Chapter 2: Basic Concepts of Probability Theory

2.1 Specifying Random Experiments

2.1

(a) $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

(b) $A = \{1, 2, 3, 4\}$ $\quad B = \{2, 3, 4, 5, 6, 7, 8\}$ $\quad D = \{1, 3, 5, 7, 9, 11\}$

(c) $A \cap B \cap D = \{3\}$ $\quad A^c \cap B = \{5, 6, 7, 8\}$

$A \cup (B \cap D^c) = \{1, 2, 3, 4, 6, 8\}$

$(A \cup B) \cap D^c = \{2, 4, 6, 8\}$

2.2 The outcome of this experiment consists of a pair of numbers (x, y) where x = number of dots in first toss and y = number of dots in second toss. Therefore, S = set of ordered pairs (x, y) where $x, y \epsilon \{1, 2, 3, 4, 5, 6\}$ which are listed in the table below:

a)

y

x	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

checkmarks indicate elements of events

b)

y

x	1	2	3	4	5	6
1	✓					
2	✓	✓				
3	✓	✓	✓			
4	✓	✓	✓	✓		
5	✓	✓	✓	✓	✓	
6	✓	✓	✓	✓	✓	✓

$A = \{N_1 < N_2\}^c = \{N_1 \geq N_2\}$

c)

y

x	1	2	3	4	5	6
1						
2						
3						
4						
5						
6	✓	✓	✓	✓	✓	✓

$B = \{N_1 = 6\}$

d) B is a subset of A so when B occurs then A also occurs, thus B implies A

e) $A \cap B^C = \{N_2 \leq N_1 < 6\}$

x	1	2	3	4	5	6
1	✓					
2	✓	✓				
3	✓	✓	✓			
4	✓	✓	✓	✓		
5	✓	✓	✓	✓	✓	
6						

f) C = "number of dots differ by 2"

	1	2	3	4	5	6
1			✓			
2				✓		
3	✓				✓	
4		✓				✓
5			✓			
6				✓		

Comparing the tables for A and C we see

$A \cap C = \{(3,1), (4,2), (5,3), (6,4)\}$

2.3

(a) $A = \{0, 1, 2, 3, 4, 5\}$

(b) $A = \{3\}$

(c) $\{0\} = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)\}$

$\{1\} = \{(1,2), (2,3), (3,4), (4,5), (5,6), (2,1), (3,2), (4,3), (5,4), (6,5)\}$

$\{2\} = \{(1,3), (2,4), (3,5), (4,6), (3,1), (4,2), (5,3), (6,4)\}$

$\{3\} = \{(1,4), (2,5), (3,6), (4,1), (5,2), (6,3)\}$

$\{4\} = \{(1,5), (2,6), (5,1), (6,2)\}$

$\{5\} = \{(1,6), (6,1)\}$

2.4

(a)

X \ Y	−2	−1	0	1	2
+2	—	—	(2,0)	(2,1)	(2,2)
−2	(−2,−2)	(−2,−1)	(−2,0)	—	—

(b) "X definitely +2" (based on observed Y): $\{(2,1), (2,2)\}$

(c) $\{Y=0\} = \{(2,0), (-2,0)\}$

"observed output is zero"
cannot determine input

2.5

a) Each testing of a pen has two possible outcomes: "pen good" (g) or "pen bad" b. The experiment consists of testing pens until a good pen is found. Therefore each outcome of the experiment consists of a string of "b's" ended by a "g". We assume that each pen is not put back in the drawer after being tests. Thus $\mathcal{S} = \{g, bg, bbg, bbbg, bbbbg\}$

b) We now simply record the number of pens tested, so $\mathcal{S} = \{1, 2, 3, 4, 5\}$

c) The outcome now consists of a substring of "b's" and one "g" in any order followed by a final "g". $\mathcal{S} = \{gg, bgg, gbg, gbbg, bbgg, gbbbg, bgbbg, bbgbg, bbbgg, gbbbbg, bgbbbg, bbgbbg, bbbgbg, bbbbgg\}$

d) $\mathcal{S} = \{2, 3, 4, 5, 6\}$

2.6

(a) $S = \{abc, cab, bca, acb, bac, cba\}$

(b) $A = \{abc, acb\}$ $B = \{abc, cba\}$ $C = \{abc, bac\}$

(c) $(A \cup B \cup C)^c = \{abc, acb, cba, bac\}^c = \{cab, bca\}$

(d) $A \cap B \cap C = \{abc\}$

(e) $A \cup B \cup C = \{abc, acb, cba, bac\}$.

2.7

(a) $A = \{2, 4, 6, 8, \ldots\}$

(b) $B = \{3, 6, 9, \ldots\}$

(c) $C = \{1, 2, 3, 4, 5, 6\}$

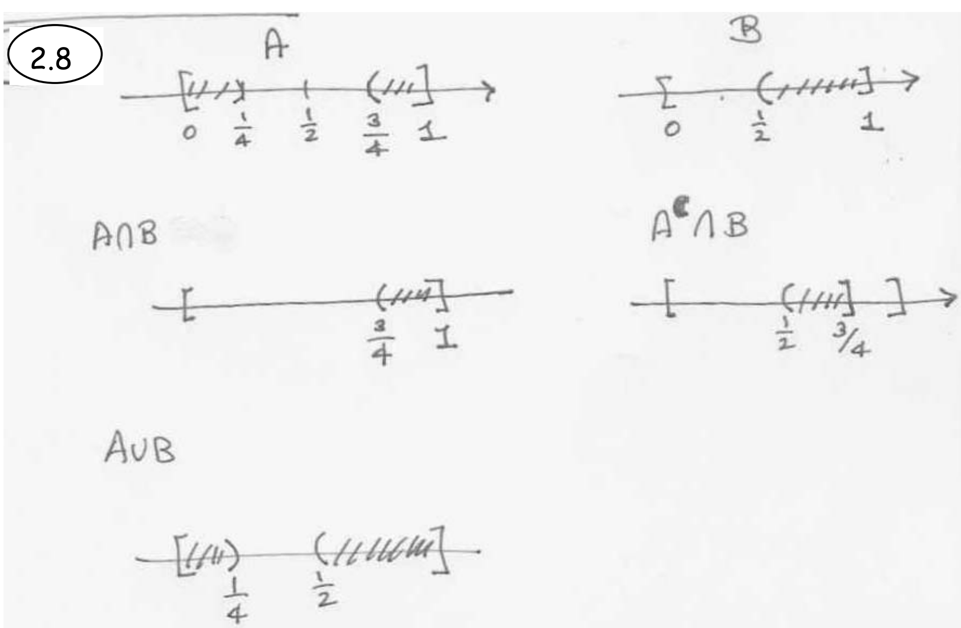
(d) $A \cap B = \{6, 12, 18, \ldots\}$ "multiple of 6"

$A - B =$ "even positive integer and not multiple of 3"

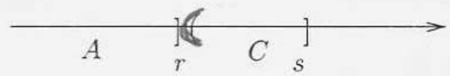
$= \{n = 2m : m$ positive integer, not multiple of $3\}$

$A \cap B \cap C = \{6\}$ "even multiple of 3 less than or equal to 6"

2.8



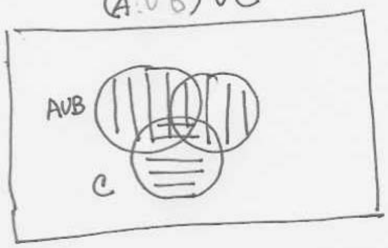
2.9 If we sketch the events A and B we see that $B = A \cup B$. We also see that the intervals corresponding to A and C have no points in common so $A \cap C =$.



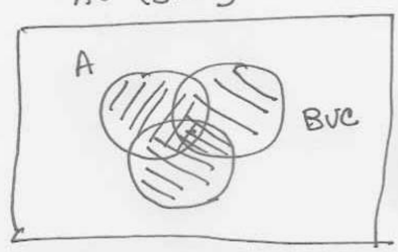
We also see that $(r, s] = (r, \infty) \cap (-\infty, s] = (-\infty, r]^C \cap (-\infty, s]$
that is $C = A^C \cap B$

2.10 ⓐ $A \cup (B \cup C) = (A \cup B) \cup C$

$(A \cup B) \cup C$

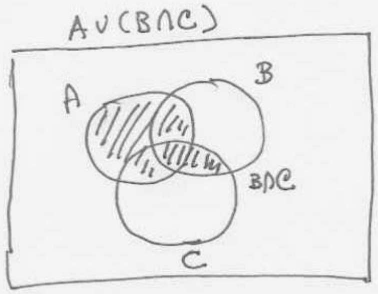


$A \cup (B \cup C)$

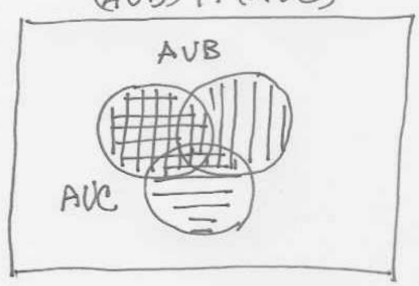


ⓑ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$A \cup (B \cap C)$

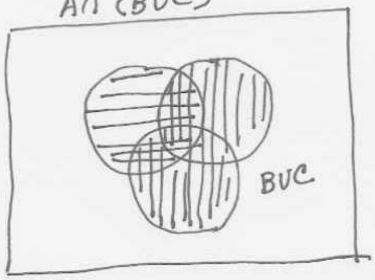


$(A \cup B) \cap (A \cup C)$

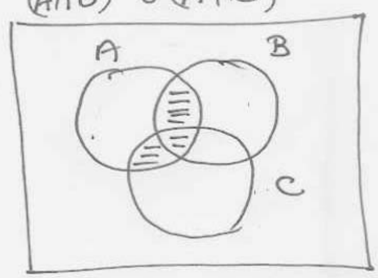


ⓒ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

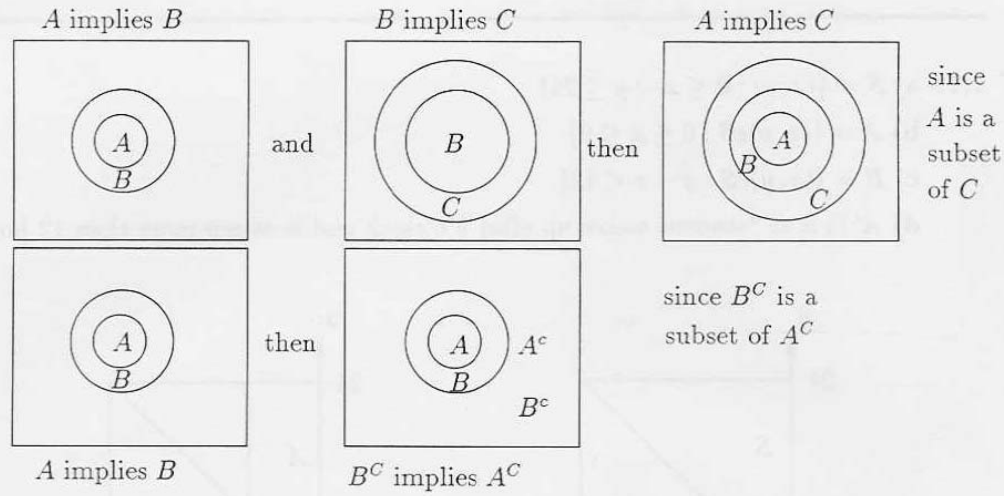
$A \cap (B \cup C)$



$(A \cap B) \cup (A \cap C)$



2.11



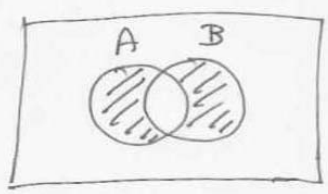
2.12 Given $A \cup B = A$ and $A \cap B = A$ claim $A = B$

Let $\xi \in A$, then $\xi \in A \cap B \Rightarrow \xi \in B \quad \therefore A \subset B$

Let $\xi \in B$ then $\xi \in A \cup B \Rightarrow \xi \in A \quad \therefore B \subset A$

$\therefore A = B$.

2.13 "Exactly one of A or B occurs" $= (A \cap B^c) \cup (A^c \cap B)$



2.14

a) $(A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^a) \cup (A^c \cap B^c \cap C)$

b) $(A \cap B \cap C^c) \cup (A \cap B^c \cap C) \cup (A^c \cap B \cap C)$

c) $A \cup B \cup C$

d) $(A \cap B \cap C^c) \cup (A \cap B^c \cap C) \cup (A^c \cap B \cap C) \cup (A \cap B \cap C)$

e) $A^c \cap B^c \cap C^c$

2.15

(a) $D = A_1 \cap A_2 \cap A_3$

(b) $D = A_1 \cup A_2 \cup A_3$

(c) $D = (A_1 \cap A_2 \cap A_3) \cup (A_1^c \cap A_2 \cap A_3) \cup (A_1 \cap A_2^c \cap A_3) \cup (A_1 \cap A_2 \cap A_3^c)$

2.16

(a) "System j is up" $= A_{j1} \cap A_{j2}$

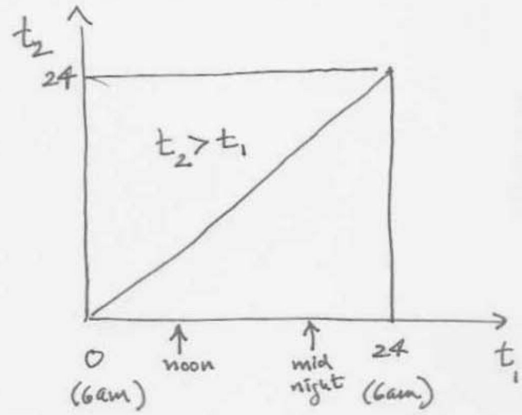
"System is up" $= (A_{11} \cap A_{12}) \cup (A_{21} \cap A_{22}) \cup (A_{31} \cap A_{32})$

(b) "jth level connection active" if $A_{j1} \cap A_{j2}$

"connection active" if any of 3 connections is active

2.17

(a)



