1 Solutions to Odd Numbered Problems *Random Processes for Engineers*

1.1 Simple events (a) $\Omega = \{0,1\}^8$, or $\Omega = \{x_1x_2x_3x_4x_5x_6x_7x_8 : x_i \in \{0,1\}$ for each $i\}$. It is natural to let $\mathcal{F}$ be the set of all subsets of Ω. Finally, let $P(A) = \frac{|A|}{256}$, where $|A|$ denotes the cardinality of a set $|A|$.
(b) $E_1 = \{01010101, 10101010\}$ and $P(E_1) = \frac{2}{256} = \frac{1}{128}$.
$E_2 = \{00110011, 01100110, 11001100, 10011001\}$ and $P(E_2) = 4/256 = 1/64$.
$E_3 = \{x \in \Omega : x_1 + \cdots + x_8 = 4\}$ and $P(E_2) = \binom{8}{4}/256 = 70/256 = 35/128$.
$E_4 = \{11111111, 11111110, 11111101, 10111111, 01111111, 00111111, 01111110, 11111100\}$ and $P(E_4) = 8/256 = 1/32$.
(c) $E_1 \subset E_3$, so $P(E_1|E_3) = |E_1|/|E_3| = 2/70 = 1/35$.
$E_2 \subset E_3$, so $P(E_2|E_3) = |E_2|/|E_3| = 4/70 = 2/35$.

1.3 Ordering of three random variables $P\{X < u < Y\} = P\{X < u\}P\{u < Y\} = (1 - e^{-\lambda u})e^{-\lambda u} = e^{-\lambda u} - e^{-2\lambda u}$. Averaging over the choices of u using the pdf of U yields,

$$P\{X < U < Y\} = \int_0^1 e^{-\lambda u} - e^{-2\lambda u} du = \frac{0.5 - e^{-\lambda} + 0.5e^{-2\lambda}}{\lambda}.$$

1.5 Congestion at output ports (a) One possibility is $\Omega = \{1, 2, \ldots, 8\}^4 = \{(d_1, d_2, d_3, d_4) : 1 \le d_i \le 8 \text{ for } 1 \le i \le 4\}$, where the packets are assumed to be numbered one through four, and d_i is the output port of packet i. Let $\mathcal{F}$ be all the subsets of Ω, and for any event A, let $P(A) = \frac{|A|}{8^4}$.
(b)

$$P\{X_1 = k_1, \ldots, X_8 = k_8\} = \frac{1}{8^4}\binom{4}{k_1 k_2 \cdots k_8}$$

where $\binom{4}{k_1 k_2 \cdots k_8} = \frac{4!}{k_1! k_2! \cdots k_8!}$ is the multinomial coefficient.
(c) One way to do this problem is to note that $X_j = \sum_{i=1}^4 X_{ij}$, where $X_{ij} = 1$ if packet i is routed to output port j, and $X_{ij} = 0$ otherwise. Suppose $j \neq j'$. Then $X_{ij}X_{ij'} \equiv 0$, and so also, $E[X_{ij}X_{ij'}] = 0$. Thus, $\mathrm{Cov}(X_{ij}, X_{ij'}) = 0 - \frac{1}{8}^2 = -\frac{1}{64}$.

Also, $\mathrm{Cov}(X_{ij}, X_{i'j'}) = 0$ if $i \neq i'$. Thus,

$$\begin{aligned}\mathrm{Cov}(X_j, X_{j'}) &= \mathrm{Cov}(\sum_{i=1}^{4} X_{ij}, \sum_{i'=1}^{4} X_{i'j'}) \\ &= \sum_{i=1}^{4}\sum_{i'=1}^{4} \mathrm{Cov}(X_{ij}, X_{i'j'}) \\ &= \sum_{i=1}^{4} \mathrm{Cov}(X_{ij}, X_{ij'}) = 4(-\frac{1}{64}) = -\frac{1}{16}.\end{aligned}$$

(d) Consider the packets one at a time in order. The first packet is routed to a random output port. The second is routed to a different output port with probability $\frac{7}{8}$. Given the first two packets are routed to different output ports, the third packet is routed to yet another output port with probability $\frac{6}{8}$. Similarly, given the first three packets are routed to distinct output ports, the fourth packet is routed to yet another output port with probability $\frac{5}{8}$. The answer is thus $\frac{8\cdot 7\cdot 6\cdot 5}{8^4} = \frac{105}{256} \approx 0.410$.

