# Contents

# About this manual

This manual contains solutions to the problems set at the end of each chapter of "Quantum Mechanics". It is divided into sections corresponding to the chapters in the text and titled accordingly. Bracketed numbers refer to equations in the main text.

CHAPTER 1

# The Physics and Mathematics of Waves

**1.1** Use Euler's formula to find a purely real expression for $i^i$.

**Solution**

$$i^i = \left(e^{i\pi/2}\right)^i = e^{-\pi/2}$$

**1.2** Show that (1.3) can be written as

$$x(t) = A\cos(\omega t + \phi)$$

and derive expressions for $A$ and $\phi$ in terms of $B$ and $C$. Assuming the oscillator starts out at position $x(0) = x_0$ with velocity $v(0) = v_0$, determine $A$ and $\phi$ in terms of $x_0$ and $v_0$. Note: We call $A$ the *amplitude* and $\phi$ the *phase* of the oscillation.

**Solution**

Replace the constants $B$ and $C$ in $x(t) = B\cos\omega t + C\sin\omega t$ with two different constants $A$ and $\phi$ which solve $B = A\cos\phi$ and $C = -A\sin\phi$. This results in $x(t) = A\cos(\omega t + \phi)$. Now $x(0) = A\cos\phi = x_0$ and $\dot{x}(0) = -\omega A\sin\phi = v_0$ so $A = (x_0^2 + v_0^2/\omega^2)^{1/2}$ and $\phi = -\tan^{-1}(v_0/x_0\omega)$.

**1.3** A spring with stiffness $k$ hangs vertically from point on the ceiling. A mass $m$ is attached to the lower end of the spring without stretching it, and then is released from rest. Show that when the gravitational force $mg$ is taken into account, the motion is still sinusoidal with $\omega = (k/m)^{1/2}$ but with an equilibrium position shifted to a lower point. Find the new equilibrium position in terms of $m$, $k$, and $g$.

**Solution**

Let $y$ measure the vertical position of the mass, with $y = 0$ the unstretched string. Then $m\ddot{y} = -ky - mg = -k(y + mg/k)$. Defining $x \equiv y + mg/k$, get $m\ddot{x} = -kx$, so once again $\omega^2 = k/m$. The equilibrium point is $x = 0$ or $y = -mg/k$.

**1.4** Consider a system of a mass and spring, such as in Figure 1.1, but with an additional force $F_{\text{damp}} = -bv$ proportional to velocity but acting in the direction opposite to the motion. Reformulate the equation of motion, and find the solution for $x(0) = x_0$ and $v(0) = v_0$. Use the ansatz $x(t) = \exp(i\alpha t)$ to solve for $\alpha$. You may assume that $b^2/m^2$ is less than $4k/m$.

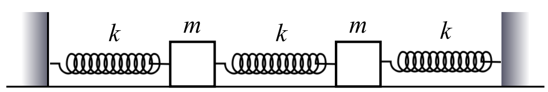
**Solution**

The equation of motion is $m\ddot{x} = -kx - b\dot{x}$ so $-m\alpha^2 = -k - ib\alpha$. Defining $\omega_0^2 \equiv k/m$ and $\beta \equiv b/2m$, yields the equation $\alpha^2 - 2i\beta\alpha - \omega_0^2 =$. Solving this,

$$\alpha = \frac{1}{2}\left[2i\beta \pm \left(-4\beta^2 + 4\omega_0^2\right)^{1/2}\right] = i\beta \pm \omega$$

where $\omega^2 \equiv \omega_0^2 - \beta^2$. (Note that $\beta^2/\omega_0^2 = (b^2/m^2)/(4k/m) < 1$ so $\omega$ is real.) The solution becomes $x(t) = Ae^{-\beta t}\cos(\omega t + \phi)$ where $A\cos\phi = x_0$ and $-A(\beta\cos\phi + \omega\sin\phi) = v_0$. These can be solved in principle, but the interesting solution is when $\beta \ll \omega$, i.e. “lightly damped” motion. The result in oscillation which slowly damps.

**1.5** Two equal masses $m$ move in one dimension and are each connected to fixed walls by springs with stiffness $k$. The masses are also connected to each other by a third, identical spring, as shown:




Write the (differential) equations of motion for the positions $x_1(t)$ and $x_2(t)$ of the two masses. Solve those equations with the ansatz $x_1(t) = A_1\exp(i\alpha t)$ and $x_2(t) = A_2\exp(i\alpha t)$; you will discover nontrivial solutions only for two values of $\omega^2$. (Those two values are called *eigenfrequencies.*) What kind of motion corresponds to each of these two eigenfrequencies?

**Solution**

Label the two masses #1 and #2 from left to right. The force on $m$ #1 is $-kx_1 + k(x_2 - x_1) = -2kx_1 + kx_2$, and the force on $m$ #2 is $-kx_2 - k(x_2 - x_1) = -2kx_2 + kx_1$, so, defining $\omega_0^2 \equiv k/m$, the equations of motion are

$$\ddot{x}_1 = -2\omega_0^2 x_1 + \omega_0^2 x_2 \qquad \text{and} \qquad \ddot{x}_2 = -2\omega_0^2 x_2 + \omega_0^2 x_1$$

Now insert the ansatz solution. After a little rearranging, you find

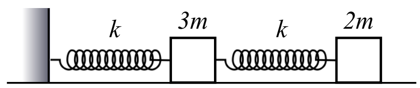
$$(2\omega_0^2 - \omega^2)A_1 - \omega_0^2 A_2 = 0 \qquad \text{and} \qquad -\omega_0^2 A_1 + (2\omega_0^2 - \omega^2)A_2 = 0$$

These are two homogenous equations for $A_1$ and $A_2$. The only solution is $A_1 = A_2 = 0$, that is no motion, unless the determinant vanishes:

$$\omega_0^4 = (\omega^2 - 2\omega_0^2)^2 \qquad \text{so} \qquad \omega^2 = \omega_0^2 \qquad \text{or} \qquad \omega^2 = 3\omega_0^2$$

For $\omega^2 = \omega_0^2$, find $A_1 = A_2$ so the two masses oscillate in phase with the same amplitude. For $\omega^2 = 3\omega_0^2$, find $A_1 = -A_2$ so the two masses oscillate out of phase with the same amplitude.

