Complete Solutions Manual to Accompany

Numerical Analysis

TENTH EDITION

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Preface

This Instructor's Manual for the Tenth edition of Numerical Analysis by Burden, Faires, and Burden contains solutions to all the exercises in the book. Although the answers to the odd exercises are also in the back of the text, we have found that users of the book appreciate having all the solutions in one source. In addition, the results listed in this Instructor's Manual often go beyond those given in the back of the book. For example, we do not place the long solutions to theoretical and applied exercises in the book. You will find them here.

A Student Study Guide for the Tenth edition of Numerical Analysis is also available and the solutions given in the Guide are generally more detailed than those in the Instructor's Manual.

We have added a number of exercises to the text that can be implemented in any generic computer algebra system such as Maple, Matlab, Mathematica, Sage, and FreeMat. In our recent teaching of the course we found that students understood the concepts better when they worked through the algorithms step-by-step, but let the computer algebra system do the tedious computation.

It has been our practice to include in our Numerical Analysis book structured algorithms of all the techniques discussed in the text. The algorithms are given in a form that can be coded in any appropriate programming language, by students with even a minimal amount of programming expertise.

At our companion website for the book,

https://sites.google.com/site/numericalanalysis1burden/

you will find all the algorithms written in the programming languages FORTRAN, Pascal, C, Java, and in the Computer Algebra Systems, Maple, MATLAB, and Mathematica. For the Tenth edition, we have added new Maple programs to reflect the *NumericalAnalysis* package.

The companion website also contains additional information about the book and will be updated regularly to reflect any modifications that might be made. For example, we will place there any responses to questions from users of the book concerning interpretations of the exercises and appropriate applications of the techniques. We also have a set of PowerPoint files for many of the methods in the book. Many of these files were created by Professor John Carroll of Dublin City University and several were developed by Dr. Annette M. Burden of Youngstown State University.

We hope our supplement package provides flexibility for instructors teaching Numerical Analysis. If you have any suggestions for improvements that can be incorporated into future editions of the book or the supplements, we would be most grateful to receive your comments. We can be most easily contacted by electronic mail at the addresses listed below.

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Mathematical Preliminaries

Exercise Set 1.1, page 14

1. For each part, $f \in C[a,b]$ on the given interval. Since $f(a)$ and $f(b)$ are of opposite sign, the Intermediate Value Theorem implies that a number c exists with $f(c) = 0$.

2. (a) $f(x) = \sqrt{(}x) - \cos x; f(0) = -1 < 0, f(1) = 1 - \cos 1 > 0.45 > 0$; Intermediate Value Theorem implies there is a c in $(0,1)$ such that $f(c) = 0$.
   (b) $f(x) = e^x - x^2 + 3x - 2; f(0) = -1 < 0, f(1) = e > 0$; Intermediate Value Theorem implies there is a c in $(0,1)$ such that $f(c) = 0$.
   (c) $f(x) = -3\tan(2x) + x; f(0) = 0$ so there is a c in $[0,1]$ such that $f(c) = 0$.
   (d) $f(x) = \ln x - x^2 + \frac{5}{2}x - 1; f(\frac{1}{2}) = -\ln 2 < 0, f(1) = \frac{1}{2} > 0$; Intermediate Value Theorem implies there is a c in $(\frac{1}{2}, 1)$ such that $f(c) = 0$.

3. For each part, $f \in C[a,b]$, f' exists on (a,b) and $f(a) = f(b) = 0$. Rolle's Theorem implies that a number c exists in (a,b) with $f'(c) = 0$. For part (d), we can use $[a,b] = [-1,0]$ or $[a,b] = [0,2]$.

4. (a) $[0,1]$
   (b) $[0,1]$, $[4,5]$, $[-1,0]$
   (c) $[-2,-2/3]$, $[0,1]$, $[2,4]$
   (d) $[-3,-2]$, $[-1,-0.5]$, and $[-0.5,0]$

5. The maximum value for $|f(x)|$ is given below.
   (a) 0.4620981
   (b) 0.8
   (c) 5.164000
   (d) 1.582572

6. (a) $f(x) = \frac{2x}{x^2+1}; 0 \le x \le 2; f(x) \ge 0$ on $[0,2], f'(1) = 0, f(0) = 0, f(1) = 1, f(2) = \frac{4}{5}, \max_{0 \le x \le 2} |f(x)| = 1$.
   (b) $f(x) = x^2\sqrt{4-x}; 0 \le x \le 4; f'(0) = 0, f'(3.2) = 0, f(0) = 0, f(3.2) = 9.158934436, f(4) = 0, \max_{0 \le x \le 4} |f(x)| = 9.158934436$.
   (c) $f(x) = x^3 - 4x + 2; 1 \le x \le 2; f'(\frac{2\sqrt{3}}{3}) = 0, f'(1) = -1, f(\frac{2\sqrt{3}}{3}) = -1.079201435, f(2) = 2, \max_{1 \le x \le 2} |f(x)| = 2$.

(d) $f(x) = x\sqrt{3-x^2}; 0 \le x \le 1; f'(\sqrt{\frac{3}{2}}) = 0, \sqrt{\frac{3}{2}}$ not in $[0,1]$, $f(0) = 0, f(1) = \sqrt{2}, \max_{0\le x\le 1} |f(x)| = \sqrt{2}$.

7. For each part, $f \in C[a,b]$, f' exists on (a,b) and $f(a) = f(b) = 0$. Rolle's Theorem implies that a number c exists in (a,b) with $f'(c) = 0$. For part (d), we can use $[a,b] = [-1,0]$ or $[a,b] = [0,2]$.

8. Suppose p and q are in $[a,b]$ with $p \ne q$ and $f(p) = f(q) = 0$. By the Mean Value Theorem, there exists $\xi \in (a,b)$ with

$$f(p) - f(q) = f'(\xi)(p-q).$$

But, $f(p) - f(q) = 0$ and $p \ne q$. So $f'(\xi) = 0$, contradicting the hypothesis.

9. (a) $P_2(x) = 0$
   (b) $R_2(0.5) = 0.125$; actual error $= 0.125$
   (c) $P_2(x) = 1 + 3(x-1) + 3(x-1)^2$
   (d) $R_2(0.5) = -0.125$; actual error $= -0.125$

10. $P_3(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$

x	0.5	0.75	1.25	1.5
$P_3(x)$	1.2265625	1.3310547	1.5517578	1.6796875
$\sqrt{x+1}$	1.2247449	1.3228757	1.5	1.5811388
$\lvert\sqrt{x+1} - P_3(x)\rvert$	0.0018176	0.0081790	0.0517578	0.0985487

11. Since

$$P_2(x) = 1 + x \quad \text{and} \quad R_2(x) = \frac{-2e^{\xi}(\sin\xi + \cos\xi)}{6}x^3$$

for some ξ between x and 0, we have the following:

   (a) $P_2(0.5) = 1.5$ and $|f(0.5) - P_2(0.5)| \le 0.0932$;
   (b) $|f(x) - P_2(x)| \le 1.252$;
   (c) $\int_0^1 f(x)\,dx \approx 1.5$;
   (d) $|\int_0^1 f(x)\,dx - \int_0^1 P_2(x)\,dx| \le \int_0^1 |R_2(x)|\,dx \le 0.313$, and the actual error is 0.122.

12. $P_2(x) = 1.461930 + 0.617884\left(x - \frac{\pi}{6}\right) - 0.844046\left(x - \frac{\pi}{6}\right)^2$ and $R_2(x) = -\frac{1}{3}e^{\xi}(\sin\xi + \cos\xi)\left(x - \frac{\pi}{6}\right)^3$ for some ξ between x and $\frac{\pi}{6}$.

   (a) $P_2(0.5) = 1.446879$ and $f(0.5) = 1.446889$. An error bound is 1.01×10^{-5}, and the actual error is 1.0×10^{-5}.
   (b) $|f(x) - P_2(x)| \le 0.135372$ on $[0,1]$
   (c) $\int_0^1 P_2(x)\,dx = 1.376542$ and $\int_0^1 f(x)\,dx = 1.378025$
   (d) An error bound is 7.403×10^{-3}, and the actual error is 1.483×10^{-3}.

13. $P_3(x) = (x-1)^2 - \frac{1}{2}(x-1)^3$
    (a) $P_3(0.5) = 0.312500,\ f(0.5) = 0.346574$. An error bound is $0.291\overline{6}$, and the actual error is 0.034074.
    (b) $|f(x) - P_3(x)| \leq 0.291\overline{6}$ on $[0.5, 1.5]$
    (c) $\int_{0.5}^{1.5} P_3(x)\ dx = 0.08\overline{3},\ \int_{0.5}^{1.5}(x-1)\ln x\ dx = 0.088020$
    (d) An error bound is $0.058\overline{3}$, and the actual error is 4.687×10^{-3}.

14. (a) $P_3(x) = -4 + 6x - x^2 - 4x^3$; $P_3(0.4) = -2.016$
    (b) $|R_3(0.4)| \leq 0.05849$; $|f(0.4) - P_3(0.4)| = 0.013365367$
    (c) $P_4(x) = -4 + 6x - x^2 - 4x^3$; $P_4(0.4) = -2.016$
    (d) $|R_4(0.4)| \leq 0.01366$; $|f(0.4) - P_4(0.4)| = 0.013365367$

15. $P_4(x) = x + x^3$
    (a) $|f(x) - P_4(x)| \leq 0.012405$
    (b) $\int_0^{0.4} P_4(x)\ dx = 0.0864,\ \int_0^{0.4} xe^{x^2}\ dx = 0.086755$
    (c) 8.27×10^{-4}
    (d) $P_4'(0.2) = 1.12,\ f'(0.2) = 1.124076$. The actual error is 4.076×10^{-3}.

16. First we need to convert the degree measure for the sine function to radians. We have $180° = \pi$ radians, so $1° = \frac{\pi}{180}$ radians. Since,

$$f(x) = \sin x, \quad f'(x) = \cos x, \quad f''(x) = -\sin x, \quad \text{and} \quad f'''(x) = -\cos x,$$

we have $f(0) = 0$, $f'(0) = 1$, and $f''(0) = 0$.

The approximation $\sin x \approx x$ is given by

$$f(x) \approx P_2(x) = x, \quad \text{and} \quad R_2(x) = -\frac{\cos\xi}{3!}x^3.$$

If we use the bound $|\cos\xi| \leq 1$, then

$$\left|\sin\frac{\pi}{180} - \frac{\pi}{180}\right| = \left|R_2\left(\frac{\pi}{180}\right)\right| = \left|\frac{-\cos\xi}{3!}\left(\frac{\pi}{180}\right)^3\right| \leq 8.86 \times 10^{-7}.$$

17. Since $42° = 7\pi/30$ radians, use $x_0 = \pi/4$. Then

$$\left|R_n\left(\frac{7\pi}{30}\right)\right| \leq \frac{\left(\frac{\pi}{4} - \frac{7\pi}{30}\right)^{n+1}}{(n+1)!} < \frac{(0.053)^{n+1}}{(n+1)!}.$$

For $|R_n(\frac{7\pi}{30})| < 10^{-6}$, it suffices to take $n = 3$. To 7 digits,

$$\cos 42° = 0.7431448 \quad \text{and} \quad P_3(42°) = P_3(\frac{7\pi}{30}) = 0.7431446,$$

so the actual error is 2×10^{-7}.

