Chapter 1 Limits and Continuity

1.1 Limits of Functions Using Numerical and Graphical Techniques

Concepts and Vocabulary

1. The limit as x approaches c of a function f is written symbolically as $\boxed{\textbf{(c)}\ \lim_{x \to c} f(x)}$.

2. $\boxed{\text{True}}$. The tangent line to the graph of f at a point $P = (c, f(c))$ is the limiting position of the secant lines passing through P and a point $(x, f(x))$, $x \neq c$, as x moves closer to c.

3. $\boxed{\text{False}}$. If f is not defined at $x = c$, the $\lim_{x \to c} f(x)$ **may** exist.

4. $\boxed{\text{False}}$. The limit L of a function $y = f(x)$ as x approaches the number c **does not** depend on the value of f at c.

5. $\boxed{\text{False}}$. If $\lim_{x \to c} \dfrac{f(x) - f(c)}{x - c}$ exists, it equals the **slope** of the tangent line to the graph of f at the point $(c, f(c))$.

6. $\boxed{\text{False}}$. The limit of a function $y = f(x)$ as x approaches a number c equals L **if and only if both** of the one-sided limits as x approaches c equal L.

Skill Building

7. The values in the table below suggest that the value of $f(x) = 2x$ can be made "as close as we please" to 2 by choosing x "sufficiently close" to 1. It therefore appears that

$$\boxed{\lim_{x \to 1} 2x = 2}.$$

x	0.9	0.99	0.999	$\to 1 \leftarrow$	1.001	1.01	1.1
$f(x) = 2x$	1.8	1.98	1.998	$f(x)$ approaches 2	2.002	2.02	2.2

8. The values in the table below suggest that the value of $f(x) = x + 3$ can be made "as close as we please" to 5 by choosing x "sufficiently close" to 2. It therefore appears that

$$\lim_{x \to 2} (x + 3) = \boxed{5}.$$

x	1.9	1.99	1.999	$\to 2 \leftarrow$	2.001	2.01	2.1
$f(x) = x + 3$	4.9	4.99	4.999	$f(x)$ approaches 5	5.001	5.01	5.1

9. The values in the table below suggest that the value of $f(x) = x^2 + 2$ can be made "as close as we please" to 2 by choosing x "sufficiently close" to 0. It therefore appears that

$$\boxed{\lim_{x \to 0}(x^2 + 2) = 2}.$$

x	-0.1	-0.01	-0.001	$\to 0 \leftarrow$	0.001	0.01	0.1
$f(x) = x^2 + 2$	2.01	2.0001	2.000001	$f(x)$ approaches 2	2.000001	2.0001	2.01

10. The values in the table below suggest that the value of $f(x) = x^2 - 2$ can be made "as close as we please" to -1 by choosing x "sufficiently close" to -1. It therefore appears that

$$\lim_{x \to -1}(x^2 - 2) = \boxed{-1}.$$

x	-1.1	-1.01	-1.001	$\to -1 \leftarrow$	-0.999	-0.99	-0.9
$f(x) = x^2 - 2$	-0.79	-0.9799	-0.997999	$f(x)$ approaches -1	-1.001999	-1.0199	-1.19

11. The values in the table below suggest that the value of $f(x) = \dfrac{x^2 - 9}{x + 3}$ can be made "as close as we please" to -6 by choosing x "sufficiently close" to -3. It therefore appears that

$$\boxed{\lim_{x \to -3} \frac{x^2 - 9}{x + 3} = -6}.$$

x	-3.5	-3.1	-3.01	$\to -3 \leftarrow$	-2.99	-2.9	-2.5
$f(x) = \dfrac{x^2 - 9}{x + 3}$	-6.5	-6.1	-6.01	$f(x)$ approaches -6	-5.99	-5.9	-5.5

12. The values in the table below suggest that the value of $f(x) = \dfrac{x^3 + 1}{x + 1}$ can be made "as close as we please" to 3 by choosing x "sufficiently close" to -1. It therefore appears that

$$\lim_{x \to -1} \frac{x^3 + 1}{x + 1} = \boxed{3}.$$

x	-1.1	-1.01	-1.001	$\to -1 \leftarrow$	-0.999	-0.99	-0.9
$f(x) = \dfrac{x^3 + 1}{x + 1}$	3.31	3.0301	3.003001	$f(x)$ approaches 3	2.997001	2.9701	2.71

13. The values in the table below, which have been rounded to five decimal places for display purposes, suggest that the value of $f(x) = \dfrac{2 - 2e^x}{x}$ can be made "as close as we please" to -2 by choosing x "sufficiently close" to 0. It therefore appears that

$$\boxed{\lim_{x \to 0} \frac{2 - 2e^x}{x} = -2}.$$

x	-0.2	-0.1	-0.01	$\to 0 \leftarrow$	0.01	0.1	0.2
$f(x) = \dfrac{2 - 2e^x}{x}$	-1.81269	-1.90325	-1.99003	$f(x)$ approaches -2	-2.01003	-2.10342	-2.21403

14. The values in the table below, which have been rounded to five decimal places for display purposes, suggest that the value of $f(x) = \dfrac{\ln x}{x-1}$ can be made "as close as we please" to 1 by choosing x "sufficiently close" to 1. It therefore appears that

$$\lim_{x\to 1} \frac{\ln x}{x-1} = \boxed{1}.$$

x	0.9	0.99	0.999	$\to 1 \leftarrow$	1.001	1.01	1.1
$f(x) = \dfrac{\ln x}{x-1}$	1.05361	1.00503	1.00050	$f(x)$ approaches 1	0.99950	0.99503	0.95310

15. The values in the table below, which have been rounded to five decimal places for display purposes, suggest that the value of $f(x) = \dfrac{1-\cos x}{x}$ can be made "as close as we please" to 0 by choosing x "sufficiently close" to 0. It therefore appears that

$$\boxed{\lim_{x\to 0} \frac{1-\cos x}{x} = 0}.$$

x	-0.2	-0.1	-0.01	$\to 0 \leftarrow$	0.01	0.1	0.2
$f(x) = \dfrac{1-\cos x}{x}$	-0.09967	-0.04996	-0.00500	$f(x)$ approaches 0	0.00500	0.04996	0.09967

16. The values in the table below, which have been rounded to five decimal places for display purposes, suggest that the value of $f(x) = \dfrac{\sin x}{1+\tan x}$ can be made "as close as we please" to 0 by choosing x "sufficiently close" to 0. It therefore appears that

$$\lim_{x\to 0} \frac{\sin x}{1+\tan x} = \boxed{0}.$$

x	-0.2	-0.1	-0.01	$\to 0 \leftarrow$	0.01	0.1	0.2
$f(x) = \dfrac{\sin x}{1+\tan x}$	-0.24918	-0.11097	-0.01010	$f(x)$ approaches 0	0.00990	0.09073	0.16518

17. The graph suggests that the value of f approaches 2 as x approaches 2 from the left and as x approaches 2 from the right. Thus,

(a) $\displaystyle\lim_{x\to 2^-} f(x) = \boxed{2}$;

(b) $\displaystyle\lim_{x\to 2^+} f(x) = \boxed{2}$; and

(c) $\displaystyle\lim_{x\to 2} f(x) = \boxed{2}$.

18. The graph suggests that the value of f approaches 4 as x approaches 2 from the left and as x approaches 2 from the right. Thus,

(a) $\displaystyle\lim_{x\to 2^-} f(x) = \boxed{4}$;

(b) $\displaystyle\lim_{x\to 2^+} f(x) = \boxed{4}$; and

(c) $\displaystyle\lim_{x\to 2} f(x) = \boxed{4}$.

19. The graph suggests that the value of f approaches 3 as x approaches 2 from the left but the value of f approaches 6 as x approaches 2 from the right. Thus,

    (a) $\lim_{x\to 2^-} f(x) = \boxed{3}$;

    (b) $\lim_{x\to 2^+} f(x) = \boxed{6}$; and

    (c) $\lim_{x\to 2} f(x)$ $\boxed{\text{does not exist}}$ because there is no single number that the values of f approach when x is close to 2.

20. The graph suggests that the value of f approaches 4 as x approaches 2 from the left but the value of f approaches 2 as x approaches 2 from the right. Thus,

    (a) $\lim_{x\to 2^-} f(x) = \boxed{4}$;

    (b) $\lim_{x\to 2^+} f(x) = \boxed{2}$; and

    (c) $\lim_{x\to 2} f(x)$ $\boxed{\text{does not exist}}$ because there is no single number that the values of f approach when x is close to 2.

21. The graph suggests that, as x approaches c from the left,

$$\lim_{x\to c^-} f(x) = 1,$$

while, as x approaches c from the right,

$$\lim_{x\to c^+} f(x) = 1.$$

Because the two one-sided limits are equal, it follows that

$$\lim_{x\to c} f(x) = \boxed{1}.$$

22. The graph suggests that, as x approaches c from the left,

$$\lim_{x\to c^-} f(x) = 1,$$

while, as x approaches c from the right,

$$\lim_{x\to c^+} f(x) = 1.$$

Because the two one-sided limits are equal, it follows that

$$\lim_{x\to c} f(x) = \boxed{1}.$$

23. The graph suggests that, as x approaches c from the left,

$$\lim_{x\to c^-} f(x) = 1,$$

while, as x approaches c from the right,

$$\lim_{x\to c^+} f(x) = 1.$$

Because the two one-sided limits are equal, it follows that

$$\lim_{x\to c} f(x) = \boxed{1}.$$

24. The graph suggests that, as x approaches c from the left,
$$\lim_{x \to c^-} f(x) = 1,$$
while, as x approaches c from the right,
$$\lim_{x \to c^+} f(x) = 2.$$
$\boxed{\text{Because the two one-sided limits are not equal}}$ (that is, there is no single number that the values of f approach when x is close to c), it follows that
$$\lim_{x \to c} f(x) \boxed{\text{does not exist}}.$$

25. The graph suggests that, as x approaches c from the left,
$$\lim_{x \to c^-} f(x) = -1,$$
while, as x approaches c from the right,
$$\lim_{x \to c^+} f(x) = 1.$$
$\boxed{\text{Because the two one-sided limits are not equal}}$ (that is, there is no single number that the values of f approach when x is close to c), it follows that
$$\lim_{x \to c} f(x) \boxed{\text{does not exist}}.$$

26. The graph suggests that, as x approaches c from the left,
$$\lim_{x \to c^-} f(x) = 1,$$
while, as x approaches c from the right,
$$\lim_{x \to c^+} f(x) = 3.$$
$\boxed{\text{Because the two one-sided limits are not equal}}$ (that is, there is no single number that the values of f approach when x is close to c), it follows that
$$\lim_{x \to c} f(x) \boxed{\text{does not exist}}.$$

27. The graph suggests that, as x approaches c from the left,
$$\lim_{x \to c^-} f(x) = 2,$$
while, as x approaches c from the right,
$$\lim_{x \to c^+} f(x) = 1.$$
$\boxed{\text{Because the two one-sided limits are not equal}}$ (that is, there is no single number that the values of f approach when x is close to c), it follows that
$$\lim_{x \to c} f(x) \boxed{\text{does not exist}}.$$

28. The graph suggests that, as x approaches c from the left,

$$\lim_{x \to c^-} f(x) = 1,$$

while, as x approaches c from the right,

$$\lim_{x \to c^+} f(x) = 2.$$

$\boxed{\text{Because the two one-sided limits are not equal}}$ (that is, there is no single number that the values of f approach when x is close to c), it follows that

$$\lim_{x \to c} f(x) \; \boxed{\text{does not exist}}.$$

29. The graph of f shown below suggests that, as x approaches 2 from the left,

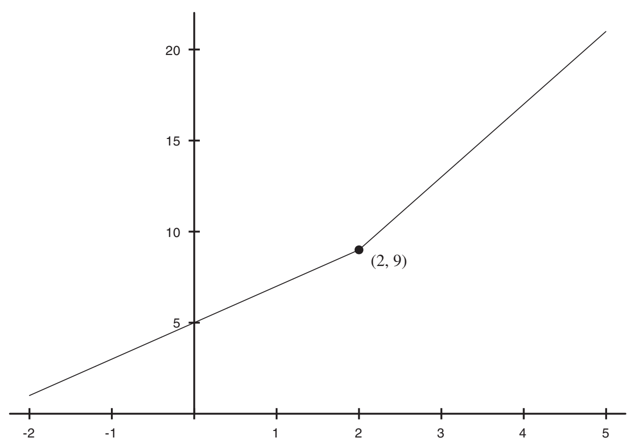
$$\lim_{x \to 2^-} f(x) = 9,$$

while, as x approaches 2 from the right,

$$\lim_{x \to 2^+} f(x) = 9.$$

Because the two one-sided limits are equal, it follows that

$$\lim_{x \to 2} f(x) = \boxed{9}.$$



30. The graph of f shown below suggests that, as x approaches 0 from the left,

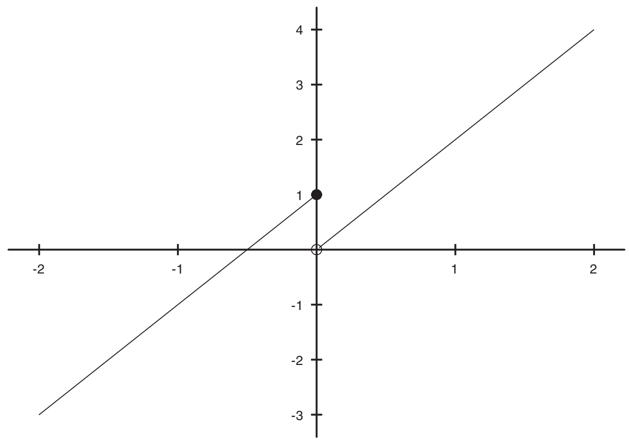
$$\lim_{x \to 0^-} f(x) = 1,$$

while, as x approaches 0 from the right,

$$\lim_{x \to 0^+} f(x) = 0.$$

Because the two one-sided limits are not equal, it follows that

$$\lim_{x \to 0} f(x) \; \boxed{\text{does not exist}}.$$



31. The graph of f shown below suggests that, as x approaches 1 from the left,

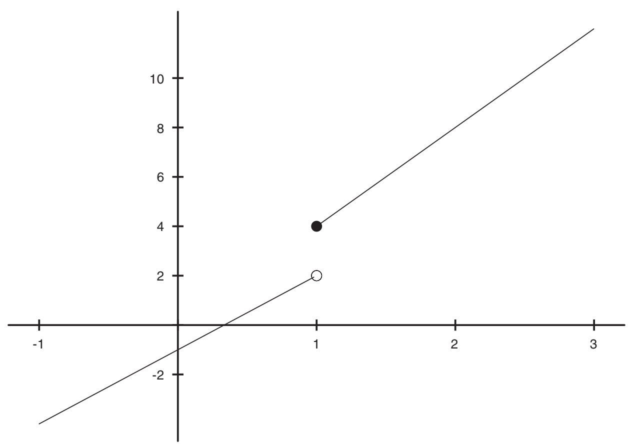
$$\lim_{x \to 1^-} f(x) = 2,$$

while, as x approaches 1 from the right,

$$\lim_{x \to 1^+} f(x) = 4.$$

Because the two one-sided limits are not equal, it follows that

$$\lim_{x \to 1} f(x) \boxed{\text{does not exist}}.$$



32. The graph of f shown below suggests that, as x approaches 2 from the left,

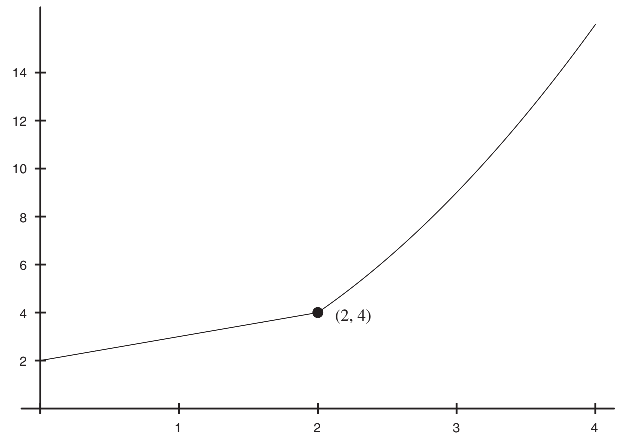
$$\lim_{x \to 2^-} f(x) = 4,$$

while, as x approaches 2 from the right,

$$\lim_{x \to 2^+} f(x) = 4.$$

Because the two one-sided limits are equal, it follows that

$$\lim_{x \to 2} f(x) = \boxed{4}.$$



33. The graph of f shown below suggests that, as x approaches 1 from the left,

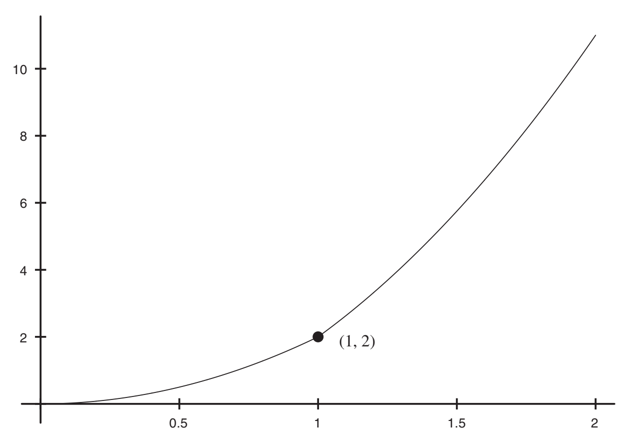
$$\lim_{x\to 1^-} f(x) = 2,$$

while, as x approaches 1 from the right,

$$\lim_{x\to 1^+} f(x) = 2.$$

Because the two one-sided limits are equal, it follows that

$$\lim_{x\to 1} f(x) = \boxed{2}.$$



34. The graph of f shown below suggests that, as x approaches -1 from the left,

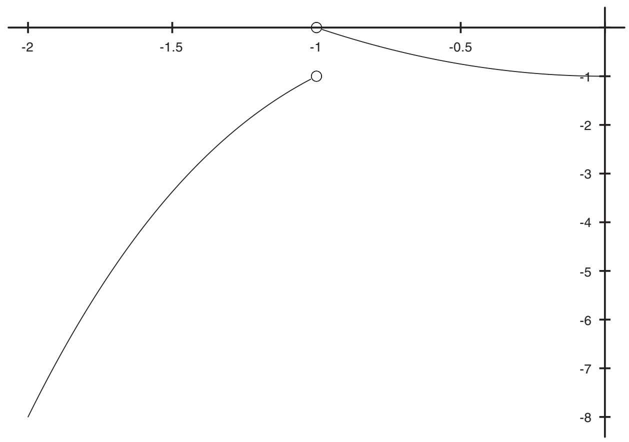
$$\lim_{x\to -1^-} f(x) = -1,$$

while, as x approaches -1 from the right,

$$\lim_{x\to -1^+} f(x) = 0.$$

Because the two one-sided limits are not equal, it follows that

$$\lim_{x\to -1} f(x) \ \boxed{\text{does not exist}}.$$



35. The graph of f shown below suggests that, as x approaches 0 from the left,

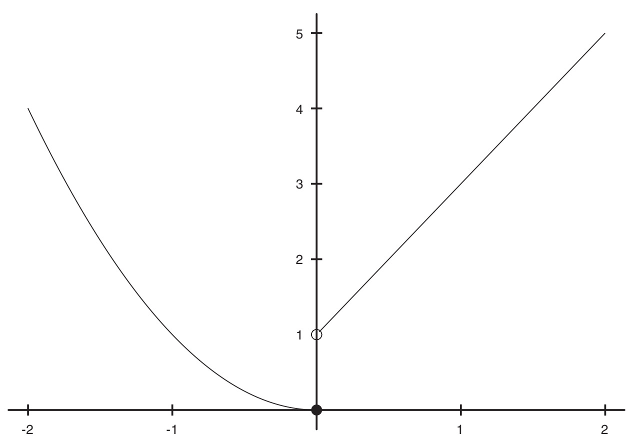
$$\lim_{x \to 0^-} f(x) = 0,$$

while, as x approaches 0 from the right,

$$\lim_{x \to 0^+} f(x) = 1.$$

Because the two one-sided limits are not equal, it follows that

$$\lim_{x \to 0} f(x) \boxed{\text{does not exist}}.$$



36. The graph of f shown below suggests that, as x approaches 1 from the left,

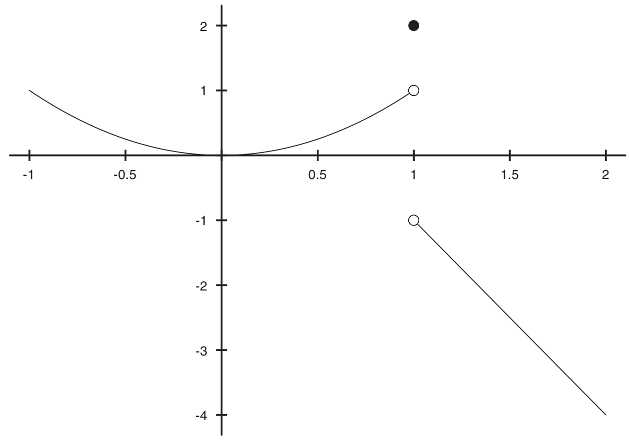
$$\lim_{x \to 1^-} f(x) = 1,$$

while, as x approaches 1 from the right,

$$\lim_{x \to 1^+} f(x) = -1.$$

Because the two one-sided limits are not equal, it follows that

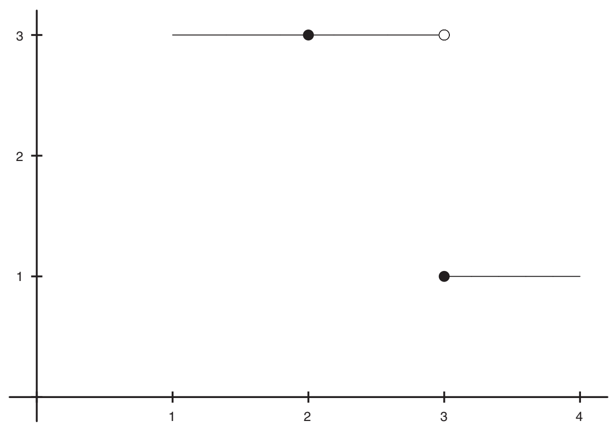
$$\lim_{x \to 1} f(x) \boxed{\text{does not exist}}.$$



Applications and Extensions

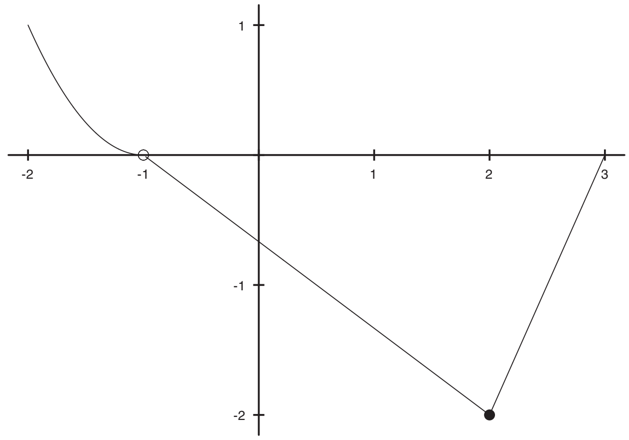
37. Answers will vary. Below is the graph of a function f for which

$$\lim_{x\to 2} f(x) = 3; \quad \lim_{x\to 3^-} f(x) = 3; \quad \lim_{x\to 3^+} f(x) = 1; \quad f(2) = 3; \quad f(3) = 1.$$



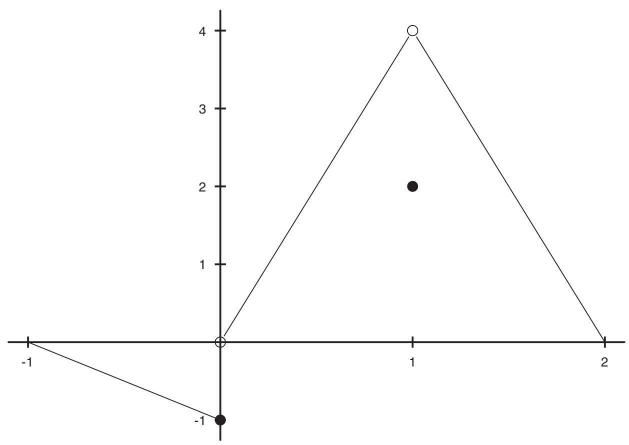
38. Answers will vary. Below is the graph of a function f for which

$$\lim_{x\to -1} f(x) = 0; \quad \lim_{x\to 2^-} f(x) = -2; \quad \lim_{x\to 2^+} f(x) = -2; \quad f(-1) \text{ is not defined}; \quad f(2) = -2.$$



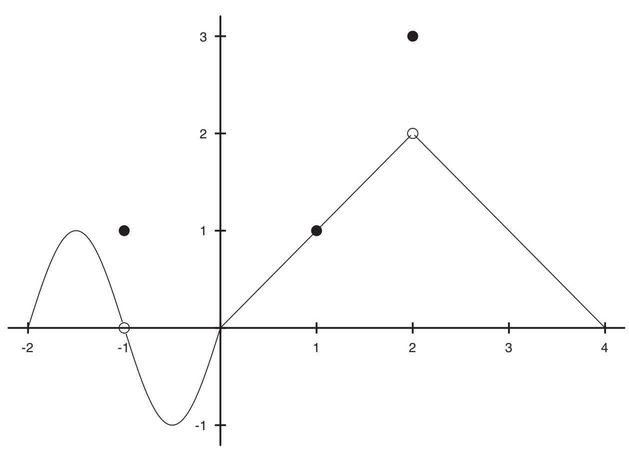
39. Answers will vary. Below is the graph of a function f for which

$$\lim_{x\to 1} f(x) = 4; \quad \lim_{x\to 0^-} f(x) = -1; \quad \lim_{x\to 0^+} f(x) = 0; \quad f(0) = -1; \quad f(1) = 2.$$



40. Answers will vary. Below is the graph of a function f for which

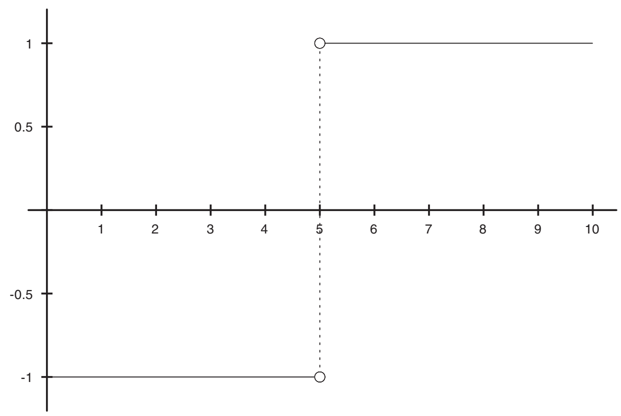
$$\lim_{x\to 2} f(x) = 2; \quad \lim_{x\to -1} f(x) = 0; \quad \lim_{x\to 1} f(x) = 1; \quad f(-1) = 1; \quad f(2) = 3.$$



41. The table of values below suggests $\lim_{x\to 5^+} \frac{|x-5|}{x-5} = \boxed{1}$.

x	$5 \leftarrow$	5.001	5.01	5.1
$f(x) = \frac{\lvert x-5\rvert}{x-5}$	$f(x)$ approaches 1	1	1	1

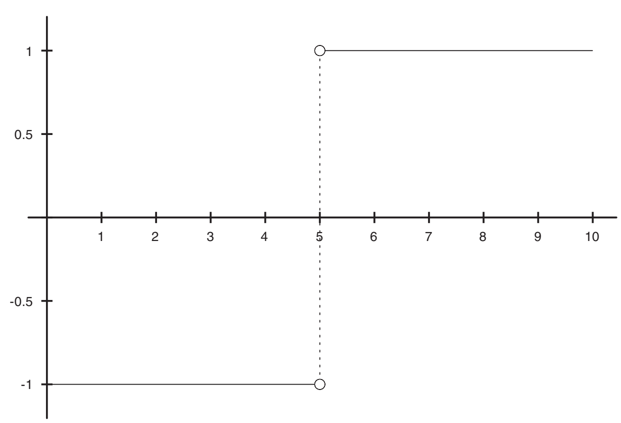
Alternately, the graph below suggests $\lim_{x\to 5^+} \frac{|x-5|}{x-5} = \boxed{1}$.



42. The table of values below suggests $\lim_{x\to 5^-} \frac{|x-5|}{x-5} = \boxed{-1}$.

x	4.9	4.99	4.999	$\rightarrow 5$
$f(x) = \frac{\lvert x-5\rvert}{x-5}$	-1	-1	-1	$f(x)$ approaches -1

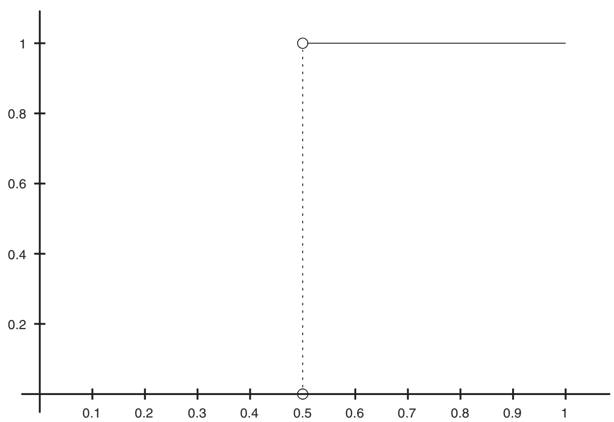
Alternately, the graph below suggests $\lim_{x\to 5^-} \frac{|x-5|}{x-5} = \boxed{-1}$.



