Instructor's Solutions Manual for

Photonics

Optical Electronics in Modern Communications

SIXTH EDITION

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To the Instructor:

In the present (sixth) edition of *Photonics,* we have added several new topics on related technology in optical electronics and communications. We have also deleted some less important topics in the main text. Reflecting the revision, the solutions manual is also being revised accordingly. Many of the problems in this manual are based on actual problems assigned to the Applied Physics 130 class at Caltech and the ECE 26 class at UC Santa Barbara through 2004. The solutions have thus been tested and debugged by the students.

The current solutions manual is based on the last (fifth) edition. We have added solutions for the new problems and deleted the problems that are associated with the less important topics deleted in the present edition. The authors acknowledge contributions to the solutions manual of the fifth edition by students and co-workers, including Huey-Daw Wu, Frank Barnes, Bin Zhao, Bin Zhao, Shu Wu Wu, Randolf Hofmeister, Boaz Salik, and George Barbastathis.

Good luck with the teaching of this course.

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Pasadena, California
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Contents

Chapter 1

1.1 *Solution*:
Taking " $\nabla\cdot$ " operation on both sides of Eq. (1.1-2) and using $\nabla\cdot(\nabla\times\mathbf{H})=0$, we obtain

$$-\frac{\partial}{\partial t}\nabla\cdot\mathbf{D}=\nabla\cdot\mathbf{J}$$

This proves the conservation of charge after using Eq. (1.1-3).

1.2 *Solution*:
Using $\int\nabla\cdot\mathbf{D}dV=\oint\mathbf{D}\cdot d\mathbf{S}=\oint\mathbf{D}\cdot\mathbf{n}da$, where **n** is an outward normal unit vector. The surface of the pillbox shown in Figure 1.1 can be divided into three parts: a top circle, a bottom circle and a ring.

$$\oint\mathbf{D}\cdot\mathbf{n}da=\int_{S_1}\mathbf{D}\cdot\mathbf{n}da+\int_{S_2}\mathbf{D}\cdot\mathbf{n}da+\int_{\text{ring}}\mathbf{D}\cdot\mathbf{n}da$$

$$=A(\mathbf{D}_1\cdot\mathbf{n}_1+\mathbf{D}_2\cdot\mathbf{n}_2)=A(-\mathbf{D}_1\cdot\mathbf{n}_2+\mathbf{D}_2\cdot\mathbf{n}_2)=A(-\mathbf{D}_1\cdot\mathbf{n}+\mathbf{D}_2\cdot\mathbf{n})=Q$$

where Q is the total charge within the pillbox, A is the area of the circle, $\mathbf{n}_2 = -\mathbf{n}_1 = \mathbf{n}$. The integral over the ring approaches zero. The proof for **B** is similar.

1.3 *Solution*:
Using $\int(\nabla\times\mathbf{H})\cdot d\mathbf{S}=\oint\mathbf{H}\cdot d\vec{\ell}$ and the rectangular path shown in Figure 1.1, we obtain

$\int\mathbf{J}\cdot d\mathbf{S}=\Delta I=\oint\mathbf{H}\cdot d\vec{\ell}=(\mathbf{t}\times\mathbf{n})\cdot(\mathbf{H}_2-\mathbf{H}_1)\Delta L$, where ΔI is the current flowing through the rectangular area, ΔL is the length of the rectangular path, **t** is a unit tangent vector perpendicular to the rectangle and parallel to the interface, **n** is a unit normal to the interface. Since $\mathbf{K}\cdot\mathbf{t}=\Delta I/\Delta L$ and $(\mathbf{t}\times\mathbf{n})\cdot(\mathbf{H}_2-\mathbf{H}_1)=\mathbf{t}\cdot[\mathbf{n}\times(\mathbf{H}_2-\mathbf{H}_1)]$, we obtain (1.1-11). The proof for **E** is similar.

1.4 *Solution*: Direct substitution into Maxwell's equations.

1.5 *Solution*:
Take divergence of the stress tensor

$$\nabla\cdot\mathbf{T}=\nabla\cdot(\varepsilon\mathbf{EE}+\mu\mathbf{HH})-\nabla(\varepsilon E^2+\mu H^2)/2$$
$$=\varepsilon(\nabla\cdot\mathbf{E})\mathbf{E}+\mu(\nabla\cdot\mathbf{H})\mathbf{H}+\varepsilon(\mathbf{E}\cdot\nabla)\mathbf{E}+\mu(\mathbf{H}\cdot\nabla)\mathbf{H}$$
$$-\varepsilon(\mathbf{E}\cdot\nabla)\mathbf{E}-\mu(\mathbf{H}\cdot\nabla)\mathbf{H}-\varepsilon\mathbf{E}\times(\nabla\times\mathbf{E})-\mu\mathbf{H}\times(\nabla\times\mathbf{H})$$
$$=\rho\mathbf{E}-\varepsilon\mathbf{E}\times(\nabla\times\mathbf{E})-\mu\mathbf{H}\times(\nabla\times\mathbf{H})$$

where we assume ε and μ are constants.
Then take the time derivative of the momentum density

$$\frac{\partial\mathbf{P}}{\partial t}=\mu\varepsilon(\frac{\partial}{\partial t}\mathbf{E})\times\mathbf{H}+\mu\varepsilon\mathbf{E}\times\frac{\partial}{\partial t}\mathbf{H}=-\mu\mathbf{H}\times\nabla\times\mathbf{H}-\varepsilon\mathbf{E}\times\nabla\times\mathbf{E}-\mathbf{J}\times\mathbf{B}$$

We note $\rho\mathbf{E}+\mathbf{J}\times\mathbf{B}$ is the Lorentz force.
This proves the equation of motion.

1.6 *Solution*:

(a) Work done = Force x distance. $dW = \mathbf{F}\cdot d\mathbf{x} = q\mathbf{E}\cdot d\mathbf{x} = q\mathbf{E}\cdot \mathbf{v}dt$

(b) $dW = \sum_i \mathbf{F}_i \cdot d\mathbf{x}_i = \sum_i q_i \mathbf{E}\cdot d\mathbf{x}_i$

(c) $dW == \sum_i d\mathbf{x}_i \cdot q_i \mathbf{E} = \mathbf{E}\cdot \sum_i d\mathbf{x}_i q_i = \mathbf{E}\cdot d\mathbf{P}$

1.7 *Solution*:

(a) The flight time is given by

$$\tau = L / \mathrm{v}_g = Ldk/d\omega = L\frac{d}{d\omega}\left(\frac{\omega}{c}n\right) = L\left(\frac{n}{c}+\frac{\omega}{c}\frac{dn}{d\omega}\right).$$

Taking the differential on both sides of $\lambda\omega = 2\pi c$, we obtain $\lambda d\omega + \omega d\lambda = 0$.

Thus, $L\left(\frac{n}{c}+\frac{\omega}{c}\frac{dn}{d\omega}\right) = L\left(\frac{n}{c}-\frac{\lambda}{c}\frac{dn}{d\lambda}\right)$.

(b) Using the result from (a), we can write $D = \frac{d}{d\lambda}\left(\frac{n}{c}-\frac{\lambda}{c}\frac{dn}{d\lambda}\right) = -\frac{\lambda}{c}\frac{d^2n}{d\lambda^2} = -\frac{1}{c\lambda}\lambda^2\frac{d^2n}{d\lambda^2}$

(c) $(\mathrm{v}_2-\mathrm{v}_g)(\mathrm{v}_1-\mathrm{v}_g) = \left(\frac{\omega_2}{k_2}-\frac{\omega_2-\omega_1}{k_2-k_1}\right)\left(\frac{\omega_1}{k_1}-\frac{\omega_2-\omega_1}{k_2-k_1}\right) = \frac{(\omega_1 k_2-\omega_2 k_1)^2}{k_1 k_2 (k_2-k_1)^2} > 0$

1.8 *Solution*:

Develop a simple computer program to plot the dispersion curves.

1.9 *Solution*:

Without loss of generality, we assume

$$E_x = A_x \cos(\omega t - \delta/2)$$
$$E_y = A_y \cos(\omega t + \delta/2)$$

Expand the cosine functions, we obtain

$E_x / A_x = \cos(\omega t - \delta/2) = \cos(\omega t)\cos(\delta/2) + \sin(\omega t)\sin(\delta/2)$

$E_y / A_y = \cos(\omega t + \delta/2) = \cos(\omega t)\cos(\delta/2) - \sin(\omega t)\sin(\delta/2)$

Addition and subtraction of above equations lead to

$E_x / A_x + E_y / A_y = 2\cos(\omega t)\cos(\delta/2)$

$E_x / A_x - E_y / A_y = 2\sin(\omega t)\sin(\delta/2)$

and then

$\sin(\delta/2)[E_x / A_x + E_y / A_y] = 2\cos(\omega t)\sin(\delta/2)\cos(\delta/2) = \cos(\omega t)\sin\delta$

$\cos(\delta/2)[E_x / A_x - E_y / A_y] = 2\sin(\omega t)\sin(\delta/2)\cos(\delta/2) = \sin(\omega t)\sin\delta$

We now add the square of the above equations and obtain

$\sin^2(\delta/2)[E_x / A_x + E_y / A_y]^2 + \cos^2(\delta/2)[E_x / A_x - E_y / A_y]^2 = \sin^2\delta$

Using $\sin^2(\delta/2) + \cos^2(\delta/2) = 1$ and $\cos^2(\delta/2) - \sin^2(\delta/2) = \cos\delta$ lead to Eq. (1.6-12).

1.10 *Solution*:

Using the coordinate rotation of Figure 1.4, we have

$$E_x = E_{x'} \cos\phi - E_{y'} \sin\phi$$

$$E_y = E_{x'} \sin\phi + E_{y'} \cos\phi$$

Substitution of above equation into (1.6-12), we obtain, after multiplying both sides by $A_x^2 A_y^2$

$$A_y^2(E_{x'}\cos\phi - E_{y'}\sin\phi)^2 + A_x^2(E_{x'}\sin\phi + E_{y'}\cos\phi)^2$$

$$-2A_xA_y\cos\delta(E_{x'}\cos\phi - E_{y'}\sin\phi)(E_{x'}\sin\phi + E_{y'}\cos\phi) = A_x^2A_y^2\sin^2\delta$$

or

$$E_{x'}^2(A_x^2\sin^2\phi + A_y^2\cos^2\phi - 2A_xA_y\cos\delta\cos\phi\sin\phi) +$$

$$E_{y'}^2(A_x^2\cos^2\phi + A_y^2\sin^2\phi + 2A_xA_y\cos\delta\cos\phi\sin\phi) +$$

$$E_{x'}E_{y'}(-2A_y^2\cos\phi\sin\phi + 2A_x^2\cos\phi\sin\phi + 2A_xA_y\cos\delta\sin^2\phi - 2A_xA_y\cos\delta\cos^2\phi) = A_x^2A_y^2\sin^2\delta$$

In the principal coordinate, the equation must be of the following form

$E_{x'}^2/a^2 + E_{y'}^2/b^2 = 1$, or equivalently $b^2E_{x'}^2 + a^2E_{y'}^2 = a^2b^2$

Thus, we obtain

$$\tan 2\phi = \frac{2A_xA_y}{A_x^2 - A_y^2}\cos\delta$$

$$a^2 = A_x^2\cos^2\phi + A_y^2\sin^2\phi + 2A_xA_y\cos\delta\cos\phi\sin\phi$$

$$b^2 = A_x^2\sin^2\phi + A_y^2\cos^2\phi - 2A_xA_y\cos\delta\cos\phi\sin\phi$$

The equality $a^2b^2 = A_x^2A_y^2\sin^2\delta$ can be proven by using the above three equations.

We obtain

Eq.1 $\quad a^2 + b^2 = A_x^2 + A_y^2$

Eq.2 $\quad a^2 - b^2 = (A_x^2 - A_y^2)\cos 2\phi + 2A_xA_y\cos\delta\sin 2\phi$

Eq.3 $\quad 0 = (A_x^2 - A_y^2)\sin 2\phi - 2A_xA_y\cos\delta\cos 2\phi$

We now calculate (Eq. 1)^2 - (Eq. 2)^2 - (Eq. 3)^2. This leads to $a^2b^2 = A_x^2A_y^2\sin^2\delta$.

1.11 *Solution*:
Without loss of generality, we assume

$$E_x = A_x \cos(\omega t), \qquad E_y = A_y \cos(\omega t + \delta)$$

We now examine the electric field vector at ωt=π/2−δ, and ωt= π/2−δ + Δt. We obtain

ωt=π/2−δ: $\quad E_x = A_x \sin\delta, \qquad E_y = 0$

ωt= π/2−δ + Δt: $\quad E_x = A_x \sin(\delta - \Delta t), \qquad E_y = -A_y \sin(\Delta t)$

We see that the polarization revolve in a clockwise direction if $\sin\delta > 0$.

1.12 *Solution*:
(b) We find the inclination angle of the major axis of the polarization ellipse.

$$\tan 2\phi_1 = \frac{2\cos\psi\sin\psi}{\cos^2\psi - \sin^2\psi}\cos\delta = \tan 2\psi\cos\delta$$

$$\tan 2\phi_2 = \frac{2\sin\psi\cos\psi}{-\cos^2\psi + \sin^2\psi}\cos(\pi+\delta) = \tan 2\psi\cos\delta = \tan 2\phi_1$$

So, $2\phi_2 = 2\phi_1 + m\pi$, where m is an integer. In other words, the major axes are either parallel or perpendicular. To show that the major axes of the polarization ellipses of the two states are mutually orthogonal, examine some special cases (e.g., ψ=0 or δ=0) and calculate the length *a* for the two states.
The length of the major axes and minor axes can be calculated by using Eq. (1.6-14).

$$a^2 = A_x^2\cos^2\phi + A_y^2\sin^2\phi + 2A_xA_y\cos\delta\cos\phi\sin\phi$$

$$b^2 = A_x^2\sin^2\phi + A_y^2\cos^2\phi - 2A_xA_y\cos\delta\cos\phi\sin\phi$$

$$a_1^2 = \cos^2\psi\cos^2\phi + \sin^2\psi\sin^2\phi + 2\cos\psi\sin\psi\cos\delta\cos\phi\sin\phi$$

$$b_1^2 = \cos^2\psi\sin^2\phi + \sin^2\psi\cos^2\phi - 2\cos\psi\sin\psi\cos\delta\cos\phi\sin\phi$$

$$a_2^2 = \sin^2\psi\cos^2\phi + \cos^2\psi\sin^2\phi + 2\cos\psi\sin\psi\cos(\delta+\pi)\cos\phi\sin\phi$$
$$= \sin^2\psi\cos^2\phi + \cos^2\psi\sin^2\phi - 2\cos\psi\sin\psi\cos\delta\cos\phi\sin\phi$$

$$b_2^2 = \sin^2\psi\sin^2\phi + \cos^2\psi\cos^2\phi - 2\cos\psi\sin\psi\cos(\delta+\pi)\cos\phi\sin\phi$$
$$= \sin^2\psi\sin^2\phi + \cos^2\psi\cos^2\phi + 2\cos\psi\sin\psi\cos\delta\cos\phi\sin\phi$$

We note $a_1^2 = b_2^2$ and $a_2^2 = b_1^2$. Thus, the major axes are indeed orthogonal.
The senses of revolution are opposite since $\sin\delta\sin(\pi+\delta) = -\sin^2\delta < 0$.

