Principles Of Digital Communication
A Top-Down Approach
Solution Manual

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Contents

Chapter 1

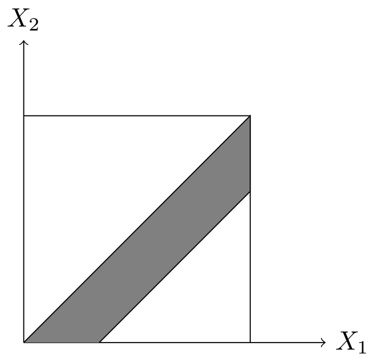
Introduction and Objectives

Solution 1. (Probabilities of basic events)

In each case, the shaded region represents the (X_1, X_2) values satisfying the corresponding inequalities. Since X_1 and X_2 are independent and uniformly distributed, the area of the shaded region gives the probability of the inequality being satisfied. We use $Pr\{\cdot\}$ to denote the probability of an event.

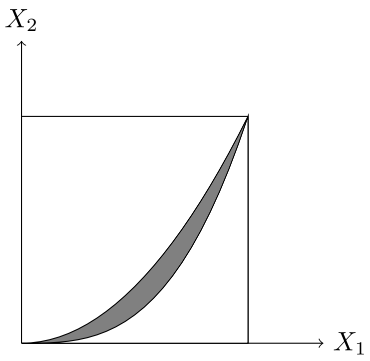
(a)

$$Pr\left\{0 \le X_1 - X_2 \le \frac{1}{3}\right\} = \frac{1}{2} - \frac{1}{2} \times \left(\frac{2}{3} \times \frac{2}{3}\right) = \frac{5}{18}.$$



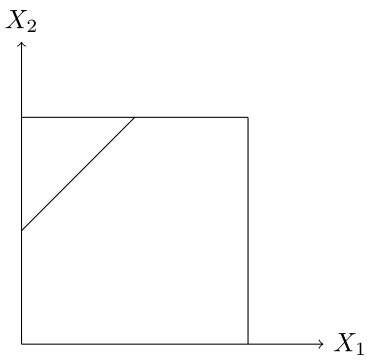
(b)

$$Pr\left\{X_1^3 \le X_2 \le X_1^2\right\} = \int_0^1 \left(x^2 - x^3\right) dx = \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{1}{12}.$$



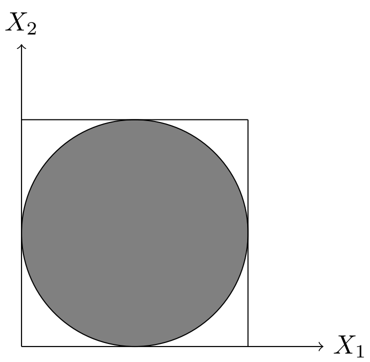
(c)

$$Pr\left\{X_2 - X_1 = \frac{1}{2}\right\} = 0.$$



(d)

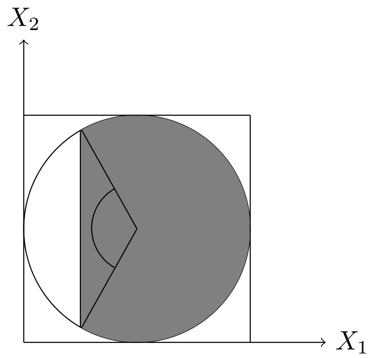
$$Pr\left\{\left(X_1 - \frac{1}{2}\right)^2 + \left(X_2 - \frac{1}{2}\right)^2 \le \left(\frac{1}{2}\right)^2\right\} = \pi\left(\frac{1}{2}\right)^2 = \frac{\pi}{4}.$$



(e) *In this part we have*

$$Pr\left\{\left(X_1-\frac{1}{2}\right)^2+\left(X_2-\frac{1}{2}\right)^2\le\left(\frac{1}{2}\right)^2 \;\middle|\; X_1\ge\frac{1}{4}\right\}$$
$$=\frac{Pr\left\{\left(X_1-\frac{1}{2}\right)^2+\left(X_2-\frac{1}{2}\right)^2\le\left(\frac{1}{2}\right)^2\;,\;X_1\ge\frac{1}{4}\right\}}{Pr\left\{X_1\ge\frac{1}{4}\right\}}$$
$$=\frac{\frac{\pi}{6}+\frac{\sqrt{3}}{16}}{\frac{3}{4}}.$$

It can easily be seen that the probability term in the numerator is equal to the area of the shaded region in the figure below. We can divide the shaded area into two parts, triangular and sub circular. It is easy to show that the angle of the triangle on the picture is $120°$ *so the sub circular part consists of* $\frac{2}{3}$ *of the circle area. So the sub circular part's area is* $\frac{2}{3}\,\pi(\frac{1}{2})^2=\frac{\pi}{6}$ *and the triangular part's area is* $\frac{\sqrt{3}}{16}$*. Summing the area of these two parts, we reach the final result.*



Solution 2. (Basic probabilities)

(a) *First, we find the probability of the complement of the event, namely the probability of drawing only black balls. This probability is equal to*

$$Pr\left\{\text{All } k \text{ balls are black}\right\}=\frac{\binom{n}{k}}{\binom{m+n}{k}}.$$

Therefore the probability of drawing at least one white ball is equal to

$$Pr\left\{\text{At least one ball is white}\right\}=1-\frac{\binom{n}{k}}{\binom{m+n}{k}}.$$

(b) *Define the following random variables*

$$X=\begin{cases}0 & \text{if the chosen coin is fair,}\\ 1 & \text{otherwise,}\end{cases}$$

and

$$Y = \begin{cases} 00 & \textit{if both outcomes are tail,} \\ 01 & \textit{if the first one is tail, the second one is head,} \\ 10 & \textit{if the first one is head, the second one is tail,} \\ 11 & \textit{if both outcomes are head.} \end{cases}$$

So having these two random variables defined, we want to compute $Pr\{X = 0|Y = 11\}$*. So we can write*

$$\begin{aligned} Pr\{X = 0|Y = 11\} &= \frac{Pr\{Y = 11|X = 0\}Pr\{X = 0\}}{Pr\{Y = 11\}} \\ &= \frac{1/4 \times 1/2}{Pr\{Y = 11\}} \\ &= \frac{1/8}{Pr\{Y = 11\}}. \end{aligned}$$

Then for $Pr\{Y = 11\}$ *we have*

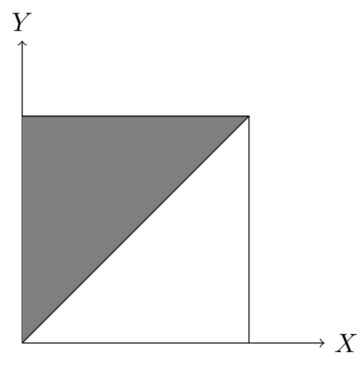
$$\begin{aligned} Pr\{Y = 11\} &= Pr\{X = 0\} \cdot Pr\{Y = 11|X = 0\} + Pr\{X = 1\} \cdot Pr\{Y = 11|X = 1\} \\ &= 1/2 \times 1/4 + 1/2 \times 1 \\ &= 5/8. \end{aligned}$$

So, finally we have

$$Pr\{X = 0|Y = 11\} = \frac{1/8}{5/8} = \frac{1}{5}.$$

Solution 3. (Conditional distribution)

The probability mass has been distributed uniformly on the upper triangular area according to the shape below:



(a) If X *and* Y *were independent then the distribution of* X *would not depend on* Y*. This is clearly not the case. In fact, the range of values taken by* X *is between* 0 *and* Y*.*

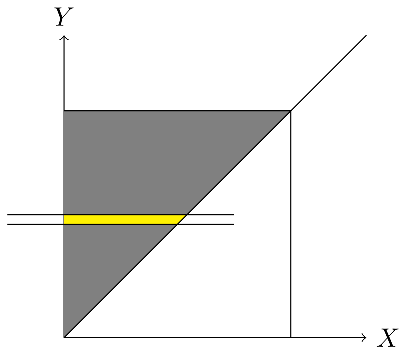
(b) The integral of $f_{X,Y}(x, y)$ *must be* 1*. Hence* $A \times \frac{1}{2} = 1$ *and so* $A = 2$*.*

(c) *We know that* $f_Y(y)\ dy = Pr\{y < Y < y + dy\}$, *but for a special* y *as can be seen from the figure below, this probability mass is equal to* A *times the area of a rectangle with length* y *and width* dy *when* $0 \le y \le 1$.

$$f_Y(y) = \begin{cases} 2y & 0 < y < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Or more formally

$$f_Y(y) = \int_0^1 f_{X,Y}(x,y)\ dx = \int_0^y 2\ dx = 2y.$$



(d) *Under the condition* $Y = y$, *the random variable* X *is uniformly distributed between* 0 *and* y *and so* $f(y) = \mathbb{E}[X|Y = y] = \frac{y}{2}$.

(e) $f(Y)$ *is a function of* Y *so it is a random variable and we can compute its expected value.*

$$\mathbb{E}[f(Y)] = \int_0^1 f(y)f_Y(y)\ dy = \int_0^1 y^2\ dy = \frac{1}{3}.$$

(f) *We compute* $\mathbb{E}[X]$ *using the definition.*

$$\mathbb{E}[X] = \iint x f_{X,Y}(x,y)\ dx\ dy = \int_0^1 \left[\int_0^y 2x\ dx\right]\ dy = \frac{1}{3},$$

and it is seen that $\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|Y]]$. *This result, which holds in general, is named the law of total expectation.*

Solution 4. (Playing darts)

(a) $X = ZX_1 + (1 - Z)X_2$.

(b) *Note that* $\mathbb{E}[X] = 0$, *because expectation is linear and* Z *is independent from* X_1 *and* X_2. *Thus,*

$$\begin{aligned} \mathrm{Var}(X) &= \mathbb{E}\left[X^2\right] - \mathbb{E}[X]^2 \\ &= \mathbb{E}\left[X^2\right] \\ &= \mathbb{E}\left[X^2|Z = 1\right]p + \mathbb{E}\left[X^2|Z = 0\right](1 - p) \\ &= p\sigma_1^2 + (1 - p)\sigma_2^2. \end{aligned}$$

X is not Gaussian. In fact X is not a linear combination of two Gaussians, it is rather a mixture *of two Gaussians. One can use the characteristic function to show rigorously that X is not a Gaussian, but this is outside the scope of this class.*

(c)

$$\begin{aligned}\mathbb{E}[S] &= p\int_{-\infty}^{\infty}|x|\,\frac{1}{\sigma_1\sqrt{2\pi}}e^{-\frac{x^2}{2\sigma_1^2}}\;dx + (1-p)\int_{-\infty}^{\infty}|x|\,\frac{1}{\sigma_2\sqrt{2\pi}}e^{-\frac{x^2}{2\sigma_2^2}}\;dx\\ &= 2p\int_0^{\infty}x\frac{1}{\sigma_1\sqrt{2\pi}}e^{-\frac{x^2}{2\sigma_1^2}}\;dx + 2(1-p)\int_0^{\infty}x\frac{1}{\sigma_2\sqrt{2\pi}}e^{-\frac{x^2}{2\sigma_2^2}}\;dx.\end{aligned}$$

With the change of variables $u_1 = \frac{x^2}{2\sigma_1^2}$ and $u_2 = \frac{x^2}{2\sigma_2^2}$, we obtain

$$\begin{aligned}\mathbb{E}[S] &= 2p\frac{\sigma_1}{\sqrt{2\pi}}\int_0^{\infty}e^{-u_1}\;du_1 + 2(1-p)\frac{\sigma_2}{\sqrt{2\pi}}\int_0^{\infty}e^{-u_2}\;du_2\\ &= \frac{2}{\sqrt{2\pi}}\left[p\sigma_1 + (1-p)\sigma_2\right].\end{aligned}$$

Solution 5. (Uncorrelated vs. independent random variables)

Note:

- *By definition, X and Y are uncorrelated if and only if*

$$0 = \operatorname{cov}(X,Y) = \mathbb{E}[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y].$$

  Hence $\operatorname{cov}(X,Y) = 0$ is equivalent to the the condition $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$.

- *X and Y are independent when $f_{XY} = f_X f_Y$.*

(a) Assume that the random variables X and Y are independent. Then

$$\begin{aligned}\mathbb{E}[XY] &= \iint xyf_{X,Y}(x,y)\;dx\;dy = \iint xyf_X(x)f_Y(y)\;dx\;dy\\ &= \int xf_X(x)\;dx\int yf_Y(y)\;dy = \mathbb{E}[X]\mathbb{E}[Y],\end{aligned}$$

where the second equality follows from the assumption that X and Y are independent. Hence, if X and Y are independent, they are also uncorrelated.

(b) X and Y are obviously dependent. For example, $X = 0$ implies $U = 0$ and $V = 0$. Hence it implies also $Y = 0$. The marginals of X and Y are

$$X = \begin{cases}0 & \textit{with prob. } \frac{1}{4},\\ 1 & \textit{with prob. } \frac{1}{2},\\ 2 & \textit{with prob. } \frac{1}{4},\end{cases}$$

$$Y = \begin{cases}0 & \textit{with prob. } \frac{1}{2},\\ 1 & \textit{with prob. } \frac{1}{2}.\end{cases}$$

The mean for X *is* $\mathbb{E}[X] = 1$ *and for* Y *it is* $\mathbb{E}[Y] = \frac{1}{2}$. *Finally, we have that*

$$\mathbb{E}[XY] = \left(\frac{1}{4} \times 0 \times 0\right) + \left(\frac{1}{4} \times 1 \times 1\right) + \left(\frac{1}{4} \times 1 \times 1\right) + \left(\frac{1}{4} \times 0 \times 2\right) = \frac{1}{2}.$$

From the above we obtain

$$\operatorname{cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = 0.$$

Therefore, we see that X *and* Y *are uncorrelated, even though they are dependent.*

Solution 6. (Monty Hall)

(a) $Pr\{A \text{ contains one million Swiss francs}\} = 1/3$.

(b) Observe that B *contains the money if and only if* A *does not contain the money, thus*

$$Pr\{B \text{ contains one million Swiss francs}\} = Pr\{A \text{ contains nothing}\} = 2/3.$$

(c) A reasonable person will choose B *since it has a larger probability of containing the money.*

Chapter 2

Receiver Design for Discrete-Time Observations: First Layer

Solution 1. (Hypothesis testing: Uniform and uniform)

(a) Let $l(y)$ be the number of 0's in the sequence y.

$$P_{Y|H}(y|0) = \frac{1}{2^{2k}}$$
$$P_{Y|H}(y|1) = \begin{cases} \frac{1}{\binom{2k}{k}}, & \textit{if } l = k \\ 0, & \textit{otherwise} \end{cases}$$

(b) The ML decision rule is:

$$P_{Y|H}(y|1) \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} P_{Y|H}(y|0)$$

Because $\frac{1}{\binom{2k}{k}} > \frac{1}{2^{2k}}$ for any value of k, the ML decision rule becomes

$$\hat{H} = \begin{cases} 0, & \textit{if } l(y) \neq k \\ 1, & \textit{if } l(y) = k. \end{cases}$$

The single number needed is $l(y)$, the number of 0's in the sequence y.

(c) The decision rule that minimizes the error probability is the MAP rule:

$$P_{Y|H}(y|1)P_H(1) \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} P_{Y|H}(y|0)P_H(0).$$

The MAP decision rule gives $\hat{H} = 0$ whenever $l(y) \neq k$. When $l(y) = k$:

$$\hat{H} = \begin{cases} 0, & \textit{if } \frac{\binom{2k}{k}}{2^{2k}} \geq \frac{P_H(1)}{P_H(0)} \\ 1, & \textit{otherwise.} \end{cases}$$

(d) *Trivial solution: If* $P_H(1) = 1$ *then* $\hat{H} = 1$ *for all* y *(In this case,* $l(y) = k$ *is guaranteed). Similarly, if* $P_H(0) = 1$ *then* $\hat{H} = 0$ *for all* y*.*

Now assume $P_H(1) \neq 1$*. Then there is a nonzero probability that* $l(y) \neq k$*, in which case* $\hat{H} = 0$*. The MAP decision rule always chooses* $\hat{H} = 0$ *if*

$$\frac{\binom{2k}{k}}{2^{2k}} \geq \frac{P_H(1)}{P_H(0)} \Longleftrightarrow P_H(0) \geq \frac{\frac{1}{\binom{2k}{k}}}{\frac{1}{\binom{2k}{k}} + \frac{1}{2^{2k}}}.$$

Solution 2. (The "Wetterfrosch")

(a) *A and B must be chosen such that the suggested functions become valid probability density functions, i.e.* $\int_0^1 f_{Y|H}(y|i)dy = 1$ *for* $i = 0, 1$*. This yields* $A = 4/3$ *and* $B = 6/7$*. (A quicker way is to draw the functions and find the area by looking at the drawings.)*

(b) *Let us first find the marginal of* Y*, i.e.*

$$f_Y(y) \quad = \quad f_{Y|H}(y|0)P_H(0) + f_{Y|H}(y|1)P_H(1) = C - Dy,$$

where we find $C = 23/21$ *and* $D = 4/21$*. Then, applying Bayes' rule gives*

$$P_{H|Y}(0|y) \quad = \quad \frac{f_{Y|H}(y|0)P_H(0)}{f_Y(y)} = \frac{1}{2}\frac{A - \frac{A}{2}y}{C - Dy} = \frac{1}{2}\frac{4/3 - 2/3y}{23/21 - 4/21y},$$

and similarly

$$P_{H|Y}(1|y) \quad = \quad \frac{f_{Y|H}(y|1)P_H(1)}{f_Y(y)} = \frac{1}{2}\frac{B + \frac{B}{3}y}{C - Dy} = \frac{1}{2}\frac{6/7 + 2/7y}{23/21 - 4/21y}.$$

(c) *The threshold is where the two a posteriori probabilities are equal,*

$$\frac{1}{2}\frac{4/3 - 2/3y}{23/21 - 4/21y} \quad = \quad \frac{1}{2}\frac{6/7 + 2/7y}{23/21 - 4/21y},$$

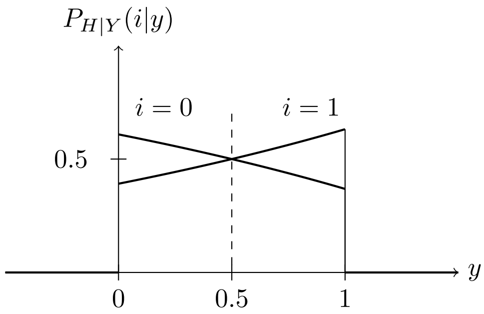
or equivalently,

$$4/3 - 2/3y \quad = \quad 6/7 + 2/7y.$$

The y *that satisfies this equation is our threshold* θ*, thus* $\theta = 0.5$*.*

(d) *The probability that we decide* $\hat{H}_\gamma(y) = 1$ *when in reality* $H = 0$ *is just the probability that* y *is* larger *than the threshold given that* $H = 0$*, which is*

$$\begin{aligned} Pr\{Y > \gamma | H = 0\} &= \int_\gamma^1 f_{Y|H}(y|0)dy = \int_\gamma^1 \left(A - \frac{A}{2}y\right) dy \\ &= A(1-\gamma) - \frac{A}{2}\frac{1-\gamma^2}{2} \\ &= \frac{4(1-\gamma)}{3} - \frac{1-\gamma^2}{3}. \end{aligned}$$



(e) *By analogy to the previous question,*

$$\begin{aligned} Pr\{Y<\gamma|H=1\} &= \int_0^\gamma f_{Y|H}(y|1)dy = \int_0^\gamma \left(B+\frac{B}{3}y\right)dy \\ &= B\gamma + \frac{B}{3}\frac{\gamma^2}{2} \\ &= \frac{6\gamma}{7}+\frac{\gamma^2}{7}. \end{aligned}$$

$$\begin{aligned} P_e(\gamma) &= Pr\{Y>\gamma|H=0\}P_H(0)+Pr\{Y<\gamma|H=1\}P_H(1) \\ &= \frac{1}{2}\left(\frac{4(1-\gamma)}{3}-\frac{1-\gamma^2}{3}+\frac{6\gamma}{7}+\frac{\gamma^2}{7}\right). \end{aligned}$$

For $\gamma=\theta=0.5$, we find $P_e(\theta)=0.44$.