18. $P_n(x) = \sum_{k=0}^{n} x^k,\ n \geq 19$

19. $P_n(x) = \displaystyle\sum_{k=0}^{n} \frac{1}{k!}x^k,\ n \geq 7$

20. For n odd, $P_n(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \cdots + \frac{1}{n}(-1)^{(n-1)/2}x^n$. For n even, $P_n(x) = P_{n-1}(x)$.

21. A bound for the maximum error is 0.0026.

22. For $x < 0$, $f(x) < 2x + k < 0$, provided that $x < -\frac{1}{2}k$. Similarly, for $x > 0$, $f(x) > 2x + k > 0$, provided that $x > -\frac{1}{2}k$. By Theorem 1.11, there exists a number c with $f(c) = 0$. If $f(c) = 0$ and $f(c') = 0$ for some $c' \neq c$, then by Theorem 1.7, there exists a number p between c and c' with $f'(p) = 0$. However, $f'(x) = 3x^2 + 2 > 0$ for all x.

23. Since $R_2(1) = \frac{1}{6}e^{\xi}$, for some ξ in $(0, 1)$, we have $|E - R_2(1)| = \frac{1}{6}|1 - e^{\xi}| \leq \frac{1}{6}(e - 1)$.

24. (a) Use the series

$$e^{-t^2} = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{k!} \quad \text{to integrate} \quad \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt,$$

and obtain the result.

(b) We have

$$\begin{aligned}
\frac{2}{\sqrt{\pi}} e^{-x^2} \sum_{k=0}^{\infty} \frac{2^k x^{2k+1}}{1 \cdot 3 \cdots (2k+1)} =& \frac{2}{\sqrt{\pi}} \left[1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^7 + \frac{1}{24}x^8 + \cdots \right] \\
&\cdot \left[x + \frac{2}{3}x^3 + \frac{4}{15}x^5 + \frac{8}{105}x^7 + \frac{16}{945}x^9 + \cdots \right] \\
=& \frac{2}{\sqrt{\pi}} \left[x - \frac{1}{3}x^3 + \frac{1}{10}x^5 - \frac{1}{42}x^7 + \frac{1}{216}x^9 + \cdots \right] = \text{erf}\,(x)
\end{aligned}$$

(c) 0.8427008

(d) 0.8427069

(e) The series in part (a) is alternating, so for any positive integer n and positive x we have the bound

$$\left| \text{erf}(x) - \frac{2}{\sqrt{\pi}} \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)k!} \right| < \frac{x^{2n+3}}{(2n+3)(n+1)!}\ .$$

We have no such bound for the positive term series in part (b).

25. (a) $P_n^{(k)}(x_0) = f^{(k)}(x_0)$ for $k = 0, 1, \ldots, n$. The shapes of P_n and f are the same at x_0.

(b) $P_2(x) = 3 + 4(x - 1) + 3(x - 1)^2$.

26. (a) The assumption is that $f(x_i) = 0$ for each $i = 0, 1, \ldots, n$. Applying Rolle's Theorem on each on the intervals $[x_i, x_{i+1}]$ implies that for each $i = 0, 1, \ldots, n - 1$ there exists a number z_i with $f'(z_i) = 0$. In addition, we have

$$a \leq x_0 < z_0 < x_1 < z_1 < \cdots < z_{n-1} < x_n \leq b.$$

(b) Apply the logic in part (a) to the function $g(x) = f'(x)$ with the number of zeros of g in $[a, b]$ reduced by 1. This implies that numbers w_i, for $i = 0, 1, \dots, n-2$ exist with

$$g'(w_i) = f''(w_i) = 0, \quad \text{and} \quad a < z_0 < w_0 < z_1 < w_1 < \cdots w_{n-2} < z_{n-1} < b.$$

(c) Continuing by induction following the logic in parts (a) and (b) provides $n+1-j$ distinct zeros of $f^{(j)}$ in $[a, b]$.

(d) The conclusion of the theorem follows from part (c) when $j = n$, for in this case there will be (at least) $(n+1) - n = 1$ zero in $[a, b]$.

27. First observe that for $f(x) = x - \sin x$ we have $f'(x) = 1 - \cos x \geq 0$, because $-1 \leq \cos x \leq 1$ for all values of x.

(a) The observation implies that $f(x)$ is non-decreasing for all values of x, and in particular that $f(x) > f(0) = 0$ when $x > 0$. Hence for $x \geq 0$, we have $x \geq \sin x$, and $|\sin x| = \sin x \leq x = |x|$.

(b) When $x < 0$, we have $-x > 0$. Since $\sin x$ is an odd function, the fact (from part (a)) that $\sin(-x) \leq (-x)$ implies that $|\sin x| = -\sin x \leq -x = |x|$.

As a consequence, for all real numbers x we have $|\sin x| \leq |x|$.

28. (a) Let x_0 be any number in $[a, b]$. Given $\epsilon > 0$, let $\delta = \epsilon/L$. If $|x - x_0| < \delta$ and $a \leq x \leq b$, then $|f(x) - f(x_0)| \leq L|x - x_0| < \epsilon$.

(b) Using the Mean Value Theorem, we have

$$|f(x_2) - f(x_1)| = |f'(\xi)||x_2 - x_1|,$$

for some ξ between x_1 and x_2, so

$$|f(x_2) - f(x_1)| \leq L|x_2 - x_1|.$$

(c) One example is $f(x) = x^{1/3}$ on $[0, 1]$.

29. (a) The number $\frac{1}{2}(f(x_1) + f(x_2))$ is the average of $f(x_1)$ and $f(x_2)$, so it lies between these two values of f. By the Intermediate Value Theorem 1.11 there exist a number ξ between x_1 and x_2 with

$$f(\xi) = \frac{1}{2}(f(x_1) + f(x_2)) = \frac{1}{2}f(x_1) + \frac{1}{2}f(x_2).$$

(b) Let $m = \min\{f(x_1), f(x_2)\}$ and $M = \max\{f(x_1), f(x_2)\}$. Then $m \leq f(x_1) \leq M$ and $m \leq f(x_2) \leq M$, so

$$c_1 m \leq c_1 f(x_1) \leq c_1 M \quad \text{and} \quad c_2 m \leq c_2 f(x_2) \leq c_2 M.$$

Thus

$$(c_1 + c_2)m \leq c_1 f(x_1) + c_2 f(x_2) \leq (c_1 + c_2)M$$

and

$$m \leq \frac{c_1 f(x_1) + c_2 f(x_2)}{c_1 + c_2} \leq M.$$

By the Intermediate Value Theorem 1.11 applied to the interval with endpoints x_1 and x_2, there exists a number ξ between x_1 and x_2 for which

$$f(\xi) = \frac{c_1 f(x_1) + c_2 f(x_2)}{c_1 + c_2}.$$

(c) Let $f(x) = x^2 + 1$, $x_1 = 0$, $x_2 = 1$, $c_1 = 2$, and $c_2 = -1$. Then for all values of x,

$$f(x) > 0 \quad \text{but} \quad \frac{c_1 f(x_1) + c_2 f(x_2)}{c_1 + c_2} = \frac{2(1) - 1(2)}{2 - 1} = 0.$$

30. (a) Since f is continuous at p and $f(p) \neq 0$, there exists a $\delta > 0$ with

$$|f(x) - f(p)| < \frac{|f(p)|}{2},$$

for $|x - p| < \delta$ and $a < x < b$. We restrict δ so that $[p - \delta, p + \delta]$ is a subset of $[a, b]$. Thus, for $x \in [p - \delta, p + \delta]$, we have $x \in [a, b]$. So

$$-\frac{|f(p)|}{2} < f(x) - f(p) < \frac{|f(p)|}{2} \quad \text{and} \quad f(p) - \frac{|f(p)|}{2} < f(x) < f(p) + \frac{|f(p)|}{2}.$$

If $f(p) > 0$, then

$$f(p) - \frac{|f(p)|}{2} = \frac{f(p)}{2} > 0, \quad \text{so} \quad f(x) > f(p) - \frac{|f(p)|}{2} > 0.$$

If $f(p) < 0$, then $|f(p)| = -f(p)$, and

$$f(x) < f(p) + \frac{|f(p)|}{2} = f(p) - \frac{f(p)}{2} = \frac{f(p)}{2} < 0.$$

In either case, $f(x) \neq 0$, for $x \in [p - \delta, p + \delta]$.

(b) Since f is continuous at p and $f(p) = 0$, there exists a $\delta > 0$ with

$$|f(x) - f(p)| < k, \quad \text{for} \quad |x - p| < \delta \quad \text{and} \quad a < x < b.$$

We restrict δ so that $[p - \delta, p + \delta]$ is a subset of $[a, b]$. Thus, for $x \in [p - \delta, p + \delta]$, we have

$$|f(x)| = |f(x) - f(p)| < k.$$

Exercise Set 1.2, page 28

1. We have

	Absolute error	Relative error
(a)	0.001264	4.025×10^{-4}
(b)	7.346×10^{-6}	2.338×10^{-6}
(c)	2.818×10^{-4}	1.037×10^{-4}
(d)	2.136×10^{-4}	1.510×10^{-4}

2. We have

	Absolute error	Relative error
(a)	2.647×10^{1}	1.202×10^{-3}arule
(b)	1.454×10^{1}	1.050×10^{-2}
(c)	420	1.042×10^{-2}
(d)	3.343×10^{3}	9.213×10^{-3}

3. The largest intervals are
   (a) (149.85,150.15)
   (b) (899.1, 900.9)
   (c) (1498.5, 1501.5)
   (d) (89.91,90.09)

4. The largest intervals are:
   (a) $(3.1412784, 3.1419068)$
   (b) $(2.7180100, 2.7185536)$
   (c) $(1.4140721, 1.4143549)$
   (d) $(1.9127398, 1.9131224)$

5. The calculations and their errors are:
   (a) (i) 17/15 (ii) 1.13 (iii) 1.13 (iv) both 3×10^{-3}
   (b) (i) 4/15 (ii) 0.266 (iii) 0.266 (iv) both 2.5×10^{-3}
   (c) (i) 139/660 (ii) 0.211 (iii) 0.210 (iv) 2×10^{-3}, 3×10^{-3}
   (d) (i) 301/660 (ii) 0.455 (iii) 0.456 (iv) 2×10^{-3}, 1×10^{-4}

6. We have

	Approximation	Absolute error	Relative error
(a)	134	0.079	5.90×10^{-4}
(b)	133	0.499	3.77×10^{-3}
(c)	2.00	0.327	0.195
(d)	1.67	0.003	1.79×10^{-3}

7. We have

	Approximation	Absolute error	Relative error
(a)	1.80	0.154	0.0786
(b)	-15.1	0.0546	3.60×10^{-3}
(c)	0.286	2.86×10^{-4}	10^{-3}
(d)	23.9	0.058	2.42×10^{-3}

8. We have

	Approximation	Absolute error	Relative error
(a)	1.986	0.03246	0.01662
(b)	-15.16	0.005377	3.548×10^{-4}
(c)	0.2857	1.429×10^{-5}	5×10^{-5}
(d)	23.96	1.739×10^{-3}	7.260×10^{-5}

9. We have

	Approximation	Absolute error	Relative error
(a)	3.55	1.60	0.817
(b)	-15.2	0.0454	0.00299
(c)	0.284	0.00171	0.00600
(d)	0	0.02150	1

10. We have

	Approximation	Absolute error	Relative error
(a)	1.983	0.02945	0.01508
(b)	-15.15	0.004622	3.050×10^{-4}
(c)	0.2855	2.143×10^{-4}	7.5×10^{-4}
(d)	23.94	0.018261	7.62×10^{-4}

11. We have

	Approximation	Absolute error	Relative error
(a)	3.14557613	3.983×10^{-3}	1.268×10^{-3}
(b)	3.14162103	2.838×10^{-5}	9.032×10^{-6}

12. We have

	Approximation	Absolute error	Relative error
(a)	2.7166667	0.0016152	5.9418×10^{-4}
(b)	2.718281801	2.73×10^{-8}	1.00×10^{-8}

13. (a) We have

$$\lim_{x\to 0} \frac{x\cos x - \sin x}{x - \sin x} = \lim_{x\to 0} \frac{-x\sin x}{1-\cos x} = \lim_{x\to 0} \frac{-\sin x - x\cos x}{\sin x} = \lim_{x\to 0} \frac{-2\cos x + x\sin x}{\cos x} = -2$$

(b) $f(0.1) \approx -1.941$

(c) $\dfrac{x(1-\frac{1}{2}x^2) - (x - \frac{1}{6}x^3)}{x - (x - \frac{1}{6}x^3)} = -2$

(d) The relative error in part (b) is 0.029. The relative error in part (c) is 0.00050.

