Principles Of Digital Communication
A Top-Down Approach
Solution Manual

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Contents

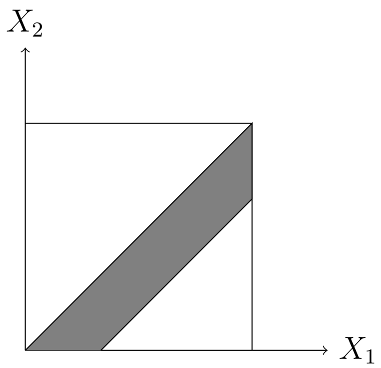
1 Introduction and objectives

Solution 1. (Probabilities of basic events)

In each case, the shaded region represents the (X_1, X_2) values satisfying the corresponding inequalities. Since X_1 and X_2 are independent and uniformly distributed, the area of the shaded region gives the probability of the inequality being satisfied. We use $Pr\{\cdot\}$ to denote the probability of an event.

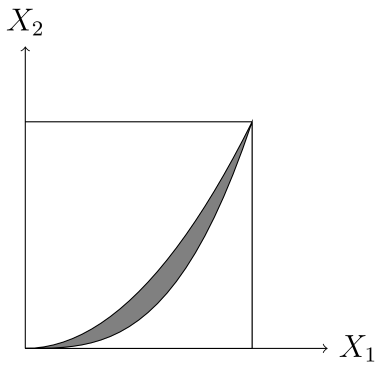
(a)

$$Pr\left\{0 \le X_1 - X_2 \le \frac{1}{3}\right\} = \frac{1}{2} - \frac{1}{2} \times \left(\frac{2}{3} \times \frac{2}{3}\right) = \frac{5}{18}.$$



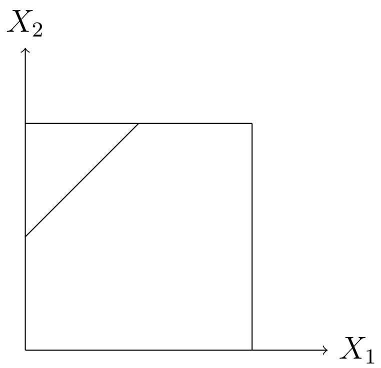
(b)

$$Pr\left\{X_1^3 \le X_2 \le X_1^2\right\} = \int_0^1 (x^2 - x^3)\ dx = \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{1}{12}.$$



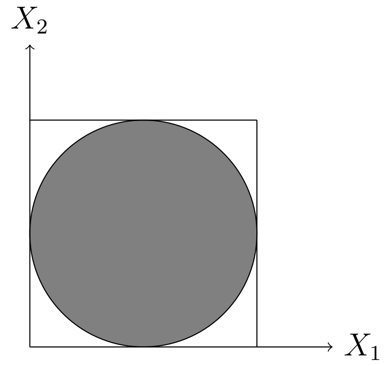
(c)

$$Pr\left\{X_2 - X_1 = \frac{1}{2}\right\} = 0.$$



(d)

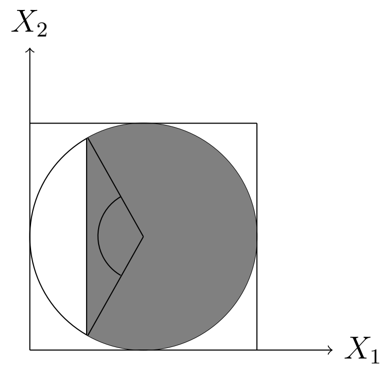
$$Pr\left\{\left(X_1-\frac{1}{2}\right)^2+\left(X_2-\frac{1}{2}\right)^2\leq\left(\frac{1}{2}\right)^2\right\}=\pi\left(\frac{1}{2}\right)^2=\frac{\pi}{4}.$$



(e) In this part we have

$$\begin{aligned}
&Pr\left\{\left(X_1-\frac{1}{2}\right)^2+\left(X_2-\frac{1}{2}\right)^2\leq\left(\frac{1}{2}\right)^2 \;\middle|\; X_1\geq\frac{1}{4}\right\}\\
&=\frac{Pr\left\{\left(X_1-\frac{1}{2}\right)^2+\left(X_2-\frac{1}{2}\right)^2\leq\left(\frac{1}{2}\right)^2\ ,\ X_1\geq\frac{1}{4}\right\}}{Pr\left\{X_1\geq\frac{1}{4}\right\}}\\
&=\frac{\frac{\pi}{6}+\frac{\sqrt{3}}{16}}{\frac{3}{4}}.
\end{aligned}$$

It can easily be seen that the probability term in the numerator is equal to the area of the shaded region in the figure below. We can divide the shaded area into two parts, triangular and sub circular. It is easy to show that the angle of the triangle on the picture is $120°$ *so the sub circular part consists of* $\frac{2}{3}$ *of the circle area. So the sub circular part's area is* $\frac{2}{3}\ \pi(\frac{1}{2})^2=\frac{\pi}{6}$ *and the triangular part's area is* $\frac{\sqrt{3}}{16}$*. Summing the area of these two parts, we reach the final result.*



Solution 2. (Basic probabilities)

(a) *First, we find the probability of the complement of the event, namely the probability of drawing only black balls. This probability is equal to*

$$Pr\{\text{All } k \text{ balls are black}\} = \frac{\binom{n}{k}}{\binom{m+n}{k}}.$$

Therefore the probability of drawing at least one white ball is equal to

$$Pr\{\text{At least one ball is white}\} = 1 - \frac{\binom{n}{k}}{\binom{m+n}{k}}.$$

(b) *Define the following random variables*

$$X = \begin{cases} 0 & \text{if the chosen coin is fair,} \\ 1 & \text{otherwise,} \end{cases}$$

and

$$Y = \begin{cases} 00 & \text{if both outcomes are tail,} \\ 01 & \text{if the first one is tail, the second one is head,} \\ 10 & \text{if the first one is head, the second one is tail,} \\ 11 & \text{if both outcomes are head.} \end{cases}$$

