to t=1.01 (1/100th of a year later from year 1), the value will drop approximately .02. Here is the calculation. Label the change in time by Δt and the resulting change in price by Δs. Then Δt=1.01-1=.01, and note that the rate

$$\frac{\Delta s}{\Delta t} \approx s'(1) = -2, \text{ and so } \Delta s \approx -2\Delta t = -2(.01) = -.02$$

b. $s'(3) = 10\, per\ year$

The average rate of change of the stumpage value over a time interval does not require calculus, it simply equals the change in value divided by the change in time.

c. Average rate = $\frac{s(3)-s(1)}{3-1} = \frac{13-9}{3-1}$= 2 per year.
We calculated that the value increase to be 4 (from 9 to 13) over the 2-year period from t=1 to t=3 to arrive at an annual rate of increase of 2. Of course, the (instantaneous) rate of change of value over all of the infinite number of moments of time from time t=1 to t=3, measured by $s'(t)$ at each of these moments of time t, averages to 2 per year.

d. It is not clear. But, we might propose the following: The $(-3t^2)$ part of the function is the only part of the of the value function that is declining. We might propose that the fungus will continue to diminish the value of the trees, but this decrease will be more than offset by increases from healthy growth.

1.14. Each term in a polynomial can be written as $y = cx^n$. The derivative of this term with respect to x is:

$$\frac{dy}{dx} = f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{c(x+h)^n - cx^n}{h}$$

We now rewrite using the Binomial Theorem and simplify:

$$\frac{dy}{dx} = \lim_{\Delta x\to 0}\frac{c\left(\binom{n}{0}x^n(\Delta x)^0 + \binom{n}{1}x^{n-1}(\Delta x)^1 + \binom{n}{2}x^{n-2}(\Delta x)^2 + \cdots + \binom{n}{n-1}x^1(\Delta x)^{n-1} + \binom{n}{n}x^0(\Delta x)^n\right) - cx^n}{\Delta x}$$

$$\frac{dy}{dx} = \lim_{\Delta x\to 0}\frac{c\left(\binom{n}{1}x^{n-1}(\Delta x)^1 + \binom{n}{2}x^{n-2}(\Delta x)^2 + \cdots + \binom{n}{n-1}x^1(\Delta x)^{n-1} + \binom{n}{n}x^0(\Delta x)^n\right)}{\Delta x}$$

$$\frac{dy}{dx} = \lim_{\Delta x\to 0} c\left(\binom{n}{1}x^{n-1} + \binom{n}{2}x^{n-2}(\Delta x)^{2-1} + \cdots + \binom{n}{n-1}x^1(\Delta x)^{n-2} + \binom{n}{n}x^0(\Delta x)^{n-1}\right) = cnx^{n-1}$$

If $y = \sum_{j=1}^{m} c_j \cdot x^{n_j}$, then by the sum rule and the derivation above, we conclude that:

$$\frac{dy}{dx} = \sum_{j=1}^{m} c_j \cdot n_j \cdot x^{n_j - 1}$$

1.16.a. dy/dx = $.05e^{.05x}$
b. dy/dx = $e^x(x-1)/x^2$
c. dy/dx = 5/x
d. dy/dx = $e^x(1/x + \ln(x))$ (using the product rule)
e. dy/dx = $x^2e^x + 2xe^x$ (using the product, power and Exponential rules)
f. $\frac{dy}{dx} = \frac{1}{5x^3+x}(15x^2+1)$ (using the power, log, and chain rules)
g. Since $y = 5x^3 - 6x^{\frac{1}{2}} + 2e^x$, then $\frac{dy}{dx} = \frac{d}{dx}[5x^3] + \frac{d}{dx}\left[-6x^{\frac{1}{2}}\right] + \frac{d}{dx}[2e^x]$ (sum rule)

$= 5\frac{d}{dx}[x^3] - 6\frac{d}{dx}\left[x^{\frac{1}{2}}\right] + 2\frac{d}{dx}[e^x]$ (constant multiple rule) $= 5(3)x^2 - 6\left(\frac{1}{2}\right)x^{-\frac{1}{2}} + 2e^x$

(power and exponential rules) $= 15x^2 - 3x^{-\frac{1}{2}} + 2e^x$

h. $\frac{dy}{dx} = x^2\left(\frac{1}{x}\right) + 2xlnx = x + 2xlnx$ (product rule, power rule and sum rule)