14. (a) $\displaystyle\lim_{x\to 0} \frac{e^x - e^{-x}}{x} = \lim_{x\to 0} \frac{e^x + e^{-x}}{1} = 2$

(b) $f(0.1) \approx 2.05$

(c) $\dfrac{1}{x}\left(\left(1+x+\dfrac{1}{2}x^2+\dfrac{1}{6}x^3\right)-\left(1-x+\dfrac{1}{2}x^2-\dfrac{1}{6}x^3\right)\right)=\dfrac{1}{x}\left(2x+\dfrac{1}{3}x^3\right)=2+\dfrac{1}{3}x^2$; using three-digit rounding arithmetic and $x = 0.1$, we obtain 2.00.

(d) The relative error in part (b) is $= 0.0233$. The relative error in part (c) is $= 0.00166$.

15.

	x_1	Absolute error	Relative error	x_2	Absolute error	Relative error
(a)	92.26	0.01542	1.672×10^{-4}	0.005419	6.273×10^{-7}	1.157×10^{-4}
(b)	0.005421	1.264×10^{-6}	2.333×10^{-4}	-92.26	4.580×10^{-3}	4.965×10^{-5}
(c)	10.98	6.875×10^{-3}	6.257×10^{-4}	0.001149	7.566×10^{-8}	6.584×10^{-5}
(d)	-0.001149	7.566×10^{-8}	6.584×10^{-5}	-10.98	6.875×10^{-3}	6.257×10^{-4}

16.

	Approximation for x_1	Absolute error	Relative error
(a)	1.903	6.53518×10^{-4}	3.43533×10^{-4}
(b)	-0.07840	8.79361×10^{-6}	1.12151×10^{-4}
(c)	1.223	1.29800×10^{-4}	1.06144×10^{-4}
(d)	6.235	1.7591×10^{-3}	2.8205×10^{-4}

	Approximation for x_2	Absolute error	Relative error
(a)	0.7430	4.04830×10^{-4}	5.44561
(b)	-4.060	3.80274×10^{-4}	9.36723×10^{-5}
(c)	-2.223	1.2977×10^{-4}	5.8393×10^{-5}
(d)	-0.3208	1.2063×10^{-4}	3.7617×10^{-4}

17.

	Approximation for x_1	Absolute error	Relative error
(a)	92.24	0.004580	4.965×10^{-5}
(b)	0.005417	2.736×10^{-6}	5.048×10^{-4}
(c)	10.98	6.875×10^{-3}	6.257×10^{-4}
(d)	-0.001149	7.566×10^{-8}	6.584×10^{-5}

	Approximation for x_2	Absolute error	Relative error
(a)	0.005418	2.373×10^{-6}	4.377×10^{-4}
(b)	-92.25	5.420×10^{-3}	5.875×10^{-5}
(c)	0.001149	7.566×10^{-8}	6.584×10^{-5}
(d)	-10.98	6.875×10^{-3}	6.257×10^{-4}

18.

	Approximation for x_1	Absolute error	Relative error
(a)	1.901	1.346×10^{-3}	7.078×10^{-4}
(b)	-0.07843	2.121×10^{-5}	2.705×10^{-4}
(c)	1.222	8.702×10^{-4}	7.116×10^{-4}
(d)	6.235	1.759×10^{-3}	2.820×10^{-4}

	Approximation for x_2	Absolute error	Relative error
(a)	0.7438	3.952×10^{-4}	5.316×10^{-4}
(b)	-4.059	6.197×10^{-4}	1.526×10^{-4}
(c)	-2.222	8.702×10^{-4}	3.915×10^{-4}
(d)	-0.3207	2.063×10^{-5}	6.433×10^{-5}

19. The machine numbers are equivalent to

   (a) 3224

   (b) -3224

   (c) 1.32421875

   (d) 1.3242187500000002220446049250313080847263336181640625

20. (a) Next Largest: 3224.00000000000045474735088646411895751953125;
   Next Smallest: 3223.99999999999954525264911353588104248046875

   (b) Next Largest: $-3224.00000000000045474735088646411895751953125$;
   Next Smallest: $-3223.99999999999954525264911353588104248046875$

   (c) Next Largest: 1.3242187500000002220446049250313080847263336181640625;
   Next Smallest: 1.3242187499999997779553950749686919152736663818359375

   (d) Next Largest: 1.324218750000000444089209850062616169452667236328125;
   Next Smallest: 1.32421875

21. (b) The first formula gives -0.00658, and the second formula gives -0.0100. The true three-digit value is -0.0116.

22. (a) -1.82

(b) 7.09×10^{-3}

(c) The formula in (b) is more accurate since subtraction is not involved.

23. The approximate solutions to the systems are

(a) $x = 2.451$, $y = -1.635$

(b) $x = 507.7$, $y = 82.00$

24. (a) $x = 2.460 \quad y = -1.634$

(b) $x = 477.0 \quad y = 76.93$

25. (a) In nested form, we have $f(x) = (((1.01e^x - 4.62)e^x - 3.11)e^x + 12.2)e^x - 1.99$.

(b) -6.79

(c) -7.07

(d) The absolute errors are

$$|-7.61 - (-6.71)| = 0.82 \quad \text{and} \quad |-7.61 - (-7.07)| = 0.54.$$

Nesting is significantly better since the relative errors are

$$\left|\frac{0.82}{-7.61}\right| = 0.108 \quad \text{and} \quad \left|\frac{0.54}{-7.61}\right| = 0.071,$$

26. Since $0.995 \leq P \leq 1.005$, $0.0995 \leq V \leq 0.1005$, $0.082055 \leq R \leq 0.082065$, and $0.004195 \leq N \leq 0.004205$, we have $287.61° \leq T \leq 293.42°$. Note that $15°\text{C} = 288.16\text{K}$.

When P is doubled and V is halved, $1.99 \leq P \leq 2.01$ and $0.0497 \leq V \leq 0.0503$ so that $286.61° \leq T \leq 293.72°$. Note that $19°\text{C} = 292.16\text{K}$. The laboratory figures are within an acceptable range.

27. (a) $m = 17$

(b) We have

$$\binom{m}{k} = \frac{m!}{k!(m-k)!} = \frac{m(m-1)\cdots(m-k-1)(m-k)!}{k!(m-k)!} = \left(\frac{m}{k}\right)\left(\frac{m-1}{k-1}\right)\cdots\left(\frac{m-k-1}{1}\right)$$

(c) $m = 181707$

(d) 2,597,000; actual error 1960; relative error 7.541×10^{-4}

28. When $d_{k+1} < 5$,

$$\left|\frac{y - fl(y)}{y}\right| = \frac{0.d_{k+1}\ldots \times 10^{n-k}}{0.d_1\ldots \times 10^n} \leq \frac{0.5 \times 10^{-k}}{0.1} = 0.5 \times 10^{-k+1}.$$

When $d_{k+1} > 5$,

$$\left|\frac{y - fl(y)}{y}\right| = \frac{(1 - 0.d_{k+1}\ldots) \times 10^{n-k}}{0.d_1\ldots \times 10^n} < \frac{(1 - 0.5) \times 10^{-k}}{0.1} = 0.5 \times 10^{-k+1}.$$

29. (a) The actual error is $|f'(\xi)\epsilon|$, and the relative error is $|f'(\xi)\epsilon| \cdot |f(x_0)|^{-1}$, where the number ξ is between x_0 and $x_0 + \epsilon$.

(b) (i) 1.4×10^{-5}; 5.1×10^{-6} (ii) 2.7×10^{-6}; 3.2×10^{-6}

(c) (i) 1.2; 5.1×10^{-5} (ii) 4.2×10^{-5}; 7.8×10^{-5}

Exercise Set 1.3, page 39

1 (a) The approximate sums are 1.53 and 1.54, respectively. The actual value is 1.549. Significant roundoff error occurs earlier with the first method.

(b) The approximate sums are 1.16 and 1.19, respectively. The actual value is 1.197. Significant roundoff error occurs earlier with the first method.

2. We have

	Approximation	Absolute Error	Relative Error
(a)	2.715	3.282×10^{-3}	1.207×10^{-3}
(b)	2.716	2.282×10^{-3}	8.394×10^{-4}
(c)	2.716	2.282×10^{-3}	8.394×10^{-4}
(d)	2.718	2.818×10^{-4}	1.037×10^{-4}

3. (a) 2000 terms

   (b) 20,000,000,000 terms

4. 4 terms

5. 3 terms

6. (a) $O\left(\frac{1}{n}\right)$

   (b) $O\left(\frac{1}{n^2}\right)$

   (c) $O\left(\frac{1}{n^2}\right)$

   (d) $O\left(\frac{1}{n}\right)$

7. The rates of convergence are:

   (a) $O(h^2)$

   (b) $O(h)$

   (c) $O(h^2)$

   (d) $O(h)$

8. (a) If $|\alpha_n - \alpha|/(1/n^p) \leq K$, then

$$|\alpha_n - \alpha| \leq K(1/n^p) \leq K(1/n^q) \quad \text{since} \quad 0 < q < p.$$

Thus

$$|\alpha_n - \alpha|/(1/n^p) \leq K \quad \text{and} \quad \{\alpha_n\}_{n=1}^{\infty} \to \alpha$$

with rate of convergence $O(1/n^p)$.

(b)

n	$1/n$	$1/n^2$	$1/n^3$	$1/n^5$
5	0.2	0.04	0.008	0.0016
10	0.1	0.01	0.001	0.0001
50	0.02	0.0004	8×10^{-6}	1.6×10^{-7}
100	0.01	10^{-4}	10^{-6}	10^{-8}

The most rapid convergence rate is $O(1/n^4)$.