1.13 *Solution*:
From Problem 1.12, we have

$$\tan 2\phi = \tan 2\psi \cos\delta = \frac{2\tan\psi}{1-\tan^2\psi}\cos\delta = \frac{\text{Re}[\chi]}{1-|\chi|^2}$$. This is Eq. (1.6-18).

Using the definition $\tan\theta = \pm b/a$, we have $\sin 2\theta = 2\sin\theta\cos\theta = 2ab/(a^2+b^2)$.
Using $a^2b^2 = A_x^2A_y^2\sin^2\delta$ and $a^2+b^2 = A_x^2+A_y^2$ from Problem 1.10, we obtain

$$\sin 2\theta = 2ab/(a^2+b^2) = -2A_xA_y\sin\delta/(A_x^2+A_y^2) = -2\,\text{Im}[\chi]/(1+|\chi|^2)$$

where the choice of " - " sign is consistent with the sense of revolution of the polarization ellipse.

1.14 *Solution*:
(a) Without loss of generality, we assume

$$\mathbf{A} = \begin{pmatrix} \cos\psi_a \\ e^{i\delta_a}\sin\psi_a \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} \cos\psi_b \\ e^{i\delta_b}\sin\psi_b \end{pmatrix}$$

$\mathbf{A}^*\cdot\mathbf{B} = 0$ leads to $\cos\psi_a\cos\psi_b + \sin\psi_a\sin\psi_b\exp[i(\delta_b-\delta_a)] = 0$ which leads to $\delta_a - \delta_b = \pm\pi$ and $\cos(\psi_a+\psi_b) = 0$.
(b) $\delta_a\delta_b < 0$ follows immediately from the condition that $-\pi < \delta < \pi$. If one of the phases is π, then the other phase must be zero. This proves $\delta_a\delta_b \le 0$.
(c)

$$\chi_a^*\chi_b = \frac{\sin\psi_a\sin\psi_b}{\cos\psi_a\cos\psi_b}\exp[i(\delta_b-\delta_a)] = -\frac{\sin\psi_a\sin\psi_b}{\cos\psi_a\cos\psi_b} = -1$$

(d) From Eq. (1.6-18), and $\delta_a - \delta_b = \pm\pi$ and $\cos(\psi_a+\psi_b) = 0$,

$$\tan 2\phi_a = \tan 2\psi_a\cos\delta_a$$

$$\tan 2\phi_b = \tan 2\psi_b\cos\delta_b = \tan 2(\pi/2-\psi_a)\cos(\delta_a-\pi) = \tan 2\psi_a\cos\delta_a = \tan 2\phi_a$$

So, the major axes are either parallel or orthogonal. To show that the major axes of the polarization ellipses of the two states are mutually orthogonal, examine some special cases (e.g., ψ=0 or δ=0) and calculate the length *a* for the two states.
Follow the same approach used in Problem 1.12(b).

1.15 *Solution*:
In the principal coordinate, the polarization ellipse can be written

$$E_{x'} = a\cos(\omega t)$$

$$E_{y'} = b\cos(\omega t \pm \pi/2) = b\sin(\omega t)$$

We note that in the principal coordinate the two orthogonal polarization components are out of phase by π/2. Align the wave plate with a phase retardation of π/2 so that its slow axis (or fast axis) is parallel (or perpendicular) to the one of the principal axes of the polarization ellipse. The output is a linear polarization state.

1.16 *Solution*:
Without loss of generality, we assume

$$E_x = A_x \cos(\omega t - kz), \qquad E_y = A_y \cos(\omega t - kz + \delta)$$

At z=0, the temporal variation is written $E_x = A_x \cos(\omega t)$, $\qquad E_y = A_y \cos(\omega t + \delta)$

At t=0, the spatial variation is written $E_x = A_x \cos(-kz)$, $\qquad E_y = A_y \cos(-kz + \delta)$

A direct comparison shows that the spatial variation is equivalent to time-reversed variation. Thus, the E-vector of right-hand circular polarized light will appear left-handed in the space domain, and vice versa.

(a) Let $\mathbf{E} = \mathbf{R} + e^{i\delta}\mathbf{L}$, where δ is an arbitrary phase shift. The (x, y) components of the complex field amplitudes can be written

$$E_x = \frac{1}{\sqrt{2}}(1 + e^{i\delta})\exp[i(\omega t - kz)], \qquad E_y = \frac{1}{\sqrt{2}}(-i + ie^{i\delta})\exp[i(\omega t - kz)]$$

We now examine the ratio of the complex amplitudes,

$$\frac{(-i + ie^{i\delta})}{(1 + e^{i\delta})} = \frac{(-i + ie^{i\delta})(1 + e^{-i\delta})}{(1 + e^{i\delta})(1 + e^{-i\delta})} = \frac{-i + i + i(e^{i\delta} - e^{-i\delta})}{2 + 2\cos\delta} = \frac{-2\sin\delta}{2 + 2\cos\delta}$$

This is a real number. In other words, the (x, y) components are in phase. So, the resultant is a linearly polarized wave, regardless of the phase shift.

(b) Let a polarized wave be written

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = c_1\mathbf{E}_1 + c_2\mathbf{E}_2 = c_1\begin{pmatrix} a \\ ib \end{pmatrix} + c_2\begin{pmatrix} b \\ -ia \end{pmatrix}$$

where we assume that both *a* and *b* are real.

The constants c_1 and c_2 can be easily obtained by using the orthogonal property of the basis. They are given by

$$c_1 = \frac{a\alpha - ib\beta}{a^2 + b^2}, \qquad c_2 = \frac{b\alpha + ia\beta}{a^2 + b^2}$$

For a beam of linearly polarized light, α, β are real. So, both c_1 and c_2 are complex.

For a beam of right-hand circularly polarized light with $\alpha = 1/\sqrt{2}$, $\beta = -i/\sqrt{2}$, the expansion coefficients are

$$c_1 = \frac{1}{\sqrt{2}}(a - b), \qquad c_2 = \frac{1}{\sqrt{2}}(a + b)$$

We note that both c_1 and c_2 are real.

1.17 *Solution*:

(a) Let the circularly polarized state be written $\mathbf{E}_C = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$, and the unpolarized state be written $\mathbf{E}_U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ e^{i\delta} \end{pmatrix}$, where δ is a random phase. The projection along the transmission axis of a polarizer (oriented at azimuth angle ψ) is given by

$$\mathbf{p}\cdot\mathbf{E}_C = \frac{1}{\sqrt{2}}(\cos\psi - i\sin\psi)\,,\quad \mathbf{p}\cdot\mathbf{E}_U = \frac{1}{\sqrt{2}}(\cos\psi + e^{i\delta}\sin\psi)$$

It follows that

$$|\mathbf{p}\cdot\mathbf{E}_C|^2 = 1/2 = \left\langle |\mathbf{p}\cdot\mathbf{E}_U|^2 \right\rangle = 1/2$$

where $\langle\ \rangle$ represents statistical average over the random phase δ.

Thus, a polarizer alone can not distinguish the difference between circularly polarized light and unpolarized light.

(b) Let the elliptically polarized state be written $\mathbf{E} = \frac{1}{\sqrt{a^2+b^2}}\begin{pmatrix} a \\ ib \end{pmatrix}$. The projection along the transmission axis of a polarizer (oriented at azimuth angle ψ) is given by $|\mathbf{p}\cdot\mathbf{E}|^2 = \frac{1}{a^2+b^2}|a\cos\psi + ib\sin\psi|^2 = \frac{a^2\cos^2\psi + b^2\sin^2\psi}{a^2+b^2}$. A measurement of $|\mathbf{p}\cdot\mathbf{E}|^2$ as a function of the azimuth angle ψ yields the major axis and the minor axis of the ellipse, provided $a \neq b$.

It is important to note, a beam of partially polarized light can yield similar result.

1.18 *Solution:*

(a) Using the orthogonal relation, $\mathbf{E}_2^* \cdot \mathbf{E}_1 = 0$, we can obtain

$c_1 = (a\cos\psi - ib\sin\psi)/(a^2+b^2)$, $c_2 = (b\cos\psi + ia\sin\psi)/(a^2+b^2)$

(b) We write a general linearly polarized light as:

$$\mathbf{E}_0 = \begin{pmatrix}\cos\psi\\ \sin\psi\end{pmatrix} = \frac{1}{2}e^{i\psi}\begin{pmatrix}1\\ -i\end{pmatrix} + \frac{1}{2}e^{-i\psi}\begin{pmatrix}1\\ i\end{pmatrix}$$

After propagating through the optically active medium, the polarization state becomes

$$\mathbf{E}_L = \frac{1}{2}e^{i\psi}\begin{pmatrix}1\\ -i\end{pmatrix}e^{i\alpha} + \frac{1}{2}e^{-i\psi}\begin{pmatrix}1\\ i\end{pmatrix}e^{-i\alpha} = \begin{pmatrix}\cos(\psi+\alpha)\\ \sin(\psi+\alpha)\end{pmatrix}$$

We note that the polarization rotates an angle of α, where

$$\alpha = \frac{\pi}{\lambda}(n_r - n_l)L$$

We can also write a general elliptical polarization state as a sum of two orthogonal linear polarization states:

$$\mathbf{E} = \begin{pmatrix}a\\ ib\end{pmatrix} = a\begin{pmatrix}1\\ 0\end{pmatrix} + ib\begin{pmatrix}0\\ 1\end{pmatrix}$$

After propagation through the medium, each linearly polarized basis is rotated by an angle α

$$\mathbf{E}' = a\begin{pmatrix}\cos\alpha\\ \sin\alpha\end{pmatrix} + ib\begin{pmatrix}-\sin\alpha\\ \cos\alpha\end{pmatrix}$$

These two rotated bases remain the principal axes of the ellipse, as the phase shift between then remains $\pi/2$.

So, the major axis is rotated by an angle of α.

1.19 *Solution:*

(a) Using table 1.5 with $\Gamma=\pi$

$$\begin{pmatrix}V_x\\ V_y\end{pmatrix} = \begin{pmatrix}-i\cos 2\psi & -i\sin 2\psi\\ -i\sin 2\psi & i\cos 2\psi\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix} = \begin{pmatrix}-i\sin 2\psi\\ i\cos 2\psi\end{pmatrix} = \begin{pmatrix}-\sin 2\psi\\ \cos 2\psi\end{pmatrix}$$

We note the polarization state is rotated by an angle of 2ψ.

(b)

$$\begin{pmatrix}V_x\\ V_y\end{pmatrix} = \begin{pmatrix}-i\cos 2\psi & -i\sin 2\psi\\ -i\sin 2\psi & i\cos 2\psi\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}-\sin 2\psi - i\cos 2\psi\\ \cos 2\psi - i\sin 2\psi\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}-i\exp(i2\psi)\\ \exp(i2\psi)\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}\exp(i2\psi)$$

We note the output polarization state is LHC, regardless of the azimuth angle ψ.

(c) $d = \dfrac{\lambda}{2\,|n_e - n_o|} = 1254\text{ nm}$, or odd integral multiples.

1.20 *Solution:*

(a) Using table 1.5

$$\begin{pmatrix} V_x \\ V_y \end{pmatrix} = \begin{pmatrix} e^{-i\Gamma/2}\cos^2\psi + e^{i\Gamma/2}\sin^2\psi & -i\sin(\Gamma/2)\sin(2\psi) \\ -i\sin(\Gamma/2)\sin(2\psi) & e^{-i\Gamma/2}\sin^2\psi + e^{i\Gamma/2}\cos^2\psi \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -i\sin(\Gamma/2)\sin(2\psi) \\ e^{-i\Gamma/2}\sin^2\psi + e^{i\Gamma/2}\cos^2\psi \end{pmatrix} = \begin{pmatrix} -i\sin 2\psi \\ 1+i\cos 2\psi \end{pmatrix}$$

(b) By definition, $\chi = \dfrac{1+i\cos 2\psi}{-i\sin 2\psi} \equiv x + iy$. This leads to $x = -\cos 2\psi / \sin 2\psi$, $y = 1/\sin 2\psi$, and then $y^2 - x^2 = 1$ which is exactly a hyperbola. When ψ varies from 0 to $\pi/2$, y remains positive, so the locus is the upper branch of the hyperbola.

(c) $d = \dfrac{\lambda}{4\,|n_e - n_o|} = 1841\ \mathrm{nm}$, or odd integral multiples.