So having these two random variables defined, we want to compute $Pr\{X = 0|Y = 11\}$. *So we can write*

$$\begin{aligned} Pr\{X = 0|Y = 11\} &= \frac{Pr\{Y = 11|X = 0\}Pr\{X = 0\}}{Pr\{Y = 11\}} \\ &= \frac{1/4 \times 1/2}{Pr\{Y = 11\}} \\ &= \frac{1/8}{Pr\{Y = 11\}}. \end{aligned}$$

Then for $Pr\{Y = 11\}$ *we have*

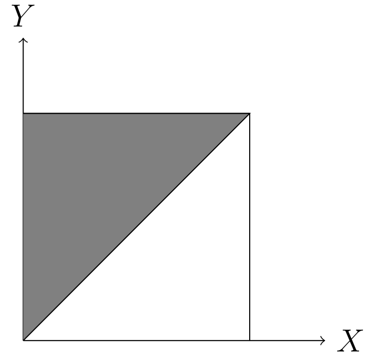
$$\begin{aligned} Pr\{Y = 11\} &= Pr\{X = 0\} \cdot Pr\{Y = 11|X = 0\} + Pr\{X = 1\} \cdot Pr\{Y = 11|X = 1\} \\ &= 1/2 \times 1/4 + 1/2 \times 1 \\ &= 5/8. \end{aligned}$$

So, finally we have

$$Pr\{X = 0|Y = 11\} = \frac{1/8}{5/8} = \frac{1}{5}.$$

Solution 3. (Conditional distribution)

The probability mass has been distributed uniformly on the upper triangular area according to the shape below:



(a) If X and Y were independent then the distribution of X would not depend on Y. This is clearly not the case. In fact, the range of values taken by X is between 0 and Y.

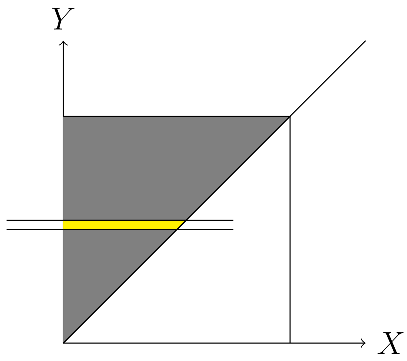
(b) The integral of $f_{X,Y}(x, y)$ must be 1. Hence $A \times \frac{1}{2} = 1$ and so $A = 2$.

(c) We know that $f_Y(y)\, dy = Pr\{y < Y < y + dy\}$, but for a special y as can be seen from the figure below, this probability mass is equal to A times the area of a rectangle with length y and width dy when $0 \leq y \leq 1$.

$$f_Y(y) = \begin{cases} 2y & 0 < y < 1, \\ 0 & \textit{otherwise}. \end{cases}$$

Or more formally

$$f_Y(y) = \int_0^1 f_{X,Y}(x, y)\, dx = \int_0^y 2\, dx = 2y.$$



(d) Under the condition $Y = y$, the random variable X is uniformly distributed between 0 and y and so $f(y) = \mathbb{E}[X|Y = y] = \frac{y}{2}$.

(e) $f(Y)$ is a function of Y so it is a random variable and we can compute its expected value.

$$\mathbb{E}[f(Y)] = \int_0^1 f(y) f_Y(y)\, dy = \int_0^1 y^2\, dy = \frac{1}{3}.$$

(f) We compute $\mathbb{E}[X]$ using the definition.

$$\mathbb{E}[X] = \iint x f_{X,Y}(x, y)\, dx\, dy = \int_0^1 \left[\int_0^y 2x\, dx \right] dy = \frac{1}{3},$$

and it is seen that $\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|Y]]$. This result, which holds in general, is named the law of total expectation.

Solution 4. (Playing darts)

(a) $X = ZX_1 + (1 - Z)X_2$.

(b) *Note that* $\mathbb{E}[X] = 0$*, because expectation is linear and* Z *is independent from* X_1 *and* X_2*. Thus,*

$$\begin{aligned}
\operatorname{Var}(X) &= \mathbb{E}\left[X^2\right] - \mathbb{E}[X]^2 \\
&= \mathbb{E}\left[X^2\right] \\
&= \mathbb{E}\left[X^2|Z=1\right]p + \mathbb{E}\left[X^2|Z=0\right](1-p) \\
&= p\sigma_1^2 + (1-p)\sigma_2^2.
\end{aligned}$$

X is not Gaussian. In fact X is not a linear combination of two Gaussians, it is rather a mixture *of two Gaussians. One can use the characteristic function to show rigorously that X is not a Gaussian, but this is outside the scope of this class.*

(c)

$$\begin{aligned}
\mathbb{E}[S] &= p\int_{-\infty}^{\infty} |x| \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma_1^2}}\ dx + (1-p)\int_{-\infty}^{\infty} |x| \frac{1}{\sigma_2\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma_2^2}}\ dx \\
&= 2p\int_{0}^{\infty} x \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma_1^2}}\ dx + 2(1-p)\int_{0}^{\infty} x \frac{1}{\sigma_2\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma_2^2}}\ dx.
\end{aligned}$$

With the change of variables $u_1 = \frac{x^2}{2\sigma_1^2}$ *and* $u_2 = \frac{x^2}{2\sigma_2^2}$*, we obtain*

$$\begin{aligned}
\mathbb{E}[S] &= 2p\frac{\sigma_1}{\sqrt{2\pi}}\int_0^{\infty} e^{-u_1}\ du_1 + 2(1-p)\frac{\sigma_2}{\sqrt{2\pi}}\int_0^{\infty} e^{-u_2}\ du_2 \\
&= \frac{2}{\sqrt{2\pi}}\left[p\sigma_1 + (1-p)\sigma_2\right].
\end{aligned}$$

Solution 5. (Uncorrelated vs. independent random variables)
Note:

- *By definition,* X *and* Y *are uncorrelated if and only if*

$$0 = \operatorname{cov}(X,Y) = \mathbb{E}\left[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])\right] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y].$$

  Hence $\operatorname{cov}(X,Y) = 0$ *is equivalent to the the condition* $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$.

- X *and* Y *are independent when* $f_{XY} = f_X f_Y$.

