OPTIMAL STATE ESTIMATION

Kalman, H_∞, and Nonlinear Approaches
Solution Manual

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INTRODUCTION

This solution manual is a companion to the text *Optimal State Estimation: Kalman, H_∞, and Nonlinear Approaches*, by Dan Simon (John Wiley & Sons, 2006). The MATLAB[1] source code for the computer exercise solutions is given at the end of this solution manual. The references in this solution manual refer to the references section in the text *Optimal State Estimation*. The equation numbers in this solution manual refer to the equations in the book *Optimal State Estimation*.

Although the MATLAB code for the solutions is not available on the Internet, MATLAB-based source code for the examples in the text is available at the author's Web site.[2] The author's e-mail address is also available on the Web site, and I eagerly invite feedback, comments, suggestions for improvements, and corrections.

A note on notation

Three dots between delimiters (parenthesis, brackets, or braces) means that the quantity between the delimiters is the same as the quantity between the previous set of identical delimiters in the same equation. For example,

$$\begin{aligned} (A + BCD) + (\cdots)^T &= (A + BCD) + (A + BCD)^T \\ A + [B(C + D)]^{-1}E[\cdots] &= A + [B(C + D)]^{-1}E[B(C + D)] \end{aligned}$$

[1] MATLAB is a registered trademark of The MathWorks, Inc.
[2] http://academic.csuohio.edu/simond/estimation – if the Web site address changes, it should be easy to find with an Internet search.

CHAPTER 1

Linear systems theory

Problems

Written exercises

1.1 Find the rank of the matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Solution

The rank of a matrix A can be defined as the dimension of the largest submatrix consisting of rows and columns of A whose determinant is nonzero. With this definition we see that the rank of the zero matrix is zero.

1.2 Find two 2×2 matrices A and B such that $A \neq B$, neither A nor B are diagonal, $A \neq cB$ for any scalar c, and $AB = BA$. Find the eigenvectors of A and B. Note that they share an eigenvector. Interestingly, every pair of commuting matrices shares at least one eigenvector [Hor85, p. 51].

Optimal State Estimation Solution Manual, First Edition. By Dan J. Simon

Solution

Suppose A and B are given as

$$\begin{aligned} A &= \begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix} \\ B &= \begin{bmatrix} b_1 & b_2 \\ b_2 & b_3 \end{bmatrix} \end{aligned}$$

Then we see that

$$\begin{aligned} AB &= \begin{bmatrix} a_1b_1 + a_2b_2 & a_1b_2 + a_2b_3 \\ a_2b_1 + a_3b_2 & a_2b_2 + a_3b_3 \end{bmatrix} \\ BA &= \begin{bmatrix} a_1b_1 + a_2b_2 & a_2b_1 + a_3b_2 \\ a_1b_2 + a_2b_3 & a_2b_2 + a_3b_3 \end{bmatrix} \end{aligned}$$

We see that $AB = BA$ if $a_1b_2 + a_2b_3 = a_2b_1 + a_3b_2$. This will be true, for example, if $a_1 = 1$, $a_2 = 2$, $a_3 = 1$, $b_1 = 1$, $b_2 = 3$, and $b_3 = 1$. This gives

$$\begin{aligned} A &= \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \\ B &= \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \end{aligned}$$

For these matrices A has the eigenvalues -1 and 3, B has the eigenvalues -2 and 4, and both A and B have the eigenvectors $\begin{bmatrix} -1 & 1 \end{bmatrix}^T$ and $\begin{bmatrix} 1 & 1 \end{bmatrix}^T$.

1.3 Prove the three identities of Equation (1.26).

Solution

a). Suppose A is an $n \times m$ matrix, and B is an $m \times p$ matrix. Then

$$\begin{aligned} (AB)^T &= \left(\begin{bmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nm} \end{bmatrix} \begin{bmatrix} B_{11} & \cdots & B_{1p} \\ \vdots & \ddots & \vdots \\ B_{m1} & \cdots & B_{mp} \end{bmatrix} \right)^T \\ &= \begin{bmatrix} \sum A_{1j}B_{j1} & \cdots & \sum A_{1j}B_{jp} \\ \vdots & \ddots & \vdots \\ \sum A_{nj}B_{j1} & \cdots & \sum A_{nj}B_{jp} \end{bmatrix}^T \\ &= \begin{bmatrix} \sum A_{1j}B_{j1} & \cdots & \sum A_{nj}B_{j1} \\ \vdots & \ddots & \vdots \\ \sum A_{1j}B_{jp} & \cdots & \sum A_{nj}B_{jp} \end{bmatrix} \\ B^TA^T &= \begin{bmatrix} B_{11} & \cdots & B_{m1} \\ \vdots & \ddots & \vdots \\ B_{1p} & \cdots & B_{mp} \end{bmatrix} \begin{bmatrix} A_{11} & \cdots & A_{n1} \\ \vdots & \ddots & \vdots \\ A_{1m} & \cdots & A_{nm} \end{bmatrix} \\ &= \begin{bmatrix} \sum B_{j1}A_{1j} & \cdots & \sum B_{j1}A_{nj} \\ \vdots & \ddots & \vdots \\ \sum B_{jp}A_{1j} & \cdots & \sum B_{jp}A_{nj} \end{bmatrix} \end{aligned}$$