43. The table of values below suggests $\lim\limits_{x \to \left(\frac{1}{2}\right)^-} \lfloor 2x \rfloor = \boxed{0}$.

x	0.4	0.49	0.499	$\to \frac{1}{2}$
$f(x) = \lfloor 2x \rfloor$	0	0	0	$f(x)$ approaches 0

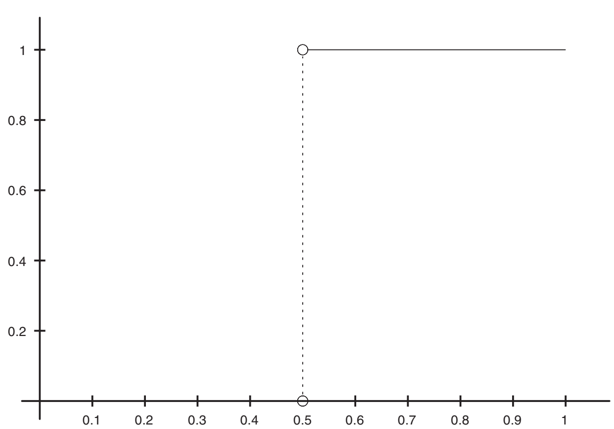
Alternately, the graph below suggests $\lim\limits_{x \to \left(\frac{1}{2}\right)^-} \lfloor 2x \rfloor = \boxed{0}$.



44. The table of values below suggests $\lim\limits_{x \to \left(\frac{1}{2}\right)^+} \lfloor 2x \rfloor = \boxed{1}$.

x	$\frac{1}{2} \leftarrow$	0.501	0.51	0.6
$f(x) = \lfloor 2x \rfloor$	$f(x)$ approaches 1	1	1	1

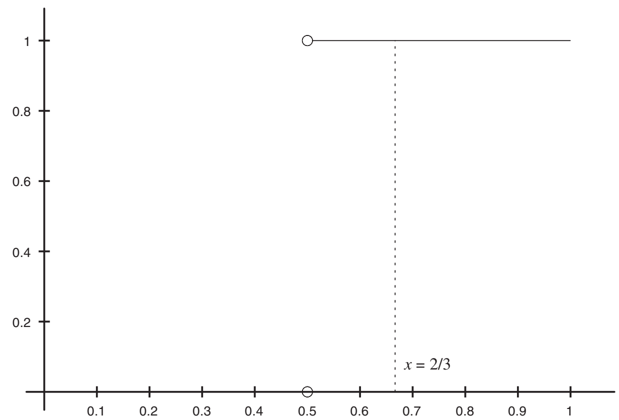
Alternately, the graph below suggests $\lim\limits_{x \to \left(\frac{1}{2}\right)^+} \lfloor 2x \rfloor = \boxed{1}$.



45. The table of values below suggests $\lim\limits_{x \to \left(\frac{2}{3}\right)^-} \lfloor 2x \rfloor = \boxed{1}$.

x	0.6	0.66	0.666	$\to \frac{2}{3}$
$f(x) = \lfloor 2x \rfloor$	1	1	1	$f(x)$ approaches 1

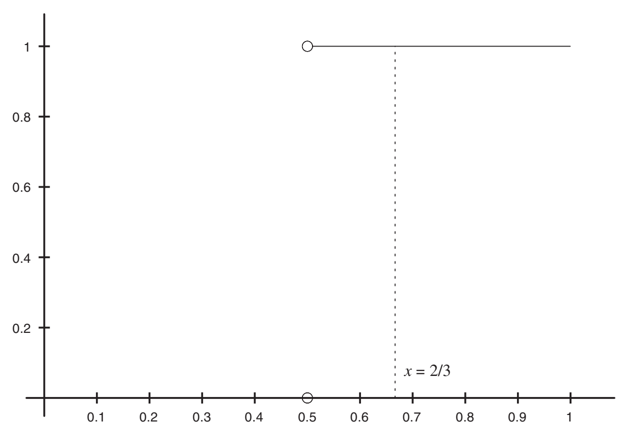
Alternately, the graph below suggests $\lim\limits_{x \to \left(\frac{2}{3}\right)^-} \lfloor 2x \rfloor = \boxed{1}$.



46. The table of values below suggests $\lim\limits_{x \to \left(\frac{2}{3}\right)^+} \lfloor 2x \rfloor = \boxed{1}$.

x	$\frac{2}{3} \leftarrow$	0.667	0.67	0.7
$f(x) = \lfloor 2x \rfloor$	$f(x)$ approaches 1	1	1	1

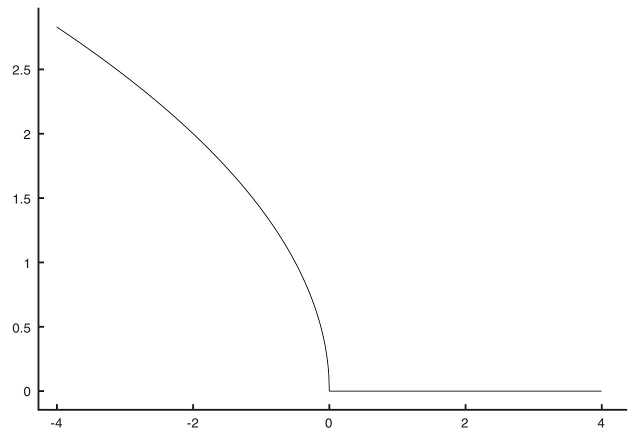
Alternately, the graph below suggests $\lim\limits_{x \to \left(\frac{2}{3}\right)^+} \lfloor 2x \rfloor = \boxed{1}$.



47. The table of values below suggests $\lim\limits_{x \to 2^+} \sqrt{|x| - x} = \boxed{0}$.

x	$2 \leftarrow$	2.001	2.01	2.1
$f(x) = \sqrt{|x| - x}$	$f(x)$ approaches 0	0	0	0

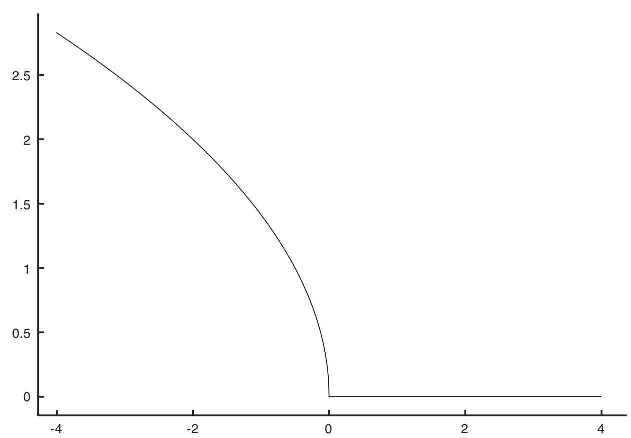
Alternately, the graph below suggests $\lim_{x \to 2^+} \sqrt{|x| - x} = \boxed{0}$.



48. The table of values below suggests $\lim_{x \to 2^-} \sqrt{|x| - x} = \boxed{0}$.

x	1.9	1.99	1.999	$\to 2$
$f(x) = \sqrt{\lvert x\rvert - x}$	0	0	0	$f(x)$ approaches 0

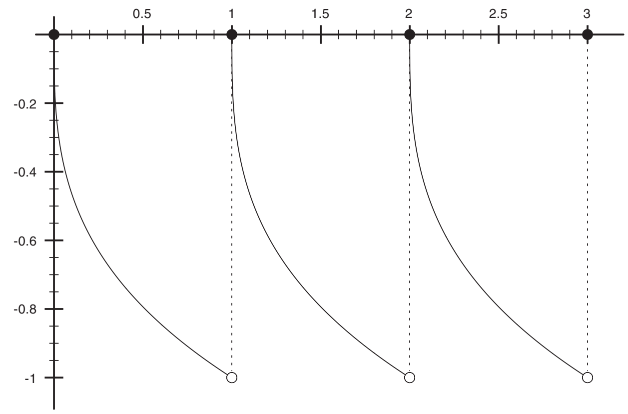
Alternately, the graph below suggests $\lim_{x \to 2^-} \sqrt{|x| - x} = \boxed{0}$.



49. The table of values below suggests $\lim_{x \to 2^+} \sqrt[3]{\lfloor x \rfloor - x} = \boxed{0}$.

x	$2 \leftarrow$	2.000000001	2.000001	2.001
$f(x) = \sqrt[3]{\lfloor x \rfloor - x}$	$f(x)$ approaches 0	-0.001	-0.01	-0.1

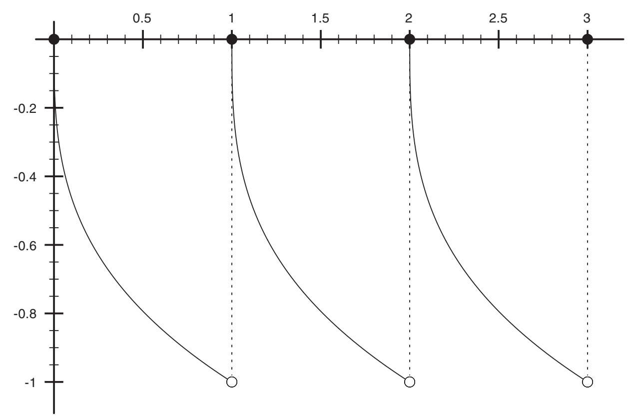
Alternately, the graph below suggests $\lim_{x \to 2^+} \sqrt[3]{\lfloor x \rfloor - x} = \boxed{0}$.



50. The table of values below, in which the function values have been rounded to five decimal places for display purposes, suggests $\lim_{x\to 2^-} \sqrt[3]{\lfloor x\rfloor - x} = \boxed{-1}$.

x	1.9	1.99	1.999	$\to 2$
$f(x) = \sqrt[3]{\lfloor x\rfloor - x}$	-0.96549	-0.99666	-0.99967	$f(x)$ approaches -1

Alternately, the graph below suggests $\lim_{x\to 2^-} \sqrt[3]{\lfloor x\rfloor - x} = \boxed{-1}$.



51. (a) The secant line containing the points $(2, 12)$ and $(3, 27)$ has a slope of

$$m_{\text{sec}} = \frac{27-12}{3-2} = \frac{15}{1} = \boxed{15}.$$

(b) The secant line containing the points $(2, 12)$ and $(x, f(x))$ for $x \neq 2$ has a slope of

$$m_{\text{sec}} = \frac{3x^2-12}{x-2} = \frac{3(x-2)(x+2)}{x-2} = \boxed{3(x+2)}.$$

(c) The values in the table below suggest that the slope of the tangent line to the graph of f at 2 is

$$\boxed{\lim_{x\to 2} m_{\text{sec}} = 12}.$$

x	1.9	1.99	1.999	$\to 2 \leftarrow$	2.001	2.01	2.1
m_{sec}	11.7	11.97	11.997	m_{sec} approaches 12	12.003	12.03	12.3

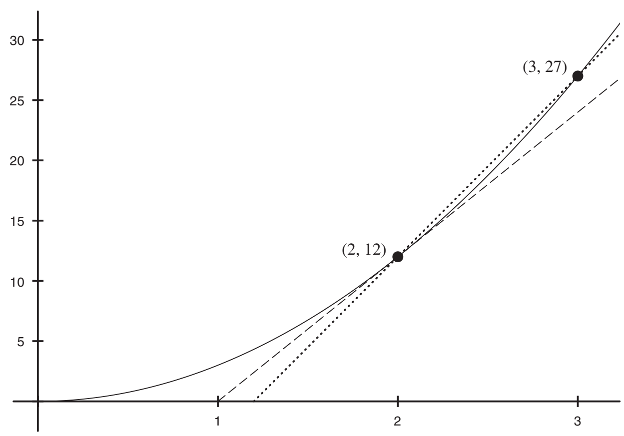
(d) The secant line from part (a) has slope 15 and passes through the point $(2, 12)$. The equation of this secant line is therefore

$$y - 12 = 15(x - 2) \quad \text{or} \quad y = 15x - 18.$$

The tangent line at $x = 2$ has slope 12 and also passes through the point $(2, 12)$; the equation of this line is

$$y - 12 = 12(x - 2) \quad \text{or} \quad y = 12x - 12.$$

The figure below displays the graph of f as the solid curve, the graph of the tangent line as the dashed curve, and the graph of the secant line as the dotted curve.



52. (a) The secant line containing the points $(2, 8)$ and $(3, 27)$ has a slope of

$$m_{\text{sec}} = \frac{27 - 8}{3 - 2} = \frac{19}{1} = \boxed{19}.$$

(b) The secant line containing the points $(2, 8)$ and $(x, f(x))$ for $x \neq 2$ has a slope of

$$m_{\text{sec}} = \frac{x^3 - 8}{x - 2} = \frac{(x - 2)(x^2 + 2x + 4)}{x - 2} = \boxed{x^2 + 2x + 4}.$$

(c) The values in the table below suggest that the slope of the tangent line to the graph of f at 2 is

$$\lim_{x \to 2} m_{\text{sec}} = \boxed{12}.$$

x	1.9	1.99	1.999	$\to 2 \leftarrow$	2.001	2.01	2.1
m_{sec}	11.41	11.9401	11.994001	m_{sec} approaches 12	12.006001	12.0601	12.61

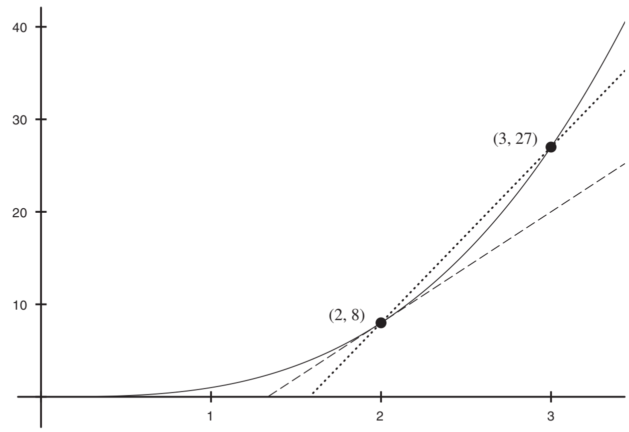
(d) The secant line from part (a) has slope 19 and passes through the point $(2, 8)$. The equation of this secant line is therefore

$$y - 8 = 19(x - 2) \quad \text{or} \quad y = 19x - 30.$$

The tangent line at $x = 2$ has slope 12 and also passes through the point $(2, 8)$; the equation of this line is

$$y - 8 = 12(x - 2) \quad \text{or} \quad y = 12x - 16.$$

The figure below displays the graph of f as the solid curve, the graph of the tangent line as the dashed curve, and the graph of the secant line as the dotted curve.



53. (a) Let $f(x) = \frac{1}{2}x^2 - 1$. The slope of the secant line containing the points $P = (2, f(2))$ and $Q = (2 + h, f(2 + h))$ is

$$\begin{aligned} m_{\text{sec}} &= \frac{f(2+h) - f(2)}{(2+h) - 2} = \frac{\frac{1}{2}(2+h)^2 - 1 - \left(\frac{1}{2}2^2 - 1\right)}{h} \\ &= \frac{\frac{1}{2}(4 + 4h + h^2) - 1 - \frac{1}{2}2^2 + 1}{h} = \frac{2 + 2h + \frac{1}{2}h^2 - 2}{h} = \frac{2h + \frac{1}{2}h^2}{h} = \boxed{2 + \frac{1}{2}h}, \end{aligned}$$

$\boxed{\text{provided } h \neq 0}$.

(b) Using the result from part (a),

h	-0.5	-0.1	-0.001	0.001	0.1	0.5
m_{sec}	1.75	1.95	1.9995	2.0005	2.05	2.25

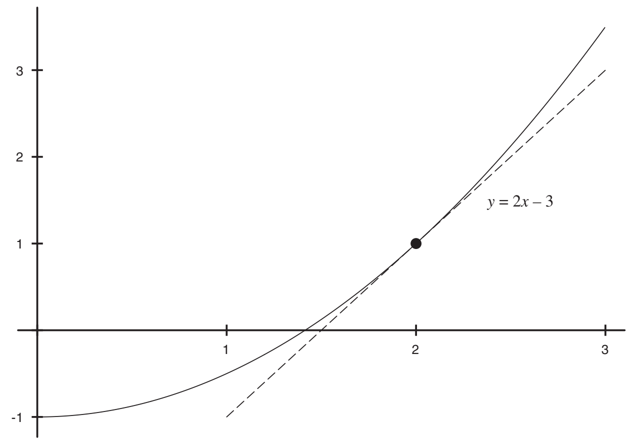
(c) The table from part (b) suggests that $\boxed{\lim_{h \to 0} m_{\text{sec}} = 2}$.

(d) Because the limit of the slope of the secant line is 2, the slope of the line tangent to the graph of f at the point $P = (2, f(2))$ is $\boxed{2}$.

(e) The tangent line to f at the point $P = (2, f(2))$ has slope 2 and contains the point $(2, f(2)) = (2, 1)$. The equation of the tangent line is therefore

$$y - 1 = 2(x - 2) \quad \text{or} \quad y = 2x - 3.$$

The figure below displays the graph of f as the solid curve and the graph of the tangent line as the dashed curve.



54. (a) Let $f(x) = x^2 - 1$. The slope of the secant line containing the points $P = (-1, f(-1))$ and $Q = (-1+h, f(-1+h))$ is

$$\begin{aligned} m_{\text{sec}} &= \frac{f(-1+h) - f(-1)}{(-1+h) - (-1)} = \frac{(-1+h)^2 - 1 - \left((-1)^2 - 1\right)}{h} \\ &= \frac{1 - 2h + h^2 - 1 - 1 + 1}{h} = \frac{-2h + h^2}{h} = \boxed{-2+h}, \end{aligned}$$

provided $h \neq 0$.

(b) Using the result from part (a),

h	-0.1	-0.01	-0.001	-0.0001	0.0001	0.001	0.01	0.1
m_{sec}	-2.1	-2.01	-2.001	-2.0001	-1.9999	-1.999	-1.99	-1.9

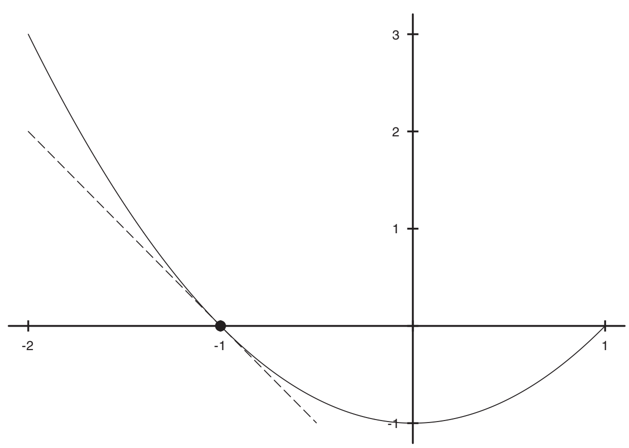
(c) The table from part (b) suggests that $\lim\limits_{h \to 0} m_{\text{sec}} = \boxed{-2}$.

(d) Because the limit of the slope of the secant line is -2, the slope of the line tangent to the graph of f at the point $P = (-1, f(-1))$ is $\boxed{-2}$.

(e) The tangent line to f at the point $P = (-1, f(-1))$ has slope -2 and contains the point $(-1, f(-1)) = (-1, 0)$. The equation of the tangent line is therefore

$$y - 0 = -2(x+1) \quad \text{or} \quad y = -2x - 2.$$

The figure below displays the graph of f as the solid curve and the graph of the tangent line as the dashed curve.



55. (a) $\boxed{\text{The values in the table below suggest that}}$

$$\boxed{\lim_{x\to 0} \cos\frac{\pi}{x} = 1}.$$

x	$-\frac{1}{2}$	$-\frac{1}{4}$	$-\frac{1}{8}$	$-\frac{1}{10}$	$-\frac{1}{12}$	$\to 0 \leftarrow$	$\frac{1}{12}$	$\frac{1}{10}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$
$f(x) = \cos\frac{\pi}{x}$	1	1	1	1	1	$f(x)$ approaches 1	1	1	1	1	1

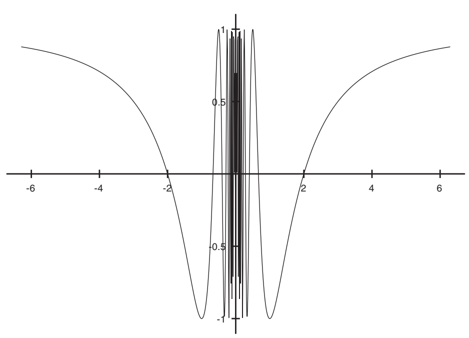
(b) The values in the table below suggest that

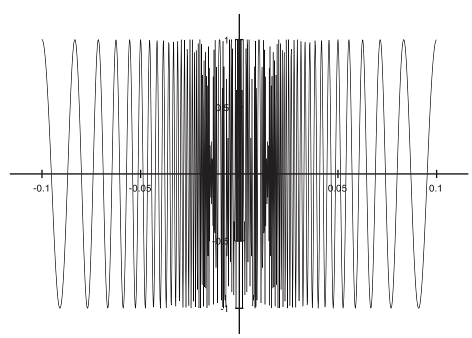
$$\lim_{x\to 0} \cos\frac{\pi}{x} = -1.$$

x	-1	$-\frac{1}{3}$	$-\frac{1}{5}$	$-\frac{1}{7}$	$-\frac{1}{9}$	$\to 0 \leftarrow$	$\frac{1}{9}$	$\frac{1}{7}$	$\frac{1}{5}$	$\frac{1}{3}$	1
$f(x) = \cos\frac{\pi}{x}$	-1	-1	-1	-1	-1	f approaches -1	-1	-1	-1	-1	-1

(c) Because the values obtained in parts (a) and (b) are not equal, we conclude $\boxed{\text{the limit does not exist}}$. As x gets closer to 0, the argument to the cosine function, $\frac{\pi}{x}$, becomes unbounded. Consequently, the cosine function oscillates repeatedly between -1 and 1 and never approaches a single value.
A table of values can be a useful tool for investigating a limit, but should only be viewed as providing evidence of a possible value for a limit. A final conclusion regarding a limit should be based on the properties of limits that will be developed in subsequent sections of this chapter.

(d) The figure below left displays the graph of f with an x-window of $(-2\pi, 2\pi)$, and the figure below right displays the graph of f with an x-window of $(-0.1, 0.1)$. Using either graph, it appears that $\boxed{\lim_{x\to 0} f(x) \text{ does not exist}}$ because the function value does not approach a single number. Instead, the function seems to oscillate more rapidly between -1 and 1 as x gets closer to 0.





56. (a) The values in the table below suggest that

$$\lim_{x\to 0} \cos\frac{\pi}{x^2} = \boxed{1}.$$

x	-0.1	-0.01	-0.001	-0.0001	$\to 0 \leftarrow$	0.0001	0.001	0.01	0.1
$f(x) = \cos\frac{\pi}{x^2}$	1	1	1	1	$f(x)$ approaches 1	1	1	1	1

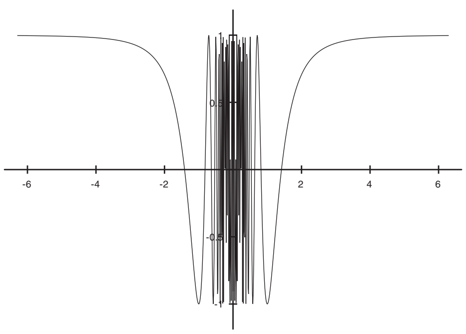
(b) The values in the table below suggest that

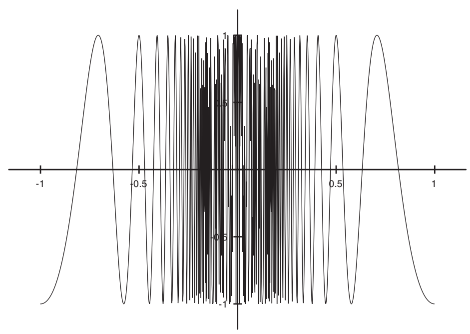
$$\lim_{x\to 0} \cos\frac{\pi}{x^2} = \boxed{\frac{\sqrt{2}}{2}}.$$

x	$-\frac{2}{3}$	$-\frac{2}{5}$	$-\frac{2}{7}$	$-\frac{2}{9}$	$\to 0 \leftarrow$	$\frac{2}{9}$	$\frac{2}{7}$	$\frac{2}{5}$	$\frac{2}{3}$
$f(x) = \cos\frac{\pi}{x^2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$f(x)$ approaches $\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$

(c) Because the values obtained in parts (a) and (b) are not equal, we conclude $\boxed{\text{the limit does not exist}}$. As x gets closer to 0, the argument to the cosine function, $\frac{\pi}{x^2}$, becomes unbounded. Consequently, the cosine function oscillates repeatedly between -1 and 1 and never approaches a single value.
A table of values can be a useful tool for investigating a limit, but should only be viewed as providing evidence of a possible value for a limit. A final conclusion regarding a limit should be based on the properties of limits that will be developed in subsequent sections of this chapter.

(d) The figure below left displays the graph of f with an x-window of $(-2\pi, 2\pi)$, and the figure below right displays the graph of f with an x-window of $(-1, 1)$. Using either graph, it appears that $\boxed{\lim_{x\to 0} f(x) \text{ does not exist}}$ because the function value does not approach a single number. Instead, the function seems to oscillate more rapidly between -1 and 1 as x gets closer to 0.





57. (a) The values in the table below suggest that

$$\boxed{\lim_{x\to 2} \frac{x-8}{2} = -3}.$$

x	1.9	1.99	1.999	$\to 2 \leftarrow$	2.001	2.01	2.1
$f(x) = \frac{x-8}{2}$	-3.05	-3.005	-3.0005	$f(x)$ approaches -3	-2.9995	-2.995	-2.95

(b) The function $f(x) = \frac{x-8}{2}$ is within 0.1 of -3 provided $|f(x) - (-3)| \le 0.1$; that is,

$$\begin{aligned} \left|\frac{x-8}{2} + 3\right| &\le 0.1 \\ |(x-8)+6| &\le 0.2 \\ |x-2| &\le 0.2. \end{aligned}$$

Thus, if $\boxed{1.8 \le x \le 2.2}$, then $f(x)$ is within 0.1 of -3.

(c) The function $f(x) = \dfrac{x-8}{2}$ is within 0.01 of -3 provided $|f(x) - (-3)| \leq 0.01$; that is,

$$\begin{aligned} \left|\frac{x-8}{2} + 3\right| &\leq 0.01 \\ |(x-8)+6| &\leq 0.02 \\ |x-2| &\leq 0.02. \end{aligned}$$

Thus, if $\boxed{1.98 \leq x \leq 2.02}$, then $f(x)$ is within 0.01 of -3.

58. (a) The values in the table below suggest that

$$\lim_{x \to 2} (5 - 2x) = \boxed{1}.$$

x	1.9	1.99	1.999	$\rightarrow 2 \leftarrow$	2.001	2.01	2.1
$f(x) = 5 - 2x$	1.2	1.02	1.002	$f(x)$ approaches 1	0.998	0.98	0.8

(b) The function $f(x) = 5 - 2x$ is within 0.1 of 1 provided $|f(x) - 1| \leq 0.1$; that is,

$$\begin{aligned} |(5-2x)-1| &\leq 0.1 \\ |4-2x| &\leq 0.1 \\ |-2(x-2)| &\leq 0.1 \\ |x-2| &\leq 0.05. \end{aligned}$$

Thus, if $\boxed{1.95 \leq x \leq 2.05}$, then $f(x)$ is within 0.1 of 1.

(c) The function $f(x) = 5 - 2x$ is within 0.01 of 1 provided $|f(x) - 1| \leq 0.01$; that is,

$$\begin{aligned} |(5-2x)-1| &\leq 0.01 \\ |4-2x| &\leq 0.01 \\ |-2(x-2)| &\leq 0.01 \\ |x-2| &\leq 0.005. \end{aligned}$$

Thus, if $\boxed{1.995 \leq x \leq 2.005}$, then $f(x)$ is within 0.01 of 1.

59. (a) Because the Postal Service rounds the weight of the letter up to the next whole number of ounces, the first-class postage charged is

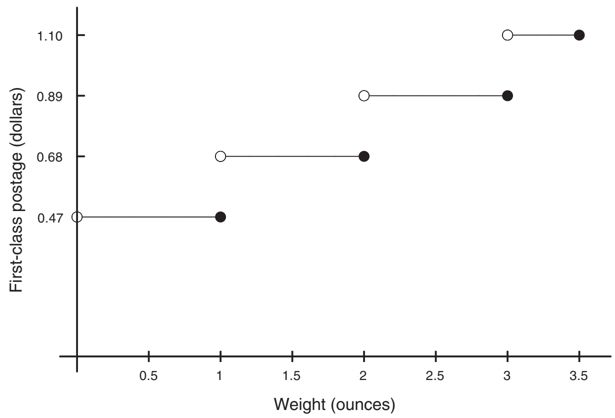
$$\boxed{C(w) = \begin{cases} 0.47, & \text{if } 0 < w \leq 1 \\ 0.68, & \text{if } 1 < w \leq 2 \\ 0.89, & \text{if } 2 < w \leq 3 \\ 1.10, & \text{if } 3 < w \leq 3.5. \end{cases}}$$

where postage is measured in dollars and weight is measured in ounces. This can be written compactly in terms of the ceiling function as

$$C(w) = 0.47 + 0.21\lceil w - 1 \rceil.$$

(b) The domain of C is the set $\boxed{\{w | 0 < w \leq 3.5\}}$.

