PROBLEM SET 1.1. PHYSICAL FOUNDATIONS: PRESSURE AND ELECTRICAL FORCES AND FLOWS

ANSWER KEY

1. Identify whether the following variables are intensive or extensive:

   A. Temperature
   B. Heat content
   C. Volume
   D. Density
   E. Mass
   F. Concentration
   G. Moles
   H. Pressure
   I. Area
   J. Flow
   K. Flux
   L. Viscosity

A.	Temperature	Intensive: independent of how much
B.	Heat content	Extensive: depends on the extent $\Delta H = C_p M \Delta T$
C.	Volume	Extensive: depends on the extent
D.	Density	Intensive: independent of extent
E.	Mass	Extensive: depends on extent
F.	Concentration	Intensive: independent of extent (for well-mixed solutions)
G.	Moles	Extensive: depends on extent
H.	Pressure	Intensive: independent of extent
I.	Area	Extensive: depends on extent
J.	Flow	Extensive: depends on area
K.	Flux	Intensive: independent of extent
L.	Viscosity	Intensive: independent of extent

2. Normal systolic blood pressure is about 120 mm Hg.

   A. Convert this to atmospheres.

   120 mm Hg = 120 mm Hg x 1 atm / 760 mm Hg = **0.1579 atm**

   B. Convert this to Pascals

   1 atm = 1.013 x 10^5 Pa; So 0.1579 atm = 0.1579 x 1.013 x 10^5 Pa =

   0.1599 x 10^5 Pa

3. Normal diastolic blood pressure is about 80 mm Hg.

A. Convert this to atmospheres.

80 mm Hg = 80 mm Hg x 1 atm / 760 mm Hg = **0.1053 atm**

B. Convert this to Pascals.

1 atm = 1.013×10^5 Pa; So 0.1053 atm = $0.1053 \times 1.013 \times 10^5$ Pa =

0.1066×10^5 Pa

4. For dialysis membranes the L_p was determined to be 6.34×10^{-7} cm min^{-1} mm Hg^{-1}. A cylindrical hole 1 cm in diameter was cut in two lexan pieces that were then bolted together with the membrane between. Fluid entered one side through a pump and a pressure transducer was affixed. The flow was adjusted until the pressure (above atmospheric) was 20000 Pascals. The pressure on the opposite side was atmospheric (zero). What was the flow through the membrane?

Here $Q_v = A\, L_p\, \Delta P$ Area = $\pi \times (0.5 \text{ cm})^2 = 0.785 \text{ cm}^2$

$L_p = 6.34 \times 10^{-7}$ cm min^{-1} mm Hg^{-1} And ΔP = 20,000 Pa x 1 mm Hg /133.29 Pa = 150 mm Hg

So Q_v = 0.785 cm^2 x 6.34×10^{-7} cm min^{-1} mm Hg^{-1} x 150 mm Hg = **7.47×10^{-5} cm^3 min^{-1}**

5. The viscosity of water at 25°C is about 0.00089 Pa s. The inner diameter of a PE160 polyethylene tubing is 1.14 mm. Assume steady-state laminar flow.

A. What pressure is necessary to get a flow of 5 mL min^{-1} through a 20 cm length of this tubing?

Here we use Poiseuille's Equation: $Q_V = \pi a^4 / 8\eta l \;\; \Delta P$. We are given that $a = 0.57 \times 10^{-3}$ m ; η = 0.00089 Pa s; l = 0.2 m and Q_V = 5 mL min^{-1} = 5×10^{-6} m^3 min^{-1} and we want to solve for

$\Delta P = Qv \times 8 \times \eta \times l / \pi a^4$.

We convert Q_V to a flow per second:

5×10^{-6} m^3 min^{-1} x 1 min / 60 s = 0.083×10^{-6} m^3 s^{-1}

ΔP = [0.083×10^{-6} m^3 s^{-1} x 8 x 0.00089 Pa s x 0.2 m] / πx $(0.57 \times 10^{-3} \text{m})^4$

= 0.0001182×10^{-6} m^4 s^{-1} Pa s / 3.316×10^{-13} m^4

= 0.0000356×10^7 Pa = **3.56×10^2 Pa**

This can be converted to mm Hg by dividing by 133.3 Pa mm Hg^{-1}:

356 Pa / 133.3 Pa mm Hg^{-1} = **2.67 mm Hg**

B. What is the velocity of the flow?

The velocity of the flow is given by $Q_V = J_V A$. The area is $\pi a^2 = \pi \times (0.57 \times 10^{-3}\ m)^2 = 1.02 \times 10^{-6}\ m^2$. Using the value of Q_V from part A, we get

$J_V = 0.083 \times 10^{-6}\ m^3 s^{-1} / 1.02 \times 10^{-6}\ m^2 = \mathbf{0.081\ m\ s^{-1}}$

C. What pressure is necessary to get a flow of 5 mL min^{-1} through the same PE160 tubing if plasma is used, with viscosity of 0.002 Pa s?