**1.6** Find the eigenfrequencies for the two-mass, two-spring system shown here:




**Solution**

Label mass $3m$ #1 and mass $2m$ #2. Then the equations of motion are

$$3m\ddot{x}_1 = -kx_1 + k(x_2 - x_1) = -2kx_1 + kx_2 \text{ and } 2m\ddot{x}_2 = -k(x_2 - x_1) = kx_1 - kx_2$$

Using the standard definitions and ansatz,

$$(3\omega^2 - 2\omega_0^2)A_1 + \omega_0^2 A_2 = 0 \qquad \text{and} \qquad \omega_0^2 A_1 + (2\omega^2 - \omega_0^2)A_2 = 0$$

Next, set the determinant equal to zero to find

$$(3\omega^2 - 2\omega_0^2)(2\omega^2 - \omega_0^2) - \omega_0^4 = 6\omega^4 - 7\omega_0^2\omega^2 + \omega_0^2 = (6\omega^2 - \omega_0^2)(\omega^2 - \omega_0^2) = 0$$

so the eigenfrequencies are $\omega^2 = \omega_0^2$, in which case the two masses oscillate with equal amplitude but out of phase, and $\omega^2 = \omega_0^2/6$, in which case the oscillations are in phase with $A_1/A_2 = 2/3$.

**1.7** For the two-mass, three-spring system discussed in Problem 1.5, find expressions for $x_1(t)$ and $x_2(t)$ subject to the initial conditions $x_1(0) = A$ and $x_2(0) = v_1(0) = v_2(0) = 0$. Make a plot of $x_1(t)$ and $x_2(t)$, and also plot the quantities $x_1(t) + x_2(t)$ and $x_1(t) - x_2(t)$. Comment on your observations.

**Solution**

Now we need to write the the general solution for the motion of the two masses:

$$\begin{aligned} x_1(t) &= ae^{i\omega_0 t} + be^{-i\omega_0 t} + ce^{\sqrt{3}i\omega_0 t} + de^{-\sqrt{3}i\omega_0 t} \\ x_2(t) &= ae^{i\omega_0 t} + be^{-i\omega_0 t} - ce^{\sqrt{3}i\omega_0 t} - de^{-\sqrt{3}i\omega_0 t} \end{aligned}$$

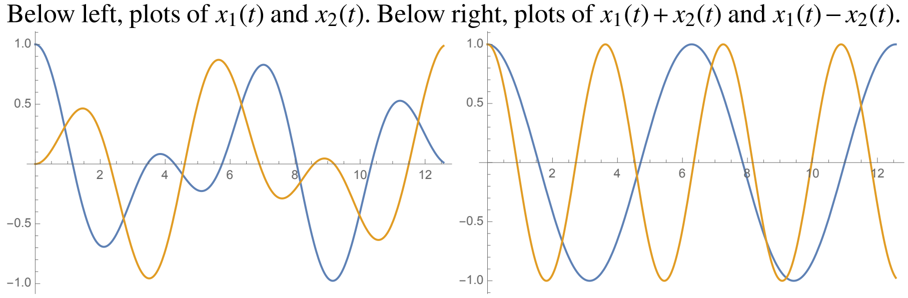
Note that we have maintained the amplitude ratios and relative phases between the different solutions for the particular eigenfrequencies. This is necessary in order to make sure that each of the four terms separately solves the coupled differential equations. Now we can apply the initial conditions:

$$\begin{aligned} A &= a+b+c+d \\ 0 &= a-b+\sqrt{3}(c-d) \\ 0 &= a+b-c-d \\ 0 &= a-b-\sqrt{3}(c-d) \end{aligned}$$

Adding first and third gives $a+b=A/2$ and adding second and fourth gives $a-b=0$, so $a=b=A/4$. Subtracting third from first gives $c+d=A/2$ and subtracting fourth from second gives $c-d=0$ so $c=d=A/4$. Therefore

$$\begin{aligned} x_1(t) &= \frac{A}{2}\left[\cos(\omega_0 t)+\cos(\sqrt{3}\omega_0 t)\right] \\ x_2(t) &= \frac{A}{2}\left[\cos(\omega_0 t)-\cos(\sqrt{3}\omega_0 t)\right] \end{aligned}$$

Below left, plots of $x_1(t)$ and $x_2(t)$. Below right, plots of $x_1(t)+x_2(t)$ and $x_1(t)-x_2(t)$.




The wiggling motion is a superposition of different eigenfrequencies, but the sum and difference show the individual isolated eigenfrequencies.

**1.8** Repeat Problem 1.7, but this time let the "coupling" spring between the two masses have a spring constant $k_c = k/100$. Show that the overall motion "oscillates" between cases were the first mass is in simple harmonic motion by itself, to one where the second mass is in simple harmonic motion, and then back again. What is the frequency of these low frequency oscillations between the two masses?