1.21 *Solution:*

(a) Using table 1.5

$$\begin{pmatrix} V_x \\ V_y \end{pmatrix} = \begin{pmatrix} e^{-i\Gamma/2}\cos^2\psi + e^{i\Gamma/2}\sin^2\psi & -i\sin(\Gamma/2)\sin(2\psi) \\ -i\sin(\Gamma/2)\sin(2\psi) & e^{-i\Gamma/2}\sin^2\psi + e^{i\Gamma/2}\cos^2\psi \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -i\sin(\Gamma/2)\sin(2\psi) \\ e^{-i\Gamma/2}\sin^2\psi + e^{i\Gamma/2}\cos^2\psi \end{pmatrix}$$

$$= \begin{pmatrix} -i\sin(\Gamma/2)\sin(2\psi) \\ \cos(\Gamma/2) + i\sin(\Gamma/2)\cos 2\psi \end{pmatrix}$$

(b), (c), (d)

By definition, $\chi = \dfrac{\cos(\Gamma/2) + i\sin(\Gamma/2)\cos 2\psi}{-i\sin(\Gamma/2)\sin 2\psi} \equiv x + iy$. This leads to $x = -\cos 2\psi / \sin 2\psi$, $y = 1/[\tan(\Gamma/2)\sin 2\psi]$, and then $y^2 - \tan^2(\Gamma/2)x^2 = 1$ which is exactly a hyperbola. When ψ varies from 0 to $\pi/2$, and Γ varies from 0 to 2π, the points (x, y) cover the entire complex plane.

(e) Using Eq. (1.9-11), the matrix is written $W = R(-\psi)W_0R(\psi)$, the Hermitian coniugate can be written

$W^\dagger = [R(-\psi)W_0R(\psi)]^\dagger = R(\psi)^\dagger W_0{}^\dagger R(-\psi)^\dagger = R(-\psi)W_0{}^\dagger R(\psi)$. Thus,

$W^\dagger W = R(-\psi)W_0{}^\dagger R(\psi)R(-\psi)W_0R(\psi) = R(-\psi)W_0{}^\dagger W_0R(\psi) = R(-\psi)R(\psi) = I$

(f) $\mathbf{V}_1^*{}' \cdot \mathbf{V}_2{}' = (W\mathbf{V}_1)^\dagger \cdot (W\mathbf{V}_2) = \mathbf{V}_1^* \cdot (W^\dagger W)\mathbf{V}_2 = \mathbf{V}_1^* \cdot \mathbf{V}_2$

Scalar product is invariant under unitary transformation.

1.22 *Solution:*

(a)

$$P_0^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = P_0$$,

$$P^2 = R(-\psi)P_0R(\psi)R(-\psi)P_0R(\psi) = R(-\psi)P_0P_0R(\psi) = R(-\psi)P_0R(\psi) = P$$

Using Dirac notation of linear algebra, a projection operator can be written $P = |p\rangle\langle p|$

(b) The transmitted state through a polarizer is obtained by operating the projection operator on the input polarization state. Thus we have

$$|E_t\rangle = P|E_i\rangle = |p\rangle\langle p|E_i\rangle$$

(c) The amplitude of transmission is given by

$\langle x|u\rangle\langle u|y\rangle = \cos\psi\cos\psi$, where $\psi = 45^\circ$.

(d) The transmitted amplitude is given by

$$\langle x|u_N\rangle\langle u_N|u_{N-1}\rangle\cdots\langle u_4|u_3\rangle\langle u_2|u_1\rangle\langle u_1|y\rangle = \cos\psi\cos\psi\cos\psi\cdots\cos\psi\cos\psi = (\cos\psi)^N$$

where $\psi=\pi/(2N)$.

For large N, $\pi/(2N) \ll 1$,

$$\cos\left(\frac{\pi}{2N}\right) \approx 1 - \frac{1}{2}\left(\frac{\pi}{2N}\right)^2$$

Using

$$\underset{N\to\infty}{\text{Lim}}\left(1\text{-}\frac{x}{N}\right)^N = \exp(-x)$$

we obtain

$$\underset{N\to\infty}{\text{Lim}}\left[\cos\left(\frac{\pi}{2N}\right)\right]^N = \underset{N\to\infty}{\text{Lim}}\left[1 - \frac{1}{2}\left(\frac{\pi}{2N}\right)^2\right]^N = \underset{N\to\infty}{\text{Lim}}\left[1 - \frac{1}{N}\frac{\pi^2}{8N}\right]^N = \underset{N\to\infty}{\text{Lim}}\exp\left(-\frac{\pi^2}{8N}\right) = 1$$

1.23 *Solution:*

(a) Using Eq. (1.9-39), the transmission of unpolarized light through the first stage (polarizer, wave plate, polarizer) is

$T=\frac{1}{2}\cos^2 x$. The transmission of polarized light through later stages is then $T=\cos^2 2^m x$, m=1, 2, 3, .. N-1. This leads to the overall transmission.

(b) Using $\cos\theta=(e^{i\theta}+e^{-i\theta})/2$, the transmission can be written

$$T=\frac{1}{2^{2N+1}}\left(e^{ix}+e^{-ix}\right)^2\left(e^{i2x}+e^{-i2x}\right)^2\left(e^{i4x}+e^{-i4x}\right)^2\ldots\left(e^{i2^{N-2}x}+e^{-i2^{N-2}x}\right)^2\left(e^{i2^{N-1}x}+e^{-i2^{N-1}x}\right)^2.$$

Carrying out the multiplications, we obtain

$$T=\frac{1}{2^{2N+1}}\left(e^{i(2^N-1)x}+e^{i(2^N-3)x}+e^{i(2^N-5)x}+.....+e^{-i(2^N-7)x}+e^{-i(2^N-5)x}+e^{-i(2^N-3)x}+e^{-i(2^N-1)x}\right)^2$$

Notice that the left side is a geometric series. Thus, we obtain

$$T=\frac{1}{2^{2N+1}}\left(\frac{e^{i(2^N-1)x}-e^{-i(2^N+1)x}}{1-e^{-i2x}}\right)^2=\frac{1}{2^{2N+1}}\left(\frac{\sin 2^N x}{\sin x}\right)^2$$

(c) For the thin plate, the transmission spectrum is $\cos^2 x$. The separation between peaks is $\Delta x=\pi$, which corresponds $\Delta\nu=\frac{c}{d(n_e-n_o)}$. The FWHM of each peak is $\Delta x_{1/2}=\pi/2$ and $\Delta\nu_{1/2}=\frac{c}{2d(n_e-n_o)}$. For the thickest plate the transmission spectrum is $\cos^2 2^{N-1}x$. The FWHM of each peak is $\Delta x_{1/2}=\pi/(2^N)$, which corresponds to $\Delta\nu_{1/2}=\frac{c}{2^N d(n_e-n_o)}$.

So, the overall transmission spectrum consists of peaks separated at $\Delta\nu=\frac{c}{d(n_e-n_o)}$, with the FWHM of each peak given by $\Delta\nu_{1/2}=\frac{c}{2^N d(n_e-n_o)}$. The finesse is thus $F\sim 2^N$.

(d) Using $\Delta\nu_{1/2}=\frac{c}{2^N d(n_e-n_o)}=\frac{c}{2D(n_e-n_o)}$, where D is the thickness of the thickest plate, we obtain

$$D=\frac{c}{2\Delta\nu_{1/2}(n_e-n_o)}=\frac{\lambda c}{2\Delta\lambda_{1/2}(n_e-n_o)\nu}=\frac{\lambda^2}{2\Delta\lambda_{1/2}(n_e-n_o)}=\frac{(6563)^2}{2(1.5506-1.5416)}\text{Angstrom}=24$$

cm

(e) The spectra feature of the function

$f(x)=\frac{\sin Mx}{\sin x}$, also appears in grating diffraction, is dominated by the numerator when M >>1.

The function is periodic with a period of 2π, and peak values of $f(0)=M$. The function drops to zero at $x=\pi/(M)$. At $x=\pi/(2M)$, the function is approximately $f=2M/\pi$, with $f^2=0.405M$. At $x=0.886\pi/(2M)$, $f^2=0.5M$.

1.24 *Solution:*

Using Table 1.7, the Jones matrix is given by (with ψ=θ)

$$\begin{pmatrix} e^{-i\Gamma/2}\cos^2\psi + e^{i\Gamma/2}\sin^2\psi & -i\sin(\Gamma/2)\sin(2\psi) \\ -i\sin(\Gamma/2)\sin(2\psi) & e^{-i\Gamma/2}\sin^2\psi + e^{i\Gamma/2}\cos^2\psi \end{pmatrix}$$

With crossed polarizers, the transmission is given by

$$T = |M_{12}|^2 / 2$$

1.25 *Solution:*

(a) Using Table 1.7, the Jones matrix for the wave plate followed by a rotator is given by

$$M = \begin{pmatrix} \cos\rho & -\sin\rho \\ \sin\rho & \cos\rho \end{pmatrix}\begin{pmatrix} e^{-i\Gamma/2}\cos^2\psi + e^{i\Gamma/2}\sin^2\psi & -i\sin(\Gamma/2)\sin(2\psi) \\ -i\sin(\Gamma/2)\sin(2\psi) & e^{-i\Gamma/2}\sin^2\psi + e^{i\Gamma/2}\cos^2\psi \end{pmatrix} \equiv RW \equiv \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

Carrying out the matrix multiplication, we obtain

$$M_{11} = \cos\rho[e^{-i\Gamma/2}\cos^2\psi + e^{i\Gamma/2}\sin^2\psi] + i\sin\rho\sin(\Gamma/2)\sin(2\psi)$$

$$M_{12} = -\sin\rho[e^{-i\Gamma/2}\sin^2\psi + e^{i\Gamma/2}\cos^2\psi] - i\cos\rho\sin(\Gamma/2)\sin(2\psi)$$

$$M_{21} = \sin\rho[e^{-i\Gamma/2}\cos^2\psi + e^{i\Gamma/2}\sin^2\psi] - i\cos\rho\sin(\Gamma/2)\sin(2\psi)$$

$$M_{22} = \cos\rho[e^{-i\Gamma/2}\sin^2\psi + e^{i\Gamma/2}\cos^2\psi] - i\sin\rho\sin(\Gamma/2)\sin(2\psi)$$

To show unitary property, we examine

$$M^\dagger M = (RW)^\dagger(RW) = W^\dagger R^\dagger RW = W^\dagger IW = W^\dagger W = I$$

(b) $a = \mathrm{Re}[M_{11}] = \cos(\Gamma/2)\cos\rho$,

$b = \mathrm{Im}[M_{11}] = \sin\rho\sin(\Gamma/2)\sin 2\psi - \cos\rho\sin(\Gamma/2)\cos 2\psi = -\cos(\rho + 2\psi)\sin(\Gamma/2)$

$c = \mathrm{Re}[M_{12}] = -\cos(\Gamma/2)\sin\rho$,

$d = \mathrm{Im}[M_{12}] = -\cos\rho\sin(\Gamma/2)\sin 2\psi - \sin\rho\sin(\Gamma/2)\cos 2\psi = -\sin(\rho + 2\psi)\sin(\Gamma/2)$

Thus, we obtain

$$\cos^2\frac{\Gamma}{2} = a^2 + c^2 \qquad \sin^2\frac{\Gamma}{2} = b^2 + d^2$$

$$\tan(\rho + 2\psi) = \frac{d}{b} \qquad \tan\rho = -\frac{c}{a}$$

1.26 *Solution:*

(a) Using the Jones matrix method, the output state can be written

$$\begin{pmatrix} V_x \\ V_y \end{pmatrix} = \begin{pmatrix} e^{-i\Gamma/2} & 0 \\ 0 & e^{+i\Gamma/2} \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \begin{pmatrix} e^{-i\Gamma/2}\cos\theta \\ e^{+i\Gamma/2}\sin\theta \end{pmatrix}$$

We now examine the Stokes parameter of this state of polarization.

$S_1 = \cos 2\theta$, $S_2 = \sin 2\theta \cos\Gamma$, $S_3 = \sin 2\theta \sin\Gamma$

We now keep Γ fixed and let θ vary from 0 to π. The points (S_1, S_2, S_3) form a circle on the plane defined by $S_3 / S_2 = \tan\Gamma$

This is a great circle formed by the intersection of the Poincare sphere(a unit circle) with the plane $S_3 / S_2 = \tan\Gamma$.

If we rotate the equatorial plane by an angle Γ around S_1-axis, we obtain the same great circle.

We now keep θ fixed and let Γ vary from 0 to 2π. The points (S_1, S_2, S_3) form a circle around S_1-axis with S_1 fixed at $\cos 2\theta$.

(b) If the wave plate is oriented at an azimuth angle ψ and the input linear polarization state maintains the same angle θ relative to the c-axis (slow axis) of the wave plate, then the output polarization ellipse can be obtained from the case of ψ=0 by a rotation of an angle of ψ. The rotation of a polarization ellipse by an angle ψ can be represented by the rotator matrix described in Problem 1.25. On the Poincare sphere, the effect of a rotator by an angle ψ in the xy-plane is a rotation around the polar axis (S_3-axis) by an angle of 2ψ. This is proven as follows. Let the rotated state be written

$$\begin{pmatrix} V_x \\ V_y \end{pmatrix} = \begin{pmatrix} \cos\rho & -\sin\rho \\ \sin\rho & \cos\rho \end{pmatrix} \begin{pmatrix} a \\ be^{i\delta} \end{pmatrix} = \begin{pmatrix} a\cos\rho - be^{i\delta}\sin\rho \\ a\sin\rho + be^{i\delta}\cos\rho \end{pmatrix}$$

where ρ is the angle of rotation. The Stokes parameters of the state before rotation is

$S_1 = a^2 - b^2$, $\quad S_2 = 2ab\cos\delta$, $\quad S_3 = 2ab\sin\delta$

whereas those of the state after the rotation is given by

$S_1' = |V_x|^2 - |V_y|^2$, $\quad S_2' = V_x V_y^* + V_x^* V_y$, $\quad S_3' = i(V_x V_y^* - V_x^* V_y)$

$$S_1' = (a\cos\rho - b\sin\rho\cos\delta)^2 + (b\sin\rho\sin\delta)^2 - (a\sin\rho + b\cos\rho\cos\delta)^2 - (b\cos\rho\sin\delta)^2$$
$$= (a^2 - b^2)\cos 2\rho - 2ab\cos\delta\sin 2\rho = S_1\cos 2\rho - S_2\sin 2\rho$$

$$S_2' = (a\cos\rho - be^{i\delta}\sin\rho)(a\sin\rho + be^{i\delta}\cos\rho)^* + (a\cos\rho - be^{i\delta}\sin\rho)^*(a\sin\rho + be^{i\delta}\cos\rho)$$
$$= 2(a\cos\rho - b\sin\rho\cos\delta)(a\sin\rho + b\cos\rho\cos\delta) - 2b^2\sin\rho\cos\rho\sin^2\delta$$
$$= (a^2 - b^2)\sin 2\rho + 2ab\cos\delta\cos 2\rho = S_1\sin 2\rho + S_2\cos 2\rho$$

$$S_3' = i(a\cos\rho - be^{i\delta}\sin\rho)(a\sin\rho + be^{i\delta}\cos\rho)^* - i(a\cos\rho - be^{i\delta}\sin\rho)^*(a\sin\rho + be^{i\delta}\cos\rho)$$
$$= 2ab\sin\delta = S_3$$

We note that (S_1', S_2', S_3') is obtained by a rotation of (S_1, S_2, S_3) by an angle 2ρ around the polar axis (S_3-axis). A great circle remains a great circle after the rotation.