(c) The graph of the postage function C is shown below.



(d) The graph of C suggests that

$$\boxed{\lim_{w\to 2^-} C(w) = 0.68 \quad \text{and} \quad \lim_{w\to 2^+} C(w) = 0.89}.$$

Because these two one-sided limits are not equal, this suggests that $\boxed{\lim_{w\to 2} C(w) \text{ does not exist}}$.

(e) The graph of C suggests that $\boxed{\lim_{w\to 0^+} C(w) = 0.47}$.

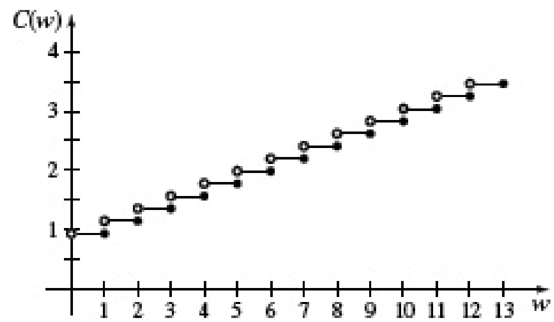
(f) The graph of C suggests that $\boxed{\lim_{w\to 3.5^-} C(w) = 1.10}$.

60. (a) Let w denote the weight (in ounces) of the envelope. For envelopes weighing less than or equal to 1 ounce, the cost is \$0.94. For $1 < w \leq 2$, the cost is \$0.94 plus an additional fee of \$0.21 for a total of \$1.15. For $2 < w \leq 3$, the cost is \$0.94 plus an additional fee of $2 \times \$0.21$ for a total of \$1.36. For $3 < w \leq 4$, the cost is \$0.94 plus an additional fee of $3 \times \$0.21$ for a total of \$1.47. Continuing in this manner gives the piecewise function defined below. This function applies to envelopes weighing up to and including 13 ounces. First-class rates do not apply to large envelopes weighing more than 13 ounces.

$$C(w) = \begin{cases} \$0.94 & \text{if} \quad 0 < w \leq 1 \\ \$1.15 & \text{if} \quad 1 < w \leq 2 \\ \$1.36 & \text{if} \quad 2 < w \leq 3 \\ \$1.57 & \text{if} \quad 3 < w \leq 4 \\ \$1.78 & \text{if} \quad 4 < w \leq 5 \\ \$1.99 & \text{if} \quad 5 < w \leq 6 \\ \$2.20 & \text{if} \quad 6 < w \leq 7 \\ \$2.41 & \text{if} \quad 7 < w \leq 8 \\ \$2.62 & \text{if} \quad 8 < w \leq 9 \\ \$2.83 & \text{if} \quad 9 < w \leq 10 \\ \$3.04 & \text{if} \quad 10 < w \leq 11 \\ \$3.25 & \text{if} \quad 11 < w \leq 12 \\ \$3.46 & \text{if} \quad 12 < w \leq 13 \end{cases}$$

(b) The domain of the function $\{w \,|\, 0 < w \leq 13\}$. The weight of these envelopes can be any positive real number up to and including 13 ounces.

(c) The graph of the piecewise function is pictured below.



(d) From the graph, $\lim_{w\to 1^-} C(w) = \$0.94$ and $\lim_{w\to 1^+} C(w) = \$1.15$. Since $\lim_{w\to 1^-} C(w) \neq \lim_{w\to 1^+} C(w)$, we conclude $\lim_{w\to 1} C(w)$ does not exist.

(e) From the graph, $\lim_{w\to 12^-} C(w) = \$3.25$ and $\lim_{w\to 12^+} C(w) = \$3.46$. Since $\lim_{w\to 12^-} C(w) \neq \lim_{w\to 12^+} C(w)$, we conclude $\lim_{w\to 1} C(w)$ does not exist.

(f) From the graph, $\lim_{w\to 0^+} C(w) = \$0.94$. As w gets closer to 0 from the right hand side, $C(w)$ is \$0.94.

(g) From the graph, $\lim_{w\to 13^-} C(w) = \$3.46$. As w gets closer to 13 ounces from the left hand side, $C(w)$ is \$3.46.

61. Let $S(t)$ denote a student's final exam score as a function of the time t that the student studies.

(a) Professor Smith's claim can be written symbolically as

$$\boxed{\lim_{t\to 7} S(t) = 100}.$$

(b) Using the ϵ-δ definition of a limit, Professor Smith's can be written as: $\boxed{\text{given any } \epsilon > 0, \text{ there is a number } \delta > 0 \text{ so that whenever } 0 < |t-7| < \delta \text{ then}}$ $\boxed{|S(t) - 100| < \epsilon}$.

62. If $h = x - c$, then $x = h + c$ and h approaches zero as x approaches c. Thus,

$$m_{\tan} = \lim_{x\to c} \frac{f(x) - f(c)}{x - c} = \lim_{h\to 0} \frac{f(c+h) - f(c)}{(c+h) - c} = \boxed{\lim_{h\to 0} \frac{f(c+h) - f(c)}{h}}.$$

63. $\boxed{\text{No, the value of the function } f \textbf{ at } x = 2 \text{ has no bearing on } \lim_{x\to 2} f(x)}$. The limit examines the value of f as x **approaches** 2 but is not equal to 2.

64. $\boxed{\text{No}}$, $\lim_{x\to 2} f(x)$ conveys no information about the value of f **at** $x = 2$. The limit examines the value of f as x **approaches** 2 but is not equal to 2.

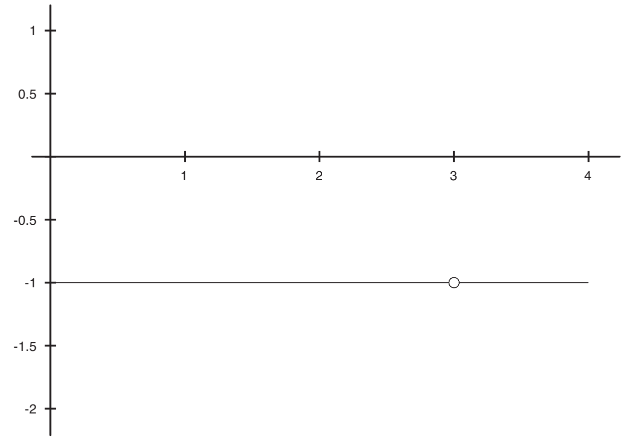
65. (a) Because

$$\frac{x-3}{3-x} = \frac{-(3-x)}{3-x} = -1$$

provided $x \neq 3$, the graph of $f(x)$ is the $\boxed{\text{horizontal line } y = -1 \text{ excluding the point}}$ $\boxed{(3, -1)}$.

(b) The graph of f (see below) suggests that

$$\boxed{\lim_{x\to 3^-} f(x) = -1 \quad \text{and} \quad \lim_{x\to 3^+} f(x) = -1}.$$



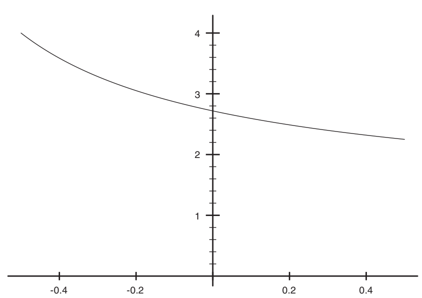
(c) Because the two one-sided limits are equal, $\boxed{\text{the graph suggests that } \lim_{x\to 3} f(x)}$ $\boxed{\text{exists and is equal to } -1}$.

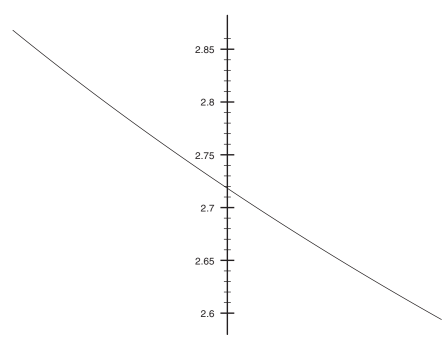
66. (a) The values in the table below, which have been rounded to five decimal places for display purposes, suggest that

$$\lim_{x\to 0}(1+x)^{1/x} \approx \boxed{2.72}.$$

x	-0.01	-0.001	-0.0001	$\to 0 \leftarrow$	0.0001	0.001	0.01
$f(x) = (1+x)^{1/x}$	2.731999	2.719642	2.718418		2.718146	2.716924	2.704814

(b) The figure below left displays the graph of g for $-0.5 \le x \le 0.5$. The figure below right displays a closer view of the graph of g, focusing on the y-intercept.





(c) The table in part (a) and the graphs in part (b) all suggest that

$$\lim_{x\to 0}(1+x)^{1/x} \approx \boxed{2.72}.$$

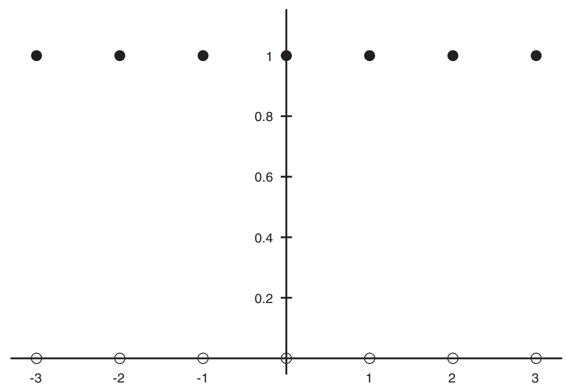
(d) Using the computer algebra system *Mathematica*,

$$\lim_{x\to 0}(1+x)^{1/x} = \boxed{e \approx 2.71828}.$$

Challenge Problems

67. The graph of the function f is shown below. Note that except when x is an integer, the graph of f coincides with the x-axis. The graph suggests that

$$\lim_{x \to 2} f(x) = \boxed{0}.$$



68. The graph of f (see Problem 67) suggests

$$\lim_{x \to 1/2} f(x) = \boxed{0}.$$

69. The graph of f (see Problem 67) suggests

$$\lim_{x \to 3} f(x) = \boxed{0}.$$

70. The graph of f (see Problem 67) suggests

$$\lim_{x \to 0} f(x) = \boxed{0}.$$

1.2 Limits of Functions Using Properties of Limits

Concepts and Vocabulary

1. (a) $\lim\limits_{x \to 4} (-3) = \boxed{-3}$.

   (b) $\lim\limits_{x \to 0} \pi = \boxed{\pi}$.

2. If $\lim\limits_{x \to c} f(x) = 3$, then $\lim\limits_{x \to c} [f(x)]^5 = 3^5 = \boxed{243}$.

3. If $\lim\limits_{x \to c} f(x) = 64$, then $\lim\limits_{x \to c} \sqrt[3]{f(x)} = \sqrt[3]{64} = \boxed{4}$.

4. (a) $\lim\limits_{x \to -1} x = \boxed{-1}$.

   (b) $\lim\limits_{x \to e} x = \boxed{e}$.

5. (a) $\lim\limits_{x \to 0} (x - 2) = 0 - 2 = \boxed{-2}$.

(b) $\lim\limits_{x \to \frac{1}{2}} (3+x) = 3 + \frac{1}{2} = \boxed{\frac{7}{2}}$.

6. (a) $\lim\limits_{x \to 2} (-3x) = -3(2) = \boxed{-6}$.

   (b) $\lim\limits_{x \to 0} (3x) = 3(0) = \boxed{0}$.

7. $\boxed{\text{True}}$. If p is a polynomial, then $\lim\limits_{x \to 5} p(x) = p(5)$.

8. If the domain of a rational function R is $\{x | x \neq 0\}$, then $\lim\limits_{x \to 2} R(x) = R(\boxed{2})$.

9. $\boxed{\text{False}}$. Properties of limits **can** be used for one-sided limits.

10. $\boxed{\text{True}}$. If $f(x) = \dfrac{(x+1)(x+2)}{x+1}$ and $g(x) = x+2$, then $\lim\limits_{x \to -1} f(x) = \lim\limits_{x \to -1} g(x) = 1$.

Skill Building

11. $\lim\limits_{x \to 3} [2(x+4)] = 2 \lim\limits_{x \to 3} (x+4) = 2\left[\lim\limits_{x \to 3} x + \lim\limits_{x \to 3} 4\right] = 2(3+4) = \boxed{14}$.

12. $\lim\limits_{x \to -2} [3(x+1)] = 3 \lim\limits_{x \to -2} (x+1) = 3\left[\lim\limits_{x \to -2} x + \lim\limits_{x \to -2} 1\right] = 3(-2+1) = \boxed{-3}$.

13.
$$\begin{aligned}
\lim_{x \to -2} [x(3x-1)(x+2)] &= \lim_{x \to -2} x \cdot \lim_{x \to -2} (3x-1) \cdot \lim_{x \to -2} (x+2) \\
&= \lim_{x \to -2} x \cdot \left(3 \lim_{x \to -2} x - \lim_{x \to -2} 1\right) \cdot \left(\lim_{x \to -2} x + \lim_{x \to -2} 2\right) \\
&= -2[3(-2)-1][-2+2] = -2(-7)(0) = \boxed{0}.
\end{aligned}$$

14.
$$\begin{aligned}
\lim_{x \to -1} [x(x-1)(x+10)] &= \lim_{x \to -1} x \cdot \lim_{x \to -1} (x-1) \cdot \lim_{x \to -1} (x+10) \\
&= \lim_{x \to -1} x \cdot \left(\lim_{x \to -1} x - \lim_{x \to -1} 1\right) \cdot \left(\lim_{x \to -1} x + \lim_{x \to -1} 10\right) \\
&= -1(-1-1)(-1+10) = -1(-2)(9) = \boxed{18}.
\end{aligned}$$

15. $\lim\limits_{t \to 1} (3t-2)^3 = \left[\lim\limits_{t \to 1} (3t-2)\right]^3 = [3(1)-2]^3 = 1^3 = \boxed{1}$.

16. $\lim\limits_{x \to 0} (-3x+1)^2 = \left[\lim\limits_{x \to 0} (-3x+1)\right]^2 = [-3(0)+1]^2 = 1^2 = \boxed{1}$.

17. $\lim\limits_{x \to 4} 3\sqrt{x} = 3\sqrt{\lim\limits_{x \to 4} x} = 3\sqrt{4} = \boxed{6}$.

18. $\lim\limits_{x \to 8} \left(\frac{1}{4}\sqrt[3]{x}\right) = \frac{1}{4}\sqrt[3]{\lim\limits_{x \to 8} x} = \frac{1}{4}\sqrt[3]{8} = \frac{1}{4} \cdot 2 = \boxed{\frac{1}{2}}$.

19. $\lim\limits_{x \to 3} \sqrt{5x-4} = \sqrt{\lim\limits_{x \to 3} (5x-4)} = \sqrt{5(3)-4} = \boxed{\sqrt{11}}$.

20. $\lim\limits_{t \to 2} \sqrt{3t+4} = \sqrt{\lim\limits_{t \to 2} (3t+4)} = \sqrt{3(2)+4} = \boxed{\sqrt{10}}$.

21.

$$\begin{aligned}\lim_{t\to 2}[t\sqrt{(5t+3)(t+4)}] &= \lim_{t\to 2} t \cdot \lim_{t\to 2} \sqrt{(5t+3)(t+4)} = 2\sqrt{\lim_{t\to 2}[(5t+3)(t+4)]} \\ &= 2\sqrt{\lim_{t\to 2}(5t+3)\cdot \lim_{t\to 2}(t+4)} = 2\sqrt{(5(2)+3)(2+4)} = \boxed{2\sqrt{78}}.\end{aligned}$$

22.

$$\begin{aligned}\lim_{t\to -1}[t\sqrt[3]{(t+1)(2t-1)}] &= \lim_{t\to -1} t \cdot \lim_{t\to -1} \sqrt[3]{(t+1)(2t-1)} = -1\sqrt[3]{\lim_{t\to -1}[(t+1)(2t-1)]} \\ &= -\sqrt[3]{\lim_{t\to -1}(t+1)\cdot \lim_{t\to -1}(2t-1)} = -\sqrt[3]{(-1+1)(2(-1)-1)} \\ &= -\sqrt[3]{0} = \boxed{0}.\end{aligned}$$

23.

$$\begin{aligned}\lim_{x\to 9}(\sqrt{x}+x+4)^{1/2} &= \left[\lim_{x\to 9}(\sqrt{x}+x+4)\right]^{1/2} = \left[\lim_{x\to 9}\sqrt{x} + \lim_{x\to 9} x + \lim_{x\to 9} 4\right]^{1/2} \\ &= \left[\sqrt{\lim_{x\to 9} x} + 9 + 4\right]^{1/2} = \left(\sqrt{9}+13\right)^{1/2} = [16]^{1/2} = \boxed{4}.\end{aligned}$$

24.

$$\begin{aligned}\lim_{t\to 2}(t\sqrt{2t}+4)^{1/3} &= \left[\lim_{t\to 2}(t\sqrt{2t}+4)\right]^{1/3} = \left[\lim_{t\to 2} t\sqrt{2t} + \lim_{t\to 2} 4\right]^{1/3} = \left[\lim_{t\to 2} t\cdot \lim_{t\to 2}\sqrt{2t}+4\right]^{1/3} \\ &= \left[2\cdot\sqrt{\lim_{t\to 2}(2t)}+4\right]^{1/3} = \left[2\sqrt{2(2)}+4\right]^{1/3} = 8^{1/3} = \boxed{2}.\end{aligned}$$

25.

$$\begin{aligned}\lim_{t\to -1}[4t(t+1)]^{2/3} &= \left[\lim_{t\to -1} 4t(t+1)\right]^{2/3} = \left[\lim_{t\to -1}(4t)\cdot \lim_{t\to -1}(t+1)\right]^{2/3} \\ &= \left[4\lim_{t\to -1} t\cdot\left(\lim_{t\to -1} t + \lim_{t\to -1} 1\right)\right]^{2/3} = [4(-1)(-1+1)]^{2/3} = 0^{2/3} = \boxed{0}.\end{aligned}$$

26. $\lim_{x\to 0}(x^2-2x)^{3/5} = \left[\lim_{x\to 0}(x^2-2x)\right]^{3/5} = \left[\left(\lim_{x\to 0} x\right)^2 - 2\lim_{x\to 0} x\right]^{3/5} = (0^2-2(0))^{3/5} = 0^{3/5} = \boxed{0}$.

27. $\lim_{t\to 1}(3t^2-2t+4) = 3(1)^2-2(1)+4 = \boxed{5}$.

28. $\lim_{x\to 0}(-3x^4+2x+1) = -3(0)^4+2(0)+1 = \boxed{1}$.

29. $\lim_{x\to \frac{1}{2}}(2x^4-8x^3+4x-5) = 2\left(\frac{1}{2}\right)^4 - 8\left(\frac{1}{2}\right)^3 + 4\left(\frac{1}{2}\right) - 5 = \frac{1}{8} - 1 + 2 - 5 = \boxed{-\frac{31}{8}}$.

30. $\lim_{x\to -\frac{1}{3}}(27x^3+9x+1) = 27\left(-\frac{1}{3}\right)^3 + 9\left(-\frac{1}{3}\right) + 1 = -1-3+1 = \boxed{-3}$.

31. Because the limit of the denominator $\lim_{x\to 4}\sqrt{x} = \sqrt{4} = 2 \neq 0$,

$$\lim_{x\to 4}\frac{x^2+4}{\sqrt{x}} = \frac{\lim_{x\to 4}(x^2+4)}{\lim_{x\to 4}\sqrt{x}} = \frac{4^2+4}{\sqrt{\lim_{x\to 4} x}} = \frac{20}{\sqrt{4}} = \frac{20}{2} = \boxed{10}.$$

32. Because the limit of the denominator $\lim_{x\to 3}\sqrt{3x} = \sqrt{9} = 3 \neq 0$,

$$\lim_{x\to 3}\frac{x^2+5}{\sqrt{3x}} = \frac{\lim_{x\to 3}(x^2+5)}{\lim_{x\to 3}\sqrt{3x}} = \frac{3^2+5}{\sqrt{\lim_{x\to 3}3x}} = \frac{14}{\sqrt{9}} = \boxed{\frac{14}{3}}.$$

33. Because $x = -2$ is in the domain of the rational function $R(x) = \dfrac{2x^3+5x}{3x-2}$,

$$\lim_{x\to -2}R(x) = R(-2) = \frac{2(-2)^3+5(-2)}{3(-2)-2} = \frac{-26}{-8} = \boxed{\frac{13}{4}}.$$

34. Because $x = 1$ is in the domain of the rational function $R(x) = \dfrac{2x^4-1}{3x^3+2}$,

$$\lim_{x\to 1}R(x) = R(1) = \frac{2(1)^4-1}{3(1)^3+2} = \boxed{\frac{1}{5}}.$$

35. Observe that the limit of the denominator is equal to zero. Factoring the numerator and simplifying yields

$$\lim_{x\to 2}\frac{x^2-4}{x-2} = \lim_{x\to 2}\frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2}(x+2) = \boxed{4}.$$

36. Observe that the limit of the denominator is equal to zero. Factoring the denominator and simplifying yields

$$\lim_{x\to -2}\frac{x+2}{x^2-4} = \lim_{x\to -2}\frac{x+2}{(x-2)(x+2)} = \lim_{x\to -2}\frac{1}{x-2} = \boxed{-\frac{1}{4}}.$$

37. Observe that the limit of the denominator is equal to zero. Factoring the numerator and simplifying yields

$$\lim_{x\to -1}\frac{x^3-x}{x+1} = \lim_{x\to -1}\frac{x(x-1)(x+1)}{x+1} = \lim_{x\to -1}[x(x-1)] = (-1)(-2) = \boxed{2}.$$

38. Observe that the limit of the denominator is equal to zero. Factoring both the numerator and the denominator and simplifying yields

$$\lim_{x\to -1}\frac{x^3+x^2}{x^2-1} = \lim_{x\to -1}\frac{x^2(x+1)}{(x-1)(x+1)} = \lim_{x\to -1}\frac{x^2}{x-1} = \frac{(-1)^2}{-2} = \boxed{-\frac{1}{2}}.$$

39. Observe that

$$\frac{2x}{x+8}+\frac{16}{x+8} = \frac{2x+16}{x+8} = \frac{2(x+8)}{x+8} = 2,$$

provided that $x \neq -8$. Therefore,

$$\lim_{x\to -8}\left(\frac{2x}{x+8}+\frac{16}{x+8}\right) = \lim_{x\to -8}2 = \boxed{2}.$$

40. Observe that

$$\frac{3x}{x-2}-\frac{6}{x-2} = \frac{3x-6}{x-2} = \frac{3(x-2)}{x-2} = 3,$$

provided that $x \neq 2$. Therefore,

$$\lim_{x\to 2}\left(\frac{3x}{x-2}-\frac{6}{x-2}\right) = \lim_{x\to 2}3 = \boxed{3}.$$

41. Observe that the limit of the denominator is equal to zero. Multiplying numerator and denominator by $\sqrt{x}+\sqrt{2}$, the conjugate of $\sqrt{x}-\sqrt{2}$, yields

$$\begin{aligned}\lim_{x\to 2}\frac{\sqrt{x}-\sqrt{2}}{x-2} &= \lim_{x\to 2}\frac{\sqrt{x}-\sqrt{2}}{x-2}\cdot\frac{\sqrt{x}+\sqrt{2}}{\sqrt{x}+\sqrt{2}} = \lim_{x\to 2}\frac{x-2}{(x-2)(\sqrt{x}+\sqrt{2})}\\ &= \lim_{x\to 2}\frac{1}{\sqrt{x}+\sqrt{2}} = \frac{1}{\sqrt{2}+\sqrt{2}} = \frac{1}{2\sqrt{2}} = \boxed{\sqrt{2}/4}.\end{aligned}$$

42. Observe that the limit of the denominator is equal to zero. Multiplying numerator and denominator by $\sqrt{x}+\sqrt{3}$, the conjugate of $\sqrt{x}-\sqrt{3}$, yields

$$\begin{aligned}\lim_{x\to 3}\frac{\sqrt{x}-\sqrt{3}}{x-3} &= \lim_{x\to 3}\frac{\sqrt{x}-\sqrt{3}}{x-3}\cdot\frac{\sqrt{x}+\sqrt{3}}{\sqrt{x}+\sqrt{3}} = \lim_{x\to 3}\frac{x-3}{(x-3)(\sqrt{x}+\sqrt{3})}\\ &= \lim_{x\to 3}\frac{1}{\sqrt{x}+\sqrt{3}} = \frac{1}{\sqrt{3}+\sqrt{3}} = \boxed{\frac{1}{2\sqrt{3}}}.\end{aligned}$$

43. Observe that the limit of the denominator is equal to zero. Multiplying numerator and denominator by $\sqrt{x+5}+3$, the conjugate of $\sqrt{x+5}-3$, yields

$$\begin{aligned}\lim_{x\to 4}\frac{\sqrt{x+5}-3}{(x-4)(x+1)} &= \lim_{x\to 4}\frac{\sqrt{x+5}-3}{(x-4)(x+1)}\cdot\frac{\sqrt{x+5}+3}{\sqrt{x+5}+3} = \lim_{x\to 4}\frac{(x+5)-9}{(x-4)(x+1)(\sqrt{x+5}+3)}\\ &= \lim_{x\to 4}\frac{x-4}{(x-4)(x+1)(\sqrt{x+5}+3)} = \lim_{x\to 4}\frac{1}{(x+1)(\sqrt{x+5}+3)}\\ &= \frac{1}{(4+1)(\sqrt{4+5}+3)} = \boxed{\frac{1}{30}}.\end{aligned}$$

44. Observe that the limit of the denominator is equal to zero. Multiplying numerator and denominator by $\sqrt{x+1}+2$, the conjugate of $\sqrt{x+1}-2$, yields

$$\begin{aligned}\lim_{x\to 3}\frac{\sqrt{x+1}-2}{x(x-3)} &= \lim_{x\to 3}\frac{\sqrt{x+1}-2}{x(x-3)}\cdot\frac{\sqrt{x+1}+2}{\sqrt{x+1}+2} = \lim_{x\to 3}\frac{(x+1)-4}{x(x-3)(\sqrt{x+1}+2)}\\ &= \lim_{x\to 3}\frac{x-3}{x(x-3)(\sqrt{x+1}+2)} = \lim_{x\to 3}\frac{1}{x(\sqrt{x+1}+2)}\\ &= \frac{1}{3(\sqrt{3+1}+2)} = \boxed{\frac{1}{12}}.\end{aligned}$$

45. $\lim_{x\to 3^-}(x^2-4) = (3)^2-4 = \boxed{5}$.

46. $\lim_{x\to 2^+}(3x^2+x) = 3(2)^2+2 = \boxed{14}$.