(e) The event is not true if and only if there are either exactly 3 packets assigned to one output port or all four packets assigned to one output port. There are $4\cdot 8\cdot 7$ possibilities for exactly three packets to be assigned to one output port, since there are four choices for which packet is not with the other three, eight choices of output port for the group of three, and given that, seven choices of output port for the fourth packet. There are 8 possibilities for all four packets to be routed to the same output port. Thus, some output port has three or more packets assigned to it with probability $\frac{4\cdot 8\cdot 7+8}{8^4} = \frac{4\cdot 7+1}{8^3} = \frac{29}{512} \approx 0.0566$. Thus, $P\{X_i \leq 2 \text{ for all } i\} = 1 - \frac{29}{512} \approx 0.9434$.

1.7 Conditional probability of failed device given failed attempts (a) P(first attempt fails)=0.2+(0.8)(0.1)=0.28

(b) P(server is working | first attempt fails) =
P(server working, first attempt fails)/P(first attempt fails) =(0.8)(0.1)/0.28≈ 0.286

(c) P(second attempt fails | first attempt fails) =P(first two attempts fail)/P(first attempt fails) = $[0.2 + (0.8)(0.1)^2]/0.28$ ≈0.783

(d) P(server is working | first and second attempts fail) =P(server is working and first two attempts fail)/P(first two attempts fail) = $(0.8)(0.1)^2/[0.2 + (0.8)(0.1)^2]$ ≈0.0385

1.9 Conditional lifetimes; memoryless property of the geometric distribution
(a) $P\{X > 3\} = 1 - p(3) = 0.8$, $P(X > 8|X > 5) = \frac{P(\{X>8\}\cap\{X>5\})}{P\{X>5\}} = \frac{P\{X>8\}}{P\{X>5\}} = \frac{0}{0.40} = 0$.
(So a five year old working battery is not equivalent to a new one!)
(b) $P\{Y > 3\} = P(\text{miss first three shots}) = (1-p)^3$. On the other hand,

$$P(Y > 8|Y > 5) = \frac{P(\{Y > 8\}\cap\{Y > 5\})}{P\{Y > 5\}} = \frac{P\{Y > 8\}}{P\{Y > 5\}} = \frac{(1-p)^8}{(1-p)^5} = (1-p)^3.$$

(A player that has missed five shots is equivalent to a player just starting to take shots.)
(c) Y has a geometric distribution. (Part (b) illustrates the fact that the geometric distribution is the memoryless lifetime distribution on the positive integers. The exponential distribution is the continuous type distribution with the same property.)
1.11 Distribution of the flow capacity of a network One way to solve this problem is to compute X for each of the 32 outcomes for the links. Another is to use divide and conquer by conditioning on the state of a key link, such as link 4.

$$\begin{aligned} P\{X=0\} &= P(((F_1F_3)\cup(F_2F_5))F_4^c) + P((F_1\cup F_2)(F_3\cup F_5)F_4) \\ &= ((0.2)^2+(0.2)^2-(0.2)^4)(0.8) + (0.2+0.2-(0.2)^2)^2(0.2) = 0.08864. \end{aligned}$$

$$\begin{aligned} P\{X=10\} &= P(F_1^cF_3^c(F_2F_5)^cF_4^c) + P(F_1^cF_2^cF_3^cF_5^cF_4) \\ &= (0.8)^3(1-(0.2)^2) + (0.8)^4(0.2) = 0.57344. \end{aligned}$$