(a) *Assume that the random variables* X *and* Y *are independent. Then*

$$\begin{aligned}
\mathbb{E}[XY] &= \iint xy f_{X,Y}(x,y)\ dx\ dy = \iint xy f_X(x) f_Y(y)\ dx\ dy \\
&= \int x f_X(x)\ dx \int y f_Y(y)\ dy = \mathbb{E}[X]\mathbb{E}[Y],
\end{aligned}$$

where the second equality follows from the assumption that X *and* Y *are independent. Hence, if* X *and* Y *are independent, they are also uncorrelated.*

(b) *X and Y are obviously dependent. For example, $X = 0$ implies $U = 0$ and $V = 0$. Hence it implies also $Y = 0$. The marginals of X and Y are*

$$X = \begin{cases} 0 & \textit{with prob. } \frac{1}{4}, \\ 1 & \textit{with prob. } \frac{1}{2}, \\ 2 & \textit{with prob. } \frac{1}{4}, \end{cases}$$

$$Y = \begin{cases} 0 & \textit{with prob. } \frac{1}{2}, \\ 1 & \textit{with prob. } \frac{1}{2}. \end{cases}$$

The mean for X is $\mathbb{E}[X] = 1$ and for Y it is $\mathbb{E}[Y] = \frac{1}{2}$. Finally, we have that

$$\mathbb{E}[XY] = \left(\frac{1}{4} \times 0 \times 0\right) + \left(\frac{1}{4} \times 1 \times 1\right) + \left(\frac{1}{4} \times 1 \times 1\right) + \left(\frac{1}{4} \times 0 \times 2\right) = \frac{1}{2}.$$

From the above we obtain

$$\operatorname{cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = 0.$$

Therefore, we see that X and Y are uncorrelated, even though they are dependent.

Solution 6. (Monty Hall)

(a) *$Pr\{A$ contains one million Swiss francs$\} = 1/3$.*

(b) *Observe that B contains the money if and only if A does not contain the money, thus*

$$Pr\{B \textit{ contains one million Swiss francs}\} = Pr\{A \textit{ contains nothing}\} = 2/3.$$

(c) *A reasonable person will choose B since it has a larger probability of containing the money.*

2 Receiver design for discrete-time observations: First layer

Solution 1. (Hypothesis testing: Uniform and uniform)

(a) *Let $l(y)$ be the number of 0's in the sequence y.*

$$P_{Y|H}(y|0) = \frac{1}{2^{2k}}$$

$$P_{Y|H}(y|1) = \begin{cases} \frac{1}{\binom{2k}{k}}, & \text{if } l = k \\ 0, & \text{otherwise} \end{cases}$$

(b) *The ML decision rule is:*

$$P_{Y|H}(y|1) \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} P_{Y|H}(y|0)$$

Because $\frac{1}{\binom{2k}{k}} > \frac{1}{2^{2k}}$ for any value of k, the ML decision rule becomes

$$\hat{H} = \begin{cases} 0, & \text{if } l(y) \neq k \\ 1, & \text{if } l(y) = k. \end{cases}$$

The single number needed is $l(y)$, the number of 0's in the sequence y.

(c) *The decision rule that minimizes the error probability is the MAP rule:*

$$P_{Y|H}(y|1)P_H(1) \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} P_{Y|H}(y|0)P_H(0).$$

The MAP decision rule gives $\hat{H} = 0$ whenever $l(y) \neq k$. When $l(y) = k$:

$$\hat{H} = \begin{cases} 0, & \text{if } \frac{\binom{2k}{k}}{2^{2k}} \geq \frac{P_H(1)}{P_H(0)} \\ 1, & \text{otherwise.} \end{cases}$$

(d) *Trivial solution: If $P_H(1) = 1$ then $\hat{H} = 1$ for all y (In this case, $l(y) = k$ is guaranteed). Similarly, if $P_H(0) = 1$ then $\hat{H} = 0$ for all y.*

Now assume $P_H(1) \neq 1$. Then there is a nonzero probability that $l(y) \neq k$, in which case $\hat{H} = 0$. The MAP decision rule always chooses $\hat{H} = 0$ if

$$\frac{\binom{2k}{k}}{2^{2k}} \geq \frac{P_H(1)}{P_H(0)} \Longleftrightarrow P_H(0) \geq \frac{\frac{1}{\binom{2k}{k}}}{\frac{1}{\binom{2k}{k}} + \frac{1}{2^{2k}}}.$$

Solution 2. (The "Wetterfrosch")

(a) *A and B must be chosen such that the suggested functions become valid probability density functions, i.e.* $\int_0^1 f_{Y|H}(y|i)dy = 1$ *for* $i = 0, 1$. *This yields* $A = 4/3$ *and* $B = 6/7$. *(A quicker way is to draw the functions and find the area by looking at the drawings.)*

(b) *Let us first find the marginal of* Y, *i.e.*

$$f_Y(y) = f_{Y|H}(y|0)P_H(0) + f_{Y|H}(y|1)P_H(1) = C - Dy,$$

where we find $C = 23/21$ *and* $D = 4/21$. *Then, applying Bayes' rule gives*

$$P_{H|Y}(0|y) = \frac{f_{Y|H}(y|0)P_H(0)}{f_Y(y)} = \frac{1}{2}\frac{A - \frac{A}{2}y}{C - Dy} = \frac{1}{2}\frac{4/3 - 2/3y}{23/21 - 4/21y},$$

and similarly

$$P_{H|Y}(1|y) = \frac{f_{Y|H}(y|1)P_H(1)}{f_Y(y)} = \frac{1}{2}\frac{B + \frac{B}{3}y}{C - Dy} = \frac{1}{2}\frac{6/7 + 2/7y}{23/21 - 4/21y}.$$

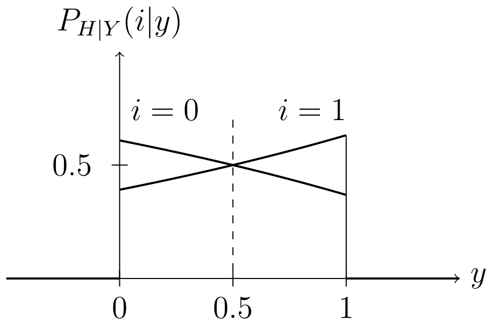
(c) *The threshold is where the two a posteriori probabilities are equal,*

$$\frac{1}{2}\frac{4/3 - 2/3y}{23/21 - 4/21y} = \frac{1}{2}\frac{6/7 + 2/7y}{23/21 - 4/21y},$$

or equivalently,

$$4/3 - 2/3y = 6/7 + 2/7y.$$

The y *that satisfies this equation is our threshold* θ, *thus* $\theta = 0.5$.