1.18.a.Let $x_0 = 5$, such that we have $f(x_0) = 125$, $f'(x_0) = 3x_0^2 = 75$, $f''(x_0) = 6x_0 = 30$, $f'''(x_0)$= 6, and all higher order derivatives are equal to zero. Now, suppose we wish to increase x by Δx = 1 to $x_1 = 6$. Our estimate for y_1 is 216, determined as follows:

$$f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\Delta x + \frac{1}{2!} f''(x_0)(\Delta x)^2 + \frac{1}{3!} f'''(x_0)(\Delta x)^3$$

$$= 125 + 75(1) + \frac{1}{2} \cdot 30(1)^2 + \frac{1}{6} \cdot 6(1)^3 = 216$$

b. This third order expansion provided an exact solution for f(5+1) = 216. This third order expansion provided an exact solution for *f(x)* because *f(x)* was differentiable only three times (higher order derivatives equal zero). In many cases, we will be able to obtain reasonable approximations where some non-zero derivatives are not used.

c. The first order approximation results in f(6) ≈ 125 +75 = 200.

d. The second order approximation results in f(6) ≈ 125 + 75 +15 = 215.

e. First, note that since $x_0 = 2$, $f(x_0) = 80$, $f'(x_0) = 30x_0^2 = 120$, $f''(x_0) = 60x_0 = 120$, $f''''(x_0)$ = 60, and all higher order derivatives are equal to zero. If we increase x by $\Delta x = 3$ to $x_1 = 5$, first order, second order and third order approximations are as follows:

$$f(2+3) \approx f(2) + f'(2) \cdot 3 = 80 + 120 \cdot 3 = 440$$

$$f(2+3) \approx f(2) + f'(2) \cdot 3 + \frac{1}{2} f''(2) \cdot 3^2 = 80 + 120 \cdot 3 + \frac{1}{2} \cdot 120 \cdot 9 = 980$$

$$f(2+3) \approx f(2) + f'(2) \cdot 3 + \frac{1}{2} f(2) \cdot 3^2 + \frac{1}{6} \text{f}(2) \cdot 3^2$$

$$= 80 + 120 \cdot 3 + \frac{1}{2} \cdot 120 \cdot 9 + \frac{1}{6} \cdot 60 \cdot 27 = 1250$$

This third order expansion provided an exact solution for f(5) = 1250.

1.20.a. First, we will work with the appropriate Lagrange function L = U(**x**) – λ(g(**x**) – w), which will be derived from the following objective and constraint functions:

$$U = 100x_1+200x_2+250x_3+350x_4-.01x_1^2-.02x_2^2-.03x_3^2-.04x_4^2-.02x_3x_4$$

$$\text{s.t.: } g(\mathbf{x}) = x_1 + 2x_2 + 3x_3 + 4x_4 = 10{,}000 = w$$

Thus, the Lagrange function is written:

$$L = 100x_1+200x_2+250x_3+350x_4-.01x_1^2-.02x_2^2-.03x_3^2-.04x_4^2-.2x_3x_4 + \lambda(10{,}000-(x_1 + 2x_2 + 3x_3 + 4x_4))$$

To solve, find first order conditions, which are written as follows in matrix format:

$$\nabla \text{L}(\mathbf{x}) = \mathbf{0}$$

$$\frac{\partial \text{L}}{\partial \lambda} = \text{g}(\mathbf{x}) - \text{w} = 0$$

The notation $\nabla L(\mathbf{x})$ means the partial derivatives of the function L with respect to each of the variables x_1, x_2, x_3, and x_4 respectively. The results of the five derivatives above all set to zero give five linear equations in the variables x_1, x_2, x_3, x_4, and λ. Rearranging slightly so that the constants are subtracted and appear on the right hand side and putting these equations in matrix form, we have:

$$\begin{bmatrix} -.02 & 0 & 0 & 0 & -1 \\ 0 & -.04 & 0 & 0 & -2 \\ 0 & 0 & -.06 & -.2 & -3 \\ 0 & 0 & -.2 & -.08 & -4 \\ -1 & -2 & -3 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \lambda \end{bmatrix} = \begin{bmatrix} -100 \\ -200 \\ -250 \\ -350 \\ -10{,}000 \end{bmatrix}$$

Now, solve this system for unknowns by inverting the coefficients matrix and rewriting:

$$\begin{vmatrix} -39.523809 & 10.4761904 & 3.333333 & 2.1428571 & -0.2095238 \\ 10.4761904 & -14.523809 & 3.333333 & 2.1428571 & -0.2095238 \\ 3.33333333 & 3.33333333 & 3.333333 & -5 & -0.0666666 \\ 2.14285714 & 2.14285714 & -5 & 2.1428571 & -0.0428571 \\ -0.2095238 & -0.2095238 & -0.06667 & -0.042857 & 0.00419047 \end{vmatrix} \times \begin{bmatrix} -100 \\ -200 \\ -250 \\ -350 \\ -10{,}000 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \lambda \end{bmatrix}$$