$P\{X=5\} = 1 - P\{X=0\} - P\{X=10\} = 0.33792.$
1.13 A CDF of mixed type (a) $F_X(0.8) = 0.5$.
(b) There is a half unit of probability mass at zero and a density of value 0.5 between 1 and 2. Thus, $E[X] = 0\times 0.5 + \int_1^2 x(0.5)dx = 3/4$ and,
(c) $E[X^2] = 0^2\times 0.5 + \int_1^2 x^2(0.5)dx = 7/6$. So $\mathrm{Var}(X) = 7/6 - (3/4)^2 = 29/48$
.
1.15 Poisson and geometric random variables with conditioning
(a) $P\{Y<Z\} = \sum_{i=0}^{\infty}\sum_{j=i+1}^{\infty}\frac{e^{-\mu}\mu^i}{i!}p(1-p)^{j-1} = \sum_{i=0}^{\infty}\frac{e^{-\mu}[\mu(1-p)]^i}{i!} = e^{-\mu p}$
(b) $P(Y<Z|Z=i) = P(Y<i|Z=i) = P\{Y<i\} = \sum_{j=0}^{i-1}\frac{e^{-\mu}\mu^j}{j!}$
(c) $P(Y=i|Y<Z) = P\{Y=i<Z\}/P\{Y<Z\} = \left(\frac{e^{-\mu}\mu^i}{i!}(1-p)^i\right)/e^{-\mu p} = \frac{e^{-\mu(1-p)}[\mu(1-p)]^i}{i!}$, which is the Poisson distribution with mean $\mu(1-p)$
(d) $\mu(1-p)$
1.17 Transformation of a random variable (a) Observe that Y takes values in the interval $[1,+\infty)$.

$$F_Y(c) = P\{\exp(X)\le c\} = \begin{cases} P\{X\le \ln c\} = 1-\exp(-\lambda\ln c) = 1-c^{-\lambda} & c\ge 1 \\ 0 & c<1 \end{cases}$$

Differentiate to obtain

$$f_Y(c) = \begin{cases} \lambda c^{-(1+\lambda)} & c\ge 1 \\ 0 & c<1 \end{cases}$$

(b) Observe that Z takes values in the interval $[0,3]$.

$$F_Z(c) = P\{\min\{X,3\}\le c\} = \begin{cases} 0 & c<0 \\ P\{X\le c\} = 1-\exp(-\lambda c) & 0\le c<3 \\ 1 & c\ge 3 \end{cases}$$

The random variable Z is neither discrete nor continuous type. Rather it is a mixture, having a density over the interval $[0,3)$ and a discrete mass at the point 3.

1.19 Moments and densities of functions of a random variable
$E[C] = 2E[L] + 2E[W] = 2$ $\quad \mathrm{Var}(C) = 4\mathrm{Var}(L) + 4\mathrm{Var}(W) = \frac{2}{3}$ $\quad$ The pdf of C is the convolution of the pdf of $2L$ with the pdf of $2W$. But $2L$ and $2W$ are each uniformly distributed over the interval $[0,2]$, so their pdfs are rectangular pulse functions. The convolution of such a function with itself is a triangular pulse function. The base of the triangle, equal to the support of f_C, is the interval $[0,4]$. The peak of the triangle is at the midpoint, and must have height 1/2 in order that the area of the triangle be one. Therefore, $f_C(x) = \begin{cases} x/4 & 0 \le x \le 2 \\ \frac{4-x}{4} & 2 \le x \le 4 \\ 0 & \text{else} \end{cases}$

$E[A] = E[L]E[W] = (\frac{1}{2})^2 = \frac{1}{4}$ $\quad E[A^2] = E[L^2]E[W^2] = (\frac{1}{3})^2 = \frac{1}{9}$ $\quad$ so $\mathrm{Var}(A) = \frac{1}{9} - (\frac{1}{4})^2 = \frac{7}{144}$.

For $0 \le c \le 1$, $P\{A \le c\}$ =area of $\{(x,y) \in [0,1]^2 : xy \le c\} = c + \int_c^1 \frac{c}{x} dx = c(1 - \ln c)$, so $f_A(c) = \begin{cases} -\ln(c) & 0 \le c \le 1 \\ 0 & \text{else} \end{cases}$

1.21 Using the Gaussian Q function (a) $P\{X \ge 16\} = P\{\frac{X-10}{3} > \frac{16-10}{3}\} = Q(\frac{16-10}{3}) = Q(2)$.
(b) $P\{X^2 \ge 16\} = P\{X \ge 4\} + P\{X \le -4\} = Q(\frac{4-10}{3}) + 1 - Q(\frac{-4-10}{3}) = Q(-2) + 1 - Q(-\frac{14}{3}) = 1 - Q(2) + Q(\frac{14}{3})$.
(c) Z is $N(0,5)$ so $P\{|Z| > 1\} = P\{Z > 1\} + P\{Z < -1\} = Q(\frac{1}{\sqrt{5}}) + 1 - Q(-\frac{1}{\sqrt{5}}) = 2Q(\frac{1}{\sqrt{5}})$.