(d) *The probability that we decide* $\hat{H}_\gamma(y) = 1$ *when in reality* $H = 0$ *is just the probability that* y *is* larger *than the threshold given that* $H = 0$, *which is*

$$\begin{aligned} Pr\{Y > \gamma | H = 0\} &= \int_\gamma^1 f_{Y|H}(y|0)dy = \int_\gamma^1 \left(A - \frac{A}{2}y\right)dy \\ &= A(1-\gamma) - \frac{A}{2}\frac{1-\gamma^2}{2} \\ &= \frac{4(1-\gamma)}{3} - \frac{1-\gamma^2}{3}. \end{aligned}$$

(e) *By analogy to the previous question,*

$$\begin{aligned} Pr\{Y < \gamma | H = 1\} &= \int_0^\gamma f_{Y|H}(y|1)dy = \int_0^\gamma \left(B + \frac{B}{3}y\right) dy \\ &= B\gamma + \frac{B}{3}\frac{\gamma^2}{2} \\ &= \frac{6\gamma}{7} + \frac{\gamma^2}{7}. \end{aligned}$$

$$\begin{aligned} P_e(\gamma) &= Pr\{Y > \gamma | H = 0\} P_H(0) + Pr\{Y < \gamma | H = 1\} P_H(1) \\ &= \frac{1}{2}\left(\frac{4(1-\gamma)}{3} - \frac{1-\gamma^2}{3} + \frac{6\gamma}{7} + \frac{\gamma^2}{7}\right). \end{aligned}$$

For $\gamma = \theta = 0.5$*, we find* $P_e(\theta) = 0.44$*.*

(f) *To minimize* P_e *over* γ*, we take the derivative of* P_e *with respect to* γ*, i.e.*

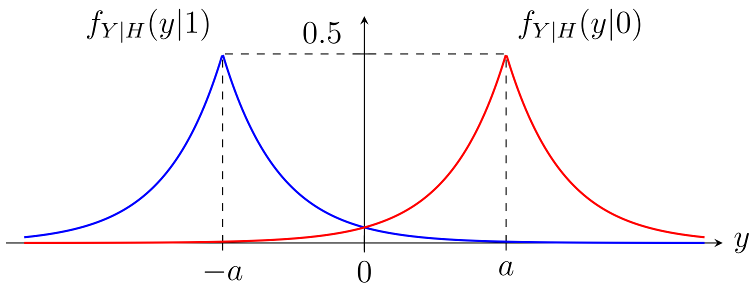
$$\frac{d}{d\gamma}P_e(\gamma) = \frac{1}{2}\left(-\frac{4}{3} + \frac{2\gamma}{3} + \frac{6}{7} + \frac{2\gamma}{7}\right).$$

Setting this equal to zero, we find $\gamma = 0.5$*. We observe that the value of* γ *which minimizes* $P_e(\gamma)$ *is equal to* θ*. This was expected, because the MAP decision rule minimizes the error probability.*

Solution 3. (Hypothesis testing in Laplacian noise)

(a) *We find the following conditional densities for the observation* Y *under hypothesis* $H = 0$ *and* $H = 1$*, respectively:*

$$\begin{aligned} f_{Y|H}(y|0) &= \frac{1}{2}e^{-|y-a|} \\ f_{Y|H}(y|1) &= \frac{1}{2}e^{-|y+a|}. \end{aligned}$$



(b) *Because the hypotheses are equally likely, the MAP rule is the same as the ML rule. Therefore, the probability of error is minimized by the following decision rule:*

$$f_{Y|H}(y|1) \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} f_{Y|H}(y|0).$$

From the picture of $f_{Y|H}(y|0)$ *and* $f_{Y|H}(y|1)$*, we see immediately that the ML decision rule decides for* $H = 0$ *when* $y > 0$ *and for* $H = 1$ *when* $y < 0$*.*

(c)

$$\begin{aligned} P_e(0) &= Pr\{y<0|H=0\} = \int_{-\infty}^{0} f_{Y|H}(y|0)dy \\ &= \int_{-\infty}^{0} \frac{1}{2}e^{-|y-a|}dy = \int_{-\infty}^{0} \frac{1}{2}e^{(y-a)}dy \\ &= \frac{e^{-a}}{2}\, e^{y}|_{-\infty}^{0} = \frac{e^{-a}}{2}. \end{aligned}$$

By symmetry, we find that

$$P_e(1) = \frac{e^{-a}}{2},$$

and thus,

$$P_e = P_e(0)P_H(0) + P_e(1)P_H(1) = \frac{e^{-a}}{2}.$$

Solution 4. (Poisson parameter estimation)

(a) We can write the MAP decision rule in the following way:

$$\frac{P_{Y|H}(y|1)}{P_{Y|H}(y|0)} \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} \frac{P_H(0)}{P_H(1)}$$

Plugging in, we find

$$\frac{\lambda_1^y e^{-\lambda_1}}{\lambda_0^y e^{-\lambda_0}} \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} \frac{p_0}{1-p_0},$$

and then

$$\left(\frac{\lambda_1}{\lambda_0}\right)^y \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} \frac{p_0}{1-p_0}e^{\lambda_1-\lambda_0}$$

Taking logarithms on both sides does not change the direction of the inequalities, therefore

$$y\log\left(\frac{\lambda_1}{\lambda_0}\right) \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} \log\left(\frac{p_0}{1-p_0}e^{\lambda_1-\lambda_0}\right)$$

Attention: the term $\log(\lambda_1/\lambda_0)$ can be negative, and if it is, then dividing by it involves changing the direction of the inequality.