### Solution

First, go back to Problem 5. Now, the force on $m$ #1 is $-kx_1 + k_c(x_2 - x_1) = -(k+k_c)x_1 + kx_2$, and the force on $m$ #2 is $-kx_2 - k_c(x_2 - x_1) = -(k+k_c)x_2 + kx_1$, so, defining $\omega_0^2 \equiv k/m$ and $\alpha^2 = 2k_c/m = 2(k_c/k)\omega_0^2$, the equations of motion are

$$\ddot{x}_1 = -(\omega_0^2+\alpha^2/2)x_1 + (\alpha^2/2)x_2 \qquad \text{and} \qquad \ddot{x}_2 = -(\omega_0^2+\alpha^2/2)x_2 + (\alpha^2/2)x_1$$

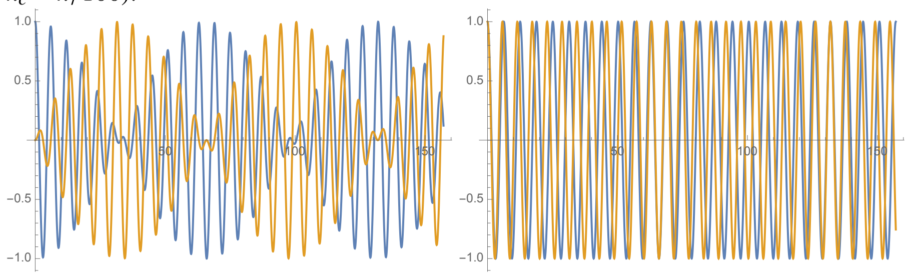
Now insert the ansatz solution. After a little rearranging, you find

$$(\omega_0^2+\alpha^2/2-\omega^2)A_1 - (\alpha^2/2)A_2 = 0 \qquad \text{and} \qquad -(\alpha^2/2)A_1 + (\omega_0^2+\alpha^2/2-\omega^2)A_2 = 0$$

so $\omega_0^2 + \alpha^2/2 - \omega^2 = \pm\alpha^2/2$ and the solutions are $\omega^2 = \omega_0^2$ and $\omega^2 = \omega_0^2 + \alpha^2$. It is easy to see, as in Problem 5, that these two solutions correspond to equal amplitude oscillations in phase and out of phase, respectively. At this point, the motions of the two masses work out just as in Problem 7, and we have

$$\begin{aligned} x_1(t) &= \frac{A}{2}\left[\cos(\omega_0 t) + \cos(\sqrt{\omega_0^2 + \alpha^2}t)\right] \\ x_2(t) &= \frac{A}{2}\left[\cos(\omega_0 t) - \cos(\sqrt{\omega_0^2 + \alpha^2}t)\right] \end{aligned}$$

When $k = k_c$, $\alpha^2 = 2\omega_0^2$ and we get the correct solution to Problem 7. Following are the same plots, but for $k_c = k/10$, that is, $\alpha^2 = \omega_0^2/5$ (which plot more nicely than $k_c = k/100$):




The right plot shows that the eigenfrequencies are very close to each other, resulting in the beat pattern shown in the left plot. With $\alpha^2 \ll \omega_0^2$, and the trigonometric identities

$$\begin{aligned} \cos u + \cos v &= 2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right) \\ \cos u - \cos v &= -2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right) \end{aligned}$$

it is clear that the plots are the product of a high frequency component

$$\frac{1}{2}\left(\omega_0 + \sqrt{\omega_0^2 + \alpha^2}\right) \approx \omega_0$$

with an envelope with low frequency

$$\frac{1}{2}\left(\sqrt{\omega_0^2 + \alpha^2} - \omega_0\right) \approx \frac{\omega_0}{2}\left(1 + \frac{\alpha^2}{2\omega_0^2} - 1\right) = \frac{\omega_0}{2}\frac{k_c}{k}$$

That is, for the left plot above, there are $\approx 20$ crests within one envelope wavelength.

**1.9** Derive the solution (1.13) to the wave equation (1.12) by going through the following steps. Consider a change of variables from $x$ and $y$ to $\xi = x - vt$ and $\eta = x + vt$. Then use the chain rule to rewrite the wave equation in terms of $\xi$ and $\eta$. You should find that

$$\frac{\partial^2 y}{\partial\xi\partial\eta} = 0$$

Then argue that this means that $y$ is a function of either $\xi$ or $\eta$, but not both at the same time. In other words, the solution is (1.13). If you are not familiar with the chain rule for partial differentiation, it means that if $w$ and $z$ are functions of $x$ and $y$, then

$$\frac{\partial}{\partial x} f(x,y) = \frac{\partial f}{\partial w}\frac{\partial w}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x}$$

and similarly for $\partial/\partial y$. You can assume that you get the same result regardless of the order in which the partial derivatives are taken.

**Solution**

As directed, apply the chain rule to the wave equation

$$\begin{aligned}
\frac{\partial y}{\partial x} &= \frac{\partial y}{\partial \xi} + \frac{\partial y}{\partial \eta} \\
\frac{\partial^2 y}{\partial x^2} &= \frac{\partial^2 y}{\partial \xi^2} + 2\frac{\partial^2 y}{\partial \xi \partial \eta} + \frac{\partial^2 y}{\partial \eta^2} \\
\frac{\partial y}{\partial t} &= -v\frac{\partial y}{\partial \xi} + v\frac{\partial y}{\partial \eta} \\
\frac{\partial^2 y}{\partial t^2} &= v^2\frac{\partial^2 y}{\partial \xi^2} - 2v^2\frac{\partial^2 y}{\partial \xi \partial \eta} + v^2\frac{\partial^2 y}{\partial \eta^2} \\
\frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} &= -4v^2\frac{\partial^2 y}{\partial \xi \partial \eta} = 0
\end{aligned}$$

leading to the expression we sought. The obvious solution to this differential equation is the sum of two "constants" in $\xi$ and $\eta$, respectively, that is $y(\xi,\eta) = f(\xi) + g(\eta)$.