47. Observe that the limit of the denominator is equal to zero. Factoring the numerator and simplifying yields

$$\lim_{x\to 3^-}\frac{x^2-9}{x-3} = \lim_{x\to 3^-}\frac{(x-3)(x+3)}{x-3} = \lim_{x\to 3^-}(x+3) = 3+3 = \boxed{6}.$$

48. Observe that the limit of the denominator is equal to zero. Factoring the numerator and simplifying yields

$$\lim_{x\to 3^+}\frac{x^2-9}{x-3} = \lim_{x\to 3^+}\frac{(x-3)(x+3)}{x-3} = \lim_{x\to 3^+}(x+3) = 3+3 = \boxed{6}.$$

49. $\displaystyle\lim_{x\to 3^-}(\sqrt{9-x^2}+x)^2 = \left(\lim_{x\to 3^-}(\sqrt{9-x^2}+x)\right)^2 = \left(\sqrt{9-3^2}+3\right)^2 = \boxed{9}.$

50. $\displaystyle\lim_{x\to 2^+}(2\sqrt{x^2-4}+3x) = 2\sqrt{2^2-4}+3(2) = \boxed{6}.$

51. $\displaystyle\lim_{x\to c}[f(x)-3g(x)] = \lim_{x\to c} f(x) - 3\lim_{x\to c} g(x) = 5-3(2) = \boxed{-1}.$

52. $\displaystyle\lim_{x\to c}[5f(x)] = 5\lim_{x\to c} f(x) = 5(5) = \boxed{25}.$

53. $\displaystyle\lim_{x\to c}[g(x)]^3 = \left[\lim_{x\to c} g(x)\right]^3 = 2^3 = \boxed{8}.$

54. Because the limit of the denominator is not equal to zero,

$$\lim_{x\to c}\frac{f(x)}{g(x)-h(x)} = \frac{\lim\limits_{x\to c} f(x)}{\lim\limits_{x\to c}(g(x)-h(x))} = \frac{\lim\limits_{x\to c} f(x)}{\lim\limits_{x\to c} g(x) - \lim\limits_{x\to c} h(x)} = \frac{5}{2-0} = \boxed{\frac{5}{2}}.$$

55. Because the limit of the denominator is not equal to zero,

$$\lim_{x\to c}\frac{h(x)}{g(x)} = \frac{\lim\limits_{x\to c} h(x)}{\lim\limits_{x\to c} g(x)} = \frac{0}{2} = \boxed{0}.$$

56. $\displaystyle\lim_{x\to c}[4f(x)\cdot g(x)] = 4\lim_{x\to c} f(x)\cdot\lim_{x\to c} g(x) = 4(5)(2) = \boxed{40}.$

57. Because the limit of the denominator is not equal to zero,

$$\lim_{x\to c}\left[\frac{1}{g(x)}\right]^2 = \left[\lim_{x\to c}\frac{1}{g(x)}\right]^2 = \left[\frac{\lim\limits_{x\to c} 1}{\lim\limits_{x\to c} g(x)}\right]^2 = \left[\frac{1}{2}\right]^2 = \boxed{\frac{1}{4}}.$$

58. $\displaystyle\lim_{x\to c}\sqrt[3]{5g(x)-3} = \sqrt[3]{\lim_{x\to c}(5g(x)-3)} = \sqrt[3]{5\lim_{x\to c} g(x) - \lim_{x\to c} 3} = \sqrt[3]{5(2)-3} = \boxed{\sqrt[3]{7}}.$

59. (a) $\displaystyle\lim_{x\to 4}[f(x)+g(x)] = \lim_{x\to 4} f(x) + \lim_{x\to 4} g(x) = 8+(-2) = \boxed{6}.$

(b) $\displaystyle\lim_{x\to 4}\{f(x)[g(x)-h(x)]\} = \lim_{x\to 4} f(x)\left(\lim_{x\to 4} g(x) - \lim_{x\to 4} h(x)\right) = 8(-2+0) = \boxed{-16}.$

(c) $\displaystyle\lim_{x\to 4}[f(x)\cdot g(x)] = \lim_{x\to 4} f(x)\cdot\lim_{x\to 4} g(x) = 8(-2) = \boxed{-16}.$

(d) $\displaystyle\lim_{x\to 4}[2h(x)] = 2\lim_{x\to 4} h(x) = 2(0) = \boxed{0}.$

(e) $\displaystyle\lim_{x\to 4}\frac{g(x)}{f(x)} = \frac{\lim\limits_{x\to 4} g(x)}{\lim\limits_{x\to 4} f(x)} = \frac{-2}{8} = \boxed{-\frac{1}{4}}.$

(f) $\displaystyle\lim_{x\to 4}\frac{h(x)}{f(x)} = \frac{\lim\limits_{x\to 4} h(x)}{\lim\limits_{x\to 4} f(x)} = \frac{0}{8} = \boxed{0}.$

60. (a) $\displaystyle\lim_{x\to 3}\{2[f(x)+h(x)]\} = 2\left(\lim_{x\to 3} f(x) + \lim_{x\to 3} h(x)\right) = 2(0+(-2)) = \boxed{-4}.$

(b) $\displaystyle\lim_{x\to 3^-}[g(x)+h(x)] = \lim_{x\to 3^-} g(x) + \lim_{x\to 3^-} h(x) = 6+(-2) = \boxed{4}.$

(c) $\displaystyle\lim_{x\to 3}\sqrt[3]{h(x)} = \sqrt[3]{\lim_{x\to 3} h(x)} = \sqrt[3]{-2} = \boxed{-\sqrt[3]{2}}.$

(d) $\displaystyle\lim_{x\to 3}\frac{f(x)}{h(x)} = \frac{\displaystyle\lim_{x\to 3} f(x)}{\displaystyle\lim_{x\to 3} h(x)} = \frac{0}{-2} = \boxed{0}.$

(e) $\displaystyle\lim_{x\to 3}[h(x)]^3 = \left(\lim_{x\to 3} h(x)\right)^3 = (-2)^3 = \boxed{-8}.$

(f) Because

$$\lim_{x\to 3}[f(x) - 2h(x)] = \lim_{x\to 3} f(x) - 2\lim_{x\to 3} h(x) = 0 - 2(-2) = 4,$$

it follows that

$$\lim_{x\to 3}[f(x) - 2h(x)]^{3/2} = \left[\lim_{x\to 3}(f(x) - 2h(x))\right]^{3/2} = 4^{3/2} = \boxed{8}.$$

61. Let $f(x) = 3x^2$ and $c = 1$. Then

$$\begin{aligned}\lim_{x\to c}\frac{f(x)-f(c)}{x-c} &= \lim_{x\to 1}\frac{f(x)-f(1)}{x-1} = \lim_{x\to 1}\frac{3x^2-3}{x-1} = \lim_{x\to 1}\frac{3(x-1)(x+1)}{x-1} \\ &= \lim_{x\to 1} 3(x+1) = 3(1+1) = \boxed{6}.\end{aligned}$$

62. Let $f(x) = 8x^3$ and $c = 2$. Then

$$\begin{aligned}\lim_{x\to c}\frac{f(x)-f(c)}{x-c} &= \lim_{x\to 2}\frac{f(x)-f(2)}{x-2} = \lim_{x\to 2}\frac{8x^3-64}{x-2} = \lim_{x\to 2}\frac{8(x-2)(x^2+2x+4)}{x-2} \\ &= 8\lim_{x\to 2}(x^2+2x+4) = 8(2^2+2(2)+4) = \boxed{96}.\end{aligned}$$

63. Let $f(x) = -2x^2 + 4$ and $c = 1$. Then

$$\begin{aligned}\lim_{x\to c}\frac{f(x)-f(c)}{x-c} &= \lim_{x\to 1}\frac{f(x)-f(1)}{x-1} = \lim_{x\to 1}\frac{-2x^2+4-2}{x-1} = \lim_{x\to 1}\frac{-2x^2+2}{x-1} \\ &= \lim_{x\to 1}\frac{-2(x-1)(x+1)}{x-1} = -2\lim_{x\to 1}(x+1) = -2(1+1) = \boxed{-4}.\end{aligned}$$

64. Let $f(x) = 20 - 0.8x^2$ and $c = 3$. Then

$$\begin{aligned}\lim_{x\to c}\frac{f(x)-f(c)}{x-c} &= \lim_{x\to 3}\frac{f(x)-f(3)}{x-3} = \lim_{x\to 3}\frac{20-0.8x^2-12.8}{x-3} = \lim_{x\to 3}\frac{-0.8x^2+7.2}{x-3} \\ &= \lim_{x\to 3}\frac{-0.8(x-3)(x+3)}{x-3} = -0.8\lim_{x\to 3}(x+3) = -0.8(3+3) = \boxed{-4.8}.\end{aligned}$$

65. Let $f(x) = \sqrt{x}$ and $c = 1$. Then

$$\begin{aligned}\lim_{x\to c}\frac{f(x)-f(c)}{x-c} &= \lim_{x\to 1}\frac{f(x)-f(1)}{x-1} = \lim_{x\to 1}\frac{\sqrt{x}-1}{x-1} = \lim_{x\to 1}\frac{\sqrt{x}-1}{x-1}\cdot\frac{\sqrt{x}+1}{\sqrt{x}+1} \\ &= \lim_{x\to 1}\frac{x-1}{(x-1)(\sqrt{x}+1)} = \lim_{x\to 1}\frac{1}{\sqrt{x}+1} = \frac{1}{1+1} = \boxed{\frac{1}{2}}.\end{aligned}$$

66. Let $f(x) = \sqrt{2x}$ and $c = 5$. Then

$$\begin{aligned}\lim_{x\to c}\frac{f(x)-f(c)}{x-c} &= \lim_{x\to 5}\frac{f(x)-f(5)}{x-5} = \lim_{x\to 5}\frac{\sqrt{2x}-\sqrt{10}}{x-5} = \lim_{x\to 5}\frac{\sqrt{2x}-\sqrt{10}}{x-5}\cdot\frac{\sqrt{2x}+\sqrt{10}}{\sqrt{2x}+\sqrt{10}} \\ &= \lim_{x\to 5}\frac{2x-10}{(x-5)(\sqrt{2x}+\sqrt{10})} = \lim_{x\to 5}\frac{2}{\sqrt{2x}+\sqrt{10}} = \frac{2}{\sqrt{10}+\sqrt{10}} = \boxed{\frac{1}{\sqrt{10}}}.\end{aligned}$$

67. Let $f(x) = 4x - 3$. Then

$$\begin{aligned}\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{4(x+h)-3-(4x-3)}{h} = \lim_{h\to 0}\frac{4x+4h-3-4x+3}{h}\\ &= \lim_{h\to 0}\frac{4h}{h} = \lim_{h\to 0} 4 = \boxed{4}.\end{aligned}$$

68. Let $f(x) = 3x + 5$. Then

$$\begin{aligned}\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{3(x+h)+5-(3x+5)}{h} = \lim_{h\to 0}\frac{3x+3h+5-3x-5}{h}\\ &= \lim_{h\to 0}\frac{3h}{h} = \lim_{h\to 0} 3 = \boxed{3}.\end{aligned}$$

69. Let $f(x) = 3x^2 + 4x + 1$. Then

$$\begin{aligned}\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{3(x+h)^2+4(x+h)+1-(3x^2+4x+1)}{h}\\ &= \lim_{h\to 0}\frac{3x^2+6xh+3h^2+4x+4h+1-3x^2-4x-1}{h}\\ &= \lim_{h\to 0}\frac{6xh+3h^2+4h}{h} = \lim_{h\to 0}(6x+3h+4)\\ &= 6x+3(0)+4 = \boxed{6x+4}.\end{aligned}$$

70. Let $f(x) = 2x^2 + x$. Then

$$\begin{aligned}\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{2(x+h)^2+(x+h)-(2x^2+x)}{h}\\ &= \lim_{h\to 0}\frac{2x^2+4xh+2h^2+x+h-2x^2-x}{h}\\ &= \lim_{h\to 0}\frac{4xh+2h^2+h}{h} = \lim_{h\to 0}(4x+2h+1)\\ &= 4x+2(0)+1 = \boxed{4x+1}.\end{aligned}$$

71. Let $f(x) = \dfrac{2}{x}$. Then

$$\begin{aligned}\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{\frac{2}{x+h}-\frac{2}{x}}{h} = \lim_{h\to 0}\frac{\frac{2}{x+h}-\frac{2}{x}}{h}\cdot\frac{x(x+h)}{x(x+h)} = \lim_{h\to 0}\frac{2x-2(x+h)}{hx(x+h)}\\ &= \lim_{h\to 0}\frac{-2h}{hx(x+h)} = -\lim_{h\to 0}\frac{2}{x(x+h)} = -\frac{2}{x(x+0)} = \boxed{-\frac{2}{x^2}}.\end{aligned}$$

72. Let $f(x) = \dfrac{3}{x^2}$. Then

$$\begin{aligned}\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{\frac{3}{(x+h)^2}-\frac{3}{x^2}}{h} = \lim_{h\to 0}\frac{\frac{3}{(x+h)^2}-\frac{3}{x^2}}{h}\cdot\frac{x^2(x+h)^2}{x^2(x+h)^2}\\ &= \lim_{h\to 0}\frac{3x^2-3(x+h)^2}{hx^2(x+h)^2} = \lim_{h\to 0}\frac{3x^2-3x^2-6xh-3h^2}{hx^2(x+h)^2}\\ &= -\lim_{h\to 0}\frac{h(6x+3h)}{hx^2(x+h)^2} = -\lim_{h\to 0}\frac{6x+3h}{x^2(x+h)^2} = -\frac{6x+3(0)}{x^2(x+0)^2}\\ &= -\frac{6x}{x^4} = \boxed{-\frac{6}{x^3}}.\end{aligned}$$

73. For $x < 1$, $f(x) = 2x - 3$, so that

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (2x-3) = 2(1) - 3 = \boxed{-1},$$

while for $x > 1$, $f(x) = 3 - x$, so that

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (3-x) = 3 - 1 = \boxed{2}.$$

Because the two one-sided limits as x approaches 1 are not equal, $\boxed{\lim_{x\to 1} f(x) \text{ does not exist}}$.

74. For $x < 2$, $f(x) = 5x + 2$, so that

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (5x+2) = 5(2) + 2 = \boxed{12},$$

while for $x > 2$, $f(x) = 1 + 3x$, so that

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (1+3x) = 1 + 3(2) = \boxed{7}.$$

Because the two one-sided limits as x approaches 2 are not equal, $\boxed{\lim_{x\to 2} f(x) \text{ does not exist}}$.

75. For $x < 1$, $f(x) = 3x - 1$, so that

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (3x-1) = 3(1) - 1 = \boxed{2},$$

while for $x > 1$, $f(x) = 2x$, so that

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 2x = 2(1) = \boxed{2}.$$

Because the two one-sided limits as x approaches 1 are equal to 2, $\boxed{\lim_{x\to 1} f(x) = 2}$.

76. For $x < 1$, $f(x) = 3x - 1$, so that

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (3x-1) = 3(1) - 1 = \boxed{2},$$

while for $x > 1$, $f(x) = 2x$, so that

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 2x = 2(1) = \boxed{2}.$$

Because the two one-sided limits as x approaches 1 are equal to 2, $\boxed{\lim_{x\to 1} f(x) = 2}$.

77. For $x < 1$, $f(x) = x - 1$, so that

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x-1) = 1 - 1 = \boxed{0},$$

while for $x > 1$, $f(x) = \sqrt{x-1}$, so that

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \sqrt{x-1} = \sqrt{1-1} = \boxed{0}.$$

Because the two one-sided limits as x approaches 1 are equal to 0, $\boxed{\lim_{x\to 1} f(x) = 0}$.

78. For $x < 3$, $f(x) = \sqrt{9 - x^2}$, so that

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \sqrt{9 - x^2} = \sqrt{9 - 3^2} = \boxed{0},$$

while for $x > 3$, $f(x) = \sqrt{x^2 - 9}$, so that

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \sqrt{x^2 - 9} = \sqrt{3^2 - 9} = \boxed{0}.$$

Because the two one-sided limits as x approaches 3 are equal to 0, $\boxed{\lim_{x\to 3} f(x) = 0}$.

79. For $x = 3$, $f(x) = \dfrac{x^2 - 9}{x - 3}$, so that

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{x^2 - 9}{x - 3} = \lim_{x\to 3^-} \frac{(x+3)(x-3)}{x - 3} = \lim_{x\to 3^-} (x+3) = 3 + 3 = \boxed{6},$$

and

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \frac{x^2 - 9}{x - 3} = \lim_{x\to 3^+} \frac{(x+3)(x-3)}{x - 3} = \lim_{x\to 3^+} (x+3) = 3 + 3 = \boxed{6}.$$

Because the two one-sided limits as x approaches 3 are equal to 6, $\boxed{\lim_{x\to 3} f(x) = 6}$.

80. For $x = 2$, $f(x) = \dfrac{x - 2}{x^2 - 4}$, so that

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{x - 2}{x^2 - 4} = \lim_{x\to 2^-} \frac{x - 2}{(x-2)(x+2)} = \lim_{x\to 2^-} \frac{1}{x + 2} = \frac{1}{2 + 2} = \boxed{\frac{1}{4}},$$

and

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{x - 2}{x^2 - 4} = \lim_{x\to 2^+} \frac{x - 2}{(x-2)(x+2)} = \lim_{x\to 2^+} \frac{1}{x + 2} = \frac{1}{2 + 2} = \boxed{\frac{1}{4}}.$$

Because the two one-sided limits as x approaches 2 are equal to $\dfrac{1}{4}$, $\boxed{\lim_{x\to 2} f(x) = \dfrac{1}{4}}$.

Applications and Extensions

81. For $t < 1$,

$$\lim_{t\to 1^-} u_1(t) = \lim_{t\to 1^-} 0 = 0,$$

while for $t > 1$,

$$\lim_{t\to 1^+} u_1(t) = \lim_{t\to 1^+} 1 = 1.$$

Because the two one-sided limits as x approaches 1 are not equal, $\boxed{\lim_{t\to 1} u_1(t) \text{ does not exist}}$.

82. For $t < 3$,

$$\lim_{t\to 3^-} u_3(t) = \lim_{t\to 3^-} 0 = 0,$$

while for $t > 3$,

$$\lim_{t\to 3^+} u_3(t) = \lim_{t\to 3^+} 1 = 1.$$

Because the two one-sided limits as x approaches 3 are not equal, $\boxed{\lim_{t\to 3} u_3(t) \text{ does not exist}}$.

83. $\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}=\lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h}=\lim_{h\to 0}\frac{h(2x+h)}{h}=\lim_{h\to 0}(2x+h)=\boxed{2x}.$

84.

$$\begin{aligned}\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h} &= \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}=\lim_{h\to 0}\frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})}\\ &= \lim_{h\to 0}\frac{h}{h(\sqrt{x+h}+\sqrt{x})}=\lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}=\boxed{\frac{1}{2\sqrt{x}}}.\end{aligned}$$

85.

$$\begin{aligned}\lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h} &= \lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h}\cdot\frac{x(x+h)}{x(x+h)}=\lim_{h\to 0}\frac{x-(x+h)}{hx(x+h)}=-\lim_{h\to 0}\frac{h}{hx(x+h)}\\ &= -\lim_{h\to 0}\frac{1}{x(x+h)}=\boxed{-\frac{1}{x^2}}.\end{aligned}$$

86.

$$\begin{aligned}\lim_{h\to 0}\frac{\frac{1}{(x+h)^3}-\frac{1}{x^3}}{h} &= \lim_{h\to 0}\frac{\frac{1}{(x+h)^3}-\frac{1}{x^3}}{h}\cdot\frac{x^3(x+h)^3}{x^3(x+h)^3}=\lim_{h\to 0}\frac{x^3-(x+h)^3}{hx^3(x+h)^3}\\ &= \lim_{h\to 0}\frac{x^3-x^3-3x^2h-3xh^2-h^3}{hx^3(x+h)^3}=-\lim_{h\to 0}\frac{h(3x^2+3xh+h^2)}{hx^3(x+h)^3}\\ &= -\lim_{h\to 0}\frac{3x^2+3xh+h^2}{x^3(x+h)^3}=-\frac{3x^2}{x^6}=\boxed{-\frac{3}{x^4}}.\end{aligned}$$

87. Observe that

$$\frac{1}{4+x}-\frac{1}{4}=\frac{4-(4+x)}{4(4+x)}=-\frac{x}{4(4+x)},$$

so that

$$\lim_{x\to 0}\left[\frac{1}{x}\left(\frac{1}{4+x}-\frac{1}{4}\right)\right]=\lim_{x\to 0}\left[\frac{1}{x}\left(-\frac{x}{4(4+x)}\right)\right]=-\lim_{x\to 0}\frac{1}{4(4+x)}=\boxed{-\frac{1}{16}}.$$

88. Observe that

$$\frac{1}{3}-\frac{1}{x+4}=\frac{(x+4)-3}{3(x+4)}=\frac{x+1}{3(x+4)},$$

so that

$$\lim_{x\to -1}\left[\frac{2}{x+1}\left(\frac{1}{3}-\frac{1}{x+4}\right)\right]=\lim_{x\to -1}\left[\frac{2}{x+1}\left(\frac{x+1}{3(x+4)}\right)\right]=\lim_{x\to -1}\frac{2}{3(x+4)}=\boxed{\frac{2}{9}}.$$

89.
$$\begin{aligned}\lim_{x\to 7}\frac{x-7}{\sqrt{x+2}-3} &= \lim_{x\to 7}\frac{x-7}{\sqrt{x+2}-3}\cdot\frac{\sqrt{x+2}+3}{\sqrt{x+2}+3}=\lim_{x\to 7}\frac{(x-7)(\sqrt{x+2}+3)}{(x+2)-9}\\ &= \lim_{x\to 7}\frac{(x-7)(\sqrt{x+2}+3)}{x-7}=\lim_{x\to 7}(\sqrt{x+2}+3)=\sqrt{9}+3=\boxed{6}.\end{aligned}$$

90.
$$\begin{aligned}\lim_{x\to 2}\frac{x-2}{\sqrt{x+2}-2} &= \lim_{x\to 2}\frac{x-2}{\sqrt{x+2}-2}\cdot\frac{\sqrt{x+2}+2}{\sqrt{x+2}+2}=\lim_{x\to 2}\frac{(x-2)(\sqrt{x+2}+2)}{(x+2)-4}\\ &= \lim_{x\to 2}\frac{(x-2)(\sqrt{x+2}+2)}{x-2}=\lim_{x\to 2}(\sqrt{x+2}+2)=\sqrt{4}+2=\boxed{4}.\end{aligned}$$

91. $\lim\limits_{x\to 1} \dfrac{x^3-3x^2+3x-1}{x^2-2x+1} = \lim\limits_{x\to 1} \dfrac{(x-1)^3}{(x-1)^2} = \lim\limits_{x\to 1}(x-1) = 1-1 = \boxed{0}$.

92. $\lim\limits_{x\to -3} \dfrac{x^3+7x^2+15x+9}{x^2+6x+9} = \lim\limits_{x\to -3} \dfrac{(x+1)(x^2+6x+9)}{x^2+6x+9} = \lim\limits_{x\to -3}(x+1) = -3+1 = \boxed{-2}$.

93. (a) Using the rate schedule provided,

$$\boxed{C(x) = \begin{cases} 9.00, & \text{if } 0 \le x \le 10 \\ 9.00 + 0.95(x-10), & \text{if } 10 < x \le 30 \\ 28.00 + 1.65(x-30), & \text{if } 30 < x \le 100 \\ 143.50 + 2.20(x-100), & \text{if } x > 100. \end{cases}}$$

(b) The domain of C is the set $\boxed{\{x|x \ge 0\}}$.

(c) Start with the one-sided limits:

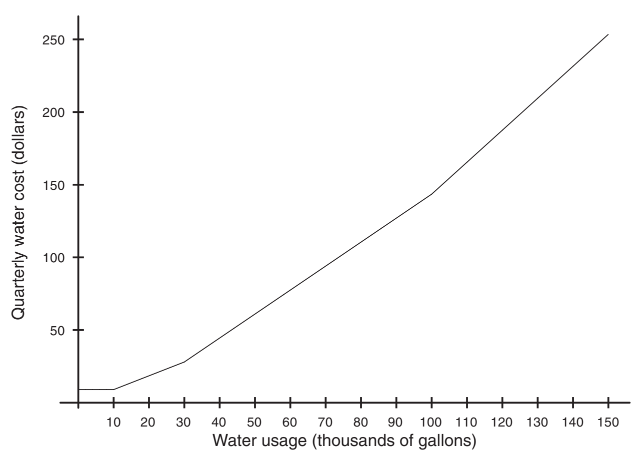
$$\begin{aligned}
\lim_{x\to 5^-} C(x) &= \lim_{x\to 5^-} 9.00 = 9.00 \\
\lim_{x\to 5^+} C(x) &= \lim_{x\to 5^+} 9.00 = 9.00 \\
\lim_{x\to 10^-} C(x) &= \lim_{x\to 10^-} 9.00 = 9.00 \\
\lim_{x\to 10^+} C(x) &= \lim_{x\to 10^+} [9.00 + 0.95(x-10)] = 9.00 + 0.95(10-10) = 9.00 \\
\lim_{x\to 30^-} C(x) &= \lim_{x\to 30^-} [9.00 + 0.95(x-10)] = 9.00 + 0.95(30-10) = 28.00 \\
\lim_{x\to 30^+} C(x) &= \lim_{x\to 30^+} [28.00 + 1.65(x-30)] = 28.00 + 1.65(30-30) = 28.00 \\
\lim_{x\to 100^-} C(x) &= \lim_{x\to 100^-} [28.00 + 1.65(x-30)] = 28.00 + 1.65(100-30) = 143.50 \\
\lim_{x\to 100^+} C(x) &= \lim_{x\to 100^+} [143.50 + 2.20(x-100)] = 143.50 + 2.20(100-100) = 143.50
\end{aligned}$$

Based on these one-sided limits, it follows that

$$\boxed{\lim_{x\to 5} C(x) = 9.00,\ \lim_{x\to 10} C(x) = 9.00,\ \lim_{x\to 30} C(x) = 28.00, \text{ and } \lim_{x\to 100} C(x) = 143.50}.$$

(d) $\lim\limits_{x\to 0^+} C(x) = \lim\limits_{x\to 0^+} 9.00 = \boxed{9.00}$.

(e) The graph of C is shown below.



94. (a) Let x denote the monthly amount of electricity (in kWH) used by a customer. Each customer is assessed a fixed charge of \$7.87 regardless of how little or how much electricity is used. In addition to the fixed charge, each customer pays a variable charge of \$0.02173 for every kWH of electricity used up to and including 1000 kWH. So, for a customer using x kWH of electricity, the variable charge is $\$0.02173x$. Again, this variable rate is applied for usage up to and including 1000 kWH. The total charge for usage x up to and including 1000 kWH ($0 \le x \le 1000$) is $\$7.87 + \$0.02173x$.

For usage over 1000 ($x > 1000$), the customer is charged \$0.03173 per kWH for every hour over 1000. Since x denotes the total usage, the quantity $(x - 1000)$ denotes the number of kWH over 1000. For these customers, the total charge for the first 1000 hours is $\$7.87 + \$0.02173(1000) = \$29.60$. For $x > 1000$, the additional fee is $0.02183(x - 1000)$.

The final piecewise function is

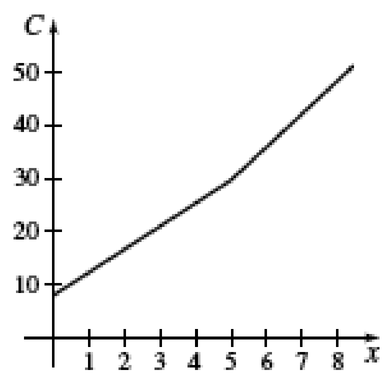
$$C(x) = \begin{cases} 7.87 + 0.02173x & \text{if } \; 0 \le x \le 1000 \\ 29.60 + 0.03173(x - 1000) & \text{if } \; x > 1000 \end{cases}.$$

(b) The domain of the function is any nonnegative real number, $\{x | x \ge 0\}$.

(c) Since $\lim_{x \to 50^-} (7.87 + 0.02173x) = 7.87 + 0.02173(50) = 8.9565$ and $\lim_{x \to 50^+} C(x) = (7.87 + 0.02173x) = 7.87 + 0.02173(50) = 8.9565$, we conclude $\lim_{x \to 50} C(x) = 8.9565$.

(d) As the usage (x) gets closer and closer to zero, the total cost gets closer and closer to the fixed cost of \$7.87. $\lim_{x \to 0^+} C(x) = 7.87$.

(e) The graph of $C(x)$ is shown below. The graph consists of two lines with different slopes.



95. (a) Solving the equation $\frac{1}{2}mv^2 = \frac{3}{2}kT$ for $v(T)$ yields $v(T) = \sqrt{\dfrac{3kT}{m}}$. Thus,

$$\lim_{T \to 0} v(T) = \lim_{T \to 0} \sqrt{\frac{3kT}{m}} = \sqrt{\frac{3k}{m}} \sqrt{\lim_{T \to 0} T} = \boxed{0}.$$

(b) Answers will vary. One interpretation is that as the temperature of a gas approaches absolute zero, the average speed of the molecules in the gas also approaches zero. In other words, $\boxed{\text{the molecules in the gas stop moving}}$.

96. (a) For $h < 0$, $2 + h < 2$, so that $f(2 + h) = 3(2 + h) + 5 = 3h + 11$. Thus,

$$\lim_{h \to 0^-} \frac{f(2 + h) - f(2)}{h} = \lim_{h \to 0^-} \frac{3h + 11 - 11}{h} = \lim_{h \to 0^-} \frac{3h}{h} = \lim_{h \to 0^-} 3 = \boxed{3}.$$

(b) For $h > 0$, $2 + h > 2$, so that $f(2+h) = 13 - (2+h) = 11 - h$. Thus,

$$\lim_{h\to 0^+} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0^+} \frac{11-h-11}{h} = \lim_{h\to 0^+} \frac{-h}{h} = \lim_{h\to 0^+} -1 = \boxed{-1}.$$

(c) Because the two one-sided limits as h approaches 0 are not equal, $\lim_{h\to 0} \dfrac{f(2+h)-f(2)}{h}$ $\boxed{\text{does not exist}}$.

97. Consider the one-sided limits:

$$\boxed{\lim_{x\to 0^-} |x| = \lim_{x\to 0^-} (-x) = 0}$$

and

$$\boxed{\lim_{x\to 0^+} |x| = \lim_{x\to 0^+} x = 0}.$$

Because the two one-sided limits as x approaches 0 are equal to 0, $\lim_{x\to 0} |x| = 0$.

98. $\lim_{x\to 0} |x| = \lim_{x\to 0} \sqrt{x^2} = \sqrt{\lim_{x\to 0} x^2} = \sqrt{0} = 0$.

99. Answers will vary. One possibility is the following. Let

$$\boxed{f(x) = \begin{cases} 1, & \text{if } x < 2 \\ 0, & \text{if } x \ge 2 \end{cases}, \quad g(x) = \begin{cases} 0, & \text{if } x < 2 \\ 1, & \text{if } x \ge 2 \end{cases}},$$

and $c = 2$. Then

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} 1 = 1 \text{ and } \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 0 = 0$$

so that $\lim_{x\to 2} f(x)$ does not exist, and

$$\lim_{x\to 2^-} g(x) = \lim_{x\to 2^-} 0 = 0 \text{ and } \lim_{x\to 2^+} g(x) = \lim_{x\to 2^+} 1 = 1$$

so that $\lim_{x\to 2} g(x)$ does not exist. However, $(f+g)(x) = 1$ for all x so that $\lim_{x\to 2}[f(x)+g(x)] = 1$.