(c) According to the results from Problem 1.25(b), a general birefringent network is equivalent to a wave plate followed by a rotator. If the input linear state is parallel to the slow (or fast) axis of the wave plate, then the output state will remain linear after transmitting through the wave plate. The rotator mere rotates the linear state by an angle ρ.

1.27 *Solution:*

From the basics of eigenvalue problem in linear algebra, the eigenvectors of the following equation

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

can be written (in terms of row vectors)

$$(x \quad y \quad z) = \left(\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \quad \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix} \quad \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\right), \quad \text{or} \quad \left(\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} \quad \begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix} \quad \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}\right), \quad \text{or}$$

$$\left(\begin{vmatrix} b_3 & c_3 \\ b_1 & c_1 \end{vmatrix} \quad \begin{vmatrix} c_3 & a_3 \\ c_1 & a_1 \end{vmatrix} \quad \begin{vmatrix} a_3 & b_3 \\ a_1 & b_1 \end{vmatrix}\right)$$

From Eq. (1.7-9),
$$\begin{pmatrix} \omega^2\mu\varepsilon_x - k_y^2 - k_z^2 & k_x k_y & k_x k_z \\ k_y k_x & \omega^2\mu\varepsilon_y - k_x^2 - k_z^2 & k_y k_z \\ k_z k_x & k_z k_y & \omega^2\mu\varepsilon_z - k_x^2 - k_y^2 \end{pmatrix}\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0$$

Note, we are interested in the direction of the eigenvectors. So, for simplicity, we evaluate the ratios of the components. This avoids having to deal with terms involving ω^4. Thus

$$x : y = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} : \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}, \qquad y : z = \begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix} : \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix},$$

From Eq. (1.7-9), we have

$$E_x : E_y = \begin{vmatrix} k_x k_y & k_x k_z \\ \omega^2\mu\varepsilon_y - k_x^2 - k_z^2 & k_y k_z \end{vmatrix} : \begin{vmatrix} k_x k_z & \omega^2\mu\varepsilon_x - k_y^2 - k_z^2 \\ k_y k_z & k_y k_x \end{vmatrix} = k_x k_z (k^2 - \omega^2\mu\varepsilon_y) : k_y k_z (k^2 - \omega^2\mu\varepsilon_x)$$

$$E_y : E_z = \begin{vmatrix} k_y k_z & k_y k_x \\ \omega^2\mu\varepsilon_z - k_x^2 - k_y^2 & k_z k_x \end{vmatrix} : \begin{vmatrix} k_y k_x & \omega^2\mu\varepsilon_y - k_x^2 - k_z^2 \\ k_z k_x & k_z k_y \end{vmatrix} = k_y k_x (k^2 - \omega^2\mu\varepsilon_z) : k_z k_x (k^2 - \omega^2\mu\varepsilon_y)$$

where $k^2 = k_x^2 + k_y^2 + k_z^2$. From the above two equations, we obtain

$$(E_x \quad E_y \quad E_z) = \left(\frac{k_x}{(k^2 - \omega^2\mu\varepsilon_x)} \quad \frac{k_y}{(k^2 - \omega^2\mu\varepsilon_y)} \quad \frac{k_z}{(k^2 - \omega^2\mu\varepsilon_z)}\right)$$

If we define $\mathbf{k} = ns\omega / c$, then

$$(E_x \quad E_y \quad E_z) = \left(\frac{s_x}{(n^2 - \varepsilon_x/\varepsilon_0)} \quad \frac{s_y}{(n^2 - \varepsilon_y/\varepsilon_0)} \quad \frac{s_z}{(n^2 - \varepsilon_z/\varepsilon_0)}\right)$$

(b) Using $\mathbf{D} = \varepsilon\mathbf{E}$, we have

$$(D_x \quad D_y \quad D_z) = \left(\frac{\varepsilon_x s_x}{(n^2 - \varepsilon_x/\varepsilon_0)} \quad \frac{\varepsilon_y s_y}{(n^2 - \varepsilon_y/\varepsilon_0)} \quad \frac{\varepsilon_z s_z}{(n^2 - \varepsilon_z/\varepsilon_0)}\right)$$

(c) Let $n_x^2 = \varepsilon_x / \varepsilon_0$, $n_y^2 = \varepsilon_y / \varepsilon_0$, $n_z^2 = \varepsilon_z / \varepsilon_0$. Let the two eigenvectors be written

$$\mathbf{E}_1 = \left(\frac{s_x}{(n_1^2 - n_x^2)} \quad \frac{s_y}{(n_1^2 - n_y^2)} \quad \frac{s_z}{(n_1^2 - n_z^2)} \right), \quad \mathbf{E}_2 = \left(\frac{s_x}{(n_2^2 - n_x^2)} \quad \frac{s_y}{(n_2^2 - n_y^2)} \quad \frac{s_z}{(n_2^2 - n_z^2)} \right)$$

$$\mathbf{D}_1 = \left(\frac{n_x^2 s_x}{(n_1^2 - n_x^2)} \quad \frac{n_y^2 s_y}{(n_1^2 - n_y^2)} \quad \frac{n_z^2 s_z}{(n_1^2 - n_z^2)} \right), \quad \mathbf{D}_2 = \left(\frac{n_x^2 s_x}{(n_2^2 - n_x^2)} \quad \frac{n_y^2 s_y}{(n_2^2 - n_y^2)} \quad \frac{n_z^2 s_z}{(n_2^2 - n_z^2)} \right)$$

Here we assume that the denominators are nonvanishing (for biaxial media). We will treat uniaxial (and isotropic media) later.

$$\mathbf{E}_1 \cdot \mathbf{E}_2 = \frac{s_x}{(n_1^2 - n_x^2)} \frac{s_x}{(n_2^2 - n_x^2)} + \frac{s_y}{(n_1^2 - n_y^2)} \frac{s_y}{(n_2^2 - n_y^2)} + \frac{s_z}{(n_1^2 - n_z^2)} \frac{s_z}{(n_2^2 - n_z^2)}$$

$$= \frac{1}{(n_2^2 - n_1^2)} \left[s_x^2 \left(\frac{1}{(n_1^2 - n_x^2)} - \frac{1}{(n_2^2 - n_x^2)} \right) + s_y^2 \left(\frac{1}{(n_1^2 - n_y^2)} - \frac{1}{(n_2^2 - n_y^2)} \right) + s_z^2 \left(\frac{1}{(n_1^2 - n_z^2)} - \frac{1}{(n_2^2 - n_z^2)} \right) \right]$$

$$= \frac{1}{(n_2^2 - n_1^2)} \left(\frac{s_x^2}{(n_1^2 - n_x^2)} + \frac{s_y^2}{(n_1^2 - n_y^2)} + \frac{s_z^2}{(n_1^2 - n_z^2)} - \frac{s_x^2}{(n_2^2 - n_x^2)} - \frac{s_y^2}{(n_2^2 - n_y^2)} - \frac{s_z^2}{(n_2^2 - n_z^2)} \right)$$

$$= \frac{1}{(n_2^2 - n_1^2)} \left(\frac{1}{n_1^2} - \frac{1}{n_2^2} \right) = \frac{1}{n_1^2 n_2^2} \neq 0$$

where we have used Fresnel equation (1.7-12).
So, in general, the **E**-vectors in biaxial media are not orthogonal. Now let's evaluate

$$\mathbf{D}_1 \cdot \mathbf{D}_2 = \frac{n_x^2 s_x}{(n_1^2 - n_x^2)} \frac{n_x^2 s_x}{(n_2^2 - n_x^2)} + \frac{n_y^2 s_y}{(n_1^2 - n_y^2)} \frac{n_y^2 s_y}{(n_2^2 - n_y^2)} + \frac{n_z^2 s_z}{(n_1^2 - n_z^2)} \frac{n_z^2 s_z}{(n_2^2 - n_z^2)}$$

$$= \left[s_x^2 \left(\frac{n_1^2}{(n_1^2 - n_x^2)} - 1 \right) \left(\frac{n_2^2}{(n_2^2 - n_x^2)} - 1 \right) + s_y^2 \left(\frac{n_1^2}{(n_1^2 - n_y^2)} - 1 \right) \left(\frac{n_2^2}{(n_2^2 - n_y^2)} - 1 \right) + s_z^2 \left(\frac{n_1^2}{(n_1^2 - n_z^2)} - 1 \right) \left(\frac{n_2^2}{(n_2^2 - n_z^2)} - 1 \right) \right]$$

$$= n_1^2 n_2^2 \left(\frac{s_x^2}{(n_1^2 - n_x^2)(n_2^2 - n_x^2)} + \frac{s_y^2}{(n_1^2 - n_y^2)(n_2^2 - n_y^2)} + \frac{s_z^2}{(n_1^2 - n_z^2)(n_2^2 - n_z^2)} \right)$$

$$- n_1^2 \left(\frac{s_x^2}{(n_1^2 - n_x^2)} + \frac{s_y^2}{(n_1^2 - n_y^2)} + \frac{s_z^2}{(n_1^2 - n_z^2)} \right) - n_2^2 \left(\frac{s_x^2}{(n_2^2 - n_x^2)} + \frac{s_y^2}{(n_2^2 - n_y^2)} + \frac{s_z^2}{(n_2^2 - n_z^2)} \right) + (s_x^2 + s_y^2 + s_z^2)$$

$= n_1^2 n_2^2 \dfrac{1}{n_1^2 n_2^2} - n_1^2 \dfrac{1}{n_1^2} - n_2^2 \dfrac{1}{n_2^2} - 1 = 0$, where we have used Fresnel equation (1.7-12).

For uniaxial media, $n_x = n_y = n_o$ and $n_z = n_e$, the electric field vector of the eigenmodes are given Eq. (1.8-5) and Eq. (1.8-6). They are mutually orthogonal.

(d) Using results from (c), we have

$$\mathbf{E}_1 \cdot \mathbf{D}_2 = \frac{n_x^2 s_x^2}{(n_1^2 - n_x^2)(n_2^2 - n_x^2)} + \frac{n_y^2 s_y^2}{(n_1^2 - n_y^2)(n_2^2 - n_y^2)} + \frac{n_z^2 s_z^2}{(n_1^2 - n_z^2)(n_2^2 - n_z^2)}$$

$$= \frac{n_1^2 n_2^2 s_x^2}{(n_2^2 - n_1^2)}\left(\frac{n_2^{-2}}{(n_1^2 - n_x^2)} - \frac{n_1^{-2}}{(n_2^2 - n_x^2)}\right) + \frac{n_1^2 n_2^2 s_y^2}{(n_2^2 - n_1^2)}\left(\frac{n_2^{-2}}{(n_1^2 - n_y^2)} - \frac{n_1^{-2}}{(n_2^2 - n_y^2)}\right) + \frac{n_1^2 n_2^2 s_z^2}{(n_2^2 - n_1^2)}\left(\frac{n_2^{-2}}{(n_1^2 - n_z^2)} - \frac{n_1^{-2}}{(n_2^2 - n_z^2)}\right)$$

$$= \frac{n_1^2 n_2^2}{(n_2^2 - n_1^2)} \frac{1}{n_1^2 n_2^2} - \frac{n_1^2 n_2^2}{(n_2^2 - n_1^2)} \frac{1}{n_1^2 n_2^2} = 0$$

where we have used Fresnel equation (1.7-12). The proof for $\mathbf{E}_2 \cdot \mathbf{D}_1 = 0$ is similar.

1.28 *Solution*:

(a) Let $\mathbf{k} = n\mathbf{s}\omega / c$ and $n_x^2 = \varepsilon_x / \varepsilon_0$, $n_y^2 = \varepsilon_y / \varepsilon_0$, $n_z^2 = \varepsilon_z / \varepsilon_0$, Eq. (1.7-10) becomes

$$\begin{vmatrix} (n_x^2 - n^2) + n^2 s_x^2 & n^2 s_x s_y & n^2 s_x s_z \\ n^2 s_y s_x & (n_y^2 - n^2) + n^2 s_y^2 & n^2 s_y s_z \\ n^2 s_z s_x & n^2 s_z s_y & (n_z^2 - n^2) + n^2 s_z^2 \end{vmatrix} = 0$$

Carrying out the multiplication, we obtain

$$[(n_x^2 - n^2) + n^2 s_x^2] \begin{vmatrix} (n_y^2 - n^2) + n^2 s_y^2 & n^2 s_y s_z \\ n^2 s_z s_y & (n_z^2 - n^2) + n^2 s_z^2 \end{vmatrix}$$

$$+ n^2 s_x s_y \begin{vmatrix} n^2 s_y s_z & n^2 s_y s_x \\ (n_z^2 - n^2) + n^2 s_z^2 & n^2 s_z s_x \end{vmatrix} + n^2 s_x s_z \begin{vmatrix} n^2 s_y s_x & (n_y^2 - n^2) + n^2 s_y^2 \\ n^2 s_z s_x & n^2 s_z s_y \end{vmatrix} = 0$$

Carrying out the operations and simplifying the equation,

$$[(n_x^2 - n^2) + n^2 s_x^2][(n_y^2 - n^2) + n^2 s_y^2][(n_z^2 - n^2) + n^2 s_z^2] - [(n_x^2 - n^2) + n^2 s_x^2] n^4 s_z^2 s_y^2$$

$$- n^4 s_x^2 s_y^2 (n_z^2 - n^2) - n^4 s_x^2 s_z^2 (n_y^2 - n^2) = 0$$

$$(n_x^2 - n^2)(n_y^2 - n^2)(n_z^2 - n^2) + n^2 s_x^2 [(n_y^2 - n^2)(n_z^2 - n^2)]$$

$$+ n^2 s_y^2 [(n_x^2 - n^2)(n_z^2 - n^2)] + n^2 s_z^2 [(n_y^2 - n^2)(n_x^2 - n^2)] = 0$$

Dividing both sides by $(n_x^2 - n^2)(n_y^2 - n^2)(n_z^2 - n^2) n^2$, we obtain the Fresnel equation.