**1.10** Prove the principle of linear superposition for the wave equation (1.12). That is, show that if $y_1(x,t)$ and $y_2(x,t)$ are solutions of the wave equation, then $y(x,t) = ay_1(x,t) + by_2(x,t)$ is also a solution, where $a$ and $b$ are arbitrary constants.

**Solution**

All you have to do is plug it in and it falls out easily:

$$\begin{aligned}
\frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} &= a\frac{1}{v^2}\frac{\partial^2 y_1}{\partial t^2} + b\frac{1}{v^2}\frac{\partial^2 y_2}{\partial t^2} - a\frac{\partial^2 y_1}{\partial x^2} - b\frac{\partial^2 y_2}{\partial x^2} \\
&= a\left[\frac{1}{v^2}\frac{\partial^2 y_1}{\partial t^2} - \frac{\partial^2 y_1}{\partial x^2}\right] + b\left[\frac{1}{v^2}\frac{\partial^2 y_2}{\partial t^2} - \frac{\partial^2 y_2}{\partial x^2}\right] \\
&= 0 + 0 = 0
\end{aligned}$$

**1.11** A string with linear mass density $\mu$ hangs motionless between two fixed points $(x,y) = (\pm a, 0)$ where $y$ measures the vertical direction. The length of the string is greater than $2a$, so the lowest point is at $(x.y) = (0,b)$. Derive the differential equation that describes the shape of the string, and solve it for $y(x)$ in terms of $a$, $b$, and the acceleration $g$ due to gravity. Unlike our derivation of the wave equation, do not make the "small displacement" assumption. Note that the shape, called a *catenary*, is not a parabola.

**Solution**

Define angles as in Fig.1.2, but for a down-hanging string in the region $x > 0$. The tension arises now from gravity, so is not a set parameter, and $T_1$ need not be the same as $T_2$. (Think of a point mass, suspended from two wires, each attached at different vertical positions on walls.) Horizontal and vertical equilibrium for a short section of string give

$$T_1 \cos\theta_1 = T_2 \cos\theta_2 \qquad \text{and} \qquad T_1 \sin\theta_1 = T_2 \sin\theta_2 + (\mu\Delta s)g$$

where $\Delta s = (\Delta x^2 + \Delta y^2)^{1/2}$ is the length of the short section of string. Now as $\Delta x \to 0$, the first equation implies that the tension is a constant, so we take $T_1 = T_2 = T$. Since

$$\sin\theta = \cos\theta \tan\theta = (1 + \tan^2\theta)^{-1/2} \tan\theta \qquad \text{and} \qquad \tan\theta = \frac{dy}{dx} \equiv y'$$

the second equation becomes

$$T \frac{d}{dx}\left[\frac{y'}{\left(1 + y'^2\right)^{1/2}}\right] = \mu g \left(1 + y'^2\right)^{1/2}$$

Taking the derivative, multiplying by $(1 + y'^2)^{3/2}$, and defining $k \equiv T/\mu g$, we have

$$y''\left(1 + y'^2\right) - y'^2 y'' = y'' = \frac{1}{k}\left(1 + y'^2\right)^2$$

The *catenary* $y = k\cosh[(x - A)/k] + B$ solves this equation. (Note that for the derivative of $\cosh u$ is $\sinh u$ and vice versa, and that $1 + \sinh^2 u = \cosh^2 u$). Since our case is symmetric, $A = 0$. Also, $B = b - k$, and the length of the string is determined by $a$, which in turn determines the tension $T$.

**1.12** Show that the standing wave solutions (1.19) are linear combinations of the traveling wave solutions $\cos[kx \pm \omega t]$ and $\sin[kx \pm \omega t]$.

**Solution**

Just use the simple expressions for sines and cosines of sums or differences:

$$\begin{aligned}
\cos(kx \pm \omega t) &= \cos(kx)\cos(\omega t) \mp \sin(kx)\sin(\omega t) \\
\sin(kx \pm \omega t) &= \sin(kx)\cos(\omega t) \pm \cos(kx)\sin(\omega t) \\
\text{so} \qquad A\cos(kx)\cos(\omega t) &= \frac{A}{2}\left[\cos(kx + \omega t) + \cos(kx - \omega t)\right] \\
\text{and} \qquad A\sin(kx)\cos(\omega t) &= \frac{A}{2}\left[\sin(kx + \omega t) + \sin(kx - \omega t)\right]
\end{aligned}$$

**1.13** A function $f(x)$ is periodic, such that $f(x+2) = f(x)$. For $-1 < x < 0$ $f(x) = -1$, and for $0 < x < 1$ $f(x) = +1$. Find the first five terms of the Fourier expansion for $f(x)$, and make a plot of the approximations based on the first term, and the sums up to the third and fifth terms, along with a plot of $f(x)$ itself.