(f) *To minimize P_e over γ, we take the derivative of P_e with respect to γ, i.e.*

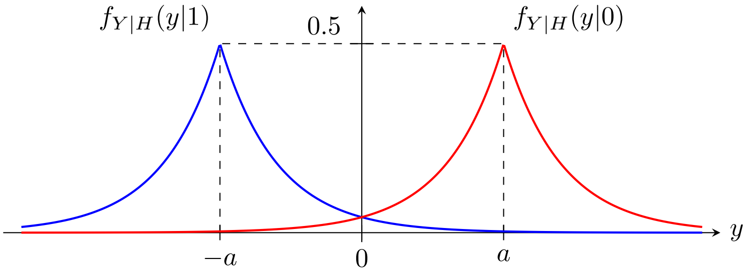
$$\frac{d}{d\gamma}P_e(\gamma) = \frac{1}{2}\left(-\frac{4}{3}+\frac{2\gamma}{3}+\frac{6}{7}+\frac{2\gamma}{7}\right).$$

Setting this equal to zero, we find $\gamma=0.5$. We observe that the value of γ which minimizes $P_e(\gamma)$ is equal to θ. This was expected, because the MAP decision rule minimizes the error probability.

Solution 3. (Hypothesis testing in Laplacian noise)

(a) *We find the following conditional densities for the observation Y under hypothesis $H=0$ and $H=1$, respectively:*

$$\begin{aligned} f_{Y|H}(y|0) &= \frac{1}{2}e^{-|y-a|} \\ f_{Y|H}(y|1) &= \frac{1}{2}e^{-|y+a|}. \end{aligned}$$



(b) *Because the hypotheses are equally likely, the MAP rule is the same as the ML rule. Therefore, the probability of error is minimized by the following decision rule:*

$$f_{Y|H}(y|1) \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} f_{Y|H}(y|0).$$

From the picture of $f_{Y|H}(y|0)$ and $f_{Y|H}(y|1)$, we see immediately that the ML decision rule decides for $H = 0$ when $y > 0$ and for $H = 1$ when $y < 0$.

(c)

$$\begin{aligned} P_e(0) &= Pr\{y < 0|H = 0\} = \int_{-\infty}^{0} f_{Y|H}(y|0)dy \\ &= \int_{-\infty}^{0} \frac{1}{2}e^{-|y-a|}dy = \int_{-\infty}^{0} \frac{1}{2}e^{(y-a)}dy \\ &= \frac{e^{-a}}{2}\, e^{y}|_{-\infty}^{0} = \frac{e^{-a}}{2}. \end{aligned}$$

By symmetry, we find that

$$P_e(1) = \frac{e^{-a}}{2},$$

and thus,

$$P_e = P_e(0)P_H(0) + P_e(1)P_H(1) = \frac{e^{-a}}{2}.$$

Solution 4. (Poisson parameter estimation)

(a) *We can write the MAP decision rule in the following way:*

$$\frac{P_{Y|H}(y|1)}{P_{Y|H}(y|0)} \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} \frac{P_H(0)}{P_H(1)}$$

Plugging in, we find

$$\frac{\lambda_1^y e^{-\lambda_1}}{\lambda_0^y e^{-\lambda_0}} \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} \frac{p_0}{1-p_0},$$

and then

$$\left(\frac{\lambda_1}{\lambda_0}\right)^y \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} \frac{p_0}{1-p_0} e^{\lambda_1-\lambda_0}$$

Taking logarithms on both sides does not change the direction of the inequalities, therefore

$$y \log\left(\frac{\lambda_1}{\lambda_0}\right) \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} \log\left(\frac{p_0}{1-p_0} e^{\lambda_1-\lambda_0}\right)$$

Attention: the term $\log(\lambda_1/\lambda_0)$ *can be negative, and if it is, then dividing by it involves changing the direction of the inequality.*

Suppose $\lambda_1 > \lambda_0$. *Then,* $\log(\lambda_1/\lambda_0) > 0$, *and the decision rule becomes*

$$y \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} \frac{\log\left(\frac{p_0}{1-p_0} e^{\lambda_1-\lambda_0}\right)}{\log\left(\frac{\lambda_1}{\lambda_0}\right)} \stackrel{\text{def}}{=} \theta$$

(b) We compute

$$\begin{aligned} P_e(0) &= Pr\left\{Y > \theta | H = 0\right\} = \sum_{y=\lceil\theta\rceil}^{\infty} P_{Y|H}(y|0) \\ &= 1 - \sum_{y=0}^{\lfloor\theta\rfloor} \frac{\lambda_0^y}{y!} e^{-\lambda_0}, \end{aligned}$$

and by analogy

$$\begin{aligned} P_e(1) &= Pr\left\{Y < \theta | H = 1\right\} = \sum_{y=0}^{\lfloor\theta\rfloor} P_{Y|H}(y|1) \\ &= \sum_{y=0}^{\lfloor\theta\rfloor} \frac{\lambda_1^y}{y!} e^{-\lambda_1} \end{aligned}$$

Thus, the probability of error becomes

$$P_e = p_0 \left(1 - \sum_{y=0}^{\lfloor\theta\rfloor} \frac{\lambda_0^y}{y!} e^{-\lambda_0}\right) + (1-p_0) \sum_{y=0}^{\lfloor\theta\rfloor} \frac{\lambda_1^y}{y!} e^{-\lambda_1}$$

Now, suppose that $\lambda_1 < \lambda_0$. *Then,* $\log(\lambda_1/\lambda_0) < 0$, *and we have to swap the inequality sign, thus*

$$y \underset{\hat{H}=1}{\overset{\hat{H}=0}{\gtrless}} \frac{\log\left(\frac{p_0}{1-p_0} e^{\lambda_1-\lambda_0}\right)}{\log\left(\frac{\lambda_1}{\lambda_0}\right)} \stackrel{\text{def}}{=} \theta$$

The rest of the analysis goes along the same lines, and finally, we obtain

$$P_e = p_0 \sum_{y=0}^{\lfloor\theta\rfloor} \frac{\lambda_0^y}{y!} e^{-\lambda_0} + (1-p_0)\left(1 - \sum_{y=0}^{\lfloor\theta\rfloor} \frac{\lambda_1^y}{y!} e^{-\lambda_1}\right)$$

The case $\lambda_0 = \lambda_1$ *yields* $\log(\lambda_1/\lambda_0) = 0$*, so the decision rule becomes* $0 \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} \theta$*, regardless of* y*. Thus, we can exclude the case* $\lambda_0 = \lambda_1$ *from our discussion.*

(c) Here, we are in the case $\lambda_1 > \lambda_0$*, and we find* $\theta \approx 4.54$*. We thus evaluate*

$$P_e = \frac{1}{3}\left(1 - \sum_{y=0}^{4} \frac{2^y}{y!} e^{-2}\right) + \frac{2}{3} \sum_{y=0}^{4} \left(\frac{10^y}{y!} e^{-10}\right) \approx 0.03705$$

(d) We find $\theta \approx 7.5163$

$$P_e = \frac{1}{3}\left(1 - \sum_{y=0}^{7} \frac{2^y}{y!} e^{-2}\right) + \frac{2}{3} \sum_{y=0}^{7} \left(\frac{20^y}{y!} e^{-20}\right) \approx 0.000885$$

The two Poisson distributions are much better separated than in (c); therefore, it becomes considerably easier to distinguish them based on one single observation y*.*

Solution 5. (Lie detector)

(a) Let $H \in \{T, L\}$*.*

$$\begin{aligned} H &= T \text{ (telling truth):} \quad f_{Y|H}(y|T) = \alpha e^{-\alpha y},\ y \geq 0 \\ H &= L \text{ (telling lie):} \quad f_{Y|H}(y|L) = \beta e^{-\beta y},\ y \geq 0. \end{aligned}$$

The MAP decision rule is

$$p\beta e^{-\beta y} \underset{\hat{H}=T}{\overset{\hat{H}=L}{\gtrless}} (1-p)\alpha e^{-\alpha y}.$$

After taking the logarithm, we obtain

$$-\beta y + \ln(p\beta) \underset{\hat{H}=T}{\overset{\hat{H}=L}{\gtrless}} -\alpha y + \ln((1-p)\alpha).$$

Or, equivalently

$$y \underset{\hat{H}=L}{\overset{\hat{H}=T}{\gtrless}} \frac{1}{\alpha - \beta} \ln\left[\frac{\alpha}{\beta} \frac{(1-p)}{p}\right] = \theta$$

(b)

$$P_{L|T} = \int_0^\theta \alpha e^{-\alpha y} dy = 1 - e^{-\alpha\theta}.$$

(c)

$$P_{T|L} = \int_\theta^\infty \beta e^{-\beta y} dy = e^{-\beta\theta}.$$

(d)

$$\begin{aligned} H = T: \quad & f_{Y|H}(y|T) = \alpha^n e^{-\alpha(y_1+\ldots+y_n)} = \alpha^n e^{-\alpha z} \\ H = L: \quad & f_{Y|H}(y|L) = \beta^n e^{-\beta(y_1+\ldots+y_n)} = \beta^n e^{-\beta z}, \end{aligned}$$

where Y is the random vector $(Y_1, ..., Y_n)$ and where $z = \sum_{i=1}^n y_i$. With this new definition, the test becomes $z \underset{\hat{H}=L}{\overset{\hat{H}=T}{\gtrless}} \theta$, with the new threshold $\theta = \frac{1}{\alpha-\beta} \ln\left[\left(\frac{\alpha}{\beta}\right)^n \frac{(1-p)}{p}\right]$.

$$P_{L|T} = \int_0^\theta f_{Z|H}(z|T) dz,$$

where $Z = \sum_{i=1}^n Y_i$ and

$$f_{Z|H}(z|T) = \frac{\alpha^n}{(n-1)!} z^{(n-1)} e^{-\alpha z}.$$

This is the density of the Erlang distribution. Putting things together, we get

$$P_{L|T} = \int_0^\theta \frac{\alpha^n}{(n-1)!} z^{(n-1)} e^{-\alpha z} dz.$$

Solution 6. (Fault detector)

$H = 1$ is the hypothesis that the box works properly and $H = 0$ the hypothesis that the box fails.

(a) *The MAP test is*

$$\frac{f_{X|H}(x|1)}{P_H(0)} \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} \frac{f_{X|H}(x|0)}{P_H(1)}.$$

If $l(x)$ is the number of zeros in the sequence x,

$$\begin{aligned} f_{X|H}(x|1) &= \begin{cases} p^{16-l}(1-p)^l, & \text{if } 0 \le l \le 16 \\ 0, & \text{otherwise} \end{cases} \\ f_{X|H}(x|0) &= \frac{1}{2^{16}} \end{aligned}$$

(b) *By substituting $l = 8$, $p = 0.25$, $P_H(0) = \frac{1}{1025}$ and $P_H(1) = \frac{1024}{1025}$ in the decision rule, we obtain*

$$\frac{3^8}{2^6} \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} 1,$$

therefore the hypothesis is $\hat{H} = 1$ — the box works properly.

Solution 7. (Multiple choice exam)

(a) *We have a binary hypothesis testing problem: The hypothesis H is the answer you will select, and your decision will be based on the observation of $\hat{H}_L$ and $\hat{H}_R$. Let H take value 1 if answer 1 is chosen, and value 2 if answer 2 is chosen. In this case, we can write the MAP decision rule as follows:*

$$Pr\left\{H = 1|\hat{H}_L = 1, \hat{H}_R = 2\right\} \underset{\hat{H}=2}{\overset{\hat{H}=1}{\gtrless}} Pr\left\{H = 2|\hat{H}_L = 1, \hat{H}_R = 2\right\}$$

From the problem setting we know the priors $Pr\{H = 1\}$ and $Pr\{H = 2\}$; we can also determine the conditional probabilities $Pr\left\{\hat{H}_L = 1|H = 1\right\}$, $Pr\left\{\hat{H}_L = 1|H = 2\right\}$, $Pr\left\{\hat{H}_R = 2|H = 1\right\}$ and $Pr\left\{\hat{H}_R = 2|H = 2\right\}$ (we have $Pr\left\{\hat{H}_L = 1|H = 1\right\} = 0.9$ and $Pr\left\{\hat{H}_L = 1|H = 2\right\} = 0.1$). Introducing these quantities and using the Bayes rule we can formulate the MAP decision rule as

$$\frac{Pr\left\{\hat{H}_L = 1, \hat{H}_R = 2|H = 1\right\}Pr\{H = 1\}}{Pr\left\{\hat{H}_L = 1, \hat{H}_R = 2\right\}} \underset{\hat{H}=2}{\overset{\hat{H}=1}{\gtrless}} \frac{Pr\left\{\hat{H}_L = 1, \hat{H}_R = 2|H = 2\right\}Pr\{H = 2\}}{Pr\left\{\hat{H}_L = 1, \hat{H}_R = 2\right\}}$$

Now, assuming that the event $\{\hat{H}_L = 1\}$ is independent of the event $\{\hat{H}_R = 2\}$ and simplifying the expression, we obtain

$$Pr\left\{\hat{H}_L = 1|H = 1\right\}Pr\left\{\hat{H}_R = 2|H = 1\right\}Pr\{H = 1\} \underset{\hat{H}=2}{\overset{\hat{H}=1}{\gtrless}}$$
$$Pr\left\{\hat{H}_L = 1|H = 2\right\}Pr\left\{\hat{H}_R = 2|H = 2\right\}Pr\{H = 2\},$$

which is our final decision rule.

(b) *Evaluating the previous decision rule, we have*

$$0.9 \times 0.3 \times 0.25 \underset{\hat{H}=2}{\overset{\hat{H}=1}{\gtrless}} 0.1 \times 0.7 \times 0.75,$$

which gives

$$0.0675 \underset{\hat{H}=2}{\overset{\hat{H}=1}{\gtrless}} 0.0525$$

This implies that the answer $\hat{H}$ is equal to 1.

Solution 8. (MAP decoding rule: Alternative derivation)

(a) *The probability of error can be written as*

$$\begin{aligned} P_e &= P_H(0)\, Pr\{Y \in \mathcal{R}_1 | H = 0\} + P_H(1)\, Pr\{Y \in \mathcal{R}_0 | H = 1\} \\ &= P_H(0) \int_{\mathcal{R}_1} f_{Y|H}(y|0) dy + P_H(1) \int_{\mathcal{R}_0} f_{Y|H}(y|1) dy \\ &= P_H(0) \int_{\mathcal{R}_1} f_{Y|H}(y|0) dy + P_H(1) \left(1 - \int_{\mathcal{R}_1} f_{Y|H}(y|1) dy \right) \\ &= P_H(1) + \int_{\mathcal{R}_1} \left(P_H(0) f_{Y|H}(y|0) - P_H(1) f_{Y|H}(y|1) \right) dy, \end{aligned} \tag{2.1}$$

where the third equality follows from the hint

$$\int_{\mathcal{R}_0 \bigcup \mathcal{R}_1} f_{Y|H}(y|1) dy = \int_{\mathcal{R}_0} f_{Y|H}(y|1) dy + \int_{\mathcal{R}_1} f_{Y|H}(y|1) dy = 1.$$

(b) *Note that P_e is smallest if the second term $\int_{\mathcal{R}_1} \left(P_H(0) f_{Y|H}(y|0) - P_H(1) f_{Y|H}(y|1) \right) dy$ in (2.1) is made as negative as possible. Note that the first term $P_H(1)$ in (2.1) is fixed and does not depend on our choices for $\mathcal{R}_0$ and $\mathcal{R}_1$. The second term can be minimized if we collect in $\mathcal{R}_1$ all $y \in \mathbb{R}$ that yield negative contribution, i.e. $y \in \mathcal{R}_1$ iff $P_H(0) f_{Y|H}(y|0) - P_H(1) f_{Y|H}(y|1) < 0$.*

Note: How does this approach compare to the one from the book? Conditioning is one of the most important tricks to make progress in computing a probability. There are two random variables involved, namely H and Y. In the notes we have conditioned on $Y = y$. Here we are conditioning on $H = i$.

Solution 9. (Independent and identically distributed vs. first-order Markov)

An explanation regarding the title of this problem: independent and identically distributed means that all $Y_1, \ldots, Y_k$ have the same probability mass function and are independent of each other. First-order Markov *means that $Y_1, \ldots, Y_k$ depend on each other in a particular way: the probability mass function Y_i depends on the value of Y_{i-1}, but given the value of Y_{i-1}, it is independent of $Y_1, \ldots, Y_{i-2}$. Thus, in this problem, we observe a binary sequence, and we want to know whether it has been generated by an* i.i.d. *(independent and identically distributed) source or by a first-order Markov source.*

(a) *Since the two hypotheses are equally likely, we find*

$$\frac{P_{Y|H}(y|1)}{P_{Y|H}(y|0)} \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} \frac{P_H(0)}{P_H(1)} = 1.$$

Plugging in, we obtain

$$\frac{1/2 \cdot (1/4)^l \cdot (3/4)^{k-l-1}}{(1/2)^k} \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} 1,$$

where l is the number of times the observed sequence changes either from zero to one or from one to zero, i.e. the number of transitions in the observed sequence.