1.23 Correlation of histogram values (a) X_1 is Bernoulli($\frac{1}{6}$), so $E[X_1] = \frac{1}{6}$ and $\mathrm{Var}(X_1) = \frac{1}{6}(1 - \frac{1}{6}) = \frac{5}{36}$.
(b) $E[X] = nE[X_1] = \frac{n}{6}$ and $\mathrm{Var}(X) = n\mathrm{Var}(X_1) = \frac{5n}{36}$.
(c) We begin by computing $\mathrm{Cov}(X_1, Y_1)$. Since $X_1 Y_1 = 0$ with probability one, $E[X_1 Y_1] = 0$. Therefore $\mathrm{Cov}(X_1, Y_1) = E[X_1 Y_1] - E[X_1]E[Y_1] = 0 - \frac{1}{6}\frac{1}{6} = \frac{-1}{36}$. So $\mathrm{Cov}(X_i, Y_i) = \frac{-1}{36}$ for any i. On the other hand, if $i \ne j$ then X_i is independent of X_j. So

$$\mathrm{Cov}(X_i, Y_j) = \begin{cases} \frac{-1}{36} & \text{if } i = j \\ 0 & \text{if } i \ne j \end{cases}$$

(d)

$$\mathrm{Cov}(X, Y) = \sum_i \sum_j \mathrm{Cov}(X_i, Y_j) = \sum_i \mathrm{Cov}(X_i, Y_i) = n\mathrm{Cov}(X_1, Y_1) = \frac{-n}{36}.$$

and

$$\rho(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}} = \frac{-1}{5}.$$

(e) Given that x of the dice show a 1, each of the remaining dice is equally likely to show 2,3,4,5, or 6. Thus, each of the remaining $n-x$ dice shows a 2 with conditional probability $\frac{1}{5}$. Therefore $E[Y|X=x] = \frac{n-x}{5}$.

1.25 A function of jointly distributed random variables The square has unit area so that the joint density is unit valued within the square. The range of X is the interval $[0,1]$, so fix c in $[0,1]$ and consider the event $\{UV \leq c\}$. The probability of this event is the area of the square minus the upper right region above the curve $v = c/u$. This area is one minus the area of the region inside the square above the curve $v = c/u$. Therefore,

$$F_X(c) = \begin{cases} 0 & c \leq 0 \\ 1 - \int_c^1 (1 - \frac{c}{u})du = c - c\ln c & 0 \leq c \leq 1 \\ 1 & c \geq 1 \end{cases}$$

Differentiating yields

$$f_X(c) = \begin{cases} -\ln c & 0 < c \leq 1 \\ 0 & \text{else} \end{cases}$$

1.27 Working with a two dimensional density (a) The parallelogram has base and height one, and thus area one, so that the density is one on the region.

(b) By inspection, $f_X(x) = \begin{cases} 0.5x & 0 \leq x \leq 1 \\ 0.5 & 1 \leq x \leq 2 \\ 0.5(3-x) & 2 \leq x \leq 3 \\ 0 & else \end{cases}$

(c) Since the density of X is symmetric about 1.5 and the mean exists, $E[X] = 1.5$. $E[X^2] = 0.5[\int_0^1 x^3dx + \int_1^2 x^2dx + \int_2^3 x^2(3-x)dx] = 0.5[\frac{1}{4} + \frac{7}{3} + \frac{11}{4}] = \frac{8}{3}$, so $\mathrm{Var}(X) = \frac{8}{3} - (\frac{3}{2})^2 = \frac{5}{12}$. A slicker way to find the variance is to observe the X has the same distribution as $U_1 + 2U_2$, where U_1 and U_2 are independent and uniformly distributed over $[0,1]$, so $\mathrm{Var}(X) = \mathrm{Var}(U_1) + 4\mathrm{Var}(U_2) = \frac{5}{12}$.