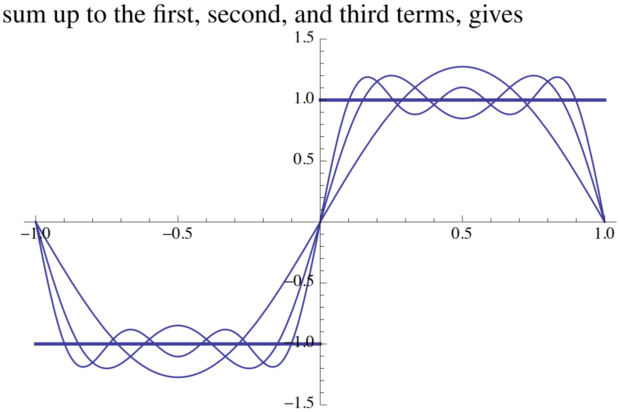
This is an odd function, with period 2, so there are only sine terms:

$$b_n = 2\int_0^1 \sin(n\pi x)dx = -\frac{2}{n\pi}\cos(n\pi x)|_0^1 = \frac{2}{n\pi}[1-\cos n\pi]$$

All the even $n$ terms vanish, so the Fourier series is

$$f(x) = \frac{4}{\pi}\sin(\pi x) + \frac{4}{3\pi}\sin(3\pi x) + \frac{4}{5\pi}\sin(5\pi x) + \cdots$$

Plotting the sum up to the first, second, and third terms, gives




**1.14** The $G$-string on a standard guitar vibrates at 196 Hz. On one particular guitar, this string is 60 cm long and has a mass of 3.1 grams per meter. Calculate the tension on the string, both in Newtons and pounds.

**Solution**

The wavelength of the fundamental is twice the string length, so the wave speed on the string is $1.2\text{ m}\times 196\text{ sec}^{-1} = 235\text{ m/sec} = \sqrt{T/\mu}$. With $\mu = 3.1\times 10^{-3}$ kg/m, we determine $T = 3.1\times 10^{-3}\times 235^2 = 171\text{ N} = 38.6$ lbweight.

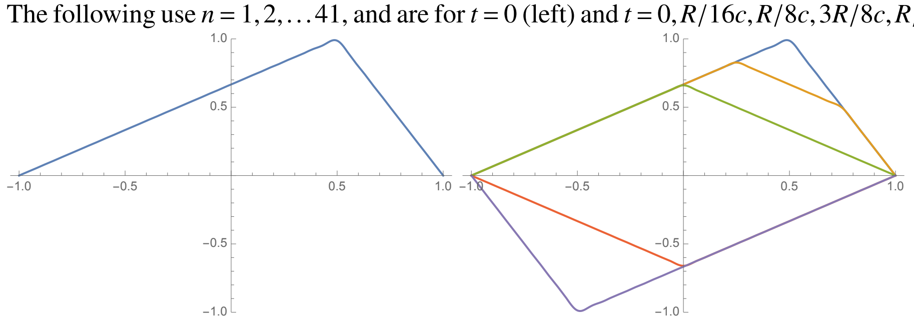
**1.15** Consider a string with an initial shape similar to that shown in Figure 1.7, but instead plucked at a position $x = a/2$ instead of $x = 0$. Find the motion of the string in this case. You might find a program like MATHEMATICA particularly useful to carry out the calculation of the Fourier components, as well as to produce an animation of the string's motion.

**Solution**

For Figure 1.7, that is Worked Example 1.3, the shape function $f(x) = y(x, t = 0)$ at zero time was an odd function of $x$, so contained only sine terms, and this guaranteed that $y(0,t) = 0$. We don't have that condition here, however, so we need to improvise. It is easiest to translate to $x' \equiv x + a$ and then make an odd function for $-R/2 \le x' \le +R/2$, with $R = 4a$. You can then carry out the calculation of the coefficients $b_n$ in (1.25b), and translate back to $x = x' - a$. One finds

$$b_n = \frac{8}{3\pi^2 n^2}\left[4\sin\left(\frac{3\pi n}{4}\right) - 3\sin(\pi n)\right]$$

The following use $n = 1, 2, \ldots 41$, and are for $t = 0$ (left) and $t = 0, R/16c, R/8c, 3R/8c, R/2c$:




The MATHEMATICA notebook used for this calculation is available upon request.

**1.16** Find the Fourier transform of the (normalized) Gaussian function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\exp(-x^2/2\sigma^2)$$

(You will need to "complete the square" of the exponent in the integrand to carry out the integration.) Using $\sigma$ as a measure of the "width" $\Delta x$ of $f(x)$, propose an analogous quantity to express with width $\Delta k$ of the Fourier transform, and evaluate $\Delta x \Delta k$.

**Solution**

We work directly from (1.31b), so

$$\begin{aligned} a(k) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sigma\sqrt{2\pi}}e^{-x^2/2\sigma^2}\right]e^{-ikx}dx = \frac{1}{2\pi\sigma}\int_{-\infty}^{\infty}\left[e^{-(x^2+2ikx\sigma^2)/2\sigma^2}\right]dx \\ &= \frac{1}{2\pi\sigma}\int_{-\infty}^{\infty}\left[e^{-(x^2+2ikx\sigma^2-k^2\sigma^4+k^2\sigma^4)/2\sigma^2}\right]dx = \frac{1}{2\pi\sigma}e^{-k^2\sigma^2/2}\int_{-\infty}^{\infty}e^{-(x+ik\sigma)^2/2\sigma^2}dx \\ &= \frac{1}{2\pi\sigma}e^{-k^2\sigma^2/2}\sigma\sqrt{2\pi} = \frac{1}{\sqrt{2\pi}}e^{-k^2\sigma^2/2} \end{aligned}$$

If the width of $f(x)$ is $\sigma$, then clearly the width of $a(k)$ is $1/\sigma$, so the product is unity.