QED

b). Suppose that $(AB)^{-1} = C$. Then $CAB = I$. Postmultiplying both sides of this equation by B^{-1} gives $CA = B^{-1}$. Postmultiplying both sides of this equation by A^{-1} gives $C = B^{-1}A^{-1}$. Hence we see that $(AB)^{-1} = B^{-1}A^{-1}$. QED

c). Suppose A is an $n \times m$ matrix, and B is an $m \times n$ matrix. Then

$$
\begin{aligned}
\mathrm{Tr}(AB) &= \mathrm{Tr}\left(\begin{bmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nm} \end{bmatrix}\begin{bmatrix} B_{11} & \cdots & B_{1n} \\ \vdots & \ddots & \vdots \\ B_{m1} & \cdots & B_{mn} \end{bmatrix}\right) \\
&= \mathrm{Tr}\left(\begin{bmatrix} \sum A_{1j}B_{j1} & \cdots & \sum A_{1j}B_{jn} \\ \vdots & \ddots & \vdots \\ \sum A_{nj}B_{j1} & \cdots & \sum A_{nj}B_{jn} \end{bmatrix}\right) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} A_{ij}B_{ji} \\
\mathrm{Tr}(BA) &= \mathrm{Tr}\left(\begin{bmatrix} B_{11} & \cdots & B_{1n} \\ \vdots & \ddots & \vdots \\ B_{m1} & \cdots & B_{mn} \end{bmatrix}\begin{bmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nm} \end{bmatrix}\right) \\
&= \mathrm{Tr}\left(\begin{bmatrix} \sum B_{1j}A_{j1} & \cdots & \sum B_{1j}A_{jm} \\ \vdots & \ddots & \vdots \\ \sum B_{mj}A_{j1} & \cdots & \sum B_{mj}A_{jm} \end{bmatrix}\right) \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n} B_{ij}A_{ji}
\end{aligned}
$$

QED

1.4 Find the partial derivative of the trace of AB with respect to A.

Solution

Suppose A is an $n \times m$ matrix, and B is an $m \times n$ matrix. Then

$$
\begin{aligned}
\frac{\partial \mathrm{Tr}(AB)}{\partial A} &= \frac{\partial}{\partial A}\sum_{i=1}^{n}\sum_{j=1}^{m} A_{ij}B_{ji} \\
&= \begin{bmatrix} \frac{\partial}{\partial A_{11}}\sum_{i=1}^{n}\sum_{j=1}^{m} A_{ij}B_{ji} & \cdots & \frac{\partial}{\partial A_{1m}}\sum_{i=1}^{n}\sum_{j=1}^{m} A_{ij}B_{ji} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial A_{n1}}\sum_{i=1}^{n}\sum_{j=1}^{m} A_{ij}B_{ji} & \cdots & \frac{\partial}{\partial A_{nm}}\sum_{i=1}^{n}\sum_{j=1}^{m} A_{ij}B_{ji} \end{bmatrix} \\
&= \begin{bmatrix} B_{11} & \cdots & B_{m1} \\ \vdots & \ddots & \vdots \\ B_{1n} & \cdots & B_{mn} \end{bmatrix} \\
&= B^T
\end{aligned}
$$

1.5 Consider the matrix

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

Recall that the eigenvalues of A are found by find the roots of the polynomial $P(\lambda) = |\lambda I - A|$. Show that $P(A) = 0$. (This is an illustration of the Cayley–Hamilton theorem [Bay99, Che99, Kai00].)

Solution

$$\begin{aligned} P(\lambda) &= |\lambda I - A| \\ &= \begin{vmatrix} \lambda - a & -b \\ -b & \lambda - c \end{vmatrix} \\ &= \lambda^2 - (a+c)\lambda + ac - b^2 \\ P(A) &= A^2 - (a+c)A + (ac - b^2)I \\ &= \begin{bmatrix} a^2+b^2 & ab+bc \\ ab+bc & b^2+c^2 \end{bmatrix} - (a+c)\begin{bmatrix} a & b \\ b & c \end{bmatrix} + \begin{bmatrix} ac-b^2 & 0 \\ 0 & ac-b^2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

1.6 Suppose that A is invertible and

$$\begin{bmatrix} A & A \\ B & A \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix}$$

Find B and C in terms of A [Lie67].