100. Answers will vary. One possibility is the following. Let

$$\boxed{f(x) = \begin{cases} 1, & \text{if } x < 2 \\ 0, & \text{if } x \ge 2 \end{cases}, \quad g(x) = \begin{cases} 0, & \text{if } x < 2 \\ 1, & \text{if } x \ge 2 \end{cases}},$$

and $c = 2$. Then

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} 1 = 1 \text{ and } \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 0 = 0$$

so that $\lim_{x\to 2} f(x)$ does not exist, and

$$\lim_{x\to 2^-} g(x) = \lim_{x\to 2^-} 0 = 0 \text{ and } \lim_{x\to 2^+} g(x) = \lim_{x\to 2^+} 1 = 1$$

so that $\lim_{x\to 2} g(x)$ does not exist. However, $(f \cdot g)(x) = 0$ for all x so that $\lim_{x\to 2}[f(x)g(x)] = 0$.

101. Answers will vary. One possibility is the following. Let

$$\boxed{f(x) = \begin{cases} 2, & \text{if } x < 1 \\ -2, & \text{if } x \geq 1 \end{cases}, \quad g(x) = \begin{cases} 1, & \text{if } x < 1 \\ -1, & \text{if } x \geq 1 \end{cases}},$$

and $c = 1$. Then

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 2 = 2 \text{ and } \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} -2 = -2$$

so that $\lim\limits_{x\to 1} f(x)$ does not exist, and

$$\lim_{x\to 1^-} g(x) = \lim_{x\to 1^-} 1 = 1 \text{ and } \lim_{x\to 1^+} g(x) = \lim_{x\to 1^+} -1 = -1$$

so that $\lim\limits_{x\to 1} g(x)$ does not exist. However, $\left(\dfrac{f}{g}\right)(x) = 2$ for all x so that $\lim\limits_{x\to 1}\left[\dfrac{f(x)}{g(x)}\right] = 2.$

102. Answers will vary. One possibility is the following. Let

$$\boxed{f(x) = \begin{cases} 1, & \text{if } x < 0 \\ -1, & \text{if } x \geq 0 \end{cases}}$$

and $c = 0$. Then

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 1 = 1 \text{ and } \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} -1 = -1$$

so that $\lim\limits_{x\to 0} f(x)$ does not exist. However, $|f(x)| = 1$ for all x so that $\lim\limits_{x\to 0} |f(x)| = 1.$

103. Let $f(x) = k$, where k is any real number, and let g be a function for which $\lim\limits_{x\to c} g(x)$ exists. Then $\lim\limits_{x\to c} f(x)$ exists and is equal to k, so that by the Limit of a Product Theorem, $\lim\limits_{x\to c}[f(x)g(x)]$ exists and $\lim\limits_{x\to c}[f(x)g(x)] = \lim\limits_{x\to c} f(x) \cdot \lim\limits_{x\to c} g(x)$. It then follows that

$$\lim_{x\to c}[kg(x)] = \lim_{x\to c}[f(x)g(x)]$$

exists, and

$$\lim_{x\to c}[kg(x)] = \lim_{x\to c}[f(x)g(x)] = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x) = k \lim_{x\to c} g(x).$$

104. Let c be a number in the domain of the rational function $R(x) = \frac{p(x)}{q(x)}$, where p and q are both polynomial functions. Because c is in the domain of R, $q(c) \neq 0$. Moreover, because q is a polynomial function, $\lim\limits_{x\to c} q(x) = q(c) \neq 0$. Therefore, by the Limit of a Quotient Theorem and the Limit of a Polynomial Theorem,

$$\lim_{x\to c} R(x) = \lim_{x\to c} \frac{p(x)}{q(x)} = \frac{\lim\limits_{x\to c} p(x)}{\lim\limits_{x\to c} q(x)} = \frac{p(c)}{q(c)} = R(c).$$

Challenge Problems

105. When n is a positive integer,

$$x^n - a^n = (x-a)(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots + a^{n-2}x + a^{n-1}),$$

so that

$$\begin{aligned} \lim_{x\to a} \frac{x^n - a^n}{x-a} &= \lim_{x\to a} \frac{(x-a)(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots + a^{n-2}x + a^{n-1})}{x-a} \\ &= \lim_{x\to a}(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots + a^{n-2}x + a^{n-1}) \\ &= a^{n-1} + aa^{n-2} + a^2a^{n-3} + \cdots + a^{n-2}a + a^{n-1} = \boxed{na^{n-1}}. \end{aligned}$$

106. If n is an even positive integer, then

$$\lim_{x\to -a}(x^n + a^n) = (-a)^n + a^n = 2a^n \quad \text{and} \quad \lim_{x\to -a}(x+a) = -a + a = 0,$$

so that

$$\lim_{x\to -a}\frac{x^n + a^n}{x+a}$$

$\boxed{\text{does not exist}}$. On the other hand, if n is an odd positive integer, then

$$\begin{aligned} x^n + a^n &= (x+a)(x^{n-1} - ax^{n-2} + a^2x^{n-3} - + \cdots - a^{n-2}x + a^{n-1}) \\ &= (x+a)(x^{n-1} + (-a)x^{n-2} + (-a)^2x^{n-3} + \cdots + (-a)^{n-2}x + (-a)^{n-1}), \end{aligned}$$

so that

$$\begin{aligned} \lim_{x\to -a}\frac{x^n + a^n}{x+a} &= \lim_{x\to -a}\frac{(x+a)(x^{n-1} + (-a)x^{n-2} + (-a)^2x^{n-3} + \cdots + (-a)^{n-2}x + (-a)^{n-1})}{x+a} \\ &= \lim_{x\to -a}(x^{n-1} + (-a)x^{n-2} + (-a)^2x^{n-3} + \cdots + (-a)^{n-2}x + (-a)^{n-1}) \\ &= (-a)^{n-1} + (-a)(-a)^{n-2} + (-a)^2(-a)^{n-3} + \cdots + (-a)^{n-2}(-a) + (-a)^{n-1} \\ &= n(-a)^{n-1} = \boxed{na^{n-1}}. \end{aligned}$$

107. When m and n are positive integers,

$$x^m - 1 = (x-1)(x^{m-1} + x^{m-2} + x^{m-3} + \cdots + x + 1)$$

and

$$x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + x^{n-3} + \cdots + x + 1).$$

Thus,

$$\begin{aligned} \lim_{x\to 1}\frac{x^m - 1}{x^n - 1} &= \lim_{x\to 1}\frac{(x-1)(x^{m-1} + x^{m-2} + x^{m-3} + \cdots + x + 1)}{(x-1)(x^{n-1} + x^{n-2} + x^{n-3} + \cdots + x + 1)} \\ &= \lim_{x\to 1}\frac{x^{m-1} + x^{m-2} + x^{m-3} + \cdots + x + 1}{x^{n-1} + x^{n-2} + x^{n-3} + \cdots + x + 1} \\ &= \frac{\overbrace{1 + 1 + 1 + \cdots + 1}^{m \text{ terms}}}{\underbrace{1 + 1 + 1 + \cdots + 1}_{n \text{ terms}}} = \boxed{\frac{m}{n}}. \end{aligned}$$

108.

$$\begin{aligned} \lim_{x\to 0}\frac{\sqrt[3]{1+x} - 1}{x} &= \lim_{x\to 0}\frac{\sqrt[3]{1+x} - 1}{x}\cdot\frac{\sqrt[3]{(1+x)^2} + \sqrt[3]{1+x} + 1}{\sqrt[3]{(1+x)^2} + \sqrt[3]{1+x} + 1} \\ &= \lim_{x\to 0}\frac{(1+x) - 1}{x(\sqrt[3]{(1+x)^2} + \sqrt[3]{1+x} + 1)} = \lim_{x\to 0}\frac{x}{x(\sqrt[3]{(1+x)^2} + \sqrt[3]{1+x} + 1)} \\ &= \lim_{x\to 0}\frac{1}{\sqrt[3]{(1+x)^2} + \sqrt[3]{1+x} + 1} = \frac{1}{\sqrt[3]{1} + \sqrt[3]{1} + 1} = \boxed{\frac{1}{3}}. \end{aligned}$$

109.

$$\begin{aligned}\lim_{x\to 0}\frac{\sqrt{(1+ax)(1+bx)}-1}{x} &= \lim_{x\to 0}\frac{\sqrt{(1+ax)(1+bx)}-1}{x}\cdot\frac{\sqrt{(1+ax)(1+bx)}+1}{\sqrt{(1+ax)(1+bx)}+1}\\ &= \lim_{x\to 0}\frac{(1+ax)(1+bx)-1}{x(\sqrt{(1+ax)(1+bx)}+1)} = \lim_{x\to 0}\frac{1+(a+b)x+abx^2-1}{x(\sqrt{(1+ax)(1+bx)}+1)}\\ &= \lim_{x\to 0}\frac{x[(a+b)+abx]}{x(\sqrt{(1+ax)(1+bx)}+1)} = \lim_{x\to 0}\frac{(a+b)+abx}{\sqrt{(1+ax)(1+bx)}+1}\\ &= \frac{a+b+ab(0)}{\sqrt{(1+0)(1+0)}+1} = \boxed{\frac{a+b}{2}}.\end{aligned}$$

110. Note that

$$\begin{aligned}(1+a_1x)(1+a_2x)\cdots(1+a_nx) &= 1+(a_1+a_2+\cdots+a_n)x\\ &\quad+\text{ terms containing } x^m \text{ where } m\geq 2,\end{aligned}$$

so that

$(1+a_1x)(1+a_2x)\cdots(1+a_nx)-1 = (a_1+a_2+\cdots+a_n)x+$ terms containing x^m where $m\geq 2$.

Thus,

$$\begin{aligned}&\lim_{x\to 0}\frac{\sqrt{(1+a_1x)(1+a_2x)\cdots(1+a_nx)}-1}{x}\\ &= \lim_{x\to 0}\frac{\sqrt{(1+a_1x)(1+a_2x)\cdots(1+a_nx)}-1}{x}\cdot\frac{\sqrt{(1+a_1x)(1+a_2x)\cdots(1+a_nx)}+1}{\sqrt{(1+a_1x)(1+a_2x)\cdots(1+a_nx)}+1}\\ &= \lim_{x\to 0}\frac{(1+a_1x)(1+a_2x)\cdots(1+a_nx)-1}{x(\sqrt{(1+a_1x)(1+a_2x)\cdots(1+a_nx)}+1)}\\ &= \lim_{x\to 0}\frac{(a_1+a_2+\cdots+a_n)x+\text{ terms containing } x^m \text{ where } m\geq 2}{x(\sqrt{(1+a_1x)(1+a_2x)\cdots(1+a_nx)}+1)}\\ &= \lim_{x\to 0}\frac{(a_1+a_2+\cdots+a_n)+\text{ terms containing } x^m \text{ where } m\geq 1}{\sqrt{(1+a_1x)(1+a_2x)\cdots(1+a_nx)}+1}\\ &= \frac{a_1+a_2+\cdots+a_n+0}{\sqrt{1}+1} = \boxed{\frac{a_1+a_2+\cdots+a_n}{2}}.\end{aligned}$$

111. Let $f(x)=x|x|$. For $x<0$, $f(x)=x(-x)=-x^2$. Thus,

$$\lim_{h\to 0^-}\frac{f(h)-f(0)}{h} = \lim_{h\to 0^-}\frac{-h^2-0}{h} = \lim_{h\to 0^-}(-h) = 0.$$

For $x\geq 0$, $f(x)=x(x)=x^2$. Thus,

$$\lim_{h\to 0^+}\frac{f(h)-f(0)}{h} = \lim_{h\to 0^+}\frac{h^2-0}{h} = \lim_{h\to 0^+} h = 0.$$

Because the two one-sided limits as h approaches 0 are equal to 0,

$$\lim_{h\to 0}\frac{f(h)-f(0)}{h} = \boxed{0}.$$

1.3 Continuity

Concepts and Vocabulary

1. $\boxed{\text{True}}$. A polynomial function is continuous at every real number.

2. $\boxed{\text{False}}$. Piecewise-defined functions **may not be** continuous at numbers where the function changes equations.

3. The three conditions necessary for a function f to be continuous at a number c are $\boxed{f(c) \text{ is defined}}$, $\boxed{\lim\limits_{x\to c} f(x) \text{ exists}}$, and $\boxed{\lim\limits_{x\to c} f(x) = f(c)}$.

4. $\boxed{\text{True}}$. If $f(x)$ is continuous at 0, then $g(x) = \frac{1}{4}f(x)$ is continuous at 0.

5. $\boxed{\text{False}}$. If f is a function defined everywhere in an open interval containing c, except possibly at c, then the number c is called a removable discontinuity of f if the function f is not continuous at c but $\lim\limits_{x\to c} f(x)$ exists.

6. $\boxed{\text{False}}$. If a function f is discontinuous at a number c, then it might be the case that $\lim\limits_{x\to c} f(x)$ does not exist. However, the function could be discontinuous at c because $\lim\limits_{x\to c} f(x)$ exists but either $f(c)$ is not defined or $\lim\limits_{x\to c} f(x) \neq f(c)$.

7. $\boxed{\text{False}}$. If a function f is continuous on an open interval (a, b), then it is continuous on the closed interval $[a, b]$ only if the function is also continuous from the right at the number a and continuous from the left at the number b.

8. $\boxed{\text{True}}$. If a function f is continuous on the closed interval $[a, b]$, then f is continuous on the open interval (a, b).

9. This function is $\boxed{\text{discontinuous}}$. When the ball lands on the ground and stops, there will be a jump discontinuity as the velocity abruptly changes from a nonzero value to zero.

10. This function is $\boxed{\text{continuous}}$. Though the temperature might change rapidly when the oven is first turned on and then again after the oven is turned off, there will be no abrupt jumps in the temperature.

11. $\boxed{\text{True}}$. If a function f is continuous on a closed interval $[a, b]$, then the Intermediate Value Theorem guarantees that the function takes on every value y between $f(a)$ and $f(b)$.

12. $\boxed{\text{False}}$. If a function f is continuous on a closed interval $[a, b]$ and $f(a) \neq f(b)$, but both $f(a) > 0$ and $f(b) > 0$, then f **may have** a zero on the open interval (a, b), but the Intermediate Value Theorem cannot guarantee that a zero exists. The Intermediate Value Theorem can never guarantee that a function does not have a zero on a particular interval.

Skill Building

13. (a) The function f is $\boxed{\text{not continuous at } c = -3}$.
    (b) Although f is defined at $c = -3$ with $f(-3) = 1$ and $\lim\limits_{x\to -3} f(x) = -2$, $\boxed{\lim\limits_{x\to -3} f(x) \neq f(-3)}$.
    (c) The discontinuity at $c = -3$ is $\boxed{\text{removable}}$ because $\lim\limits_{x\to -3} f(x)$ exists.
    (d) Redefine f at $c = -3$ so that $\boxed{f(-3) = -2} = \lim\limits_{x\to -3} f(x)$. The resulting function will then be continuous at $c = -3$.

14. (a) Because f is defined at $c = 0$ with $f(0) = 2$, $\lim_{x\to 0} f(x) = 2$, and $\lim_{x\to 0} f(x) = f(0)$, the function f is $\boxed{\text{continuous}}$ at $c = 0$.

15. (a) The function f is $\boxed{\text{not continuous at } c = 2}$.

    (b) Although f is defined at $c = 2$ with $f(2) = 3$, $\boxed{\lim_{x\to 2} f(x) \text{ does not exist}}$.

    (c) The discontinuity at $c = 2$ is $\boxed{\text{not removable}}$ because $\lim_{x\to 2} f(x)$ does not exist.

16. (a) The function f is $\boxed{\text{not continuous}}$ at $c = 3$.

    (b) Though f is defined at $c = 3$ with $f(3) = -1$, $\boxed{\lim_{x\to 3} f(x) \text{ does not exist}}$.

    (c) The discontinuity at $c = 3$ is $\boxed{\text{not removable}}$ because $\lim_{x\to 3} f(x)$ does not exist.

17. (a) Because f is defined at $c = 4$ with $f(4) = 0$, $\lim_{x\to 4} f(x) = 0$, and $\lim_{x\to 4} f(x) = f(4)$, the $\boxed{\text{function } f \text{ is continuous at } c = 4}$.

18. (a) The function f is $\boxed{\text{not continuous}}$ at $c = 5$.

    (b) The function f is discontinuous at $c = 5$ because $\boxed{f \text{ is not defined at } c = 5}$ and because
    $\lim_{x\to 5^-} f(x) = -3$ and $\lim_{x\to 5^+} f(x) = 1$ so that $\boxed{\lim_{x\to 5} f(x) \text{ does not exist}}$.

    (c) The discontinuity at $c = 5$ is $\boxed{\text{not removable}}$, because $\lim_{x\to 5} f(x)$ does not exist,

19. The domain of the function $f(x) = x^2 + 1$ is the set of all real numbers, so f is defined at $c = -1$ with $f(-1) = 2$. Next,

$$\lim_{x\to -1} f(x) = \lim_{x\to -1} (x^2 + 1) = (-1)^2 + 1 = 2,$$

so that $\lim_{x\to -1} f(x)$ exists. Finally, $\lim_{x\to -1} f(x) = f(-1)$. Because all three conditions of the definition of continuity at $c = -1$ are satisfied, the function f is $\boxed{\text{continuous at } c = -1}$.

20. The domain of the function $f(x) = x^3 - 5$ is the set of all real numbers, so f is defined at $c = 5$ with $f(5) = 120$. Next,

$$\lim_{x\to 5} f(x) = \lim_{x\to 5} (x^3 - 5) = 5^3 - 5 = 120,$$

so that $\lim_{x\to 5} f(x)$ exists. Finally, $\lim_{x\to 5} f(x) = f(5)$. Because all three conditions of the definition of continuity at $c = 5$ are satisfied, the function f is $\boxed{\text{continuous}}$ at $c = 5$.

21. Because $x^2 + 4$ is never equal to zero for any real number x, the domain of the function $f(x) = \dfrac{x}{x^2+4}$ is the set of all real numbers. Thus, f is defined at $c = -2$, with $f(-2) = -\frac{1}{4}$. Next,

$$\lim_{x\to -2} f(x) = \lim_{x\to -2} \frac{x}{x^2+4} = \frac{-2}{(-2)^2+2} = -\frac{1}{4},$$

so that $\lim_{x\to -2} f(x)$ exists. Finally, $\lim_{x\to -2} f(x) = f(-2)$. Because all three conditions of the definition of continuity at $c = -2$ are satisfied, the function f is $\boxed{\text{continuous at } c = -2}$.

22. The domain of the function $f(x) = \dfrac{x}{x-2}$ is the set $\{x|x \neq 2\}$. Because f is not defined at $c = 2$, the function f is $\boxed{\text{not continuous}}$ at $c = 2$.

23. The domain of the function f is the set of all real numbers, so f is defined at $c = 2$ with $f(2) = 2(2) + 5 = 9$. Next,
$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (2x+5) = 9 \quad \text{and} \quad \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (4x+1) = 9,$$
so that $\lim_{x\to 2} f(x)$ exists and is equal to 9. Finally, $\lim_{x\to 2} f(x) = f(2)$. Because all three conditions of the definition of continuity at $c = 2$ are satisfied, the function f is $\boxed{\text{continuous at } c = 2}$.

24. The domain of the function f is the set of all real numbers, so f is defined at $c = 0$ with $f(0) = 2(0) + 1 = 2$. Next,
$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (2x+1) = 1 \quad \text{and} \quad \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 2x = 0,$$
so that $\lim_{x\to 0} f(x)$ does not exist. Therefore, the function f is $\boxed{\text{not continuous}}$ at $c = 0$.

25. The domain of the function f is the set of all real numbers, so f is defined at $c = 1$ with $f(1) = 4$. Next,
$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (3x-1) = 2 \quad \text{and} \quad \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 2x = 2,$$
so that $\lim_{x\to 1} f(x)$ exists and is equal to 2. However, $\lim_{x\to 1} f(x) \neq f(1)$, so the function f is $\boxed{\text{not continuous at } c = 1}$.

26. The domain of the function f is the set of all real numbers, so f is defined at $c = 1$ with $f(1) = 2$. Next,
$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (3x-1) = 2 \quad \text{and} \quad \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 2x = 2,$$
so that $\lim_{x\to 1} f(x)$ exists and is equal to 2. Finally, $\lim_{x\to 1} f(x) = f(1)$. Because all three conditions of the definition of continuity at $c = 1$ are satisfied, the function f is $\boxed{\text{continuous}}$ at $c = 1$.

27. The domain of the function f is the set $\{x|x \neq 1\}$. Because f is not defined at $c = 1$, the function f is $\boxed{\text{not continuous at } c = 1}$.

28. The domain of the function f is the set of all real numbers, so f is defined at $c = 1$ with $f(1) = 2$. Next,
$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (3x-1) = 2 \quad \text{and} \quad \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 3x = 3,$$
so that $\lim_{x\to 1} f(x)$ does not exist. Therefore, the function f is $\boxed{\text{not continuous}}$ at $c = 1$.

29. The domain of the function f is the set of all real numbers, so f is defined at $c = 0$ with $f(0) = 0$. Next,
$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x^2 = 0 \quad \text{and} \quad \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 2x = 0,$$
so that $\lim_{x\to 0} f(x)$ exists and is equal to 0. Finally, $\lim_{x\to 0} f(x) = f(0)$. Because all three conditions of the definition of continuity at $c = 0$ are satisfied, the function f is $\boxed{\text{continuous at } c = 0}$.

30. The domain of the function f is the set of all real numbers, so f is defined at $c = -1$ with $f(-1) = 2$. Next,

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} x^2 = 1 \quad \text{and} \quad \lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} (-3x+2) = 5,$$

so that $\lim_{x\to -1} f(x)$ does not exist. Therefore, the function f is $\boxed{\text{not continuous}}$ at $c = -1$.

31. The domain of the function f is the set $\{x|x < 4\}$, so f is defined at $c = 0$ with $f(0) = 4$. Next,

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (4-3x^2) = 4 \quad \text{and} \quad \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \sqrt{\frac{16-x^2}{4-x}} = 2,$$

so that $\lim_{x\to 0} f(x)$ does not exist. Therefore, the function f is $\boxed{\text{not continuous at } c = 0}$.

32. The domain of the function f is the set $\{x|x \geq -4\}$, so f is defined at $c = 4$ with $f(4) = \sqrt{8} = 2\sqrt{2}$. Next,

$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} \sqrt{4+x} = 2\sqrt{2}$$

and

$$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} \sqrt{\frac{x^2-3x-4}{x-4}} = \lim_{x\to 4^+} \sqrt{\frac{(x-4)(x+1)}{x-4}} = \lim_{x\to 4+} \sqrt{x+1} = \sqrt{5},$$

so that $\lim_{x\to 4} f(x)$ does not exist. Therefore, the function f is $\boxed{\text{not continuous}}$ at $c = 4$.

33. Because

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2}(x+2) = 4,$$

$\boxed{f(2) \text{ should be assigned the value } 4}$. Then $\lim_{x\to 2} f(x) = f(2)$, and the resulting function will be continuous at $c = 2$.

34. Because

$$\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{x^2+x-12}{x-3} = \lim_{x\to 3} \frac{(x-3)(x+4)}{x-3} = \lim_{x\to 3}(x+4) = 7,$$

$\boxed{f(3) \text{ should be assigned the value } 7}$. Then $\lim_{x\to 3} f(x) = f(3)$, and the resulting function will be continuous at $c = 3$.

35. Because

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (1+x) = 2 \quad \text{and} \quad \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 2x = 2,$$

it follows that $\lim_{x\to 1} f(x) = 2$, so $\boxed{f(1) \text{ should be assigned the value } 2}$. Then $\lim_{x\to 1} f(x) = f(1)$, and the resulting function will be continuous at $c = 1$.

36. Because

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} (x^2+5x) = -4 \quad \text{and} \quad \lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} (x-3) = -4,$$

it follows that $\lim_{x\to -1} f(x) = -4$, so $\boxed{f(-1) \text{ should be assigned the value } -4}$. Then $\lim_{x\to -1} f(x) = f(-1)$, and the resulting function will be continuous at $c = -1$.

37. The domain of the function $f(x) = \dfrac{x^2-9}{x-3}$ is the set $\{x|x \neq 3\}$. Since f is not defined at 3, f is $\boxed{\text{not continuous on the interval } [-3,3]}$.

Let c be any number in the open interval $(-3,3)$. Then

$$\lim_{x\to c} f(x) = \lim_{x\to c} \frac{x^2-9}{x-3} = \frac{c^2-9}{c-3} = f(c),$$

so f is continuous on the open interval $(-3,3)$. Also,

$$\lim_{x\to -3^+} f(x) = \lim_{x\to -3^+} \frac{x^2-9}{x-3} = 0 = f(-3),$$

so f is also continuous from the right at $c = -3$. However, since $c = 3$ is not in the domain of f, f is not continuous from the left at $c = 3$. Therefore, the function f is $\boxed{\text{continuous on the interval } [-3,3)}$.

38. The domain of the function $f(x) = 1 + \dfrac{1}{x}$ is the set $\{x|x \neq 0\}$. Since f is not defined at 0, f is $\boxed{\text{not continuous on the interval } [-1,0]}$.

Let c be any number in the open interval $(-1,0)$. Then

$$\lim_{x\to c} f(x) = \lim_{x\to c} \left(1 + \frac{1}{x}\right) = 1 + \frac{1}{c} = f(c),$$

so f is continuous on the open interval $(-1,0)$. Also,

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} \left(1 + \frac{1}{x}\right) = 0 = f(-1),$$

so f is also continuous from the right at $c = -1$. However, since $c = 0$ is not in the domain of f, f is not continuous from the left at $c = 0$. Therefore, the function f is $\boxed{\text{continuous on the interval } [-1,0)}$.

39. The domain of $f(x) = \dfrac{1}{\sqrt{x^2-9}}$ is the set $\{x|\ |x| > 3\}$, so that f is not defined anywhere on the closed interval $[-3,3]$. Therefore, the function f is $\boxed{\text{not continuous at any number in the interval } [-3,3]}$.

40. The domain of the function $f(x) = \sqrt{9-x^2}$ is the set $\{x|-3 \leq x \leq 3\}$, so that f is defined on the closed interval $[-3,3]$. Let c be any number in the open interval $(-3,3)$. Then

$$\lim_{x\to c} f(x) = \lim_{x\to c} \sqrt{9-x^2} = \sqrt{9-c^2} = f(c),$$

so f is continous on the open interval $(-3,3)$. At $c = -3$,

$$\lim_{x\to -3^+} f(x) = \lim_{x\to -3^+} \sqrt{9-x^2} = 0 = f(-3),$$

so f is also continuous from the right at $c = -3$. Finally, at $c = 3$,

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \sqrt{9-x^2} = 0 = f(3),$$

and f is continuous from the left at $c = 3$. Therefore, the function f is $\boxed{\text{continuous on the closed interval } [-3,3]}$.

41. Let $g(x) = 2x^2 + 5x$ and $h(x) = \frac{1}{x}$. The domain of the polynomial function g is the set of all real numbers, and the domain of the rational function h is the set $\{x|x \neq 0\}$. Each function is continuous on its domain. Because the function f is the difference of the functions g and h, the domain of f is the intersection of the domains of g and h; that is, the $\boxed{\text{domain of } f \text{ is the set } \{x|x \neq 0\}}$. The function f is continuous for all values x at which both g and h are continuous, so that f is $\boxed{\text{continuous on the set } \{x|x \neq 0\}}$.

42. Let $g(x) = x + 1$ and $h(x) = \frac{2x}{x^2 + 5}$. The domain of the polynomial function g is the set of all real numbers, and the domain of the rational function h is also the set of all real numbers (because $x^2 + 5$ is never equal to zero for any real number x). Each function is continuous on its domain. Because the function f is the sum of the functions g and h, the domain of f is the intersection of the domains of g and h; that is, the $\boxed{\text{domain of } f \text{ is the set of all real numbers}}$. The function f is continuous for all values x at which both g and h are continuous, so that f is $\boxed{\text{continuous for all real numbers}}$.

43. Let $g(x) = x - 1$ and $h(x) = x^2 + x + 1$. The domain of the polynomial function g is the set of all real numbers, and the domain of the polynomial function h is also the set of all real numbers. Each function is continuous on its domain. Because the function f is the product of the functions g and h, the domain of f is the intersection of the domains of g and h; that is, the $\boxed{\text{domain of } f \text{ is the set of all real numbers}}$. The function f is continuous for all values x at which both g and h are continuous, so that f is $\boxed{\text{continuous for all real numbers}}$.

44. Let $g(x) = \sqrt{x}$ and $h(x) = x^3 - 5$. The domain of the function g is the set $\{x|x \geq 0\}$, and the domain of the polynomial function h is the set of all real numbers. Each function is continuous on its domain. Because the function f is the product of the functions g and h, the domain of f is the intersection of the domains of g and h; that is, the $\boxed{\text{domain of } f \text{ is the set } \{x|x \geq 0\}}$. The function f is continuous at all values x at which both g and h are continuous, so that f is $\boxed{\text{continuous on the set } \{x|x \geq 0\}}$.

45. Let $g(x) = x - 9$ and $h(x) = \sqrt{x} - 3$. The domain of the polynomial function g is the set of all real numbers, and the domain of the function h is the set $\{x|x \geq 0\}$. Each function is continuous on its domain. Because the function f is the quotient of the functions g and h, the domain of f is the intersection of the domains of g and h excluding any x for which $h(x) = 0$; that is, the $\boxed{\text{domain of } f \text{ is the set } \{x|x \geq 0, x \neq 9\}}$. The function f is continuous for all values x at which both g and h are continuous, excluding any x for which $h(x) = 0$, so that f is $\boxed{\text{continuous on the set } \{x|x \geq 0, x \neq 9\}}$.