(b) *The sufficient statistic here is simply the number of transitions l; this entirely specifies the likelihood ratio.*

(c) *In this case, the number of non-transitions is $(k-l) = s$, and the log-likelihood ratio becomes*

$$\begin{aligned}\log \frac{1/2 \cdot (1/4)^{k-s} \cdot (3/4)^{s-1}}{(1/2)^k} &= \log \frac{(1/4)^{k-s} \cdot (3/4)^{s-1}}{(1/2)^{k-1}} \\ &= (k-s)\log(1/4) + (s-1)\log(3/4) - (k-1)\log(1/2) \\ &= s \log \frac{3/4}{1/4} + k \log \frac{1/4}{1/2} + \log \frac{1/2}{3/4} \\ &= s \log 3 + k \log 1/2 + \log 2/3.\end{aligned}$$

Thus, in terms of this log-likelihood ratio, the decision rule becomes

$$s \log 3 + k \log 1/2 + \log 2/3 \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} 0.$$

That is, we have to find the smallest possible s such that this expression becomes larger or equal to zero. Therefore,

$$s \geq \left\lceil \frac{k \log 1/2 + \log 2/3}{\log 1/3} \right\rceil.$$

Solution 10. (SIMO channel with Laplacian noise)

(a) *Let the two hypotheses be $H = 0$ and $H = 1$ when c_0 and c_1 are transmitted, respectively. The ML decision rule is*

$$f_{Y_1 Y_2|H}(y_1, y_2|1) \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} f_{Y_1 Y_2|H}(y_1, y_2|0).$$

Because Z_1 and Z_2 are independent, we can write

$$\frac{1}{2}e^{-|y_1-1|}\frac{1}{2}e^{-|y_2-1|} \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} \frac{1}{2}e^{-|y_1+1|}\frac{1}{2}e^{-|y_2+1|},$$

and, after taking the logarithm,

$$|y_1+1| + |y_2+1| \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} |y_1-1| + |y_2-1|.$$

(b) *Because the hypotheses are equally likely and Z_1 and Z_2 have the same distribution, the decision region for $\hat{H} = 0$ contains the points closer to $(-1,-1)$ and the decision region for $\hat{H} = 1$ contains the points closer to $(1, 1)$. For this problem, the distance between the*

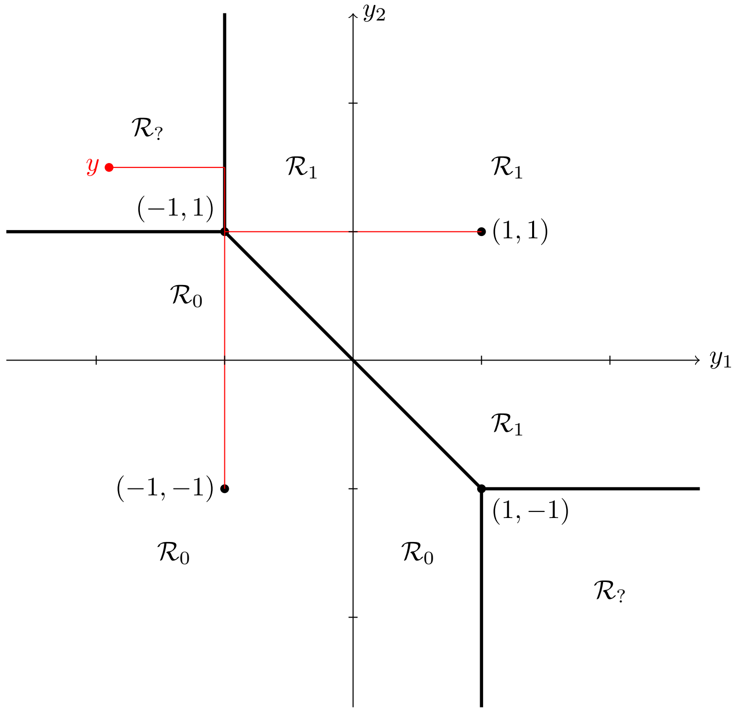


Figure 2.1: Decision regions

points (y_{11}, y_{12}) *and* (y_{21}, y_{22}) *is the Manhattan distance,* $|y_{11} - y_{21}| + |y_{12} - y_{22}|$*, and not the Euclidian distance.*

Let us first consider the points above the line $y_2 = -y_1$ *from Figure 2.1. It is easy to notice that the points in the positive quadrant are closer to* $(1,1)$ *than to* $(-1,-1)$*, therefore they belong to* $\mathcal{R}_1$ $(\hat{H} = 1)$*. This is also true if* $\{(y_1 \geq 0) \cap (y_2 \in (-1,0))\}$*, or if* $\{(y_2 \geq 0) \cap (y_1 \in (-1,0))\}$*.*

Similar reasoning can be applied to the points below the diagonal to determine $\mathcal{R}_0$*.*

The points for which $\{(y_1 \leq -1) \cap (y_2 \geq 1)\}$ *or* $\{(y_1 \geq 1) \cap (y_2 \leq -1)\}$ *are equally distanced to* $(-1,-1)$ *and* $(1,1)$*, therefore they can belong to either* $\mathcal{R}_0$ *or* $\mathcal{R}_1$ *with the same probability. This region is named* $\mathcal{R}_?$*.*

(c) The two hypotheses are equally probable for the region $\mathcal{R}_?$*. Therefore, we can split this region in any way between the decision regions and have the same error probability. Because* $\mathcal{R}_1$ *is included in the region for which* $y_2 > -y_1$ *and* $\mathcal{R}_0$ *does not intersect the region for which* $y_2 > -y_1$*, the error probability is minimized by deciding* $\hat{H} = 1$ *if* $(y_1 + y_2) > 0$*.*

(d)

$$\begin{aligned} P_e(0) &= Pr\{Y_1 + Y_2 > 0 | H = 0\} \\ &= Pr\{Z_1 + Z_2 - 2 > 0\} \\ &= \int_2^\infty \frac{e^{-w}}{4}(1+w)\,dw \\ &= \frac{-e^{-w}}{4}(w+2)\Big|_2^\infty = e^{-2}. \end{aligned}$$

By symmetry, and considering that the messages are equally likely, $P_e(0) = P_e(1) = P_e$.

Solution 11. (Q-Function on regions)

(a) One can see that the event $\{X \in Region\}$ *only depends on the first component* X_1. *Hence, we have*

$$\begin{aligned} Pr\{X \in Region\} &= Pr\{(X_1 \geq -2) \cap (X_1 \leq 1)\} \\ &= 1 - Pr\{(X_1 < -2) \cup (X_1 > 1)\} \\ &= 1 - Q\left(\frac{2}{\sigma}\right) - Q\left(\frac{1}{\sigma}\right), \end{aligned}$$

where the last equality is true because $\{X_1 < -2\}$ *and* $\{X_1 > 1\}$ *are disjoint events.*

(b) Because X_1 *and* X_2 *are independent and have the same variance, rotating the vector* X *by any angle around the origin does not change its distribution. Equivalently, we can rotate the square region in Figure (b) by* 45 *degrees, and the probability of* X *being in the rotated region is the same as for the original region. The new region is a square whose edges are parallel to the axes of the coordinate system. The points where the edges of the square intersect the axes are* $(\sqrt{2}, 0)$, $(-\sqrt{2}, 0)$, $(0, \sqrt{2})$ *and* $(0, -\sqrt{2})$. *Hence,*

$$\begin{aligned} Pr\{X \in Region\} &= Pr\left\{(-\sqrt{2} \leq X_1 \leq \sqrt{2}) \cap (-\sqrt{2} \leq X_2 \leq \sqrt{2})\right\} \\ &\overset{(1)}{=} Pr\left\{-\sqrt{2} \leq X_1 \leq \sqrt{2}\right\}^2 \\ &= \left[1 - Pr\left\{(X_1 < -\sqrt{2}) \cup (X_1 > \sqrt{2})\right\}\right]^2 \\ &= \left[1 - 2Q\left(\frac{\sqrt{2}}{\sigma}\right)\right]^2, \end{aligned}$$

where (1) *holds because* X_1 *and* X_2 *are independent and identically distributed.*

(c) We solve this part in three different ways:

(i) First Solution: *As in the previous part, we can rotate* X *such that one of its components, say* X_1, *is perpendicular to the straight line that delimits the shaded region. Then, we need to know the shortest distance* d *of that line to the origin (the length*

of a segment that starts at $(0,0)$ and is perpendicular to the line). Using standard trigonometric techniques, one finds that this length is $d = \frac{2}{\sqrt{5}}$. Then, it follows that

$$\begin{aligned} Pr\left\{X \in Region\right\} &= Pr\left\{X_1 \geq \frac{2}{\sqrt{5}}\right\} \\ &= Q\left(\frac{2}{\sqrt{5}\sigma}\right). \end{aligned}$$

(ii) Second Solution: *We are looking for the probability that $X_2 \geq 1 - \frac{1}{2}X_1$, i.e., the probability that $Z \triangleq X_2 + \frac{1}{2}X_1 - 1 \geq 0$. But $Z \sim \mathcal{N}(-1, \frac{5}{4}\sigma^2)$. Hence, $Pr\left\{X \in Region\right\} = Pr\left\{Z \geq 0\right\} = Q\left(\frac{2}{\sqrt{5}\sigma}\right)$.*

(iii) Third Solution: *We project $X = (X_1, X_2)^{\mathsf{T}}$ to the vector perpendicular to the line that delimits the shaded region. The length of the projection is $Z \sim \mathcal{N}(0, \sigma^2)$. The sought probability is $Pr\left\{Z \geq d\right\} = Q\left(\frac{d}{\sigma}\right) = Q\left(\frac{2}{\sqrt{5}\sigma}\right)$, where d is the distance from the delimiting line to the origin.*

Solution 12. (Properties of the Q function)

(a)

$$\begin{aligned} F_Z(z) =& Pr\left\{Z \leq z\right\} = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx \\ =& \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx - \int_{z}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx \\ =& 1 - Q(z). \end{aligned}$$

(b)

$$Q(0) = \frac{1}{2},$$

because we have the same area on both sides of the Gaussian bell.

$$Q(-\infty) = Pr\left\{Z \geq -\infty\right\} = 1.$$

$$Q(\infty) = Pr\left\{Z \geq \infty\right\} = 0.$$

(c) If we add $Q(-x)$ and $Q(x)$, we get 1. Refer to Figure 2.2.

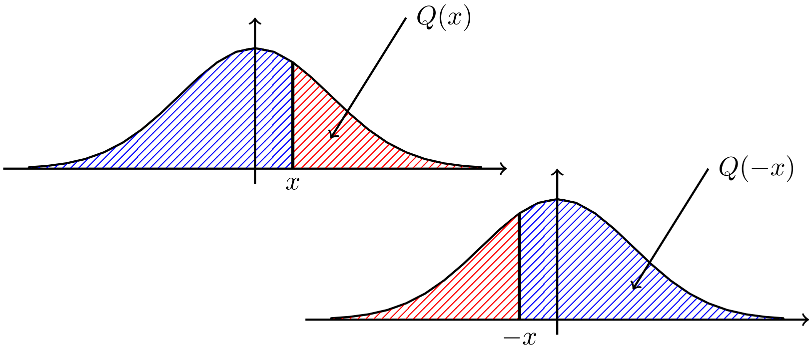


Figure 2.2: Identically shaded portions have the same area

(d) Consider the following integration by parts:

$$\begin{aligned} Q(\alpha) &= \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\infty} e^{-\frac{x^2}{2}}\, dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\infty} \frac{1}{x} x e^{-\frac{x^2}{2}}\, dx \\ &= \frac{1}{\sqrt{2\pi}} \left(-\frac{e^{-\frac{x^2}{2}}}{x} \Bigg|_{\alpha}^{\infty} - \int_{\alpha}^{\infty} \frac{1}{x^2} e^{-\frac{x^2}{2}}\, dx \right) \\ &= \frac{1}{\sqrt{2\pi}} \left(\frac{e^{-\frac{\alpha^2}{2}}}{\alpha} - \int_{\alpha}^{\infty} \frac{1}{x^2} e^{-\frac{x^2}{2}}\, dx \right). \end{aligned}$$

Since the integral on the last line is non-negative, we get an upper bound if we neglect that term. That is the upper bound we are looking for. To obtain the lower bound, we increase the integral by substituting $\frac{1}{\alpha^2}$ *for* $\frac{1}{x^2}$ *and then use the upper bound just derived. This gives*

$$\begin{aligned} Q(\alpha) &\geq \frac{1}{\sqrt{2\pi}} \frac{e^{-\frac{\alpha^2}{2}}}{\alpha} - \frac{1}{\alpha^2} \int_{\alpha}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx \\ &\geq \frac{1}{\sqrt{2\pi}} \frac{e^{-\frac{\alpha^2}{2}}}{\alpha} - \frac{1}{\alpha^2} \frac{1}{\sqrt{2\pi}} \frac{e^{-\frac{\alpha^2}{2}}}{\alpha} \\ &= \frac{1}{\sqrt{2\pi}} \frac{e^{-\frac{\alpha^2}{2}}}{\alpha} \left(1 - \frac{1}{\alpha^2} \right). \end{aligned}$$

Note: The bound that we have proved is the well-known lower bound to $Q(x)$*. A slightly* *better but less known lower bound can be obtained the following way:*

$$\begin{aligned} Q(\alpha) &\geq \frac{1}{\sqrt{2\pi}} \frac{e^{-\frac{\alpha^2}{2}}}{\alpha} - \frac{1}{\alpha^2} \int_{\alpha}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx \\ &= \frac{1}{\sqrt{2\pi}} \frac{e^{-\frac{\alpha^2}{2}}}{\alpha} - \frac{1}{\alpha^2} Q(\alpha). \end{aligned}$$

Therefore,

$$Q(\alpha) \geq \frac{1}{\sqrt{2\pi}} e^{-\frac{\alpha^2}{2}} \frac{\alpha^2}{\alpha^2+1}.$$

Solution 13. (16-PAM vs. 16-QAM)

(a) 16-PAM. *Denote the additive white Gaussian noise process by Z. Thus, Z is zero-mean Gaussian of variance σ^2, and the observation Y is also Gaussian of variance σ^2, but with mean corresponding to the particular signal point that is being transmitted. If H is the hypothesis and we label the signal points from left to right by $1, \ldots, 16$, then*

$$\begin{aligned} P_e(1) &= Pr\left\{Y \geq -7a | H = 1\right\} = Pr\left\{Z \geq \frac{a}{2}\right\} \\ &= Pr\left\{\frac{Z}{\sigma} \geq \frac{a}{2\sigma}\right\} = Q\left(\frac{a}{2\sigma}\right). \end{aligned}$$

By symmetry, $P_e(1) = P_e(16)$.

Moreover,

$$\begin{aligned} P_e(2) &= Pr\left\{(Y \leq -7a) \cup (Y \geq -6a) | H = 2\right\} \\ &= Pr\left\{\left(Z \leq -\frac{a}{2}\right) \cup \left(Z \geq \frac{a}{2}\right)\right\} = 2Pr\left\{Z \geq \frac{a}{2}\right\} \\ &= 2Q\left(\frac{a}{2\sigma}\right). \end{aligned}$$

Again, by symmetry, $P_e(i) = P_e(2)$, for $i = 3, \ldots, 15$. Putting things together, we obtain

$$\begin{aligned} P_e &= \sum_{i=1}^{16} P_H(i) P_e(i) = \sum_{i=1}^{16} \frac{1}{16} P_e(i) \\ &= \frac{1}{16}\left(2 \cdot Q\left(\frac{a}{2\sigma}\right) + 14 \cdot 2Q\left(\frac{a}{2\sigma}\right)\right) \\ &= \frac{15}{8} Q\left(\frac{a}{2\sigma}\right). \end{aligned}$$

16-QAM. *Denote the additive white Gaussian noise process in the x_1-direction by Z_1 and in the x_2-direction by Z_2. In our setup, both Z_1 and Z_2 are zero-mean Gaussian of variance σ^2. Label the signal points from left to right, top to bottom by $1, \ldots, 16$. Then, for the four corner points, we find*

$$P_e(1) = Pr\left\{(Y_1 \geq -b) \cup (Y_2 \leq b) | H = 1\right\}.$$

Notice that $\{Y_1 \geq -b\}$ *and* $\{Y_2 \leq b\}$ *are not disjoint events, so*

$$P_e(1) = Pr\{Y_1 \geq -b|H=1\} + Pr\{Y_2 \leq b|H=1\} - Pr\{(Y_1 \geq -b) \cap (Y_2 \leq b)|H=1\}.$$

An alternative (and somewhat simpler) approach is to compute the probability of the correct decision, $P_c(1)$*, and then determine* $P_e(1) = 1 - P_c(1)$*. Thus,*

$$\begin{aligned}
P_c(1) &= Pr\{(Y_1 \leq -b) \cap (Y_2 \geq b)|H=1\} \\
&= Pr\{(Y_1 \leq -b|H=1\} Pr\{Y_2 \geq b|H=1\} \\
&= Pr\left\{Z_1 \leq \frac{b}{2}\right\} Pr\left\{Z_2 \geq -\frac{b}{2}\right\} \\
&= \left(1 - Q\left(\frac{b}{2\sigma}\right)\right) Q\left(-\frac{b}{2\sigma}\right) \\
&= \left(1 - Q\left(\frac{b}{2\sigma}\right)\right)^2.
\end{aligned}$$

For the points on the edges (i.e. numbers $2, 3, 5, 8, 9, 12, 14, 15$*), we find similarly*

$$\begin{aligned}
P_c(2) &= Pr\{(-b \leq Y_1 \leq 0) \cap (Y_2 \geq b)|H=2\} \\
&= Pr\left\{-\frac{b}{2} \leq Z_1 \leq \frac{b}{2}\right\} Pr\left\{Z_2 \geq -\frac{b}{2}\right\},
\end{aligned}$$

where

$$\begin{aligned}
Pr\left\{-\frac{b}{2} \leq Z_1 \leq \frac{b}{2}\right\} &= 1 - Pr\left\{\left(Z_1 \leq -\frac{b}{2}\right) \cup \left(Z_1 \geq \frac{b}{2}\right)\right\} \\
&= 1 - 2Pr\left\{Z_1 \geq \frac{b}{2}\right\} \\
&= 1 - 2Q\left(\frac{b}{2\sigma}\right),
\end{aligned}$$

thus,

$$P_c(2) = \left(1 - 2Q\left(\frac{b}{2\sigma}\right)\right)\left(1 - Q\left(\frac{b}{2\sigma}\right)\right).$$

Finally, for the four points in the middle, we obtain

$$\begin{aligned}
P_c(6) &= Pr\{(-b \leq Y_1 \leq 0) \cap (0 \leq Y_2 \leq b)|H=6\} \\
&= Pr\left\{-\frac{b}{2} \leq Z_1 \leq \frac{b}{2}\right\} Pr\left\{-\frac{b}{2} \leq Z_2 \leq \frac{b}{2}\right\} \\
&= \left(1 - 2Q\left(\frac{b}{2\sigma}\right)\right)^2.
\end{aligned}$$

Putting things together, we find

$$\begin{aligned}
P_c &= \sum_{i=1}^{16} P_H(i) P_c(i) = \sum_{i=1}^{16} \frac{1}{16} P_c(i) \\
&= \frac{1}{16}\left[4\left(1 - Q\left(\frac{b}{2\sigma}\right)\right)^2 + 8\left(1 - Q\left(\frac{b}{2\sigma}\right)\right)\left(1 - 2Q\left(\frac{b}{2\sigma}\right)\right)\right. \\
&\qquad \left. + 4\left(1 - 2Q\left(\frac{b}{2\sigma}\right)\right)^2\right] \\
&= 1 - 3Q\left(\frac{b}{2\sigma}\right) + \frac{9}{4}\left(Q\left(\frac{b}{2\sigma}\right)\right)^2.
\end{aligned}$$

From here, we find $P_e = 1 - P_c$, thus

$$P_e = 3Q\left(\frac{b}{2\sigma}\right) - \frac{9}{4}\left(Q\left(\frac{b}{2\sigma}\right)\right)^2.$$

(b) 16-PAM. *By symmetry, we only consider the positive signals to find*

$$\begin{aligned}
\mathcal{E} &= 2\frac{1}{16}\left(\left(\frac{a}{2}\right)^2 + \left(\frac{3a}{2}\right)^2 + \ldots + \left(\frac{15a}{2}\right)^2\right) \\
&= \frac{a^2}{32}\left(1 + 3^2 + 5^2 + \ldots + 15^2\right) = \frac{85a^2}{4}.
\end{aligned}$$

16-QAM. *By symmetry, we only consider the first quadrant to find*

$$\begin{aligned}
\mathcal{E} &= 4\frac{1}{16}\left(\left[\left(\frac{b}{2}\right)^2 + \left(\frac{b}{2}\right)^2\right] + \left[\left(\frac{3b}{2}\right)^2 + \left(\frac{3b}{2}\right)^2\right] + 2\left[\left(\frac{b}{2}\right)^2 + \left(\frac{3b}{2}\right)^2\right]\right) \\
&= \frac{b^2}{16}\left(1 + 1 + 9 + 9 + 2(1 + 9)\right) = \frac{5b^2}{2}.
\end{aligned}$$

(c) 16-PAM. *We find $a/2 = \sqrt{\mathcal{E}/85}$, thus*

$$P_e = \frac{15}{8} Q\left(\sqrt{\frac{\mathcal{E}}{85\sigma^2}}\right).$$

16-QAM. *We find $b/2 = \sqrt{\mathcal{E}/10}$, thus*

$$P_e = 3Q\left(\sqrt{\frac{\mathcal{E}}{10\sigma^2}}\right) - \frac{9}{4} Q^2\left(\sqrt{\frac{\mathcal{E}}{10\sigma^2}}\right).$$

Solution 14. (QPSK decision regions)

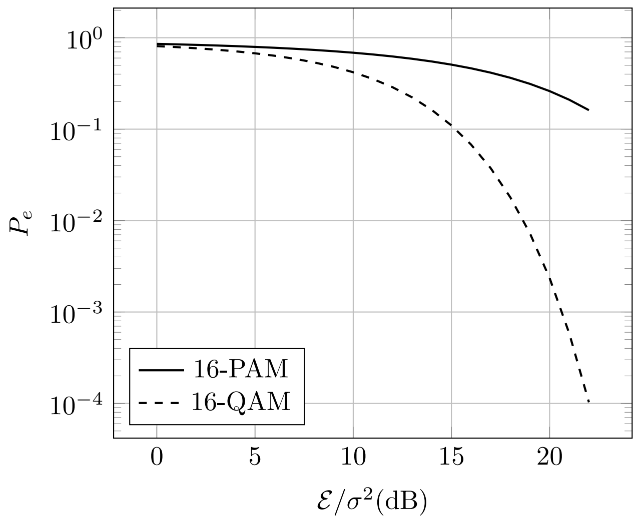


Figure 2.3: Error probability vs. average signal energy for 16-PAM (solid) and 16-QAM (dashed)

(a) *If $P_H(i)$ is the same for all i, then the decision regions are given in Figure 2.4.*

(b) *The decision boundary between two hypotheses $\hat{H} = i$ and $\hat{H} = j$ is given by*

$$||Y - c_i||^2 - ||Y - c_j||^2 = 2\sigma^2 \ln \frac{P_H(i)}{P_H(j)}.$$

This is an affine plane perpendicular to the segment that joins c_i to c_j. If $P_H(i) > P_H(j)$, then the affine plane is shifted away from c_i, to increase $\mathcal{R}_i$. The decision regions for this case are given in Figure 2.5.