Solution

Multiplying out the matrix equation gives the following two equations.

$$\begin{aligned} A^2 + AC &= 0 \\ BA + AC &= I \end{aligned}$$

Solving for B and C in terms of A gives

$$\begin{aligned} B &= A + A^{-1} \\ C &= -A \end{aligned}$$

1.7 Show that AB may not be symmetric even though both A and B are symmetric.

Solution

Suppose the symmetric matrices A and B are given as

$$\begin{aligned} A &= \begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix} \\ B &= \begin{bmatrix} b_1 & b_2 \\ b_2 & b_3 \end{bmatrix} \end{aligned}$$

Then we see that

$$AB = \begin{bmatrix} a_1b_1 + a_2b_2 & a_1b_2 + a_2b_3 \\ a_2b_1 + a_3b_2 & a_2b_2 + a_3b_3 \end{bmatrix}$$

AB is not symmetric if $a_1b_2 + a_2b_3 \neq a_2b_1 + a_3b_2$.

1.8 Consider the matrix

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

where a, b, and c are real, and a and c are nonnegative.

a) Compute the solutions of the characteristic polynomial of A to prove that the eigenvalues of A are real.

b) For what values of b is A positive semidefinite?

Solution

a). The characteristic polynomial of A is

$$\begin{aligned} P(\lambda) &= |\lambda I - A| \\ &= \begin{vmatrix} \lambda - a & -b \\ -b & \lambda - c \end{vmatrix} \\ &= \lambda^2 - (a+c)\lambda + ac - b^2 \end{aligned}$$

Finding the roots of this gives

$$\lambda = \frac{1}{2}\left[a + c \pm \sqrt{(a-c)^2 + 4b^2}\right]$$

The discriminant is non-negative so λ is real.
QED

b). In order for A to be positive semidefinite, its eigenvalues must be positive. The eigenvalues are

$$\lambda = \begin{cases} \frac{1}{2}\left[a + c + \sqrt{(a-c)^2 + 4b^2}\right] \\ \frac{1}{2}\left[a + c - \sqrt{(a-c)^2 + 4b^2}\right] \end{cases}$$

The first eigenvalue is always non-negative. The second eigenvalue is non-negative if $a + c \geq \sqrt{(a-c)^2 + 4b^2}$. Solving this equation gives $|b| \leq \sqrt{ac}$ as the condition of positive semidefiniteness.

1.9 Derive the properties of the state transition matrix given in Equation (1.72).

Solution

$$\frac{d}{dt}e^{At} = \frac{d}{dt}\sum_{j=0}^{\infty}\frac{(At)^j}{j!}$$

$$
\begin{aligned}
&= \frac{d}{dt}\left[I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots\right] \\
&= A + A^2 t + \frac{A^3 t^2}{2!} + \cdots \\
&= A\left[I + At + \frac{(At)^2}{2!} + \cdots\right] \\
&= Ae^{At}
\end{aligned}
$$

This proves the first equality. After writing the third expression of the above sequence of equations, we can bring the common factor A out to the right to obtain

$$
\begin{aligned}
\frac{d}{dt}e^{At} &= \left[I + At + \frac{(At)^2}{2!} + \cdots\right] A \\
&= e^{At} A
\end{aligned}
$$

QED

1.10 Suppose that the matrix A has eigenvalues λ_i and eigenvectors v_i $(i = 1, \cdots, n)$. What are the eigenvalues and eigenvectors of $-A$?

Solution

$Av_i = \lambda_i v_i$, therefore $-Av_i = -\lambda_i v_i$. From this we see that $-A$ has eigenvalues $-\lambda_i$ and eigenvectors v_i.

1.11 Show that $|e^{At}| = e^{|A|t}$ for any square matrix A.

Solution:

$$
\begin{aligned}
e^{At} &= I + At + \frac{A^2 t^2}{2} + \cdots \\
|e^{At}| &= |I| + |At| + \frac{|A^2 t^2|}{2} + \cdots \\
&= |I| + |A|t + \frac{|A|^2 t^2}{2} + \cdots \\
&= e^{|A|t}
\end{aligned}
$$

QED

1.12 Show that if $\dot{A} = BA$, then

$$
\frac{d|A|}{dt} = \mathrm{Tr}(B)|A|
$$

Solution:

The equation $\dot{A} = BA$ can be solved as $A = e^{Bt}A(0)$. Taking the determinant of this equation gives

$$
|A| = |e^{Bt}A(0)|
$$

$$\begin{aligned} &= |e^{Bt}||A(0)| \\ &= e^{|B|t}|A(0)| \end{aligned}$$

From this we see that

$$\begin{aligned} \frac{d|A|}{dt} &= |B|e^{|B|t}|A(0)| \\ &= \mathrm{Tr}(B)|A| \end{aligned}$$

QED

1.13 The linear position p of an object under constant acceleration is

$$p = p_0 + \dot{p}t + \frac{1}{2}\ddot{p}t^2$$

where p_0 is the initial position of the object.

a) Define a state vector as $x = \begin{bmatrix} p & \dot{p} & \ddot{p} \end{bmatrix}^T$ and write the state space equation $\dot{x} = Ax$ for this system.

b) Use all three expressions in Equation (1.71) to find the state transition matrix e^{At} for the system.

c) Prove for the state transition matrix found above that $e^{A0} = I$.