Suppose $\lambda_1 > \lambda_0$. Then, $\log(\lambda_1/\lambda_0) > 0$, and the decision rule becomes

$$y \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} \frac{\log\left(\frac{p_0}{1-p_0}e^{\lambda_1-\lambda_0}\right)}{\log\left(\frac{\lambda_1}{\lambda_0}\right)} \stackrel{\text{def}}{=} \theta$$

(b) *We compute*

$$\begin{aligned} P_e(0) &= Pr\{Y > \theta | H = 0\} = \sum_{y=\lceil \theta \rceil}^{\infty} P_{Y|H}(y|0) \\ &= 1 - \sum_{y=0}^{\lfloor \theta \rfloor} \frac{\lambda_0^y}{y!} e^{-\lambda_0}, \end{aligned}$$

and by analogy

$$\begin{aligned} P_e(1) &= Pr\{Y < \theta | H = 1\} = \sum_{y=0}^{\lfloor \theta \rfloor} P_{Y|H}(y|1) \\ &= \sum_{y=0}^{\lfloor \theta \rfloor} \frac{\lambda_1^y}{y!} e^{-\lambda_1} \end{aligned}$$

Thus, the probability of error becomes

$$P_e = p_0 \left(1 - \sum_{y=0}^{\lfloor \theta \rfloor} \frac{\lambda_0^y}{y!} e^{-\lambda_0} \right) + (1 - p_0) \sum_{y=0}^{\lfloor \theta \rfloor} \frac{\lambda_1^y}{y!} e^{-\lambda_1}$$

Now, suppose that $\lambda_1 < \lambda_0$*. Then,* $\log(\lambda_1/\lambda_0) < 0$*, and we have to swap the inequality sign, thus*

$$y \underset{\hat{H}=1}{\overset{\hat{H}=0}{\gtrless}} \frac{\log\left(\frac{p_0}{1-p_0} e^{\lambda_1 - \lambda_0}\right)}{\log\left(\frac{\lambda_1}{\lambda_0}\right)} \stackrel{\text{def}}{=} \theta$$

The rest of the analysis goes along the same lines, and finally, we obtain

$$P_e = p_0 \sum_{y=0}^{\lfloor \theta \rfloor} \frac{\lambda_0^y}{y!} e^{-\lambda_0} + (1 - p_0) \left(1 - \sum_{y=0}^{\lfloor \theta \rfloor} \frac{\lambda_1^y}{y!} e^{-\lambda_1} \right)$$

The case $\lambda_0 = \lambda_1$ *yields* $\log(\lambda_1/\lambda_0) = 0$*, so the decision rule becomes* $0 \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} \theta$*, regardless of* y*. Thus, we can exclude the case* $\lambda_0 = \lambda_1$ *from our discussion.*

(c) *Here, we are in the case* $\lambda_1 > \lambda_0$*, and we find* $\theta \approx 4.54$*. We thus evaluate*

$$P_e = \frac{1}{3}\left(1 - \sum_{y=0}^{4} \frac{2^y}{y!} e^{-2}\right) + \frac{2}{3} \sum_{y=0}^{4} \left(\frac{10^y}{y!} e^{-10}\right) \approx 0.03705$$

(d) *We find* $\theta \approx 7.5163$

$$P_e = \frac{1}{3}\left(1 - \sum_{y=0}^{7} \frac{2^y}{y!} e^{-2}\right) + \frac{2}{3} \sum_{y=0}^{7} \left(\frac{20^y}{y!} e^{-20}\right) \approx 0.000885$$

The two Poisson distributions are much better separated than in (c); therefore, it becomes considerably easier to distinguish them based on one single observation y*.*

Solution 5. (Lie detector)

(a) *Let* $H \in \{T, L\}$.

$$\begin{aligned} H &= T \text{ (telling truth):} \quad f_{Y|H}(y|T) = \alpha e^{-\alpha y},\ y \geq 0 \\ H &= L \text{ (telling lie):} \quad f_{Y|H}(y|L) = \beta e^{-\beta y},\ y \geq 0. \end{aligned}$$

The MAP decision rule is

$$p\beta e^{-\beta y} \underset{\hat{H}=T}{\overset{\hat{H}=L}{\gtreqless}} (1-p)\alpha e^{-\alpha y}.$$

After taking the logarithm, we obtain

$$-\beta y + \ln(p\beta) \underset{\hat{H}=T}{\overset{\hat{H}=L}{\gtreqless}} -\alpha y + \ln((1-p)\alpha).$$

Or, equivalently

$$y \underset{\hat{H}=L}{\overset{\hat{H}=T}{\gtreqless}} \frac{1}{\alpha-\beta} \ln\left[\frac{\alpha}{\beta}\frac{(1-p)}{p}\right] = \theta$$

(b)

$$P_{L|T} = \int_0^\theta \alpha e^{-\alpha y} dy = 1 - e^{-\alpha\theta}.$$

(c)

$$P_{T|L} = \int_\theta^\infty \beta e^{-\beta y} dy = e^{-\beta\theta}.$$

(d)

$$\begin{aligned} H = T: \quad & f_{Y|H}(y|T) = \alpha^n e^{-\alpha(y_1+\ldots+y_n)} = \alpha^n e^{-\alpha z} \\ H = L: \quad & f_{Y|H}(y|L) = \beta^n e^{-\beta(y_1+\ldots+y_n)} = \beta^n e^{-\beta z}, \end{aligned}$$

where Y *is the random vector* $(Y_1, ..., Y_n)$ *and where* $z = \sum_{i=1}^n y_i$. *With this new definition, the test becomes* $z \underset{\hat{H}=L}{\overset{\hat{H}=T}{\gtreqless}} \theta$, *with the new threshold* $\theta = \frac{1}{\alpha-\beta} \ln\left[\left(\frac{\alpha}{\beta}\right)^n \frac{(1-p)}{p}\right]$.

$$P_{L|T} = \int_0^\theta f_{Z|H}(z|T) dz,$$

where $Z = \sum_{i=1}^n Y_i$ *and*

$$f_{Z|H}(z|T) = \frac{\alpha^n}{(n-1)!} z^{(n-1)} e^{-\alpha z}.$$

This is the density of the Erlang distribution. Putting things together, we get

$$P_{L|T} = \int_0^\theta \frac{\alpha^n}{(n-1)!} z^{(n-1)} e^{-\alpha z} dz.$$

Solution 6. (Fault detector)

$H = 1$ is the hypothesis that the box works properly and $H = 0$ the hypothesis that the box fails.