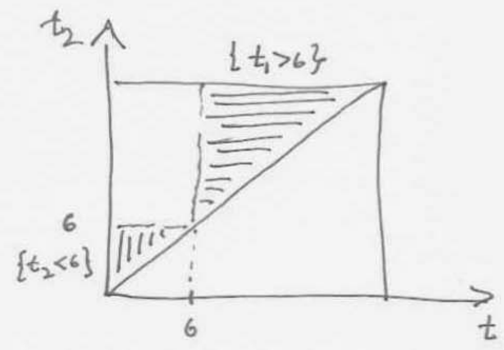
wake at t_1

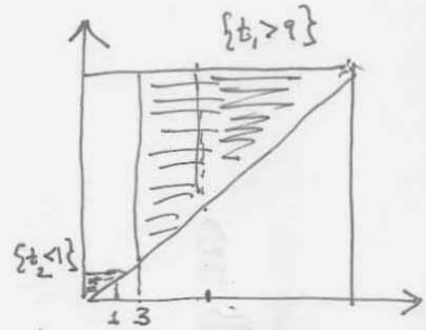
sleep at $t_2 > t_1$

(b) "Asleep at noon" = wake up after noon

or wake up & go to sleep before noon

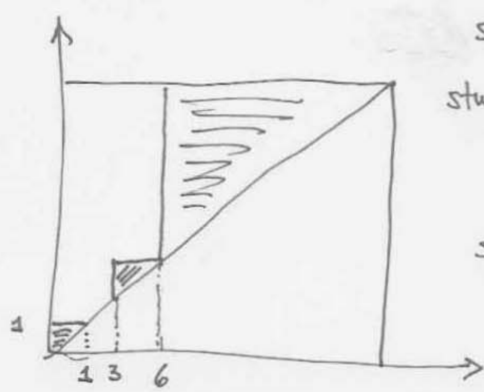
$= \{t_1 > 6\} \cup \{t_2 < 6\}$





(c) "Student sleeps through breakfast" $= \{t_1 > 3\} \cup \{t_2 < 1\}$

(d) $A \cap B$



student wakes up after noon

or

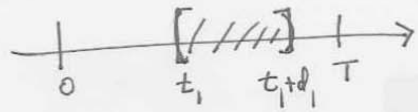
student wakes up after 9am and goes back to sleep before noon

or

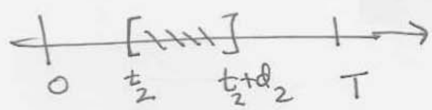
student wakes up and goes back to sleep before 7am

2.18 (a) $S = \{(t_1, t_2) : 0 < t_1 < T,\ 0 < t_2 < T\}$

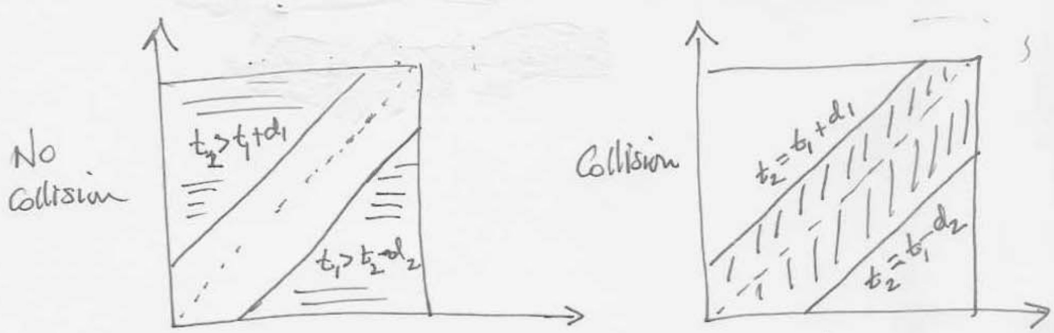
(b) A = train in crossing
$= \{t_1 < t < t_1 + d_1\}$



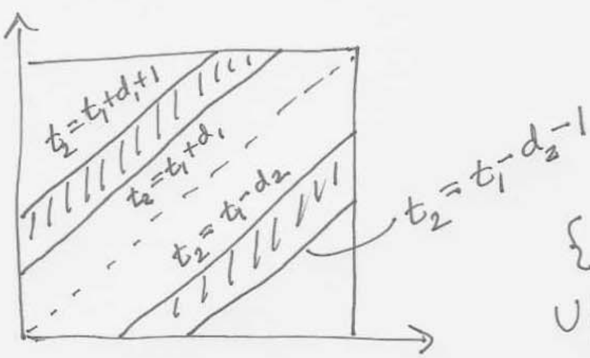
B = car in crossing
$= \{t_2 < t < t_2 + d_2\}$



No Collision occurs if $A \cap B$ is empty $\iff$ $t_1 > t_2 + d_2$ or $t_2 > t_1 + d_1$



(c) Collision missed by 1 second or less.
$= \{\text{No Collision}\} \cap \{\text{within 1 second of collision}\}$



$\{t_1 + d_1 < t_2 < t_1 + d_1 + 1\}$
$\cup \{t_2 + d_2 < t_1 < t_2 + d_2 + 1\}$

2.19 (a) ϕ, $S = \{-1, 0, +1\}$, $\{-1\}$, $\{0\}$, $\{+1\}$, $\{-1, 0\}$, $\{-1, +1\}$, $\{0, +1\}$

(b) $S = \{ (-1, 0), (-1, +1), (0, -1), (0, +1), (+1, -1), (+1, 0) \}$

power set has $2^6 = 64$ subsets.

2.20 $S = \{ HH, HT, TH, TT \}$

(a) ϕ, S, $\{HH\}$, $\{HT\}$, $\{TH\}$, $\{TT\}$, $\{HH, HT\}$, $\{HH, TH\}$, $\{HH, TT\}$,
$\{HT, TH\}$, $\{HT, TT\}$, $\{TH, TT\}$, $\{HH, HT, TH\}$, $\{HH, TH, TT\}$, $\{HH, HT, TT\}$
$\{HT, TH, TT\}$

(b) $S' = \{0, 1, 2\}$
ϕ, S, $\{0\}$, $\{1\}$, $\{2\}$, $\{0, 1\}$, $\{0, 2\}$, $\{1, 2\}$

(c) S has 2^{10} elements & its power set has $2^{2^{10}} = 2^{1024}$ subsets
S' has 11 elements & its power set has 2^{11} subsets

2.2 The Axioms of Probability

2.21) The sample space in tossing a die is $S = \{1, 2, 3, 4, 5, 6\}$. Let $p_i = P[\{i\}] = p$ since all faces are equally likely. By Axiom 1

$$\begin{aligned} 1 &= P[S] \\ &= P[\{1\} \cup \{2\} \cup \{3\} \cup \{4\} \cup \{5\} \cup \{6\}] \end{aligned}$$

The elementary events $\{i\}$ are mutually exclusive so by Corollary 4:

$$1 = p_1 + p_2 + \ldots + p_6 = 6p \Rightarrow p_i = p = \tfrac{1}{6} \text{ for } i = 1, \ldots, 6$$

(b) $P[A] = P[> 3\text{dots}] = P[\{4,5,6\}] = P[\{4\}] + P[\{5\}] + P[\{6\}] = \frac{3}{6}$

$P[B] = P[\text{odd \#}] = P[\{1,3,5\}] = P[\{1\}] + P[\{3\}] + P[\{5\}] = \frac{3}{6}$

(c) $P[A \cup B] = P[\{1, 3, 4, 5, 6\}] = \frac{5}{6}$

$P[A \cap B] = P[\{5\}] = \frac{1}{6}$

$P[A^c] = 1 - P[A] = \frac{3}{6}$

2.22

(a) In first toss, each face occurs with relative frequency 1/6

Each first toss outcome is followed by each possible face 1/6 of the time

$\therefore$ Each pair occurs with relative frequency $1/6 \times 1/6 = 1/36$.

(b) $P[A] = \frac{21}{36}$ $P[B] = \frac{6}{36}$ $P[C] = \frac{8}{36}$ $P[A \cap B^c] = \frac{15}{36}$ $P[A^c] = \frac{15}{36}$

2.23

$P[\{c,d\}] = p_c + p_d = \frac{3}{8}$ by expressing each event in terms of elementary events

$P[\{b,c\}] = p_b + p_c = \frac{6}{8}$ ~~solving this set of linear~~

$P[\{d\}] = \frac{1}{8}$

$1 = P[S] = p_a + p_b + p_c + p_d = 1$

Solving this set of linear equations gives

$p_a = \frac{1}{8} \quad p_b = \frac{4}{8} \quad p_c = \frac{2}{8} \quad p_d = \frac{1}{8}$

2.24

a) $P[A \cap B^c] = P[A] - P[A \cap B]$

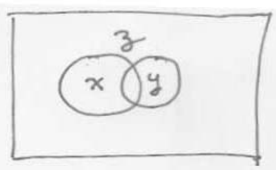
$P[A^c \cap B] = P[B] - P[A \cap B]$

b) $P[A \cap B^c \cup A^c \cap B] = P[A] + P[B] - 2P[A \cap B]$

c) $P[(A \cup B)^c] = 1 - P[A \cup B] = 1 - P[A] - P[B] + P[A \cap B]$

2.25

$z = P[A \cup B] = P[A] + P[B] - P[A \cap B] = x + y - P[A \cap B]$



$P[A \cap B] = x + y - z$

$P[A^c \cap B^c] = 1 - P[(A^c \cap B^c)^c] = 1 - P[A \cup B]$

$= 1 - z$

$P[A^c \cup B^c] = 1 - P[(A^c \cup B^c)^c] = 1 - P[A \cap B] = 1 - x - y + z$

$P[A \cap B^c] = P[A] - P[A \cap B] = x - (x + y - z) = z - y$

$P[A^c \cup B] = 1 - P[A \cap B^c] = 1 - z + y$

2.26

Identities of this type are shown by application of the axioms. We begin by treating $(A \cup B)$ as a single event, then

$$\begin{aligned}
P[A \cup B \cup C] &= P[(A \cup B) \cup C] & \\
&= P[A \cup B] + P[C] - P[(A \cup B) \cap C] & \text{by Cor. 5} \\
&= P[A] + P[B] - P[A \cap B] + P[C] & \text{by Cor. 5 on } A \cup B \\
&\quad - P[(A \cap C) \cup (B \cap C)] & \text{and by distributive property} \\
&= P[A] + P[B] + P[C] - P[A \cap B] & \\
&\quad - P[A \cap C] - P[B \cap C] & \text{by Cor. 5 on} \\
&\quad + P[(A \cap B) \cap (B \cap C)] & (A \cap C) \cup (B \cap C) \\
&= P[A] + P[B] + P[C] - P[A \cap B] - P[A \cap C] & \text{since} \\
&\quad - P[B \cap C] + P[A \cap B \cap C]. & (A \cap B) \cap (B \cap C) = A \cap B \cap C
\end{aligned}$$

2.27 Corollary 5 implies that the result is true for $n = 2$. Suppose the result is true for n, that is,

$$P\left[\bigcup_{k=1}^{n} A_k\right] = \sum_{j=1}^{n} P[A_j] - \sum_{j<k\leq n} P[A_j \cap A_k] + \sum_{j<k<l\leq n} P[A_j \cap A_k \cap A_l] + \ldots$$
$$+(-1)^{n+1} P[A_1 \cap A_2 \cap \ldots \cap A_n] \qquad (*)$$

Consider the $n+1$ case and use the argument applied in Prob. 2.18:

$$\begin{aligned} P\left[\bigcup_{k=1}^{n+1} A_k\right] &= P\left[\left(\bigcup_{k=1}^{n} A_k\right) \cup A_{n+1}\right] \\ &= P\left[\bigcup_{k=1}^{n} A_k\right] + P[A_{n+1}] - P\left[\left(\bigcup_{k=1}^{n} A_k\right) \cap A_{n+1}\right] \\ &= \sum_{j=1}^{n} P[A_j] - \sum_{j<k\leq n} P[A_j \cap A_k] + \ldots + (-1)^{n+1} P[A_1 \cap \ldots \cap A_n] \\ &\quad + P[A_{n+1}] - P\left[\bigcup_{k=1}^{n} (A_k \cap A_{n+1})\right] \text{ from } (*) \end{aligned}$$

Apply Equation (*) to the last term in the previous expression

$$\begin{aligned} P\left[\bigcup_{k=1}^{n} (A_k \cap A_{n+1})\right] &= \sum_{j=1}^{n} P[A_k \cap A_{n+1}] - \sum_{j<k\leq n} P[A_j \cap A_k \cap A_{n+1}] \\ &\quad + \ldots + (-1)^{n+1} P[A_1 \cap A_2 \cap \ldots \cap A_{n+1}] \end{aligned}$$

Thus

$$\begin{aligned} P\left[\bigcup_{k=1}^{n+1} A_k\right] &= \sum_{j=1}^{n} P[A_j] + P[A_{n+1}] + \\ &\quad - \sum_{j<k\leq n} P[A_j \cap A_k] - \sum_{j=1}^{n} P[A_k \cap A_{n+1}] \\ &\quad + \sum_{j<k\leq n} P[A_j \cap A_k \cap A_l] + \sum_{j<k\leq n} P[A_j \cap A_k \cap A_{n+1}] \\ &\quad + \ldots + (-1)^{n+2} P[A_1 \cap A_2 \cap \ldots \cap A_{n+1}] \\ &= \sum_{j=1}^{n+1} P[A_j] - \sum_{j<k\leq n+1} P[A_j \cap A_k] \\ &\quad + \sum_{j<k<l\leq n+1} P[A_j \cap A_k \cap A_l] \\ &\quad + \ldots + (-1)^{n+2} P[A_1 \cap A_2 \cap \ldots \cap A_{n+1}] \end{aligned}$$

which shows that the $n+1$ case holds. This completes the induction argument, and the result holds for $n \geq 2$.

2.28) This experiment is equivalent to tossing a coin 3 times and noting the sequence of heads and tails. There are 8 outcomes and each outcome has probability $\frac{1}{8}$.

$S = \{000, 001, 010, 100, 011, 101, 110, 111\}$

(a)

$$P[A_1] = P[\{100, 101, 110, 111\}] = \frac{4}{8} = \frac{1}{2}$$

$$P[A_1 \cap A_3] = P[\{101, 111\}] = \frac{2}{8} = \frac{1}{4}$$

$$P[A_1 \cap A_2 \cap A_3] = P[\{111\}] = \frac{1}{8}$$

$$P[A_1 \cup A_2 \cup A_3] = 1 - P[(A_1 \cup A_2 \cup A_3)^c] = 1 - P[A_1^c \cap A_2^c \cap A_3^c]$$
$$= 1 - P[\{000\}] = \frac{7}{8}.$$

(b) Let $p = P[\text{"1"}]$

$$P[A_1] = P[\{100\}] + P[\{101\}] + P[\{110\}] + P[\{111\}]$$
$$= p(1-p)^2 + 2p^2(1-p) + p^3$$

$$P[A_1 \cap A_3] = p^2(1-p) + p^3$$

$$P[A_1 \cap A_2 \cap A_3] = p^3$$

$$P[A_1 \cup A_2 \cup A_3] = 1 - (1-p)^3$$

2.29) Each transmission is equivalent to tossing a fair coin. If the outcome is heads, then the transmission is successful. If tails, then another retransmission is required.
As in Example 2.11 the probability that j transmissions are required is:

$$P[j] = \left(\tfrac{1}{2}\right)^j$$

$$P[A] = P[j \text{ even}] = \sum_{k=1}^{\infty} \left(\tfrac{1}{2}\right)^{2k} = \sum_{k=1}^{\infty} \left(\tfrac{1}{4}\right)^k = \sum_{k=0}^{\infty} \left(\tfrac{1}{4}\right)^k - 1$$

$$= \frac{1}{1-\frac{1}{4}} - 1 = \frac{1}{3}$$

$$P[B] = P[j \text{ multiple of } 3] = \sum_{k=1}^{\infty} \left(\tfrac{1}{2}\right)^{3k} = \frac{1}{1-\frac{1}{8}} - 1 = \frac{1}{7}$$

$$P[C] = \sum_{k=1}^{6} \left(\tfrac{1}{2}\right)^k = \tfrac{1}{2}\sum_{k=0}^{5} \left(\tfrac{1}{2}\right)^k = \frac{1}{2}\,\frac{1-\left(\frac{1}{2}\right)^6}{1-\frac{1}{2}} = \frac{63}{64}$$

$$P[C^c] = 1 - P[C] = \frac{1}{64}$$

$$P[A \cap B] = \sum_{k=1}^{\infty} \left(\tfrac{1}{2}\right)^{6k} = \frac{1}{1-\frac{1}{64}} - 1 = \frac{1}{63}$$

$$P[A-B] = P[A] - P[A \cap B] = \frac{1}{3} - \frac{1}{63} = \frac{20}{63}$$

$$P[A \cap B \cap C] = \left(\tfrac{1}{2}\right)^6 = \frac{1}{64}$$

2.30 **a)** Corollary 7 implies $P[A \cup B] \leq P[A] + P[B]$. (Eqn. 2.8). Applying this inequality twice, we have

$$P[(A \cup B) \cup C] \leq P[A \cup B] + P[C] \leq P[A] + P[B] + P[C]$$

b) Eqn. 2.8 implies the $n = 2$ case.
Suppose the result is true for n:

$$P\left[\bigcup_{k=1}^{n} A_k\right] \leq \sum_{k=1}^{n} P[A_k] \qquad (*)$$

Then

$$\begin{aligned}
P\left[\bigcup_{k=1}^{n+1} A_k\right] &= P\left[\left(\bigcup_{k=1}^{n} A_k\right) \cup A_{n+1}\right] \\
&= \leq P\left[\bigcup_{k=1}^{n} A_k\right] + P[A_{n+1}] \text{ by Eqn. 2.8} \\
&\leq \sum_{k=1}^{n} P[A_k] + P[A_{n+1}] \text{ by } (*) \\
&= \sum_{k=1}^{n+1} P[A_k]
\end{aligned}$$

which completes the induction argument.