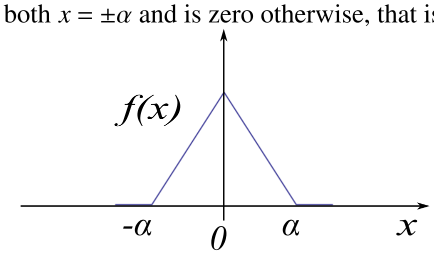
**1.17** The "width" of a localized function can have a precise definition, based in fact on the concept of standard deviation. For some normalized distribution $f(z)$, we can define the width $\Delta z$ by

$$\langle z\rangle = \int_{-\infty}^{\infty} z f(z)dz \qquad \langle z^2\rangle = \int_{-\infty}^{\infty} z^2 f(z)dz \qquad \text{and} \qquad (\Delta z)^2 = \langle z^2\rangle - \langle z\rangle^2 \quad (1.1)$$

Use this definition to find the width $\Delta k$ of the distribution function $a(k)$ in Figure 1.8. Are you surprised?

**Solution**
For Figure 1.8, that is Worked Example 1.4, $a(k) = \sin(k\alpha)/k\alpha\sqrt{2\pi}$. Therefore, $\langle k\rangle \propto \int \sin(k\alpha)dk = 0$ but $\langle k^2\rangle \propto \int k\sin(k\alpha)dk \to \infty$. Yes, this is surprising. The plot of $a(k)$ in Fig. 1.8 does appear to have a finite width.

**1.18** Consider a triangular function $f(x)$ that forms a straight line from the point at $x = 0$ to the $x$-axis at both $x = \pm\alpha$ and is zero otherwise, that is




Find the Fourier transform. Then find the width of both $f(x)$ and of its Fourier transform, using the definition (1.38) based on standard deviation.

**Solution**
The (normalized) function $f(x)$ is even with $f(x) = (\alpha - x)/\alpha^2$ for $0 < x < \alpha$ and $f(x) = 0$ for $x > \alpha$. Clearly, $\langle x\rangle = 0$. Therefore

$$(\Delta x)^2 = 2\int_0^{\alpha} x^2 \frac{1}{\alpha^2}(\alpha - x)dx = \frac{2}{\alpha^2}\left[\frac{\alpha^3}{3}\alpha - \frac{\alpha^4}{4}\right] = \frac{\alpha^2}{6} \qquad \text{and} \qquad \Delta x = \frac{\alpha}{\sqrt{6}}$$

Since $e^{-ikx} = \cos kx - i\sin kx$ and $f(x)$ is even, we calculate the Fourier transform as

$$a(k) = 2\frac{1}{\sqrt{2\pi}}\int_0^{\alpha} \frac{1}{\alpha^2}(\alpha - x)\cos(kx)dx = \sqrt{\frac{2}{\pi}}\frac{1-\cos(\alpha k)}{\alpha^2 k^2} = \sqrt{\frac{2}{\pi}}\frac{2}{\alpha^2 k^2}\sin^2\left(\frac{\alpha k}{2}\right)$$

This is also an even function, so $\langle k\rangle = 0$, and

$$(\Delta k)^2 = \int_{-\infty}^{\infty} k^2 a(k)dk \propto \int_{-\infty}^{\infty} \sin^2\left(\frac{\alpha k}{2}\right)dk \to \infty$$

Once again, as in Prob. 17, the "width" of the distribution is technically infinite.

There are mathematics theorems that discuss the properties of such functions, mostly having to do with cases where $f(x)$ has sharp corners.

**A better problem** would be to offer something with rounded edges so that the Fourier transform has a finite width. I worked it through with (the area-normalized function)

$$f(x) = \frac{15}{16}\frac{1}{\alpha^5}(x-\alpha)^2(x+\alpha)^2 = \frac{15}{16}\frac{1}{\alpha^5}(x^2-\alpha^2)^2$$

I find $\Delta x = \alpha/\sqrt{7}$ and for the Fourier transform

$$a(k) = -\frac{15\left(\left(\alpha^2k^2-3\right)\sin(\alpha k)+3\alpha k\cos(\alpha k)\right)}{\sqrt{2\pi}\alpha^5k^5}$$

After area-normalizing the Fourier Transform, find $\Delta a = 2/\alpha$, so $\Delta x\Delta a = 2/\sqrt{7} = 0.756$.

**1.19** Evaluate the integral

$$\int_{-\infty}^{\infty}\delta(ax)dx$$

where $a$ is a positive constant.

**Solution**

Simply make the substitution $w = ax$ so that $dx = dw/a$ and

$$\int_{-\infty}^{\infty}\delta(ax)dx = \frac{1}{a}\int_{-\infty}^{\infty}\delta(w)dw = \frac{1}{a}$$

**1.20** The "step function" $H(x)$ is defined so that $H(x) = 0$ for $x < 0$, and $H(x) = 1$ for $x > 0$. Show that $dH/dx$ has the correct properties to claim that $dH/dx = \delta(x)$.

**Solution**

Since $H(x)$ is a constant everywhere except at $x = 0$, $dH/dx = 0$ everywhere except at $x = 0$, just as for $\delta(x)$. Then, for some constant $a > 0$,

$$\int_{-a}^{a}\frac{dH}{dx}dx = H(a) - H(-a) = 1 - 0 = 1$$

so the integral also has the properties of $\delta(x)$. That's all we need to show $dH/dx = \delta(x)$.