Pressure goes up linearly with the viscosity. So we can set up a proportion here rather than do the calculation all over again. The new pressure would be

$\Delta P = 3.56 \times 10^2$ Pa x 0.002 / 0.00089 = $\mathbf{8 \times 10^2}$ **Pa**

Which in turn is converted to **6 mm Hg**

D. What pressure would be needed for the same flow of water through a 20 cm length of PE60 tubing with i.d. = 0.76 mm?

We repeat the calculation in part A:

$\Delta P = [\ 0.083 \times 10^{-6} m^3 s^{-1} \times 8 \times 0.00089\ Pa\ s \times 0.2\ m\ ] / \pi \times (0.38 \times 10^{-3} m)^4$

$= 0.0001182 \times 10^{-6}\ m^4\ s^{-1}\ Pa\ s / 0.0655 \times 10^{-12}\ m^4$

$= 0.00180 \times 10^6\ Pa = \mathbf{1.8 \times 10^3\ Pa}$

Which is converted to mm Hg as 1.80×10^3 Pa / 133.3 Pa mm Hg^{-1} = **13.51 mm Hg**

Note that the pressure rises quite a lot for a small change in diameter.

6. After the end of a normal inspiration, the volume of air in the lungs is about 2.8 L. Normally quiet inspiration is driven by a pressure difference of about 2 mm Hg. The air in the lungs is at 37° C and after normal expiration it is at atmospheric pressure. Quiet inspiration is driven by the expansion of the chest cavity by contraction of the diaphragm which expands the air in the lungs. How much is the air expanded to produce an decrease of 2 mm Hg in pressure? Use the ideal gas equation, $PV = nRT$, where P is the pressure, V is the volume, n is the number of moles, R =0.082 L atm $mol^{-1}\ °K^{-1}$ is the gas constant and T is the absolute temperature.

Here the number of molecules, or moles, in the gas does not change. Thus we write

$P_1 V_1 = n_1 RT = P_2 V_2$

We ask, what is V_2 relative to V_1 when $P_2 = P_1 - 2$ mm Hg? Here T = 310° K and R= 0.082 L atm $mol^{-1}\ °K^{-1}$. We do not know V_1 or n_1 but we do know that P_1 = 1 atm = 760 mm Hg; Thus we can calculate

V_2 / V_1 = 760 mm Hg / 758 mm Hg = **1.00264** Thus, expansion of the volume by a scant 0.26% is sufficient to lower its pressure by 2 mm Hg and drive inspiration.

7. A Burette is a vertical, right circular cylinder that is open at the top and has a stopcock valve at the bottom to let fluid out. Assume that you have a burette that has an inner diameter of 1 cm and a height of 100 cm.

A. Assume that you fill the burette with water to some height, h. What is the relation between h and the pressure at the base of the burette?

The pressure is the weight of the column of fluid per unit area. The weight is m**g** where m is the mass of the column and **g** is the acceleration due to gravity. The mass, m, is just the volume times the density = A x h. So the mass m = ρ Ah. The pressure is thus given as

$P = \rho Ah\, \mathbf{g} / A = \boldsymbol{\rho g h}$

B. What is the relation between volume of water in the burette and the pressure?

This is actually answered in part A: $P = \rho V \mathbf{g} / A$ or $\mathbf{V = P\, A / \rho g}$

C. How does this relation map onto the relation between charge and voltage on a capacitor?

Volume / Pressure = A /ρg maps onto Charge / Voltage = Capacitance

So volume maps onto charge; pressure maps on voltage, and Area divided by ρg maps onto $\kappa\epsilon_0 A / \delta$ - both capacitances are given in terms of the dimensions of the capacitor and physical constants of the medium.

D. What is the hydraulic analogue of voltage?

Pressure

E. What is the hydraulic analogue of charge?

Volume

F. Suppose you open the stopcock fully. The fluid will drain out. Assuming a constant diameter of the opening of the stopcock that provides a resistance, R. Derive an equation that describes the time course of draining the burette.

$dV/dt = - P / R = - (\rho \mathbf{g} / AR)\ V$

$dV / V = - \rho \mathbf{g} / AR\ dt$

$\int dV / V = \int -\rho \mathbf{g} / AR\ dt$

$\ln V / V_0 = -\rho \mathbf{g} / AR\ t$

$V = V_0\, e^{-\rho \mathbf{g} / AR\ t} = \mathbf{V_0\, e^{-t / (AR/\rho g)}}$

G. Identify the time constant for the burette emptying.

The time constant is **AR/ρg**.

8. Consider a capacitor with a capacitance of 10μf. You connect it to a variable DC voltage source with a switch in the circuit and a resistor of 1000 Ω in series with the capacitor.