Solution

a).

$$\frac{d}{dt}\begin{bmatrix} p \\ \dot{p} \\ \ddot{p} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} p \\ \dot{p} \\ \ddot{p} \end{bmatrix}$$

b). From the first expression in Equation (1.72) we obtain

$$\begin{aligned} e^{At} &= \sum_{j=0}^{\infty} \frac{(At)^j}{j!} \\ &= \frac{(At)^0}{0!} + \frac{At}{1!} + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots \\ &= I + \begin{bmatrix} 0 & t & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & t^2/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + 0 + \cdots \\ &= \begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

From the second expression in Equation (1.72) we obtain

$$\begin{aligned} e^{At} &= \mathcal{L}^{-1}[(sI - A)^{-1}] \\ &= \mathcal{L}^{-1}\begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 0 & 0 & s \end{bmatrix}^{-1} \end{aligned}$$

$$= \mathcal{L}^{-1}\begin{bmatrix} 1/s & 1/s^2 & 1/s^3 \\ 0 & 1/s & 1/s^2 \\ 0 & 0 & 1/s \end{bmatrix}$$
$$= \begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

From the third expression in Equation (1.72) we obtain

$$e^{At} = Qe^{\hat{A}t}Q^{-1}$$

The eigendata of A are found to be

$$\lambda = \{0, 0, 0\}$$
$$v = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Actually we can note that A is already in Jordan form, which means that its eigenvalues are on the diagonal, and its eigenvectors form the identity matrix when augmented together. Recall for a third order Jordan block that

$$e^{At} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2}e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{bmatrix}$$

In our case $\lambda = 0$ so

$$e^{At} = \begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

c). From the above expression for e^{At}, if we substitute $t = 0$ we see that $e^{At} = I$. QED

1.14 Consider the following system matrix.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Show that the matrix

$$S(t) = \begin{bmatrix} e^t & 0 \\ 0 & 2e^{-t} \end{bmatrix}$$

satisfies the relation $\dot{S}(t) = AS(t)$, but $S(t)$ is not the state transition matrix of the system.

Solution

$$\dot{S}(t) = \begin{bmatrix} e^t & 0 \\ 0 & -2e^{-t} \end{bmatrix}$$

$$\begin{aligned} AS(t) &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & 2e^{-t} \end{bmatrix} \\ &= \begin{bmatrix} e^t & 0 \\ 0 & -2e^{-t} \end{bmatrix} \end{aligned}$$

We see that $\dot{S}(t) = AS(t)$. However, the state transition matrix is found to be

$$e^{At} = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}$$

QED

1.15 Give an example of a discrete-time system that is marginally stable but not asymptotically stable.

Solution

The system $x_{k+1} = x_k$ is marginally stable, because the state is bounded for any initial bounded state, but it is not asymptotically stable, because it is not true that the state approaches zero for all initial states.

1.16 Show (H, F) is an observable matrix pair if and only if (H, F^{-1}) is observable (assuming that F is nonsingular).

Solution

If (H, F) is observable, then $Qx \neq 0$ for all nonzero x, where

$$Q = \begin{bmatrix} H \\ HF \\ \vdots \\ HF^{n-1} \end{bmatrix}$$

Since F is nonsingular $F^{-(n-1)}x$ spans the entire n-dimensional space. (That is, any n-element vector can be written as $F^{-(n-1)}x$ for some n-element vector x.) So the observability of (H, F) is equivalent to $QF^{-(n-1)}x \neq 0$ for all nonzero x. This is equivalent to $Q'x \neq 0$ for all nonzero x, where

$$Q' = \begin{bmatrix} HF^{-(n-1)} \\ HF^{-(n-2)} \\ \vdots \\ H \end{bmatrix}$$

which is the observability matrix of (H, F^{-1}).
QED

Computer exercises

1.17 The dynamics of a DC motor can be described as

$$J\ddot{\theta} + F\dot{\theta} = T$$

where θ is the angular position of the motor, J is the moment of inertia, F is the coefficient of viscous friction, and T is the torque applied to the motor.

a) Generate a two-state linear system equation for this motor in the form

$$\dot{x} = Ax + Bu$$

b) Simulate the system for 5 s and plot the angular position and velocity. Use $J = 10$ kg m^2, $F = 100$ kg m^2/s, $x(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$, and $T = 10$ N m. Use rectangular integration with a step size of 0.05 s. Do the output plots look correct? What happens when you change the step size Δt to 0.2 s? What happens when you change the step size to 0.5 s? What are the eigenvalues of the A matrix, and how can you relate their magnitudes to the step size that is required for a correct simulation?