CHAPTER 2

# Maxwell's equation and Electromagnetic Waves

**2.1** Consider a vector field $\mathbf{V}(\mathbf{r}) = 4x\mathbf{x}_0 + 5y\mathbf{y}_0 + 6z\mathbf{z}_0$ and a closed, cubical $S$ surface with side length $L$ and one corner at the origin, lying in the first octant. Evaluate the integral $\oint_S \mathbf{V} \cdot d\mathbf{A}$ by first carrying out the dot product and integral on each of the six faces of the cube, and adding them up. Check your answer by using the divergence theorem, which you are likely able to do in your head.

**Solution**
Make a table to explicitly evaluate the surface integral:

| Surface | $\int \mathbf{V} \cdot d\mathbf{A}$ |
|---|---|
| $xy$ @ $z = 0$ | 0 |
| $xy$ @ $z = L$ | $6L \cdot L^2$ |
| $yz$ @ $x = 0$ | 0 |
| $yz$ @ $x = L$ | $4L \cdot L^2$ |
| $zx$ @ $y = 0$ | 0 |
| $zx$ @ $y = L$ | $5L \cdot L^2$ |
| Sum | $15L^3$ |

On the other hand, $\int \nabla \cdot \mathbf{V} d\tau = 15 \int d\tau = 15L^3$.

**2.2** Show that the integral form of Coulomb's law can be derived from Gauss's law. First, argue why rotational symmetry implies that the electric field from a point charge $q$ has to be isotropic in all directions, and can only depend on the distance

$r$ from the charge. Next, use this to choose an appropriate "Gaussian surface" $S$ so that the integral in Equation (2.4a) is simple to evaluate. Finally, use Equation (2.5) to show that the force $F$ on another charge $q'$ is

$$F = \frac{1}{4\pi\epsilon_0}\frac{qq'}{r^2}$$

**Solution**

It is obvious that the field can only depend on $r$ because there is no preferred direction. Similarly, it can only be radially outward (or inward), so choose a Gaussian surface that is a sphere of radius $r$ centered at the origin. The magnitude $E$ of the electric field is given by (2.4a) as

$$E \cdot 4\pi r^2 = \frac{q}{\epsilon_0} \qquad \text{so} \qquad E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}$$

and the force on a charge $q'$ is just $q'E$.

**2.3** A "parallel plate capacitor" is made from two plane conducting sheets, each with area $A$, separated by a distance $d$. The plates carry equal but opposite charges $\pm Q$, uniformly distributed over their surface, and this creates a potential difference $V$ between them. Infer the (constant) electric field between the plates, and use Gauss's law to show that $Q = CV$, where $C$ depends only $A$ and $d$ (and $\epsilon_0$).

**Solution**

The surface charge density is $\sigma = Q/A$, so a "pillbox" Gaussian surface with one flat surface inside the metal plate (where the field is zero) and the other flat surface in the gap, gives $E = \sigma/\epsilon_0$. The potential difference for this (constant) electric field is just $V = Ed = \sigma d/\epsilon_0 = Qd/\epsilon_0 A$ so that $C = Q/V = \epsilon_0 A/d$.

**2.4** Use the concept of a parallel plate capacitor to find the energy density in an electric field. Charge is added in small increments $dQ'$ to an initially uncharged capacitor giving a potential difference $V'$. Each increment changes the stored energy by $V'dQ' = (Q'/C)dQ'$ where $C$ is the capacitance. (See Problem 2.3.) Integrate to find the total energy when charge $Q$ is stored in the capacitor. Divide by the volume of the capacitor to find the electric field energy density

$$u_E = \frac{1}{2}\epsilon_0 E^2$$

where $E$ is the electric field inside the capacitor.

**Solution**

Just do as the problem statement tells you to do:

$$u_e = \frac{1}{Ad}U = \frac{1}{Ad}\int_0^Q \frac{1}{C}Q'dQ' = \frac{1}{Ad}\frac{Q^2}{2C} = \frac{1}{Ad}\frac{(\epsilon_0 EA)^2}{2\epsilon_0 A/d} = \frac{1}{2}\epsilon_0 E^2$$

**2.5** Calculate the magnetic field at a distance $r$ from an infinitely long straight wire which carries a current $I$. First, using Gauss’s law for magnetism, explain why the field must be tangential to a circle of radius $r$, centered on the wire and lying in a plan perpendicular to the wire. Then use Ampére’s law to show that the magnitude of the magnetic field is

$$B = \frac{\mu_0 I}{2\pi r}$$

**Solution**

The problem has cylindrical symmetry, but there is a handedness set by the direction of the current. By Gauss’ Law for magnetism, there can be no radial component, as a cylindrical Gaussian surface can pass no no flux. With a circular loop at radius $r$, the line integral of Ampere’s Law (2.4d) is just $B \cdot 2\pi r$ for an azimuthal field $\mathbf{B}$, hence $B = \mu_0 I/2\pi r$.

**2.6** A long cylindrical coil of wire is called a *solenoid* and can be used to store a magnetic field. If the coil is infinitely long, there is a uniform magnetic field in the axial direction inside the coil, and no field outside the coil. Use an “Amperian Loop” that is a rectangle enclosing some length of the coil, with one leg inside and one leg outside, to show that the magnetic field is

$$B = \mu_0 I n$$

where $I$ is the current in the wire and there are $n$ turns per unit length in the coil.

**Solution**

The current enclosed in the rectangular loop is $nI\ell$ where $\ell$ is the length of the loop. There is no field outside the solenoid, and the field inside is parallel to the axis, so the line integral just gives $B\ell$, hence $B = \mu_0 nI$.