A. What is the relation between the steady-state voltage across the capacitor and the charge?

Charge / Voltage = Capacitance

B. Before you close the switch, there is no charge on the capacitor. When you close the switch, current begins to flow from the voltage source at E volts. What is the relation between current and the voltage drop across the resistor in terms of the current and the resistance?

This is Ohm's Law. The voltage drop across the resistor is V = IR = **R dq/dt**

C. What is the voltage drop across the capacitor in terms of charge and capacitance?

The voltage drop across the capacitor is V = charge / capacitance = **q / C**

D. Kirchoff's voltage law says that the voltage drop around any loop must be zero. Write the equation for the voltage drop across the resistor, capacitor and voltage source.

E - R dq/dt - q /C = 0

E. Solve the equation in part D to derive the time course of charging of the capacitor.

$RC\, dq/dt + q = EC$

$RC\, dq/dt = EC - q$

$dq / (q - EC) = - dt / RC$

$\int dq / (q - EC) = \int - dt / RC$

$\ln (q - EC)/(q_0 - EC) = - t/ RC$

$q = EC + (q_0 - EC)\, e^{-t / RC}$

Since $q_0 = 0$, this becomes $\mathbf{q = q\ (1 - e^{-t/RC})}$

F. Solve the equation in part C to derive the time course of the current.

$I = dq /dt = q_\infty (- - 1 / RC\, e^{-t/RC}) = (q_\infty / C) \times (1/ R) \times e^{-t/RC} = E / R\, e^{-t/RC} = \mathbf{I_0\, e^{-t/RC}}$

G. Identify the time constant for the charging of the capacitor.

The time constant is **RC**.

9. You are given a long vertical tube filled with fluid of viscosity η at sea level where the acceleration due to gravity is $g = 9.81\ m\ s^{-2}$. You have a steel ball or radius r and density ρ_{steel} that you carefully drop into the fluid. Assume that the drag force on the ball obeys Stokes Law: the drag coefficient is 6πrη where $\mathbf{F}_{drag} = - 6\pi r\eta\, \mathbf{v}$. Remember that the steel ball is subjected to a bouyant force equal to the volume of the ball times the density of the fluid, ρ_{fluid}, times the acceleration of gravity.

A. Derive an expression for the time of approach of the steel ball to terminal velocity.

We need to know the acceleration of the steel ball towards its terminal velocity. The forces acting on it are: the acceleration due to gravity, the buoyant force, and the drag force. These sum as:

$(m - \rho_{water} v)\,\mathbf{g} - \beta\,\mathbf{v} = m\,dv/dt$ Where m is the mass of the ball, v is its volume, ρ_{water} is the density of the water, **g** is the acceleration due to gravity, β is the drag coefficient = 6πrη, and **v** is the velocity. We can write $(m - \rho_{water} v) = v(\rho - \rho_{water}) = m'$, the apparent mass of the object suspended in water. Then we have

$m\,dv/dt = m'\mathbf{g} - \beta\,\mathbf{v}$

$dv/dt = m'/m\ \mathbf{g} - \beta/m\ \mathbf{v}$

$dv / (m'/m\ \mathbf{g} - \beta / m\, v) = dt$ Integrating this between 0 and t, where v = 0 at t =0 and v=v at t,

$-m/\beta\ \ln(m'/m\ \mathbf{g} - \beta/m\, v) / \ln(m'/m\ \mathbf{g}) = t$

$\ln(m'/m\ \mathbf{g} - \beta/m\, v) / (m'/m\ \mathbf{g})) = -\beta/m\, t$

$\ln(1 - \beta/m'\mathbf{g}\ v) = -\beta/m\, t$

$1 - \beta/m'\mathbf{g}\, v = e^{-\beta/m\,t}$

$\mathbf{v = m'g/\beta\ (1 - e^{-\beta/m\,t})}$

This equation describes an exponential approach to the terminal velocity.

B. Derive an expression for the viscosity of the fluid as a function of the terminal velocity.

The terminal velocity occurs at t = ∞ in the above equation, at which time

$v = m'\,\mathbf{g}/\beta = 4/3\pi r^3 (\Delta\rho)\,\mathbf{g} / 6\pi r\eta$

Solving for η, we obtain: $\boldsymbol{\eta = 4/3\pi r^3 (\Delta\rho)\, g / 6\pi r\, v = 4/18\, r^2 (\Delta\rho)\, g / v}$

where Δρ is the difference in density between the steel ball and the fluid medium whose viscosity we are measuring.

10. An unmyelinated axon can be considered to be a long right circular cylinder. Consider that an axon is 10 cm long with a diameter of 1.0μ (1 μ = 10^{-6} m).