9. (a) If $F(h) = L + O(h^p)$, there is a constant $k > 0$ such that

$$|F(h) - L| \leq kh^p,$$

for sufficiently small $h > 0$. If $0 < q < p$ and $0 < h < 1$, then $h^q > h^p$. Thus, $kh^p < kh^q$, so

$$|F(h) - L| \leq kh^q \quad \text{and} \quad F(h) = L + O(h^q).$$

(b) For various powers of h we have the entries in the following table.

h	h^2	h^3	h^4
0.5	0.25	0.125	0.0625
0.1	0.01	0.001	0.0001
0.01	0.0001	0.00001	10^{-8}
0.001	10^{-6}	10^{-9}	10^{-12}

The most rapid convergence rate is $O\left(h^4\right)$.

10. Suppose that for sufficiently small $|x|$ we have positive constants K_1 and K_2 independent of x, for which

$$|F_1(x) - L_1| \leq K_1|x|^\alpha \quad \text{and} \quad |F_2(x) - L_2| \leq K_2|x|^\beta.$$

Let $c = \max(|c_1|, |c_2|, 1)$, $K = \max(K_1, K_2)$, and $\delta = \max(\alpha, \beta)$.

(a) We have

$$\begin{aligned}|F(x) - c_1L_1 - c_2L_2| &= |c_1(F_1(x) - L_1) + c_2(F_2(x) - L_2)| \\ &\leq |c_1|K_1|x|^\alpha + |c_2|K_2|x|^\beta \leq cK[|x|^\alpha + |x|^\beta] \\ &\leq cK|x|^\gamma[1 + |x|^{\delta-\gamma}] \leq \tilde{K}|x|^\gamma,\end{aligned}$$

for sufficiently small $|x|$ and some constant $\tilde{K}$. Thus, $F(x) = c_1L_1 + c_2L_2 + O(x^\gamma)$.

(b) We have

$$\begin{aligned}|G(x) - L_1 - L_2| &= |F_1(c_1x) + F_2(c_2x) - L_1 - L_2| \\ &\leq K_1|c_1x|^\alpha + K_2|c_2x|^\beta \leq Kc^\delta[|x|^\alpha + |x|^\beta] \\ &\leq Kc^\delta|x|^\gamma[1 + |x|^{\delta-\gamma}] \leq \tilde{K}|x|^\gamma,\end{aligned}$$

for sufficiently small $|x|$ and some constant $\tilde{K}$. Thus, $G(x) = L_1 + L_2 + O(x^\gamma)$.

11. Since

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = x \quad \text{and} \quad x_{n+1} = 1 + \frac{1}{x_n},$$

we have

$$x = 1 + \frac{1}{x}, \quad \text{so} \quad x^2 - x - 1 = 0.$$

The quadratic formula implies that

$$x = \frac{1}{2}\left(1 + \sqrt{5}\right).$$

This number is called the *golden ratio*. It appears frequently in mathematics and the sciences.

12. Let $F_n = C^n$. Substitute into $F_{n+2} = F_n + F_{n+1}$ to obtain $C^{n+2} = C^n + C^{n+1}$ or $C^n[C^2 - C - 1] = 0$. Solving the quadratic equation $C^2 - C - 1 = 0$ gives $C = \frac{1\pm\sqrt{5}}{2}$. So $F_n = a(\frac{1+\sqrt{5}}{2})^n + b(\frac{1-\sqrt{5}}{2})$ satisfies the recurrence relation $F_{n+2} = F_n + F_{n+1}$. For $F_0 = 1$ and $F_1 = 1$ we need $a = \frac{1+\sqrt{5}}{2}\frac{1}{\sqrt{5}}$ and $b = -(\frac{1-\sqrt{5}}{2})\frac{1}{\sqrt{5}}$. Hence, $F_n = \frac{1}{\sqrt{5}}((\frac{1+\sqrt{5}}{2})^{n+1} - (\frac{1-\sqrt{5}}{2})^{n+1})$.

13. $SUM = \sum_{i=1}^{N} x_i$. This saves one step since initialization is $SUM = x_1$ instead of $SUM = 0$. Problems may occur if $N = 0$.

14. (a) OUTPUT is PRODUCT $= 0$ which is correct only if $x_i = 0$ for some i.
    (b) OUTPUT is PRODUCT $= x_1 x_2 \ldots x_N$.
    (c) OUTPUT is PRODUCT $= x_1 x_2 \ldots x_N$ but exists with the correct value 0 if one of $x_i = 0$.

15. (a) $n(n+1)/2$ multiplications; $(n+2)(n-1)/2$ additions.
    (b) $\displaystyle\sum_{i=1}^{n} a_i \left(\sum_{j=1}^{i} b_j\right)$ requires n multiplications; $(n+2)(n-1)/2$ additions.

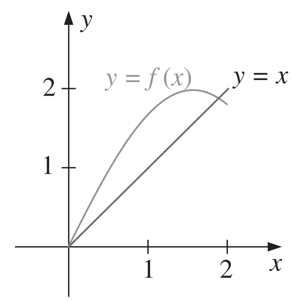
Solutions of Equations of One Variable

Exercise Set 2.1, page 54

1. $p_3 = 0.625$
2. (a) $p_3 = -0.6875$
   (b) $p_3 = 1.09375$
3. The Bisection method gives:
   (a) $p_7 = 0.5859$
   (b) $p_8 = 3.002$
   (c) $p_7 = 3.419$
4. The Bisection method gives:
   (a) $p_7 = -1.414$
   (b) $p_8 = 1.414$
   (c) $p_7 = 2.727$
   (d) $p_7 = -0.7265$
5. The Bisection method gives:
   (a) $p_{17} = 0.641182$
   (b) $p_{17} = 0.257530$
   (c) For the interval $[-3, -2]$, we have $p_{17} = -2.191307$, and for the interval $[-1, 0]$, we have $p_{17} = -0.798164$.
   (d) For the interval $[0.2, 0.3]$, we have $p_{14} = 0.297528$, and for the interval $[1.2, 1.3]$, we have $p_{14} = 1.256622$.
6. (a) $p_{17} = 1.51213837$
   (b) $p_{18} = 1.239707947$
   (c) For the interval $[1, 2]$, we have $p_{17} = 1.41239166$, and for the interval $[2, 4]$, we have $p_{18} = 3.05710602$.

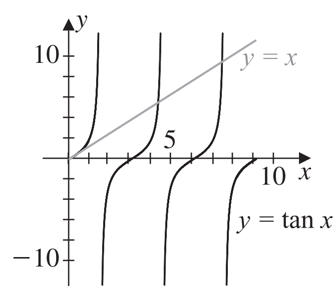
(d) For the interval $[0, 0.5]$, we have $p_{16} = 0.20603180$, and for the interval $[0.5, 1]$, we have $p_{16} = 0.68196869$.

7. (a)



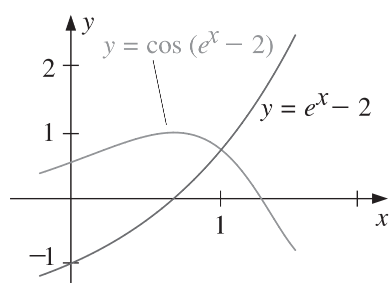
(b) Using $[1.5, 2]$ from part (a) gives $p_{16} = 1.89550018$.

8. (a)



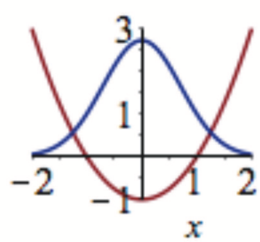
(b) Using $[4.2, 4.6]$ from part (a) gives $p_{16} = 4.4934143$.

9. (a)



(b) $p_{17} = 1.00762177$

10. (a)



(b) $p_{11} = -1.250976563$

11. (a) 2
    (b) -2
    (c) -1
    (d) 1

12. (a) 0
    (b) 0
    (c) 2
    (d) -2

13. The cube root of 25 is approximately $p_{14} = 2.92401$, using $[2, 3]$.

14. We have $\sqrt{3} \approx p_{14} = 1.7320$, using $[1, 2]$.

15. The depth of the water is 0.838 ft.

16. The angle θ changes at the approximate rate $w = -0.317059$.

17. A bound is $n \geq 14$, and $p_{14} = 1.32477$.

18. A bound for the number of iterations is $n \geq 12$ and $p_{12} = 1.3787$.

19. Since $\lim_{n\to\infty}(p_n - p_{n-1}) = \lim_{n\to\infty} 1/n = 0$, the difference in the terms goes to zero. However, p_n is the nth term of the divergent harmonic series, so $\lim_{n\to\infty} p_n = \infty$.

20. For $n > 1$,
$$|f(p_n)| = \left(\frac{1}{n}\right)^{10} \leq \left(\frac{1}{2}\right)^{10} = \frac{1}{1024} < 10^{-3},$$
so
$$|p - p_n| = \frac{1}{n} < 10^{-3} \Leftrightarrow 1000 < n.$$

21. Since $-1 < a < 0$ and $2 < b < 3$, we have $1 < a + b < 3$ or $1/2 < 1/2(a + b) < 3/2$ in all cases. Further,
$$\begin{aligned} f(x) < 0, &\quad \text{for } -1 < x < 0 \quad \text{and} \quad 1 < x < 2; \\ f(x) > 0, &\quad \text{for } 0 < x < 1 \quad \text{and} \quad 2 < x < 3. \end{aligned}$$
Thus, $a_1 = a$, $f(a_1) < 0$, $b_1 = b$, and $f(b_1) > 0$.
    (a) Since $a + b < 2$, we have $p_1 = \frac{a+b}{2}$ and $1/2 < p_1 < 1$. Thus, $f(p_1) > 0$. Hence, $a_2 = a_1 = a$ and $b_2 = p_1$. The only zero of f in $[a_2, b_2]$ is $p = 0$, so the convergence will be to 0.
    (b) Since $a + b > 2$, we have $p_1 = \frac{a+b}{2}$ and $1 < p_1 < 3/2$. Thus, $f(p_1) < 0$. Hence, $a_2 = p_1$ and $b_2 = b_1 = b$. The only zero of f in $[a_2, b_2]$ is $p = 2$, so the convergence will be to 2.
    (c) Since $a + b = 2$, we have $p_1 = \frac{a+b}{2} = 1$ and $f(p_1) = 0$. Thus, a zero of f has been found on the first iteration. The convergence is to $p = 1$.

Exercise Set 2.2, page 64

1. For the value of x under consideration we have

   (a) $x = (3 + x - 2x^2)^{1/4} \Leftrightarrow x^4 = 3 + x - 2x^2 \Leftrightarrow f(x) = 0$

   (b) $x = \left(\dfrac{x + 3 - x^4}{2}\right)^{1/2} \Leftrightarrow 2x^2 = x + 3 - x^4 \Leftrightarrow f(x) = 0$

   (c) $x = \left(\dfrac{x + 3}{x^2 + 2}\right)^{1/2} \Leftrightarrow x^2(x^2 + 2) = x + 3 \Leftrightarrow f(x) = 0$

   (d) $x = \dfrac{3x^4 + 2x^2 + 3}{4x^3 + 4x - 1} \Leftrightarrow 4x^4 + 4x^2 - x = 3x^4 + 2x^2 + 3 \Leftrightarrow f(x) = 0$

2. (a) $p_4 = 1.10782$; (b) $p_4 = 0.987506$; (c) $p_4 = 1.12364$; (d) $p_4 = 1.12412$;

   (b) Part (d) gives the best answer since $|p_4 - p_3|$ is the smallest for (d).