**2.7** Find the vector potential $\mathbf{A}(\mathbf{r})$ which gives the magnetic field for the long straight wire in Problem 2.5. It is easiest to let the wire lie along the $z$-axis and express your result in terms of $r = (x^2 + y^2)^{1/2}$, and to carry out the calculation in cylindrical polar coordinates $(r, \theta, z)$.

**Solution**

$\mathbf{B} = B\hat{\theta} = \mathbf{\nabla} \times \mathbf{A} = -(\partial A_z/\partial r)\hat{\theta}$, so $\partial A_z/\partial r = -\mu_0 I/2\pi r$ and $\mathbf{A} = -(\mu_0 I/2\pi) \log r \, \hat{\mathbf{z}}$.

**2.8** Follow this guide to convince yourself that the second term on the right in (2.4d) is needed for the whole equation to make sense. First, imagine a long straight current-carrying wire, with associated magnetic field given in Problem 2.5. Now “cut” the wire, and insert a very thin capacitor, with plates perpendicular to the direction of the wire. Current continues to flow through the wire while the capacitor charges up, but no current flows between the capacitor plates, so it would seem there should be no magnetic field there. But that doesn’t make sense: how could the magnetic field just stop at the capacitor? Intuitively, you expect it to be continuous right through it.

Finally, show that the second term, called a *displacement current*, in fact gives you the same $B$ inside the capacitor.

**Solution** The capacitor plates are circular with area $A = \pi r^2$. The electric field flux through this area is $A\sigma/\epsilon_0 = Q/\epsilon_0$, since $\sigma = Q/A$. The left side of (2.4d) must equal $\mu_0 I$, where $I = dQ/dt$, so

$$\mu_0 I = K\frac{d}{dt}\left[\frac{Q}{\epsilon_0}\right] = K\frac{I}{\epsilon_0}$$

which implies that $K = \epsilon_0\mu_0$, establishing (2.4d).

**2.9** A " $1/r^2$" vector field, such as the electric field from a point charge or the gravitational field from a point mass, takes the form

$$\mathbf{V}(\mathbf{r}) = \frac{k}{r^2}\mathbf{r}_0 = \frac{k}{r^3}\mathbf{r}$$

Show by an explicit calculation in Cartesian coordinate coordinates, that $\nabla\cdot\mathbf{V} = 0$ everywhere, except at the origin. Then, using a spherical surface about the origin, show that $\oint \mathbf{V}\cdot d\mathbf{A} = 4\pi k$. Hence argue that the charge density for a point charge $q$ located at the origin is $q\delta^3(\mathbf{r}) = q\delta(x)\delta(y)\delta(z)$, where $\delta(x)$ is a Dirac $\delta(x)$ function as defined in Chapter 1.

**Solution**
The calculation is straightforward, although a bit tedious:

$$\begin{aligned}\boldsymbol{\nabla}\cdot\frac{k}{r^3}\mathbf{r} &= k\left[\frac{\partial}{\partial x}\frac{x}{(x^2+y^2+z^2)^{3/2}} + \frac{\partial}{\partial y}\frac{y}{(x^2+y^2+z^2)^{3/2}} + \frac{\partial}{\partial z}\frac{z}{(x^2+y^2+z^2)^{3/2}}\right] \\ &= \frac{1}{r^3} - \frac{3x^2}{r^5} + \frac{1}{r^3} - \frac{3y^2}{r^5} + \frac{1}{r^3} - \frac{3z^2}{r^5} = \frac{3r^2 - 3(x^2+y^z+z^2)}{r^5} = 0\end{aligned}$$

for $r \neq 0$. For a sphere of radius $r$ about the origin, $\oint \mathbf{V}\cdot d\mathbf{A} = (k/r^2)\cdot 4\pi r^2 = 4\pi k$. So, by Gauss' Theorem for this spherical volume, $\int \boldsymbol{\nabla}\cdot\mathbf{V} = 4\pi k$, but since $\boldsymbol{\nabla}\cdot\mathbf{V} = 0$ everywhere except the origin, consider a small cube around the origin, and it is clear that $\boldsymbol{\nabla}\cdot\mathbf{V}$ satisfies the properties of the 3D $\delta$-function, i.e. $\boldsymbol{\nabla}\cdot\mathbf{V} = 4\pi k\delta(x)\delta(y)\delta(z) = 4\pi k\delta^{(3)}(\mathbf{r})$. For Coulomb's Law, $\mathbf{V} = \mathbf{E}$ and $k = q/4\pi\epsilon_0$, so Gauss' Law takes the form $\boldsymbol{\nabla}\cdot\mathbf{E} = q\delta^{(3)}(\mathbf{r})/\epsilon_0$. Comparing to (2.19a), this implies $\rho(\mathbf{r}) = q\delta^{(3)}(\mathbf{r})$ for a point charge $q$.

**2.10** A " $1/r$" vector field, such as the magnetic field from an infinitely long current carrying wire, takes the form

$$\mathbf{V}(\mathbf{r}) = \frac{k}{r}\phi_0 = \frac{k}{r^z}\left[-y\mathbf{x}_0 + x\mathbf{y}_0\right]$$

Show by an explicit calculation in Cartesian coordinate coordinates, that $\nabla\times\mathbf{V} = 0$ everywhere, except at the origin. Then, using a circular curve about the z-axis, show that $\oint \mathbf{V}\cdot d\mathbf{l} = 2\pi k$. Hence argue that the current density for an infinitely long current carrying wire of zero thickness located along the $z$-axis is $I\delta(x)\delta(y)$.