(b) Carrying out the multiplications in the last equation, we note the n^6 terms cancel out due to $s_x^2 + s_y^2 + s_z^2 = 1$.

We obtain

$$n^4 (n_x^2 s_x^2 + n_y^2 s_y^2 + n_z^2 s_z^2) + n^2 [-n_x^2 n_y^2 (s_x^2 + s_y^2) - n_z^2 n_y^2 (s_z^2 + s_y^2) - n_x^2 n_z^2 (s_x^2 + s_z^2)] + n_x^2 n_y^2 n_z^2 = 0$$

This is a quadratic equation in n^2, with

$$A = n_x^2 s_x^2 + n_y^2 s_y^2 + n_z^2 s_z^2$$

$$B = -n_x^2 n_y^2 (s_x^2 + s_y^2) - n_z^2 n_y^2 (s_z^2 + s_y^2) - n_x^2 n_z^2 (s_x^2 + s_z^2)$$

$$C = n_x^2 n_y^2 n_z^2$$

(c)We now evaluate $B^2 - 4AC$. For simplicity in the proof, let $a = n_x^2$, $b = n_y^2$, $c = n_z^2$, $\alpha = s_x^2$, $\beta = s_y^2$, $\gamma = s_z^2$.

$$B^2 - 4AC = (ab\alpha + ab\beta + bc\beta + bc\gamma + ca\gamma + ca\alpha)^2 - 4abc(a\alpha + b\beta + c\gamma)(\alpha + \beta + \gamma)$$

where we have added $(\alpha + \beta + \gamma) = 1$ in the last term. Carrying out the multiplication and rearranging the terms, we obtain

$$B^2 - 4AC = a^2\alpha^2 (b - c)^2 + b^2\beta^2 (c - a)^2 + c^2\gamma^2 (a - b)^2$$

$$+ 2ab\alpha\beta(a - c)(b - c) + 2bc\beta\gamma(b - a)(c - a) + 2ca\gamma\alpha(c - b)(a - b)$$

Without loss of generality, we assume $a < b < c$. We note all terms are positive except the last term. We rewrite the last equation as

$$B^2-4AC=[a\alpha(c-b)+c\gamma(a-b)]^2+b^2\beta^2(c-a)^2+2ab\alpha\beta(a-c)(b-c)+2bc\beta\gamma(b-a)(c-a)>0$$

(d) From (b) we have

$$n^4(n_x^2s_x^2+n_y^2s_y^2+n_z^2s_z^2)+n^2[-n_x^2n_y^2(s_x^2+s_y^2)-n_z^2n_y^2(s_z^2+s_y^2)-n_x^2n_z^2(s_x^2+s_z^2)]+n_x^2n_y^2n_z^2=0$$

For uniaxial media, $n_x=n_y=n_o$ and $n_z=n_e$, the equation becomes

$$n^4(n_o^2s_x^2+n_o^2s_y^2+n_e^2s_z^2)+n^2[-n_o^2n_o^2(s_x^2+s_y^2)-n_e^2n_o^2(s_z^2+s_y^2)-n_o^2n_e^2(s_x^2+s_z^2)]+n_o^2n_o^2n_e^2=0$$

Dividing both sides of the above equation by $n_o^2n_o^2n_e^2$, we obtain

$$\frac{n^4}{n_o^2}\left(\frac{s_x^2+s_y^2}{n_e^2}+\frac{s_z^2}{n_o^2}\right)+n^2\left(-\frac{s_x^2+s_x^2}{n_e^2}-\frac{s_z^2+1}{n_o^2}\right)+1=0$$

or, equivalently

$$\frac{n^2}{n_o^2}\left(\frac{n^2s_x^2+n^2s_y^2}{n_e^2}+\frac{n^2s_z^2}{n_o^2}\right)-\left(\frac{n^2s_x^2+n^2s_x^2}{n_e^2}+\frac{n^2s_z^2}{n_o^2}\right)-\frac{n^2}{n_o^2}+1=0$$

or,

$$\left(\frac{n^2s_x^2+n^2s_y^2}{n_e^2}+\frac{n^2s_z^2}{n_o^2}-1\right)\left(\frac{n^2}{n_o^2}-1\right)=0$$

Using $\mathbf{k}=n\mathbf{s}\omega/c$, the above equation becomes

$$\left(\frac{k_x^2+k_y^2}{n_e^2}+\frac{k_z^2}{n_o^2}-\frac{\omega^2}{c^2}\right)\left(\frac{k^2}{n_o^2}-\frac{\omega^2}{c^2}\right)=0$$

(e) In isotropic media, $n_e=n_o=n$, the above equation becomes

$$\left(\frac{k^2}{n^2}-\frac{\omega^2}{c^2}\right)^2=0$$

1.29 *Solution:*

The group velocity and energy velocity are defined as follows:

$\mathbf{v}_g = \nabla_\mathbf{k}\omega(\mathbf{k})$, $\mathbf{v}_e = \mathbf{S}/U$, where $\omega(\mathbf{k})$ is the dispersion relation, **S** is the Poynting vector, U is the energy density. We start from

$\mathbf{k}\times\mathbf{E} = \omega\mu\mathbf{H}, \quad \mathbf{k}\times\mathbf{H} = -\omega\varepsilon\mathbf{E}$

A wave packet consists of a linear superposition of monochromatic plane wave with a spectrum peak around a central frequency ω. Consider an infinitesimal variation $\delta\mathbf{k}$ in the above equations. Let $\delta\omega$, $\delta\mathbf{E}$ and $\delta\mathbf{H}$ be the corresponding variations in ω, **E** and **H**, we have

$\delta\mathbf{k}\times\mathbf{E} + \mathbf{k}\times\delta\mathbf{E} = \delta\omega\mu\mathbf{H} + \omega\mu\delta\mathbf{H}$

$\delta\mathbf{k}\times\mathbf{H} + \mathbf{k}\times\delta\mathbf{H} = -\delta\omega\varepsilon\mathbf{E} - \omega\varepsilon\delta\mathbf{E}$

where we have treated μ and ε as constants.

Now scalar-multiplying the first equation with **H**, and the second equation above with **E** and using the vector identity

$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) - \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

we obtain

$$\delta\mathbf{k}\cdot(\mathbf{E}\times\mathbf{H}) + \mathbf{k}\cdot(\delta\mathbf{E}\times\mathbf{H}) = \delta\omega(\mathbf{H}\cdot\mu\mathbf{H}) + \omega(\mathbf{H}\cdot\mu\delta\mathbf{H})$$

$$-\delta\mathbf{k}\cdot(\mathbf{E}\times\mathbf{H}) + \mathbf{k}\cdot(\delta\mathbf{H}\times\mathbf{E}) = -\delta\omega(\mathbf{E}\cdot\varepsilon\mathbf{E}) - \omega(\mathbf{E}\cdot\varepsilon\delta\mathbf{E})$$

Subtracting the second equation from the first equation above and using the symmetry property of the dielectric tensor ε and permeability tensor μ, we obtain

$2\delta\mathbf{k}\cdot(\mathbf{E}\times\mathbf{H}) - \delta\mathbf{E}\cdot(\mathbf{k}\times\mathbf{H}) - \omega(\delta\mathbf{E}\cdot\varepsilon\mathbf{E}) + \delta\mathbf{H}\cdot(\mathbf{k}\times\mathbf{E}) - \omega(\delta\mathbf{H}\cdot\mu\mathbf{H}) = \delta\omega(\mathbf{H}\cdot\mu\mathbf{H}) + \delta\omega(\mathbf{E}\cdot\varepsilon\mathbf{E})$

which becomes, after using $\mathbf{k}\times\mathbf{E} = \omega\mu\mathbf{H}, \quad \mathbf{k}\times\mathbf{H} = -\omega\varepsilon\mathbf{E}$

$\delta\mathbf{k}\cdot(\mathbf{E}\times\mathbf{H}) = \delta\omega[(\mathbf{E}\cdot\varepsilon\mathbf{E}) + (\mathbf{H}\cdot\mu\mathbf{H})]/2$

or, equivalently

$\delta\omega = \delta\mathbf{k}\cdot\mathbf{S}/\mathrm{U} = \delta\mathbf{k}\cdot\mathbf{v}_e$

From the definition of the group velocity we also have

$\delta\omega = \delta\mathbf{k}\cdot\nabla_\mathbf{k}\omega(\mathbf{k}) = \delta\mathbf{k}\cdot\mathbf{v}_g$

Since $\delta\mathbf{k}$ is an arbitrary vector, we conclude that the energy velocity is the same as the group velocity. The equality holds provided both ε and μ are independent of the frequency ω.

For complex **E** and **H** we start from

$\delta\mathbf{k}\times\mathbf{E} + \mathbf{k}\times\delta\mathbf{E} = \delta\omega\mu\mathbf{H} + \omega\mu\delta\mathbf{H}$

$\delta\mathbf{k}\times\mathbf{H} + \mathbf{k}\times\delta\mathbf{H} = -\delta\omega\varepsilon\mathbf{E} - \omega\varepsilon\delta\mathbf{E}$

Now scalar-multiplying the first equation with **H***, and the second equation with **E*** and using the vector identity

$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

we obtain

$$\delta\mathbf{k}\cdot(\mathbf{E}\times\mathbf{H}^*) + \mathbf{k}\cdot(\delta\mathbf{E}\times\mathbf{H}^*) = \delta\omega(\mathbf{H}^*\cdot\mu\mathbf{H}) + \omega(\mathbf{H}^*\cdot\mu\delta\mathbf{H})$$

$$-\delta\mathbf{k}\cdot(\mathbf{E}\times\mathbf{H}^*) + \mathbf{k}\cdot(\delta\mathbf{H}^*\times\mathbf{E}) = -\delta\omega(\mathbf{E}\cdot\varepsilon\mathbf{E}^*) - \omega(\mathbf{E}\cdot\varepsilon\delta\mathbf{E}^*)$$

Subtracting the second equation from the first equation above and using the symmetry property of the dielectric tensor ε and permeability tensor μ, we obtain

$\delta\mathbf{k}\cdot(\mathbf{E}\times\mathbf{H}^*) = \delta\omega[(\mathbf{E}\cdot\varepsilon\mathbf{E}^*) + (\mathbf{H}\cdot\mu\mathbf{H}^*)]/2$

Taking the real part on both sides, we obtain

$\delta\omega = \delta\mathbf{k}\cdot\mathbf{S}/\mathrm{U} = \delta\mathbf{k}\cdot\mathbf{v}_e$

1.30 Solution:

(a) We start from the dispersion relation for the extraordinary wave

$$\frac{k_x^2+k_y^2}{n_e^2}+\frac{k_z^2}{n_o^2}=\frac{\omega^2}{c^2}$$

Taking differential on both sides

$$\frac{2k_x}{n_e^2}\delta k_x+\frac{2k_y}{n_e^2}\delta k_y+\frac{2k_z}{n_o^2}\delta k_z=\frac{2\omega}{c^2}\delta\omega=\frac{2\omega}{c^2}\delta\mathbf{k}\cdot\nabla_{\mathbf{k}}\omega(\mathbf{k})$$

The group velocity is thus given by

$$\mathbf{v}_g=\nabla_{\mathbf{k}}\omega(\mathbf{k})=\left(\frac{k_x}{n_e^2}\ \ \frac{k_y}{n_e^2}\ \ \frac{k_z}{n_o^2}\right)\frac{c^2}{\omega}$$

Let θ be the polar angle of the wavevector, then

$k_x=n\omega\sin\theta\cos\phi/c,\ \ k_y=n\omega\sin\theta\sin\phi/c,\ \ k_z=n\omega\cos\theta/c$

where ϕ is the azimuth angle, n is given by

$$\frac{1}{n^2}=\frac{1}{n_e^2(\theta)}=\frac{\sin^2\theta}{n_e^2}+\frac{\cos^2\theta}{n_o^2}$$

(b) To find the angle α between **k** and $\mathbf{v}_g$, we evaluate

$$\cos\alpha=\frac{\mathbf{k}\cdot\mathbf{v}_g}{|\mathbf{k}|\ |\mathbf{v}_g|}=\frac{\dfrac{\sin^2\theta}{n_e^2}+\dfrac{\cos^2\theta}{n_o^2}}{\sqrt{\dfrac{\sin^2\theta}{n_e^4}+\dfrac{\cos^2\theta}{n_o^4}}}$$

(c) For θ=0, or $\pi/2$, cos α =1, according to the above equation.

Differentiate with respect to the angle θ,

$$\frac{\partial}{\partial\theta}\cos\alpha=\frac{\dfrac{2\sin\theta\cos\theta}{n_e^2}-\dfrac{2\sin\theta\cos\theta}{n_o^2}}{\sqrt{\dfrac{\sin^2\theta}{n_e^4}+\dfrac{\cos^2\theta}{n_o^4}}}-\frac{1}{2}\frac{\dfrac{2\sin\theta\cos\theta}{n_e^4}-\dfrac{2\sin\theta\cos\theta}{n_o^4}}{\left(\dfrac{\sin^2\theta}{n_e^4}+\dfrac{\cos^2\theta}{n_o^4}\right)^{3/2}}\left(\frac{\sin^2\theta}{n_e^2}+\frac{\cos^2\theta}{n_o^2}\right)$$

$$\frac{\partial}{\partial\theta}\cos\alpha\propto 2\sin\theta\cos\theta\left(\frac{1}{n_e^2}-\frac{1}{n_o^2}\right)\left[\left(\frac{\sin^2\theta}{n_e^4}+\frac{\cos^2\theta}{n_o^4}\right)-\frac{1}{2}\left(\frac{\sin^2\theta}{n_e^2}+\frac{\cos^2\theta}{n_o^2}\right)\left(\frac{1}{n_e^2}+\frac{1}{n_o^2}\right)\right]$$

$$=\sin\theta\cos\theta\left(\frac{1}{n_e^2}-\frac{1}{n_o^2}\right)^2\left(\frac{\sin^2\theta}{n_e^2}-\frac{\cos^2\theta}{n_o^2}\right)$$

So, the maximum occurs at: $\tan\theta=n_e/n_o$.

