Contents

0.1 Solutions Ch. 1

1.

(i) The orthonormality of the states is demonstrated as follows

$< \alpha_1 |\alpha_1\rangle = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1; < \alpha_1 |\alpha_2\rangle = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$. Similarly one can show $< \alpha_2 |\alpha_1\rangle = 0$ and $< \alpha_2 |\alpha_2\rangle = 1$

(ii) The column matrix can be written as

$$\begin{bmatrix} a \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(iii) The outer products $|\alpha_i\rangle \langle\alpha_j|$ give the following matrices

$$|\alpha_1\rangle \langle\alpha_1| = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; |\alpha_1\rangle \langle\alpha_2| = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix};$$
$$|\alpha_2\rangle \langle\alpha_1| = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}; |\alpha_2\rangle \langle\alpha_2| = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

(iv) The $|\alpha_i\rangle' s$ satisfy completeness relation from the following relation

$\sum_i |\alpha_i\rangle \langle\alpha_i| = |\alpha_1\rangle \langle\alpha_1| + |\alpha_2\rangle \langle\alpha_2| = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{1}$

(v) write

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = a |\alpha_1\rangle \langle\alpha_1| + b |\alpha_1\rangle \langle\alpha_2| + c |\alpha_2\rangle \langle\alpha_1| + d |\alpha_2\rangle \langle\alpha_2|$

(vi)

$$A |\alpha_1\rangle = + |\alpha_1\rangle \text{ and } A |\alpha_2\rangle = - |\alpha_2\rangle$$

Constructing the matrix elements from the above relation and using orthonormality we find

$\{A\} = \begin{bmatrix} \langle\alpha_1| A |\alpha_1\rangle & \langle\alpha_1| A |\alpha_2\rangle \\ \langle\alpha_2| A |\alpha_1\rangle & \langle\alpha_2| A |\alpha_2\rangle \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

2.Start with the relation

$AA^{-1} = 1$

Take the derivative with respect to λ

$\frac{d}{d\lambda} \left(AA^{-1}\right) = 0$, therefore

$A\frac{dA^{-1}}{d\lambda} + \frac{dA}{d\lambda}A^{-1} = 0$, multiplying on the left by $A^{-1}$and then moving the second term to the left gives

$\frac{dA^{-1}}{d\lambda} = -A^{-1}\frac{dA}{d\lambda}A^{-1}$

3.

For an operator A,

$AA^{-1} = 1$,therefore $\left(AA^{-1}\right)^\dagger = 1$ or $\left(A^{-1}\right)^\dagger = \left(A^\dagger\right)^{-1}$

$U = \frac{\mathbf{1} + iK}{\mathbf{1} - iK} = (\mathbf{1} + iK) (\mathbf{1} - iK)^{-1} = (\mathbf{1} - iK)^{-1} (\mathbf{1} + iK)$

the last step follows from the fact that $K's$ commute among themselves

Therefore,

$U^\dagger = (\mathbf{1} - iK)^\dagger \left((\mathbf{1} + iK)^{-1}\right)^\dagger = (\mathbf{1} - iK) (\mathbf{1} + iK)^{-1}$ since K is Hermitian, and

and $UU^\dagger = (\mathbf{1} + iK) \left[(\mathbf{1} - iK)^{-1} (\mathbf{1} - iK)\right] (\mathbf{1} + iK)^{-1} = (\mathbf{1} + iK) (\mathbf{1} + iK)^{-1} = 1$

One can write

$e^{iC} = \frac{e^{iC/2}}{e^{-iC/2}} = \frac{1 + i\tan C/2)}{1 - i\tan C/2)}$

and identify

$K = \tan C/2)$

One can also show that

$U = e^{iC} = \cos C + i\sin C$

If $U = A + iB$ then identifying
$A = \cos C$, $B = \sin C$ we note that A and B commute.

4.
Let U be a unitary operator diagonalizing A, so that
$A_D = UAU^\dagger$
is a diagonal matrix. Then
$Tr(A) = Tr(A_D)$
$\det(A) = \det(A_D)$
Similarly by expanding e^A in powers of A we get
$\det(e^A) = \det(e^{A_D}) = e^{(A_D)_{11}} \cdot e^{(A_D)_{22}} \cdot e^{(A_D)_{33}} \cdot ...... = e^{Tr(A_D)} = e^{Tr(A)}$

5.
$Tr\left[|\alpha\rangle\langle\beta|\right] = \sum_n < n\,|\alpha\rangle\langle\beta|\,n >= \sum_n \langle\beta|\,n >< n\,|\alpha\rangle = \langle\beta|\,\alpha\rangle$

6.
$A = |\alpha\rangle\langle\alpha| + \lambda\,|\beta\rangle\langle\alpha| + \lambda^*\,|\alpha\rangle\langle\beta| + \mu\,|\beta\rangle\langle\beta|$
In the matrix form it can be written (take $|\alpha\rangle = |1\rangle\,, |\beta\rangle = |2\rangle$ to implement matrix notation)
$\{A\} = \begin{bmatrix} 1 & \lambda^* \\ \lambda & \mu \end{bmatrix}$
Let a be the eigenvaliues, then
$(1-a)(\mu - a) - |\lambda|^2 = 0$
The solutions are
$a = \dfrac{(1+\mu) \pm \sqrt{(1-\mu)^2 + 4\,|\lambda|^2}}{2}$
(i) $\lambda = 1$, $\mu = +1$
$a = \dfrac{2 \pm \sqrt{4\,|\lambda|^2}}{2} = 1 \pm |\lambda|^2$

$\lambda = 1$, $\mu = -1$
$a = \pm\sqrt{1 + |\lambda|^2}$

(ii)$\lambda = i$, $\mu = +1$
$a = 2{,}0$

$\lambda = i$, $\mu = -1$
$a = \pm\sqrt{2}$