A. If the specific capacitance of the membrane is 1 μf cm^{-2}, what is the capacitance of the axon membrane?

Here $C = C_m A = 1\ \mu f\ cm^{-2} \times 2\pi r l$ The surface area, $2\pi r l = 1.0 \times 10^{-4}\ cm \times \pi \times 10\ cm = 0.00314\ cm^2$. So the capacitance is $1\ \mu f\ cm^{-2} \times 0.00314\ cm^2 =$ $\mathbf{3.14 \times 10^{-9}\ f}$

B. How much charge is separated by this membrane to give a potential of 70 mV?

$C = q/V$ So $q = C \times V = 3.14 \times 10^{-9}\ f \times 0.070\ v =$ $\mathbf{0.2198 \times 10^{-9}}$ **coulomb**

11. A muscle cell approximates a right circular cylinder 10 cm long and 70 μ in diameter. The specific capacitance of the membrane is 1 μf cm^{-2}.

A. What is the capacitance of the muscle membrane?

Here $C = C_m A = 1$ μf cm^{-2} x $2\pi r l$ The surface area, $2\pi r l =$ 70 x 10^{-4} cm x π x 10 cm = 0.2199 cm^2. So the capcitance is 1 μf cm^{-2} x 0.2199 cm^2 = **0.2199 x 10^{-6} f**

B. How much charge is separated by this membrane to give a potential of -85 mV?

The charge is given as q = V C = 0.085 volts x 0.2199 x 10^{-6} f = **1.87 x 10^{-8} coulomb**

12. The Poiseuille equation that relates flow in narrow tubes to the pressure difference is analogous to Ohm's Law for current flow.

A. What is the resistance to flow in terms of the parameters of the tube?

The resistance is the inverse of the coefficient relating flow to ΔP. It is **$R = 8\eta L / \pi r^4$**

B. What happens to resistance if the radius of the tube is halved?

If the radius is halved, resistance increases by a factor of 2^4 = **16**

C. What happens to resistance if the radius of the tube is doubled?

If the radius is doubled, the resistance is decreased by a factor of 2^4 = **16**

13. A bubble is held at a radius of 250 μ. The transmural pressure difference is 2 mm Hg. What is the tension in the wall?

Here we use the law of Laplace for spheres: T = P x r / 2 Here r = 250 x 10^{-6} m, P = 2 mm Hg = 2 mm Hg x 133.29 Pa mm Hg^{-1} = 266.6 N m^{-2} and T = 266.6 Nm^{-2} x 250 x 10^{-6} m / 2 = **0.0333 N m^{-1}**

14. The thickness of a single membrane is about 7 nm and its capacitance is about 1μf cm^{-2}. A myelin sheath consists of multiple membranes produced by coils of Schwann cell or oligodendroglia cells. Suppose a myelin sheath results from 100 membranes stacked on top of each other.

A. What is the specific capacitance of the myelin?

Here we use the equation $C = \kappa \epsilon_0 A / \delta$ where κ is the dielectric constant, ϵ_0 is the electrical permittivity of space, A is the area and δ is the thickness. If the myelin sheath has 100 membranes stacked on top of each other, its capacity should be 1/100 of that of a single membrane, or

1 x 10^{-8} f cm^{-2}.

B. If the myelin sheath is a right circular cylinder 1 mm long, with a radius of 3 μm, what is its total capacitance?

The total capacitance $C = C_m \times A$ The specific capacity of the myelin sheath is given in part A as 1 x 10^{-8} f cm^{-2}. The area is the surface area = $2\pi r \times L$ = 2 x π x 3 x 10^{-4} cm x 0.1 cm = 1.88 x 10^{-4} cm^2 ; So the capacitance is 1 x 10^{-8} f cm^{-2} x 1.88 x 10^{-4} cm^2 = **1.88 x 10^{-12} f**

C. How much charge does it take to produce a voltage of -70 mV across this capacitor?

The charge is calculated as C x V = 1.88×10^{-12} f x 0.070 v = **1.32×10^{-13} coulomb**

D. If the myelin were just 1 membrane - i.e., not myelin, what would be its capacitance? How much charge would it take to produce a voltage of -70 mV across the single membrane?

If the myelin were just a single membrane, with no myelin, its capacitance would be **1.88×10^{-10} f** and the charge necessary to make a voltage of -70 mV would be **1.32×10^{-11} coulomb**.

15. The transmural pressure difference across a small vein is 20 mm Hg. The radius is 1 mm. What is the wall tension?

Here we use the Law of Laplace for cylinders: $T = P \times r$ Here $r = 1 \times 10^{-3}$ m and P = 20 mm Hg x 133.3 Pa mm Hg^{-1} = 2666 Nm^{-2}. Inserting the values gives

$T = 2666\ Nm^{-2} \times 1 \times 10^{-3}\ m = \mathbf{2.666\ N\ m^{-1}}$.