At this angle, the angle α is given by

$$\cos\alpha=\frac{2n_en_o}{n_e^2+n_o^2}$$

Fr ZLI-1646 with $n_o=1.478$, $n_e=1.558$, this corresponds to θ=46.51°, α=3.02°.

(d) For α<<1, we have $\cos\alpha\approx 1-\alpha^2/2$, so $\alpha^2=2\left(1-\dfrac{2n_en_o}{n_e^2+n_o^2}\right)=\dfrac{2(n_e-n_o)^2}{n_e^2+n_o^2}$.

For $n_e\approx n_o\equiv n$, $\alpha_{\max}\approx\Delta n/n$, where $\Delta n=n_e-n_o$.

1.31 Solution:

(a) See solution of Problem 1.28(a).

(b) Taking the gradient operation of the left side, we obtain, the following components of $\nabla_{\mathbf{k}}$

$$\frac{\partial}{\partial k_x}=\frac{2k_x}{k^2-\omega^2\mu\varepsilon_x}-2k_x\left[\frac{k_x^2}{(k^2-\omega^2\mu\varepsilon_x)^2}+\frac{k_y^2}{(k^2-\omega^2\mu\varepsilon_y)^2}+\frac{k_z^2}{(k^2-\omega^2\mu\varepsilon_z)^2}\right]$$

$$\frac{\partial}{\partial k_y}=\frac{2k_y}{k^2-\omega^2\mu\varepsilon_y}-2k_y\left[\frac{k_x^2}{(k^2-\omega^2\mu\varepsilon_x)^2}+\frac{k_y^2}{(k^2-\omega^2\mu\varepsilon_y)^2}+\frac{k_z^2}{(k^2-\omega^2\mu\varepsilon_z)^2}\right]$$

$$\frac{\partial}{\partial k_z}=\frac{2k_z}{k^2-\omega^2\mu\varepsilon_z}-2k_z\left[\frac{k_x^2}{(k^2-\omega^2\mu\varepsilon_x)^2}+\frac{k_y^2}{(k^2-\omega^2\mu\varepsilon_y)^2}+\frac{k_z^2}{(k^2-\omega^2\mu\varepsilon_z)^2}\right]$$

The gradient vector is normal to the surface of constant frequency (the normal surface). Scalar multiplication with the eigenvectors of (1.7-11), we obtain

$$\frac{2k_x^2}{(k^2-\omega^2\mu\varepsilon_x)^2}-\frac{2k_x^2}{k^2-\omega^2\mu\varepsilon_x}\left[\frac{k_x^2}{(k^2-\omega^2\mu\varepsilon_x)^2}+\frac{k_y^2}{(k^2-\omega^2\mu\varepsilon_y)^2}+\frac{k_z^2}{(k^2-\omega^2\mu\varepsilon_z)^2}\right]+$$

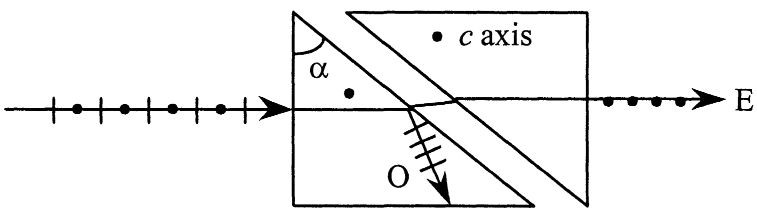
$$\frac{2k_y^2}{(k^2-\omega^2\mu\varepsilon_y)^2}-\frac{2k_y^2}{k^2-\omega^2\mu\varepsilon_y}\left[\frac{k_x^2}{(k^2-\omega^2\mu\varepsilon_x)^2}+\frac{k_y^2}{(k^2-\omega^2\mu\varepsilon_y)^2}+\frac{k_z^2}{(k^2-\omega^2\mu\varepsilon_z)^2}\right]+$$

$$\frac{2k_z^2}{(k^2-\omega^2\mu\varepsilon_z)^2}-\frac{2k_z^2}{k^2-\omega^2\mu\varepsilon_z}\left[\frac{k_x^2}{(k^2-\omega^2\mu\varepsilon_x)^2}+\frac{k_y^2}{(k^2-\omega^2\mu\varepsilon_y)^2}+\frac{k_z^2}{(k^2-\omega^2\mu\varepsilon_z)^2}\right]=$$

$$\left[\frac{k_x^2}{(k^2-\omega^2\mu\varepsilon_x)^2}+\frac{k_y^2}{(k^2-\omega^2\mu\varepsilon_y)^2}+\frac{k_z^2}{(k^2-\omega^2\mu\varepsilon_z)^2}\right]\left(2-\frac{2k_x^2}{k^2-\omega^2\mu\varepsilon_x}-\frac{2k_y^2}{k^2-\omega^2\mu\varepsilon_y}-\frac{2k_z^2}{k^2-\omega^2\mu\varepsilon_z}\right)$$

$$=\left[\frac{k_x^2}{(k^2-\omega^2\mu\varepsilon_x)^2}+\frac{k_y^2}{(k^2-\omega^2\mu\varepsilon_y)^2}+\frac{k_z^2}{(k^2-\omega^2\mu\varepsilon_z)^2}\right](2-2)=0$$

Thus, the normal vector to the normal surface is perpendicular to the eigenvector of E-field.



1.32 Solution:

(a) To ensure the transmission f the extrao-ordinary wave the internal angle of incidence is in the range:

$\sin^{-1}(1/n_o)<\theta<\sin^{-1}(1/n_e)$, or $37.1<\theta<42.3$ degrees.

(b) For the geometry shown, the internal angle of incidence is α. So, $37.1<\alpha<42.3$ degrees.

1.33 *Solution:*

(a) $\frac{I_1}{I_2} = \frac{\exp(-\alpha_1 d)}{\exp(-\alpha_2 d)}$, where d is the thickness of the medium.

(b) If ε_x, ε_y and ε_z are complex, the relative phase between the components of the eigenvectors may not be zero.

1.34 *Solution:*

(a) Define $\eta\varepsilon = \varepsilon_0$, so that $\eta = \varepsilon_0\varepsilon^{-1}$. Rewrite the equation as $\mathbf{s}\times[\mathbf{s}\times\eta\mathbf{D}] = -\frac{1}{n^2}\mathbf{D}$. Using the vector identity

$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

We have $\mathbf{s}\times[\mathbf{s}\times\eta\mathbf{D}] = \mathbf{s}(\mathbf{s}\cdot\eta\mathbf{D}) - \eta\mathbf{D}(\mathbf{s}\cdot\mathbf{s}) = \mathbf{s}(\mathbf{s}\cdot\eta\mathbf{D}) - \eta\mathbf{D} = -\frac{1}{n^2}\mathbf{D}$. So,

$$\mathbf{s}(\mathbf{s}\cdot\eta\mathbf{D}_1) - \eta\mathbf{D}_1 = -\frac{1}{n_1^2}\mathbf{D}_1$$

$$\mathbf{s}(\mathbf{s}\cdot\eta\mathbf{D}_2) - \eta\mathbf{D}_2 = -\frac{1}{n_2^2}\mathbf{D}_2$$

Scalar-multiplying the first equation with $\mathbf{D}_1$*, and the second equation with $\mathbf{D}_2$*, and using $\mathbf{D}_1\cdot\mathbf{s} = \mathbf{D}_2\cdot\mathbf{s} = 0$ we obtain

$$\left(\frac{\varepsilon_0}{\varepsilon}\right)_{11} \equiv \mathbf{D}_1^*\cdot\eta\mathbf{D}_1 = \frac{1}{n_1^2},\quad \left(\frac{\varepsilon_0}{\varepsilon}\right)_{22} \equiv \mathbf{D}_2^*\cdot\eta\mathbf{D}_2 = \frac{1}{n_2^2}$$

(b) Scalar-multiplying the first equation with $\mathbf{D}_2$*, and the second equation with $\mathbf{D}_1$*, and using $\mathbf{D}_1\cdot\mathbf{s} = \mathbf{D}_2\cdot\mathbf{s} = 0$ we obtain

$$\left(\frac{\varepsilon_0}{\varepsilon}\right)_{12} \equiv \mathbf{D}_1^*\cdot\eta\mathbf{D}_2 = \mathbf{D}_1^*\cdot\mathbf{D}_2/n_1^2$$

$$\left(\frac{\varepsilon_0}{\varepsilon}\right)_{21} \equiv \mathbf{D}_2^*\cdot\eta\mathbf{D}_1 = \mathbf{D}_2^*\cdot\mathbf{D}_1/n_2^2$$

Since η is Hermitian and $n_1 \neq n_2$, $\mathbf{D}_1^*\cdot\mathbf{D}_2 = 0$.

At the same time, we obtain

$$\left(\frac{\varepsilon_0}{\varepsilon}\right)_{12} \equiv \mathbf{D}_1^*\cdot\eta\mathbf{D}_2 = 0$$

1.35 Solution:

(a) Let the displacement field be written $\mathbf{D} = \mathbf{D}_1 e^{i(\omega t - k_1 \varsigma)} + \mathbf{D}_2 e^{i(\omega t - k_2 \varsigma)}$. The electric field and magnetic field are written

$\mathbf{E} = \mathbf{E}_1 e^{i(\omega t - k_1 \varsigma)} + \mathbf{E}_2 e^{i(\omega t - k_2 \varsigma)}$

$\mathbf{H} = \mathbf{H}_1 e^{i(\omega t - k_1 \varsigma)} + \mathbf{H}_2 e^{i(\omega t - k_2 \varsigma)}$

The power flow along the direction **s** is given by

$\mathbf{s} \cdot \mathbf{S} = \mathbf{s} \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{s} \cdot (\mathbf{E}_1 \times \mathbf{H}_1) + \mathbf{s} \cdot (\mathbf{E}_2 \times \mathbf{H}_2)$

Using $\mathbf{H} = \mathbf{s} \times \mathbf{E} n / \mu c$, the power flow becomes

$$\mathbf{s} \cdot \mathbf{S} = \frac{n_1}{\mu c}[\mathbf{E}_1 \cdot \mathbf{E}_1 - (\mathbf{E}_1 \cdot \mathbf{s})^2] + \frac{n_2}{\mu c}[\mathbf{E}_2 \cdot \mathbf{E}_2 - (\mathbf{E}_2 \cdot \mathbf{s})^2] \equiv \frac{n_1}{\mu c}(\mathbf{A}_1 \cdot \mathbf{A}_1) + \frac{n_2}{\mu c}(\mathbf{A}_2 \cdot \mathbf{A}_2)$$

Using $\mathbf{D} = -\mathbf{s} \times \mathbf{H} n / c$ and $\mathbf{H} = \mathbf{s} \times \mathbf{E} n / \mu c$, we have

$$\mathbf{D} = -\frac{n^2}{\mu c^2} \mathbf{s} \times (\mathbf{s} \times \mathbf{E}) = \frac{n^2}{\mu c^2}[\mathbf{E} - \mathbf{s}(\mathbf{E} \cdot \mathbf{s})] \equiv \frac{n^2}{\mu c^2} \mathbf{A}$$

Thus, $\mathbf{s} \cdot \mathbf{S} = \dfrac{\mu c^3}{n_1^3}(\mathbf{D}_1 \cdot \mathbf{D}_1) + \dfrac{\mu c^3}{n_2^3}(\mathbf{D}_2 \cdot \mathbf{D}_2) = \dfrac{c}{\varepsilon_0 n_1^3}(\mathbf{D}_1 \cdot \mathbf{D}_1) + \dfrac{c}{\varepsilon_0 n_2^3}(\mathbf{D}_2 \cdot \mathbf{D}_2)$

For complex representation, the above is written

$$\mathbf{s} \cdot \mathbf{S} = \frac{1}{2} \mathrm{Re}\left(\frac{c}{\varepsilon_0 n_1^3}(\mathbf{D}_1^* \cdot \mathbf{D}_1) + \frac{c}{\varepsilon_0 n_2^3}(\mathbf{D}_2^* \cdot \mathbf{D}_2) \right)$$

(b) In (a) we obtain $\mathbf{s} \cdot \mathbf{S} = \dfrac{n_1}{\mu c}(\mathbf{A}_1 \cdot \mathbf{A}_1) + \dfrac{n_2}{\mu c}(\mathbf{A}_2 \cdot \mathbf{A}_2)$. For complex representation, a similar result is obtained.

(c) In a lossless medium, the power flow is a constant.

1.36 *Solution:*

(a) For a field with a transverse dimension, a longitudinal must be present in order to satisfy the divergence condition

$\nabla \cdot \mathbf{E} = 0$ (assuming free space)

$$\nabla \cdot \mathbf{E} = \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right) \approx \left(\frac{\partial E_0}{\partial x} - i\frac{\partial E_0}{\partial y} - ikE_z \right) = 0$$

Thus, the longitudinal component is given approximately as $E_z \approx -\frac{i}{k}\left(\frac{\partial E_0}{\partial x} - i\frac{\partial E_0}{\partial y} \right)$.

The magnetic field is related to the electric field by $-i\omega\mu\mathbf{H} = \nabla \times \mathbf{E}$. In other words

$$-i\omega\mu\mathbf{H} = \nabla \times \mathbf{E} = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right)\hat{x} + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right)\hat{y} + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right)\hat{z}$$

$$= \left(\frac{\partial}{\partial y}[-\frac{i}{k}\left(\frac{\partial E_0}{\partial x} - i\frac{\partial E_0}{\partial y} \right)] + kE_0 \right)\hat{x} + \left(-ikE_0 - \frac{\partial}{\partial x}[-\frac{i}{k}\left(\frac{\partial E_0}{\partial x} - i\frac{\partial E_0}{\partial y} \right)] \right)\hat{y} + \left(-i\frac{\partial E_0}{\partial x} - \frac{\partial E_0}{\partial y} \right)\hat{z}$$

Neglecting the second order derivatives, the above becomes

$$= (kE_0)\hat{x} + (-ikE_0)\hat{y} + \left(-i\frac{\partial E_0}{\partial x} - \frac{\partial E_0}{\partial y} \right)\hat{z} = k\mathbf{E}$$

This proves $\mathrm{H}(x,y,z,t) \approx i\frac{k}{\omega\mu}\mathrm{E}(x,y,z,t)$.