46. Let $g(x) = x - 4$ and $h(x) = \sqrt{x} - 2$. The domain of the polynomial function g is the set of all real numbers, and the domain of the function h is the set $\{x|x \geq 0\}$. Each function is continuous on its domain. Because the function f is the quotient of the functions g and h, the domain of f is the intersection of the domains of g and h excluding any x for which $h(x) = 0$; that is, the $\boxed{\text{domain of } f \text{ is the set } \{x|x \geq 0, x \neq 4\}}$. The function f is continuous for all values x at which both g and h are continuous, excluding any x for which $h(x) = 0$, so that f is $\boxed{\text{continuous on the set } \{x|x \geq 0, x \neq 4\}}$.

47. Let $g(x) = \sqrt{x}$ and $h(x) = \frac{x^2 + 1}{2 - x}$. The domain of the function g is the set $\{x|x \geq 0\}$, and the domain of the rational function h is the set $\{x|x \neq 2\}$. Each function is continuous on its domain. Moreover, the solution of the inequality $h(x) \geq 0$ is the set $\{x|x < 2\}$. Because f is the composition $g(h(x))$ and $h(x)$ is in the domain of g for $x < 2$, it follows that the $\boxed{\text{domain of } f \text{ is the set } \{x|x < 2\}}$. Finally, the function f is

continuous at c provided h is continuous at c and g is continuous at $h(c)$; thus, f is $\boxed{\text{continuous on the set } \{x|x < 2\}}$.

48. Let $g(x) = \sqrt{x}$ and $h(x) = \dfrac{4}{x^2 - 1}$. The domain of the function g is the set $\{x|x \geq 0\}$, and the domain of the rational function h is the set $\{x|x \neq \pm 1\}$. Each function is continuous on its domain. Moreover, the solution of the inequality $h(x) \geq 0$ is the set $\{x|\ |x| > 1\}$. Because f is the composition $g(h(x))$ and $h(x)$ is in the domain of g for $|x| > 1$, it follows that the $\boxed{\text{domain of } f \text{ is the set } \{x|\ |x| > 1\}}$. Finally, the function f is continuous at c provided h is continuous at c and g is continuous at $h(c)$; thus, f is $\boxed{\text{continuous on the set } \{x|\ |x| > 1\}}$.

49. Let $g(x) = x^{2/3}$ and $h(x) = 2x^2 + 5x - 3$. The domain of the function g is the set of all real numbers, and the domain of the polynomial function h is also the set of all real numbers. Each function is continuous on its domain. Because f is the composition $g(h(x))$ and $h(x)$ is always in the domain of g, it follows that the $\boxed{\text{domain of } f \text{ is the set of all real numbers}}$. Finally, the function f is continuous at c provided h is continuous at c and g is continuous at $h(c)$; thus, f is $\boxed{\text{continuous on the set of all real numbers}}$.

50. Let $g(x) = x^{1/2}$ and $h(x) = x + 2$. The domain of the function g is the set $\{x|x \geq 0\}$, and the domain of the polynomial function h is the set of all real numbers. Each function is continuous on its domain. Moreover, the solution of the inequality $h(x) \geq 0$ is the set $\{x|x \geq -2\}$. Because f is the composition $g(h(x))$ and $h(x)$ is in the domain of g for $x \geq -2$, it follows that the $\boxed{\text{domain of } f \text{ is the set } \{x|x \geq -2\}}$. Finally, the function f is continuous at c provided h is continuous at c and g is continuous at $h(c)$; thus, f is $\boxed{\text{continuous on the set } \{x|x \geq -2\}}$.

51. Because f is defined at $c = 0$ with $f(0) = \sqrt{15}$ and

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \sqrt{15 - 3x} = \sqrt{15} = f(0),$$

the function f is $\boxed{\text{continuous}}$ at $c = 0$.

52. Because

$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} \lfloor x - 2 \rfloor = 1 \quad \text{but} \quad \lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} \lfloor x - 2 \rfloor = 2,$$

it follows that $\boxed{\lim_{x\to 4} f(x) \text{ does not exist}}$ and f is $\boxed{\text{not continuous}}$ at $c = 4$.

53. Because

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (9 - x^2) = 0 \quad \text{but} \quad \lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \lfloor x - 2 \rfloor = 1,$$

it follows that $\boxed{\lim_{x\to 3} f(x) \text{ does not exist}}$ and f is $\boxed{\text{not continuous}}$ at $c = 3$.

54. Because

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \sqrt{15 - 3x} = 3 \quad \text{but} \quad \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (9 - x^2) = 5,$$

it follows that $\boxed{\lim_{x\to 2} f(x) \text{ does not exist}}$ and f is $\boxed{\text{not continuous}}$ at $c = 2$.

55. Because f is defined at $c = 1$ with $f(1) = \sqrt{12} = 2\sqrt{3}$ and

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \sqrt{15 - 3x} = \sqrt{12} = 2\sqrt{3} = f(1),$$

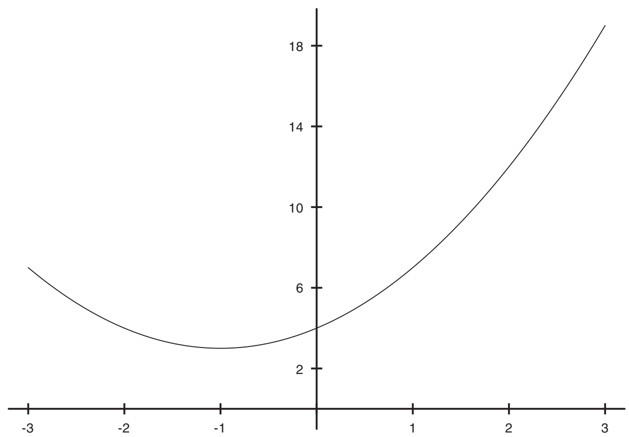
the function f is $\boxed{\text{continuous}}$ at $c = 1$.

56. Because f is defined at $c = 2.5$ with $f(2.5) = 2.75$ and

$$\lim_{x\to 2.5} f(x) = \lim_{x\to 2.5} (9 - x^2) = 9 - 2.5^2 = 2.75 = f(2.5),$$

the function f is $\boxed{\text{continuous}}$ at $c = 2.5$.

57. (a) A graph of the function f is shown below

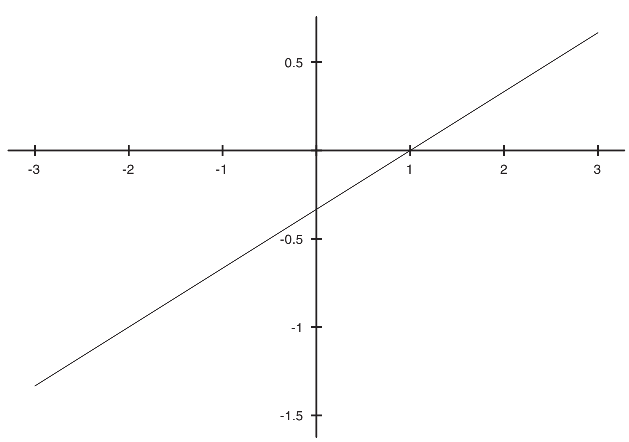


(b) Based on the graph from part (a), the function f appears to be $\boxed{\text{continuous on the set of all real numbers}}$.

(c) The polynomial functions x^3-8 and $x-2$ are continuous on the set of all real numbers. Because the function f is the quotient of these two polynomials, f is continuous on the set of all real numbers excluding any values for x at which $x - 2 = 0$. Thus, f is $\boxed{\text{continuous on the set } \{x|x \neq 2\}}$.

(d) Answers will vary. One possible response is that graphing technology can be a useful tool to suggest where a function is continuous, but "conclusions" drawn from graphing technology should always be confirmed using some basic analysis.

58. (a) A graph of the function f is shown below



(b) Based on the graph from part (a), the function f appears to be $\boxed{\text{continuous on the set of all real numbers}}$.

(c) The polynomial functions $x^2 - 3x + 2$ and $3x - 6$ are continuous on the set of all real numbers. Because the function f is the quotient of these two polynomials, f is continuous on the set of all real numbers excluding any values for x at which $3x - 6 = 0$. Thus, f is $\boxed{\text{continuous on the set } \{x|x \neq 2\}}$.

(d) Answers will vary. One possible response is that graphing technology can be a useful tool to suggest where a function is continuous, but "conclusions" drawn from graphing technology should always be confirmed using some basic analysis.

59. The polynomial function $f(x) = x^3 - 3x$ is continuous for all real numbers, so it is continuous on the closed interval $[-2, 2]$. Because $f(-2) = (-2)^3 - 3(-2) = -2 < 0$ and $f(2) = 2^3 - 3(2) = 2 > 0$, the Intermediate Value Theorem guarantees that $\boxed{f \text{ must have a zero on the interval } (-2, 2)}$.

60. The polynomial function $f(x) = x^4 - 1$ is continuous for all real numbers, so it is continuous on the closed interval $[-2, 2]$. Because $f(-2) = (-2)^4 - 1 = 15 > 0$ and $f(2) = 2^4 - 1 = 15 > 0$, the $\boxed{\text{Intermediate Value Theorem gives no information}}$ about the presence of a zero of f on the interval $(-2, 2)$.

61. The domain of the function $f(x) = \dfrac{x}{(x+1)^2} - 1$ is the set $\{x|x \neq -1\}$. Because f is the difference of a rational function and a polynomial function, it is continuous on its domain. It follows that f is continuous on the closed interval $[10, 20]$. Now,

$$f(10) = \frac{10}{11^2} - 1 = -\frac{111}{121} < 0 \quad \text{and} \quad f(20) = \frac{20}{21^2} - 1 = -\frac{421}{441} < 0,$$

so the $\boxed{\text{Intermediate Value Theorem gives no information}}$ about the presence of a zero of f on the interval $(10, 20)$

62. The polynomial function $f(x) = x^3 - 2x^2 - x + 2$ is continuous for all real numbers, so it is continuous on the closed interval $[3, 4]$. Because $f(3) = 3^3 - 2(3)^2 - 3 + 2 = 8 > 0$ and $f(4) = 4^3 - 2(4)^2 - 4 + 2 = 30 > 0$, the $\boxed{\text{Intermediate Value Theorem gives no information}}$ about the presence of a zero of f on the interval $(3, 4)$.

63. The domain of the function $f(x) = \dfrac{x^3 - 1}{x - 1}$ is the set $\{x|x \neq 1\}$. Because the closed interval $[0, 2]$ contains $x = 1$, f is not continuous on this interval, so the Intermediate Value Theorem does not apply. Therefore, the $\boxed{\text{Intermediate Value Theorem gives no information}}$ about the presence of a zero of f on the interval $(0, 2)$.

64. The domain of the function $f(x) = \dfrac{x^2 + 3x + 2}{x^2 - 1}$ is the set $\{x|x \neq \pm 1\}$. Because the closed interval $[-3, 0]$ contains $x = -1$, f is not continuous on this interval, so the Intermediate Value Theorem does not apply. Therefore, the $\boxed{\text{Intermediate Value Theorem gives no information}}$ about the presence of a zero of f on the interval $(-3, 0)$.

65. The polynomial function $f(x) = x^3 + 3x - 5$ is continuous for all real numbers, so it is continuous on the closed interval $[1, 2]$. Because $f(1) = 1^3 + 3(1) - 5 = -1 < 0$ and $f(2) = 2^3 + 3(2) - 5 = 9 > 0$, the Intermediate Value Theorem guarantees that f must have a zero on the interval $(1, 2)$. To approximate this zero, subdivide the interval $[1, 2]$ into 10 subintervals, each of length 0.1, and evaluate f at each endpoint. The results are shown in the first two columns of the table below. Because $f(1.1) = -0.369 < 0$ and

$f(1.2) = 0.328 > 0$, the Intermediate Value Theorem guarantees the zero lies in the interval $(1.1, 1.2)$. Repeating the process by subdividing the interval $[1.1, 1.2]$ into 10 subintervals of length 0.01 yields the results in the middle two columns of the table, where the function values have been rounded to five decimal places for display purposes. The zero has now been bracketed in the interval $(1.15, 1.16)$. Repeating the subdivision process once more, the results in the last two columns of the table are produced, again with the function values rounded to five decimal places. Examining the function values in the last column, it follows that the zero of the function f is $\boxed{1.154}$, correct to three decimal places.

$[1, 2]$		$[1.1, 1.2]$		$[1.15, 1.16]$	
x	$f(x)$	x	$f(x)$	x	$f(x)$
1.0	−1.000	1.10	−0.36900	1.150	−0.02913
1.1	−0.369	1.11	−0.30237	1.151	−0.02215
1.2	0.328	1.12	−0.23507	1.152	−0.01518
1.3	1.097	1.13	−0.16710	1.153	−0.00819
1.4	1.944	1.14	−0.09846	1.154	−0.00120
1.5	2.875	1.15	−0.02913	1.155	0.00580
1.6	3.896	1.16	0.04090	1.156	0.01280
1.7	5.013	1.17	0.11161	1.157	0.01982
1.8	6.232	1.18	0.18303	1.158	0.02684
1.9	7.559	1.19	0.25516	1.159	0.03386
2.0	9.000	1.20	0.32800	1.160	0.04090

66. The polynomial function $f(x) = x^3 - 4x + 2$ is continuous for all real numbers, so it is continuous on the closed interval $[1, 2]$. Because $f(1) = 1^3 - 4(1) + 2 = -1 < 0$ and $f(2) = 2^3 - 4(2) + 2 = 2 > 0$, the Intermediate Value Theorem guarantees that f must have a zero on the interval $(1, 2)$. To approximate this zero, subdivide the interval $[1, 2]$ into 10 subintervals, each of length 0.1, and evaluate f at each endpoint. The results are shown in the first two columns of the table below. Because $f(1.6) = -0.304 < 0$ and $f(1.7) = 0.113 > 0$, the Intermediate Value Theorem guarantees the zero lies in the interval $(1.6, 1.7)$. Repeating the process by subdividing the interval $[1.6, 1.7]$ into 10 subintervals of length 0.01 yields the results in the middle two columns of the table, where the function values have been rounded to five decimal places for display purposes. The zero has now been bracketed in the interval $(1.67, 1.68)$. Repeating the subdivision process once more, the results in the last two columns of the table are produced, again with the function values rounded to five decimal places. Examining the function values in the last column, it follows that the zero of the function f is $\boxed{1.675}$, correct to three decimal places.

$[1, 2]$		$[1.6, 1.7]$		$[1.67, 1.68]$	
x	$f(x)$	x	$f(x)$	x	$f(x)$
1.0	−1.000	1.60	−0.30400	1.670	−0.02254
1.1	−1.069	1.61	−0.26672	1.671	−0.01817
1.2	−1.072	1.62	−0.22847	1.672	−0.01378
1.3	−1.003	1.63	−0.18925	1.673	−0.00939
1.4	−0.856	1.64	−0.14906	1.674	−0.00499
1.5	−0.625	1.65	−0.10788	1.675	−0.00058
1.6	−0.304	1.66	−0.06570	1.676	0.00384
1.7	0.133	1.67	−0.02254	1.677	0.00828
1.8	0.632	1.68	0.02163	1.678	0.01272
1.9	1.259	1.69	0.06681	1.679	0.01717
2.0	2.000	1.70	0.11300	1.680	0.02163

67. The polynomial function $f(x) = 2x^3 + 3x^2 + 4x - 1$ is continuous for all real numbers, so it is continuous on the closed interval $[0, 1]$. Because $f(0) = -1 < 0$ and $f(1) = 2+3+4-1 = 8 > 0$, the Intermediate Value Theorem guarantees that f must have a zero on the interval $(0, 1)$. To approximate this zero, subdivide the interval $[0, 1]$ into 10 subintervals, each of length 0.1, and evaluate f at each endpoint. The results are shown in the first two columns of the table below. Because $f(0.2) = -0.064 < 0$ and $f(0.3) = 0.524 > 0$, the Intermediate Value Theorem guarantees the zero lies in the interval $(0.2, 0.3)$. Repeating the process by subdividing the interval $[0.2, 0.3]$ into 10 subintervals of length 0.01 yields the results in the middle two columns of the table, where the function values have been

rounded to five decimal places for display purposes. The zero has now been bracketed in the interval $(0.21, 0.22)$. Repeating the subdivision process once more, the results in the last two columns of the table are produced, again with the function values rounded to five decimal places. Examining the function values in the last column, it follows that the zero of the function f is $\boxed{0.211}$, correct to three decimal places.

$[0,1]$		$[0.2,0.3]$		$[0.21,0.22]$	
x	$f(x)$	x	$f(x)$	x	$f(x)$
0.0	−1.000	0.20	−0.06400	0.210	−0.00918
0.1	−0.568	0.21	−0.00918	0.211	−0.00365
0.2	−0.064	0.22	0.04650	0.212	0.00189
0.3	0.524	0.23	0.10303	0.213	0.00743
0.4	1.208	0.24	0.16045	0.214	0.01299
0.5	2.000	0.25	0.21875	0.215	0.01855
0.6	2.912	0.26	0.27795	0.216	0.02412
0.7	3.956	0.27	0.33807	0.217	0.02970
0.8	5.144	0.28	0.39910	0.218	0.03529
0.9	6.488	0.29	0.46108	0.219	0.04089
1.0	8.000	0.30	0.52400	0.220	0.04650

68. The polynomial function $f(x) = x^3 - x^2 - 2x + 1$ is continuous for all real numbers, so it is continuous on the closed interval $[0, 1]$. Because $f(0) = 1 > 0$ and $f(1) = 1 - 1 - 2 + 1 = -1 < 0$, the Intermediate Value Theorem guarantees that f must have a zero on the interval $(0, 1)$. To approximate this zero, subdivide the interval $[0, 1]$ into 10 subintervals, each of length 0.1, and evaluate f at each endpoint. The results are shown in the first two columns of the table below. Because $f(0.4) = 0.104 > 0$ and $f(0.5) = -0.125 < 0$, the Intermediate Value Theorem guarantees the zero lies in the interval $(0.4, 0.5)$. Repeating the process by subdividing the interval $[0.4, 0.5]$ into 10 subintervals of length 0.01 yields the results in the middle two columns of the table, where the function values have been rounded to five decimal places for display purposes. The zero has now been bracketed in the interval $(0.44, 0.45)$. Repeating the subdivision process once more, the results in the last two columns of the table are produced, again with the function values rounded to five decimal places. Examining the function values in the last column, it follows that the zero of the function f is $\boxed{0.445}$, correct to three decimal places.

$[0,1]$		$[0.4,0.5]$		$[0.44,0.45]$	
x	$f(x)$	x	$f(x)$	x	$f(x)$
0.0	1.000	0.40	0.10400	0.440	0.01158
0.1	0.791	0.41	0.08082	0.441	0.00929
0.2	0.568	0.42	0.05769	0.442	0.00699
0.3	0.337	0.43	0.03461	0.443	0.00469
0.4	0.104	0.44	0.01158	0.444	0.00239
0.5	−0.125	0.45	−0.01138	0.445	0.00010
0.6	−0.344	0.46	−0.03426	0.446	−0.00220
0.7	−0.547	0.47	−0.05708	0.447	−0.00449
0.8	−0.728	0.48	−0.07981	0.448	−0.00679
0.9	−0.881	0.49	−0.10245	0.449	−0.00908
1.0	−1.000	0.50	−0.12500	0.450	−0.01138

69. The polynomial function $f(x) = x^3 - 6x - 12$ is continuous for all real numbers, so it is continuous on the closed interval $[3, 4]$. Because $f(3) = 3^3 - 6(3) - 12 = -3 < 0$ and $f(4) = 4^3 - 6(4) - 12 = 28 > 0$, the Intermediate Value Theorem guarantees that f must have a zero on the interval $(3, 4)$. To approximate this zero, subdivide the interval $[3, 4]$ into 10 subintervals, each of length 0.1, and evaluate f at each endpoint. The results are shown in the first two columns of the table below. Because $f(3.1) = -0.809 < 0$ and $f(3.2) = 1.568 > 0$, the Intermediate Value Theorem guarantees the zero lies in the interval $(3.1, 3.2)$. Repeating the process by subdividing the interval $[3.1, 3.2]$ into 10 subintervals of length 0.01 yields the results in the middle two columns of the table, where the function values have been rounded to five decimal places for display purposes. The zero has now been bracketed in the interval $(3.13, 3.14)$. Repeating the subdivision process once more, the results in the last two columns of the table are produced, again with the function

values rounded to five decimal places. Examining the function values in the last column, it follows that the zero of the function f is $\boxed{3.134}$, correct to three decimal places.

[3, 4]		[3.1, 3.2]		[3.13, 3.14]	
x	$f(x)$	x	$f(x)$	x	$f(x)$
3.0	−3.000	3.10	−0.80900	3.130	−0.11570
3.1	−0.809	3.11	−0.57977	3.131	−0.09230
3.2	1.568	3.12	−0.34867	3.132	−0.06888
3.3	4.137	3.13	−0.11570	3.133	−0.04545
3.4	6.904	3.14	0.11914	3.134	−0.02199
3.5	9.875	3.15	0.35587	3.135	0.00149
3.6	13.056	3.16	0.59450	3.136	0.02498
3.7	16.453	3.17	0.83501	3.137	0.04849
3.8	20.072	3.18	1.07743	3.138	0.07202
3.9	23.919	3.19	1.32176	3.139	0.09557
4.0	28.000	3.20	1.56800	3.140	0.11914

70. The polynomial function $f(x) = 3x^3 + 5x - 40$ is continuous for all real numbers, so it is continuous on the closed interval $[2, 3]$. Because $f(2) = 3(2)^3 + 5(2) - 40 = -6 < 0$ and $f(3) = 3(3)^3 + 5(3) - 40 = 56 > 0$, the Intermediate Value Theorem guarantees that f must have a zero on the interval $(2, 3)$. To approximate this zero, subdivide the interval $[2, 3]$ into 10 subintervals, each of length 0.1, and evaluate f at each endpoint. The results are shown in the first two columns of the table below. Because $f(2.1) = -1.717 < 0$ and $f(2.2) = 2.2944 > 0$, the Intermediate Value Theorem guarantees the zero lies in the interval $(2.1, 2.2)$. Repeating the process by subdividing the interval $[2.1, 2.2]$ into 10 subintervals of length 0.01 yields the results in the middle two columns of the table, where the function values have been rounded to five decimal places for display purposes. The zero has now been bracketed in the interval $(2.13, 2.14)$. Repeating the subdivision process once more, the results in the last two columns of the table are produced, again with the function values rounded to five decimal places. Examining the function values in the last column, it follows that the zero of the function f is $\boxed{2.137}$, correct to three decimal places.

[2, 3]		[2.1, 2.2]		[2.13, 2.14]	
x	$f(x)$	x	$f(x)$	x	$f(x)$
2.0	−6.000	2.10	−1.71700	2.130	−0.35921
2.1	−1.717	2.11	−1.26281	2.131	−0.31336
2.2	2.944	2.12	−0.81562	2.132	−0.26747
2.3	8.001	2.13	−0.35921	2.133	−0.22154
2.4	13.472	2.14	0.10103	2.134	−0.17557
2.5	19.375	2.15	0.56512	2.135	−0.12957
2.6	25.728	2.16	1.03309	2.136	−0.08353
2.7	32.549	2.17	1.50494	2.137	−0.03744
2.8	39.856	2.18	1.98070	2.138	0.00868
2.9	47.667	2.19	2.46038	2.139	0.05483
3.0	56.000	2.20	2.94400	2.140	0.10103

71. The polynomial function $f(x) = x^4 - 2x^3 + 21x - 23$ is continuous for all real numbers, so it is continuous on the closed interval $[1, 2]$. Because $f(1) = 1 - 2 + 21 - 23 = -3 < 0$ and $f(2) = 2^4 - 2(2)^3 + 21(2) - 23 = 19 > 0$, the Intermediate Value Theorem guarantees that f must have a zero on the interval $(1, 2)$. To approximate this zero, subdivide the interval $[1, 2]$ into 10 subintervals, each of length 0.1, and evaluate f at each endpoint. The results are shown in the first two columns of the table below. Because $f(1.1) = -1.0979 < 0$ and $f(1.2) = 0.8176 > 0$, the Intermediate Value Theorem guarantees the zero lies in the interval $(1.1, 1.2)$. Repeating the process by subdividing the interval $[1.1, 1.2]$ into 10 subintervals of length 0.01 yields the results in the middle two columns of the table, where the function values have been rounded to five decimal places for display purposes. The zero has now been bracketed in the interval $(1.15, 1.16)$. Repeating the subdivision process once more, the results in the last two columns of the table are produced, again with the function values rounded to five decimal places. Examining the function values in the last column, it follows that the zero of the function f is $\boxed{1.157}$, correct to three decimal places.

[1, 2]		[1.1, 1.2]		[1.15, 1.16]	
x	$f(x)$	x	$f(x)$	x	$f(x)$
1.0	−3.0000	1.10	−1.09790	1.150	−0.14274
1.1	−1.0979	1.11	−0.90719	1.151	−0.12359
1.2	0.8176	1.12	−0.71634	1.152	−0.10444
1.3	2.7621	1.13	−0.52532	1.153	−0.08529
1.4	4.7536	1.14	−0.33413	1.154	−0.06613
1.5	6.8125	1.15	−0.14274	1.155	−0.04698
1.6	8.9616	1.16	0.04885	1.156	−0.02781
1.7	11.2261	1.17	0.24066	1.157	−0.00865
1.8	13.6336	1.18	0.43271	1.158	0.01051
1.9	16.2141	1.19	0.62502	1.159	0.02968
2.0	19.0000	1.20	0.81760	1.160	0.04885

72. The polynomial function $f(x) = x^4 - x^3 + x - 2$ is continuous for all real numbers, so it is continuous on the closed interval $[1, 2]$. Because $f(1) = 1 - 1 + 1 - 2 = -1 < 0$ and $f(2) = 2^4 - 2^3 + 2 - 2 = 8 > 0$, the Intermediate Value Theorem guarantees that f must have a zero on the interval $(1, 2)$. To approximate this zero, subdivide the interval $[1, 2]$ into 10 subintervals, each of length 0.1, and evaluate f at each endpoint. The results are shown in the first two columns of the table below. Because $f(1.3) = -0.0409 < 0$ and $f(1.4) = 0.4976 > 0$, the Intermediate Value Theorem guarantees the zero lies in the interval $(1.3, 1.4)$. Repeating the process by subdividing the interval $[1.3, 1.4]$ into 10 subintervals of length 0.01 yields the results in the middle two columns of the table, where the function values have been rounded to five decimal places for display purposes. The zero has now been bracketed in the interval $(1.30, 1.31)$. Repeating the subdivision process once more, the results in the last two columns of the table are produced, again with the function values rounded to five decimal places. Examining the function values in the last column, it follows that the zero of the function f is $\boxed{1.308}$, correct to three decimal places.

[1, 2]		[1.3, 1.4]		[1.30, 1.31]	
x	$f(x)$	x	$f(x)$	x	$f(x)$
1.0	−1.0000	1.30	−0.04090	1.300	−0.04090
1.1	−0.7669	1.31	0.00691	1.301	−0.03618
1.2	−0.4544	1.32	0.05599	1.302	−0.03144
1.3	−0.0409	1.33	0.10637	1.303	−0.02669
1.4	0.4976	1.34	0.15808	1.304	−0.02193
1.5	1.1875	1.35	0.21113	1.305	−0.01715
1.6	2.0576	1.36	0.26556	1.306	−0.01237
1.7	3.1391	1.37	0.32140	1.307	−0.00757
1.8	4.4656	1.38	0.37867	1.308	−0.00275
1.9	6.0731	1.39	0.43739	1.309	0.00207
2.0	8.0000	1.40	0.49760	1.310	0.00691

73. (a) The polynomial function $x^2 + 4x$ is continuous on the set of all real numbers and is non-negative on the set $\{x|x \le -4\} \cup \{x|x \ge 0\}$. The function $f(x) = \sqrt{x^2 + 4x} - 2$ is therefore continuous on the set $\{x|x \le -4\} \cup \{x|x \ge 0\}$, which contains the closed interval $[0, 1]$. Because $f(0) = \sqrt{0} - 2 = -2 < 0$ and $f(1) = \sqrt{5} - 2 \approx 0.236 > 0$, the Intermediate Value Theorem guarantees that f must have a zero on the interval $(0, 1)$.

    (b) Using the `FindRoot` command in the computer algebra system *Mathematica* produces the zero $\boxed{x \approx 0.828}$, rounded to three decimal places.

74. (a) The polynomial function $f(x) = x^3 - x + 2$ is continuous for all real numbers, so it is continuous on the closed interval $[-2, 0]$. Because $f(-2) = (-2)^3 - (-2) + 2 = -4 < 0$ and $f(0) = 2 > 0$, the Intermediate Value Theorem guarantees that f must have a zero on the interval $(-2, 0)$.

    (b) Using the `FindRoot` command in the computer algebra system *Mathematica* produces the zero $\boxed{x \approx -1.521}$, rounded to three decimal places.