(a) The MAP test is

$$\frac{f_{X|H}(x|1)}{P_H(0)} \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} \frac{f_{X|H}(x|0)}{P_H(1)}.$$

If $l(x)$ is the number of zeros in the sequence x,

$$f_{X|H}(x|1) = \begin{cases} p^{16-l}(1-p)^l, & \text{if } 0 \leq l \leq 16 \\ 0, & \text{otherwise} \end{cases}$$

$$f_{X|H}(x|0) = \frac{1}{2^{16}}$$

(b) By substituting $l = 8$, $p = 0.25$, $P_H(0) = \frac{1}{1025}$ and $P_H(1) = \frac{1024}{1025}$ in the decision rule, we obtain

$$\frac{3^8}{2^6} \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} 1,$$

therefore the hypothesis is $\hat{H} = 1$ — the box works properly.

Solution 7. (Multiple choice exam)

(a) We have a binary hypothesis testing problem: The hypothesis H is the answer you will select, and your decision will be based on the observation of $\hat{H}_L$ and $\hat{H}_R$. Let H take value 1 if answer 1 is chosen, and value 2 if answer 2 is chosen. In this case, we can write the MAP decision rule as follows:

$$Pr\left\{H = 1|\hat{H}_L = 1, \hat{H}_R = 2\right\} \underset{\hat{H}=2}{\overset{\hat{H}=1}{\gtrless}} Pr\left\{H = 2|\hat{H}_L = 1, \hat{H}_R = 2\right\}$$

From the problem setting we know the priors $Pr\{H = 1\}$ and $Pr\{H = 2\}$; we can also determine the conditional probabilities $Pr\left\{\hat{H}_L = 1|H = 1\right\}$, $Pr\left\{\hat{H}_L = 1|H = 2\right\}$, $Pr\left\{\hat{H}_R = 2|H = 1\right\}$ and $Pr\left\{\hat{H}_R = 2|H = 2\right\}$ (we have $Pr\left\{\hat{H}_L = 1|H = 1\right\} = 0.9$ and $Pr\left\{\hat{H}_L = 1|H = 2\right\} = 0.1$). Introducing these quantities and using the Bayes rule we can formulate the MAP decision rule as

$$\frac{Pr\left\{\hat{H}_L = 1, \hat{H}_R = 2|H = 1\right\}Pr\{H = 1\}}{Pr\left\{\hat{H}_L = 1, \hat{H}_R = 2\right\}} \underset{\hat{H}=2}{\overset{\hat{H}=1}{\gtrless}} \frac{Pr\left\{\hat{H}_L = 1, \hat{H}_R = 2|H = 2\right\}Pr\{H = 2\}}{Pr\left\{\hat{H}_L = 1, \hat{H}_R = 2\right\}}$$

Now, assuming that the event $\{\hat{H}_L = 1\}$ is independent of the event $\{\hat{H}_R = 2\}$ and simplifying the expression, we obtain

$$Pr\left\{\hat{H}_L = 1|H = 1\right\}Pr\left\{\hat{H}_R = 2|H = 1\right\}Pr\{H = 1\} \underset{\hat{H}=2}{\overset{\hat{H}=1}{\gtrless}}$$
$$Pr\left\{\hat{H}_L = 1|H = 2\right\}Pr\left\{\hat{H}_R = 2|H = 2\right\}Pr\{H = 2\},$$

which is our final decision rule.

(b) Evaluating the previous decision rule, we have

$$0.9 \times 0.3 \times 0.25 \quad \underset{\hat{H}=2}{\overset{\hat{H}=1}{\gtrless}} \quad 0.1 \times 0.7 \times 0.75,$$

which gives

$$0.0675 \quad \underset{\hat{H}=2}{\overset{\hat{H}=1}{\gtrless}} \quad 0.0525$$

This implies that the answer $\hat{H}$ is equal to 1.

Solution 8. (MAP decoding rule: Alternative derivation)

(a) The probability of error can be written as

$$\begin{aligned}
P_e &= P_H(0)\,Pr\{Y \in \mathcal{R}_1 | H = 0\} + P_H(1)\,Pr\{Y \in \mathcal{R}_0 | H = 1\} \\
&= P_H(0) \int_{\mathcal{R}_1} f_{Y|H}(y|0)dy + P_H(1) \int_{\mathcal{R}_0} f_{Y|H}(y|1)dy \\
&= P_H(0) \int_{\mathcal{R}_1} f_{Y|H}(y|0)dy + P_H(1) \left(1 - \int_{\mathcal{R}_1} f_{Y|H}(y|1)dy\right) \\
&= P_H(1) + \int_{\mathcal{R}_1} \left(P_H(0) f_{Y|H}(y|0) - P_H(1) f_{Y|H}(y|1)\right) dy, \qquad (1)
\end{aligned}$$

where the third equality follows from the hint

$$\int_{\mathcal{R}_0 \bigcup \mathcal{R}_1} f_{Y|H}(y|1)dy = \int_{\mathcal{R}_0} f_{Y|H}(y|1)dy + \int_{\mathcal{R}_1} f_{Y|H}(y|1)dy = 1.$$

(b) Note that P_e is smallest if the second term $\int_{\mathcal{R}_1} \left(P_H(0) f_{Y|H}(y|0) - P_H(1) f_{Y|H}(y|1)\right) dy$ in (1) is made as negative as possible. Note that the first term $P_H(1)$ in (1) is fixed and does not depend on our choices for $\mathcal{R}_0$ and $\mathcal{R}_1$. The second term can be minimized if we collect in $\mathcal{R}_1$ all $y \in \mathbb{R}$ that yield negative contribution, i.e. $y \in \mathcal{R}_1$ iff $P_H(0) f_{Y|H}(y|0) - P_H(1) f_{Y|H}(y|1) < 0$.

Note: How does this approach compare to the one from the book? Conditioning is one of the most important tricks to make progress in computing a probability. There are two random variables involved, namely H and Y. In the notes we have conditioned on $Y = y$. Here we are conditioning on $H = i$.