(b) The time-averaged momentum density is

$$\langle \mathbf{P} \rangle = \frac{1}{2}\mathrm{Re}(\mu\varepsilon\mathbf{E} \times \mathbf{H}) = \frac{\varepsilon}{\omega}\left(E_0\frac{\partial}{\partial y}E_0, \quad -E_0\frac{\partial}{\partial x}E_0, \quad kE_0^2 \right)$$

The time-averaged angular momentum density is

$$\langle \mathbf{L} \rangle = \langle \mathbf{r} \times \mathbf{P} \rangle = \frac{\varepsilon}{\omega}\left(ykE_0^2 + zE_0\frac{\partial}{\partial x}E_0, \quad zE_0\frac{\partial}{\partial y}E_0 - xE_0^2, \quad -xE_0\frac{\partial}{\partial x}E_0 - yE_0\frac{\partial}{\partial y}E_0 \right)$$

The z-component is

$$L_z = \frac{\varepsilon}{2\omega}\left(-\frac{\partial}{\partial x}xE_0^2 - \frac{\partial}{\partial y}yE_0^2 + E_0^2 + E_0^2 \right)$$

Now, integrating over the entire space

$$\iiint L_z dxdydz = \iiint dxdydz\frac{\varepsilon}{2\omega}\left(-\frac{\partial}{\partial x}xE_0^2 - \frac{\partial}{\partial y}yE_0^2 + E_0^2 + E_0^2 \right) = \iiint dxdydz\frac{\varepsilon}{\omega}E_0^2$$

The integral over the first two terms vanishes due to the condition of finite transverse dimension.

The time-averaged energy density is

$$\langle U \rangle = \frac{1}{4}\mathrm{Re}(\varepsilon\mathbf{E} \cdot \mathbf{D}^* + \mu\mathbf{B} \cdot \mathbf{H}^*) = \frac{\varepsilon}{2}(\mathbf{E} \cdot \mathbf{E}^*) = \frac{\varepsilon}{2}\left(E_0^2 + E_0^2 + \frac{1}{k^2}[\frac{\partial E_0}{\partial y}]^2 + \frac{1}{k^2}[\frac{\partial E_0}{\partial x}]^2 \right) \approx \varepsilon E_0^2$$

Let

$\iiint \langle U \rangle dxdydz = \iiint dxdydz\varepsilon E_0^2 = N\hbar\omega$, where N is the total number of photons, then

$$\iiint L_z dxdydz = \iiint dxdydz \frac{\varepsilon}{\omega} E_0^2 = N\hbar$$

Thus, a single photon of circularly polarized light carries an angular momentum of $\hbar$.

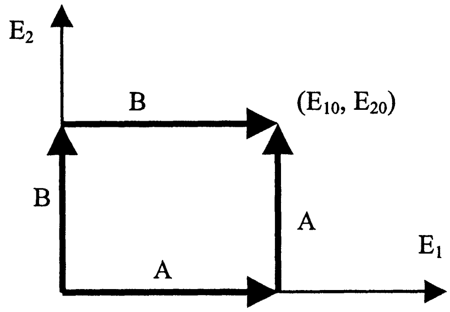
(c) For simplicity, assume a cylindrically symmetric field $E_0(x, y)$.

$$L_x = \frac{\varepsilon}{\omega}\left(ykE_0^2 + zE_0 \frac{\partial}{\partial x} E_0 \right)$$

$$L_y = \frac{\varepsilon}{\omega}\left(zE_0 \frac{\partial}{\partial y} E_0 - xE_0^2 \right)$$

We note that both L_x and L_y are odd functions. The integrals are zero.

1.37 *Solution:*



(a) We write the components of **P** in a linear medium as

$P_1 = \varepsilon_0\chi_{11}E_1 + \varepsilon_0\chi_{12}E_2, \qquad P_2 = \varepsilon_0\chi_{21}E_1 + \varepsilon_0\chi_{22}E_2$

and their differential as

$dP_1 = \varepsilon_0\chi_{11}dE_1 + \varepsilon_0\chi_{12}dE_2, \quad dP_2 = \varepsilon_0\chi_{21}dE_1 + \varepsilon_0\chi_{22}dE_2$

For path A, the integral can be written

$$W = \int_A \mathbf{E}\cdot d\mathbf{P} = \int_{(0,0)}^{(E_{10},0)} E_1 dP_1 + \int_{(E_{10},0)}^{(E_{10},E_{20})} (E_1 dP_1 + E_2 dP_2)$$

$$= \int_{(0,0)}^{(E_{10},0)} E_1\varepsilon_0\chi_{11}dE_1 + \int_{(E_{10},0)}^{(E_{10},E_{20})} (E_1\varepsilon_0\chi_{12}dE_2 + E_2\varepsilon_0\chi_{22}dE_2)$$

$$= \frac{1}{2}\varepsilon_0\chi_{11}E_{10}^2 + \varepsilon_0\chi_{12}E_{10}E_{20} + \frac{1}{2}\varepsilon_0\chi_{22}E_{20}^2$$

(b) For path B, the integral can be written

$$W = \int_B \mathbf{E}\cdot d\mathbf{P} = \int_{(0,0)}^{(0,E_{20})} E_2 dP_2 + \int_{(0,E_{20})}^{(E_{10},E_{20})} (E_1 dP_1 + E_2 dP_2)$$

$$= \int_{(0,0)}^{(E_{10},0)} E_2\varepsilon_0\chi_{22}dE_2 + \int_{(E_{10},0)}^{(E_{10},E_{20})} (E_1\varepsilon_0\chi_{11}dE_1 + E_2\varepsilon_0\chi_{21}dE_1)$$

$$= \frac{1}{2}\varepsilon_0\chi_{22}E_{20}^2 + \varepsilon_0\chi_{21}E_{20}E_{10} + \frac{1}{2}\varepsilon_0\chi_{11}E_{10}^2$$

Chapter 2

2.1 *Solution:*

(a) The eigenvalues satisfy the equation $\begin{vmatrix} A-\lambda & B \\ C & D-\lambda \end{vmatrix} = 0$, or $(A-\lambda)(D-\lambda) - BC = 0$. This is a quadratic equation. The solutions are:

$$\lambda = \frac{A+D}{2} \pm \frac{1}{2}\sqrt{(A+D)^2 - 4(AD-BC)} = \frac{A+D}{2} \pm \frac{1}{2}\sqrt{(A+D)^2 - 4}.$$

If we define $(A+D) = 2\cos\theta$, then the eigenvalues become: $\lambda = \cos\theta \pm i\sin\theta = \exp(\pm i\theta)$.

(b) From the basics of linear algebra, the eigenvectors are: $\begin{pmatrix} B \\ \lambda - A \end{pmatrix}$. We can also examine

$$\begin{pmatrix} A-\lambda & B \\ C & D-\lambda \end{pmatrix}\begin{pmatrix} B \\ \lambda - A \end{pmatrix} = \begin{pmatrix} (A-\lambda)B + B(\lambda - A) \\ (BC) + (D-\lambda)(\lambda - A) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0$$

To see if they are orthogonal, we examine

$$\begin{pmatrix} B \\ e^{-i\theta} - A \end{pmatrix}^{*}\begin{pmatrix} B \\ e^{i\theta} - A \end{pmatrix} = \begin{pmatrix} B^{*}B \\ (e^{i\theta} - A^{*})(e^{i\theta} - A) \end{pmatrix} = \begin{pmatrix} B^2 \\ (e^{i\theta} - A)^2 \end{pmatrix}$$

where we assume that both A and B are real. We see that the scalar product is not generally zero. Thus the two eigenvectors are not generally orthogonal.
They, however, are dependent when B=0, or θ=0.

(c) For a 2x2 matrix with non-zero determinant, the inverse matrix is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$ so

$$M^{-1} = \begin{pmatrix} B & B \\ e^{i\theta} - A & e^{-i\theta} - A \end{pmatrix}^{-1} = \frac{1}{-2iB\sin\theta}\begin{pmatrix} e^{-i\theta} - A & -B \\ -e^{i\theta} + A & B \end{pmatrix}$$

We now carry out the matrix multiplications

$$M^{-1}\begin{pmatrix} A & B \\ C & D \end{pmatrix}M = M^{-1} = \frac{1}{-2iB\sin\theta}\begin{pmatrix} B & B \\ e^{i\theta} - A & e^{-i\theta} - A \end{pmatrix}^{-1}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} B & B \\ e^{i\theta} - A & e^{-i\theta} - A \end{pmatrix}$$

$$= \frac{1}{-2iB\sin\theta}\begin{pmatrix} e^{-i\theta} - A & -B \\ -e^{i\theta} + A & B \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} B & B \\ e^{i\theta} - A & e^{-i\theta} - A \end{pmatrix}$$

the columns of matrix M are eigenvectors, so

$$= \frac{1}{-2iB\sin\theta}\begin{pmatrix} e^{-i\theta} - A & -B \\ -e^{i\theta} + A & B \end{pmatrix}\begin{pmatrix} e^{i\theta}B & e^{-i\theta}B \\ e^{i\theta}(e^{i\theta} - A) & e^{-i\theta}(e^{-i\theta} - A) \end{pmatrix}$$

$$= \frac{1}{-2iB\sin\theta}\begin{pmatrix} B(e^{2i\theta} - 1) & 0 \\ 0 & B(e^{-2i\theta} - 1) \end{pmatrix} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

2.2 Solution:

(a) From the above equation, we obtain

$r_0 = (\alpha_0 + \beta_0)B, \quad r_0' = \alpha_0(e^{i\theta} - A) + \beta_0(e^{-i\theta} - A)$.

Rearranging the terms, we obtain

$(\alpha_0 + \beta_0) = r_0 / B, \qquad i\sin\theta(\alpha_0 - \beta_0) = r_0' + r_0(A - \cos\theta)/B$.

Solving for α_0 and β_0, we obtain

$\alpha_0 = (r_0' + Ar_0/B - r_0 e^{-i\theta}/B)/(2i\sin\theta)$

$\beta_0 = (-r_0' - Ar_0/B + r_0 e^{+i\theta}/B)/(2i\sin\theta)$

They are indeed a conjugate pair, provided both A and B are real.

(b) $\begin{pmatrix} r_m \\ r_m' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^m \begin{pmatrix} r_0 \\ r_0' \end{pmatrix} = \alpha_0 \begin{pmatrix} A & B \\ C & D \end{pmatrix}^m \begin{pmatrix} B \\ e^{i\theta} - A \end{pmatrix} + \beta_0 \begin{pmatrix} A & B \\ C & D \end{pmatrix}^m \begin{pmatrix} B \\ e^{-i\theta} - A \end{pmatrix}$

$= \alpha_0 e^{im\theta} \begin{pmatrix} B \\ e^{i\theta} - A \end{pmatrix} + \beta_0 e^{-im\theta} \begin{pmatrix} B \\ e^{-i\theta} - A \end{pmatrix}$. Thus

$r_m = r_{\max}\sin(m\theta + \alpha) = B(\alpha_0 e^{im\theta} + \beta_0 e^{-im\theta}) = B[(\alpha_0 + \beta_0)\cos m\theta + i(\alpha_0 - \beta_0)\sin m\theta]$

Since, $r_{\max}\sin(m\theta + \alpha) = r_{\max}\sin\alpha\cos m\theta + r_{\max}\cos\alpha\sin m\theta$ and the above equation is valid for an arbitrary m, we must have

$r_{\max}\sin\alpha = B(\alpha_0 + \beta_0) = r_0$ and $r_{\max}\cos\alpha = iB(\alpha_0 - \beta_0) = [Br_0' + r_0(A - \cos\theta)]/\sin\theta$

Thus, we have

$$r_{\max}^2 = (r_0)^2 + [Br_0' + r_0(A - \cos\theta)]^2 / \sin^2\theta$$

$$\tan\alpha = \frac{r_0 \sin\theta}{Br_0' + r_0(A - \cos\theta)}$$

(c) From Eqs. (2.1-10, 11)

$$\begin{pmatrix} r_N \\ r'_N \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^N \begin{pmatrix} r_0 \\ r'_0 \end{pmatrix} = \begin{pmatrix} AU_{N-1} - U_{N-2} & BU_{N-1} \\ CU_{N-1} & DU_{N-1} - U_{N-2} \end{pmatrix} \begin{pmatrix} r_0 \\ r'_0 \end{pmatrix}$$

where

$$U_N = \frac{\sin(N+1)\theta}{\sin\theta}$$

So,

$$r_N = (AU_{N-1} - U_{N-2})r_0 + BU_{N-1}r'_0 = r_0 A \frac{\sin N\theta}{\sin\theta} - r_0 \frac{\sin(N-1)\theta}{\sin\theta} + r'_0 B \frac{\sin N\theta}{\sin\theta}$$

Using $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$, the above becomes

$$r_N = r_0 A \frac{\sin N\theta}{\sin\theta} - r_0 \frac{\sin N\theta\cos\theta - \cos N\theta\sin\theta}{\sin\theta} + r'_0 B \frac{\sin N\theta}{\sin\theta}$$

Rearranging the terms, we have

$$r_N = (r_0 A\frac{1}{\sin\theta} - r_0\frac{\cos\theta}{\sin\theta} + r'_0 B\frac{1}{\sin\theta})\sin N\theta + r_0\cos N\theta$$

The above can be written

$$r_N = r_{\max}\sin(N\theta+\alpha) = r_{\max}\sin\alpha\cos N\theta + r_{\max}\cos\alpha\sin N\theta$$

provided

$$r_{\max}\sin\alpha == r_0 \text{ and } r_{\max}\cos\alpha = [Br_0' + r_0(A-\cos\theta)]/\sin\theta$$