(d) If $0 \leq x \leq 1$, the conditional density of Y given $X = x$ is the uniform density over the interval $[0, \frac{x}{2}]$. That is, for $0 < x \leq 1$: $f_{Y|X}(y|x) = \begin{cases} \frac{2}{x} & 0 \leq y \leq \frac{x}{2} \\ 0 & \text{else} \end{cases}$

(e) By inspection, if $1 \leq x \leq 2$, the conditional density of Y given $X = x$ is the uniform density over the interval $[\frac{x-1}{2}, \frac{x}{2}]$. That is, for $1 < x \leq 2$: $f_{Y|X}(y|x) = \begin{cases} 2 & \frac{x-1}{2} \leq y \leq \frac{x}{2} \\ 0 & \text{else} \end{cases}$

(f) $E[Y|X=x]$ is well defined over the support of f_X, namely, over the interval $[0,3]$. For each X in this interval, the conditional density of Y give $X = x$ is a uniform density, so the conditional mean is the midpoint of the interval. Therefore. $E[Y|X=x] = \begin{cases} x/4 & 0 \leq x \leq 1 \\ (x-0.5)/2 & 1 \leq x \leq 2 \\ (x+1)/4 & 2 \leq x \leq 3 \\ \text{undefined} & x \notin [0,3] \end{cases}$

1.29 Uniform density over a union of two square regions (a) Region has area 2 so the density function is 1/2 in the region and zero outside.

(b) $f_X(x) = \begin{cases} 0.5 & \text{if } |x| \leq 1 \\ 0 & \text{else} \end{cases}$

(c) If $0 < a \leq 1$, $f_{Y|X}(y|a) = \begin{cases} 1 & \text{if } 0 \leq y \leq 1 \\ 0 & \text{else} \end{cases}$

(d) If $-1 \leq a < 0$, $f_{Y|X}(y|a) = \begin{cases} 1 & \text{if } -1 \leq y \leq 0 \\ 0 & \text{else} \end{cases}$

(e) $E[Y|X = a] = \begin{cases} -0.5 & \text{if } -1 \leq a < 0 \\ 0.5 & \text{if } 0 < a < 1 \end{cases}$

(f) $E[X] = E[Y] = 0$, $\mathrm{Var}(X) = E[X^2] = 1/3$, $\mathrm{Var}(Y) = 1/3$,
$E[XY] = \frac{1}{2}\int_0^1\int_0^1 xydxdy + \frac{1}{2}\int_{-1}^0\int_{-1}^0 xydxdy = \int_0^1\int_0^1 xydxdy = 1/4$. So $\rho_{XY} = \frac{1/4}{\sqrt{1/3 \times 1/3}} = \frac{3}{4}$.

(g) No, because $f_{XY}(x, y)$ doesn't factor into the product of a function of x and a function of y.

(h) The range of Z is $[-2, 2]$. $f_Z(z) = \begin{cases} |z|/2 & \text{if } -0 \leq |z| \leq 1 \\ 1 - |z|/2 & \text{if } 1 \leq |z| \leq 2 \\ 0 & \text{else} \end{cases}$ (Shape is two triangles.)

1.31 Transformation of densities (a) $\int_0^1\int_0^1 (u - v)^2 dudv = \int_0^1\int_0^1 (u^2 - 2uv + v^2)dudv = \frac{1}{6}$, so $c = 6$.

(b) The map from the u, v plane to the x, y plane given by $x = u^2$ and $y = u^2v^2$ maps the unit square $[0, 1] \times [0, 1]$ into the triangular region $0 \leq y \leq x \leq 1$ in one-to-one fashion. The inverse mapping is given by $u = v^{1/2}$ and $v = (y/x)^{1/2}$. Also, $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\begin{array}{cc} 2u & 0 \\ 2uv^2 & 2u^2v \end{array}\right| = 4u^3v = 4xy^{1/2}$. Therefore,

$$\begin{aligned} f_{XY}(x, y) &= f_{UV}(u, v)\left|\frac{\partial(x, y)}{\partial(u, v)}\right|^{-1} \\ &= \begin{cases} 6(x^{1/2} - (y/x)^{1/2})^2 \frac{1}{4xy^{1/2}} & \text{if } 0 \leq y \leq x \leq 1 \\ 0 & \text{else} \end{cases} \end{aligned}$$