Solution 9. (Independent and identically distributed vs. first-order Markov)

An explanation regarding the title of this problem: independent and identically distributed means that all $Y_1, \ldots, Y_k$ have the same probability mass function and are independent of each other. First-order Markov *means that $Y_1, \ldots, Y_k$ depend on each other in a particular way: the probability mass function Y_i depends on the value of Y_{i-1}, but given the value of Y_{i-1}, it is independent of $Y_1, \ldots, Y_{i-2}$. Thus, in this problem, we observe a binary sequence, and we want to know whether it has been generated by an* i.i.d. *(independent and identically distributed) source or by a first-order Markov source.*

(a) *Since the two hypotheses are equally likely, we find*

$$\frac{P_{Y|H}(y|1)}{P_{Y|H}(y|0)} \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} \frac{P_H(0)}{P_H(1)} = 1.$$

Plugging in, we obtain

$$\frac{1/2 \cdot (1/4)^l \cdot (3/4)^{k-l-1}}{(1/2)^k} \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} 1,$$

where l is the number of times the observed sequence changes either from zero to one or from one to zero, i.e. the number of transitions in the observed sequence.

(b) *The sufficient statistic here is simply the number of transitions l; this entirely specifies the likelihood ratio.*

(c) *In this case, the number of non-transitions is $(k-l) = s$, and the log-likelihood ratio becomes*

$$\begin{aligned}
\log \frac{1/2 \cdot (1/4)^{k-s} \cdot (3/4)^{s-1}}{(1/2)^k} &= \log \frac{(1/4)^{k-s} \cdot (3/4)^{s-1}}{(1/2)^{k-1}} \\
&= (k-s)\log(1/4) + (s-1)\log(3/4) - (k-1)\log(1/2) \\
&= s \log \frac{3/4}{1/4} + k \log \frac{1/4}{1/2} + \log \frac{1/2}{3/4} \\
&= s \log 3 + k \log 1/2 + \log 2/3.
\end{aligned}$$

Thus, in terms of this log-likelihood ratio, the decision rule becomes

$$s \log 3 + k \log 1/2 + \log 2/3 \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} 0.$$

That is, we have to find the smallest possible s such that this expression becomes larger or equal to zero. Therefore,

$$s \geq \left\lceil \frac{k \log 1/2 + \log 2/3}{\log 1/3} \right\rceil.$$

Solution 10. (SIMO channel with Laplacian noise)

(a) *Let the two hypotheses be $H = 0$ and $H = 1$ when c_0 and c_1 are transmitted, respectively. The ML decision rule is*

$$f_{Y_1 Y_2|H}(y_1, y_2|1) \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} f_{Y_1 Y_2|H}(y_1, y_2|0).$$

Because Z_1 and Z_2 are independent, we can write

$$\frac{1}{2}e^{-|y_1-1|}\frac{1}{2}e^{-|y_2-1|} \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} \frac{1}{2}e^{-|y_1+1|}\frac{1}{2}e^{-|y_2+1|},$$

and, after taking the logarithm,

$$|y_1+1| + |y_2+1| \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} |y_1-1| + |y_2-1|.$$

(b) Because the hypotheses are equally likely and Z_1 and Z_2 have the same distribution, the decision region for $\hat{H} = 0$ contains the points closer to $(-1, -1)$ and the decision region for $\hat{H} = 1$ contains the points closer to $(1, 1)$. For this problem, the distance between the points (y_{11}, y_{12}) and (y_{21}, y_{22}) is the Manhattan distance, $|y_{11} - y_{21}| + |y_{12} - y_{22}|$, and not the Euclidian distance.

Let us first consider the points above the line $y_2 = -y_1$ from Figure 1. It is easy to notice that the points in the positive quadrant are closer to $(1, 1)$ than to $(-1, -1)$, therefore they belong to $\mathcal{R}_1$ ($\hat{H} = 1$). This is also true if $\{(y_1 \geq 0) \cap (y_2 \in (-1, 0))\}$, or if $\{(y_2 \geq 0) \cap (y_1 \in (-1, 0))\}$.

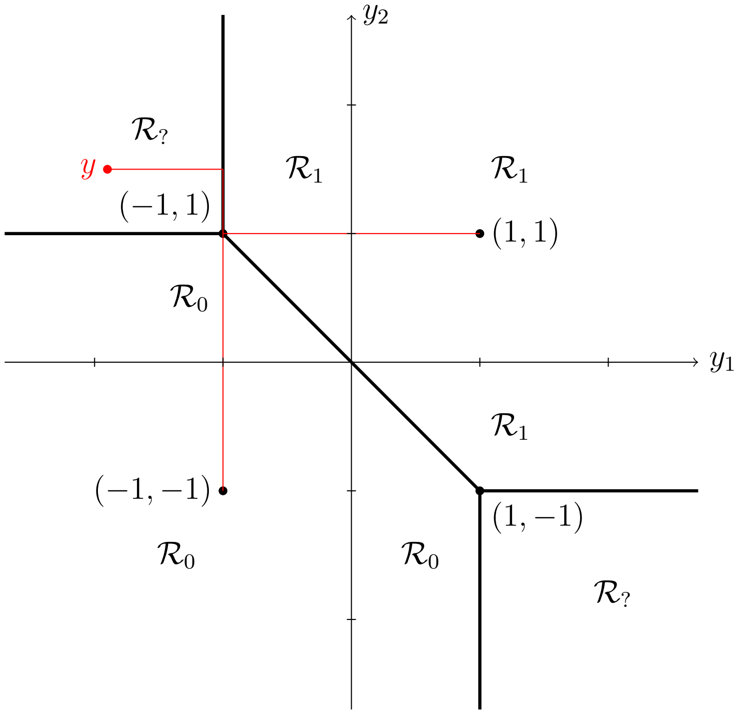


Figure 1: Decision regions

Similar reasoning can be applied to the points below the diagonal to determine $\mathcal{R}_0$.

The points for which $\{(y_1 \leq -1) \cap (y_2 \geq 1)\}$ or $\{(y_1 \geq 1) \cap (y_2 \leq -1)\}$ are equally distanced to $(-1, -1)$ and $(1, 1)$, therefore they can belong to either $\mathcal{R}_0$ or $\mathcal{R}_1$ with the same probability. This region is named $\mathcal{R}_?$.