The results are in agreement with (b). Thus, we also have

$$r_{\max}^2 = (r_0)^2 + [Br_0' + r_0(A-\cos\theta)]^2/\sin^2\theta$$

$$\tan\alpha = \frac{r_0\sin\theta}{Br_0' + r_0(A-\cos\theta)}$$

(d) From Eq. (2.1-21), *A=1, B=d, C=-1/f, D=1-d/f*, we have

$$\cos\theta = 1 - d/(2f),\ \sin\theta = \sqrt{1-\cos^2\theta} = \left(\frac{d}{f} - \frac{d^2}{4f^2}\right)^{1/2}$$

$$r_{\max}^2 = (r_0)^2 + [dr_0' + r_0(1-\cos\theta)]^2/\sin^2\theta = (r_0)^2 + [dr_0' + r_0(d/(2f))]^2 / \left(\frac{d}{f} - \frac{d^2}{4f^2}\right)$$

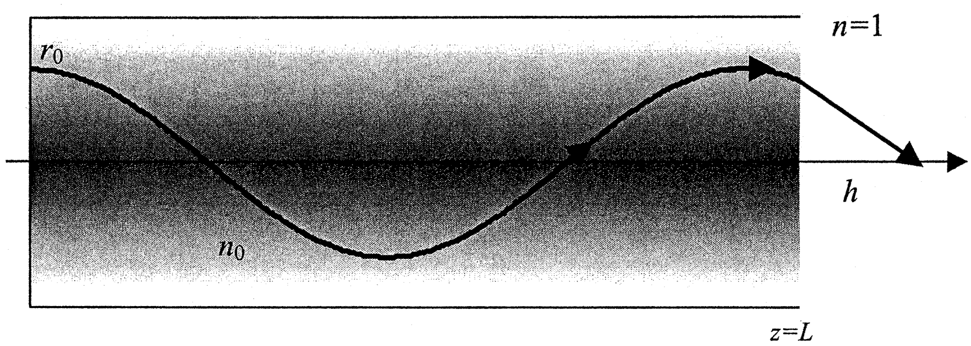
$$= (r_0)^2 + \frac{4f^2[dr_0' + r_0(d/(2f))]^2}{4fd-d^2} = \frac{4fdr_0^2 - d^2r_0^2 + 4f^2d^2r_0'^2 + 4fd^2r_0r_0' + d^2r_0^2}{4fd-d^2}$$

$$= \frac{4fdr_0^2 + 4f^2d^2r_0'^2 + 4fd^2r_0r_0'}{4fd-d^2} = 4f\frac{r_0^2 + fdr_0'^2 + dr_0r_0'}{4f-d}$$

From (c), we have

$$\tan\alpha = \frac{r_0\sin\theta}{Br_0' + r_0(A-\cos\theta)} = \frac{r_0}{dr_0' + dr_0/(2f)}\left(\frac{d}{f} - \frac{d^2}{4f^2}\right)^{1/2} = \frac{r_0}{2fr_0' + r_0}\left(\frac{4f}{d} - 1\right)^{1/2}$$

2.3 Solution:



For an incident ray parallel to the axis with an initial position of r_0 at $z = 0$, the ray path is given by

$r = r_0 \cos(gz)$

At the exit face ($z = L$), the ray position and slope are given by

$r = r_0 \cos(gL)$

$r' = -gr_0 \sin(gL)$

As a result of refraction at the interface between the medium and air, the slope in the air is given by (Snell's law)

$r_{out}' = -n_0 g r_0 \sin(gL)$

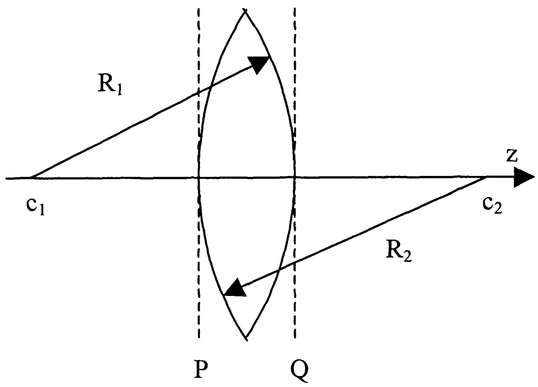
For paraxial rays, the magnitude of the slope is the angle of the ray. So, we have

$$r_{out}' = -n_0 g r_0 \sin(gL) = -\frac{r_0 \cos(gL)}{h}$$

Solving for h, we obtain

$$h = \frac{1}{n_0 g \tan(gL)}$$

2.4 Solution:



First we calculate the thickness d(x,y) of the lens at location (x,y).
Consider a parallel ray at position (x, y). Let z_1 and z_2 be the longitudinal position of the lens surface where the ray traverses. These positions governed by the spherical surfaces and are given by

$$\sqrt{x^2 + y^2 + (z_1 - c_1)^2} = R_1, \qquad \sqrt{x^2 + y^2 + (z_2 - c_2)^2} = R_2$$

where c_1 and c_2 are location of the centers of curvature.

$$z_1 - c_1 + \frac{x^2 + y^2}{2(z_1 - c_1)} \approx R_1, \qquad - z_2 + c_2 + \frac{x^2 + y^2}{2|z_2 - c_2|} \approx R_2$$

Further approximation yields,

$$z_1 \approx +c_1 - \frac{x^2 + y^2}{2R_1} + R_1, \qquad z_2 \approx c_2 + \frac{x^2 + y^2}{2R_2} - R_2$$

The lens thickness function is thus

$$d(x, y) = z_1 - z_2 \approx d(0,0) - \frac{x^2 + y^2}{2R_1} - \frac{x^2 + y^2}{2R_2} \equiv d_0 - \frac{x^2 + y^2}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

For paraxial rays, the phase shift upon transmission through the thin lens (between planes z=P and z=Q) is given by

$$\psi(x, y) = \int_P^Q n(z)dz = nkd(x, y) + k[d_0 - d(x, y)]$$

where $k=2\pi/\lambda$, the first term on the right side is due to transmission through the lens medium with index n, and the second term is due to transmission through air. Substituting the expression for d(x,y), we obtain

$$\psi(x, y) = k(n-1)d(x, y) + kd_0 = -k(n-1)\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{x^2 + y^2}{2} + knd_0$$

$$\psi(x, y) = -k\frac{1}{f}\frac{x^2 + y^2}{2} + \phi$$

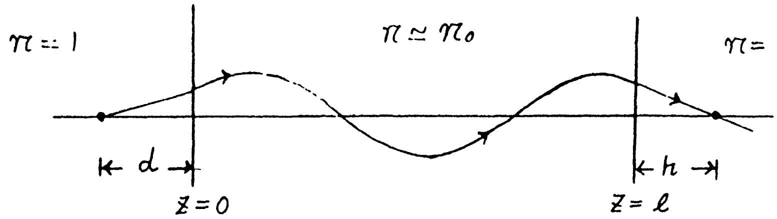
where ϕ is the phase shift knd_0 for ray at the center of the lens (x=y=0) and f is given by

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

Assuming a thin lens, the transmitted wave is thus given by

$$E_{out}(x, y) = E_{in}(x, y)\exp[-i\psi(x, y)] = E_{in}(x, y)\exp[ik\frac{x^2 + y^2}{2f}]\exp(-i\phi)$$

2.5



Note: $g^2 = \frac{k_2}{k}$

For a point on the axis at $z = -d$, $\begin{bmatrix} Y_{in} \\ Y_{in}' \end{bmatrix} = \begin{bmatrix} 0 \\ Y_{in}' \end{bmatrix}$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & n_0 \end{bmatrix}\begin{bmatrix} \cos(\sqrt{\frac{K_2}{K}}\ell) & \sqrt{\frac{K}{K_2}}\sin(\sqrt{\frac{K_2}{K}}\ell) \\ -\sqrt{\frac{K_2}{K}}\sin(\sqrt{\frac{K_2}{K}}\ell) & \cos(\sqrt{\frac{K_2}{K}}\ell) \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{n_0} \end{bmatrix}\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\sqrt{\frac{K_2}{K}}\ell) - n_0 h\sqrt{\frac{K_2}{K}}\sin(\sqrt{\frac{K_2}{K}}\ell) & d\cos(\sqrt{\frac{K_2}{K}}\ell) - n_0 h d\sqrt{\frac{K_2}{K}}\sin(\sqrt{\frac{K_2}{K}}\ell) + \frac{1}{n_0}\sqrt{\frac{K}{K_2}}\sin(\sqrt{\frac{K_2}{K}}\ell) + h\cos(\sqrt{\frac{K_2}{K}}\ell) \\ -n_0\sqrt{\frac{K_2}{K}}\sin(\sqrt{\frac{K_2}{K}}\ell) & -n_0 d\sqrt{\frac{K_2}{K}}\sin(\sqrt{\frac{K_2}{K}}\ell) + \cos(\sqrt{\frac{K_2}{K}}\ell) \end{bmatrix}$$

$$\begin{bmatrix} Y_{out} \\ Y_{out}' \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} Y_{in} \\ Y_{in}' \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} 0 \\ Y_{in}' \end{bmatrix} = \begin{bmatrix} B\,Y_{in}' \\ D\,Y_{in}' \end{bmatrix}$$

$\Rightarrow$ $Y_{out} = 0$ (imaging onto a single point) if $B = 0$

$\therefore$ $d\cos(\sqrt{\frac{K_2}{K}}\ell) - n_0 h d\sqrt{\frac{K_2}{K}}\sin(\sqrt{\frac{K_2}{K}}\ell) + \frac{1}{n_0}\sqrt{\frac{K}{K_2}}\sin(\sqrt{\frac{K_2}{K}}\ell) + h\cos(\sqrt{\frac{K_2}{K}}\ell) = 0$

$\Rightarrow$ $h\left[n_0 d\sqrt{\frac{K_2}{K}}\sin(\sqrt{\frac{K_2}{K}}\ell) - \cos(\sqrt{\frac{K_2}{K}}\ell)\right] = d\cos(\sqrt{\frac{K_2}{K}}\ell) + \frac{1}{n_0}\sqrt{\frac{K}{K_2}}\sin(\sqrt{\frac{K_2}{K}}\ell)$

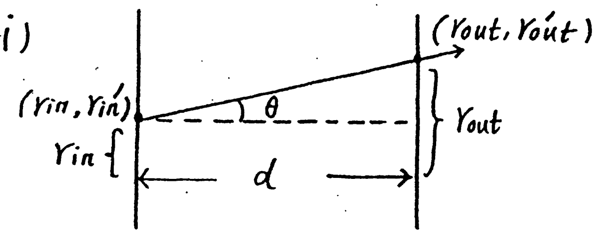
$$\Rightarrow h = \frac{d\cos(\sqrt{\frac{K_2}{K}}\ell) + \frac{1}{n_0}\sqrt{\frac{K}{K_2}}\sin(\sqrt{\frac{K_2}{K}}\ell)}{n_0 d\sqrt{\frac{K_2}{K}}\sin(\sqrt{\frac{K_2}{K}}\ell) - \cos(\sqrt{\frac{K_2}{K}}\ell)}$$

$$\therefore \begin{cases} h = -\frac{1}{n_0}\sqrt{\frac{K}{K_2}}\tan(\sqrt{\frac{K_2}{K}}\ell) & \text{if } d = 0 \\ h \simeq \frac{1}{n_0}\sqrt{\frac{K}{K_2}}\cot(\sqrt{\frac{K_2}{K}}\ell) & \text{as } d \to \infty \end{cases}$$

A lenslike medium occupying the region $0 \le z \le \ell$ will image a point on the axis at $z = -d$ onto a single point at $z = \ell + h$. If $h > 0$, the image is real. If $h < 0$, the image is virtual.

2.6 (i)

Note: $g^2 = \frac{k_2}{k}$



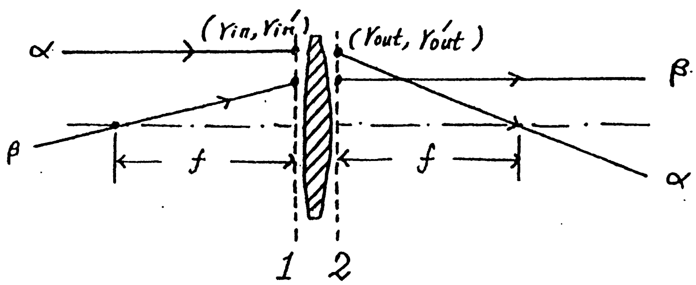
Since Y_{in}' (slope) $= \tan\theta = \frac{Y_{out} - Y_{in}}{d}$

$\therefore\ Y_{out} = Y_{in} + d\,Y_{in}'$ ---------- (1)

$Y_{in}' = Y_{out}' = \tan\theta$ (same slope) ------ (2)

$\therefore$ From Eq. (1) (2) $\Rightarrow \begin{bmatrix} Y_{out} \\ Y_{out}' \end{bmatrix} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} Y_{in} \\ Y_{in}' \end{bmatrix}$

(ii) Thin Lens of focal length f



Assume the lens is so thin that there is negligible distance between the entrance [1] and the exit [2] planes. Thus,

$Y_{in} = Y_{out}$ ---------- (3)

Consider ray α in the diagram. For this special case, the input slope $Y_{in}' = 0$, yet it is obvious that the output slope is $\frac{-Y_{out}}{f}\left(= \frac{-Y_{in}}{f}\right)$

Assume $Y_{out}' = C\,Y_{in} + D\,Y_{in}'$

$\Rightarrow -\frac{Y_{in}}{f} = C\,Y_{in} + D \cdot 0 \quad \Rightarrow \quad C = \frac{-1}{f}$ ------ (4)

In the orther case, ray β comes in with a slope of $+\frac{Y_{in}}{f}$ and obviously exits parallel to the axis. Thus, $Y_{out}' = 0$

$\Rightarrow \quad 0 = C\,Y_{in} + D\left(\frac{Y_{in}}{f}\right) \quad \Rightarrow D = -fC = (-f)\left(\frac{-1}{f}\right) = 1$ ----- (5)

From Eq (4) (5), $\Rightarrow Y_{out}' = \left(-\frac{1}{f}\right) Y_{in} + Y_{in}'$ ----- (6)

$\therefore$ From Eq. (3) (6), we have $\begin{bmatrix} Y_{out} \\ Y_{out}' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} Y_{in} \\ Y_{in}' \end{bmatrix}$