3. (a) Solve for $2x$ then divide by 2. $p_1 = 0.5625, p_2 = 0.58898926, p_3 = 0.60216264, p_4 = 0.60917204$

   (b) Solve for x^3, divide by x^2. $p_1 = 0, p_2$ undefined

   (c) Solve for x^3, divide by x, then take positive square root. $p_1 = 0, p_2$ undefined

   (d) Solve for x^3, then take negative of the cubed root. $p_1 = 0, p_2 = -1, p_3 = -1.4422496, p_4 = -1.57197274$. Parts (a) and (d) seem promising.

4. (a) $x^4 + 3x^2 - 2 = 0 \Leftrightarrow 3x^2 = 2 - x^4 \Leftrightarrow x = \sqrt{\frac{2-x^4}{3}}$; $p_0 = 1$, $p_1 = 0.577350269$,$p_2 = 0.79349204$,$p_3 = 0.73111023$, $p_4 = 0.75592901$.

   (b) $x^4 + 3x^2 - 2 = 0 \Leftrightarrow x^4 = 2 - 3x^2 \Leftrightarrow x = \sqrt[4]{2 - 3x^2}$; $p_0 = 1$, p_1 undefined.

   (c) $x^4 + 3x^2 - 2 = 0 \Leftrightarrow 3x^2 = 2 - x^4 \Leftrightarrow x = \frac{2-x^4}{3x}$; $p_0 = 1$, $p_1 = \frac{1}{3}$, $p_2 = 1.9876543$,$p_3 = -2.2821844$, $p_4 = 3.6700326$.

   (d) $x^4 + 3x^2 - 2 = 0 \Leftrightarrow x^4 = 2 - 3x^2 \Leftrightarrow x^3 = \frac{2-3x^2}{x} \Leftrightarrow x = \sqrt[3]{\frac{2-3x^2}{x}}$; $p_0 = 1$, $p_1 = -1$, $p_2 = 1$, $p_3 = -1$, $p_4 = 1$.

   Only the method of part (a) seems promising.

5. The order in descending speed of convergence is (b), (d), and (a). The sequence in (c) does not converge.

6. The sequence in (c) converges faster than in (d). The sequences in (a) and (b) diverge.

7. With $g(x) = (3x^2 + 3)^{1/4}$ and $p_0 = 1$, $p_6 = 1.94332$ is accurate to within 0.01.

8. With $g(x) = \sqrt{1 + \frac{1}{x}}$ and $p_0 = 1$, we have $p_4 = 1.324$.

9. Since $g'(x) = \frac{1}{4}\cos\frac{x}{2}$, g is continuous and g' exists on $[0, 2\pi]$. Further, $g'(x) = 0$ only when $x = \pi$, so that $g(0) = g(2\pi) = \pi \leq g(x) =\leq g(\pi) = \pi + \frac{1}{2}$ and $|g'(x)| \leq \frac{1}{4}$, for $0 \leq x \leq 2\pi$. Theorem 2.3 implies that a unique fixed point p exists in $[0, 2\pi]$. With $k = \frac{1}{4}$ and $p_0 = \pi$, we have $p_1 = \pi + \frac{1}{2}$. Corollary 2.5 implies that

$$|p_n - p| \leq \frac{k^n}{1-k}|p_1 - p_0| = \frac{2}{3}\left(\frac{1}{4}\right)^n.$$

For the bound to be less than 0.1, we need $n \geq 4$. However, $p_3 = 3.626996$ is accurate to within 0.01.

10. Using $p_0 = 1$ gives $p_{12} = 0.6412053$. Since $|g'(x)| = 2^{-x} \ln 2 \leq 0.551$ on $\left[\frac{1}{3}, 1\right]$ with $k = 0.551$, Corollary 2.5 gives a bound of 16 iterations.

11. For $p_0 = 1.0$ and $g(x) = 0.5(x + \frac{3}{x})$, we have $\sqrt{3} \approx p_4 = 1.73205$.

12. For $g(x) = 5/\sqrt{x}$ and $p_0 = 2.5$, we have $p_{14} = 2.92399$.

13. (a) With $[0, 1]$ and $p_0 = 0$, we have $p_9 = 0.257531$.
    (b) With $[2.5, 3.0]$ and $p_0 = 2.5$, we have $p_{17} = 2.690650$.
    (c) With $[0.25, 1]$ and $p_0 = 0.25$, we have $p_{14} = 0.909999$.
    (d) With $[0.3, 0.7]$ and $p_0 = 0.3$, we have $p_{39} = 0.469625$.
    (e) With $[0.3, 0.6]$ and $p_0 = 0.3$, we have $p_{48} = 0.448059$.
    (f) With $[0, 1]$ and $p_0 = 0$, we have $p_6 = 0.704812$.

14. The inequalities in Corollary 2.4 give $|p_n - p| < k^n \max(p_0 - a, b - p_0)$. We want

$$k^n \max(p_0 - a, b - p_0) < 10^{-5} \quad \text{so we need} \quad n > \frac{\ln(10^{-5}) - \ln(\max(p_0 - a, b - p_0))}{\ln k}.$$

    (a) Using $g(x) = 2 + \sin x$ we have $k = 0.9899924966$ so that with $p_0 = 2$ we have $n > \ln(0.00001)/\ln k = 1144.663221$. However, our tolerance is met with $p_{63} = 2.5541998$.
    (b) Using $g(x) = \sqrt[3]{2x + 5}$ we have $k = 0.1540802832$ so that with $p_0 = 2$ we have $n > \ln(0.00001)/\ln k = 6.155718005$. However, our tolerance is met with $p_6 = 2.0945503$.
    (c) Using $g(x) = \sqrt{e^x/3}$ and the interval $[0, 1]$ we have $k = 0.4759448347$ so that with $p_0 = 1$ we have $n > \ln(0.00001)/\ln k = 15.50659829$. However, our tolerance is met with $p_{12} = 0.91001496$.
    (d) Using $g(x) = \cos x$ and the interval $[0, 1]$ we have $k = 0.8414709848$ so that with $p_0 = 0$ we have $n > \ln(0.00001)/\ln k > 66.70148074$. However, our tolerance is met with $p_{30} = 0.73908230$.

15. For $g(x) = (2x^2 - 10 \cos x)/(3x)$, we have the following:

$$p_0 = 3 \Rightarrow p_8 = 3.16193; \quad p_0 = -3 \Rightarrow p_8 = -3.16193.$$

For $g(x) = \arccos(-0.1x^2)$, we have the following:

$$p_0 = 1 \Rightarrow p_{11} = 1.96882; \quad p_0 = -1 \Rightarrow p_{11} = -1.96882.$$

16. For $g(x) = \dfrac{1}{\tan x} - \dfrac{1}{x} + x$ and $p_0 = 4$, we have $p_4 = 4.493409$.

17. With $g(x) = \dfrac{1}{\pi} \arcsin\left(-\dfrac{x}{2}\right) + 2$, we have $p_5 = 1.683855$.

18. With $g(t) = 501.0625 - 201.0625e^{-0.4t}$ and $p_0 = 5.0$, $p_3 = 6.0028$ is within 0.01 s of the actual time.

19. Since g' is continuous at p and $|g'(p)| > 1$, by letting $\epsilon = |g'(p)| - 1$ there exists a number $\delta > 0$ such that $|g'(x) - g'(p)| < |g'(p)| - 1$ whenever $0 < |x - p| < \delta$. Hence, for any x satisfying $0 < |x - p| < \delta$, we have

$$|g'(x)| \geq |g'(p)| - |g'(x) - g'(p)| > |g'(p)| - (|g'(p)| - 1) = 1.$$

If p_0 is chosen so that $0 < |p - p_0| < \delta$, we have by the Mean Value Theorem that

$$|p_1 - p| = |g(p_0) - g(p)| = |g'(\xi)||p_0 - p|,$$

for some ξ between p_0 and p. Thus, $0 < |p - \xi| < \delta$ so $|p_1 - p| = |g'(\xi)||p_0 - p| > |p_0 - p|$.

20. (a) If fixed-point iteration converges to the limit p, then

$$p = \lim_{n\to\infty} p_n = \lim_{n\to\infty} 2p_{n-1} - Ap_{n-1}^2 = 2p - Ap^2.$$

Solving for p gives $p = \dfrac{1}{A}$.

(b) Any subinterval $[c, d]$ of $\left(\dfrac{1}{2A}, \dfrac{3}{2A}\right)$ containing $\dfrac{1}{A}$ suffices.

Since

$$g(x) = 2x - Ax^2, \quad g'(x) = 2 - 2Ax,$$

so $g(x)$ is continuous, and $g'(x)$ exists. Further, $g'(x) = 0$ only if $x = \dfrac{1}{A}$.

Since

$$g\left(\frac{1}{A}\right) = \frac{1}{A}, \quad g\left(\frac{1}{2A}\right) = g\left(\frac{3}{2A}\right) = \frac{3}{4A}, \quad \text{and we have} \quad \frac{3}{4A} \leq g(x) \leq \frac{1}{A}.$$

For x in $\left(\frac{1}{2A}, \frac{3}{2A}\right)$, we have

$$\left|x - \frac{1}{A}\right| < \frac{1}{2A} \quad \text{so} \quad |g'(x)| = 2A\left|x - \frac{1}{A}\right| < 2A\left(\frac{1}{2A}\right) = 1.$$

21. One of many examples is $g(x) = \sqrt{2x - 1}$ on $\left[\frac{1}{2}, 1\right]$.

22. (a) The proof of existence is unchanged. For uniqueness, suppose p and q are fixed points in $[a, b]$ with $p \neq q$. By the Mean Value Theorem, a number ξ in (a, b) exists with

$$p - q = g(p) - g(q) = g'(\xi)(p - q) \leq k(p - q) < p - q,$$

giving the same contradiction as in Theorem 2.3.

(b) Consider $g(x) = 1 - x^2$ on $[0, 1]$. The function g has the unique fixed point

$$p = \frac{1}{2}\left(-1 + \sqrt{5}\right).$$

With $p_0 = 0.7$, the sequence eventually alternates between 0 and 1.

23. (a) Suppose that $x_0 > \sqrt{2}$. Then

$$x_1 - \sqrt{2} = g(x_0) - g\left(\sqrt{2}\right) = g'(\xi)\left(x_0 - \sqrt{2}\right),$$

where $\sqrt{2} < \xi < x$. Thus, $x_1 - \sqrt{2} > 0$ and $x_1 > \sqrt{2}$. Further,

$$x_1 = \frac{x_0}{2} + \frac{1}{x_0} < \frac{x_0}{2} + \frac{1}{\sqrt{2}} = \frac{x_0 + \sqrt{2}}{2}$$

and $\sqrt{2} < x_1 < x_0$. By an inductive argument,

$$\sqrt{2} < x_{m+1} < x_m < \ldots < x_0.$$

Thus, $\{x_m\}$ is a decreasing sequence which has a lower bound and must converge. Suppose $p = \lim_{m\to\infty} x_m$. Then

$$p = \lim_{m\to\infty}\left(\frac{x_{m-1}}{2} + \frac{1}{x_{m-1}}\right) = \frac{p}{2} + \frac{1}{p}. \quad \text{Thus} \quad p = \frac{p}{2} + \frac{1}{p},$$

which implies that $p = \pm\sqrt{2}$. Since $x_m > \sqrt{2}$ for all m, we have $\lim_{m\to\infty} x_m = \sqrt{2}$.