(c) $P\left[\bigcap_{k=1}^{n} A_k\right] = 1 - P\left[\left(\bigcap_{k=1}^{n} A_k\right)^c\right] = 1 - P\left[\bigcup_{k=1}^{n} A_k^c\right]$

$\geq 1 - \sum_{k=1}^{n} P[A_k^c]$ using the result of part b.

2.31 Let $A_i = \{i\text{th character is in error}\}$

$$P[\text{any error in document}] = P\left[\bigcup_{i=1}^{n} A_i\right] \leq \sum_{i=1}^{n} P[A_i] = np$$

2.32

a) $p_1 = p_3 = p_5 = p \qquad p_2 = p_4 = p_6 = 2p$

$$1 = p_1 + p_2 + p_3 + p_4 + p_5 + p_6 = 9p \qquad p = \frac{1}{9}$$

b) $P[A] = p_4 + p_5 + p_6 = \frac{4}{9} + \frac{1}{9} = \frac{5}{9}$

$P[B] = p_1 + p_3 + p_5 = \frac{3}{9}$

c) $P[A \cup B] = p_1 + p_3 + p_4 + p_5 + p_6 = 1 - p_2 = \frac{7}{9}$

$P[A \cap B] = p_5 = \frac{1}{9}$

$P[A^c] = 1 - \frac{5}{9} = \frac{4}{9}.$

2.33

$S = \{1, 2, \ldots, 59, 60\}$

a) $P[k] = \frac{1}{60} \qquad k \in S$

b) $p_2 = \frac{1}{2} p_1 \quad p_3 = \frac{1}{3} p_1 \quad \ldots \quad p_{60} = \frac{1}{60} p_1$

$$1 = p_1 + p_2 + \cdots + p_{60} = p_1 \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{60}\right) = 4.68\, p_1$$

$p_1 = 0.2137$

c) $p_2 = \frac{1}{2} p_1 \quad p_3 = \frac{1}{4} p_1 \quad p_3 = \frac{1}{8} p_1 \quad \ldots \quad p_{60} = \left(\frac{1}{2}\right)^{59} p_1$

$$1 = p_1 \left(1 + \frac{1}{2} + \frac{1}{4} + \cdots + \left(\frac{1}{2}\right)^{59}\right) \approx 2 p_1$$

$p_1 = \frac{1}{2}$

d) for a: $p[60] = \frac{1}{60} \qquad$ b: $p[60] = \frac{0.2137}{60} = 0.00356 \qquad$ c: $p[60] = 0.86 \times 10^{-18}$

2.34 Assume that the probability of any subinterval I of $[-1,2]$ is proportional to its length, then

$$P[I] = k \text{ length } (I).$$

If we let $I = [-1,2]$ then we must have that

$$1 = P[S] = P[[-1,2]] = k \text{ length } ([-1,2]) = 3k \Rightarrow k = \frac{1}{3}.$$

a)
$$P[A] = \frac{1}{3} \text{ length } ([-1,0)) = \frac{1}{3}(1) = \frac{1}{3}$$
$$P[B] = \frac{1}{3} \text{ length } ((-0.5,1)) = \frac{1}{3}\,\frac{3}{2} = \frac{1}{2}$$
$$P[C] = \frac{1}{3} \text{ length } ((0.75,2)) = \frac{1}{3}\,\frac{5}{4} = \frac{5}{12}$$
$$P[A \cap B] = \frac{1}{3} \text{ length } ((-0.5,0)) = \frac{1}{3}\,\frac{1}{2} = \frac{1}{6}$$
$$P[A \cap C] = P[\emptyset] = 0$$

b)
$$P[A \cup B] = \cancel{P[S]} \; \frac{1}{3} \text{ length}([-1,1)) = \frac{2}{3}$$
$$P[A \cup C] = \frac{1}{3} \text{ length } (A \cup C) = \frac{1}{3}\left(1 + \frac{5}{4}\right) = \frac{3}{4}$$
$$P[A \cup B \cup C] = P[S] = 1$$

Now use axioms and corollaries:

$$P[A \cup B] = P[A] + P[B] - P[A \cap B] \quad \text{by Cor. 5}$$
$$= \frac{1}{3} + \frac{1}{2} - \frac{1}{6} = \frac{2}{3} \quad \checkmark$$
$$P[A \cup C] = P[A] + P[C] - P[\underbrace{A \cap C}_{\emptyset}] = \frac{1}{3} + \frac{5}{12} = \frac{3}{4} \quad \checkmark \quad \text{by Cor. 5}$$
$$P[A \cup B \cup C] = P[A] + P[B] + P[C]$$
$$-P[A \cap B] - P[A \cap C] - P[B \cap C]$$
$$+P[A \cap B \cap C] \quad \text{by Eq. (2.7)}$$
$$= \frac{1}{3} + \frac{1}{2} + \frac{5}{12} - \frac{1}{6} - 0 - \frac{1}{12} + 0$$
$$= 1 \quad \checkmark$$

2.35 a) Let I be a subinterval of [-1,2] then

$$P[I] = \cancel{2}k \text{ length } (I \cap [0,2]) + 2k \text{ length } (I \cap [-1,0])$$

Letting $I = [-1,2]$ we have

$$1 = P[[-1,2]] = 2k + 2k = 4k \Rightarrow k = \frac{1}{4}$$

b) $P[A] = \frac{2}{4}(1) = \frac{1}{2}$

$P[B] = \frac{2}{4}(\frac{1}{2}) + \frac{1}{4}(1) = \frac{1}{2}$

$P[C] = \frac{1}{4}(\frac{5}{4}) = \frac{5}{16}$

$P[A \cap B] = \frac{2}{4}(\frac{1}{2}) = \frac{1}{4}$

$P[A \cap C] = P[\emptyset] = 0$

$P[A \cup B] = \cancel{P[S] = 1}$ $\frac{2}{4}(1) + \frac{1}{4}(1) = \frac{3}{4}$

$P[A \cup C] = \frac{2}{4}(1) + \frac{1}{4}\left(\frac{5}{4}\right) = \cancel{\frac{1}{2}}\ \frac{13}{16}$

$P[A \cup B \cup C] = P[S] = 1$

Now use axioms and corollaries

$$\begin{aligned}
P[A \cup B] &= P[A] + P[B] - P[A \cap B] \\
&= \frac{1}{2} + \frac{1}{2} - \frac{1}{4} = \cancel{1}\checkmark \ \ 3/4 \\
P[A \cup C] &= P[A] + P[C] - P[\underbrace{A \cap C}_{\emptyset}] \\
&= \frac{1}{2} + \frac{5}{16} = \cancel{\frac{1}{2}}\ \frac{13}{16} \\
P[A \cup B \cup C] &= P[A] + P[B] + P[C] + \\
&\quad -P[A \cap B] - P[A \cap C] - P[B \cap C] + P[A \cap B \cap C] \\
&= \cancel{\frac{1}{3} + \frac{5}{6} + \frac{1}{6} - \frac{1}{6} - 0 - \frac{1}{6} + 0 = 1\checkmark} \\
&= \frac{1}{2} + \frac{1}{2} + \frac{5}{16} - \frac{1}{4} - 0 - (\frac{1}{4})(\frac{1}{4}) = 1 \checkmark
\end{aligned}$$

2.36 Let x denote the lifetime, then

$$A = \{x > \cancel{5}\,4\} \text{ and } B = \{x > \cancel{10}\,8\}$$

a) $P[A \cap B] = P[\{x > \cancel{5}\,4\} \cap \{x > \cancel{10}\,8\}] = P[\{x > \cancel{10}\,8\}] = \cancel{\ }\ 1/8$

$P[A \cup B] = P[\{x > \cancel{5}\,4\} \cup \{x > \cancel{10}\,8\}] = P[\{x > \cancel{5}\,4\}] = \cancel{\ }\ 1/4$

b)

$$\begin{aligned} P[\{x > 5\}] &= P[\{\cancel{5}\,6 < x \le \cancel{10}\,12\} \cup \{x > \cancel{10}\,12\}] \\ &= P[\{\cancel{5}\,6 < x \le \cancel{10}\,12\}] + P[\{x > \cancel{10}\,12\}] \\ \Rightarrow P[\{\cancel{5}\,6 < x \le \cancel{10}\,12\}] &= P[\{x > \cancel{5}\,6\}] - P[\{x > \cancel{10}\,12\}] = 1/6 - 1/12 = 1/12 \end{aligned}$$

2.37 a) Since $(-\infty, r] \subset (-\infty, s]$ when $r < s$

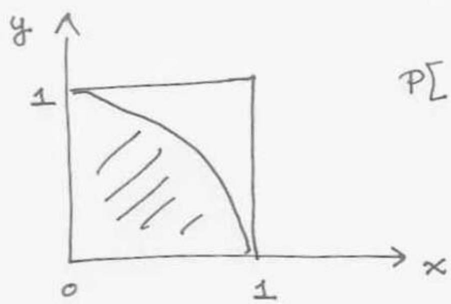
$$P[(-\infty, r]] \le P[(-\infty, s]] \text{ by Corollary 7.}$$

b)

$$\begin{aligned} P[(-\infty, s]] &= P[(-\infty, r] \cup (r, s]] \\ &= P[(-\infty, r]] + P[(r, s]] \\ \Rightarrow P[(r, s]] &= P[(-\infty, s]] - P[(-\infty, r]] \end{aligned}$$

2.38

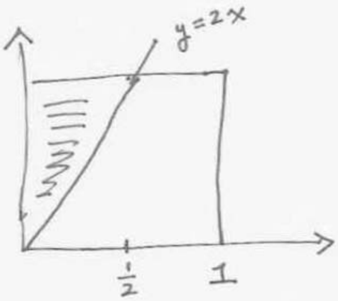
a)



$P[x^2 + y^2 < 1] = \frac{\pi(1)^2}{4} = \frac{\pi}{4}$

Area inside circle

b)



$P[y > 2x] = \frac{1}{4}$

Area in right triangle

2.3 *Computing Probabilities Using Counting Methods

2.39 The number of distinct ordered triplets $= 60 \cdot 60 \cdot 60 = 60^3$

2.40 The number of distinct 7-tuples $= 8 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 8(10^6)$

2.41 The number of distinct ordered triplets $= 6 \cdot 2 \cdot 52 = 624$

2.42 # sequences of length 8 $= 2^8 = 256$

P[arbitrary sequence = correct sequence] $= \frac{1}{256}$

P[success in two tries] = 1 − P[failure in both tries]

$= 1 - \frac{255}{256} \cdot \frac{254}{255}$

2.43 8, 9, or 10 characters long

– at least 1 special character from set of size 24

– numbers from size 10

– upper & lower case letters $26 \times 2 = 52$

} 62 choices

For length n:

– pick position of required special character & pick character

n positions $\times$ 24 characters.

– pick number/letter/special character for remaining $n-1$ positions

86^{n-1}

Total # passwords $= n \cdot 24 \cdot 86^{n-1}$

length 8, 9, or 10 $= 8 \cdot 24 \cdot 86^7 + 9 \cdot 24 \cdot 86^8 + 10 \cdot 24 \cdot 86^9 = 6.24 \times 10^{19}$

Time to try all passwords $= 6.24 \times 10^{13}$ seconds $= 2(10^6)$ years

2.44 $3^{10} = 59049$ possible answers

Assuming each paper selects answers at random

P[two papers are identical] $= \frac{1}{3^{10}} \times \frac{1}{3^{10}} = \frac{1}{3^{20}} = 2.87 \times 10^{-10}$

2.45 (a) # combinations = $5 \times 3 = 15$

(b) The table below shows the 15 combinations and a schedule that allows all combinations without using the same t-shirt on consecutive days.

jeans \ t-shirts	1	2	3	4	5
1	1	4	7	10	13
2	14	2	5	8	11
3	12	15	3	6	9

2.46 The order in which the 4 toppings are selected does not matter so we have sampling without ordering.

If toppings may not be repeated, Eqn. (2.22) gives

$$\binom{15}{4} = 1365 \text{ possible deluxe pizzas}$$

If toppings may be repeated, we have sampling with replacement and without ordering. The number of such arrangements is

$$\binom{14+4}{4} = 3060 \text{ possible deluxe pizzas.}$$

2.47 # student seat selections $= 60 \cdot 59 \cdot 58 \cdot \ldots \cdot 16 = \dfrac{60!}{15!}$

2.48

ab ba $\Rightarrow 2 = 2!$

abc cab bca acb bac cba $\Rightarrow 6 = 3!$

abcd dabc cdab bcda
acbd dacb bdac cbda
adbc cadb bcad dbca
abdc cabd dcab bdca
acdb bacd dbac cdba
adcb badc cbad dcba

$\Rightarrow 24 = 4!$

2.49 There are 3! permutations of which only one corresponds to the correct order; assuming equiprobable permutations:

$$P[\text{correct order}] = \frac{1}{3!} = \frac{1}{6}$$

2.50 # ways to cover all buckets = $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5!$

placement of 5 balls w 5 buckets = 5^5

probability all buckets covered = $5!/5^5 = 0.0384$

2.51 Combinations of 2 from 2 objects : ab $\binom{2}{2} = 1$

Combinations of 2 " 3 objects : ab ac bc $\binom{3}{2} = \frac{3!}{2!} = 3$

Combinations of 2 " 4 objects : ab ac ad bc bd cd $\binom{4}{2} = \frac{4!}{2!2!} = 6$

2.52 8! arrangements of people around a table = 40320

Experiment: select male or female for first spot: 2

select first spot gender x 4

" 2nd spot gender x+1 4

" 3rd spot gender x 3

⋮ ⋮

$2 \times 4! \times 4! = 1152$

2.53 Number ways of picking one out of $\cancel{5}\,6 = \binom{6}{1} = 6$

Number ways of picking two out of $\cancel{5}\,6 = \binom{6}{2} = \cancel{10}\ 15$

Number ways of picking none, some or all of $\cancel{5}\,6 = \sum_{j=0}^{6} \binom{6}{j} = 2^6 = 64$

2.54a The number of ways of choosing M out of 100 is $\binom{100}{M}$. This is the total number of equiprobable outcomes in the sample space.