(c) *Define a new observarion $\tilde{Y} = (Y_1, Y_2/2)$. The new observation $\tilde{Y}$ is a sufficient statistic because we can determine Y from $\tilde{Y}$. Thus the receiver observes $\tilde{Y} = \tilde{c}_i + \tilde{Z}$, where $\tilde{c}_i = (c_{i1}, c_{i2}/2)$ and $\tilde{Z} = (Z_1, Z_2/2)$. Note that in this new setup we have $\tilde{c}_0 = c_0$, $\tilde{c}_1 = c_1/2$, $\tilde{c}_2 = c_2$, $\tilde{c}_3 = c_3/2$ and $\tilde{Z} \sim \mathcal{N}(0, \sigma^2 I_2)$. The decision regions for this case are given in Figure 2.6.*

Solution 15. (Antenna array)

Since Z_1 and Z_2 don't have the same variance, the noise is not white, and so we cannot directly apply the results for discrete time AWGN channels which we are familiar with. A smart way to solve this problem is to apply a transformation on $Y = (Y_1, Y_2)^\mathsf{T}$ to get a sufficient statistic $\tilde{Y} = (\tilde{Y}_1, \tilde{Y}_2)^\mathsf{T}$ that can be seen as the output of a discrete time AWGN channel.

Since Z_1 and Z_2 are independent and have variance σ_1^2 and σ_2^2 respectively, $\frac{Z_1}{\sigma_1}$ and $\frac{Z_2}{\sigma_2}$ are independent and have variance 1. Thus, $\left(\frac{Z_1}{\sigma_1}, \frac{Z_2}{\sigma_2}\right)^\mathsf{T} \sim \mathcal{N}(0, I_2)$ which is a white noise of power

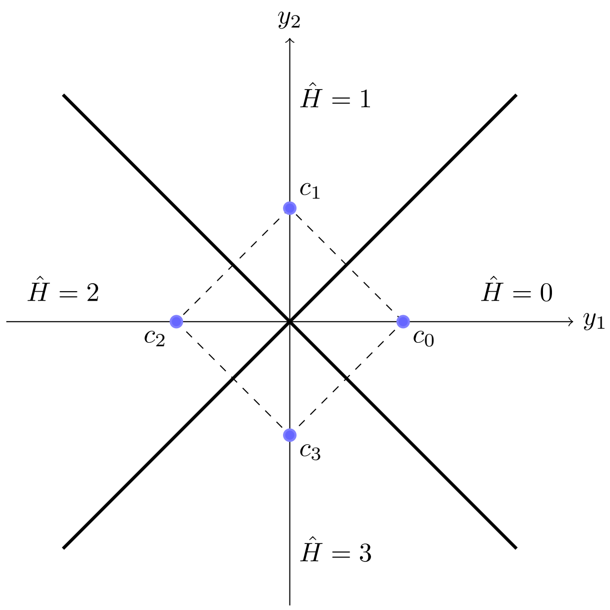


Figure 2.4: Decision regions for equally likely hypotheses

1. *Therefore, if we define* $\tilde{Y} = (\tilde{Y}_1, \tilde{Y}_2)^\mathsf{T} = \left(\frac{Y_1}{\sigma_1}, \frac{Y_2}{\sigma_2}\right)^\mathsf{T}$ *and* $\tilde{Z} = (\tilde{Z}_1, \tilde{Z}_2)^\mathsf{T} = \left(\frac{Z_1}{\sigma_1}, \frac{Z_2}{\sigma_2}\right)^\mathsf{T}$, *we will have* $\tilde{Y} = \tilde{c}_0 + \tilde{Z}$ *if* $H = 0$ *and* $\tilde{Y} = \tilde{c}_1 + \tilde{Z}$ *if* $H = 1$, *where* $\tilde{c}_0 = \left(\frac{A}{\sigma_1}, \frac{A}{\sigma_2}\right)^\mathsf{T}$, $\tilde{c}_1 = \left(-\frac{A}{\sigma_1}, -\frac{A}{\sigma_2}\right)^\mathsf{T}$ *and* $\tilde{Z} \sim \mathcal{N}(0, I_2)$. *It is clear that* $\tilde{Y}$ *can be seen as the output of a discrete time AWGN channel (with two observations), which is a situation we are familiar with and know very well how to handle.*

Another solution for the problem is to start from the basic principles, i.e., computing the probability densities $f_{Y|H}$ *and probabilities* $P_{H|Y}$, *then computing the decision regions and error probabilities without relying on the results of discrete time AWGN channels.*

We provide the two solutions here. While the second solution starts from the basic principles, the first one builds on results and intuitions that we have already developed.

First solution:

(a) *Since* $\tilde{Z} \sim \mathcal{N}(0, I_2)$, *the line that separates the two decision regions in the* $\tilde{y}$*-plane is the perpendicular bisector of the segment* $[\tilde{c}_0\ \tilde{c}_1]$ *(i.e, the line that has* $\tilde{c}_0 - \tilde{c}_1$ *as a normal vector and passes through the midpoint of* $\tilde{c}_0$ *and* $\tilde{c}_1$ *— which is the origin). Therefore, the MAP decision regions in the* $\tilde{y}$*-plane are given by*

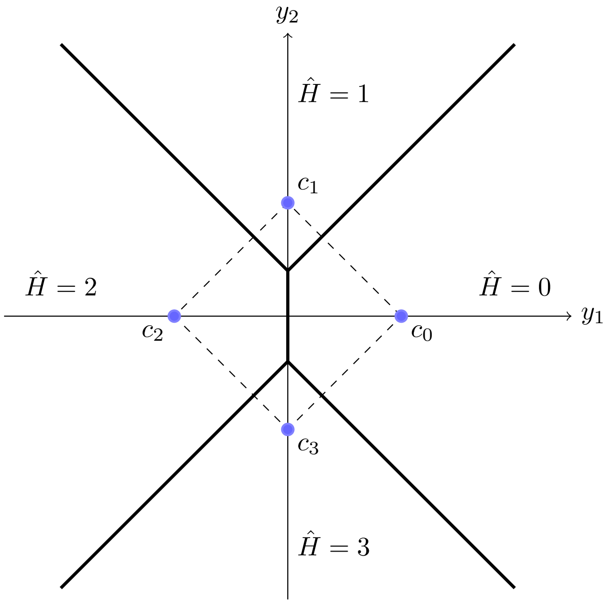


Figure 2.5: Decision regions for hypotheses with different prior probabilities

$$\begin{aligned}
\langle \tilde{y}, \tilde{c}_0 - \tilde{c}_1 \rangle \quad &\underset{\hat{H}=1}{\overset{\hat{H}=0}{\gtrless}} \quad 0, \textit{ or equivalently,} \\
\tilde{y}_1 \frac{2A}{\sigma_1} + \tilde{y}_2 \frac{2A}{\sigma_2} \quad &\underset{\hat{H}=1}{\overset{\hat{H}=0}{\gtrless}} \quad 0, \\
\frac{\tilde{y}_1}{\sigma_1} + \frac{\tilde{y}_2}{\sigma_2} \quad &\underset{\hat{H}=1}{\overset{\hat{H}=0}{\gtrless}} \quad 0.
\end{aligned}$$

Now since $\tilde{y}_1 = \dfrac{y_1}{\sigma_1}$ *and* $\tilde{y}_2 = \dfrac{y_2}{\sigma_2}$, *the MAP decision regions in the* y*-plane are given by*

$$\begin{aligned}
\frac{y_1}{\sigma_1^2} + \frac{y_2}{\sigma_2^2} \quad &\underset{\hat{H}=1}{\overset{\hat{H}=0}{\gtrless}} \quad 0, \textit{ or equivalently,} \\
\sigma_2^2 y_1 + \sigma_1^2 y_2 \quad &\underset{\hat{H}=1}{\overset{\hat{H}=0}{\gtrless}} \quad 0.
\end{aligned}$$

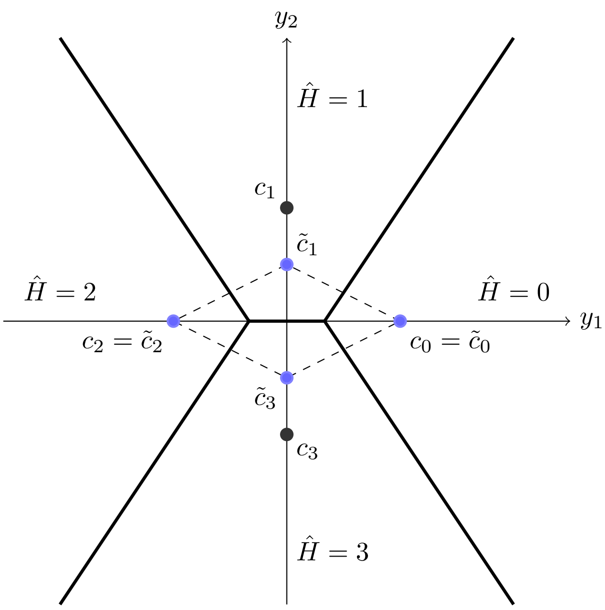


Figure 2.6: Decision regions for noise with different variance in each component

(b) *When $\sigma_1 = 2\sigma_2$, the decision rule becomes*

$$\sigma_2^2 y_1 + 4\sigma_2^2 y_2 \underset{\hat{H}=1}{\overset{\hat{H}=0}{\gtrless}} 0, \text{ or equivalently,}$$

$$y_2 \underset{\hat{H}=1}{\overset{\hat{H}=0}{\gtrless}} -\frac{y_1}{4}.$$

The decision regions are sketched in Figure 2.7.

(c) *We compute the probability of error based on $\tilde{Y}$ and $\tilde{Z}$. The distance between $\tilde{c}_0$ and the separator line is equal to*

$$||\tilde{c}_0|| = A\sqrt{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}.$$

Since $\tilde{Z} \sim \mathcal{N}(0, I_2)$, we have

$$P_e(0) = Q\left(A\sqrt{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}\right).$$

Similarly, we have

$$P_e(1) = Q\left(A\sqrt{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}\right).$$

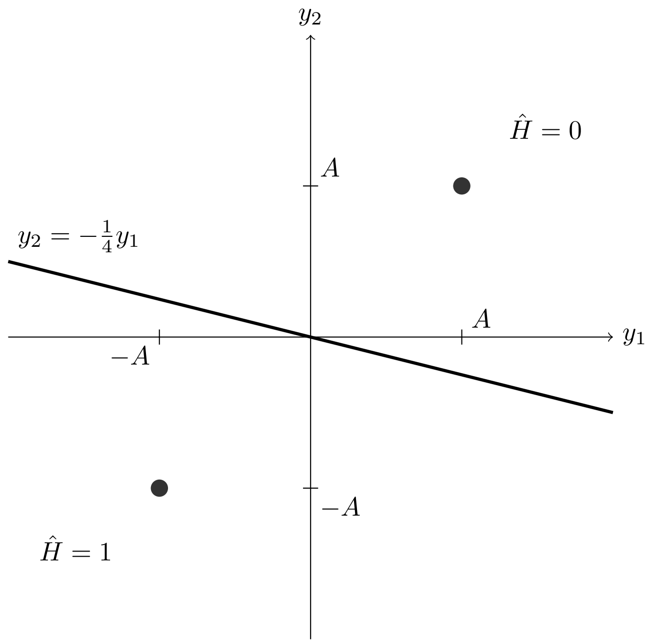


Figure 2.7: Decision regions

Therefore,

$$P_e = P_e(0)P_H(0) + P_e(1)P_H(1) = Q\left(A\sqrt{\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2}}\right).$$

Second solution:

(a) *We have*

$$\begin{aligned} f_{Y|H}(y|0) &= \frac{1}{2\pi\sigma_1\sigma_2}\exp\left[-\frac{(y_1-A)^2}{2\sigma_1^2}-\frac{(y_2-A)^2}{2\sigma_2^2}\right] \\ f_{Y|H}(y|1) &= \frac{1}{2\pi\sigma_1\sigma_2}\exp\left[-\frac{(y_1+A)^2}{2\sigma_1^2}-\frac{(y_2+A)^2}{2\sigma_2^2}\right]. \end{aligned}$$

The MAP decision rule is

$$\frac{f_{Y|H}(y|1)}{f_{Y|H}(y|0)} \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} \frac{P_H(0)}{P_H(1)}$$

or, by taking the logarithm,

$$\ln\left[\frac{f_{Y|H}(y|1)}{f_{Y|H}(y|0)}\right] \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} \ln\left[\frac{P_H(0)}{P_H(1)}\right], \text{ or equivalently,}$$

$$\frac{2Ay_1}{\sigma_1^2}+\frac{2Ay_2}{\sigma_2^2} \underset{\hat{H}=1}{\overset{\hat{H}=0}{\gtrless}} 0,$$

$$\sigma_2^2 y_1+\sigma_1^2 y_2 \underset{\hat{H}=1}{\overset{\hat{H}=0}{\gtrless}} 0.$$

(b) *Refer to the first solution.*

(c) *We first determine the probability of error when $H=1$:*

$$P_e(1)=Pr\left\{\sigma_2^2Y_1+\sigma_1^2Y_2>0|H=1\right\}.$$

If $H=1$, $\sigma_2^2Y_1+\sigma_1^2Y_2=\sigma_2^2(-A+Z_1)+\sigma_1^2(-A+Z_2)$. We see immediately that this is normally distributed, $\sim\mathcal{N}(-A(\sigma_2^2+\sigma_1^2),(\sigma_2^4\sigma_1^2+\sigma_1^4\sigma_2^2))$. Hence,

$$\begin{aligned} P_e(1) &= Q\left(\frac{A(\sigma_2^2+\sigma_1^2)}{\sqrt{\sigma_2^4\sigma_1^2+\sigma_1^4\sigma_2^2}}\right) \\ &= Q\left(A\sqrt{\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2}}\right). \end{aligned}$$

Similarly,

$$P_e(0)=Q\left(A\sqrt{\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2}}\right),$$

and

$$P_e=P_e(0)P_H(0)+P_e(1)P_H(1)=Q\left(A\sqrt{\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2}}\right).$$

Solution 16. (Multi-antenna receiver)

(a) *We have a binary hypothesis testing problem with V as the observable:*

$$\begin{aligned} &\textit{if } B=1: && V=\langle g,w\rangle+\langle Z,w\rangle=a+Z_t \\ &\textit{if } B=-1: && V=-\langle g,w\rangle+\langle Z,w\rangle=-a+Z_t, \end{aligned}$$

where

$$\begin{aligned} Z_t &= \langle Z,w\rangle\sim\mathcal{N}(0,\sigma^2\|w\|^2) \\ a &= \langle g,w\rangle. \end{aligned}$$

The ML decision rule is

$$e^{-\frac{|v-a|^2}{2\sigma^2\|w\|^2}} \underset{\hat{B}=-1}{\overset{\hat{B}=1}{\gtrless}} e^{-\frac{|v+a|^2}{2\sigma^2\|w\|^2}}$$

If $a > 0$, *this leads to*

$$\hat{B} = 1 \text{ if } v \geq 0$$
$$\hat{B} = -1 \text{ if } v < 0.$$

If $a < 0$, *then the decision is reversed.*

(b) *By symmetry, and assuming that* $a > 0$,

$$P_e(1) = P_e(-1) = \int_a^\infty \frac{1}{\sqrt{2\pi\sigma^2\|w\|^2}} e^{-\frac{|v-a|^2}{2\sigma^2\|w\|^2}}\, dv = Q\left(\frac{|\langle g, w\rangle|}{\sigma\|w\|}\right).$$

Because the hypotheses are equiprobable, $P_e = Q\left(\frac{|\langle g,w\rangle|}{\sigma\|w\|}\right)$.

The same result is obtained for $a < 0$.

(c)

$$P_e = Q\left(\frac{\beta\|g\|}{\sigma}\right).$$

(d) $\beta_{max} = 1$, *achieved when* g *and* w *are collinear. (This is the Cauchy-Schwarz inequality, but it is obvious from a drawing of the two vectors.)* $\beta_{min} = 0$, *achieved when* g *and* w *are orthogonal.*

(e) $P_{e,min} = Q\left(\frac{\|g\|}{\sigma}\right)$, *achieved when* β *is maximum.*

By using Y *instead of* V, *the ML rule becomes*

$$-\frac{\|y-g\|^2}{2\sigma^2} \underset{\hat{B}=-1}{\overset{\hat{B}=1}{\gtrless}} -\frac{\|y+g\|^2}{2\sigma^2},$$

and the error probability is

$$P_e = Q\left(\frac{\|g\|}{\sigma}\right).$$

Therefore, we cannot reduce $P_{e,min}$ *by operating directly on the observation* Y.