(b) We have

$$0 < \left(x_0 - \sqrt{2}\right)^2 = x_0^2 - 2x_0\sqrt{2} + 2,$$

so $2x_0\sqrt{2} < x_0^2 + 2$ and $\sqrt{2} < \frac{x_0}{2} + \frac{1}{x_0} = x_1$.

(c) Case 1: $0 < x_0 < \sqrt{2}$, which implies that $\sqrt{2} < x_1$ by part (b). Thus,

$$0 < x_0 < \sqrt{2} < x_{m+1} < x_m < \ldots < x_1 \quad \text{and} \quad \lim_{m\to\infty} x_m = \sqrt{2}.$$

Case 2: $x_0 = \sqrt{2}$, which implies that $x_m = \sqrt{2}$ for all m and $\lim_{m\to\infty} x_m = \sqrt{2}$.
Case 3: $x_0 > \sqrt{2}$, which by part (a) implies that $\lim_{m\to\infty} x_m = \sqrt{2}$.

24. (a) Let

$$g(x) = \frac{x}{2} + \frac{A}{2x}.$$

Note that $g\left(\sqrt{A}\right) = \sqrt{A}$. Also,

$$g'(x) = 1/2 - A/\left(2x^2\right) \text{ if } x \neq 0 \quad \text{and} \quad g'(x) > 0 \text{ if } x > \sqrt{A}.$$

If $x_0 = \sqrt{A}$, then $x_m = \sqrt{A}$ for all m and $\lim_{m\to\infty} x_m = \sqrt{A}$.
If $x_0 > A$, then

$$x_1 - \sqrt{A} = g(x_0) - g\left(\sqrt{A}\right) = g'(\xi)\left(x_0 - \sqrt{A}\right) > 0.$$

Further,

$$x_1 = \frac{x_0}{2} + \frac{A}{2x_0} < \frac{x_0}{2} + \frac{A}{2\sqrt{A}} = \frac{1}{2}\left(x_0 + \sqrt{A}\right).$$

Thus, $\sqrt{A} < x_1 < x_0$. Inductively,

$$\sqrt{A} < x_{m+1} < x_m < \ldots < x_0$$

and $\lim_{m\to\infty} x_m = \sqrt{A}$ by an argument similar to that in Exercise 23(a).
If $0 < x_0 < \sqrt{A}$, then

$$0 < \left(x_0 - \sqrt{A}\right)^2 = x_0^2 - 2x_0\sqrt{A} + A \quad \text{and} \quad 2x_0\sqrt{A} < x_0^2 + A,$$

which leads to

$$\sqrt{A} < \frac{x_0}{2} + \frac{A}{2x_0} = x_1.$$

Thus

$$0 < x_0 < \sqrt{A} < x_{m+1} < x_m < \ldots < x_1,$$

and by the preceding argument, $\lim_{m\to\infty} x_m = \sqrt{A}$.

(b) If $x_0 < 0$, then $\lim_{m\to\infty} x_m = -\sqrt{A}$.

25. Replace the second sentence in the proof with: "Since g satisfies a Lipschitz condition on $[a, b]$ with a Lipschitz constant $L < 1$, we have, for each n,

$$|p_n - p| = |g(p_{n-1}) - g(p)| \le L|p_{n-1} - p|."$$

The rest of the proof is the same, with k replaced by L.

26. Let $\varepsilon = (1 - |g'(p)|)/2$. Since g' is continuous at p, there exists a number $\delta > 0$ such that for $x \in [p-\delta, p+\delta]$, we have $|g'(x) - g'(p)| < \varepsilon$. Thus, $|g'(x)| < |g'(p)| + \varepsilon < 1$ for $x \in [p-\delta, p+\delta]$. By the Mean Value Theorem

$$|g(x) - g(p)| = |g'(c)||x - p| < |x - p|,$$

for $x \in [p-\delta, p+\delta]$. Applying the Fixed-Point Theorem completes the problem.

Exercise Set 2.3, page 75

1. $p_2 = 2.60714$

2. $p_2 = -0.865684$; If $p_0 = 0$, $f'(p_0) = 0$ and p_1 cannot be computed.

3. (a) 2.45454
   (b) 2.44444
   (c) Part (a) is better.

4. (a) -1.25208
   (b) -0.841355

5. (a) For $p_0 = 2$, we have $p_5 = 2.69065$.
   (b) For $p_0 = -3$, we have $p_3 = -2.87939$.
   (c) For $p_0 = 0$, we have $p_4 = 0.73909$.

(d) For $p_0 = 0$, we have $p_3 = 0.96434$.

6. (a) For $p_0 = 1$, we have $p_8 = 1.829384$.
   (b) For $p_0 = 1.5$, we have $p_4 = 1.397748$.
   (c) For $p_0 = 2$, we have $p_4 = 2.370687$; and for $p_0 = 4$, we have $p_4 = 3.722113$.
   (d) For $p_0 = 1$, we have $p_4 = 1.412391$; and for $p_0 = 4$, we have $p_5 = 3.057104$.
   (e) For $p_0 = 1$, we have $p_4 = 0.910008$; and for $p_0 = 3$, we have $p_9 = 3.733079$.
   (f) For $p_0 = 0$, we have $p_4 = 0.588533$; for $p_0 = 3$, we have $p_3 = 3.096364$; and for $p_0 = 6$, we have $p_3 = 6.285049$.

7. Using the endpoints of the intervals as p_0 and p_1, we have:
   (a) $p_{11} = 2.69065$
   (b) $p_7 = -2.87939$
   (c) $p_6 = 0.73909$
   (d) $p_5 = 0.96433$

8. Using the endpoints of the intervals as p_0 and p_1, we have:
   (a) $p_7 = 1.829384$
   (b) $p_9 = 1.397749$
   (c) $p_6 = 2.370687; p_7 = 3.722113$
   (d) $p_8 = 1.412391; p_7 = 3.057104$
   (e) $p_6 = 0.910008; p_{10} = 3.733079$
   (f) $p_6 = 0.588533; p_5 = 3.096364; p_5 = 6.285049$

9. Using the endpoints of the intervals as p_0 and p_1, we have:
   (a) $p_{16} = 2.69060$
   (b) $p_6 = -2.87938$
   (c) $p_7 = 0.73908$
   (d) $p_6 = 0.96433$

10. Using the endpoints of the intervals as p_0 and p_1, we have:
   (a) $p_8 = 1.829383$
   (b) $p_9 = 1.397749$
   (c) $p_6 = 2.370687; p_8 = 3.722112$
   (d) $p_{10} = 1.412392; p_{12} = 3.057099$
   (e) $p_7 = 0.910008; p_{29} = 3.733065$
   (f) $p_9 = 0.588533; p_5 = 3.096364; p_5 = 6.285049$

11. (a) Newton's method with $p_0 = 1.5$ gives $p_3 = 1.51213455$.
   The Secant method with $p_0 = 1$ and $p_1 = 2$ gives $p_{10} = 1.51213455$.
   The Method of False Position with $p_0 = 1$ and $p_1 = 2$ gives $p_{17} = 1.51212954$.

(b) Newton's method with $p_0 = 0.5$ gives $p_5 = 0.976773017$.
The Secant method with $p_0 = 0$ and $p_1 = 1$ gives $p_5 = 10.976773017$.
The Method of False Position with $p_0 = 0$ and $p_1 = 1$ gives $p_5 = 0.976772976$.

12. (a) We have

	Initial Approximation	Result	Initial Approximation	Result
Newton's	$p_0 = 1.5$	$p_4 = 1.41239117$	$p_0 = 3.0$	$p_4 = 3.05710355$
Secant	$p_0 = 1, p_1 = 2$	$p_8 = 1.41239117$	$p_0 = 2, p_1 = 4$	$p_{10} = 3.05710355$
False Position	$p_0 = 1, p_1 = 2$	$p_{13} = 1.41239119$	$p_0 = 2, p_1 = 4$	$p_{19} = 3.05710353$

(b) We have

	Initial Approximation	Result	Initial Approximation	Result
Newton's	$p_0 = 0.25$	$p_4 = 0.206035120$	$p_0 = 0.75$	$p_4 = 0.681974809$
Secant	$p_0 = 0, p_1 = 0.5$	$p_9 = 0.206035120$	$p_0 = 0.5, p_1 = 1$	$p_8 = 0.681974809$
False Position	$p_0 = 0, p_1 = 0.5$	$p_{12} = 0.206035125$	$p_0 = 0.5, p_1 = 1$	$p_{15} = 0.681974791$

13. (a) For $p_0 = -1$ and $p_1 = 0$, we have $p_{17} = -0.04065850$, and for $p_0 = 0$ and $p_1 = 1$, we have $p_9 = 0.9623984$.

(b) For $p_0 = -1$ and $p_1 = 0$, we have $p_5 = -0.04065929$, and for $p_0 = 0$ and $p_1 = 1$, we have $p_{12} = -0.04065929$.

(c) For $p_0 = -0.5$, we have $p_5 = -0.04065929$, and for $p_0 = 0.5$, we have $p_{21} = 0.9623989$.

14. (a) The Bisection method yields $p_{10} = 0.4476563$.

(b) The method of False Position yields $p_{10} = 0.442067$.

(c) The Secant method yields $p_{10} = -195.8950$.

15. Newton's method for the various values of p_0 gives the following results.

(a) $p_0 = -10, p_{11} = -4.30624527$

(b) $p_0 = -5, p_5 = -4.30624527$

(c) $p_0 = -3, p_5 = 0.824498585$

(d) $p_0 = -1, p_4 = -0.824498585$

(e) $p_0 = 0$, p_1 cannot be computed because $f'(0) = 0$

(f) $p_0 = 1, p_4 = 0.824498585$

(g) $p_0 = 3, p_5 = -0.824498585$

(h) $p_0 = 5, p_5 = 4.30624527$

(i) $p_0 = 10, p_{11} = 4.30624527$

16. Newton's method for the various values of p_0 gives the following results.

    (a) $p_8 = -1.379365$
    (b) $p_7 = -1.379365$
    (c) $p_7 = 1.379365$
    (d) $p_7 = -1.379365$
    (e) $p_7 = 1.379365$
    (f) $p_8 = 1.379365$

17. For $f(x) = \ln(x^2 + 1) - e^{0.4x} \cos \pi x$, we have the following roots.

    (a) For $p_0 = -0.5$, we have $p_3 = -0.4341431$.
    (b) For $p_0 = 0.5$, we have $p_3 = 0.4506567$.
    For $p_0 = 1.5$, we have $p_3 = 1.7447381$.
    For $p_0 = 2.5$, we have $p_5 = 2.2383198$.
    For $p_0 = 3.5$, we have $p_4 = 3.7090412$.
    (c) The initial approximation $n - 0.5$ is quite reasonable.
    (d) For $p_0 = 24.5$, we have $p_2 = 24.4998870$.

18. Newton's method gives $p_{15} = 1.895488$, for $p_0 = \frac{\pi}{2}$; and $p_{19} = 1.895489$, for $p_0 = 5\pi$. The sequence does not converge in 200 iterations for $p_0 = 10\pi$. The results do not indicate the fast convergence usually associated with Newton's method.