We are interested in the outcomes in which m of the chosen items are defective and $M-m$ are nondefective.

The number of ways of choosing m defectives out of k is $\binom{k}{m}$.

The number of ways of choosing $M-m$ nondefectives out of 100 k is $\binom{100-k}{M-m}$.

The number of ways of choosing m defectives out of k <u>*and*</u> $M-m$ non-defectives out of $100-k$ is

$$\binom{k}{m}\binom{100-k}{M-m}$$

$$\begin{aligned} P[m \text{ defectives in } M \text{ samples}] &= \frac{\#\text{ outcomes with } k \text{ defective}}{\text{Total }\#\text{ of outcomes}} \\ &= \frac{\binom{k}{m}\binom{100-k}{M-m}}{\binom{100}{M}} \end{aligned}$$

This is called the <u>Hypergeometric</u> distribution.

(b) $$P[\text{lot accepted}] = P[m=0 \text{ or } m=1] = \frac{\binom{100-k}{M}}{\binom{100}{M}} + \frac{k\binom{100-k}{M-1}}{\binom{100}{M}}.$$

2.55 Number ways of picking 20 raccoons out of $N = \binom{N}{20}$

Number ways of picking 4 ~~5~~ tagged raccoons out of ~~10~~ 8 and ~~15~~ 16 untagged raccoons out of $N - \cancel{10}\,8$ $= \binom{\cancel{10}\,8}{\cancel{5}\,4}\binom{N-\cancel{10}\,8}{\cancel{15}\,16}$

$$P[5 \text{ tagged out of 20 samples}] = \frac{\binom{\cancel{10}\,8}{\cancel{5}\,4}\binom{N-\cancel{10}\,8}{\cancel{15}\,16}}{\binom{N}{20}} \triangleq p(N)$$

$p(N)$ increases with N as long as $p(N)/p(N-1) > 1$

$$\frac{p(N)}{p(N-1)} = \frac{\binom{N-\cancel{10}\,9}{\cancel{15}\,16}\binom{N-1}{20}}{\binom{N}{20}\binom{N-\cancel{11}\,9}{\cancel{15}\,16}} = \frac{(N-\cancel{10}\,8)(N-20)}{(N-\cancel{25}\,4)N} \geq 1$$

$(N-\cancel{10}\,8)(N-20) \geq (N-\cancel{25}\,4)N \Rightarrow 40 \geq N$
$p(40) = p(39) = \cancel{.284}$ 0.305 maxima of $p(N)$.

2.56

(b) $P[X=k] = \dfrac{\binom{10}{k}\binom{40}{5-k}}{\binom{50}{5}}$ $k=0,1,\ldots,5$ without replacement

Hypergeometric probabilities

(a) With replacement:

pick k defective balls then pick $5-k$ nondefective balls

$$\underbrace{\text{pick } k \text{ defective balls}}_{10^k} \quad \underbrace{\text{then pick } 5-k \text{ nondefective balls}}_{40^{5-k}}$$

There are $\binom{5}{k}$ arrangements of this composition

$$\#\text{ ways of obtaining } k \text{ defective in 5 tested} = \frac{\binom{5}{k}10^k 40^{5-k}}{50^5}$$

$$= \binom{5}{k}\left(\frac{10}{50}\right)^k\left(\frac{40}{50}\right)^{5-k} \quad k=0,1,\ldots,5$$

Binomial probabilities.

2.57 $\dfrac{9!}{4!2!3!} = 1260$

2.58

# forward combinations	$\binom{6}{3}$	assuming forwards do not have assigned position (left, center, right) and similarly for defensemen
# defense combinations	$\binom{4}{2}$	
# goalie combinations	$\binom{2}{1}$	

$$\#\text{teams} = \binom{6}{3}\binom{4}{2}\binom{2}{1} = 240$$

If forwards + defensemen have assigned positions

$$\#\text{teams} = \binom{6}{3}\times 3! \times \binom{4}{2}\times 2! \times \binom{2}{1} = 4760$$

2.59 Suppose each student is viewed as selecting one of the 7 days (e.g. placing a ball in one of 7 urns) then there are 7^{28} possible sequences of choices.

Of the sequences that have 4 choices for each day there are

$$\frac{28!}{4!\,4!\,4!\,4!\,4!\,4!\,4!}$$ such sequences.

$$\therefore\ P[\text{4 students at each day}] = \frac{28!}{(4!)^7}\,\frac{1}{7^{28}}$$

2.60

$$\binom{n}{k} = \frac{n!}{k(n-k)!}$$

$$\binom{n}{n-k} = \frac{n!}{(n-k)!\,(n-(n-k))!} = \frac{n!}{(n-k)!\,k!}$$

2.61 **a)** Since N_i denotes the number of possible outcomes of the ith subset after $i-1$ subsets have been selected, it can be considered as the number of subpopulations of size k_i from a population of size $n - k_1 - k_2 - \ldots - k_{i-1}$, hence

$$N_i = \binom{n - k_1 - \ldots - k_{i-1}}{k_i} \qquad i = 1, \ldots, J-1$$

Note that after $J-1$ subsets arae selected, the set B_J is determined, i.e. $N_J = 1$.

b) The number of possible outcomes for B_1 is B_1, B_2 is N_2, etc. hence

$$\#\text{ partitions} = N_1 N_2 \ldots N_{J-1} = \prod_{i=1}^{J-1} \frac{(n - k_1 - \ldots - k_{i-1})!}{k_1!(n - k_1 - \ldots - k_i)!} = \frac{n!}{k_1! k_2! \ldots k_J!}$$

2.4 Conditional Probability

2.62

$A = \{N_1 \geq N_2\}$ $B = \{N_1 = 6\}$

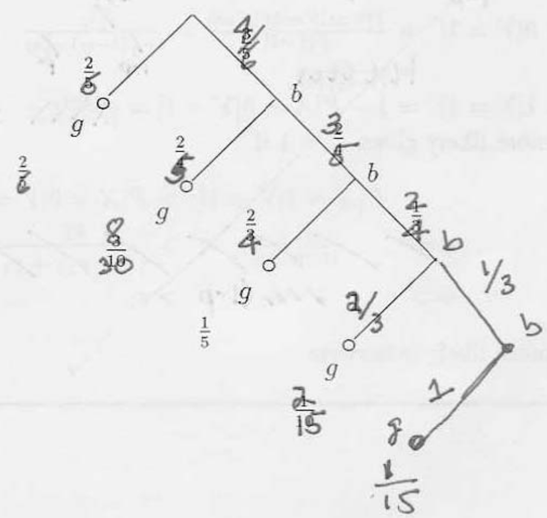
From problem 2.2 we have that $A \supset B$, therefore

$$P[A|B] = \frac{P[A \cap B]}{P[B]} = \frac{P[B]}{P[B]} = 1$$

and

$$P[B|A] = \frac{P[A \cap B]}{P[A]} = \frac{P[B]}{P[A]} = \frac{6/36}{21/36} = \frac{2}{7}$$

2.63a



$$P[g] = \frac{2}{6}$$

$$P[bg] = P[b]P[g|b] = \cancel{\frac{3}{5} \cdot \frac{2}{4} = \frac{3}{10}} \quad \frac{4}{6} \cdot \frac{2}{5} = \frac{8}{30}$$

$$P[bbg] = \cancel{\frac{3}{5} \cdot \frac{2}{4} \cdot \frac{2}{3} = \frac{1}{5}} = \frac{4}{6} \, \frac{3}{5} \, \frac{2}{4} = \frac{1}{5}$$

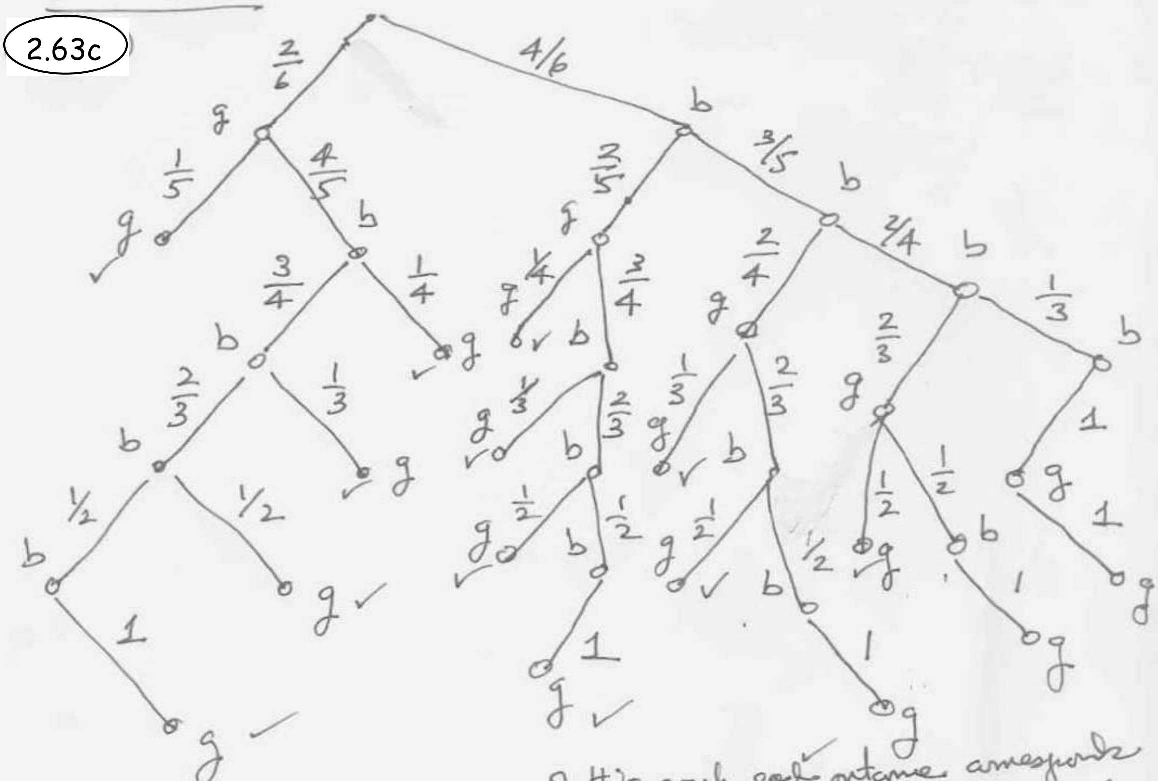
$$P[bbbg] = \cancel{\frac{3}{5} \cdot \frac{2}{4} \cdot \frac{1}{3} = \frac{1}{10}} = \frac{4}{6} \, \frac{3}{5} \, \frac{2}{4} \, \frac{2}{3} = \frac{2}{15}$$

$$P[bbbbg] = \frac{4}{6} \cdot \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{1}{3} \cdot 1 = \frac{1}{15}$$

b) $P[1 \text{ pen tested}] = P[g] = \frac{2}{6}$

$P[2] = P[bg]$ $P[3] = P[bbg]$ $P[4] = P[bbbg]$ $P[5] = P[bbbbg]$

2.63c



In this graph each outcome corresponds to a distinct arrangement of 4 b's and 2 w's. There are $\binom{6}{2}=15$ arrangements.

$$P[2\text{ tests}] = \frac{2}{6}\cdot\frac{1}{5} = \frac{1}{15}$$

$$P[3\text{ tests}] = \frac{2}{6}\frac{4}{5}\frac{1}{4}+\frac{4}{6}\frac{2}{5}\frac{1}{4} = \frac{1}{15}+\frac{1}{15} = \frac{2}{15}$$

$$P[4\text{ tests}] = \frac{2}{6}\frac{4}{5}\frac{3}{4}\frac{1}{3}+\frac{4}{6}\left(\frac{2}{5}\right)\frac{3}{4}\frac{1}{3}+\frac{4}{6}\frac{3}{5}\frac{2}{4}\frac{1}{3} = \frac{3}{15}$$

$$P[5\text{ tests}] = \frac{2}{6}\frac{4}{5}\frac{3}{4}\frac{2}{3}\frac{1}{2}+\frac{4}{6}\frac{2}{5}\frac{3}{4}\frac{2}{3}\frac{1}{2}+\frac{4}{6}\frac{3}{5}\frac{2}{4}\frac{2}{3}\frac{1}{2}+\frac{4}{6}\frac{3}{5}\frac{2}{4}\frac{2}{3}\frac{1}{2} = \frac{4}{15}$$

$$P[6\text{ tests}] = \frac{2}{6}\frac{4}{5}\frac{3}{4}\frac{2}{3}\frac{1}{2}+\frac{4}{6}\frac{2}{5}\frac{3}{4}\frac{2}{3}\frac{1}{2}+\frac{4}{6}\frac{3}{5}\frac{2}{4}\frac{2}{3}\frac{1}{2}+\frac{4}{6}\frac{3}{5}\frac{2}{4}\frac{2}{3}\frac{1}{2}+\frac{4}{6}\frac{3}{5}\frac{2}{4}\frac{1}{3} = \frac{5}{15}$$

2.64

$P[B\cap C|A] = P[$Bob & Chris pick their names $|$ Al picked his name$]$

$$= \frac{P[B\cap C\cap A]}{P[A]} = \frac{P[\{abc\}]}{P[\{abc, acb\}]} = \frac{1/6}{2/6} = \frac{1}{2}.$$

$P[C|A\cap B] = P[$Chris picks his name $|$ Al & Bob picked their name$]$

$$= \frac{P[A\cap B\cap C]}{P[A\cap B]} = \frac{P[\{abc\}]}{P[\{abc\}]} = 1.$$

2.65 $P[B|A] = \frac{P[A \cap B]}{P[A]} = \frac{P[\text{multiple of } 6]}{P[\text{even}]} = \frac{1/63}{1/3} = \frac{1}{21}$

$P[A|B] = \frac{P[A \cap B]}{P[A]} = \frac{P[\text{multiple of } 6]}{P[\text{multiple of } 6]} = 1.$

2.66 from problem 2.8:

$P[B|A] = \frac{P[A \cap B]}{P[A]} = \frac{P[\frac{3}{4} < U \le 1]}{P[|U - \frac{1}{2}| > \frac{1}{4}]} = \frac{1/4}{1/2} = \frac{1}{2}$

$P[B|A] = \frac{P[A \cap B]}{P[B]} = \frac{P[\frac{3}{4} < U \le 1]}{P[\frac{1}{2} < U \le 1]} = \frac{1/4}{1/2} = \frac{1}{2}.$