(f) *If we let* $\tilde{Y}_k = \frac{Y_k}{\sigma_k} = B\tilde{g}_k + \tilde{Z}_{tk}$ *then we are back to the original problem except that the* k^{th} *antenna gain is now* $\tilde{g}_k = \frac{g_k}{\sigma_k}$ *and the noise variance is* 1.

Solution 17. (Signal constellation)

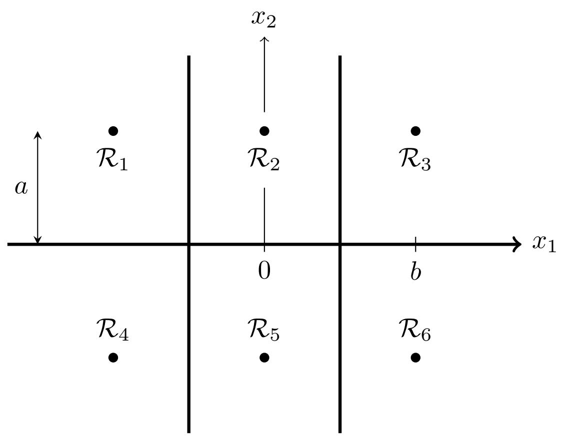


Figure 2.8: Decision regions

(a) *Label the signal points from left to right, top to bottom by* $1, \ldots, 6$. *The decision regions are shown in Figure 2.8.*

(b) *Denote the additive white Gaussian noise process in the* x_1 *direction by* Z_1 *and in the* x_2 *direction by* Z_2. *In our setup, both* Z_1 *and* Z_2 *are zero-mean Gaussian of variance* σ^2. *The observations* Y_1 *and* Y_2 *are also Gaussian of variance* σ^2, *but with mean corresponding to the particular signal point that is being transmitted.*

If we denote the hypothesis by H, *for the four corner points (numbers* 1, 3, 4 *and* 6*), we find*

$$P_e(1) = Pr\left\{\left(Y_1 \geq -\frac{b}{2}\right) \cup (Y_2 \leq 0) | H = 1\right\}.$$

To determine this, we first compute the probability of the correct decision, $P_c(1)$, *and then determine* $P_e(1) = 1 - P_c(1)$. *Thus,*

$$\begin{aligned}
P_c(1) &= Pr\left\{\left(Y_1 \leq -\frac{b}{2}\right) \cap (Y_2 \geq 0) | H = 1\right\} \\
&= Pr\left\{Y_1 \leq -\frac{b}{2} | H = 1\right\} Pr\left\{Y_2 \geq 0 | H = 1\right\} \\
&= Pr\left\{Z_1 \leq \frac{b}{2}\right\} Pr\left\{Z_2 \geq -a\right\} \\
&= \left(1 - Q\left(\frac{b}{2\sigma}\right)\right) Q\left(-\frac{a}{\sigma}\right) \\
&= \left(1 - Q\left(\frac{b}{2\sigma}\right)\right)\left(1 - Q\left(\frac{a}{\sigma}\right)\right).
\end{aligned}$$

For the other two points (numbers 2 and 5), we obtain

$$\begin{aligned}
P_c(2) &= Pr\left\{\left(-\frac{b}{2} \le Y_1 \le \frac{b}{2}\right) \cap (Y_2 \ge 0)|H=1\right\} \\
&= Pr\left\{-\frac{b}{2} \le Y_1 \le \frac{b}{2}|H=1\right\} Pr\left\{Y_2 \ge 0|H=1\right\} \\
&= Pr\left\{-\frac{b}{2} \le Z_1 \le \frac{b}{2}\right\} Pr\left\{Z_2 \ge -a\right\} \\
&= \left(1 - 2Q\left(\frac{b}{2\sigma}\right)\right)\left(1 - Q\left(\frac{a}{\sigma}\right)\right).
\end{aligned}$$

Putting things together, we find

$$\begin{aligned}
P_c &= \sum_{i=1}^{6} P_H(i)P_c(i) = \sum_{i=1}^{6} \frac{1}{6} P_c(i) \\
&= \frac{1}{6}\left[4\left(1 - Q\left(\frac{b}{2\sigma}\right)\right)\left(1 - Q\left(\frac{a}{\sigma}\right)\right) + 2\left(1 - 2Q\left(\frac{b}{2\sigma}\right)\right)\left(1 - Q\left(\frac{a}{\sigma}\right)\right)\right] \\
&= 1 - \frac{4}{3}Q\left(\frac{b}{2\sigma}\right) - Q\left(\frac{a}{\sigma}\right) + \frac{4}{3}Q\left(\frac{b}{2\sigma}\right)Q\left(\frac{a}{\sigma}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
P_e &= 1 - P_c \\
&= \frac{4}{3}Q\left(\frac{b}{2\sigma}\right) + Q\left(\frac{a}{\sigma}\right) - \frac{4}{3}Q\left(\frac{b}{2\sigma}\right)Q\left(\frac{a}{\sigma}\right).
\end{aligned}$$

(c) *The average energy per symbol is*

$$\begin{aligned}
\mathcal{E} &= \frac{1}{6}\left[4(a^2 + b^2) + 2a^2\right] \\
&= a^2 + \frac{2b^2}{3}.
\end{aligned}$$

Solution 18. (Hypothesis testing and fading)

(a) *Our observation is $Y = AX + Z$. The conditional pdf of Y under the hypothesis $H = 0$ can be computed in the following manner:*

$$\begin{aligned}
f_{Y|H}(y|0) &= f_{Y|H,A}(y|0,0)P_A(0) + f_{Y|H,A}(y|0,1)P_A(1) \\
&= \frac{1}{2}f_Z(y) + \frac{1}{2}f_Z(y+b) \\
&= \frac{1}{2}\frac{1}{\sqrt{2\pi\sigma^2}}\left(e^{-\frac{y^2}{2\sigma^2}} + e^{-\frac{(y+b)^2}{2\sigma^2}}\right).
\end{aligned}$$

In the same way, we have

$$f_{Y|H}(y|1) = \frac{1}{2}\frac{1}{\sqrt{2\pi\sigma^2}}\left(e^{-\frac{y^2}{2\sigma^2}} + e^{-\frac{(y-b)^2}{2\sigma^2}}\right).$$

Writing the ML decision rule in this case, we get

$$\frac{1}{2}\frac{1}{\sqrt{2\pi\sigma^2}}\left(e^{-\frac{y^2}{2\sigma^2}} + e^{-\frac{(y+b)^2}{2\sigma^2}}\right) \underset{\hat{H}=1}{\overset{\hat{H}=0}{\gtrless}} \frac{1}{2}\frac{1}{\sqrt{2\pi\sigma^2}}\left(e^{-\frac{y^2}{2\sigma^2}} + e^{-\frac{(y-b)^2}{2\sigma^2}}\right),$$

which is equivalent to

$$e^{-\frac{(y+b)^2}{2\sigma^2}} \underset{\hat{H}=1}{\overset{\hat{H}=0}{\gtrless}} e^{-\frac{(y-b)^2}{2\sigma^2}}, \textit{ or, after taking the logarithm,}$$

$$0 \underset{\hat{H}=1}{\overset{\hat{H}=0}{\gtrless}} y.$$

Thus, we get a familiar problem and we see immediately that our ML rule decides for $H = 0$ when $y \leq 0$ and for $H = 1$ when $y > 0$ (can be easily seen from Figure 2.9).

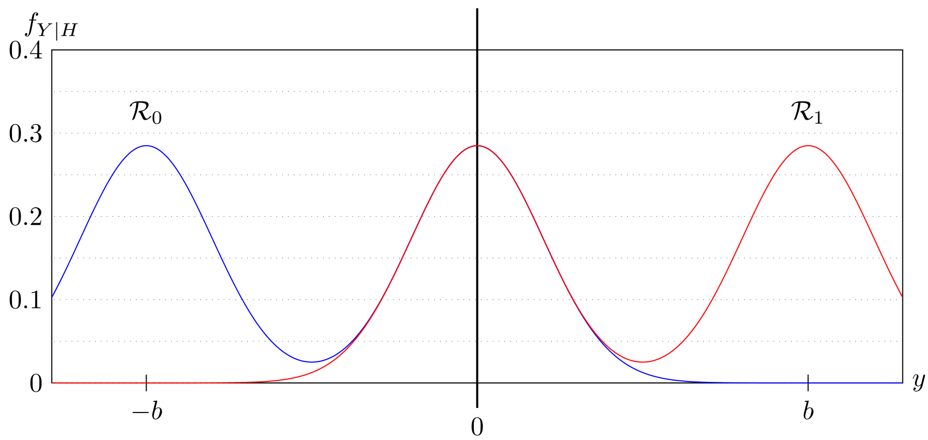


Figure 2.9: Decision regions

(b) By symmetry, we have

$$\begin{aligned} P_e &= P_e(0) = P_e(1) \\ &= Pr\{y > 0 | H = 0\} \\ &= \int_0^\infty f_{Y|H}(y|0)\ dy \\ &= \int_0^\infty \frac{1}{2}\frac{1}{\sqrt{2\pi\sigma^2}}\left(e^{-\frac{y^2}{2\sigma^2}} + e^{-\frac{(y+b)^2}{2\sigma^2}}\right)\ dy \\ &= \frac{1}{2}Q(0) + \frac{1}{2}Q\left(\frac{b}{\sigma}\right) \\ &= \frac{1}{4} + \frac{1}{2}Q\left(\frac{b}{\sigma}\right). \end{aligned}$$

Solution 19. (MAP decoding regions)

(a) The resulting decision region is shown in Figure 2.10.

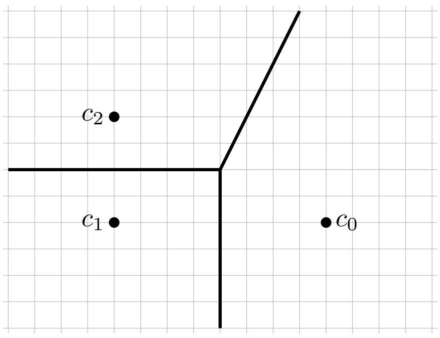


Figure 2.10: Decision region for ML

(b) As the probability of $H = 2$ increases, the corresponding region for $H = 2$ expands as well. However, the boundary of the decision regions are still lines parallel to the corresponding lines of the ML decision region. Moreover, as the probabilities of $H = 0$ and $H = 1$ remain equal, the separating line between c_0 and c_1 does not change. The result is depicted in Figure 2.11. (The three separating planes have to meet in one point. To see why, first notice that $P_{H|Y}(0|y) = P_{H|Y}(1|y)$ on the plane separating the decoding region for $H = 0$ and $H = 1$. Reasoning similarly, we see that where this plane meets the plane separating $H = 1$ and $H = 2$, $P_{H|Y}(0|y) = P_{H|Y}(1|y) = P_{H|Y}(2|y)$. Hence the contact point is also on the plane separating $H = 0$ and $H = 2$, namely where $P_{H|Y}(0|y) = P_{H|Y}(2|y)$.)

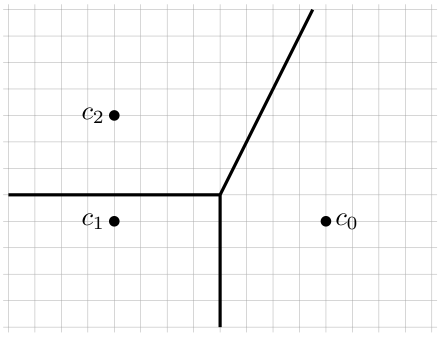


Figure 2.11: Decision region for MAP

(c) *The MAP receiver considers both the initial probabilities (prior information) and the information received via the observations (posterior information). When the noise variance increases, the prior information is more reliable than the posterior one. Thus, the "trend" of the previous figure is further "amplified". See Figure 2.12.*

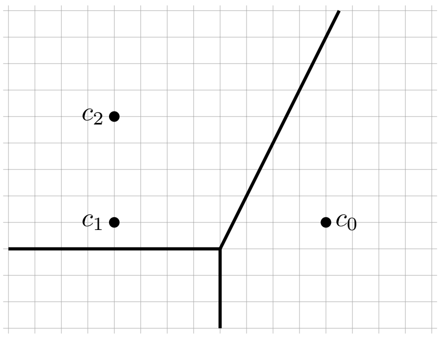


Figure 2.12: Decision region for MAP for higher noise variance

Solution 20. (Sufficient statistic)

If $H = 0$, we have $Y_2 = Z_1Z_2 = Y_1Z_2$, and if $H = 1$, we have $Y_2 = -Z_1Z_2 = Y_1Z_2$. Therefore, $Y_2 = Y_1Z_2$ in all cases. Now since Z_2 is independent of H, we clearly have $H \to Y_1 \to (Y_1, Y_1Z_2)$. Hence, Y_1 is a sufficient statistic.

Solution 21. (More on sufficient statistic)

(a) *The MAP decoder $\hat{H}(y)$ is given by*

$$\hat{H}(y) = \arg\max_i P_{Y|H}(y|i) = \begin{cases} 0 & \textit{if } y = 0 \textit{ or } y = 1 \\ 1 & \textit{if } y = 2 \textit{ or } y = 3. \end{cases}$$

$T(Y)$ takes two values with the conditional probabilities

$$P_{T|H}(t|0) = \begin{cases} 0.7 & \text{if } t = 0 \\ 0.3 & \text{if } t = 1 \end{cases} \qquad P_{T|H}(t|1) = \begin{cases} 0.3 & \text{if } t = 0 \\ 0.7 & \text{if } t = 1. \end{cases}$$

Therefore, the MAP decoder $\hat{H}(T(y))$ is

$$\hat{H}(T(y)) = \arg\max_i P_{T(Y)|H}(t|i) = \begin{cases} 0 & \text{if } t = 0 \quad (y = 0 \text{ or } y = 1) \\ 1 & \text{if } t = 1 \quad (y = 2 \text{ or } y = 3). \end{cases}$$

Hence, the two decoders are equivalent.

(b) We have

$$Pr\{Y = 0|T(Y) = 0, H = 0\} = \frac{Pr\{Y = 0, T(Y) = 0|H = 0\}}{Pr\{T(Y) = 0|H = 0\}} = \frac{0.4}{0.7} = \frac{4}{7}$$

and

$$Pr\{Y = 0|T(Y) = 0, H = 1\} = \frac{Pr\{Y = 0, T(Y) = 0|H = 1\}}{Pr\{T(Y) = 0|H = 1\}} = \frac{0.1}{0.3} = \frac{1}{3}.$$

Thus $Pr\{Y = 0|T(Y) = 0, H = 0\} \neq Pr\{Y = 0|T(Y) = 0, H = 1\}$, hence $H \to T(Y) \to Y$ is not true, although the MAP decoders are equivalent.

Solution 22. (Fisher–Neyman factorization theorem)

(a) The MAP decision rule can always be written as

$$\begin{aligned} \hat{H}(y) &= \arg\max_i f_{Y|H}(y|i)P_H(i) \\ &= \arg\max_i g_i(T(y))h(y)P_H(i) \\ &= \arg\max_i g_i(T(y))P_H(i). \end{aligned}$$

The last step is valid because $h(y)$ is a non-negative constant which is independent of i and thus does not give any further information for our decision.

(b) Let us define the event $\mathcal{B} = \{y : T(y) = t\}$. Then,

$$\begin{aligned} f_{Y|H,T(Y)}(y|i,t) &= \frac{f_{Y,T(Y)|H}(y,t|i)P_H(i)}{f_{T(Y)|H}(t|i)P_H(i)} \\ &= \frac{Pr\{Y = y, T(Y) = t|H = i\}}{Pr\{T(Y) = t|H = i\}} = \frac{Pr\{Y = y, Y \in \mathcal{B}|H = i\}}{Pr\{Y \in \mathcal{B}|H = i\}} \\ &= \frac{f_{Y|H}(y|i)\mathbb{1}_{\mathcal{B}}(y)}{\int_{\mathcal{B}} f_{Y|H}(y|i)dy}. \end{aligned}$$

If $f_{Y|H}(y|i) = g_i(T(y))h(y)$*, then*

$$\begin{aligned} f_{Y|H,T(Y)}(y|i,t) &= \frac{g_i(T(y))h(y)\mathbb{1}_{\mathcal{B}}(y)}{\int_{\mathcal{B}} g_i(T(y))h(y)dy} \\ &= \frac{g_i(t)h(y)\mathbb{1}_{\mathcal{B}}(y)}{g_i(t)\int_{\mathcal{B}} h(y)dy} \\ &= \frac{h(y)\mathbb{1}_{\mathcal{B}}(y)}{\int_{\mathcal{B}} h(y)dy}. \end{aligned}$$

Hence, we see that $f_{Y|H,T(Y)}(y|i,t)$ *does not depend on* i*, so* $H \to T(Y) \to Y$*.*

(c) *Note that* $P_{Y_k|H}(1|i) = p_i, P_{Y_k|H}(0|i) = 1 - p_i$ *and*

$$P_{Y_1,\dots,Y_n|H}(y_1,\dots,y_n|i) = P_{Y_1|H}(y_1|i)\dots P_{Y_n|H}(y_n|i).$$

Thus, we have

$$P_{Y_1,\dots,Y_n|H}(y_1,\dots,y_n|i) = p_i^t(1-p_i)^{(n-t)},$$

where $t = \sum_k y_k$*.*

Choosing $g_i(t) = p_i^t(1-p_i)^{(n-t)}$ *and* $h(y) = 1$*, we see that* $P_{Y_1,\dots,Y_n|H}(y_1,\dots,y_n|i)$ *fulfills the condition in the question.*

(d) *Because* $Y_1,\dots,Y_n$ *are independent,*

$$\begin{aligned} f_{Y_1,\dots,Y_n|H}(y_1,\dots,y_n|i) &= \prod_{k=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-\frac{(y_k-m_i)^2}{2}} \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\sum_{k=1}^{n} \frac{(y_k-m_i)^2}{2}} \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{\sum_{k=1}^{n} y_k^2}{2}} \; e^{nm_i(\frac{1}{n}\sum_{k=1}^{n} y_k \; - \; \frac{m_i}{2})}. \end{aligned}$$

Choosing $g_i(t) = e^{nm_i(t-\frac{m_i}{2})}$ *and* $h(y_1,\dots,y_n) = \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{\sum_{k=1}^{n} y_k^2}{2}}$*, we see that*

$$f_{Y_1,\dots,Y_n|H}(y_1,\dots,y_n|i) = g_i(T(y_1,\dots,y_n))h(y_1,\dots,y_n).$$

Hence the condition in the question is fulfilled.