Solution

a). Let $x_1 = \theta$ and $x_2 = \dot{\theta}$. Then

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -F/J \end{bmatrix} + \begin{bmatrix} 0 \\ 1/J \end{bmatrix} T$$

b). Output plots for various simulation step sizes are shown in Figures 1.1–1.3. With $\Delta t = 0.05$ the simulation works fine. With $\Delta t = 0.2$ the simulation results are obviously incorrect, although the simulation is still stable. With $\Delta t = 0.5$ the simulation blows up. The eigenvalues of A are 0 and -10. The simulation step size should be appreciably smaller than $1/|\lambda|_{max}$, which implies that the step size should be smaller than $1/10$, which is consistent with our experimental results.

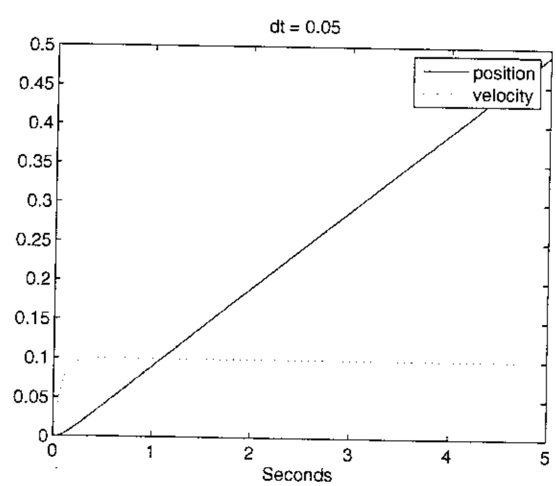

Figure 1.1 Problem 1.17 simulation with $\Delta t = 0.05$. Good simulation

1.18 The dynamic equations for a series RLC circuit can be written as

$$u = IR + L\dot{I} + V_c$$

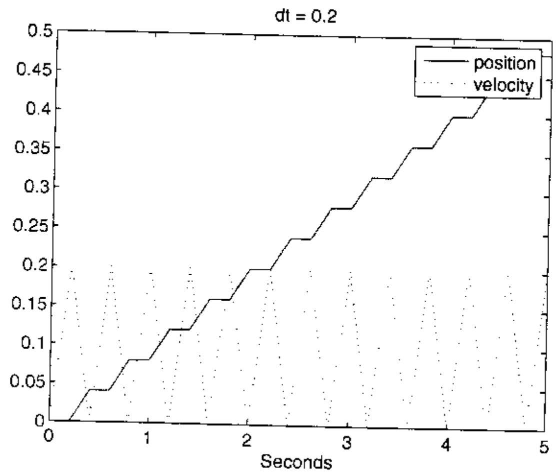

Figure 1.2 Problem 1.17 simulation with $\Delta t = 0.2$. Poor simulation

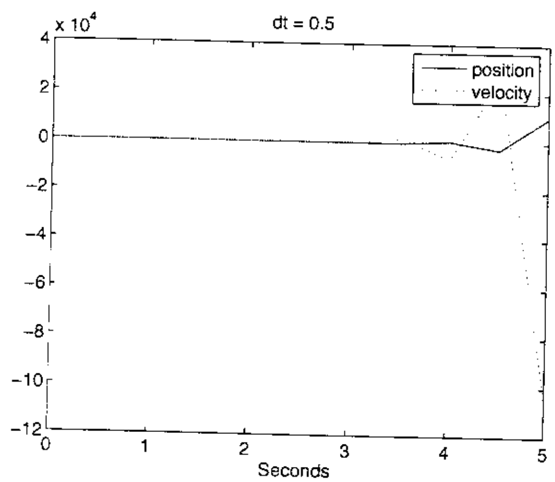

Figure 1.3 Problem 1.17 simulation with $\Delta t = 0.5$. Unstable simulation

$$I = C\dot{V}_c$$

where u is the applied voltage, I is the current through the circuit, and V_c is the voltage across the capacitor.

a) Write a state-space equation in matrix form for this system with x_1 as the capacitor voltage and x_2 as the current.

b) Suppose that $R = 3$, $L = 1$, and $C = 0.5$. Find an analytical expression for the capacitor voltage for $t \geq 0$, assuming that the initial state is zero, and the input voltage is $u(t) = e^{-2t}$.

c) Simulate the system using rectangular, trapezoidal, and fourth-order Runge–Kutta integration to obtain a numerical solution for the capacitor voltage.

Simulate from $t = 0$ to $t = 5$ using step sizes of 0.1 and 0.2. Tabulate the RMS value of the error between the numerical and analytical solutions for the capacitor voltage for each of your six simulations.