1.33 Transformation of joint densities To be definite, assume $\binom{X}{Y}$ takes values in the positive quadrant of the $u - v$ plane and $\binom{W}{Z}$ takes values in the $\alpha - \beta$ plane. We have $\binom{W}{Z} = g\binom{X}{Y}$ where the transformation g is given by $\alpha = u - v$ and $\beta = u^2 + u - v$. The transformation is invertible. In fact, we see that $u = \sqrt{\beta - \alpha}$ and $v = \sqrt{\beta - \alpha} - \alpha$, for (α, β) in the range of g, which is the set $\{(\alpha, \beta) : \beta > \alpha + (\max\{0, \alpha\})^2\}$. (To understand the geometry of the function better, note as u varies over $u > 0$ with $v = 0$ the function $g(u, v)$ traces out the curve $\beta = \alpha^2 + \alpha$ for $\alpha > 0$. Then for any u fixed with $u > 0$, the function $g(u, v)$ traces out a half line of slope one as v ranges over $v > 0$.) The determinant of the Jacobian of g is given by

$$\det(J) = \det\begin{pmatrix} 1 & -1 \\ 2u & -1 \end{pmatrix} = u = \sqrt{\beta - \alpha}.$$

Therefore,

$$f_{W,Z}(\alpha,\beta) = \frac{f_{X,Y}(u.v)}{\det J} = \begin{cases} \frac{\exp\left(-\lambda\left(2\sqrt{\beta-\alpha}-\alpha\right)\right)}{\sqrt{\beta-\alpha}} & \beta > \alpha + (\max\{0,\alpha\})^2 \\ 0 & else \end{cases}$$

1.35 Conditional densities and expectations
(a)

$$\begin{aligned} E[XY] &= \int_0^1 \int_0^u uv(4u^2)dvdu \\ &= \int_0^1 4u^3 \left(\int_0^u vdv \right) du \\ &= \int_0^1 2u^5 du = \frac{1}{3}. \end{aligned}$$

(b)

$$f_Y(v) = \int_v^1 4u^2 \; du = \begin{cases} \frac{4}{3}(1-v^3), & 0 \le v \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

(c)

$$f_{X|Y}(u|v) = \begin{cases} 0, & 0 < v < 1, \;\; 0 < u < v \\ \\ \frac{4u^2}{\frac{4}{3}(1-v^3)} = \frac{3u^2}{1-v^3}, & 0 < v < 1, \;\; v < u < 1 \\ \\ \text{undefined}, & v < 0 \text{ or } v > 1 \end{cases}$$

(d) For $0 < v < 1$, $E[X^2|Y=v] = \int_v^1 \; u^2 \; \frac{3u^2}{1-v^3} \; du = \frac{3}{5}\frac{1-v^5}{1-v^3}$

2.1 Limits and infinite sums for deterministic sequences (a) Before beginning the proof we observe that $|\cos(\theta)| \le 1$, so $|\theta(1+\cos(\theta))| \le 2|\theta|$. Now, for the proof. Given an arbitrary ϵ with $\epsilon > 0$, let $\delta = \epsilon/2$. For any θ with $|\theta - 0| \le \delta$, the following holds: $|\theta(1+\cos(\theta)) - 0| \le 2|\theta| \le 2\delta = \epsilon$. Since ϵ was arbitrary the convergence is proved.
(b) Before beginning the proof we observe that if $0 < \theta < \pi/2$, then $\cos(\theta) \ge 0$ and $\frac{1+\cos(\theta)}{\theta} \ge 1/\theta$. Now, for the proof. Given an arbitrary positive number K, let $\delta = \min\{\frac{\pi}{2}, \frac{1}{K}\}$. For any θ with $0 < \theta < \delta$, the following holds: $\frac{1+\cos(\theta)}{\theta} \ge 1/\theta \ge 1/\delta \ge K$. Since K was arbitrary the convergence is proved.
(c) The sum is by definition equal to $\lim_{N\to\infty} s_N$ where $s_N = \sum_{n=1}^N \frac{1+\sqrt{n}}{1+n^2}$. The sequence S_N is increasing in N. Note that the $n=1$ term of the sum is 1 and for any $n \ge 1$ the n^{th} term of the sum can be bounded as follows:

$$\frac{1+\sqrt{n}}{1+n^2} \le \frac{2\sqrt{n}}{n^2} = 2n^{-3/2}.$$

Therefore, comparing the partial sum with an integral, yields

$$s_N \le 1 + \sum_{n=2}^N 2n^{-3/2} \le 1 + \int_1^N 2x^{-3/2}dx = 5 - 4N^{-1/2} \le 5.$$