Solution 23. (Irrelevance and operational irrelevance)

(a) *By assumption,* V *and* H *are not independent. This means that* $P_{V|H}(\cdot|i)$ *does depend on* i*. Specifically, it means that for at least one* $k \in \mathcal{V}$ *and a pair* $i,j \in \mathcal{H}$*,* $P_{V|H}(k|i) \neq P_{V|H}(k|j)$*. Without loss of generality, we can assume that* $P_{V|H}(k|i) > P_{V|H}(k|j)$*. We know that probabilities sum up to 1, i.e.* $\sum_{m\in\mathcal{V}} P_{V|H}(m|i) = 1$ *and* $\sum_{m\in\mathcal{V}} P_{V|H}(m|j) = 1$*. Since* $P_{V|H}(\cdot|i)$ *puts more probability on* $k \in \mathcal{V}$ *than on* $P_{V|H}(\cdot|j)$*, there exists another symbol* $l \in \mathcal{V}$ *for which* $P_{V|H}(l|i) < P_{V|H}(l|j)$*.*

(b) *Let i and j be as above and choose $P_H(i) = P_H(j) = \frac{1}{2}$ and $P_H(l) = 0$ for $l \neq i, j$. Now suppose that we observe $V = k$. From the previous part of the problem we know that $P_{V|H}(k|i) > P_{V|H}(k|j)$ and so the MAP decision rule selects $\hat{H} = i$. On the contrary, if $V = l$, the MAP decision rule decides $\hat{H} = j$. Hence, V affects the MAP decision rule.*

(c) *Now we have two observables, U and V, which take values in $\mathcal{U}$ and $\mathcal{V}$ respectively. We know that the probabilistic relation $H \to U \to V$ does not hold, which means that there exists $u^\star \in \mathcal{U}$ such that H and V are dependent when $U = u^\star$. Now, given that $U = u^\star$, we are back to the situation from the previous part of the problem. Therefore, conditioned on $U = u^\star$, there exists a distribution of H for which V affects the decision.*

Solution 24. (Antipodal signaling)

(a) *Assume for instance that $P_H(0) = P_H(1) = \frac{1}{2}$. Then, the decision regions are:*

$$\begin{aligned} \mathcal{R}_0 &= \{(y_1, y_2) : y_2 < -y_1\}, \\ \mathcal{R}_1 &= \{(y_1, y_2) : y_2 \geq -y_1\}. \end{aligned}$$

If now, for instance, $Y_1 = a$, then for values of Y_2 that are larger than $-a$, we decide $\hat{H} = 1$, whereas for values of Y_2 that are smaller than $-a$, we decide $\hat{H} = 0$. Hence, we still need Y_2, and the knowledge of Y_1 is not sufficient.

(b) *A new constellation for which Y_1 is a sufficient statistic is for instance*

$$\begin{aligned} \tilde{c}_0 &= (-a, 0), \\ \tilde{c}_1 &= (a, 0). \end{aligned}$$

Solution 25. (Is it a sufficient statistic?)

(a) *An ML decoder is a minimum distance decoder in the AWGN channel. In this case the two decoding regions are separated by the line $y_1 + y_2 = 0$. Hence the ML decoder decides as follows:*

$$Y_1 + Y_2 \underset{\hat{H}=1}{\overset{\hat{H}=0}{\gtrless}} 0.$$

So the answer is no.

(b) *By the first hint, to prove $H \to (Y_1 + Y_2) \to Y$, it suffices to prove $H \to (Y_1 + Y_2) \to (Y_1 + Y_2, Y_1 - Y_2)$ or, equivalently, that $H \to (Y_1 + Y_2) \to (Y_1 - Y_2)$.*

Since $Y_1 + Y_2$ and $Y_1 - Y_2$ are orthogonal, knowing that $Y_1 + Y_2 = a$ changes nothing to the distribution of $Y_1 - Y_2 = Z_1 - Z_2$: It remains $\sim \mathcal{N}(0, 2\sigma^2)$, independently of the value of H. Hence, we have the Markov chain $H \to (Y_1 + Y_2) \to (Y_1 - Y_2)$.

Solution 26. (Union bound)

Let $W \sim \mathcal{N}(0, \sigma^2 I_2)$ be the zero mean Gaussian noise, where $Z = W + c$. Let $\mathcal{A}$ be the area on the left of the vertical dividing line, and let $\mathcal{B}$ be the area below the slanted dividing line. The boundary of $\mathcal{A}$ is at distance $d_1 = 2$ from c, whereas the boundary of $\mathcal{B}$ is at distance $d_2 = \sqrt{2}$ from c. Hence

$$Pr\{Z \in \mathcal{A} \cup \mathcal{B}\} \leq Pr\{Z \in \mathcal{A}\} + Pr\{Z \in \mathcal{B}\} = Q\left(\frac{d_1}{\sigma}\right) + Q\left(\frac{d_2}{\sigma}\right) = Q\left(\frac{2}{\sigma}\right) + Q\left(\frac{\sqrt{2}}{\sigma}\right).$$

Solution 27. (QAM with erasure)

$$\begin{aligned} P_{00} &= Pr\{(N_1 \geq -a) \cap (N_2 \geq -a)\} \\ &= Pr\{(N_1 \leq a)\} Pr\{(N_2 \leq a)\} \\ &= \left[1 - Q\left(\frac{a}{\sigma}\right)\right]^2. \end{aligned}$$

By symmetry:

$$\begin{aligned} P_{01} = P_{03} &= Pr\{(N_1 \leq -(2b-a)) \cap (N_2 \geq -a)\} \\ &= Pr\{N_1 \geq 2b-a\} Pr\{N_2 \leq a\} \\ &= Q\left(\frac{2b-a}{\sigma}\right)\left[1 - Q\left(\frac{a}{\sigma}\right)\right]. \end{aligned}$$

$$\begin{aligned} P_{02} &= Pr\{(N_1 \leq -(2b-a)) \cap (N_2 \leq -(2b-a))\} \\ &= Pr\{N_1 \geq 2b-a\} Pr\{N_2 \geq 2b-a\} \\ &= \left[Q\left(\frac{2b-a}{\sigma}\right)\right]^2. \end{aligned}$$

$$\begin{aligned} P_{0\delta} &= 1 - Pr\{(Y \in \mathcal{R}_0) \cup (Y \in \mathcal{R}_1) \cup (Y \in \mathcal{R}_2) \cup (Y \in \mathcal{R}_3) | c_0 \textit{ was sent}\} \\ &= 1 - P_{00} - P_{01} - P_{02} - P_{03} \\ &= 1 - \left[1 - Q\left(\frac{a}{\sigma}\right)\right]^2 - 2Q\left(\frac{2b-a}{\sigma}\right)\left[1 - Q\left(\frac{a}{\sigma}\right)\right] - \left[Q\left(\frac{2b-a}{\sigma}\right)\right]^2 \\ &= 1 - \left[1 - Q\left(\frac{a}{\sigma}\right) + Q\left(\frac{2b-a}{\sigma}\right)\right]^2. \end{aligned}$$

Equivalently,

$$\begin{aligned} P_{0\delta} &= Pr\{(N_1 \in [a, 2b-a]) \cup (N_2 \in [a, 2b-a])\} \\ &= Pr\{N_1 \in [a, 2b-a]\} + Pr\{N_2 \in [a, 2b-a]\} - Pr\{(N_1 \in [a, 2b-a]) \cap (N_2 \in [a, 2b-a])\} \\ &= 2\left[Q\left(\frac{a}{\sigma}\right) - Q\left(\frac{2b-a}{\sigma}\right)\right] - \left[Q\left(\frac{a}{\sigma}\right) - Q\left(\frac{2b-a}{\sigma}\right)\right]^2, \end{aligned}$$

which gives the same result as before.

Solution 28. (Repeat codes and Bhattacharyya bound)

(a) *Let $X_i = c_{H,i}$ be the i-th symbol that was sent, i.e., $X_i = 1$ if $H = 0$ and $X_i = -1$ if $H = 1$. We have:*

$$P_{W_i|X_i}(1|-1) = Pr\{Y_i > 0|H = 1\} = Pr\{-1 + Z > 0\} = Q\left(\frac{1}{\sigma}\right).$$

Similarly, we can show that $P_{W_i|X_i}(-1|-1) = 1 - Q\left(\frac{1}{\sigma}\right)$, $P_{W_i|X_i}(-1|1) = Q\left(\frac{1}{\sigma}\right)$ *and* $P_{W_i|X_i}(1|1) = 1 - Q\left(\frac{1}{\sigma}\right)$.

The overall system between X_i and W_i may be viewed as a channel with input 1 or -1 and output also 1 or -1. There is a certain probability ϵ (called transition *or* crossover *probability, and which is equal to $Q\left(\frac{1}{\sigma}\right)$ in our case) that the channel converts 1 into -1 or vice versa. (see Figure 2.13.)*

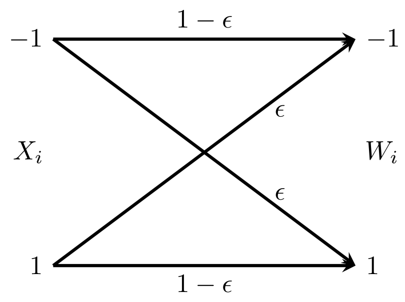


Figure 2.13: Binary Symmetric Channel (BSC) with crossover probability ϵ

This particular channel is called the Binary Symmetric Channel. *Various results can be found easily from Figure 2.13. For instance, it is clear that if we put n consecutive 1's into the channel, the probability of getting, at the output, a particular sequence $(w_1, \dots, w_n)$ which contains exactly k 1's is simply $(1-\epsilon)^k \epsilon^{n-k}$. Similarly, the probability of getting, at the output,* any *sequence that contains exactly k 1's is $\binom{n}{k}(1-\epsilon)^k\epsilon^{n-k}$ because there are $\binom{n}{k}$ distinct sequences with exactly k ones each, and every one of them has probability $(1-\epsilon)^k\epsilon^{n-k}$.*

The MAP decision rule is

$$\frac{P_{W_1 \dots W_n|H}(w_1, \dots, w_n|1)}{P_{W_1 \dots W_n|H}(w_1, \dots, w_n|0)} \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} \frac{P_H(0)}{P_H(1)} = 1 \quad \textit{or,}$$

$$\frac{\epsilon^k (1-\epsilon)^{n-k}}{(1-\epsilon)^k \epsilon^{n-k}} = \left(\frac{\epsilon}{1-\epsilon}\right)^{2k-n} \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} 1.$$

The expression only depends on k, therefore the number of ones in the received sequence is a sufficient statistic.

Taking the logarithm, we obtain

$$(2k-n)\log\left(\frac{\epsilon}{1-\epsilon}\right) \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} 0.$$

Since $\epsilon < 1/2$, $\log\left(\frac{\epsilon}{1-\epsilon}\right) < 0$, *and thus, when we divide by this term, the direction of the inequality is changed. Using this, the decision rule can be written as*

$$k \underset{\hat{H}=1}{\overset{\hat{H}=0}{\gtrless}} \frac{n}{2}.$$

That is, the best decision rule is simply majority voting*: if the majority of the received values is* 1*, we decide for hypothesis* $H = 0$ *(i.e. the transmitted value was* 1*). If the majority of the received values is* -1*, we decide for hypothesis* $H = 1$ *(i.e. the transmitted value was* -1*).*

(b) *Let us assume that* n *is odd. Then,*

$$\begin{aligned} P_e(0) &= Pr\{k < n/2 | H = 0\} \\ &= \sum_{m=0}^{(n-1)/2} \tbinom{n}{m}(1-\epsilon)^m \epsilon^{n-m}. \end{aligned}$$

By the symmetry of the problem, $P_e(1)$ *has the same value. Thus,*

$$\tilde{P}_e = \sum_{m=0}^{(n-1)/2} \tbinom{n}{m}(1-\epsilon)^m \epsilon^{n-m}.$$

If n *is even, we introduce a slight asymmetry because the term for* $n/2$ *has to be assigned to either* $H = 0$ *or* $H = 1$*.*

Because this sum cannot be evaluated explicitly, in the following, we bound it using the Bhattacharyya bound*.*

(c) *The general formula for the Bhattacharyya bound is*

$$\tilde{P}_e \le \sum_i \sum_{j:j\neq i} \sqrt{P_H(i)P_H(j)} \int_{w\in\mathbb{R}^n} \sqrt{f_{W|H}(w|i) f_{W|H}(w|j)}\, dw.$$

In our case, this becomes

$$\begin{aligned} \tilde{P}_e &\le 2\frac{1}{2}\sum_w \sqrt{P_{W|H}(w|0)P_{W|H}(w|1)} \\ &= \sum_w \sqrt{(1-\epsilon)^{k(w)}\epsilon^{n-k(w)}\ \epsilon^{k(w)}(1-\epsilon)^{n-k(w)}} \\ &= \sum_w \sqrt{\epsilon^n(1-\epsilon)^n} \\ &= 2^n\sqrt{\epsilon^n(1-\epsilon)^n}. \end{aligned}$$

(d) *Again, we assume that n is odd; note however that the case when n is even would not add much insight. Figure 2.14 shows the error probabilities for various values of n (the plot was created from the* **matlab** *program that follows).*

```
n = 1:2:30;
sigma = 1;

Pe = qfunc(sqrt(n)/sigma);
epsilon = qfunc(1/sigma);
Pet = zeros(1, length(n));
for ic = 1:length(n),
  for m = 0:(n(ic)-1)/2,
    Pet(ic) = Pet(ic) + nchoosek(n(ic),m) * (1-epsilon)^m * epsilon^(n(ic)-m);
  end;
end;
PetBhatt = (2*sqrt(epsilon*(1-epsilon))).^n;

semilogy(n, Pe, '-o', n, PetBhatt, '-^', n, Pet, '-s');
```

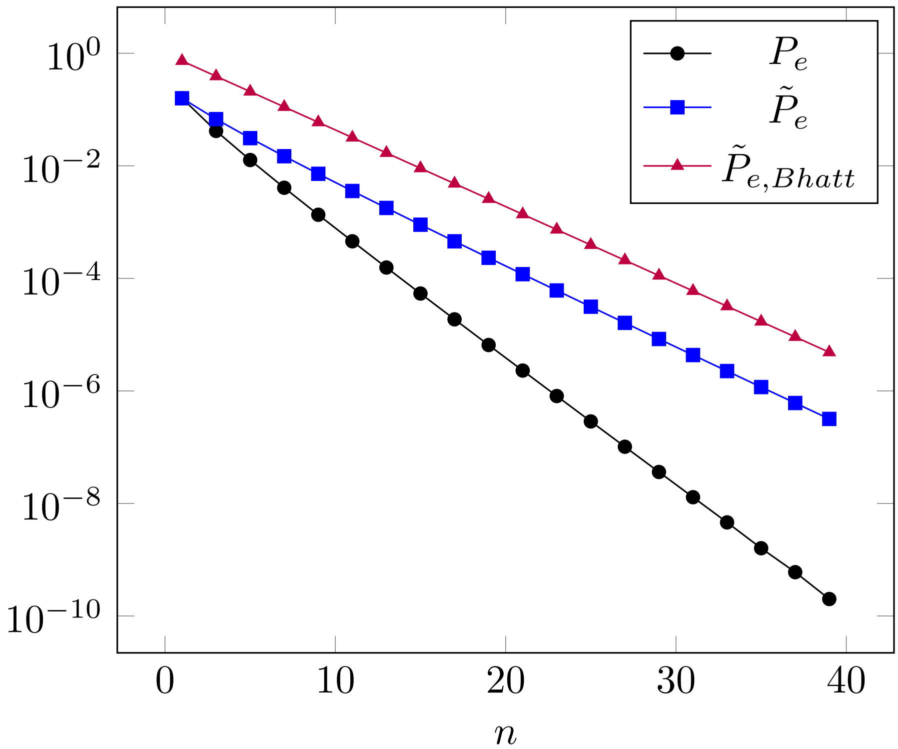


Figure 2.14: Error probability as a function of repetition length n

Solution 29. (Tighter union Bhattacharyya bound: Binary case)

(a) *From the definition of the decision region* $\mathcal{R}_i$*,*

$$\mathcal{R}_i = \{y : P_H(i) f_{Y|H}(y|i) \geq P_H(j) f_{Y|H}(y|j)\} \qquad i \neq j,$$

it is easy to see that in region $\mathcal{R}_0$

$$P_H(0) f_{Y|H}(y|0) \geq P_H(1) f_{Y|H}(y|1)$$

and vice-versa. Thus we can write

$$\begin{aligned}
P_e &= P_H(0) \int_{\mathcal{R}_1} f_{Y|H}(y|0)\ dy + P_H(1) \int_{\mathcal{R}_0} f_{Y|H}(y|1)\ dy \\
&= \int_{\mathcal{R}_1} \min\{P_H(0) f_{Y|H}(y|0), P_H(1) f_{Y|H}(y|1)\}\ dy \\
&\quad + \int_{\mathcal{R}_0} \min\{P_H(0) f_{Y|H}(y|0), P_H(1) f_{Y|H}(y|1)\}\ dy \\
&= \int_{\mathcal{R}_0+\mathcal{R}_1} \min\{P_H(0) f_{Y|H}(y|0), P_H(1) f_{Y|H}(y|1)\}\ dy \\
&= \int_y \min\{P_H(0) f_{Y|H}(y|0), P_H(1) f_{Y|H}(y|1)\}\ dy.
\end{aligned}$$

(b) *Without loss of generality, let us assume that* $a \leq b$*. Then* $\sqrt{b/a} \geq 1$ *and* $\min(a,b) = a \leq a\sqrt{b/a} = \sqrt{ab}$*.*

To show that for $a, b \geq 0, \sqrt{ab} \leq \frac{a+b}{2}$*, we proceed as follows. Let* $m = (a+b)/2$ *be the midpoint of an imaginary segment of the real line that goes from* a *to* b*. Let* $d = (b-a)/2$ *be half the distance between* a *and* b*. Writing* a *and* b *in terms of* m *and* d *we obtain* $ab = (m-d)(m+d) = m^2 - d^2 \leq m^2$*, which is the desired result.*

Considering this, we can write

$$\begin{aligned}
P_e &= \int_y \min\left\{P_H(0) f_{Y|H}(y|0), P_H(1) f_{Y|H}(y|1)\right\}\ dy \\
&\leq \int_y \sqrt{P_H(0) f_{Y|H}(y|0) P_H(1) f_{Y|H}(y|1)}\ dy \\
&= \sqrt{P_H(0) P_H(1)} \int_y \sqrt{f_{Y|H}(y|0) f_{Y|H}(y|1)}\ dy \\
&\leq \frac{P_H(0) + P_H(1)}{2} \int_y \sqrt{f_{Y|H}(y|0) f_{Y|H}(y|1)}\ dy \\
&= \frac{1}{2} \int_y \sqrt{f_{Y|H}(y|0) f_{Y|H}(y|1)}\ dy.
\end{aligned}$$

(c) *In the book, we upper bound* $P_e(i)$ *individually instead of upperbounding the final result,* $P_e = \sum_i P_H(i) P_e(i)$*. For the binary case, this is equivalent to*

$$\begin{aligned} P_e(0) &= \int_{\mathcal{R}_1} f_{Y|H}(y|0)\ dy \\ &= \int_{\mathcal{R}_1} \min\left\{f_{Y|H}(y|0), f_{Y|H}(y|1)\right\}\ dy \\ &\le \int_{\mathcal{R}_1} \sqrt{f_{Y|H}(y|0) f_{Y|H}(y|1)}\ dy \\ &\le \int_{y} \sqrt{f_{Y|H}(y|0) f_{Y|H}(y|1)}\ dy. \end{aligned}$$

The last step, which further loosens the bound, is necessary to find a bound of $P_e(0)$ that does not depend on $\mathcal{R}_1$. This "overbounding" is avoided in (b) by finding the bound over the whole P_e.

Solution 30. (Tighter union Bhattacharyya bound: M-ary case)

(a)

$$\mathcal{B}_{j,i} = \begin{cases} y : P_H(i) f_{Y|H}(y|i) \ge P_H(j) f_{Y|H}(y|j), & i < j \\ y : P_H(i) f_{Y|H}(y|i) > P_H(j) f_{Y|H}(y|j), & i > j \end{cases}$$

Therefore,

$$\mathcal{B}^c_{j,i} = \begin{cases} y : P_H(i) f_{Y|H}(y|i) < P_H(j) f_{Y|H}(y|j), & i < j \\ y : P_H(i) f_{Y|H}(y|i) \le P_H(j) f_{Y|H}(y|j), & i > j, \end{cases}$$

which is the same as $\mathcal{B}_{i,j}$.