Solution

a).

$$\begin{aligned} u &= x_2 R + L\dot{x}_2 + x_1 \\ x_2 &= C\dot{x}_1 \end{aligned}$$

Putting this in matrix form gives

$$\dot{x} = \begin{bmatrix} 0 & 1/C \\ -1/L & -R/L \end{bmatrix} x + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} u$$

b).

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} Bu(\tau)\, d\tau$$

Plugging in the values of R, L, and C into the A matrix and computing e^{At} gives

$$e^{At} = e^{-t}\begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix} + e^{-2t}\begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix}$$

Substituting everything into the expression for $x(t)$ and computing the first element of $x(t)$ gives

$$\begin{aligned} x_1(t) &= 2\int_0^t \left[e^{-(t-\tau)} - e^{-2(t-\tau)}\right] e^{-2\tau}\, d\tau \\ &= 2(e^{-t} - e^{-2t} - te^{-2t}) \end{aligned}$$

c). Table 1.1 shows the RMS error of the numerical integration methods.

Table 1.1 Solution to Problem 1.18. RMS errors when numerically integrating the series RLC circuit, for various integration algorithms, and for various time step sizes T.

	$T = 0.1$	$T = 0.2$
Rectangular	0.016	0.035
Trapezoidal	0.012	0.024
Fourth order Runge Kutta	0.0018	0.0035

1.19 The vertical dimension of a hovering rocket can be modeled as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{Ku - gx_2}{x_3} - \frac{GM}{(R + x_1)^2} \\ \dot{x}_3 &= -u \end{aligned}$$

where x_1 is the vertical position of the rocket, x_2 is the vertical velocity, x_3 is the mass of the rocket, u is the control input (the flow rate of rocket propulsion), $K = 1000$ is the thrust constant of proportionality, $g = 50$ is the drag constant, $G = 6.673E-11$ m^3/kg/s^2 is the universal gravitational constant, $M = 5.98E24$ kg is the mass of the earth, and $R = 6.37E6$ m is the radius of the earth radius.

a) Find $u(t) = u_0(t)$ such that the system is in equilibrium at $x_1(t) = 0$ and $x_2(t) = 0$.

b) Find $x_3(t)$ when $x_1(t) = 0$ and $x_2(t) = 0$.

c) Linearize the system around the state trajectory found above.

d) Simulate the nonlinear system for five seconds and the linearized system for five seconds with $u(t) = u_0(t) + \Delta u \cos(t)$. Plot the altitude of the rocket for the nonlinear simulation and the linear simulation (on the same plot) when $\Delta u = 10$. Repeat for $\Delta u = 100$ and $\Delta u = 300$. Hand in your source code and your three plots. What do you conclude about the accuracy of your linearization?

Solution

a).

$$\begin{aligned}\dot{x}_2 &= \frac{Ku - gx_2}{x_3} - \frac{GM}{(R+x_1)^2} \\ &= 0 \\ x_1 &= 0\end{aligned}$$

Solving the above for $u(t)$ gives

$$u(t) = \frac{GMx_3}{KR^2}$$

b). From the third state equation and the equilibrium point obtained above we get

$$\dot{x}_3 = \frac{-GMx_3}{KR^2}$$

Solving for $x_3(t)$ gives

$$x_3(t) = x_3(0)\exp\left(\frac{-GMt}{KR^2}\right)$$

c). Use the notation $x_{10}(t) = 0$, $x_{20}(t) = 0$, $x_{30}(t) = x_3(0)\exp\left(\frac{-GMt}{KR^2}\right)$, and $u_0(t) = \frac{GMx_3}{KR^2}$ to denote the nominal trajectory.

$$\begin{aligned}\Delta\dot{x}_1 &= \left.\frac{\partial \dot{x}_1}{\partial x_1}\Delta x_1 + \frac{\partial \dot{x}_1}{\partial x_2}\Delta x_2 + \frac{\partial \dot{x}_1}{\partial x_3}\Delta x_3 + \frac{\partial \dot{x}_1}{\partial u}\Delta u\right|_{x_{10}(t),x_{20}(t),x_{30}(t),u_0(t)} \\ &= \Delta x_2 \\ \Delta\dot{x}_2 &= \left.\frac{\partial \dot{x}_2}{\partial x_1}\Delta x_1 + \frac{\partial \dot{x}_2}{\partial x_2}\Delta x_2 + \frac{\partial \dot{x}_2}{\partial x_3}\Delta x_3 + \frac{\partial \dot{x}_2}{\partial u}\Delta u\right|_{x_{10}(t),x_{20}(t),x_{30}(t),u_0(t)} \\ &= \frac{2GM}{R^3}\Delta x_1 - \frac{g}{x_{30}(t)}\Delta x_2 - \frac{GM}{R^2 x_{30}(t)}\Delta x_3 + \frac{K}{x_{30}(t)}\Delta u\end{aligned}$$

$$\begin{aligned}\Delta\dot{x}_3 &= \left.\frac{\partial\dot{x}_3}{\partial x_1}\Delta x_1 + \frac{\partial\dot{x}_3}{\partial x_2}\Delta x_2 + \frac{\partial\dot{x}_3}{\partial x_3}\Delta x_3 + \frac{\partial\dot{x}_3}{\partial u}\Delta u\right|_{x_{10}(t),x_{20}(t).x_{30}(t),u_0(t)} \\ &= -\Delta u\end{aligned}$$

d). Figure 1.4 shows simulation results for various values of Δu. As Δu increases the linearized simulation becomes less accurate (i.e., the linearized simulation does not track the nonlinear simulation as accurately).