2.67 from problem 2.36

$P[B|A] = \frac{P[A \cap B]}{P[A]} = \frac{P[x > 8]}{P[x > 4]} = \frac{1/8}{1/4} = \frac{1}{2}$

$P[A|B] = \frac{P[x > 8]}{P[x > 8]} = 1.$

2.68

(a) $P[A] = P[\text{hand rests in last 10 minutes}]$

$$P[A] = p_{51} + p_{52} + \cdots + p_{60} = \frac{10}{60} = \frac{1}{6}$$

$$P[B] = p_{56} + p_{57} + p_{58} + p_{59} + p_{60} = \frac{5}{60} = \frac{1}{12}$$

$$P[B|A] = \frac{P[A \cap B]}{P[A]} = \frac{1/12}{1/6} = \frac{1}{2}$$

(b)

$$P[A] = p_1\left(\frac{1}{51} + \frac{1}{52} + \cdots + \frac{1}{60}\right)$$

$$P[B] = p_1\left(\frac{1}{56} + \frac{1}{57} + \cdots + \frac{1}{60}\right)$$

$$P[B|A] = \frac{P[A \cap B]}{P[A]} = \frac{\frac{1}{56} + \frac{1}{57} + \cdots + \frac{1}{60}}{\frac{1}{51} + \frac{1}{52} + \cdots + \frac{1}{60}} = 0.477$$

(c)

$$P[A] = \frac{1}{2}\left(\left(\frac{1}{2}\right)^{50} + \left(\frac{1}{2}\right)^{56} + \cdots + \left(\frac{1}{2}\right)^{59}\right)$$

$$P[B] = \frac{1}{2}\left(\left(\frac{1}{2}\right)^{55} + \cdots + \left(\frac{1}{2}\right)^{59}\right)$$

$$P[B|A] = \frac{P[A \cap B]}{P[A]} = \frac{\left(\frac{1}{2}\right)^{56} + \cdots \left(\frac{1}{2}\right)^{60}}{\left(\frac{1}{2}\right)^{51} + \cdots + \left(\frac{1}{2}\right)^{60}} = 0.030$$

2.69 Proceeding as in Problem 2.84

$$P[A|B] = \frac{P[A \cap B]}{P[B]} = \frac{P[(-0.5,0)]}{P[(-0.5,1)]} = \frac{1/6}{1/2} = \frac{1}{3}$$

$$P[B|C] = \frac{P[B \cap C]}{P[C]} = \frac{P[(0.75,1)]}{P[(0.75,2)]} = \frac{1/12}{5/12} = \frac{1}{5}$$

$$P[A|C^c] = \frac{P[A \cap C^c]}{P[C^c]} = \frac{P[(-1,0)]}{P[[-1,0.75]]} = \frac{1/3}{7/12} = \frac{4}{7}$$

$$P[B|C^c] = \frac{P[B \cap C^c]}{P[C^c]} = \frac{P[(-0.5,0.75)]}{P[[-1,0.75]]} = \frac{5/12}{7/12} = \frac{5}{7}.$$

2.70

$$P[X>2t / X>t] = \frac{P[\{X>2t\} \cap \{X>t\}]}{P[X>t]} = \frac{P[X>2t]}{P[X>t]}$$

$$= \frac{1/2t}{1/t} = \frac{1}{2} \qquad t>1$$

This conditional probability does not depend on t.
The corresponding probability law is said to be scale-invariant.

2.71 P[2 or more students have some birthday]

$$= 1 - P[\text{all students have different birthdays}]$$

P[all students have different birthdays]

$$= \frac{365}{365}\frac{364}{365}\frac{363}{365}\cdots\frac{346}{365} = 0.588$$

P[2 or more have same birthday] $= 0.412$

P[2 or more have same birthday in class of 23] $= 0.507$

2.72 # of fingerprints $= 2^L$ $\quad L=64$ or $L=128$

Pick hash at random until we find a repeat.

Same as birthday problem (problem 2.7~~1~~)

$$P[\text{all hashes different given } N \text{ tries}] = \frac{2^L}{2^L}\,\frac{2^L-1}{2^L}\cdots\frac{2^L-N+1}{2^L}$$

Find N so that

$$\frac{1}{2} = 1 - \prod_{j=0}^{N-1} \frac{2^L-j}{2^L} = 1 - p(N)$$

$$\ln p(N) = \sum_{j=0}^{N-1} \ln\left(1-\frac{j}{2^L}\right) \approx \sum_{j=0}^{N-1} -\frac{j}{2^L} = -\frac{1}{2^L}\sum_{j=0}^{N-1} j$$

$$\approx -\frac{1}{2^L}\,\frac{N(N-1)}{2}$$

$$p(N) = e^{-\frac{N(N-1)}{2}\frac{1}{2^L}} \approx e^{-\frac{N^2}{2}\frac{1}{2^L}} = \frac{1}{2}$$

$$N \approx \sqrt{(2\ln 2)\,2^L} = 1.17\, 2^{L/2}$$

for $L=64$ $\quad 2^{32}$ attempts required

$L=128$ $\quad 2^{64}$ "

2.73 a) The results follow directly from the definition of conditional probability. $P[A|B] = \frac{P[A \cap B]}{P[B]}$

If $A \cap B = \emptyset$ then $P[A \cap B] = 0$ by Corollary 3 and thus $P[A|B] = 0$.

If $A \subset B$ then $A \cap B = A$ and $P[A|B] = \frac{P[A]}{P[B]}$.

If $A \supset B \Rightarrow A \cap B = B$ and $P[A|B] = \frac{P[B]}{P[B]} = 1.$

b) If $P[A|B] = \frac{P[A \cap B]}{P[B]} > P[A]$ then multiplying both sides by $P[B]$ we have: $P[A \cap B] > P[A]P[B]$

We then also have that $P[B|A] = \frac{P[A \cap B]}{P[A]} > \frac{P[A]P[B]}{P[A]} = P[B].$

We conclude that if $P[A|B] > P[A]$ then B and A tend to occur jointly.

2.74

$$P[A|B] = \frac{P[A \cap B]}{P[B]} \quad \text{for } P[B] > 0.$$

(i) $P[A \cap B] \geq 0 \Rightarrow P[A|B] \geq 0.$ ✓

$A \cap B \subset B \Rightarrow P[A \cap B] \leq P[B] \Rightarrow P[A|B] \leq 1.$ ✓

(ii) $$P[S|B] = \frac{P[B \cap S]}{P[B]} = \frac{P[B]}{P[B]} = 1$$ ✓

(iii) If $A \cap C = \phi$ then

$$P[A \cup C|B] = \frac{P[(A \cup C) \cap B]}{P[B]} = \frac{P[(A \cap B) \cup (C \cap B)]}{P[B]}$$

$$= \frac{P[A \cap B] + P[C \cap B]}{P[B]} \quad \text{since } (A \cap B) \cap (C \cap B) = A \cap B \cap C = \phi$$

$$= P[A|B] + P[C|B]$$ ✓

2.75

$$P[A \cap B \cap C] = P[A|B \cap C]P[B \cap C] = P[A|B \cap C]P[B|C]P[C]$$

2.76

a) We use conditional probability to solve this problem. Let A_i = {nondefective item found in ith test}. A lot is accepted if the items in tests 1 and 2 are nondefective, that is, if $A_1 \cap A_2$ occurs. Therefore

$$P[\text{lot accepted}] = P[A_2 \cap A_1] = P[A_2|A_1]P[A_1] \quad \text{by Eqn. 2.2}\cancel{5}8$$

This equation simply states that we must have A_1 occur, and then A_2 occur given that A_1 already occurred. If the lot of 100 items contains ~~5~~ k defective items then

$$P[A_1] = \frac{\cancel{95}\ 100-k}{100} \text{ and}$$

$$P[A_2|A_1] = \frac{\cancel{94}\ 99-k}{99} \text{ since } \cancel{94}\ 99-k \text{ of the many 99 items are now defective.}$$

Thus

$$P[\text{lot accepted}] = \cancel{\frac{94}{99} \cdot \frac{95}{100}} \cdot \frac{99-k}{99} \cdot \frac{100-k}{100}$$

(b) $P[$1 or more items in m tested are defective$] > 99\%$

$\Leftrightarrow P[$no items in m are defective$] < 1\%$

$$P[A_m A_{m-1} \ldots A_1] = \frac{50}{100} \cdot \frac{49}{99} \cdots \frac{50-m+1}{100-m+1} = 0.01$$

for $m=6$ we have

$$P[A_6 A_5 A_4 A_3 A_2 A_1] = \frac{50}{100} \cdots \frac{45}{95} = 0.0133$$

2.77 Let X denote the input and Y the output

(a) $$P[Y=0] = P[Y=0|X=0]P[X=0] + P[Y=0|X=1]P[X=1]$$
$$= (1-\varepsilon_1)q + \varepsilon_2 p.$$

Similarly
$$P[Y=1] = (1-\varepsilon_2)p + \varepsilon_1 q$$

(b) $$P[X=0|Y=1] = \frac{P[Y=1|X=0]P[X=0]}{P[Y=1]} = \frac{\varepsilon_1 q}{(1-\varepsilon_2)p + \varepsilon_1 q}$$

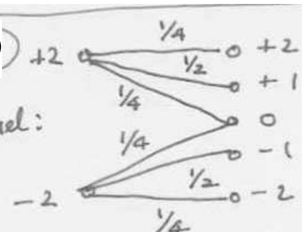
$$P[X=1|Y=1] = \frac{(1-\varepsilon_2)p}{(1-\varepsilon_2)p + \varepsilon_1 q}$$

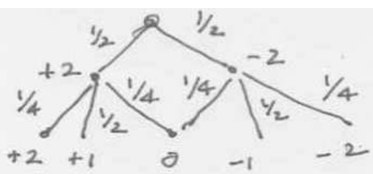
$$P[X=1|Y=1] > P[X=0|Y=1]$$

$$\Leftrightarrow \quad (1-\varepsilon_2)p > \varepsilon_1 q = \varepsilon_1(1-p)$$

$$\Leftrightarrow \quad p > \frac{\varepsilon_1}{1-\varepsilon_2+\varepsilon_1}$$

2.78 channel:





(b) $$P[X=+2, Y=+2] = P[Y=2|X=2]P[X=2]$$
$$= \tfrac{1}{4}\tfrac{1}{2} = \tfrac{1}{8}$$

$$P[X=2, Y=1] = \tfrac{1}{2}\tfrac{1}{2} = \tfrac{1}{4}$$
$$P[X=2, Y=0] = \tfrac{1}{2}\tfrac{1}{4} = \tfrac{1}{8}$$
$$P[X=-2, Y=0] = \tfrac{1}{2}\tfrac{1}{4} = \tfrac{1}{8}$$
$$P[X=-2, Y=1] = \tfrac{1}{2}\tfrac{1}{2} = \tfrac{1}{4}$$
$$P[X=-2, Y=-2] = \tfrac{1}{2}\tfrac{1}{4} = \tfrac{1}{8}$$

(c) $$P[Y=+2] = \tfrac{1}{2}\,\tfrac{1}{4} = \tfrac{1}{8} = P[Y=-2]$$
$$P[Y=+1] = \tfrac{1}{2}\,\tfrac{1}{2} = \tfrac{1}{4} = P[Y=-1]$$
$$P[Y=0] = 2(\tfrac{1}{2}\cdot\tfrac{1}{4}) = \tfrac{2}{8} = P[Y=0]$$

(d) $$P[X=2|Y=k] = \frac{P[Y=k|X=2]P[X=2]}{P[Y=k]}$$
$$= \begin{cases} \frac{1}{8}/\frac{1}{8} = 1 & k=2 \\ \frac{1}{4}/\frac{1}{4} = 1 & k=1 \\ \frac{1}{8}/\frac{1}{4} = \frac{1}{2} & k=0 \\ 0 & \text{other } k \end{cases}$$

2.79

(a) $P[N=k] = P[N=k \mid \text{coin } 1]\, P[\text{coin } 1] + P[N=k \mid \text{coin } 2]\, P[\text{coin } 2]$

$$= \binom{3}{k} p_1^k (1-p_1)^{3-k} \frac{1}{2} + \binom{3}{k} p_2^k (1-p_2)^{3-k} \frac{1}{2} \qquad k = 0,1,2,3$$

(b) $$P[\text{coin } 1 \mid N=k] = \frac{P[N=k \mid \text{coin } 1]\, P[\text{coin } 1]}{P[N=k]}$$

$$= \frac{\binom{3}{k} p_1^k (1-p_1)^{3-k} \frac{1}{2}}{\binom{3}{k} p_1^k (1-p_1)^{3-k} \frac{1}{2} + \binom{3}{k} p_2^k (1-p_2)^{3-k} \frac{1}{2}}$$

(c) Coin 1 is more probable if

$$\binom{3}{k} p_1^k (1-p_1)^{3-k} \frac{1}{2} > \binom{3}{k} p_2^k (1-p_2)^{3-k} \frac{1}{2}$$

$$1 > \left(\frac{p_2}{p_1}\right)^k \left(\frac{1-p_2}{1-p_1}\right)^{3-k} = 2^k \left(\frac{1}{2}\right)^{3-k} = \left(\frac{1}{8}\right) 4^k$$

$$0 > \ln\frac{1}{8} + k \ln 4$$

$$1.5 = \frac{\ln 8}{\ln 4} > k$$

Coin 1 more probable if $N = 0$ or 1
Coin 2 more probable otherwise.

(d) In general coin 1 is more probable if

$$\binom{n}{k} p_1^k (1-p_1)^{n-k} \frac{1}{2} > \binom{n}{k} p_2^k (1-p_2)^{n-k} \frac{1}{2}$$

$$1 > \left(\frac{p_2}{p_1}\right)^k \left(\frac{1-p_2}{1-p_1}\right)^{n-k} = \left(\frac{p_2(1-p_1)}{p_1(1-p_2)}\right)^k \left(\frac{1-p_2}{1-p_1}\right)^n$$

$$T = \frac{n \ln\left(\frac{1-p_1}{1-p_2}\right)}{\ln\left(\frac{p_2(1-p_1)}{p_1(1-p_2)}\right)} > k$$

(e) If $p_2 = 1$ then $P[N=k \mid \text{coin } 2] = \begin{cases} 1 & \text{if } k=m \\ 0 & \text{otherwise} \end{cases}$

We cannot determine coin with certainty only if all tosses are heads.