(b) The probability of error is

$$\begin{aligned} P_e &= \sum_i \sum_{j \ne i} Pr\left\{Y \in \mathcal{R}_j | H = i\right\} P_H(i) \\ &= \sum_i \sum_{j > i} \left[Pr\left\{Y \in \mathcal{R}_j | H = i\right\} P_H(i) + Pr\left\{Y \in \mathcal{R}_i | H = j\right\} P_H(j)\right], \end{aligned}$$

where $\mathcal{R}_i$ is the decision region for hypothesis i. Beause $\mathcal{R}_j \subset \mathcal{B}_{i,j}$ and $\mathcal{R}_i \subset \mathcal{B}_{j,i}$, we can write

$$
\begin{aligned}
P_e &\le \sum_i \sum_{j>i} \left[Pr\left\{Y \in \mathcal{B}_{i,j} | H = i\right\} P_H(i) + Pr\left\{Y \in \mathcal{B}_{j,i} | H = j\right\} P_H(j) \right] \\
&= \sum_i \sum_{j>i} \left[Pr\left\{Y \in \mathcal{B}_{i,j} | H = i\right\} P_H(i) + Pr\left\{Y \in \mathcal{B}_{i,j}^c | H = j\right\} P_H(j) \right] \\
&= \sum_i \sum_{j>i} \left[\int_{\mathcal{B}_{i,j}} f_{Y|H}(y|i) P_H(i) \, dy + \int_{\mathcal{B}_{i,j}^c} f_{Y|H}(y|j) P_H(j) \, dy \right] \\
&\overset{(\star)}{=} \sum_i \sum_{j>i} \left[\int_{\mathcal{B}_{i,j}} \min\{f_{Y|H}(y|i) P_H(i), f_{Y|H}(y|j) P_H(j)\} \, dy \right. \\
&\quad \left. + \int_{\mathcal{B}_{i,j}^c} \min\{f_{Y|H}(y|i) P_H(i), f_{Y|H}(y|j) P_H(j)\} \, dy \right] \\
&= \sum_i \sum_{j>i} \left[\int_y \min\{f_{Y|H}(y|i) P_H(i), f_{Y|H}(y|j) P_H(j)\} \, dy \right].
\end{aligned}
$$

Relation $(\star)$ *follows from the definition of* $\mathcal{B}_{i,j}$.

(c) Using the hint, we obtain

$$
\begin{aligned}
P_e &\le \sum_i \sum_{j>i} \left[\int_y \sqrt{f_{Y|H}(y|i) P_H(i) f_{Y|H}(y|j) P_H(j)} \, dy \right] \\
&= \sum_i \sum_{j>i} \left[\sqrt{P_H(i) P_H(j)} \int_y \sqrt{f_{Y|H}(y|i) f_{Y|H}(y|j)} \, dy \right] \\
&\le \sum_i \sum_{j>i} \left[\frac{P_H(i) + P_H(j)}{2} \int_y \sqrt{f_{Y|H}(y|i) f_{Y|H}(y|j)} \, dy \right].
\end{aligned}
$$

Solution 31. (Applying the tight Bhattacharyya bound)

(a) Using the tight bhattacharyya bound*, we get*

$$
\begin{aligned}
P_e &\le \frac{1}{2} \int_y \sqrt{f_{Y|H}(y|0) f_{Y|H}(y|1)} \, dy \\
&= \frac{1}{2} \int_y \sqrt{\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y+a)^2}{2\sigma^2}\right) \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-a)^2}{2\sigma^2}\right)} \, dy \\
&= \frac{1}{2} \int_y \frac{1}{\sqrt{2\pi\sigma^2}} \sqrt{\exp\left(-\frac{y^2 + a^2}{\sigma^2}\right)} \, dy \\
&= \frac{1}{2} \exp\left(-\frac{a^2}{2\sigma^2}\right) \int_y \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{y^2}{2\sigma^2}\right) \, dy \\
&= \frac{1}{2} \exp\left(-\frac{a^2}{2\sigma^2}\right).
\end{aligned}
$$

(b) *The above bound is the same as the one from the problem statement, which was obtained working specifically with the expression for the Q-function. It is surprising that the* Bhattacharyya bound, *which applies to arbitrary channels, yields the same result.*

Solution 32. (Bhattacharyya bound for DMCs)

(a) *Inequality (a) follows from the* Bhattacharyya Bound.

Using the definition of DMC, it is straightforward to see that

$$\begin{aligned} P_{Y|X}(y|c_0) &= \prod_{i=1}^{n} P_{Y|X}(y_i|c_{0,i}) \qquad \textit{and} \\ P_{Y|X}(y|c_1) &= \prod_{i=1}^{n} P_{Y|X}(y_i|c_{1,i}). \end{aligned}$$

(b) follows by substituting the above values in (a).

Equality (c) is obtained by observing that $\sum_y$ *is the same as* $\sum_{y_1,\ldots,y_n}$ *(the first one being a vector notation for the sum over all possible* $y_1, \ldots, y_n$*).*

In (c), we see that we want the sum of all possible products. This is the same as summing over each y_i *and taking the product of the resulting sum for all* y_i*. This results in equality (d). We obtain (e) by writing (d) in a more concise form.*

When $c_{0,i} = c_{1,i}$, $\sqrt{P_{Y|X}(y|c_{0,i})P_{Y|X}(y|c_{1,i})} = P_{Y|X}(y|c_{0,i})$*. Therefore,*

$$\sum_y \sqrt{P_{Y|X}(y|c_{0,i})P_{Y|X}(y|c_{1,i})} = \sum_y P_{Y|X}(y|c_{0,i}) = 1.$$

This does not affect the product, so we are only interested in the terms where $c_{0,i} \neq c_{1,i}$*. We form the product of all such sums where* $c_{0,i} \neq c_{1,i}$*. We then look out for terms where* $c_{0,i} = a$ *and* $c_{1,i} = b, a \neq b$*, and raise the sum to the appropriate power. (Eg. If we have the product* $prpqrpqrr$*, we would write it as* $p^3q^2r^4$*). Hence equality (f).*

(b) *For a binary input channel, we have only two source symbols* $\mathcal{X} = \{a, b\}$*. Thus,*

$$\begin{aligned} P_e &\leq z^{n(a,b)} z^{n(b,a)} \\ &= z^{n(a,b)+n(b,a)} \\ &= z^{d_H(c_0,c_1)}. \end{aligned}$$

(c) *The value of* z *is:*

(i) *For a binary input Gaussian channel,*

$$\begin{aligned} z &= \int_y \sqrt{f_{Y|X}(y|0) f_{Y|X}(y|1)}\, dy \\ &= \exp\left(-\frac{E}{2\sigma^2}\right). \end{aligned}$$

(ii) For the Binary Symmetric Channel (BSC),

$$\begin{aligned} z &= \sqrt{Pr\{y=0|x=0\}Pr\{y=0|x=1\}} + \sqrt{Pr\{y=1|x=0\}Pr\{y=1|x=1\}} \\ &= 2\sqrt{\delta(1-\delta)}. \end{aligned}$$

(iii) For the Binary Erasure Channel (BEC),

$$\begin{aligned} z &= \sqrt{Pr\{y=0|x=0\}Pr\{y=0|x=1\}} + \sqrt{Pr\{y=E|x=0\}Pr\{y=E|x=1\}} \\ &\quad + \sqrt{Pr\{y=1|x=0\}Pr\{y=1|x=1\}} \\ &= 0+\delta+0 \\ &= \delta. \end{aligned}$$

Solution 33. (Bhattacharyya bound and Laplacian noise)

The Bhattacharyya Bound for the binary case with equally likely hypotheses is

$$P_e \le \int_y \sqrt{f_{Y|H}(y|0) f_{Y|H}(y|1)}\ dy = B(a).$$

By replacing the density functions

$$\begin{aligned} f_{Y|H}(y|0) &= \frac{1}{2}e^{-|y+a|} \\ f_{Y|H}(y|1) &= \frac{1}{2}e^{-|y-a|}, \end{aligned}$$

we obtain

$$B(a) = \int_{-\infty}^{\infty} \sqrt{\frac{1}{4}e^{-|y+a|}e^{-|y-a|}}\ dy = T_1 + T_2 + T_3,$$

$$\begin{aligned} T_1 &= \int_{-\infty}^{-a} \frac{1}{2}\sqrt{e^{(y+a)+(y-a)}}\ dy = \frac{1}{2}\int_{-\infty}^{-a} e^{y}\ dy = \frac{1}{2}e^{-a} \\ T_2 &= \int_{-a}^{a} \frac{1}{2}\sqrt{e^{-(y+a)+(y-a)}}\ dy = \frac{1}{2}\int_{-a}^{a} e^{-a} = ae^{-a} \\ T_3 &= \int_{a}^{-\infty} \frac{1}{2}\sqrt{e^{-(y+a)-(y-a)}}\ dy = T_1. \end{aligned}$$

Therefore, $B(a) = (1+a)e^{-a}$.

Solution 34. (Dice tossing)

(a) *Let $H = 0$ be the hypothesis for a fair dice, and $H = 1$ the hypothesis for a loaded dice. Then,*

$$\begin{aligned} P_{Y|H}(y|1) &= (1/4)^s (3/20)^{n-s}, \text{ and} \\ P_{Y|H}(y|0) &= (1/6)^n, \end{aligned}$$

where s is the number of 6's in the observed sequence.

The MAP decision rule is

$$\frac{P_{Y|H}(y|1)}{P_{Y|H}(y|0)} \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} \frac{P_H(0)}{P_H(1)} = 1.$$

Plugging in, we obtain

$$\frac{(1/4)^s \cdot (3/20)^{n-s}}{(1/6)^n} \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} 1.$$

By taking the logarithm, this becomes

$$\begin{aligned} n \log \frac{9}{10} + s \log \frac{5}{3} &\underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} 0, \text{ or} \\ s &\underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} \frac{n \log \frac{9}{10}}{\log \frac{3}{5}}. \end{aligned}$$

(b) *The sufficient statistic here is simply s, the number of 6's in the observed sequence.*

(c) *The Bhattacharyya bound can be found in three different ways.*

First solution: *Here we work with the observation $Y = (Y_1, \ldots, Y_n)$. We get*

$$\begin{aligned} Pr\left\{Y \in B_{0,1} | H = 0\right\} &\le \sum_{y \in \{1,2,\ldots,6\}^n} \sqrt{P_{Y|H}(y|1) P_{Y|H}(y|0)} \\ &= \sum_{y \in \{1,2,\ldots,6\}^n} \sqrt{(1/4)^{s(y)} \cdot (3/20)^{n-s(y)} \cdot (1/6)^n} \\ &\overset{(a)}{=} \sum_{s=0}^{n} \binom{n}{s} 5^{n-s} \sqrt{(1/4)^s \cdot (3/20)^{n-s} \cdot (1/6)^n} \\ &= \left(\sqrt{5/8}\right)^n \sum_{s=0}^{n} \binom{n}{s} \left(\sqrt{1/15}\right)^s \\ &\overset{(b)}{=} (\sqrt{5/8})^n \left(1 + \sqrt{1/15}\right)^n \\ &= \left(\sqrt{5/8} + \sqrt{1/24}\right)^n. \end{aligned}$$

In (a) we use the fact that for every $s \in \{0, \ldots, n\}$, there are $\binom{n}{s} 5^{n-s}$ different sequences in $\{1, \ldots, 6\}^n$ that have s sixes. In (b) we use the hint given in the assignment. The same *bound applies for $H = 1$. Hence,*

$$\begin{aligned} P_e &\le \frac{1}{2}\left(\sqrt{\frac{5}{8}}+\sqrt{\frac{1}{24}}\right)^n + \frac{1}{2}\left(\sqrt{\frac{5}{8}}+\sqrt{\frac{1}{24}}\right)^n \\ &= \left(\sqrt{\frac{5}{8}}+\sqrt{\frac{1}{24}}\right)^n. \end{aligned}$$

Second solution: *Here, we work with the observation $Z = (Z_1, \ldots, Z_n)$, where Z_i is defined as*

$$Z_i = \begin{cases} 1 & \text{if } Y_i = 6 \\ 0 & \text{otherwise.} \end{cases}$$

Note that the random variables Z_i are i.i.d., all distributed like Z, where Z is a Bernoulli random variable with $Pr\{Z = 1|H = 0\} = 1/6$ and $Pr\{Z = 1|H = 1\} = 1/4$.
The Bhattacharyya bound is

$$\begin{aligned} Pr\{Z \in B_{0,1}|H = 0\} &\le \sum_{z\in\{0,1\}^n} \sqrt{P_{Z|H}(z|1)P_{Z|H}(z|0)} \\ &= \sum_{z_1\in\{0,1\}}\sum_{z_2\in\{0,1\}}\cdots\sum_{z_n\in\{0,1\}}\prod_{i=1}^{n}\sqrt{P_{Z|H}(z_i|1)P_{Z|H}(z_i|0)} \\ &= \left(\sum_{z\in\{0,1\}}\sqrt{P_{Z|H}(z|1)P_{Z|H}(z|0)}\right)^n \\ &= \left(\sqrt{P_{Z|H}(0|1)P_{Z|H}(0|0)}+\sqrt{P_{Z|H}(1|1)P_{Z|H}(1|0)}\right)^n \\ &= \left(\sqrt{(3/4)\cdot(5/6)}+\sqrt{(1/4)\cdot(1/6)}\right)^n \\ &= \left(\sqrt{5/8}+\sqrt{1/24}\right)^n. \end{aligned}$$

Again, by symmetry, we find that

$$P_e \le \left(\sqrt{5/8}+\sqrt{1/24}\right)^n.$$

Third solution: *Here we work with the observation S, which is the number of sixes in $Y = (Y_1, \ldots, Y_n)$. Note that when $H = 0$ (when the dice is fair), S has a binomial distribution with parameters n and $(1/6)$. On the other hand, when $H = 1$ (when the dice is loaded), S has a binomial distribution with parameters n and $(1/4)$. Hence, we obtain*

$$\begin{aligned} Pr\{S \in B_{0,1}|H = 0\} &\le \sum_{s=0}^{n}\sqrt{P_{S|H}(s|1)P_{S|H}(s|0)} \\ &= \sum_{s=0}^{n}\sqrt{\binom{n}{s}(1/4)^s\cdot(3/4)^{n-s}\cdot\binom{n}{s}\cdot(1/6)^s\cdot(5/6)^{n-s}} \\ &= \ldots \\ &= \left(\sqrt{\frac{5}{8}}+\sqrt{\frac{1}{24}}\right)^n, \end{aligned}$$

where all remaining steps are the same as in the first solution, when using Y. Again, by symmetry, we find that

$$P_e \leq \left(\sqrt{5/8}+\sqrt{1/24}\right)^n.$$

Solution 35. (ML receiver and union bound for orthogonal signaling)

(a) *The ML decision rule is*

$$\begin{aligned}
\hat{H}_{ML}(y) &= \arg\max_i f_{Y|H}(y|i) \\
&= \arg\max_i \frac{1}{(2\pi\sigma^2)^{m/2}} \exp\left(-\frac{\|y-c_i\|^2}{2\sigma^2}\right) \\
&= \arg\min_i \|y-c_i\|.
\end{aligned}$$

Hence, the ML decision rule is a minimum distance decision rule.

(b)

$$\begin{aligned}
\|c_i-c_j\| &= \sqrt{\mathcal{E}}\|e_i-e_j\| \\
&= \sqrt{2\mathcal{E}},
\end{aligned}$$

where the second equality comes from the fact that e_i and e_j differ only in two positions, where one is equal to 0 and the other is equal to 1.

(c) *The upper bound for the error probability when $H=i$ is*

$$\begin{aligned}
P_e(i) &\leq \sum_{j:j\neq i} Pr\left\{Y\in\mathcal{B}_{i,j}|H=i\right\} \\
&= \sum_{j:j\neq i} \int_{\mathcal{B}_{i,j}} f_{Y|H}(y|i)\ dy \\
&= \sum_{j:j\neq i} Q\left(\frac{\|c_i-c_j\|}{2\sigma}\right) \\
&= (m-1)Q\left(\frac{\sqrt{2\mathcal{E}}}{2\sigma}\right).
\end{aligned}$$

Solution 36. (Uniform polar to Cartesian)

(a) *At first look it may seem that the probability is uniformly distributed over the disk, but in the next part we will show that this is not true.*

(b) *We know that* R *is uniformly distributed in* $[0,1]$ *and* Φ *is uniformly distributed in* $[0,2\pi)$, *so we have* $f_R(r) = 1$ *if* $0 \le r \le 1$ *and* $f_\Phi(\phi) = \frac{1}{2\pi}$ *if* $0 \le \phi < 2\pi$.

As these two random variables are independent, we have

$$f_{R,\Phi}(r,\phi) = \begin{cases} \frac{1}{2\pi} & 0 \le r \le 1 \textit{ and } 0 \le \phi < 2\pi \\ 0 & \textit{otherwise.} \end{cases}$$

It can be easily shown that the Jacobian determinant is $\det J = r = \sqrt{x^2+y^2}$. *Therefore, the probability distribution in cartesian coordinates is*

$$\begin{aligned} f_{X,Y}(x,y) &= \frac{1}{|\det J|} f_{R,\Phi}(r,\phi) \\ &= \begin{cases} \frac{1}{2\pi\sqrt{x^2+y^2}} & x^2+y^2 \le 1 \\ 0 & \textit{otherwise.} \end{cases} \end{aligned}$$

(c) *We see that the probability distribution is not distributed uniformly. This makes sense because rings of equal width have the same probability but not the same area.*

Solution 37. (Real-valued Gaussian random variables)

(a) *We have to find the* marginal *of* X *when* X *and* Y *are jointly distributed as in the first equation of this problem. The marginal of* X *is found by integrating* $f_{XY}(x,y)$ *over* y, *i.e.*

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(x,y)dy \\ &= \int_{-\infty}^{\infty} \frac{1}{2\pi\sqrt{\sigma_X^2\sigma_Y^2 - \sigma_{XY}^2}} \exp\left(-\frac{1}{2}\frac{\sigma_Y^2 x^2 + \sigma_X^2 y^2 - 2\sigma_{XY}xy}{\sigma_X^2\sigma_Y^2 - \sigma_{XY}^2}\right) dy \\ &= \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_X^2} - \frac{2\rho xy}{\sigma_X\sigma_Y} + \frac{y^2}{\sigma_Y^2}\right)\right) dy, \end{aligned}$$

where $\rho = \frac{\sigma_{XY}}{\sigma_X\sigma_Y}$.

We can rewrite

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{x^2}{2\sigma_X^2}\right) \exp\left(-\frac{1}{2(1-\rho^2)\sigma_Y^2}\left(y - \frac{\rho\sigma_Y x}{\sigma_X}\right)^2\right) dy \\ &= \frac{1}{\sqrt{2\pi\sigma_X^2}} \exp\left(-\frac{x^2}{2\sigma_X^2}\right) \\ &\quad \times \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma_y^2(1-\rho^2)}} \exp\left(-\frac{1}{2(1-\rho^2)\sigma_Y^2}\left(y - \frac{\rho\sigma_Y x}{\sigma_X}\right)^2\right) dy. \end{aligned}$$

The expression under the integral is a Gaussian density in the variable y, which integrates to one. Therefore,

$$f_X(x) \quad = \quad \frac{1}{\sqrt{2\pi\sigma_X^2}} \exp\left(-\frac{x^2}{2\sigma_X^2}\right),$$

showing that X is a zero-mean Gaussian random variable with variance σ_X^2. By symmetry, the computations for Y are the same.