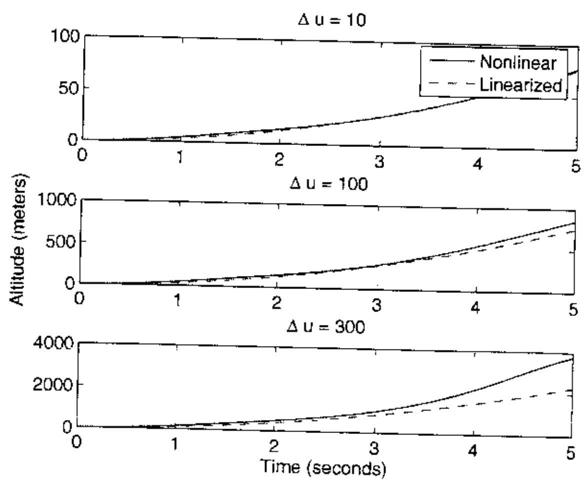

Figure 1.4 Rocket simulations for Problem 1.19

CHAPTER 2

Probability theory

Problems

Written exercises

2.1 What is the 0th moment of an RV? What is the 0th central moment of an RV?

Solution:

$$\begin{aligned}
i\text{th moment of } x &= E(x^i) \\
0\text{th moment of } x &= E(x^0) \\
&= E(1) \\
&= 1 \\
i\text{th central moment of } x &= E[(x-\bar{x})^i] \\
0\text{th central moment of } x &= E[(x-\bar{x})^0] \\
&= E(1) \\
&= 1
\end{aligned}$$

2.2 Suppose a deck of 52 cards is randomly divided into four piles of 13 cards each. Find the probability that each pile contains exactly one ace [Gre01].

Solution:

Consider the first pile. There are a total of 52-choose-13 possible first piles. There are a total of 48-choose-12 different ways of selecting 12 non-Aces from the remaining 48 non-Ace cards. The odds that the first pile has exactly one Ace is therefore 4(48-choose-12)/(52-choose-13). If this event occurred we see that there are 39 cards remaining to be dealt, including three Aces. Therefore, the odds that the second pile has exactly one Ace is 3(36-choose-12)/(39-choose-13). If both of the previous events occurred we see that there are 26 cards remaining to be dealt, including two Aces. Therefore, the odds that the third pile has exactly one Ace is 2(24-choose-12)/(26-choose-13). Given that all three of the previous events occurred, the odds that the fourth pile has exactly one Ace is 1. Multiplying these odds together gives the total probability of 10.55%.

2.3 Determine the value of a in the function

$$f_X(x) = \begin{cases} ax(1-x) & x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

so that $f_X(x)$ is a valid probability density function [Lie67].

Solution:

In order for $f_X(x)$ to be a valid pdf its integral from $-\infty$ to $+\infty$ must be equal to 1.

$$\begin{aligned} \int_{-\infty}^{\infty} f_X(x)\,dx &= \int_{-\infty}^{\infty} ax(1-x)\,dx \\ &= \frac{a}{6} \end{aligned}$$

Therefore $a = 6$.

2.4 Determine the value of a in the function

$$f_X(x) = \frac{a}{e^x + e^{-x}}$$

so that $f_X(x)$ is a valid probability density function. What is the probability that $|X| \leq 1$?

Solution:

In order for $f_X(x)$ to be a valid pdf its integral from $-\infty$ to $+\infty$ must be equal to 1.

$$\begin{aligned} \int_{-\infty}^{\infty} f_X(x)\,dx &= \int_{-\infty}^{\infty} \frac{a}{e^x + e^{-x}}\,dx \\ &= a\tan^{-1}(e^x)\Big|_{-\infty}^{\infty} \\ &= a\pi/2 \end{aligned}$$

Therefore $a = 2/\pi$. The probability that $|X| \leq 1$ is computed as

$$\begin{aligned} P(|X| \leq 1) &= a \tan^{-1}(e^x)\Big|_{-1}^{1} \\ &= \frac{2}{\pi}\left(\tan^{-1} e - \tan^{-1} e^{-1}\right) \\ &\approx 0.55 \end{aligned}$$

2.5 The probability density function of an exponentially distributed random variable is defined as follows.

$$f_X(x) = \begin{cases} ae^{-ax} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

where $a \geq 0$.

a) Find the probability distribution function of an exponentially distributed random variable.

b) Find the mean of an exponentially distributed random variable.

c) Find the second moment of an exponentially distributed random variable.

d) Find the variance of an exponentially distributed random variable.

e) What is the probability that an exponentially distributed random variable takes on a value within one standard deviation of its mean?