19. For $p_0 = 1$, we have $p_5 = 0.589755$. The point has the coordinates $(0.589755, 0.347811)$.

20. For $p_0 = 2$, we have $p_2 = 1.866760$. The point is $(1.866760, 0.535687)$.

21. The two numbers are approximately 6.512849 and 13.487151.

22. We have $\lambda \approx 0.100998$ and $N(2) \approx 2{,}187{,}950$.

23. The borrower can afford to pay at most 8.10%.

24. The minimal annual interest rate is 6.67%.

25. We have $P_L = 363432$, $c = -1.0266939$, and $k = 0.026504522$. The 1990 population is $P(30) = 248{,}319$, and the 2020 population is $P(60) = 300{,}528$.

26. We have $P_L = 446505$, $c = 0.91226292$, and $k = 0.014800625$. The 1990 population is $P(30) = 248{,}707$, and the 2020 population is $P(60) = 306{,}528$.

27. Using $p_0 = 0.5$ and $p_1 = 0.9$, the Secant method gives $p_5 = 0.842$.

28. (a) $\frac{1}{3}e, t = 3$ hours
    (b) 11 hours and 5 minutes
    (c) 21 hours and 14 minutes

29. (a) We have, approximately,

$$A = 17.74, \quad B = 87.21, \quad C = 9.66, \quad \text{and} \quad E = 47.47$$

With these values we have

$$A \sin \alpha \cos \alpha + B \sin^2 \alpha - C \cos \alpha - E \sin \alpha = 0.02.$$

(b) Newton's method gives $\alpha \approx 33.2°$.

30. This formula involves the subtraction of nearly equal numbers in both the numerator and denominator if p_{n-1} and p_{n-2} are nearly equal.

31. The equation of the tangent line is

$$y - f(p_{n-1}) = f'(p_{n-1})(x - p_{n-1}).$$

To complete this problem, set $y = 0$ and solve for $x = p_n$.

32. For some ξ_n between p_n and p,

$$f(p) = f(p_n) + (p - p_n)f'(p_n) + \frac{(p - p_n)^2}{2} f''(\xi_n)$$

$$0 = f(p_n) + (p - p_n)f'(p_n) + \frac{(p - p_n)^2}{2} f''(\xi_n)$$

Since $f'(p_n) \neq 0$,

$$0 = \frac{f(p_n)}{f'(p_n)} + p - p_n + \frac{(p - p_n)^2}{2f'(p_n)} f''(\xi_n)$$

we have

$$p - [p_n - \frac{f(p_n)}{f'(p_n)}] = -\frac{(p - p_n)^2}{2f'(p_n)} f''(\xi_n)$$

and

$$p - p_{n+1} = -\frac{(p - p_n)^2}{2f'(p_n)} f''(p_n).$$

So

$$|p - p_{n+1}| \leq \frac{M^2}{2|f'(p_n)|}(p - p_n)^2.$$

Exercise Set 2.4, page 85

1. (a) For $p_0 = 0.5$, we have $p_{13} = 0.567135$.
   (b) For $p_0 = -1.5$, we have $p_{23} = -1.414325$.
   (c) For $p_0 = 0.5$, we have $p_{22} = 0.641166$.
   (d) For $p_0 = -0.5$, we have $p_{23} = -0.183274$.

2. (a) For $p_0 = 0.5$, we have $p_{15} = 0.739076589$.
   (b) For $p_0 = -2.5$, we have $p_9 = -1.33434594$.
   (c) For $p_0 = 3.5$, we have $p_5 = 3.14156793$.
   (d) For $p_0 = 4.0$, we have $p_{44} = 3.37354190$.

3. Modified Newton's method in Eq. (2.11) gives the following:
   (a) For $p_0 = 0.5$, we have $p_3 = 0.567143$.
   (b) For $p_0 = -1.5$, we have $p_2 = -1.414158$.
   (c) For $p_0 = 0.5$, we have $p_3 = 0.641274$.
   (d) For $p_0 = -0.5$, we have $p_5 = -0.183319$.

4. (a) For $p_0 = 0.5$, we have $p_4 = 0.739087439$.
   (b) For $p_0 = -2.5$, we have $p_{53} = -1.33434594$.
   (c) For $p_0 = 3.5$, we have $p_5 = 3.14156793$.
   (d) For $p_0 = 4.0$, we have $p_3 = -3.72957639$.

5. Newton's method with $p_0 = -0.5$ gives $p_{13} = -0.169607$. Modified Newton's method in Eq. (2.11) with $p_0 = -0.5$ gives $p_{11} = -0.169607$.

6. (a) Since
   $$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|} = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1} = 1,$$
   we have linear convergence. To have $|p_n - p| < 5 \times 10^{-2}$, we need $n \geq 20$.
   (b) Since
   $$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 = 1,$$
   we have linear convergence. To have $|p_n - p| < 5 \times 10^{-2}$, we need $n \geq 5$.

7. (a) For $k > 0$,
   $$\lim_{n\to\infty} \frac{|p_{n+1} - 0|}{|p_n - 0|} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^k}}{\frac{1}{n^k}} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^k = 1,$$
   so the convergence is linear.
   (b) We need to have $N > 10^{m/k}$.

8. (a) Since
   $$\lim_{n\to\infty} \frac{|p_{n+1} - 0|}{|p_n - 0|^2} = \lim_{n\to\infty} \frac{10^{-2^{n+1}}}{(10^{-2^n})^2} = \lim_{n\to\infty} \frac{10^{-2^{n+1}}}{10^{-2^{n+1}}} = 1,$$
   the sequence is quadratically convergent.

(b) We have

$$\lim_{n\to\infty}\frac{|p_{n+1}-0|}{|p_n-0|^2}=\lim_{n\to\infty}\frac{10^{-(n+1)^k}}{\left(10^{-n^k}\right)^2}=\lim_{n\to\infty}\frac{10^{-(n+1)^k}}{10^{-2n^k}}$$
$$=\lim_{n\to\infty}10^{2n^k-(n+1)^k}=\lim_{n\to\infty}10^{n^k(2-\left(\frac{n+1}{n}\right)^k)}=\infty,$$

so the sequence $p_n=10^{-n^k}$ does not converge quadratically.

9. Typical examples are

(a) $p_n=10^{-3^n}$

(b) $p_n=10^{-\alpha^n}$

10. Suppose $f(x)=(x-p)^m q(x)$. Since

$$g(x)=x-\frac{m(x-p)q(x)}{mq(x)+(x-p)q'(x)},$$

we have $g'(p)=0$.

11. This follows from the fact that

$$\lim_{n\to\infty}\frac{\left|\dfrac{b-a}{2^{n+1}}\right|}{\left|\dfrac{b-a}{2^n}\right|}=\frac{1}{2}.$$

12. If f has a zero of multiplicity m at p, then f can be written as

$$f(x)=(x-p)^m q(x),$$

for $x\neq p$, where

$$\lim_{x\to p}q(x)\neq 0.$$

Thus,

$$f'(x)=m(x-p)^{m-1}q(x)+(x-p)^m q'(x)$$

and $f'(p)=0$. Also,

$$f''(x)=m(m-1)(x-p)^{m-2}q(x)+2m(x-p)^{m-1}q'(x)+(x-p)^m q''(x)$$

and $f''(p)=0$. In general, for $k\le m$,

$$f^{(k)}(x)=\sum_{j=0}^{k}\binom{k}{j}\frac{d^j(x-p)^m}{dx^j}q^{(k-j)}(x)=\sum_{j=0}^{k}\binom{k}{j}m(m-1)\cdots(m-j+1)(x-p)^{m-j}q^{(k-j)}(x).$$

Thus, for $0\le k\le m-1$, we have $f^{(k)}(p)=0$, but $f^{(m)}(p)=m!\lim_{x\to p}q(x)\neq 0$.

Conversely, suppose that

$$f(p)=f'(p)=\ldots=f^{(m-1)}(p)=0\quad\text{and}\quad f^{(m)}(p)\neq 0.$$

Consider the $(m-1)$th Taylor polynomial of f expanded about p:

$$\begin{aligned} f(x) =& f(p) + f'(p)(x-p) + \ldots + \frac{f^{(m-1)}(p)(x-p)^{m-1}}{(m-1)!} + \frac{f^{(m)}(\xi(x))(x-p)^m}{m!} \\ =& (x-p)^m \frac{f^{(m)}(\xi(x))}{m!}, \end{aligned}$$

where $\xi(x)$ is between x and p.

Since $f^{(m)}$ is continuous, let

$$q(x) = \frac{f^{(m)}(\xi(x))}{m!}.$$

Then $f(x) = (x-p)^m q(x)$ and

$$\lim_{x\to p} q(x) = \frac{f^{(m)}(p)}{m!} \neq 0.$$

Hence f has a zero of multiplicity m at p.

13. If

$$\frac{|p_{n+1}-p|}{|p_n-p|^3} = 0.75 \quad \text{and} \quad |p_0-p| = 0.5, \quad \text{then} \quad |p_n-p| = (0.75)^{(3^n-1)/2}|p_0-p|^{3^n}.$$

To have $|p_n-p| \leq 10^{-8}$ requires that $n \geq 3$.

14. Let $e_n = p_n - p$. If

$$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^\alpha} = \lambda > 0,$$

then for sufficiently large values of n, $|e_{n+1}| \approx \lambda|e_n|^\alpha$. Thus,

$$|e_n| \approx \lambda|e_{n-1}|^\alpha \quad \text{and} \quad |e_{n-1}| \approx \lambda^{-1/\alpha}|e_n|^{1/\alpha}.$$

Using the hypothesis gives

$$\lambda|e_n|^\alpha \approx |e_{n+1}| \approx C|e_n|\lambda^{-1/\alpha}|e_n|^{1/\alpha}, \quad \text{so} \quad |e_n|^\alpha \approx C\lambda^{-1/\alpha-1}|e_n|^{1+1/\alpha}.$$

Since the powers of $|e_n|$ must agree,

$$\alpha = 1 + 1/\alpha \ \text{ and } \ \alpha = \frac{1+\sqrt{5}}{2} \approx 1.62.$$

The number α is the *golden ratio* that appeared in Exercise 11 of section 1.3.

Exercise Set 2.5, page 90

1. The results are listed in the following table.

	(a)	(b)	(c)	(d)
$\hat{p}_0$	0.258684	0.907859	0.548101	0.731385
$\hat{p}_1$	0.257613	0.909568	0.547915	0.736087
$\hat{p}_2$	0.257536	0.909917	0.547847	0.737653
$\hat{p}_3$	0.257531	0.909989	0.547823	0.738469
$\hat{p}_4$	0.257530	0.910004	0.547814	0.738798
$\hat{p}_5$	0.257530	0.910007	0.547810	0.738958

2. Newton's Method gives $p_{16} = -0.1828876$ and $\hat{p}_7 = -0.183387$.

3. Steffensen's method gives $p_0^{(1)} = 0.826427$.

4. Steffensen's method gives $p_0^{(1)} = 2.152905$ and $p_0^{(2)} = 1.873464$.

5. Steffensen's method gives $p_1^{(0)} = 1.5$.

6. Steffensen's method gives $p_2^{(0)} = 1.73205$.

7. For $g(x) = \sqrt{1 + \frac{1}{x}}$ and $p_0^{(0)} = 1$, we have $p_0^{(3)} = 1.32472$.

8. For $g(x) = 2^{-x}$ and $p_0^{(0)} = 1$, we have $p_0^{(3)} = 0.64119$.

9. For $g(x) = 0.5(x + \frac{3}{x})$ and $p_0^{(0)} = 0.5$, we have $p_0^{(4)} = 1.73205$.

10. For $g(x) = \frac{5}{\sqrt{x}}$ and $p_0^{(0)} = 2.5$, we have $p_0^{(3)} = 2.92401774$.

11. (a) For $g(x) = \left(2 - e^x + x^2\right)/3$ and $p_0^{(0)} = 0$, we have $p_0^{(3)} = 0.257530$.
    (b) For $g(x) = 0.5(\sin x + \cos x)$ and $p_0^{(0)} = 0$, we have $p_0^{(4)} = 0.704812$.
    (c) With $p_0^{(0)} = 0.25$, $p_0^{(4)} = 0.910007572$.
    (d) With $p_0^{(0)} = 0.3$, $p_0^{(4)} = 0.469621923$.