(b) *For independent random variables,*

$$\begin{aligned} f_{XY}(x,y) &= f_X(x)f_Y(y) \\ &= \frac{1}{\sqrt{2\pi\sigma_X^2}} \exp\left(-\frac{x^2}{2\sigma_X^2}\right) \frac{1}{\sqrt{2\pi\sigma_Y^2}} \exp\left(-\frac{y^2}{2\sigma_Y^2}\right) \\ &= \frac{1}{2\pi\sqrt{\sigma_X^2\sigma_Y^2}} \exp\left(-\frac{1}{2}\left(\frac{x^2}{\sigma_X^2}+\frac{y^2}{\sigma_Y^2}\right)\right). \end{aligned}$$

Because X and Y are independent, $\sigma_{XY} = 0$. Thus, by setting Σ to be a diagonal matrix with entries σ_X^2 and σ_Y^2, we obtain the joint density function as in the problem statement.

(c) *Let $X \sim \mathcal{N}(0,1)$ be a Gaussian variable. Let U be a variable that is 1 with probability 0.5 and -1 with probability 0.5. Then, $Y = UX$ is not independent of X, but it is Gaussian. X and Y are not jointly Gaussian (e.g. $f_{X,Y}(1,2) = 0$).*

(d) *Any linear combination of independent Gaussian random variables is a Gaussian random variable. All we need to describe are the mean m and the variance σ_Z^2 of $X+Y$. They are $m_Z = m_X + m_Y = 0$ and $\sigma_Z^2 = \sigma_X^2 + \sigma_Y^2$. Hence*

$$f_{X+Y}(z) = \frac{1}{\sqrt{2\pi(\sigma_X^2+\sigma_Y^2)}} \exp\left(-\frac{z^2}{2(\sigma_X^2+\sigma_Y^2)}\right).$$

Alternatively, we can compute $f_Z = f_{X+Y} = \int f_X(x)f_Y(z-x)dx$.

Solution 38. (Correlation vs. independence)

(a) *We have,*

$$\begin{aligned} \mathbb{E}[X] &= \int_{-\infty}^{\infty} x f_X(x)dx = \int_{-1}^{1} z f_Z(z)dz = \int_{-1}^{1} z\frac{1}{2}dz = 0, \\ \mathbb{E}[Y] &= \int_{-\infty}^{\infty} y f_Y(y)dy = \int_{-1}^{1} z^2 f_Z(z)dz = \int_{-1}^{1} z^2\frac{1}{2}dz = \frac{1}{3}, \\ \mathbb{E}[XY] &= \int_{-\infty}^{\infty} z^3 f_Z(z)dz = \int_{-1}^{1} z^3\frac{1}{2}dz = 0. \end{aligned}$$

The covariance between X and Y is,

$$Cov(X,Y) = \mathbb{E}[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = 0.$$

Since $Cov(X,Y) = 0$, X and Y are uncorrelated.

(b) *X and Y are not independent because $Y = X^2$.*

(c) *Because X and Y are uncorrelated with variances σ_X^2 and σ_Y^2 respectively, the covariance matrix Σ is,*

$$\Sigma = \begin{pmatrix} \sigma_X^2 & 0 \\ 0 & \sigma_Y^2 \end{pmatrix}.$$

The joint distribution of X and Y is given by,

$$\begin{aligned}
f_{XY}(x,y) &= \frac{1}{2\pi\sqrt{\det\Sigma}} \exp\left(-\frac{1}{2}\begin{pmatrix} x, & y \end{pmatrix}\Sigma^{-1}\begin{pmatrix} x \\ y \end{pmatrix}\right) \\
&= \frac{1}{2\pi\sqrt{\sigma_X^2\sigma_Y^2}} \exp\left(-\frac{x^2}{2\sigma_X^2} - \frac{y^2}{2\sigma_Y^2}\right) \\
&= \frac{1}{2\pi\sqrt{\sigma_X^2\sigma_Y^2}} \exp\left(-\frac{x^2}{2\sigma_X^2}\right) \exp\left(-\frac{y^2}{2\sigma_Y^2}\right) \\
&= \frac{1}{\sqrt{2\pi\sigma_X^2}} \exp\left(-\frac{x^2}{2\sigma_X^2}\right) \frac{1}{\sqrt{2\pi\sigma_Y^2}} \exp\left(-\frac{y^2}{2\sigma_Y^2}\right) \\
&= f_X(x) f_Y(y).
\end{aligned}$$

Since $f_{XY}(x,y) = f_X(x)f_Y(y)$, X and Y are independent. In general for jointly Gaussian random variables, uncorrelation implies independence.

Solution 39. (Data-storage channel)

(a) *The density functions are a Gaussian centered at 0 and a Gaussian of larger variance centered at 1. They intersect in two points, say y_1 and y_2 ($y_1 < y_2$). The decoding region for $\hat{H} = 0$ is $\mathcal{R}_0 = [y_1, y_2]$. This is depicted in Figure 2.15.*

(b) *Formally, $\mathcal{R}_0$ is the set of y for which*

$$\frac{1}{\sqrt{2\pi\sigma_0^2}} e^{-\frac{y^2}{2\sigma_0^2}} \geq \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{(y-1)^2}{2\sigma_1^2}}.$$

Let us find the two points y_1 and y_2 for which equality holds. They are the solution to

$$\frac{(y-1)^2}{2\sigma_1^2} - \frac{y^2}{2\sigma_0^2} = \ln\frac{\sigma_0}{\sigma_1}.$$

Hence we are looking for the roots of $ay^2 + by + c$ for

$$\begin{aligned}
a &= \frac{\sigma_0^2 - \sigma_1^2}{2\sigma_0^2\sigma_1^2} \\
b &= -\frac{1}{\sigma_1^2} \\
c &= \frac{1}{2\sigma_1^2} - \ln\frac{\sigma_0}{\sigma_1}.
\end{aligned}$$

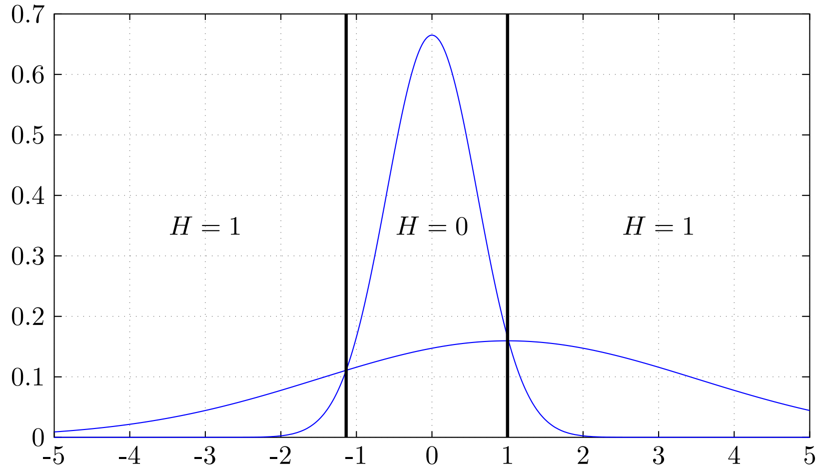


Figure 2.15: Decision regions

The solution is $y_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ *and, after a few steps, one obtains*

$$y_{1,2} = \frac{\sigma_0}{\sigma_0^2 - \sigma_1^2}\left[\sigma_0 \pm \sigma_1 \sqrt{1 + 2(\sigma_0^2 - \sigma_1^2)\ln\frac{\sigma_0}{\sigma_1}}\,\right].$$

Therefore, the optimal receiver decides $\hat{H} = 0$ *if* $y_1 \le y \le y_2$*, and* $\hat{H} = 1$ *otherwise.*

(c) The probability of error when $H = 0$ *is*

$$P_e(0) = 1 - Q\left(\frac{y_1}{\sigma_0}\right) + Q\left(\frac{y_2}{\sigma_0}\right)$$

and when $H = 1$ *is*

$$P_e(1) = Q\left(\frac{y_1 - 1}{\sigma_1}\right) - Q\left(\frac{y_2 - 1}{\sigma_1}\right).$$

Therefore,

$$P_e = \frac{1}{2}[P_e(0) + P_e(1)].$$

Solution 40. (A simple multiple-access scheme)

(a) Because the prior probabilities are the same, the MAP and ML decision rules are the identical. The optimal decision rule is

$$\hat{H}(y) \;= arg\max_{h_1 h_2} f_{Y|H_1 H_2}(y|h_1, h_2) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(y - x_1 - x_2)^2}{2\sigma^2}\right).$$

(b) For $\sigma^2 = 0$*, the signal for each hypothesis is*

H_1H_2	00	01	10	11
$X = X_1 + X_2$	3	-1	1	-3

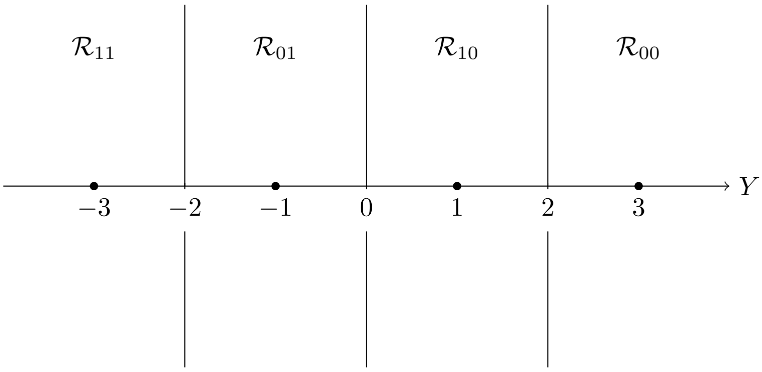


Figure 2.16: Optimal decision regions

(c) *The optimal decision regions are presented in Figure 2.16.*

(d) *Because of the symmetry,* $P_c(11) = P_c(00)$ *and* $P_c(01) = P_c(10)$*.*

$$\begin{aligned}
P_c(11) &= Pr\{Y \le -2 | H_1 = 1 \cap H_2 = 1\} \\
&= Pr\{Z \le 1\} \\
&= Q\left(-\frac{1}{\sigma}\right).
\end{aligned}$$

$$\begin{aligned}
P_c(01) &= Pr\{-2 \le Y \le 0 | H_1 = 0 \cap H_2 = 1\} \\
&= Pr\{-1 \le Z \le 1\} \\
&= Q\left(-\frac{1}{\sigma}\right) - Q\left(\frac{1}{\sigma}\right).
\end{aligned}$$

Putting things together, we obtain

$$\begin{aligned}
Pr\left\{\hat{H}_1 = H_1, \hat{H}_2 = H_2\right\} &= \frac{1}{4}\left(P_c(00) + P_c(01) + P_c(10) + P_c(11)\right) \\
&= Q\left(-\frac{1}{\sigma}\right) - \frac{1}{2}Q\left(\frac{1}{\sigma}\right).
\end{aligned}$$

(e) *The receiver decides* $\hat{H}_2 = 1$ *if* $y \le 0$*. Because of the symmetry,* $Pr\left\{\hat{H}_2 = H_2\right\} = P_{cH_2}(0) = P_{cH_2}(1)$*.*

$$\begin{aligned}
P_{cH_2}(1) &= Pr\{H_1 = 0\} Pr\{Y \le 0 | H_1 = 0 \cap H_2 = 1\} \\
&\quad + Pr\{H_1 = 1\} Pr\{Y \le 0 | H_1 = 1 \cap H_2 = 1\} \\
&= \frac{1}{2} Pr\{Z \le 1\} + \frac{1}{2} Pr\{Z \le 3\} \\
&= \frac{1}{2}\left(Q\left(-\frac{3}{\sigma}\right) + Q\left(-\frac{1}{\sigma}\right)\right).
\end{aligned}$$

Solution 41. (Data-dependent noise)

(a) The optimal decision rule is

$$f_{Y|H}(y|1) \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} f_{Y|H}(y|0), \text{ or}$$

$$\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-1)^2}{2\sigma^2}\right) \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(y+1)^2}{2}\right).$$

After taking the logarithm we obtain

$$0 \underset{\hat{H}=0}{\overset{\hat{H}=1}{\gtrless}} \frac{(y-1)^2}{2\sigma^2} - \frac{(y+1)^2}{2} + \ln\sigma$$

(b) If $\sigma^2 = e^4$, the equality sign is obtained for

$$\begin{aligned}
y_1 &= -\frac{3e^4+1}{e^4-1} \\
y_2 &= 1.
\end{aligned}$$

The receiver decides $\hat{H} = 0$ if $y_1 \le y \le y_2$, and $\hat{H} = 1$ otherwise.

(c)

$$\begin{aligned}
P_e(0) &= Pr\{Y < y_1 \cup Y > y_2 | H = 0\} \\
&= \int_{-\infty}^{y_1} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(y+1)^2}{2}\right) dy + \int_{y_2}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(y+1)^2}{2}\right) dy \\
&= 1 - Q(y_1+1) + Q(y_2+1).
\end{aligned}$$

$$\begin{aligned}
P_e(1) &= Pr\{y_1 < Y < y_2 | H = 1\} \\
&= \int_{y_1}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-1)^2}{2\sigma^2}\right) dy - \int_{y_2}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-1)^2}{2\sigma^2}\right) dy \\
&= Q\left(\frac{y_1-1}{\sigma}\right) - Q\left(\frac{y_2-1}{\sigma}\right) \\
&= Q\left(\frac{y_1-1}{e^2}\right) - Q\left(\frac{y_2-1}{e^2}\right).
\end{aligned}$$

Solution 42. (Correlated noise)

(a) *Let $\hat{Z} = BZ$. The covariance matrix of $\hat{Z}$ is*

$$\hat{\Sigma} = B\Sigma B^{\mathsf{T}}.$$

Using the hint, we set $B = \frac{1}{4}\begin{pmatrix} 2 & 0 \\ -1 & 2 \end{pmatrix}$ and obtain $\hat{\Sigma} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Because it is obtained by a linear combination of zero-mean Gaussian random variables, the vector $\hat{Z}$ is also zero-mean Gaussian. Because its covariance matrix is the unity matrix, its components are independent, and have variance equal to 1.

(b) *For $H = i$, the covariance matrix of $\hat{Y}$ is $\hat{\Sigma}$, and the probability density function is*

$$\begin{aligned} f_{\hat{Y}|H}(\hat{y}|i) &= \frac{1}{2\pi\sqrt{\det\hat{\Sigma}}}\exp\left(-\frac{1}{2}(y - Bc_i)^{\mathsf{T}}\hat{\Sigma}^{-1}(y - Bc_i)\right) \\ &= \frac{1}{2\pi}\exp\left(-\frac{1}{2}(y - Bc_i)^{\mathsf{T}}(y - Bc_i)\right). \end{aligned}$$

The hypothesis testing problem becomes

$$\begin{aligned} \hat{H}(y) &= \arg\max_i \frac{1}{2\pi}\exp\left(-\frac{1}{2}(y - Bc_i)^{\mathsf{T}}(y - Bc_i)\right) \\ &= \arg\min_i (y - Bc_i)^{\mathsf{T}}(y - Bc_i) \\ &= \arg\min_i \|y - \hat{c}_i\|, \end{aligned}$$

where $\hat{c}_i = Bc_i$.

The decision regions are depicted (qualitatively) in Figure 2.17.

(c) *We determine the union bound for the probability of error. Recall that*

$$P_e(i) \le \sum_{j:j\neq i}\int_{\mathcal{B}_{i,j}} f_{\hat{Y}|H}(\hat{y}|i)dy.$$

For our problem, this becomes

$$\begin{aligned} P_e(i) &\le \sum_{j:j\neq i} Q\left(\frac{\|\hat{c}_j - \hat{c}_i\|}{2\sigma}\right) \\ &= \sum_{j:j\neq i} Q\left(\frac{\|\hat{c}_j - \hat{c}_i\|}{2}\right). \end{aligned}$$

The bound for the error probability is

$$\begin{aligned} P_e &= \sum_i P_H(i)P_e(i) \\ &\le \sum_i \frac{1}{4}\sum_{j:j\neq i} Q\left(\frac{\|\hat{c}_j - \hat{c}_i\|}{2}\right) \\ &= \frac{1}{2}\sum_i\sum_{j:j>i} Q\left(\frac{\|\hat{c}_j - \hat{c}_i\|}{2}\right). \end{aligned}$$

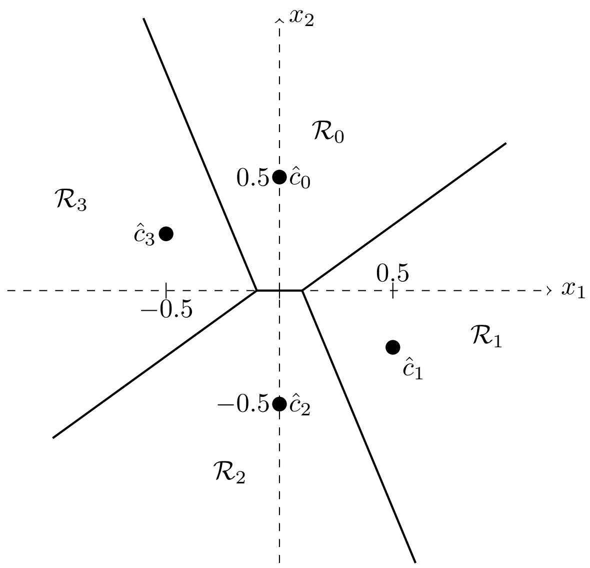


Figure 2.17: Decision regions

Chapter 3

Receiver Design for Continuous-Time AWGN Channels: Second Layer

Solution 1. (Gram–Schmidt procedure on tuples)

We start by normalizing β_1:

$$\begin{aligned} \|\beta_1\| &= \sqrt{\langle \beta_1, \beta_1 \rangle} = \sqrt{3} \\ \psi_1 &= \frac{\beta_1}{\|\beta_1\|} = (\frac{1}{\sqrt{3}}, 0, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}). \end{aligned}$$

We get the next basis vectors as follows:

$$\begin{aligned} \langle \psi_1, \beta_2 \rangle &= \sqrt{3} \\ \phi_2 &= \beta_2 - \sqrt{3}\psi_1 = (1, 1, -1, 0) \\ \|\phi_2\| &= \sqrt{3} \\ \psi_2 &= \frac{\phi_2}{\|\phi_2\|} = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, 0). \end{aligned}$$

We compute

$$\begin{aligned} \langle \psi_1, \beta_3 \rangle &= 0 \\ \langle \psi_2, \beta_3 \rangle &= 0. \end{aligned}$$

Thus,

$$\begin{aligned} \phi_3 &= \beta_3 - 0\psi_1 - 0\psi_2 = (1, 0, 1, -2) \\ \|\phi_3\| &= \sqrt{1+1+4} = \sqrt{6} \\ \psi_3 &= \frac{\phi_3}{\|\phi_3\|} = (\frac{1}{\sqrt{6}}, 0, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}). \end{aligned}$$