Solution:

a).

$$\begin{aligned} P(x) &= \int_0^x ae^{-az}\, dz \\ &= \begin{cases} 1 - e^{-ax} & x > 0 \\ 0 & x \leq 0 \end{cases} \end{aligned}$$

b).

$$\bar{x} = \int_0^\infty xae^{-ax}\, dx$$

Using integration by parts we obtain

$$\begin{aligned} \bar{x} &= xe^{-ax}\Big|_0^\infty + \int_0^\infty e^{-ax}\, dx \\ &= \frac{1}{a} \end{aligned}$$

c).

$$E(x^2) = \int_0^\infty x^2 ae^{-ax}\, dx$$

Using integration by parts we obtain

$$\begin{aligned} E(x^2) &= x^2 e^{-ax}\Big|_0^\infty + \int_0^\infty 2xe^{-ax}\,dx \\ &= \frac{2}{a}\int_0^\infty axe^{-ax}\,dx \\ &= \frac{2}{a}\bar{x} \\ &= \frac{2}{a^2} \end{aligned}$$

d).

$$\begin{aligned} \sigma^2 &= E(x^2) - \bar{x}^2 \\ &= \frac{1}{a^2} \end{aligned}$$

e).

$$\begin{aligned} P(\bar{x} - \sigma \le x \le \bar{x} + \sigma) &= \int_0^{2/a} ae^{-ax}\,dx \\ &= 1 - e^{-2} \\ &\approx 0.86 \end{aligned}$$

2.6 Derive an expression for the skew of a random variable as a function of its first, second, and third moments.

Solution:

$$\begin{aligned} \text{skew} &= E[(x-\bar{x})^3] \\ &= E[x^3 - 3x^2\bar{x} + 3x\bar{x}^2 - \bar{x}^3] \\ &= E(x^3) - 3\bar{x}E(x^2) + 2\bar{x}^3 \end{aligned}$$

2.7 Consider the following probability density function:

$$f_X(x) = \frac{ab}{b^2 + x^2}, \qquad b > 0$$

a) Determine the value of a in the so that $f_X(x)$ is a valid probability density function. (The correct value of a makes $f_X(x)$ a Cauchy pdf.)

b) Find the mean of a Cauchy random variable.

Solution:

a). In order for $f_X(x)$ to be a valid pdf its integral from $-\infty$ to $+\infty$ must be equal to 1.

$$\int_{-\infty}^{\infty} \frac{ab}{b^2 + x^2}\,dx = a\tan^{-1}(x)\Big|_{-\infty}^{\infty}$$

$$
\begin{aligned}
&= a\left(\frac{\pi}{2}+\frac{\pi}{2}\right) \\
&= a\pi
\end{aligned}
$$

Therefore $a = 1/\pi$.

b).

$$
\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} x f_X(x)\,dx \\
&= \frac{b}{\pi}\int_{-\infty}^{\infty} \frac{x}{b^2+x^2}\,dx \\
&= \frac{b}{2\pi}\ln\left|b^2+x^2\right|_{-\infty}^{\infty} \\
&= \infty
\end{aligned}
$$

This indicates that the mean is ∞, but what it really says is that the integral does not converge to a real number, so the mean of a Cauchy random variable does not exist.

2.8 Consider two zero-mean uncorrelated random variables W and V with standard deviations σ_w and σ_v, respectively. What is the standard deviation of the random variable $X = W + V$?

Solution:

$$
\begin{aligned}
\sigma_x^2 &= E[(X-\bar{x})^2 \\
&= E[(W+V-\bar{w}-\bar{v})^2 \\
&= E[(W+V)^2] \\
&= E(W^2)+E(V^2)+2E(WV) \\
&= \sigma_w^2+\sigma_v^2+0 \\
\sigma_x &= \sqrt{\sigma_w^2+\sigma_v^2}
\end{aligned}
$$

2.9 Consider two scalar RVs X and Y.

a) Prove that if X and Y are independent, then their correlation coefficient $\rho = 0$.

b) Find an example of two RVs that are not independent but that have a correlation coefficient of zero.

c) Prove that if Y is a linear function of X then $\rho = \pm 1$.

Solution:

a). If X and Y are independent then $E[(X-\bar{X})(Y-\bar{Y})] = E(X-\bar{X})E(Y-\bar{Y})$. Therefore

$$
\rho = \frac{C_{XY}}{\sigma_x \sigma_y}
$$