Applications and Extensions

75. Note: $f(c) = (-1) - 1 = -2$ and $f(d) = (1) - 1 = 0$.

    (a) We have $\lim_{x\to c^-} f(x) = \lim_{x\to -1^-} (x-1) = -2 = f(c)$. $\boxed{f \text{ is left continuous at } c = -1.}$
    We have $\lim_{x\to d^-} f(x) = \lim_{x\to 1^-} x^2 = 1 \neq f(d)$. $\boxed{f \text{ is not left continuous at } d = 1.}$

    (b) We have $\lim_{x\to c^+} f(x) = \lim_{x\to -1^+} x^2 = 1 \neq f(c)$. $\boxed{f \text{ is not right continuous at } c = -1.}$
    We have $\lim_{x\to d^+} f(x) = \lim_{x\to 1^+} (x-1) = 0 = f(d)$. $\boxed{f \text{ is right continuous at } d = 1.}$

76. Note: $f(c) = f(d) = (1)^2 - 1 = 0$

    (a) We have $\lim_{x\to c^-} f(x) = \lim_{x\to -1^-} |x+1| = 0 = f(c)$. $\boxed{f \text{ is left continuous at } c = -1.}$
    We have $\lim_{x\to d^-} f(x) = \lim_{x\to 1^-} (x^2-1) = 0 = f(d)$. $\boxed{f \text{ is left continuous at } d = 1.}$

    (b) We have $\lim_{x\to c^+} f(x) = \lim_{x\to -1^+} (x^2-1) = 0 = f(c)$. $\boxed{f \text{ is right continuous at } c = -1.}$
    We have $\lim_{x\to d^+} f(x) = \lim_{x\to 1^+} |x+1| = 2 \neq f(d)$. $\boxed{f \text{ is not right continuous at } d = 1.}$

77. Note that the domain of f is $\{x|x \leq -1\} \bigcup \{x|x \geq 5\}$. From this fact we can immediately see that $\lim_{x\to -1^+} f(x)$ and $\lim_{x\to 5^-} f(x)$ do not exist. Also, $f(c) = f(d) = 0$.

    (a) We have $\lim_{x\to c^-} f(x) = \lim_{x\to -1^-} \sqrt{(x+1)(x-5)} = 0 = f(c)$. $\boxed{f \text{ is left continuous at } c = -1.}$ $\lim_{x\to d^-} f(x)$ does not exist. $\boxed{f \text{ is not left continuous at } d = 5.}$

    (b) $\lim_{x\to c^+} f(x)$ does not exist. $\boxed{f \text{ is not right continuous at } c = -1.}$
    We have $\lim_{x\to d^+} f(x) = \lim_{x\to 5^+} \sqrt{(x+1)(x-5)} = 0 = f(d)$. $\boxed{f \text{ is right continuous at } d = 5.}$

78. Note that the domain of f is $\{x|x \leq 1\} \bigcup \{x|x \geq 2\}$. From this fact we can immediately see that $\lim_{x\to 1^+} f(x)$ and $\lim_{x\to 2^-} f(x)$ do not exist. Also, $f(c) = f(d) = 0$.

    (a) We have $\lim_{x\to c^-} f(x) = \lim_{x\to 1^-} \sqrt{(x-1)(x-2)} = 0 = f(c)$. $\boxed{f \text{ is left continuous at } c = 1.}$ $\lim_{x\to d^-} f(x)$ does not exist. $\boxed{f \text{ is not left continuous at } d = 2.}$

    (b) $\lim_{x\to c^+} f(x)$ does not exist. $\boxed{f \text{ is not right continuous at } c = 1.}$
    We have $\lim_{x\to d^+} f(x) = \lim_{x\to 2^+} \sqrt{(x-1)(x-2)} = 0 = f(d)$. $\boxed{f \text{ is right continuous at } d = 2.}$

79. (a) Because the Postal Service rounds the weight of the letter up to next whole number of ounces, the first-class postage charged is

$$\boxed{C(w) = \begin{cases} 0.47, & \text{if } 0 < w \leq 1 \\ 0.68, & \text{if } 1 < w \leq 2 \\ 0.89, & \text{if } 2 < w \leq 3 \\ 1.10, & \text{if } 3 < w \leq 3.5. \end{cases}}$$

where postage is measured in dollars and weight is measured in ounces. This can be written compactly in terms of the ceiling function as

$$C(w) = 0.47 + 0.21\lceil w - 1 \rceil.$$

(b) The domain of C is the set $\boxed{\{w \mid 0 < w \le 3.5\}}$.

(c) The function C is continuous on the intervals $(0,1)$, $(1,2)$, $(2,3)$, and $(3,3.5)$ because the function is a constant (polynomial) on each of these intervals. At $w = 1$,

$$\lim_{w\to 1^-} C(w) = \lim_{w\to 1^-} 0.47 = 0.47 \quad \text{and} \quad \lim_{w\to 1^+} = \lim_{w\to 1^+} 0.68 = 0.68,$$

so that $\lim_{w\to 1} C(w)$ does not exist. Similarly, at $w = 2$ and $w = 3$,

$$\lim_{w\to 2^-} C(w) = \lim_{w\to 2^-} 0.68 = 0.68 \quad \text{and} \quad \lim_{w\to 2^+} = \lim_{w\to 2^+} 0.89 = 0.89,$$

and

$$\lim_{w\to 3^-} C(w) = \lim_{w\to 3^-} 0.89 = 0.89 \quad \text{and} \quad \lim_{w\to 3^+} = \lim_{w\to 3^+} 1.10 = 1.10,$$

respectively, so that $\lim_{w\to 2} C(w)$ does not exist and $\lim_{w\to 3} C(w)$ does not exist. Therefore, C is not continuous at $w = 1$, $w = 2$, or $w = 3$. However,

$$\lim_{w\to 1^-} C(w) = 0.47 = C(1), \; \lim_{w\to 2^-} C(w) = 0.68 = C(2), \text{ and } \lim_{w\to 3^-} C(w) = 0.89 = C(3),$$

so C is continuous from the left at $w = 1$, $w = 2$, and $w = 3$. Additionally,

$$\lim_{w\to 3.5^-} C(w) = \lim_{w\to 3.5^-} 1.10 = 1.10 = C(3.5),$$

so C is continuous from the left at $w = 3.5$. Thus, C is continuous on the intervals $\boxed{(0,1],\ (1,2],\ (2,3],\ \text{and } (3,3.5]}$.

(d) At each number where C is not continuous ($w = 1$, $w = 2$, and $w = 3$), the two one-sided limits exist but are not equal, so each discontinuity is a $\boxed{\text{jump discontinuity}}$.

(e) Answers will vary. One possible response is that because any fraction of an ounce results in a charge for a full ounce, it is in the consumer's best interest to have letters weigh as close as possible to a whole number of ounces, without going over.

80. (a) From exercise 60 from Section 1.1, the piecewise function C that models the first-class postage charged for a large envelope weighing w ounces is

$$C(w) = \begin{cases} \$0.94 & \text{if } 0 < w \le 1 \\ \$1.15 & \text{if } 1 < w \le 2 \\ \$1.36 & \text{if } 2 < w \le 3 \\ \$1.57 & \text{if } 3 < w \le 4 \\ \$1.78 & \text{if } 4 < w \le 5 \\ \$1.99 & \text{if } 5 < w \le 6 \\ \$2.20 & \text{if } 6 < w \le 7 \\ \$2.41 & \text{if } 7 < w \le 8 \\ \$2.62 & \text{if } 8 < w \le 9 \\ \$2.83 & \text{if } 9 < w \le 10 \\ \$3.04 & \text{if } 10 < w \le 11 \\ \$3.25 & \text{if } 11 < w \le 12 \\ \$3.46 & \text{if } 12 < w \le 13 \end{cases}$$

(b) The domain of the function $\{w \mid 0 < w \le 13\}$. The weight of these envelopes can be any positive real number up to and including 13 ounces.

(c) The function $C(x)$ is continuous on the intervals (0, 1], (1, 2], (2, 3], (3, 4], (4, 5], (5, 6], (6, 7], (7, 8], (8, 9], (9, 10], (10, 11], (11, 12], and (12, 13].

(d) The function is discontinuous at $x = 1$, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, and 13. For any of these values, the limit does not exist. For example, for $x = 5$, $\lim_{w\to 5^-} C(w) = \$1.78$ and $\lim_{w\to 5^+} C(w) = \$1.99$. Since $\lim_{w\to 5^-} C(w) \ne \lim_{w\to 5^+} C(w)$, we conclude $\lim_{w\to 5} C(w)$ and that $C(w)$ is discontinuous at $x = 5$. Since the left hand limits and the right hand limits are different for $x = 1$, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, and 13, the discontinuities are jump discontinuities.

(e) The answers may vary. One possible answer is that it is in the customer's best interest to have packages that weigh as close as possible to a whole number of ounces, without going over. This avoids paying the extra \$0.21 for the first-class rate.

81. (a) From exercise 94 from Section 1.2, the piecewise function C that models the monthly cost of using x kWH of electricity is $\boxed{C(x) = \begin{cases} 7.87 + 0.02173x & \text{if } 0 \le x \le 1000 \\ -2.13 + 0.03173 & \text{if } x > 1000 \end{cases}}$.

(b) The domain of the function is any nonnegative real number, $\boxed{\{x | x \ge 0\}}$. Customers can use as little ($x = 0$) or as much ($x \to \infty$) electricity as they desire.

(c) $\boxed{C \text{ is continuous on its domain}}$. In particular, for $x = 1000$, $\lim\limits_{x\to 1000^-} C(x) = \lim\limits_{x\to 1000^-} (7.87 + 0.02173x) = \$29.60$ and $\lim\limits_{x\to 1000^+} C(x) = \lim\limits_{x\to 1000^+} (29.60 + 0.03173(x - 1000)) = \$29.60$. Thus, $\lim\limits_{x\to 1000^+} C(x) = \$29.60$.

Since $C(1000)$ is also \$29.60, we conclude that function $C(x)$ is continuous at $x = 1000$.

(d) There are no numbers where C is not continuous.

(e) The answers may vary. One possible answer: To minimize the monthly cost of electricity, it is in the consumer's best interest to minimize the amount of electricity used.

82. (a) Using the rate schedule provided,

$$\boxed{C(x) = \begin{cases} 9.00, & \text{if } 0 \le x \le 10 \\ 9.00 + 0.95(x - 10), & \text{if } 10 < x \le 30 \\ 28.00 + 1.65(x - 30), & \text{if } 30 < x \le 100 \\ 143.50 + 2.20(x - 100), & \text{if } x > 100. \end{cases}}$$

(b) The domain of C is the set $\boxed{\{x | x \ge 0\}}$.

(c) The function C is continuous on the intervals $(0, 10)$, $(10, 30)$, $(30, 100)$, and $(100, \infty)$, because it is a polynomial on each of these intervals. At $x = 10$,

$$\lim_{x\to 10^-} C(x) = \lim_{x\to 10^-} 9.00 = 9.00$$

and

$$\lim_{x\to 10^+} C(x) = \lim_{x\to 10^+} [9.00 + 0.95(x - 10)] = 9.00 + 0.95(10 - 10) = 9.00,$$

so that $\lim\limits_{x\to 10} C(x)$ exists and is equal to 9.00. As $C(10) = 9.00$, it follows that C is continuous at $x = 10$. Similarly,

$$\lim_{x\to 30^-} C(x) = \lim_{x\to 30^-} [9.00 + 0.95(x - 10)] = 9.00 + 0.95(30 - 10) = 28.00$$

and

$$\lim_{x\to 30^+} C(x) = \lim_{x\to 30^+} [28.00 + 1.65(x - 30)] = 28.00 + 1.65(30 - 30) = 28.00,$$

so that $\lim\limits_{x\to 30} C(x) = 28.00 = C(30)$ and C is continuous at $x = 30$. At $w = 100$,

$$\lim_{x\to 100^-} C(x) = \lim_{x\to 100^-} [28.00 + 1.65(x - 30)] = 28.00 + 1.65(100 - 30) = 143.50$$

and

$$\lim_{x\to 100^+} C(x) = \lim_{x\to 100^+} [143.50 + 2.20(x - 100)] = 143.50 + 2.20(100 - 100) = 143.50,$$

so that $\lim_{x\to 100} C(x)$ exists and is equal to $143.50 = C(100)$. Thus, C is continuous at $x = 100$. Finally,

$$\lim_{x\to 0^+} C(x) = \lim_{x\to 0^+} 9.00 = 9.00 = C(0),$$

so C is continuous from the right at $x = 0$. Thus, C is continuous on $\boxed{[0,\infty)}$.

(d) The function C is $\boxed{\text{continuous on its domain}}$.

(e) Answers will vary. One possible response is that there is no "penalty" to the consumer who goes just a little over 10,000 or 30,000 or 100,000 gallons rather than trying to keep consumption at or a little below these amounts.

83. (a) Because

$$\lim_{r\to R^-} g(r) = \lim_{r\to R^-} \frac{Gm}{R^3} r = \frac{Gm}{R^2}$$

and

$$\lim_{r\to R^+} g(r) = \lim_{r\to R^+} \frac{Gm}{r^2} = \frac{Gm}{R^2}$$

are equal, $\boxed{g(R) \text{ must equal } \frac{Gm}{R^2}}$ in order for the gravitational field of Europa to be continuous at its surface.

(b) With $G = 6.67\times 10^{-11}$ m^3 kg^{-1} s^{-2}, $m = 4.8\times 10^{22}$ kg and $R = 1.569\times 10^6$ m,

$$g(R) = \frac{6.67\times 10^{-11}\ \text{m}^3\ \text{kg}^{-1}\ \text{s}^{-2}\cdot 4.8\times 10^{22}\ \text{kg}}{(1.569\times 10^6\ \text{m})^2} \approx \boxed{1.3\ \text{m/s}^2}.$$

(c) Europa's gravity is $\boxed{\text{less than}}$ that on Earth.

84. The function f is continuous on the intervals $(-\infty, 0)$, $(0, 1)$, and $(1, \infty)$, because it is a polynomial on each of these intervals. At $x = 0$,

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (x-1)^2 = 1$$

and

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (A-x)^2 = A^2 = f(0).$$

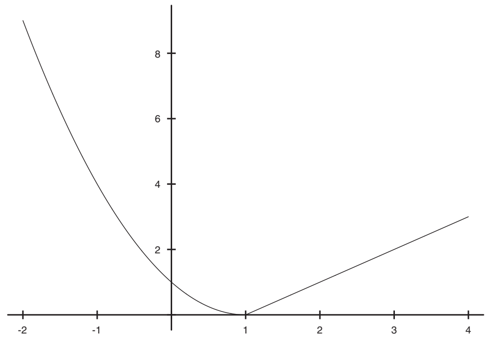
For f to be continuous at $x = 0$, the constant A must satisfy $A^2 = 1$, or $A = \pm 1$. At $x = 1$,

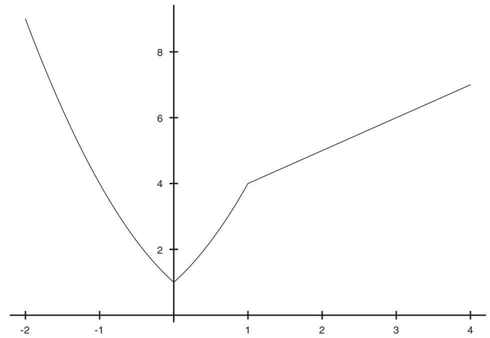
$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (A-x)^2 = (A-1)^2$$

and

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x+B) = 1 + B = f(1).$$

For f to be continuous at $x = 1$, the constant B must satisfy $1 + B = (A-1)^2$, or $B = A^2 - 2A$. There are therefore two sets of values for A and B for which the function f is continuous for all x: $\boxed{\{A = 1, B = -1\} \text{ and } \{A = -1, B = 3\}}$. The figure below left displays the graph with $A = 1$ and $B = -1$; the figure below right displays the graph with $A = -1$ and $B = 3$.





85. The function f is continuous on the intervals $(-\infty, 4)$, $(4, 9)$, and $(9, \infty)$, because it is a polynomial on each of these intervals. At $x = 4$,

$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} (x + A) = 4 + A$$

and

$$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} (x - 1)^2 = 9 = f(4).$$

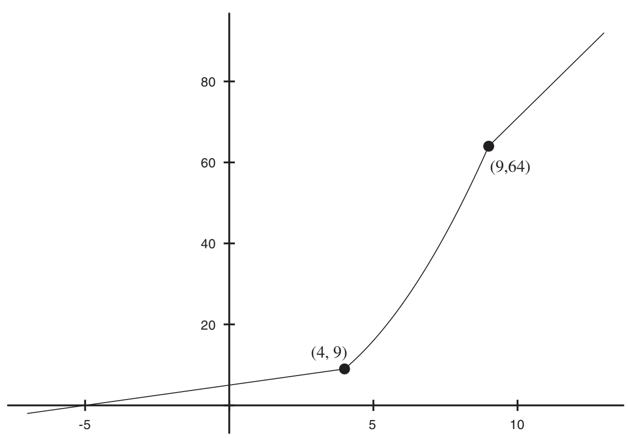
For f to be continuous at $x = 4$, the constant A must satisfy $4 + A = 9$, or $A = 5$. At $x = 9$,

$$\lim_{x\to 9^-} f(x) = \lim_{x\to 9^-} (x - 1)^2 = 64 = f(9)$$

and

$$\lim_{x\to 9^+} f(x) = \lim_{x\to 9^+} (Bx + 1) = 9B + 1.$$

For f to be continuous at $x = 9$, the constant B must satisfy $9B + 1 = 64$, or $B = 7$. Therefore, the function f will be continuous for all x provided $\boxed{A = 5 \text{ and } B = 7}$. The graph of the resulting function is shown below



86. In order to make f continuous at $x = 2$, k should be set equal to

$$\begin{aligned}
\lim_{x\to 2} f(x) &= \lim_{x\to 2} \frac{\sqrt{2x+5} - \sqrt{x+7}}{x-2} = \lim_{x\to 2} \frac{\sqrt{2x+5} - \sqrt{x+7}}{x-2} \cdot \frac{\sqrt{2x+5} + \sqrt{x+7}}{\sqrt{2x+5} + \sqrt{x+7}} \\
&= \lim_{x\to 2} \frac{(2x+5) - (x+7)}{(x-2)(\sqrt{2x+5} + \sqrt{x+7})} = \lim_{x\to 2} \frac{x-2}{(x-2)(\sqrt{2x+5} + \sqrt{x+7})} \\
&= \lim_{x\to 2} \frac{1}{\sqrt{2x+5} + \sqrt{x+7}} = \frac{1}{\sqrt{9} + \sqrt{9}} = \boxed{\frac{1}{6}}.
\end{aligned}$$

87. Let

$$f(x) = \frac{x^2-6x-16}{(x^2-7x-8)\sqrt{x^2-4}} = \frac{(x-8)(x+2)}{(x-8)(x+1)\sqrt{x^2-4}}.$$

(a) The function f is defined for all values x for which the denominator is not equal to zero and $x^2-4>0$. Thus, the domain of f is the set $\boxed{\{x|x<-2\}\cup\{x|x>2, x\neq 8\}}$. Note that the condition $x\neq -1$ need not be explicitly included because -1 does not have an absolute value greater than 2, and so is already eliminated by virtue of this condition.

(b) Because the function f is the product, quotient and composition of functions that are continuous on their domains, f is continuous on its domain. Thus, f is $\boxed{\text{discontinuous at } x=8 \text{ and on the interval } [-2,2]}$.

(c) Because

$$\lim_{x\to 8} f(x) = \lim_{x\to 8}\frac{(x-8)(x+2)}{(x-8)(x+1)\sqrt{x^2-4}} = \lim_{x\to 8}\frac{(x+2)}{(x+1)\sqrt{x^2-4}} = \frac{10}{9\sqrt{60}} = \frac{5}{9\sqrt{15}}$$

exists, the $\boxed{\text{discontinuity at } x=8 \text{ is removable}}$.

88. (a) Because the function $f(x)=\sin x+x-3$ is the sum of the sine function and a polynomial function, both of which are continuous on the set of all real numbers, f is continuous on the set of all real numbers and is therefore continuous on the closed interval $[0,\pi]$. Now, $f(0)=\sin 0+0-3=-3<0$ and $f(\pi)=\sin\pi+\pi-3=\pi-3>0$, so the Intermediate Value Theorem guarantees that f has a zero on the interval $(0,\pi)$.

(b) Using a TI-84 Plus calculator, the zero is $\boxed{x\approx 2.180}$, rounded to three decimal places.

89. (a) Because the function $f(x)=e^x+x-2$ is the sum of an exponential function and a polynomial function, both of which are continuous on the set of all real numbers, f is continuous on the set of all real numbers and is therefore continuous on the closed interval $[0,2]$. Now, $f(0)=e^0+0-2=-1<0$ and $f(2)=e^2+2-2=e^2>0$, so the Intermediate Value Theorem guarantees that f has a zero on the interval $(0,2)$.

(b) Using a TI-84 Plus calculator, the zero is $\boxed{x\approx 0.443}$, rounded to three decimal places.

90. The graph of the function $f(x)=x^3-2x^2-1$ intersects the line $y=-1$ at $x=c$ for which $f(c)=-1$. Noting that $f(1)=-2$ is less than -1 and $f(4)=31$ is more than -1, we use the Intermediate Value Theorem to conclude that $f(c)=-1$ for at least one number c in the interval (1, 4).

Using the TABLE feature on a graphing utility, we subdivide the interval [1, 4] into 10 subintervals, each of length 0.3. Then we find the subinterval whose endpoints have values on either side of $y=-1$, or the endpoint whose value equals -1 (in which case, the exact value is found). From Figure 1, since $f(1.9)=-1.3610$ and $f(2.2)=-0.0320$, by the Intermediate Value Theorem, a solution to $f(c)=-1$ lies in the interval (1.9, 2.2).

Repeat the process by subdividing the interval [1.9, 2.2] into 10 subintervals, each of length 0.03. See Figure 2. We conclude that the solution to $f(c)=-1$ lies in the interval (1.99, 2.02).

Repeat the process by subdividing the interval [1.99, 2.02] into 10 subintervals, each of length 0.003. See Figure 3. We conclude that the solution to $f(c)=-1$ lies in the interval (1.999, 2.002).

Repeat the process by subdividing the interval [1.999, 2.002] into 10 subintervals, each of length 0.0003. See Figure 4. We conclude that the solution to $f(c)=-1$ lies in the interval (1.9999, 2.0002).

Correct to 3 decimals, the solution is $c=2.000$.

x	$f(x)$	x	$f(x)$	x	$f(x)$	x	$f(x)$
1.0	−2.0000	1.90	−1.3610	1.990	−1.0396	1.9990	−1.0040
1.3	−2.1830	1.93	−1.2607	1.993	−1.0278	1.9993	−1.0028
1.6	−2.0240	1.96	−1.1537	1.996	−1.0159	1.9996	−1.0016
1.9	−1.3610	1.99	−1.0396	1.999	−1.0040	1.9999	−1.0004
2.2	−0.0320	2.02	−0.9184	2.002	−0.9920	2.0002	−0.9992
2.5	2.1250	2.05	−0.7899	2.005	−0.9799	2.0005	−0.9980
2.8	5.2720	2.08	−0.6539	2.008	−0.9677	2.0008	−0.9968
3.1	9.5710	2.11	−0.5103	2.011	−0.9555	2.0011	−0.9956
3.4	15.1840	2.14	−0.3589	2.014	−0.9432	2.0014	−0.9944
3.7	22.2730	2.17	−0.1995	2.017	−0.9308	2.0017	−0.9932
4.0	31.0000	2.20	−0.0320	2.020	−0.9184	2.0020	−0.9920
Figure 1		Figure 2		Figure 3		Figure 4	

91. The graph of the function $g(x) = -x^4 + 3x^2 + 3$ intersects the line $y = 3$ at $x = c$ for which $g(c) = 3$. Noting that $g(1) = 5$ is more than 3 and $g(2) = -1$ is less than 3, we use the Intermediate Value Theorem to conclude that $g(c) = 3$ for at least one number c in the interval (1, 2).

Using the TABLE feature on a graphing utility, we subdivide the interval [1, 2] into 10 subintervals, each of length 0.1. Then we find the subinterval whose endpoints have values on either side of $y = 3$, or the endpoint whose value equals 3 (in which case, the exact value is found). From Figure 1, since $g(1.7) = 3.3179$ and $g(1.8) = 2.2224$, by the Intermediate Value Theorem, a solution to $g(c) = 3$ lies in the interval (1.7, 1.8).

Repeat the process by subdividing the interval [1.7, 1.8] into 10 subintervals, each of length 0.01. See Figure 2. We conclude that the solution to $g(c) = 3$ lies in the interval (1.73, 1.74).

Repeat the process by subdividing the interval [1.73, 1.74] into 10 subintervals, each of length 0.001. See Figure 3. We conclude that the solution to $g(c) = 3$ lies in the interval (1.732, 1.733).

Repeat the process by subdividing the interval [1.732, 1.733] into 10 subintervals, each of length 0.0001. See Figure 4. We conclude that the solution to $g(c) = 3$ lies in the interval (1.7320, 1.7321).

Correct to 3 decimals, the solution is $\boxed{c = 1.732}$.

x	$g(x)$	x	$g(x)$	x	$g(x)$	x	$g(x)$
1.0	5.0000	1.70	3.3179	1.730	3.0212	1.7320	3.0005
1.1	5.1659	1.71	3.2219	1.731	3.0109	1.7321	2.9995
1.2	5.2464	1.72	3.1231	1.732	3.0005	1.7322	2.9984
1.3	5.2139	1.73	3.0212	1.733	2.9901	1.7323	2.9974
1.4	5.0384	1.74	2.9164	1.734	2.9797	1.7324	2.9964
1.5	4.6875	1.75	2.8086	1.735	2.9692	1.7325	2.9953
1.6	4.1264	1.76	2.6977	1.736	2.9587	1.7326	2.9943
1.7	3.3179	1.77	2.5836	1.737	2.9482	1.7327	2.9932
1.8	2.2224	1.78	2.4664	1.738	2.9376	1.7328	2.9922
1.9	0.7979	1.79	2.3460	1.739	2.9271	1.7329	2.9912
2.0	−1.0000	1.80	2.2224	1.740	2.9164	1.7330	2.9901
Figure 1		Figure 2		Figure 3		Figure 4	

92. The graph of the function $h(x) = \dfrac{x^3 - 5}{x^2 + 1}$ intersects the line $y = 1$ at $x = c$ for which $f(c) = 1$. Noting that $h(1) = -2$ is less than 1 and $h(3) = 2.2$ is more than 1, we use the Intermediate Value Theorem to conclude that $h(c) = 1$ for at least one number c in the interval (1, 3).

Using the TABLE feature on a graphing utility, we subdivide the interval [1, 3] into 10 subintervals, each of length 0.2. Then we find the subinterval whose endpoints have values on either side of $y = 1$, or the endpoint whose value equals 1 (in which case, the exact value is found). From Figure 1, since $h(2.2) = 0.9671$ and $h(2.4) = 1.3053$, by the Intermediate Value Theorem, a solution to $h(c) = 1$ lies in the interval (2.2, 2.4).

Repeat the process by subdividing the interval [2.2, 2.4] into 10 subintervals, each of length 0.02. See Figure 2. We conclude that the solution to $h(c) = 1$ lies in the interval (2.20, 2.22). Correct to one decimal, the solution is $c = 2.2$.

Repeat the process by subdividing the interval [2.20, 2.22] into 10 subintervals, each of length 0.002. See Figure 3. We conclude that the solution to $h(c) = 1$ lies in the interval (2.218, 2.220). Correct to two decimals, the solution is $c = 2.21$.

Repeat the process by subdividing the interval [2.218, 2.220] into 10 subintervals, each of length 0.0002. See Figure 4. We conclude that the solution to $h(c) = 1$ lies in the interval (2.2186, 2.2188).

Correct to 3 decimals, the solution is $c = 2.218$.

x	$h(x)$	x	$h(x)$	x	$h(x)$	x	$h(x)$
1.0	−2.0000	2.20	0.9671	2.200	0.9671	2.2180	0.9986
1.2	−1.3410	2.22	1.0021	2.202	0.9706	2.2182	0.9990
1.4	−0.7622	2.24	1.0369	2.204	0.9741	2.2184	0.9993
1.6	−0.2539	2.26	1.0713	2.206	0.9777	2.2186	0.9997
1.8	0.1962	2.28	1.1055	2.208	0.9812	2.2188	1.0000
2.0	0.6000	2.30	1.1394	2.210	0.9847	2.2190	1.0004
2.2	0.9671	2.32	1.1731	2.212	0.9882	2.2192	1.0007
2.4	1.3053	2.34	1.2065	2.214	0.9917	2.2194	1.0011
2.6	1.6206	2.36	1.2397	2.216	0.9952	2.2196	1.0014
2.8	1.9176	2.38	1.2726	2.218	0.9986	2.2198	1.0018
3.0	2.2000	2.40	1.3053	2.220	1.0021	2.2200	1.0021
Figure 1		Figure 2		Figure 3		Figure 4	

93. The graph of the function $r(x) = \dfrac{x-6}{x^2+2}$ intersects the line $y = -1$ at $x = c$ for which $r(c) = -1$. Noting that $r(0) = -3$ is less than -1 and $r(3) = -\dfrac{3}{11}$ is more than -1, we use the Intermediate Value Theorem to conclude that $r(c) = -1$ for at least one number c in the interval (0, 3).