$P[\text{coin } 1 \mid m \text{ heads}] = (1-p_1)^m / [\, 1 + (1-p_1)^m \,]$

2.80

$$P[\text{chip defective}] = P[\text{def.}|A]P[A] + P[\text{def.}|B]P[B] + P[\text{def.}|C]P[C]$$
$$= 5(10^{-3})p_A + 1(10^{-3})p_B + 10(10^{-3})p_C = 6.6\times 10^{-3}$$

$$P[A|\text{chip defective}] = \frac{P[\text{def.}|A]P[A]}{P[\text{def.}]} = \frac{5\,10^{-3}\,\cancel{p_A}\,0.5}{\cancel{10^{-3}p_A + 5(10^{-3})p_B + 10(10^{-3})p_C}\ 6.6\times10^{-3}} = 0.3788$$
$$= \cancel{\frac{p_A}{p_A + 5p_B + 10p_C}}$$

Similarly

$$P[C|\text{chip defective}] = \frac{10(10^{-3})(0.4)\ \cancel{10p_C}}{\cancel{p_A + 5p_B + 10p_C}\ 6.6\times10^{-3}} = 0.6061$$

2.81

Let X denote the input and Y the output.

a) $P[Y=0] = P[Y=0|X=0]P[X=0] + P[Y=0|X=1]P[X=0]$
$+ P[Y=0|X=2]P[X=2]$
$= \cancel{(1-\varepsilon)\tfrac{1}{2} + 0\cdot\tfrac{1}{4} + \varepsilon\tfrac{1}{4}} = (1-\varepsilon)\tfrac{1}{3} + \varepsilon\tfrac{1}{3} = \tfrac{1}{3}$
$\cancel{= \tfrac{1}{2} + \tfrac{\varepsilon}{4}}$

Similarly

$$P[Y=1] = \cancel{\varepsilon\frac{1}{2} + (1-\varepsilon)\frac{1}{4} + 0\frac{1}{4} = \frac{1}{4} + \frac{\varepsilon}{4}}\ \frac{1}{3}$$
$$P[Y=2] = \cancel{0\cdot\frac{1}{2} + \varepsilon\cdot\frac{1}{4} + (1-\varepsilon)\frac{1}{4} = \frac{1}{4}}\ \frac{1}{3}$$

b) Using Bayes' Rule

$$P[X=0|Y=1] = \frac{P[Y=1|X=0]P[X=0]}{P[Y=1]} = \frac{\frac{1}{3}\varepsilon}{\cancel{\frac{1}{4}+\frac{\varepsilon}{4}}\ \frac{1}{3}} = \cancel{\frac{2\varepsilon}{1+\varepsilon}}\ \varepsilon$$
$$P[X=1|Y=1] = \frac{P[Y=1|X=1]P[X=1]}{P[Y=1]} = \frac{(1-\varepsilon)\frac{1}{3}}{\cancel{\frac{1}{4}+\frac{\varepsilon}{4}}\ \frac{1}{3}} = \cancel{\frac{1-\varepsilon}{1+\varepsilon}}\ 1-\varepsilon$$
$$P[X=2|Y=1] = 0$$

2.5 Independence of Events

2.82

$P[A \cap B] = P[\{1\}] = \frac{1}{4} = P[A]P[B] = \frac{1}{2}\frac{1}{2}$ ✓

$P[A \cap C] = P[\{1\}] = \frac{1}{4} = P[A]P[C] = \frac{1}{2}\frac{1}{2}$ ✓

$P[B \cap C] = P[\{1\}] = \frac{1}{4} = P[B]P[C] = \frac{1}{2}\frac{1}{2}$ ✓

$P[A \cap B \cap C] = P[\{1\}] = \frac{1}{4} \neq P[A]P[B]P[C] = \frac{1}{2}\frac{1}{2}\frac{1}{2} = \frac{1}{8}$

$\Rightarrow$ Not independent

2.83

$P[A \cap B] = P[\frac{1}{4} < U < \frac{1}{2}] = \frac{1}{4} = P[A]P[B] = \frac{1}{2}\frac{1}{2}$ ✓ A & B indep

$P[A \cap C] = 0 \neq P[A]P[C] = \frac{1}{2}\frac{1}{2}$ $\Rightarrow$ A & C Not Indep.

$P[B \cap C] = P[\frac{1}{2} < U < \frac{3}{4}] = \frac{1}{4} = P[B]P[C] = \frac{1}{2} \cdot \frac{1}{2}$ ✓ B & C indep.

2.84

Let $A = \{$Alice makes shot$\}$ $M = \{$Mary makes shot$\}$

We assume that A and M are independent

$P[A] = p_a$

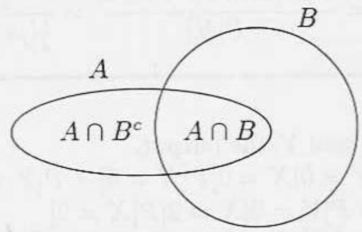
$P[\text{one makes a shot}] = P[AM^c \cup A^cM] = P[AM^c] + P[A^cM]$ since $AM^c \cap A^cM = \phi$

$= p_a(1-p_m) + (1-p_a)p_m$ by independence

$P[AM] = p_a p_m$

$P[A^cM^c] = (1-p_a)(1-p_m)$.

2.85 The event A is the union of the mutually exclusive events $A \cap B$ and $A \cap B^c$, thus



$$\begin{aligned}
P[A] &= P[A \cap B] + P[A \cap B^c] && \text{by Corollary 1} \\
\Longrightarrow P[A \cap B^c] &= P[A] - P[A \cap B] \\
&= P[A] - P[A]P[B] && \text{since } A \text{ and } B \text{ are independent} \\
&= P[A](1 - P[B]) \\
&= P[A]P[B^c] \Longrightarrow && A \text{ and } B^c \text{ are independent}
\end{aligned}$$

Similarly

$$P[B] = P[A \cap B] + P[A^c \cap B] = P[A]P[A] + P[A^c \cap B]$$

$$\begin{aligned}
\Longrightarrow P[A^c \cap B] = P[B](1 - P[A]) &= P[B]P[A^c] \\
&\Rightarrow A \text{ and } B \text{ are independent}
\end{aligned}$$

Finally

$$P[A^c] = P[A^c \cap B] + P[A^c \cap B^c] = P[A^c]P[B] + P[A^c \cap B^c]$$

$$\begin{aligned}
\Longrightarrow P[A^c \cap B^c] = P[A^c](1 - P[B]) &= P[A^c]P[B^c] \\
&\Rightarrow A^c \text{ and } B^c \text{ are independent}
\end{aligned}$$

2.86

$$P[A|B] = P[A|B^c] \Longrightarrow \frac{P[A \cap B]}{P[B]} = \frac{P[A \cap B^c]}{P[B^c]}$$

$$\begin{aligned}
\Longrightarrow P[A \cap B]P[B^c] &= P[A \cap B^c]P[B] \\
&= (P[A] - P[A \cap B])P[B] \quad \text{see Prob. 2.58 solution}
\end{aligned}$$

$$\Longrightarrow P[A \cap B]\underbrace{(P[B^c] + P[B])}_{1} = P[A]P[B]$$

$$\Longrightarrow P[A \cap B] = P[A]P[B]$$

2.87

(a) $P[A \cup B] = P[A] + P[B] - P[A \cap B] = p_A + p_B - p_A p_B$

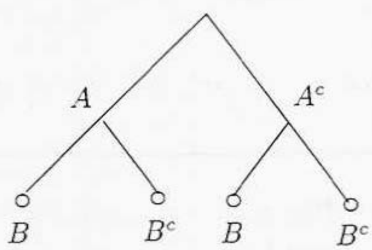
(b) $P[A \cup B] = P[A] + P[B] = p_A + p_B$

(c) $P[A \cup B \cup C] = P[A] + P[B] + P[C] - P[A \cap B] - P[A \cap C] - P[B \cap C] + P[A \cap B \cap C]$
$= p_A + p_B + p_C - p_A p_B - p_A p_C - p_B p_C + p_A p_B p_C$

(d) $P[A \cup B \cup C] = p_A + p_B + p_C$

2.88

We use a tree diagram to show the sequence of events. First we choose an urn, so A or A^c occurs. We then select a ball, so B or B^c occurs:



Now A and B are independent events if

$$P[B|A] = P[B]$$

But

$$P[B|A] = P[B] = P[B|A]P[A] + P[B|A^c]P[A^c]$$

$$\begin{aligned} \Longrightarrow P[B|A](1 - P[A]) &= P[B|A^c]P[A^c] \\ \Longrightarrow P[B|A] = P[B|A^c] & \quad \text{prob. of } B \text{ is the same given } A \text{ or } A^c\text{, that is,} \\ & \quad \text{the probability of } B \text{ is the same for both urns.} \end{aligned}$$

2.89

a) $P[A]P[B^c]P[C^c] + P[A^c]P[B]P[C^c] + P[A^c]P[B^c]P[C]$

b) $P[A]P[B]P[C^c] + P[A^c]P[B]P[C] + P[A]P[B^c]P[C]$

c) $1 - P[A^c]P[B^c]P[C^c]$

d) $P[A]P[B]P[C^c] + P[A]P[B^c]P[C] + P[A^c]P[B]P[C] + P[A]P[B]P[C]$

e) $P[A^c]P[B^c]P[C^c]$

2.90

Series $P[D_a] = P[A_1 \cap A_2 \cap A_3] = P[A_1]P[A_2]P[A_3]$

Parallel
$$P[D_a] = P[A_1 \cup A_2 \cup A_3]$$
$$= P[A_1] + P[A_2] + P[A_3] - P[A_1 \cap A_2] - P[A_1 \cap A_3] - P[A_2 \cap A_3] + P[A_1 \cap A_2 \cap A_3]$$
$$= p_{A_1} + p_{A_2} + p_{A_3} - p_{A_1}p_{A_2} - p_{A_1}p_{A_3} - p_{A_2}p_{A_3} + p_{A_1}p_{A_2}p_{A_3}$$

2-of-3
$$P[D_a] = P[A_1 \cap A_2 \cap A_3] + P[A_1^c \cap A_2 \cap A_3] + P[A_1 \cap A_2^c \cap A_3] + P[A_1 \cap A_2 \cap A_3^c]$$
$$= p_{A_1}p_{A_2}p_{A_3} + (1-p_{A_1})p_{A_2}p_{A_3} + p_{A_1}(1-p_{A_2})p_{A_3} + p_{A_1}p_{A_2}(1-p_{A_3})$$

2.91

$$P[\text{System Up}] = P[(A_{11} \cap A_{12}) \cup (A_{21} \cap A_{22}) \cup (A_{31} \cap A_{32})]$$
$$= P[A_{11} \cap A_{12}] + P[A_{21} \cap A_{22}] + P[A_{31} \cap A_{32}] - P[A_{11} \cap A_{12} \cap A_{21} \cap A_{22}]$$
$$- P[A_{11} \cap A_{12} \cap A_{31} \cap A_{32}] - P[A_{21} \cap A_{22} \cap A_{31} \cap A_{32}]$$
$$+ P[A_{11} \cap A_{12} \cap A_{21} \cap A_{22} \cap A_{31} \cap A_{32}]$$
$$= p_{A_{11}}p_{A_{12}} + p_{A_{21}}p_{A_{22}} + p_{A_{31}}p_{A_{32}} - p_{A_{11}}p_{A_{12}}p_{A_{21}}p_{A_{22}}$$
$$- p_{A_{11}}p_{A_{12}}p_{A_{31}}p_{A_{32}} - p_{A_{21}}p_{A_{22}}p_{A_{31}}p_{A_{32}}$$
$$+ p_{A_{11}}p_{A_{12}}p_{A_{21}}p_{A_{22}}p_{A_{31}}p_{A_{32}}$$

2.92

Events A and B are independent iff

$$P[A \cap B] = P[A]P[B]$$

In terms of relative frequencies we expect

$$\underbrace{f_{A\cap B} n}_{\substack{\text{rel. freq. if} \\ \text{joint occurrence} \\ \text{of } A \text{ and } B}} = \underbrace{f_A(n) f_B(n)}_{\text{rel. freq.'s of } A \text{ and } B}$$

2.93

Let the jth bit in the hex character be B_j

To test independence we need:

All pairs ~~of~~ should satisfy $f_{B_j \cap B_k} \approx f_{B_j} f_{B_k}$

All triplets should satisfy $f_{B_j \cap B_k \cap B_l} \approx f_{B_j} f_{B_k} f_{B_l}$

Quadruplets should satisfy $f_{B_1 \cap B_2 \cap B_3 \cap B_4} \approx f_{B_1} f_{B_2} f_{B_3} f_{B_4}$

__Note__ Relative frequencies for different B_j need not be the same.

2.94

$$\begin{aligned} P[\text{System Up}] &= P[\text{at least one controller is working}] \times \\ &\quad\; P[\text{at least two peripherals are working}] \end{aligned}$$

$$\begin{aligned} P[\text{at least one controller working}] &= 1 - P[\text{both not working}] \\ &= 1 - p^2 \end{aligned}$$

$$\therefore P[\text{System Up}] = (1-p^2)\{(1-a)^3 + 3(1-a)^2 a\}$$

2.95)

$$P[A_0 \cap B_0] = (1-p)(1-\varepsilon)$$

$$P[B_0] = (1-p)(1-\varepsilon) + p\varepsilon$$

$$P[A_0] = (1-p)$$

$$P[A_0 \cap B_0] = P[B_0]P[A_0]$$

$$\Leftrightarrow \quad (1-p)(1-\varepsilon) = [(1-p)(1-\varepsilon) + p\varepsilon](1-p)$$

$$\Leftrightarrow \quad (1-\varepsilon) = (1-p)(1-\varepsilon) + p\varepsilon$$

$$\Leftrightarrow \quad (1-\varepsilon)p = p\varepsilon$$

$$\Leftrightarrow \quad \varepsilon = \frac{1}{2}.$$

Channel cannot transmit information if output is independent of the input.

2.96) Regardless of the value of ε, we always have

$$P[X=2 \mid Y=1] = 0 \neq P[X=2] = \frac{1}{3}$$

$\therefore$ the output cannot be independent of the input.

2.6 Sequential Experiments

2.97

a) $P[\text{0 or 1 errors}] = (1-p)^{100} + 100(1-p)^{99}p \qquad p = 10^{-2}$

$= 0.3660 + .3697$

$= 0.7357$

b) $p_R = P[\text{Retransmission Required}] = 1 - P[\text{0 or 1 error}] = 0.2642$

$P[M \text{ transmissions in total}] = (1-p_R)p_R^{m} \qquad M = 1, 2, \ldots$

$P[M \text{ or more transmissions required}] = \sum_{j=M}^{\infty}(1-p)p_R^{j} = p_R^{M}\sum_{j=0}^{\infty}(1-p)p_R^{j}$

$= p_R^{m}$

2.98

a) $P[N > 1] = 1 - P[N=0 \text{ or } N=1]$

$= 1 - (1-p)^n - n(1-p)^{n-1}p.$

b)

$P[N > 0] = 0.99 = 1 - (1-0.1)^n$

$0.01 = (0.9)^n$

$n = \dfrac{\ln 100}{\ln 1/0.9} = 44$

2.99 $p = \text{prob. of success} = \frac{95}{100} = \frac{19}{20}$

Pick n so that $P[k \geq \cancel{10}\ 8] \geq 0.9$

$$P[k \geq \cancel{10}\ 8] = \sum_{k=8}^{n} \cancel{k=10} \binom{n}{k} p^k (1-p)^{n-k}$$

for

$$0.95^9 + 9(0.95)^8(.05) = 0.92$$

$n = \cancel{11}\ 9 \qquad P[k \geq \cancel{10}\ 8] = \cancel{0.89811}\ 0.92$

~~$n = 12 \qquad P[k \geq 10] = 0.98093 \Rightarrow \text{pick } n = 12$~~

1 extra chip is enough.