12. (a) For $g(x) = 2 + \sin x$ and $p_0^{(0)} = 2$, we have $p_0^{(4)} = 2.55419595$.
    (b) For $g(x) = \sqrt[3]{2x+5}$ and $p_0^{(0)} = 2$, we have $p_0^{(2)} = 2.09455148$.
    (c) With $g(x) = \sqrt{\frac{e^x}{3}}$ and $p_0^{(0)} = 1$, we have $p_0^{(3)} = 0.910007574$.
    (d) With $g(x) = \cos x$, and $p_0^{(0)} = 0$, we have $p_0^{(4)} = 0.739085133$.

13. Aitken's Δ^2 method gives:
    (a) $\hat{p}_{10} = 0.0\overline{45}$
    (b) $\hat{p}_2 = 0.0363$

14. (a) A positive constant λ exists with

$$\lambda = \lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha}.$$

Hence

$$\lim_{n\to\infty}\left|\frac{p_{n+1}-p}{p_n-p}\right| = \lim_{n\to\infty}\frac{|p_{n+1}-p|}{|p_n-p|^{\alpha}}\cdot|p_n-p|^{\alpha-1} = \lambda\cdot 0 = 0 \quad\text{and}\quad \lim_{n\to\infty}\frac{p_{n+1}-p}{p_n-p} = 0.$$

(b) One example is $p_n = \frac{1}{n^n}$.

15. We have

$$\frac{|p_{n+1}-p_n|}{|p_n-p|} = \frac{|p_{n+1}-p+p-p_n|}{|p_n-p|} = \left|\frac{p_{n+1}-p}{p_n-p}-1\right|,$$

so

$$\lim_{n\to\infty}\frac{|p_{n+1}-p_n|}{|p_n-p|} = \lim_{n\to\infty}\left|\frac{p_{n+1}-p}{p_n-p}-1\right| = 1.$$

16.

$$\frac{\hat{p}_n-p}{p_n-p} = \frac{\lambda\left(\delta_n+\delta_{n+1}\right)-2\delta_n+\delta_n\delta_{n+1}-2\delta_n(\lambda-1)-\delta_n^2}{(\lambda-1)^2+\lambda\left(\delta_n+\delta_{n+1}\right)-2\delta_n+\delta_n\delta_{n+1}}$$

17. (a) Since $p_n = P_n(x) = \displaystyle\sum_{k=0}^{n}\frac{1}{k!}x^k$, we have

$$p_n - p = P_n(x) - e^x = \frac{-e^{\xi}}{(n+1)!}x^{n+1},$$

where ξ is between 0 and x. Thus, $p_n - p \neq 0$, for all $n \geq 0$. Further,

$$\frac{p_{n+1}-p}{p_n-p} = \frac{\frac{-e^{\xi_1}}{(n+2)!}x^{n+2}}{\frac{-e^{\xi}}{(n+1)!}x^{n+1}} = \frac{e^{(\xi_1-\xi)}x}{n+2},$$

where ξ_1 is between 0 and 1. Thus, $\lambda = \lim_{n\to\infty}\frac{e^{(\xi_1-\xi)}x}{n+2} = 0 < 1$.

(b)

n	p_n	$\hat{p}_n$
0	1	3
1	2	2.75
2	2.5	$2.7\overline{2}$
3	$2.\overline{6}$	2.71875
4	$2.708\overline{3}$	$2.718\overline{3}$
5	$2.71\overline{6}$	2.7182870
6	$2.7180\overline{5}$	2.7182823
7	2.7182539	2.7182818
8	2.7182787	2.7182818
9	2.7182815	
10	2.7182818	

(c) Aitken's Δ^2 method gives quite an improvement for this problem. For example, $\hat{p}_6$ is accurate to within 5×10^{-7}. We need p_{10} to have this accuracy.

Exercise Set 2.6, page 100

1. (a) For $p_0 = 1$, we have $p_{22} = 2.69065$.

   (b) For $p_0 = 1$, we have $p_5 = 0.53209$; for $p_0 = -1$, we have $p_3 = -0.65270$; and for $p_0 = -3$, we have $p_3 = -2.87939$.

   (c) For $p_0 = 1$, we have $p_5 = 1.32472$.

   (d) For $p_0 = 1$, we have $p_4 = 1.12412$; and for $p_0 = 0$, we have $p_8 = -0.87605$.

   (e) For $p_0 = 0$, we have $p_6 = -0.47006$; for $p_0 = -1$, we have $p_4 = -0.88533$; and for $p_0 = -3$, we have $p_4 = -2.64561$.

   (f) For $p_0 = 0$, we have $p_{10} = 1.49819$.

2. (a) For $p_0 = 0$, we have $p_9 = -4.123106$; and for $p_0 = 3$, we have $p_6 = 4.123106$. The complex roots are $-2.5 \pm 1.322879i$.

   (b) For $p_0 = 1$, we have $p_7 = -3.548233$; and for $p_0 = 4$, we have $p_5 = 4.38111$. The complex roots are $0.5835597 \pm 1.494188i$.

   (c) The only roots are complex, and they are $\pm\sqrt{2}i$ and $-0.5 \pm 0.5\sqrt{3}i$.

   (d) For $p_0 = 1$, we have $p_5 = -0.250237$; for $p_0 = 2$, we have $p_5 = 2.260086$; and for $p_0 = -11$, we have $p_6 = -12.612430$. The complex roots are $-0.1987094 \pm 0.8133125i$.

   (e) For $p_0 = 0$, we have $p_8 = 0.846743$; and for $p_0 = -1$, we have $p_9 = -3.358044$. The complex roots are $-1.494350 \pm 1.744219i$.

   (f) For $p_0 = 0$, we have $p_8 = 2.069323$; and for $p_0 = 1$, we have $p_3 = 0.861174$. The complex roots are $-1.465248 \pm 0.8116722i$.

   (g) For $p_0 = 0$, we have $p_6 = -0.732051$; for $p_0 = 1$, we have $p_4 = 1.414214$; for $p_0 = 3$, we have $p_5 = 2.732051$; and for $p_0 = -2$, we have $p_6 = -1.414214$.

   (h) For $p_0 = 0$, we have $p_5 = 0.585786$; for $p_0 = 2$, we have $p_2 = 3$; and for $p_0 = 4$, we have $p_6 = 3.414214$.

3. The following table lists the initial approximation and the roots.

	p_0	p_1	p_2	Approximate roots	Complex Conjugate roots
(a)	-1	0	1	$p_7 = -0.34532 - 1.31873i$	$-0.34532 + 1.31873i$
	0	1	2	$p_6 = 2.69065$	
(b)	0	1	2	$p_6 = 0.53209$	
	1	2	3	$p_9 = -0.65270$	
	-2	-3	-2.5	$p_4 = -2.87939$	
(c)	0	1	2	$p_5 = 1.32472$	
	-2	-1	0	$p_7 = -0.66236 - 0.56228i$	$-0.66236 + 0.56228i$
(d)	0	1	2	$p_5 = 1.12412$	
	2	3	4	$p_{12} = -0.12403 + 1.74096i$	$-0.12403 - 1.74096i$
	-2	0	-1	$p_5 = -0.87605$	
(e)	0	1	2	$p_{10} = -0.88533$	
	1	0	-0.5	$p_5 = -0.47006$	
	-1	-2	-3	$p_5 = -2.64561$	
(f)	0	1	2	$p_6 = 1.49819$	
	-1	-2	-3	$p_{10} = -0.51363 - 1.09156i$	$-0.51363 + 1.09156i$
	1	0	-1	$p_8 = 0.26454 - 1.32837i$	$0.26454 + 1.32837i$

4. The following table lists the initial approximation and the roots.

	p_0	p_1	p_2	Approximate roots	Complex Conjugate roots
(a)	0	1	2	$p_{11} = -2.5 - 1.322876i$	$-2.5 + 1.322876i$
	1	2	3	$p_6 = 4.123106$	
	-3	-4	-5	$p_5 = -4.123106$	
(b)	0	1	2	$p_7 = 0.583560 - 1.494188i$	$0.583560 + 1.494188i$
	2	3	4	$p_6 = 4.381113$	
	-2	-3	-4	$p_5 = -3.548233$	
(c)	0	1	2	$p_{11} = 1.414214i$	$-1.414214i$
	-1	-2	-3	$p_{10} = -0.5 + 0.866025i$	$-0.5 - 0.866025i$
(d)	0	1	2	$p_7 = 2.260086$	
	3	4	5	$p_{14} = -0.198710 + 0.813313i$	$-0.198710 + 0.813313i$
	11	12	13	$p_{22} = -0.250237$	
	-9	-10	-11	$p_6 = -12.612430$	
(e)	0	1	2	$p_6 = 0.846743$	
	3	4	5	$p_{12} = -1.494349 + 1.744218i$	$-1.494349 - 1.744218i$
	-1	-2	-3	$p_7 = -3.358044$	
(f)	0	1	2	$p_6 = 2.069323$	
	-1	0	1	$p_5 = 0.861174$	
	-1	-2	-3	$p_8 = -1.465248 + 0.811672i$	$-1.465248 - 0.811672i$
(g)	0	1	2	$p_6 = 1.414214$	
	-2	-1	0	$p_7 = -0.732051$	
	0	-2	-1	$p_7 = -1.414214$	
	2	3	4	$p_6 = 2.732051$	
(h)	0	1	2	$p_8 = 3$	
	-1	0	1	$p_5 = 0.585786$	
	2.5	3.5	4	$p_6 = 3.414214$	

5. (a) The roots are 1.244, 8.847, and -1.091, and the critical points are 0 and 6.
   (b) The roots are 0.5798, 1.521, 2.332, and -2.432, and the critical points are 1, 2.001, and -1.5.
6. We get convergence to the root 0.27 with $p_0 = 0.28$. We need p_0 closer to 0.29 since $f'(0.28\overline{3}) = 0$.
7. The methods all find the solution 0.23235.
8. The width is approximately $W = 16.2121$ ft.
9. The minimal material is approximately 573.64895 cm^2.
10. Fibonacci's answer was 1.3688081078532, and Newton's Method gives 1.36880810782137 with a tolerance of 10^{-16}, so Fibonacci's answer is within 4×10^{-11}. This accuracy is amazing for the time.