Thus, the optimal consumption levels are \$2,369.047619 in year 1, \$2,369.047619 in year 2, 583.3333333 in year 3 and 285.7142857 in year 4. The LaGrange multiplier is 52.61904762

b. Substitute into the following utility function optimal consumption levels as follows (note rounding of some values):

$$U = 100\times2369+200\times2369+250\times583+350\times285 - .01\times2369^2 - .02\times2369^2 - .03\times583^2 - .04\times285^2 - .02\times583\times285 = 771{,}066.$$

1.22.a. F(x) = C

b. F(x) = 7x + C

c. $F(x) = x^2 + C$

d. $F(x) = 7x^3 + C$

e. $F(x) = 7x^3 + 5x + C$

f. $F(x) = e^x + C$

g. $F(x) = e^{.5x} + C$

h. $F(x) = 5^x + C$ since $\int a^x dx = \frac{1}{\ln a} a^x + C$ for any positive constant *a*.

i. $F(x) = \ln|x| + C$

j. $F(x) = 5\ln|x| + 3e^x + (4/3)x^3 - .5x^2 + C$

1.24. We will divide the area under the curve into a number of rectangles, reduce widths of these rectangles and allow their numbers to approach infinity ($n \to \infty$). Next, we will find the area of each rectangle. Each of these rectangles, which are numbered sequentially, will have a width of $\Delta x = (b-a)/n = (1-0)/n = 1/n \to 0$ and a height of $f(x_i^*)$ where x_i^* is some value between x_{i-1} and x_i (for sake of simplicity here, assume $x_i^* = x_i$). In this example $x_i = i/n$. Since the number of these rectangles under the curve will approach infinity, we have the width of each of these rectangles to approach (though not quite equal) zero. The area of each of these rectangles (where the product is non-negative) is simply the product of its height and width. The area of each box and the sum of these areas is computed as follows:

$$\sum_{i=1}^{n} f(x_i)\Delta x_i = \sum_{i=1}^{n}[10x_i - x_i^2]\frac{1}{n} = \sum_{i=1}^{n}\left[\frac{10i}{n} - \left(\frac{i}{n}\right)^2\right]\frac{1}{n}$$

So the exact area is

$$\int_0^1 (10x - x^2)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\left[\frac{10i}{n} - \left(\frac{i}{n}\right)^2\right]\frac{1}{n} = \lim_{n\to\infty}\left[\frac{10}{n^2}\sum_{i=1}^{n} i - \frac{1}{n^3}\sum_{i=1}^{n} i^2\right].$$

The following sums are well-known and can be found after some algebraic calculations:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \;\; and \;\; \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Now, substitute these results into the right hand side of the above equation to obtain

$$\int_0^1 (10x - x^2)\,dx = \lim_{n\to\infty}[\frac{10n(n+1)}{2n^2} - \frac{n(n+1)(2n+1)}{6n^3}] = \lim_{n\to\infty}\left[5 + \frac{5}{n} - \left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right)\right]$$

Obviously, as $n \to \infty$, this area approaches 5- $1/3$ = $4\,2/3$. We obtain the same result through antidifferentiation:

$$\int_0^1 (10x - x^2)\,dx = 5x^2 - \frac{1}{3}x^3\Big|_0^1 = 5 - \frac{1}{3} = 4\frac{2}{3}$$

1.26. Suppose that *z*, *y* and *x* are all functions of *t* such that:

$$\frac{1}{z}\frac{dz}{dt} = x\frac{dx}{dt} + 5\frac{dy}{dt}.$$

We first multiply both sides by dt:

$$\frac{dz}{z} = xdx + 5dy$$

Next, we integrate the left side over z and the right side over x then y, find anti-logs and z:

$$\int \frac{dz}{z} = \int xdx + \int 5dy$$

$$lnz = \frac{1}{2}x^2 + 5y + C$$

$$z = Ke^{\frac{1}{2}x^2+5y}$$

with $K = e^C$.

Chapter 2

2.2.a. The individual outcomes are all of the possible 3 coin outcomes. The sample space Ω={HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}, where H means heads and T means tails.

b. There are 8 outcomes, meaning that there are 2^8 = 256 possible events including all possible combinations of outcomes and ∅.

c. Yes; 1/8+1/8 = 1/4

d. Yes; 0

2.4.

a. $E[R_{Mc}]$ = .062