2.100

(a) $P[\text{1 of } n \text{ terminals transmits}] = n(1-p)^{n-1}p$

(b) Take derivative with respect to p:

$$0 = -n(n-1)(1-p)^{n-2}p + n(1-p)^{n-1}$$

$$\Rightarrow (n-1)p = (1-p) \Rightarrow np = 1-p+p \Rightarrow p = \frac{1}{n}$$

(c) $P_{\text{success}} = n\left(1-\frac{1}{n}\right)^{n-1}\frac{1}{n} = \left(1-\frac{1}{n}\right)^{n-1} \to e^{-1} = \frac{1}{e}$ as $n \to \infty$.

$= 0.3678$

2.101 $P[N \geq 2] = 1 - P[N=0] - P[N=1]$

$$P\left[X \leq \frac{2}{\lambda}\right] = 1 - e^{-(\lambda \frac{2}{\lambda})^2} = 1 - e^{-4} = 0.9816$$

$$P[N \geq 2] = 1 - (1-e^{-4})^8 - 8(1-e^{-4})^7 e^{-4}$$

$$= 1 - 0.8625 - 0.1287 = 8.7 \times 10^{-3}$$

2.102

a) $P[k \text{ errors}] = \binom{n}{k} p^k (1-p)^{n-k}$

b) Type 1 errors occur with problem $p\alpha$ and do not occur with problem $1 - p\alpha$

$$P[k_1 \text{ type 1 errors}] = \binom{n}{k_1} (p\alpha)^{k_1} (1-p\alpha)^{n-k_1}$$

c) $P[k_2 \text{ type 2 errors}] = \binom{n}{k_2} (p(1-\alpha))^{k_2} (1-p(1-\alpha))^{n-k_2}$

d) Three outcomes: type 1 error, type 2 error, no error

$$P[k_1, k_2, n-k_1-k_2] = \frac{n!}{k_1! k_2! (n-k_1-k_2)!} (p\alpha)^{k_1} (p(1-\alpha))^{k_2} (1-p)^{n-k_1-k_2}$$

2.103 $P[EF] = 0.10$ $P[AF] = 0.30$ $P[BE] = 0.60$

a) $P[k \text{ are not } EF] = P[N-k \text{ are } EF] = \binom{N}{N-k} (0.10)^{N-k} (0.90)^{k}$

b) $P[k \text{ until } EF] = (1 - P(EF))^{k-1} P[EF] = 0.9^{k-1} (0.1)$

c) $P[k_{EF} = 4, k_{AF} = 6, k_{BE} = 10] = \frac{20!}{4!\,6!\,10!} (0.1)^4 (0.3)^6 (0.6)^{10}$

2.104

~~2.78~~ a)

$$\begin{aligned} P[k=0] &= p \\ P[k=1] &= (1-p)p \\ P[k=2] &= (1-p)^2 p \\ P[k=3] &= 1 - P[k=0] - P[k=1] - P[k=2] = (1-p)^3 \end{aligned}$$

b)

$$\begin{aligned} P[k] &= (1-p)^k p \quad 0 \le k < m \\ P[m] &= 1 - \sum_{k=0}^{m-1} P[k] \\ &= 1 - \sum_{k=0}^{m-1} (1-p)^k p \\ &= 1 - p\frac{1-(1-p)^m}{1-(1-p)} = (1-p)^m \end{aligned}$$

2.105

a) $P[k \text{ halfhours}] = \left(\frac{1}{2}\right)^k \qquad k = 1, 2, \ldots$

$P[k \text{ dollars paid}] = \left(\frac{1}{2}\right)^k$

b) $P[k \text{ dollars}] = \left(\frac{1}{2}\right)^k \quad k = 1, 2, 3, 4, 5$

$P[6] = 1 - \left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^3 - \left(\frac{1}{2}\right)^4 - \left(\frac{1}{2}\right)^5 = \frac{1}{32}$

2.106

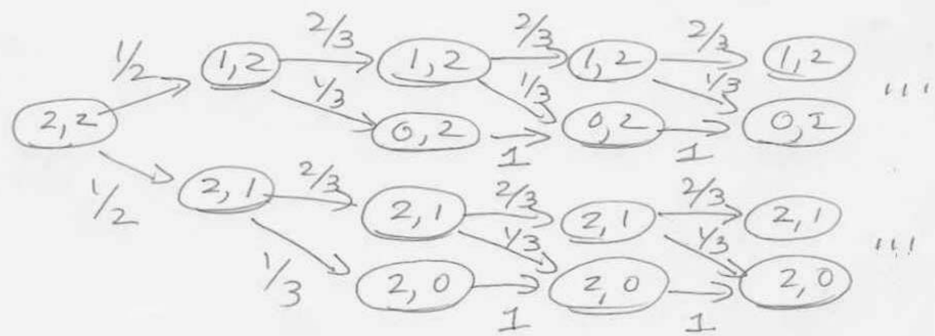
2.80 $P[k$ tosses required until heads comes up ~~twice~~ three times$] = P[$heads in kth toss $|$ 2 heads in $k-1$ tosses$]P[$2 heads in $k-1$ tosses$] = P[A|B]P[B]$.

Now $P[A|B] = P[$2 heads in first $k-1$ tosses$] = \binom{k-1}{2} p^2(1-p)^{k-3}$

Thus $P[A|B]P[B] = P[A|B]p = \binom{k-1}{2} p^3(1-p)^{k-3}$ $\quad k=3,4,\ldots$

2.107 The first draw is key since that ball is not put back.

Let (j,k) be a state where j = # black balls in urn k = # white balls in urn



(b) $P[bb] = \frac{1}{2}\frac{1}{3} = \frac{1}{6}$ $\quad P[bw] = \frac{1}{2}\frac{2}{3} = \frac{1}{3}$

$P[ww] = \frac{1}{2}\frac{1}{3} = \frac{1}{6}$ $\quad P[wb] = \frac{1}{2}\frac{2}{3} = \frac{1}{3}$

$P[bbw] = \frac{1}{2}\frac{1}{3}1 = \frac{1}{6}$ $\quad P[bww] = \frac{1}{2}\frac{2}{3}\frac{2}{3} = \frac{2}{9}$ $\quad P[bwb] = \frac{1}{2}\frac{2}{3}\frac{1}{3} = \frac{1}{9}$

$P[wwb] = \frac{1}{2}\frac{1}{3}1 = \frac{1}{6}$ $\quad P[wbb] = \frac{1}{2}\frac{2}{3}\frac{2}{3} = \frac{2}{9}$ $\quad P[wbw] = \frac{1}{2}\frac{2}{3}\frac{1}{3} = \frac{1}{9}$

(c) $P[(0,2)$ after 3 draws$] = P[bbw] + P[bwb] = \frac{1}{6} + \frac{1}{9} = \frac{5}{18}$

Similarly $P[(2,0)$ after 3 draws$] = \frac{5}{16}$

2.107

(d) $P[(2,0)$ after $n] = P[$1st draw is white and at least one white in $n-1]$

$$= \frac{1}{2}\Big[1 - \underbrace{\left(\tfrac{2}{3}\right)^{n-1}}_{\text{all blacks}}\Big]$$

2.108

a) $p_0(1) = \frac{1}{2} \qquad p_1(1) = \frac{1}{2}$

b) $p_0(n+1) = \frac{2}{3}p_0(n) + \frac{1}{6}p_1(n)$

$p_1(n+1) = \frac{1}{3}p_0(n) + \frac{5}{6}p_1(n)$

In matrix notation, we have

$$[p_0(n+1), p_1(n+1)] = [p_0(n), p_1(n)] \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{5}{6} \end{bmatrix}$$

or

$$\underline{p}(n+1) = \underline{p}(n)\mathbb{P}$$

c) $\underline{p}(0) = \left[\frac{1}{2}, \frac{1}{2}\right]$

$\underline{p}(1) = \underline{p}(0)\mathbb{P}$

$\underline{p}(2) = \underline{p}(1)\mathbb{P} = \underline{p}(0)\mathbb{P}\mathbb{P} = \underline{p}(0)\mathbb{P}^2$

in general

$$\underline{p}(n) = \underline{p}(0)\mathbb{P}^n$$

To find $\mathbb{P}^n$ we note that if $\mathbb{P}$ has eigenvalues λ_1, λ_2 and eigenvectors $\underline{e}_1$, $\underline{e}_1$ then

$$\mathbb{P} = \mathbb{E}\underbrace{\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}}_{\Lambda}\mathbb{E}^{-1} \qquad \text{where } \mathbb{E} \text{ has } \underline{e}_1 \text{ and } \underline{e}_2 \text{ as columns}$$

and

$$\begin{aligned} \mathbb{P}^n &= (\mathbb{E}\Lambda E^{-1})(E\Lambda E^{-1})...(E\Lambda E^{-1}) \qquad n \text{ times} \\ &= E\Lambda(E^{-1}E)\Lambda...(E^{-1}E)\Lambda E^{-1} \\ &= E\Lambda^n E^{-1} \end{aligned}$$

Now $\mathbb{P} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{5}{6} \end{bmatrix}$ has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = \frac{1}{2}$ and eigenvector $\underline{e}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\underline{e}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

Thus

$$\begin{aligned} \mathbb{P}^n &= \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \left(\frac{1}{2}\right)^n \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} \frac{1}{3} + \frac{1}{3}\left(\frac{1}{2}\right)^{n-1} & \frac{2}{3} - \frac{1}{3}\left(\frac{1}{2}\right)^{n-1} \\ \frac{1}{3} - \frac{1}{3}\left(\frac{1}{2}\right)^n & \frac{2}{3} + \frac{1}{3}\left(\frac{1}{2}\right)^n \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} \underline{p}(n) &= \underline{p}(0)\mathbb{P}^n \\ &= \left[\frac{1}{2}, \frac{1}{2}\right]\begin{bmatrix} \frac{1}{3} + \frac{1}{3}\left(\frac{1}{2}\right)^{n-1} & \frac{2}{3} - \frac{1}{3}\left(\frac{1}{2}\right)^{n-1} \\ \frac{1}{3} - \frac{1}{3}\left(\frac{1}{2}\right)^n & \frac{2}{3} + \frac{1}{3}\left(\frac{1}{2}\right)^n \end{bmatrix} \\ &= \left[\frac{1}{3} + \frac{1}{3}\left(\frac{1}{2}\right)^{n+1}, \frac{2}{3} - \frac{1}{3}\left(\frac{1}{2}\right)^{n+1}\right] \end{aligned}$$

c) $\underline{p}(n) \to \left[\frac{1}{3}, \frac{2}{3}\right]$ as $n \to \infty$

2.7 *Synthesizing Randomness: Random Number Generators

2.109

$p_1 = \frac{1}{3}$ $p_2 = \frac{1}{5}$ $p_3 = \frac{1}{4}$ $p_4 = \frac{1}{7}$ $p_5 = 1 - \sum_{i=1}^{4} p_i = 1 - \frac{140+84+105+60}{420}$

$= \frac{31}{420}$

Use an urn with 420 identical balls labeled as follows

140 labeled 1
84 " 2
105 " 3
60 " 4
31 " 5

By finding least common multiple of denominators of rational probabilities we can define an equivalent urn experiment.

2.110

~~2.84~~ Three tosses of a fair coin result in eight equiprobable outcomes:

000	→ 0		100	→ 4	
001	→ 1		101	→ 5	
010	→ 2		101	→ No output	
011	→ 3		111	→ No output	

a)

$$P[\text{a number is output in step 1}] = 1 - P[\text{no output}] = 1 - \frac{2}{8} = \frac{3}{4}$$

b) Let $A_i = \{\text{output number } i\}$ $\quad i = 0, ..., 5$
and $B = \{\text{a number is output in step 1}\}$
then

$$P[A_i|B] = \frac{P[A_i \cap B]}{P[B]} = \frac{P[\text{binary string corresponds to } i]}{\frac{3}{4}} = \frac{\frac{1}{8}}{\frac{3}{4}} = \frac{1}{6}$$

c) Suppose we want to an urn experiment with N equiprobable outcomes. Let n be the smallest integer such that $2^n \geq N$. We can simulate the urn experiment by tossing a fair coin n times and outputting a number when the binary string for the numbers $0, ..., N-1$ occur and not outputting a number otherwise.

2.111

```
> X=rand(1000,1)
> Y=rand(1000,1)
> plot(X,Y,"+")
```

This program will produce a 2D scattergram in unit square

2.112

```
> X=rand(1100,1);
> Y=rand(1100,1);
> Xacc = zeros(500,1);
> Yacc = zeros(500,1);
> n=0
> j=0
> while n<500
      j=j+1
      if X(j)<Y(j)
          n=n+1
          Xacc(n)=X(j);
          Yacc(n)=Y(j);
      end
end
plot(Xacc, Yacc, "+")
```

This program will plot 500 points in the upper diagonal region of the unit square.

2.113

a) Assume that $X(j)$ assumes values from the sample space $S = \{x_1, x_2, \ldots, x_{,}\}$, and let $N_k(n)$ be the number of tries x_k occurs in n repetitions of the experiment, then

$$\begin{aligned} < X^2 >_n &= \frac{1}{n}\sum_{j=1}^{n} X^2(j) \\ &= \frac{1}{N}\sum_{k=1}^{K} x_k^2 N_k(n) \\ &\to \sum_{k=1}^{K} x_k^2 f_k(n) \end{aligned}$$

Thus we expect that $< x^2 >_n \to \sum_{k=1}^{K} x_k^2 p_k$.

b) The same derivation of Problem 1.~~7~~9, gives

$$< X^2 >_n = < X^2 >_{n-1} + \frac{X_n^2 - < X^2 >_{n-1}}{n}$$

2.114

a)

$$
\begin{aligned}
< V^2 >_n &= \frac{1}{n}\sum_{j=1}^{n}\{X(j) - < X >_n\}^2 \\
&= \frac{1}{n}\sum_{j=1}^{n}\{X^2(j) - 2X(j) < X >_n + < X >_n^2\} \\
&= \frac{1}{n}\sum_{j=1}^{n} X^2(j) - 2\left(\frac{1}{n}\sum_{j=1}^{n} X(j)\right) < X >_n + < X >_n^2 \\
&= < X^2 >_n - < X >_n^2
\end{aligned}
$$

9

b) From the next to last line in solution to Problem 1.7, we have:

$$
\begin{aligned}
< V^2 >_n &= \overbrace{\frac{n-1}{n} < X^2 >_{n-1} + \frac{X^2(n)}{n}}^{< X^2 >_n} - \overbrace{\left\{\frac{n-1}{n} < X >_{n-1} + \frac{X(n)}{n}\right\}^2}^{< X >_n^2} \\
&= \frac{n-1}{n}(< V^2 >_{n-1} + < X >_{n-1}^2) + \frac{X^2(n)}{n} \\
&\quad - \left(\frac{n-1}{n}\right)^2 < X >_{n-1}^2 - 2\frac{1}{n}\left(\frac{n-1}{n}\right) < X >_{n-1} X(n) \\
&\quad - \frac{X^2(n)}{n^2} \\
&= \frac{n-1}{n} < V^2 >_{n-1} + \frac{n-1}{n}\left(1 - \frac{n-1}{n}\right) < X >_{n-1}^2 \\
&\quad - 2\frac{1}{n}\left(\frac{n-1}{n}\right) < X >_{n-1} X(n) + \frac{1}{n}\left(1 - \frac{1}{n}\right) X^2(n) \\
&= \left(1 - \frac{1}{n}\right) < V^2 >_{n-1} + \frac{1}{n}\left(1 - \frac{1}{n}\right)\{< X >_{n-1}^2 \\
&\quad - 2 < X >_{n-1} X(n) + X^2(n)\} \\
&= \left(1 - \frac{1}{n}\right) < V^2 >_{n-1} + \frac{1}{n}\left(1 - \frac{1}{n}\right)\{X(n) - < X >_{n-1}\}^2
\end{aligned}
$$

2.115) $Y_n = \alpha U_n + \beta$ should map onto $[a, b]$

(a) when $U_n = 0$ we want $Y_n = \beta = a$
when $U_n = 1$ we want $Y_n = \alpha + \beta = b$ $\Big\} \Rightarrow \alpha = b - \beta = b - a$

$\alpha = b - a \quad \beta = a$

$\Rightarrow Y_n = (b-a) U_n + a$

(b)
```
> a=-5
> b=15
> Y = (b-a)*rand(1000,1) + a*ones(1000,1);
> mean(Y)            % computes sample mean
> cov(Y,Y)           % computes sample variance
```

In a test we obtained

mean$(Y) = 5.2670$ vs $\frac{b-a}{2} = 5$

cov$(Y,Y) = 34.065$ vs $\frac{(b-a)^2}{12} = 33.333$

2.116) (a) This problem uses the code in Example 2.47
(b) The ~~pdf~~ histogram will change with different values of p.