(c) The two hypotheses are equally probable for the region $\mathcal{R}_?$. Therefore, we can split this region in any way between the decision regions and have the same error probability. Because $\mathcal{R}_1$ is included in the region for which $y_2 > -y_1$ and $\mathcal{R}_0$ does not intersect the region for which $y_2 > -y_1$, the error probability is minimized by deciding $\hat{H} = 1$ if $(y_1 + y_2) > 0$.

(d)

$$\begin{aligned} P_e(0) &= Pr\left\{Y_1 + Y_2 > 0 | H = 0\right\} \\ &= Pr\left\{Z_1 + Z_2 - 2 > 0\right\} \\ &= \int_2^\infty \frac{e^{-w}}{4}(1+w)\ dw \\ &= \frac{-e^{-w}}{4}(w+2)\Big|_2^\infty = e^{-2}. \end{aligned}$$

By symmetry, and considering that the messages are equally likely, $P_e(0) = P_e(1) = P_e$.

Solution 11. (Q-Function on regions)

(a) One can see that the event $\{X \in \mathit{Region}\}$ *only depends on the first component* X_1. *Hence, we have*

$$\begin{aligned} Pr\left\{X \in \mathit{Region}\right\} &= Pr\left\{(X_1 \geq -2) \cap (X_1 \leq 1)\right\} \\ &= 1 - Pr\left\{(X_1 < -2) \cup (X_1 > 1)\right\} \\ &= 1 - Q\left(\frac{2}{\sigma}\right) - Q\left(\frac{1}{\sigma}\right), \end{aligned}$$

where the last equality is true because $\{X_1 < -2\}$ *and* $\{X_1 > 1\}$ *are disjoint events.*

(b) Because X_1 *and* X_2 *are independent and have the same variance, rotating the vector* X *by any angle around the origin does not change its distribution. Equivalently, we can rotate the square region in Figure (b) by* 45 *degrees, and the probability of* X *being in the rotated region is the same as for the original region. The new region is a square whose edges are parallel to the axes of the coordinate system. The points where the edges of the square intersect the axes are* $(\sqrt{2}, 0)$, $(-\sqrt{2}, 0)$, $(0, \sqrt{2})$ *and* $(0, -\sqrt{2})$. *Hence,*

$$\begin{aligned} Pr\left\{X \in \mathit{Region}\right\} &= Pr\left\{(-\sqrt{2} \leq X_1 \leq \sqrt{2}) \cap (-\sqrt{2} \leq X_2 \leq \sqrt{2})\right\} \\ &\overset{(1)}{=} Pr\left\{-\sqrt{2} \leq X_1 \leq \sqrt{2}\right\}^2 \\ &= \left[1 - Pr\left\{(X_1 < -\sqrt{2}) \cup (X_1 > \sqrt{2})\right\}\right]^2 \\ &= \left[1 - 2Q\left(\frac{\sqrt{2}}{\sigma}\right)\right]^2, \end{aligned}$$

where (1) *holds because* X_1 *and* X_2 *are independent and identically distributed.*

(c) We solve this part in three different ways:

(i) First Solution: *As in the previous part, we can rotate* X *such that one of its components, say* X_1, *is perpendicular to the straight line that delimits the shaded region. Then, we need to know the shortest distance* d *of that line to the origin (the length of a segment that starts at* $(0, 0)$ *and is perpendicular to the line).*

Using standard trigonometric techniques, one finds that this length is $d = \frac{2}{\sqrt{5}}$. *Then, it follows that*

$$\begin{aligned} Pr\{X \in Region\} &= Pr\left\{X_1 \geq \frac{2}{\sqrt{5}}\right\} \\ &= Q\left(\frac{2}{\sqrt{5}\sigma}\right). \end{aligned}$$

(ii) Second Solution: *We are looking for the probability that* $X_2 \geq 1 - \frac{1}{2}X_1$, *i.e., the probability that* $Z \triangleq X_2 + \frac{1}{2}X_1 - 1 \geq 0$. *But* $Z \sim \mathcal{N}(-1, \frac{5}{4}\sigma^2)$. *Hence,* $Pr\{X \in Region\} = Pr\{Z \geq 0\} = Q\left(\frac{2}{\sqrt{5}\sigma}\right)$.

(iii) Third Solution: *We project* $X = (X_1, X_2)^\mathsf{T}$ *to the vector perpendicular to the line that delimits the shaded region. The length of the projection is* $Z \sim \mathcal{N}(0, \sigma^2)$. *The sought probability is* $Pr\{Z \geq d\} = Q\left(\frac{d}{\sigma}\right) = Q\left(\frac{2}{\sqrt{5}\sigma}\right)$, *where* d *is the distance from the delimiting line to the origin.*

Solution 12. (Properties of the Q function)

(a)

$$\begin{aligned} F_Z(z) =& Pr\{Z \leq z\} = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\ dx \\ =& \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\ dx - \int_{z}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\ dx \\ =& 1 - Q(z). \end{aligned}$$

(b)

$$Q(0) = \frac{1}{2},$$

because we have the same area on both sides of the Gaussian bell.

$$Q(-\infty) = Pr\{Z \geq -\infty\} = 1.$$

$$Q(\infty) = Pr\{Z \geq \infty\} = 0.$$

(c) If we add $Q(-x)$ *and* $Q(x)$, *we get* 1. *Refer to Figure 2.*

(d) Consider the following integration by parts:

$$\begin{aligned} Q(\alpha) &= \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\infty} e^{-\frac{x^2}{2}}\ dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\infty} \frac{1}{x} x e^{-\frac{x^2}{2}}\ dx \\ &= \frac{1}{\sqrt{2\pi}} \left(-\frac{e^{-\frac{x^2}{2}}}{x}\Bigg|_{\alpha}^{\infty} - \int_{\alpha}^{\infty} \frac{1}{x^2} e^{-\frac{x^2}{2}}\ dx \right) \\ &= \frac{1}{\sqrt{2\pi}} \left(\frac{e^{-\frac{\alpha^2}{2}}}{\alpha} - \int_{\alpha}^{\infty} \frac{1}{x^2} e^{-\frac{x^2}{2}}\ dx \right). \end{aligned}$$