2.8 *Fine Points: Event Classes

2.117) $f(r)=R \quad f(g)=G \quad f(t)=G$

Homer's events are quite simple:

$\phi, \{R\}, \{G\}, \{R,G\}=S_H$

(a) $f^{-1}(\{R\}\cup\{G\}) = f^{-1}(\{R,G\}) = \{r,g,t\}$

and $f^{-1}(\{R\})\cup f^{-1}(\{G\}) = \{r\}\cup\{g,t\} = \{r,g,t\}$ same.

(b) $f^{-1}(\{R\}\cap\{R,G\}) = f^{-1}(\{R\}) = \{r\}$

$f^{-1}(\{R\})\cap f^{-1}(\{R,G\}) = \{r\}\cap\{r,t,g\} = \{r\}$ same

(c) $f^{-1}(\{G\}^c) = f^{-1}(\{R\}) = \{r\}$

$f^{-1}(\{G\})^c = \{g,t\}^c = \{r\}$ same

(d) (i) $f^{-1}(A\cup B) = f^{-1}(A)\cup f^{-1}(B)$

If $\xi\in f^{-1}(A\cup B)$ then $f(\xi)\in A\cup B \Rightarrow f(\xi)\in A$ and/or $f(\xi)\in B$

$\Rightarrow \xi\in f^{-1}(A)$ and/or $\xi\in f^{-1}(B)$

$\Rightarrow \xi\in f^{-1}(A)\cup f^{-1}(B) \qquad \therefore\ f^{-1}(A\cup B)\subset f^{-1}(A)\cup f^{-1}(B)$.

If $\xi\in f^{-1}(A)\cup f^{-1}(B) \Rightarrow \xi\in f^{-1}(A)$ and/or $\xi\in f^{-1}(B)$

$\Rightarrow f(\xi)\in A$ and/or $f(\xi)\in B$

$\Rightarrow f(\xi)\in A\cup B$

$\Rightarrow \xi\in f^{-1}(A\cup B) \Rightarrow f^{-1}(A\cup B)\supset f^{-1}(A)\cup f^{-1}(B)$

$\Rightarrow$ equality.

(d) $f^{-1}(A\cap B) = f^{-1}(A)\cap f^{-1}(B)$

If $\xi \in f^{-1}(A\cap B) \Rightarrow f(\xi)\in A\cap B \Rightarrow f(\xi)\in A$ and $f(\xi)\in B$

$\Rightarrow \xi\in f^{-1}(A)$ and $\xi \in f^{-1}(B) \Rightarrow \xi\in f^{-1}(A)\cap f^{-1}(B)$.

$\Rightarrow f^{-1}(A\cup B)\subset f^{-1}(A)\cap f^{-1}(B)$.

If $\xi\in f^{-1}(A)\cap f^{-1}(B) \Rightarrow \xi\in f^{-1}(A)$ and $\xi\in f^{-1}(B)$

$\Rightarrow f(\xi)\in A$ and $f(\xi)\in B \Rightarrow f(\xi)\in A\cap B$

$\Rightarrow \xi \in f^{-1}(A\cap B)$

$\Rightarrow f^{-1}(A\cup B)\supset f^{-1}(A)\cap f^{-1}(B)$ ✓

$f^{-1}(A^c) = f^{-1}(A)^c$

If $\xi\in f^{-1}(A^c) \Rightarrow f(\xi)\in A^c \Rightarrow f(\xi)\notin A \Rightarrow \xi\notin f^{-1}(A)$

$\Rightarrow \xi\in f^{-1}(A)^c$

$\Rightarrow f^{-1}(A^c)\subset f^{-1}(A)^c$

If $\xi\in f^{-1}(A)^c \Rightarrow \xi\notin f^{-1}(A) \Rightarrow f(\xi)\notin A$

$\Rightarrow f(\xi)\in A^c$

$\Rightarrow \xi\in f^{-1}(A^c)$

$\Rightarrow f^{-1}(A)^c \subset f^{-1}(A^c)$. ✓

2.118

(a) Show that $A_1 \ldots A_n$ forms a partition of S, that is,

$$A_i \cap A_j = \phi \quad i \neq j \quad \text{and} \quad \bigcup_{i=1}^{n} A_i = S$$

(i) For $i \neq j$ consider $A_i \cap A_j$

$$A_i \cap A_j = \{\xi : \xi \in A_i \text{ and } \xi \in A_j\} = \{\xi : f(\xi) = y_i \text{ and } f(\xi) = y_j\}$$

but if $y_i \neq y_j$ then we cannot have $f(\xi) = y_i$ and $f(\xi) = y_j$ since each ξ is mapped into a single value

$\therefore A_i \cap A_j = \phi$.

(ii) Now consider $\bigcup_{i=1}^{n} A_i$

Suppose $\xi \in S$, then $f(\xi) \in S' = \{y_1, \ldots, y_n\}$

$\Rightarrow \xi \in A_j$ for some j

$\Rightarrow \xi \in \bigcup_{i=1}^{n} A_i \Rightarrow \bigcup_{i=1}^{n} A_i \supset S$.

But S contains all subsets

$\Rightarrow \bigcup_{i=1}^{n} A_i \subset S$ ✓ .

(b) any $B \subset S'$ has form $B = \{y_{i_1}\} \cup \{y_{i_2}\} \ldots \cup \{y_{i_m}\}$

From problem 2.117 (d)

$$f^{-1}(B) = f^{-1}(\{y_{i_1}\} \cup \{y_{i_2}\} \cup \ldots \cup \{y_{i_m}\})$$

$$= f^{-1}(\{y_{i_1}\}) \cup f^{-1}(\{y_{i_2}\}) \cup \ldots \cup f^{-1}(\{y_{i_m}\})$$

$$= A_{i_1} \cup A_{i_2} \ldots \cup A_{i_m}.$$

$\therefore$ Inverse image of B is a union of sets from the partition.

2.119 $\mathcal{F} = \{\phi, A, A^c, S\}$

(i) $\phi \in \mathcal{F}$ ✓

(ii) if $A, B \in \mathcal{F}$ then $A \cup B \in \mathcal{F}$?

$$A \cup A^c = S \in \mathcal{F}$$

and any other union of events in $\mathcal{F}$ yields an event in $\mathcal{F}$ ✓

(iii) if $B \in \mathcal{F}$ then $B^c \in \mathcal{F}$

$$A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$$

$$A^c \in \mathcal{F} \Rightarrow A \in \mathcal{F}$$

and similarly for other events in $\mathcal{F}$ ✓

$\therefore \mathcal{F}$ is a field.

2.9 *Fine Points: Probabilities of Sequences of Events

2.120

(a) $\bigcup_n A_n = \bigcup_n \left[a+\frac{1}{n}, b-\frac{1}{n}\right] = (a, b)$

(b) $\bigcup_n B_n = \bigcup_n \left(-n, b-\frac{1}{n}\right] = (-\infty, b)$

(c) $\bigcup_n C_n = \bigcup_n \left[a+\frac{1}{n}, b\right) = (a, b)$

2.121

(a) $\bigcap_n \left(a-\frac{1}{n}, b+\frac{1}{n}\right) = [a, b]$

(b) $\bigcap_n \left[a, b+\frac{1}{n}\right) = [a, b]$

(c) $\bigcap_n \left[a-\frac{1}{n}, b\right] = [a, b]$

2.122

(a) Starting with open sets (a, b)

$(-\infty, b)^c = [b, \infty) \in \mathcal{B}$

then $(-\infty, b) \cap [a, \infty) = [a, b)$ for $a<b$

$\therefore$ We can use semi-infinite intervals as in the chapter to show that all elements in the Borel field can be generated.

(b) Closed intervals of the form $[a, b]$ can also be used to generate the Borel field.

2.123

(a) $\lim_{n\to\infty} P[A_n] = P[\lim_{n\to\infty} A_n] = P[a < x < b]$

(b) $\lim_{n\to\infty} P[B_n] = P[\lim_{n\to\infty} B_n] = P[-\infty < x < b]$

(c) $\lim_{n\to\infty} P[C_n] = P[\lim_{n\to\infty} C_n] = P[a < x < b]$

2.124

(a) $\lim_{n\to\infty} P[A_n] = P[\lim_{n\to\infty} A_n] = P[a \le x \le b]$

(b) $\lim_{n\to\infty} P[B_n] = P[\lim_{n\to\infty} B_n] = P[a \le x \le b]$

(c) $\lim_{n\to\infty} P[C_n] = P[\lim_{n\to\infty} C_n] = P[a \le x \le b]$.

Problems Requiring Cumulative Knowledge

2.125

(a)

$$P_H[k \text{ defective of 10 tested}] = \begin{cases} \dfrac{\binom{5}{k}\binom{15}{10-k}}{\binom{20}{10}} & k=0,1,2,3,4,5 \\ 0 & k>5 \end{cases}$$

$$P_B[k \text{ defective}] = \binom{10}{k}(.25)^k(.75)^{10-k} \quad k=0,1,2,\ldots,10$$

See Table of values:

Probabilities for hypergeometric and binomial are very different.

k	Hypergeometric	Binomial
0	0.01625	0.18771
1	0.13545	0.28157
2	0.34830	0.25028
3	0.34830	0.14600
4	0.13545	0.058399
5	0.01625	0.016222
6	0	0.003089
7	0	0.00003

(b)

$$P_H[k \text{ defective}] = \frac{\binom{250}{k}\binom{750}{10-k}}{\binom{1000}{10}} \quad k=0,1,\ldots,10$$

$$P_B[k \text{ defective}] = \binom{10}{k}(.25)^k(.75)^{10-k} \quad k=0,\ldots,10$$

See Table:

k	Hypergeometric
0	0.18714
1	0.28260
2	0.25154
3	0.14614
4	0.057907
5	0.015848
6	0.0028581
7	0.00036

These are very close to the binomial probabilities

Because of the large population size sampling without replacement is almost the same as sampling with replacement.

2.126

$P[\text{both in error}] = q_1 q_2$

(a) $P[k \text{ transmsn needed}] = (q_1 q_2)^{k-1}(1-q_1 q_2) \quad k=1,2,\ldots$

$P[\text{more than } k \text{ transmissions required}]$

$$= \sum_{j=k+1}^{\infty} (q_1 q_2)^{j-1}(1-q_1 q_2) = (q_1 q_2)^k \sum_{j=0}^{\infty} (1-q_1 q_2)(q_1 q_2)^j$$

$$= (q_1 q_2)^k$$

(b) $P[\text{link 2 errorfree} \mid \text{one or more errorfree}]$

$$= \frac{P[\text{one or more errorfree, link 2 errorfree}]}{1-q_1 q_2}$$

$$= \frac{q_1(1-q_2) + (1-q_1)(1-q_2)}{1-q_1 q_2} = \frac{1-q_2}{1-q_1 q_2}$$

2.127

(a) $P_b = P[N_c \geq 7] = P[N=7] + P[N=8] = 7(1-p)^7 p + (1-p)^8$

(b) $P[N_b \geq 1] = 1 - P[N_b = 0] = 1 - (1-P_b)^n = 0.99$

$$0.01 = (1-P_b)^n \Rightarrow \ln 100 = n \ln \frac{1}{1-P_b}$$

$$n = \frac{\ln 100}{\ln \frac{1}{1-P_b}} = \frac{\ln 100}{-\ln(1-7(1-p)^7 p - (1-p)^8)}$$

2.128

(a) $P[\text{ace}] = \frac{4}{52} = \frac{1}{13}$

(b) Let A = ace in 1st draw
B = ace in 2nd draw

$P[A] = \frac{1}{13} \quad P[A^c] = \frac{12}{13}$

If we look at 1st draw:

$P[B|A] = \frac{3}{51} \quad P[B|A^c] = \frac{4}{51}$

Suppose we don't look

$$P[B] = P[B|A]P[A] + P[B|A^c]P[A^c]$$
$$= \frac{3}{51}\frac{1}{13} + \frac{4}{51}\frac{12}{13} = \frac{3+48}{51(13)} = \frac{1}{13}$$

$\Rightarrow$ 2nd Draw has same probability of ace as 1st draw

(c)

$$P[\underbrace{3 \text{ aces in } 7 \text{ cards}}_{A}] = \frac{\binom{4}{3}\binom{48}{4}}{\binom{52}{7}} = 0.00582$$

$$P[\underbrace{2 \text{ kings in } 7 \text{ cards}}_{B}] = \frac{\binom{4}{2}\binom{48}{5}}{\binom{52}{7}} = 0.07679$$

$$P[A \cup B] = P[A] + P[B] - P[A \cap B]$$

$$P[A \cap B] = \frac{\binom{4}{3}\binom{4}{2}\binom{44}{2}}{\binom{52}{7}} = 0.00017$$

$$P[A \cup B] = 0.00582 + 0.07679 - 0.00017 = 0.0824$$

(d) 52! permutations

13! | 13! | 13! | 13!

within each hand order does not matter

$\therefore \frac{52!}{(13!)^4}$ possible hands

2.128 (d) – continued –

There are $4! = 24$ of arraying the 4 aces and allotting one to each player.

There are $\frac{48!}{(12!)^4}$ ways of distributing the other 48 cards

$$\therefore P[\text{1 ace to each player}] = \frac{4! \frac{48!}{(12!)^4}}{\frac{52!}{(13!)^4}} = \frac{24(48!)\,13^4}{52!}$$

$$= 0.1055$$