Using the TABLE feature on a graphing utility, we subdivide the interval [0, 3] into 10 subintervals, each of length 0.3. Then we find the subinterval whose endpoints have values on either side of $y = -1$, or the endpoint whose value equals -1 (in which case, the exact value is found). From Figure 1, since $r(1.5) = -1.0588$ and $r(1.8) = -0.8015$, by the Intermediate Value Theorem, a solution to $r(c) = -1$ lies in the interval (1.5, 1.8).

Repeat the process by subdividing the interval [1.5, 1.8] into 10 subintervals, each of length 0.03. See Figure 2. We conclude that the solution to $r(c) = -1$ lies in the interval (1.56, 1.59).

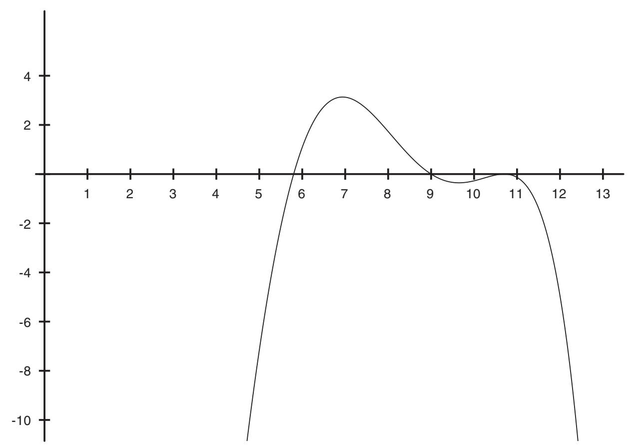
Repeat the process by subdividing the interval [1.56, 1.59] into 10 subintervals, each of length 0.003. See Figure 3. We conclude that the solution to $r(c) = -1$ lies in the interval (1.560, 1.563).

Repeat the process by subdividing the interval [1.560, 1.563] into 10 subintervals, each of length 0.0003. See Figure 4. We conclude that the solution to $r(c) = -1$ lies in the interval (1.5615, 1.5618).

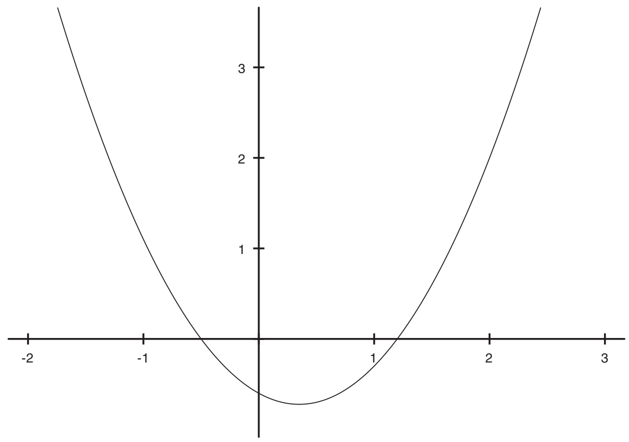
Correct to 3 decimals, the solution is $\boxed{c = 1.561}$.

x	$r(x)$	x	$r(x)$	x	$r(x)$	x	$r(x)$
0.0	−3.0000	1.50	−1.0588	1.560	−1.0014	1.5600	−1.0014
0.3	−2.7273	1.53	−1.0297	1.563	−0.9987	1.5603	−1.0012
0.6	−2.2881	1.56	−1.0014	1.566	−0.9959	1.5606	−1.0009
0.9	−1.8149	1.59	−0.9739	1.569	−0.9931	1.5609	−1.0006
1.2	−1.3953	1.62	−0.9471	1.572	−0.9903	1.5612	−1.0003
1.5	−1.0588	1.65	−0.9211	1.575	−0.9876	1.5615	−1.0000
1.8	−0.8015	1.68	−0.8958	1.578	−0.9848	1.5618	−0.9998
2.1	−0.6084	1.71	−0.8712	1.581	−0.9821	1.5621	−0.9995
2.4	−0.4639	1.74	−0.8473	1.584	−0.9794	1.5624	−0.9992
2.7	−0.3552	1.77	−0.8241	1.587	−0.9766	1.5627	−0.9989
3.0	−0.2727	1.80	−0.8015	1.590	−0.9739	1.5630	−0.9987
Figure 1		Figure 2		Figure 3		Figure 4	

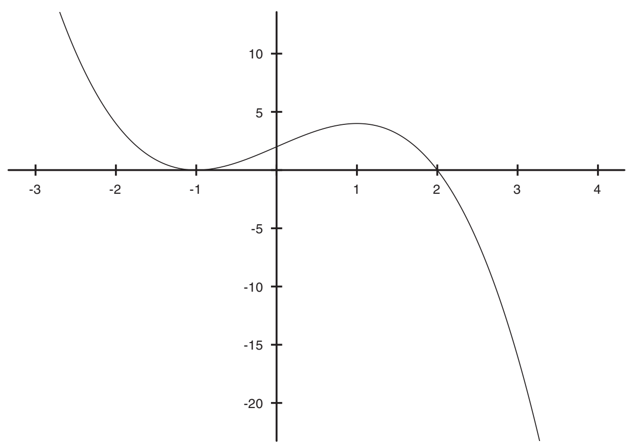
94. Answers will vary. The figure below displays the graph of a function that is continuous on $[5, 12]$, that is negative at both endpoints, and has exactly three zeros in the interval. This $\boxed{\text{does not contradict}}$ the Intermediate Value Theorem, because the theorem provides no information about zeros of a function when the endpoint values are the same sign.



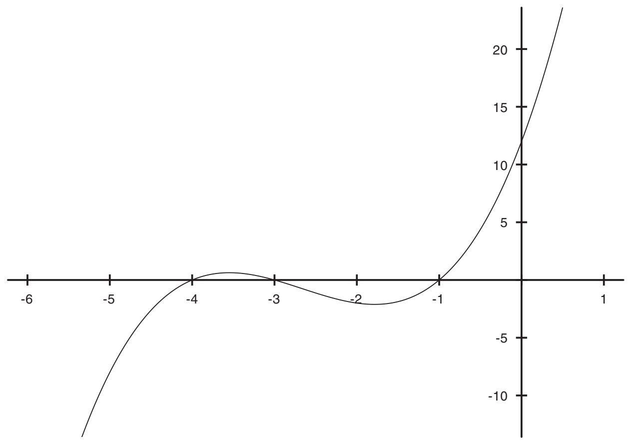
95. Answers will vary. The figure below displays the graph of a function that is continuous on $[-1, 2]$, that is positive at both endpoints, and has exactly two zeros in the interval. This $\boxed{\text{does not contradict}}$ the Intermediate Value Theorem, because the theorem provides no information about zeros of a function when the endpoint values are the same sign.



96. Answers will vary. The figure below displays the graph of a function that is continuous on $[-2, 3]$, that is positive at -2 and negative at 3, and has exactly two zeros in the interval. This $\boxed{\text{does not contradict}}$ the Intermediate Value Theorem, because the theorem guarantees that the function has **at least** one zero on the interval $(-2, 3)$.

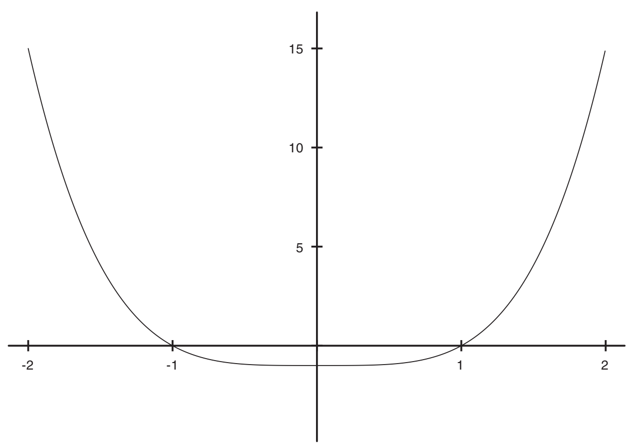


97. Answers will vary. The figure below displays the graph of a function that is continuous on $[-5, 0]$, that is negative at -5 and positive at 0, and has exactly three zeros in the interval. This $\boxed{\text{does not contradict}}$ the Intermediate Value Theorem, because the theorem guarantees that the function has **at least** one zero on the interval $(-5, 0)$.



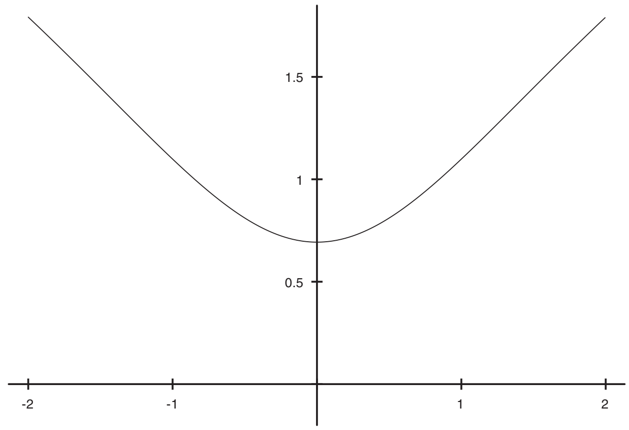
98. (a) Although the function $f(x) = x^4 - 1$ is continuous on the closed interval $[-2, 2]$, $f(-2) = (-2)^4 - 1 = 15 > 0$ and $f(2) = 2^4 - 1 = 15 > 0$. Because $\boxed{\text{the function has the same sign at both endpoints}}$, the Intermediate Value Theorem gives no information about the zeros of f on the interval $(-2, 2)$.

    (b) The graph of f shown below indicates that $\boxed{f \text{ has two zeros on the interval } (-2, 2)}$: one at $x = -1$, the other at $x = 1$.



99. (a) Although the function $f(x) = \ln(x^2 + 2)$ is continuous on the closed interval $[-2, 2]$, $f(-2) = \ln 6 > 0$ and $f(2) = \ln 6 > 0$. Because $\boxed{\text{the function has the same sign at both endpoints}}$, the Intermediate Value Theorem gives no information about the zeros of f on the interval $(-2, 2)$.

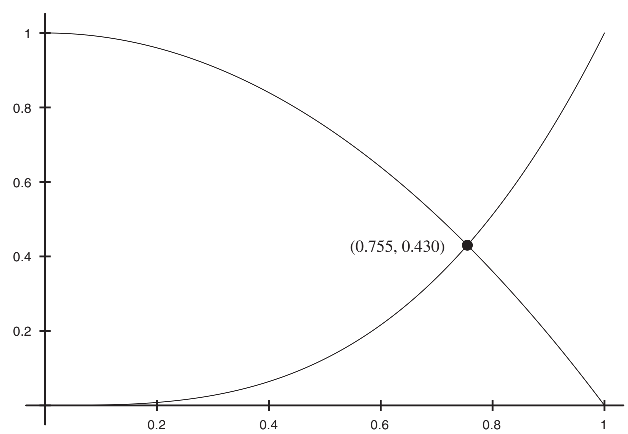
    (b) $\boxed{\text{The graph of } f \text{ shown below indicates that } f \text{ has no zero on the interval } [-2, 2]}$.



100. (a) If the graphs of the functions $y = x^3$ and $y = 1-x^2$ intersect, then the x-coordinate of the point of intersection must be a solution of the equation $x^3 = 1-x^2$, or $x^3+x^2-1 = 0$. Let $f(x) = x^3 + x^2 - 1$. This function is continuous on the closed interval $[0, 1]$ with $f(0) = -1 < 0$ and $f(1) = 1 > 0$. The Intermediate Value Theorem therefore guarantees that f has a zero on the interval $(0, 1)$. Hence, the graphs of the functions $y = x^3$ and $y = 1 - x^2$ do intersect somewhere between $x = 0$ and $x = 1$.

(b) Using a TI-84 Plus calculator, the point of intersection, rounded to three decimal places, is $\boxed{(0.755, 0.430)}$.

(c) The figure below displays the graphs of both functions with the point of intersection labeled to three decimal places.



101. Let $v(t)$ denote the speed of the airplane as a function of time. Further, let t_1 denote a time when the speed of the airplane was 620 miles per hour, $t_2 > t_1$ denote a time when the speed of the airplane had slowed to 608 miles per hour, and $t_3 > t_2$ denote a time when the speed of the airplane had increased to 614 miles per hour. Now, consider the function $f(t) = v(t) - 610$. Assuming $v(t)$ is continuous for all t, f is also continuous for all t. Because

$$f(t_1) = v(t_1) - 610 = 620 - 610 = 10 > 0$$

and

$$f(t_2) = v(t_2) - 610 = 608 - 610 = -2 < 0,$$

the Intermediate Value Theorem guarantees that $f(t) = 0$, or $v(t) = 610$, for some time between t_1 and t_2. Similarly,

$$f(t_2) = v(t_2) - 610 = 608 - 610 = -2 < 0$$

and

$$f(t_3) = v(t_3) - 610 = 614 - 610 = 4 > 0,$$

so the Intermediate Value Theorem guarantees that $f(t) = 0$, or $v(t) = 610$, for some time between t_2 and t_3. Thus, the airplane's speed is 610 miles per hour on at least two different occasions during the flight.

102. Let f be a function that is defined and continuous on the closed interval $[a, b]$. The function

$$h(x) = \frac{1}{f(x)}$$

will therefore be continuous on the closed interval $[a, b]$, except for those values x at which $f(x) = 0$. Thus, if f is never zero on the closed interval $[a, b]$, then h will be continuous on the closed interval $[a, b]$.

103. (a) Factoring, $f(x) = x^3 - 3x^2 - 4x + 12 = (x-3)(x^2-4) = (x-3)(x-2)(x+2)$. Thus, the zeros of the function f are $\boxed{x = -2,\ x = 2, \text{ and } x = 3}$.

(b) The function h will be continuous at $x = 3$ provided $p = \lim_{x\to 3} h(x)$. Now,

$$\lim_{x\to 3} h(x) = \lim_{x\to 3} \frac{(x-3)(x^2-4)}{x-3} = \lim_{x\to 3}(x^2-4) = 5.$$

Therefore, h is continuous at $x = 3$ when $\boxed{p = 5}$.

(c) With $p = 5$, the function h reduces to $x^2 - 4$ for all x. Because

$$h(-x) = (-x)^2 - 4 = x^2 - 4 = h(x),$$

$h(x)$ is an $\boxed{\text{even}}$ function.

104. Consider the one-sided limits as x approaches 0:

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{-x}{x} = \lim_{x\to 0^-} -1 = -1$$

and

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{|x|}{x} = \lim_{x\to 0^+} \frac{x}{x} = \lim_{x\to 0^+} 1 = 1.$$

Because the two one-sided limits as x approaches 0 are not equal, $\boxed{\lim_{x\to 0} f(x) \text{ does not exist}}$. Therefore, the discontinuity at $x = 0$ is not removable; e.g., it is impossible to define $f(0)$ so that f is continuous at $x = 0$.

105. Answers will vary. One possible response is the following. The polynomial functions $\boxed{f(x) = x^2 - 1 \text{ and } g(x) = x - 3}$ are continuous at $c = 3$; however, because $g(3) = 0$, the function $\dfrac{f}{g}$ is not continuous at $c = 3$.

106. Answers will vary. One possible response is the following. A discontinuity at $x = c$ is removable when the limit as x approaches c exists but that limit is not equal to the function value at $x = c$; a discontinuity at $x = c$ is nonremovable when the limit as x approaches c does not exist. An example of a removable discontinuity is $x = 3$ for the function $f(x) = \dfrac{x^2 - 9}{x - 3}$. Because

$$\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{x^2-9}{x-3} = \lim_{x\to 3} \frac{(x-3)(x+3)}{x-3} = \lim_{x\to 3}(x+3) = 6,$$

the discontinuity at $x = 3$ can be removed by defining $f(3) = 6$. An example of a nonremovable discontinuity is $x = 3$ for the function

$$g(x) = \begin{cases} x+3, & x \le 3 \\ 9-x^2, & x > 3. \end{cases}$$

Here,

$$\lim_{x\to 3^-} g(x) = \lim_{x\to 3^-}(x+3) = 6 \quad \text{but} \quad \lim_{x\to 3^+} g(x) = \lim_{x\to 3^+}(9-x^2) = 0.$$

Because the two one-sided limits as x approaches 3 are not equal, $\lim_{x\to 3} g(x)$ does not exist.

107. Let $f(x) = x^3 + 3x - 5$, and note that $f(1) = -1 < 0$ and $f(2) = 9 > 0$. Set $m_1 = 1.5$, the midpoint of the interval $(1, 2)$, and then calculate $f(m_1) = f(1.5) = 2.875 > 0$. The sign of $f(1.5)$ is opposite that of $f(1)$, so the zero lies in the subinterval $(1, 1.5)$. Now repeat the process and set $m_2 = 1.25$, the midpoint of the interval $(1, 1.5)$. Next, calculate $f(m_2) = f(1.25) = 0.703125 > 0$. The sign of $f(1.25)$ is opposite that of $f(1)$, so the zero lies in the subinterval $(1, 1.25)$. Finally, set $m_3 = 1.125$, the midpoint of the interval $(1, 1.25)$, and calculate $f(m_3) = f(1.125) \approx -0.201172 < 0$. The sign of $f(1.125)$ is opposite that of $f(1.25)$, so the zero lies in the subinterval $(1.125, 1.25)$ and is given approximately by the midpoint of this subinterval $\boxed{m_4 = 1.1875}$.

108. Let $f(x) = x^3 - 4x + 2$, and note that $f(1) = -1 < 0$ and $f(2) = 2 > 0$. Set $m_1 = 1.5$, the midpoint of the interval $(1, 2)$, and then calculate $f(m_1) = f(1.5) = -0.625 < 0$. The sign of $f(1.5)$ is opposite that of $f(2)$, so the zero lies in the subinterval $(1.5, 2)$. Now repeat the process and set $m_2 = 1.75$, the midpoint of the interval $(1.5, 2)$. Next, calculate $f(m_2) = f(1.75) = 0.359375 > 0$. The sign of $f(1.75)$ is opposite that of $f(1.5)$, so the zero lies in the subinterval $(1.5, 1.75)$. Finally, set $m_3 = 1.625$, the midpoint of the interval $(1.5, 1.75)$, and calculate $f(m_3) = f(1.625) \approx -0.208984 < 0$. The sign of $f(1.625)$ is opposite that of $f(1.75)$, so the zero lies in the subinterval $(1.625, 1.75)$ and is given approximately by the midpoint of this subinterval $\boxed{m_4 = 1.6875}$.

109. Let $f(x) = 2x^3 + 3x^2 + 4x - 1$, and note that $f(0) = -1 < 0$ and $f(1) = 8 > 0$. Set $m_1 = 0.5$, the midpoint of the interval $(0, 1)$, and then calculate $f(m_1) = f(0.5) = 2 > 0$. The sign of $f(0.5)$ is opposite that of $f(0)$, so the zero lies in the subinterval $(0, 0.5)$. Now repeat the process and set $m_2 = 0.25$, the midpoint of the interval $(0, 0.5)$. Next, calculate $f(m_2) = f(0.25) = 0.21875 > 0$. The sign of $f(0.25)$ is opposite that of $f(0)$, so the zero lies in the subinterval $(0, 0.25)$. Finally, set $m_3 = 0.125$, the midpoint of the interval $(0, 0.25)$, and calculate $f(m_3) = f(0.125) \approx -0.449219 < 0$. The sign of $f(0.125)$ is opposite that of $f(0.25)$, so the zero lies in the subinterval $(0.125, 0.25)$ and is given approximately by the midpoint of this subinterval $\boxed{m_4 = 0.1875}$.

110. Let $f(x) = x^3 - x^2 - 2x + 1$, and note that $f(0) = 1 > 0$ and $f(1) = -1 < 0$. Set $m_1 = 0.5$, the midpoint of the interval $(0, 1)$, and then calculate $f(m_1) = f(0.5) = -0.125 < 0$. The sign of $f(0.5)$ is opposite that of $f(0)$, so the zero lies in the subinterval $(0, 0.5)$. Now repeat the process and set $m_2 = 0.25$, the midpoint of the interval $(0, 0.5)$. Next, calculate $f(m_2) = f(0.25) = 0.453125 > 0$. The sign of $f(0.25)$ is opposite that of $f(0.5)$, so the zero lies in the subinterval $(0.25, 0.5)$. Finally, set $m_3 = 0.375$, the midpoint of the interval $(0.25, 0.5)$, and calculate $f(m_3) = f(0.375) \approx 0.162109 > 0$. The sign of $f(0.375)$ is opposite that of $f(0.5)$, so the zero lies in the subinterval $(0.375, 0.5)$ and is given approximately by the midpoint of this subinterval $\boxed{m_4 = 0.4375}$.

111. Let $f(x) = x^3 - 6x - 12$, and note that $f(3) = -3 < 0$ and $f(4) = 28 > 0$. Set $m_1 = 3.5$, the midpoint of the interval $(3, 4)$, and then calculate $f(m_1) = f(3.5) = 9.875 > 0$. The sign of $f(3.5)$ is opposite that of $f(3)$, so the zero lies in the subinterval $(3, 3.5)$. Now repeat the process and set $m_2 = 3.25$, the midpoint of the interval $(3, 3.5)$. Next, calculate $f(m_2) = f(3.25) = 2.828125 > 0$. The sign of $f(3.25)$ is opposite that of $f(3)$,

so the zero lies in the subinterval $(3, 3.25)$. Finally, set $m_3 = 3.125$, the midpoint of the interval $(3, 3.25)$, and calculate $f(m_3) = f(3.125) \approx -0.232422 < 0$. The sign of $f(3.125)$ is opposite that of $f(3.25)$, so the zero lies in the subinterval $(3.125, 3.25)$ and is given approximately by the midpoint of this subinterval $\boxed{m_4 = 3.1875}$.

112. Let $f(x) = 3x^3 + 5x - 40$, and note that $f(2) = -6 < 0$ and $f(3) = 56 > 0$. Set $m_1 = 2.5$, the midpoint of the interval $(2, 3)$, and then calculate $f(m_1) = f(2.5) = 19.375 > 0$. The sign of $f(2.5)$ is opposite that of $f(2)$, so the zero lies in the subinterval $(2, 2.5)$. Now repeat the process and set $m_2 = 2.25$, the midpoint of the interval $(2, 2.5)$. Next, calculate $f(m_2) = f(2.25) = 5.421875 > 0$. The sign of $f(2.25)$ is opposite that of $f(2)$, so the zero lies in the subinterval $(2, 2.25)$. Finally, set $m_3 = 2.125$, the midpoint of the interval $(2, 2.25)$, and calculate $f(m_3) = f(2.125) \approx -0.587891 < 0$. The sign of $f(2.125)$ is opposite that of $f(2.25)$, so the zero lies in the subinterval $(2.125, 2.25)$ and is given approximately by the midpoint of this subinterval $\boxed{m_4 = 2.1875}$.

113. Let $f(x) = x^4 - 2x^3 + 21x - 23$, and note that $f(1) = -3 < 0$ and $f(2) = 19 > 0$. Set $m_1 = 1.5$, the midpoint of the interval $(1, 2)$, and then calculate $f(m_1) = f(1.5) = 6.8125 > 0$. The sign of $f(1.5)$ is opposite that of $f(1)$, so the zero lies in the subinterval $(1, 1.5)$. Now repeat the process and set $m_2 = 1.25$, the midpoint of the interval $(1, 1.5)$. Next, calculate $f(m_2) = f(1.25) \approx 1.785156 > 0$. The sign of $f(1.25)$ is opposite that of $f(1)$, so the zero lies in the subinterval $(1, 1.25)$. Finally, set $m_3 = 1.125$, the midpoint of the interval $(1, 1.25)$, and calculate $f(m_3) = f(1.125) \approx -0.620850 < 0$. The sign of $f(1.125)$ is opposite that of $f(1.25)$, so the zero lies in the subinterval $(1.125, 1.25)$ and is given approximately by the midpoint of this subinterval $\boxed{m_4 = 1.1875}$.

114. Let $f(x) = x^4 - x^3 + x - 2$, and note that $f(1) = -1 < 0$ and $f(2) = 8 > 0$. Set $m_1 = 1.5$, the midpoint of the interval $(1, 2)$, and then calculate $f(m_1) = f(1.5) = 1.1875 > 0$. The sign of $f(1.5)$ is opposite that of $f(1)$, so the zero lies in the subinterval $(1, 1.5)$. Now repeat the process and set $m_2 = 1.25$, the midpoint of the interval $(1, 1.5)$. Next, calculate $f(m_2) = f(1.25) \approx -0.261719 < 0$. The sign of $f(1.25)$ is opposite that of $f(1.5)$, so the zero lies in the subinterval $(1.25, 1.5)$. Finally, set $m_3 = 1.375$, the midpoint of the interval $(1.25, 1.5)$, and calculate $f(m_3) = f(1.375) \approx 0.349854 > 0$. The sign of $f(1.375)$ is opposite that of $f(1.25)$, so the zero lies in the subinterval $(1.25, 1.375)$ and is given approximately by the midpoint of this subinterval $\boxed{m_4 = 1.3125}$.

115. The polynomial function $x^2 + 4x$ is continuous on the set of all real numbers and is nonnegative on the set $\{x|x \leq -4\} \cup \{x|x \geq 0\}$. The function $f(x) = \sqrt{x^2 + 4x} - 2$ is therefore continuous on the set $\{x|x \leq -4\} \cup \{x|x \geq 0\}$, which contains the closed interval $[0, 1]$. Because $f(0) = \sqrt{0} - 2 = -2 < 0$ and $f(1) = \sqrt{5} - 2 \approx 0.236 > 0$, the Intermediate Value Theorem guarantees that f must have a zero on the interval $(0, 1)$.

    To approximate this zero, subdivide the interval $[0, 1]$ into 10 subintervals, each of length 0.1, and evaluate f at each endpoint, looking for two successive function values with opposite signs. This yields $f(0.8) \approx -0.040408$ and $f(0.9) = 0.1$, indicating that the zero lies in the subinterval $(0.8, 0.9)$. Thus, correct to one decimal place, the zero is $\boxed{x = 0.8}$.

116. The polynomial function $f(x) = x^3 - x + 2$ is continuous for all real numbers, so it is continuous on the closed interval $[-2, 0]$. Because $f(-2) = (-2)^3 - (-2) + 2 = -4 < 0$ and $f(0) = 2 > 0$, the Intermediate Value Theorem guarantees that f must have a zero on the interval $(-2, 0)$.

    To approximate this zero, subdivide the interval $[-2, 0]$ into 20 subintervals, each of length 0.1, and evaluate f at each endpoint, looking for two successive function values with opposite signs. This yields $f(-1.6) = -0.496$ and $f(-1.5) = 0.125$, indicating that the zero lies in the subinterval $(-1.6, -1.5)$. Next, subdivide the interval $[-1.6, -1.5]$ into 10 subintervals, each of length 0.01. The two successive function values that are of opposite sign are $f(-1.53) - 0.051577$ and $f(-1.52) = 0.008192$, so the zero has now been isolated to the interval $(-1.53, -1.52)$. Thus, correct to two decimal places, the zero is $\boxed{x = -1.52}$.

117. Let f and g be functions that are continuous at c. Then,

$$\lim_{x\to c} f(x) = f(c) \quad \text{and} \quad \lim_{x\to c} g(x) = g(c).$$

To prove that $f+g$ is continuous at c, it must be shown that $\lim_{x\to c}[f(x)+g(x)] = f(c)+g(c)$. Using the Limit of a Sum Property, it follows that

$$\lim_{x\to c}[f(x)+g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = f(c)+g(c),$$

as required.

118. Let f and g be functions that are continuous on the closed interval $[a, b]$, with $f(a) < g(a)$ and $f(b) > g(b)$. Define the function $h(x) = f(x) - g(x)$. Because f and g are both continuous on $[a, b]$, it follows that h is also continuous on $[a, b]$. Now,

$$h(a) = f(a) - g(a) < 0 \quad \text{and} \quad h(b) = f(b) - g(b) > 0.$$

Thus, by the Intermediate Value Theorem, there is a number c between a and b such that $h(c) = f(c) - g(c) = 0$, or $f(c) = g(c)$. This implies that the graphs of $y = f(x)$ and $y = g(x)$ intersect at $x = c$; that is, the graphs intersect somewhere between $x = a$ and $x = b$.

Challenge Problems

119. Let $f(x) = \dfrac{1}{x-1} + \dfrac{1}{x-2}$. Because f is continuous on the interval $(1, 2)$, it is continuous on any closed interval contained within $(1, 2)$, say $[1.1, 1.9]$. With

$$f(1.1) = \frac{1}{1.1-1} + \frac{1}{1.1-2} = 10 - \frac{10}{9} = \frac{80}{9} > 0$$

and

$$f(1.9) = \frac{1}{1.9-1} + \frac{1}{1.9-2} = \frac{10}{9} - 10 = -\frac{80}{9} < 0,$$

the Intermediate value Theorem guarantees there exists a number c between 1.1 and 1.9, and hence between 1 and 2, such that $f(c) = 0$.

120. Let $f(x) = x^2 - 7$. This polynomial function is continuous on the closed interval $[2.64, 2.65]$. With

$$f(2.64) = 2.64^2 - 7 = -0.0304 < 0 \quad \text{and} \quad f(2.65) = 2.65^2 - 7 = 0.0225 > 0,$$

the Intermediate Value Theorem guarantees there exists a number c between 2.64 and 2.65 such that $f(c) = c^2 - 7 = 0$, or $c^2 = 7$.

121. Let f be a function for which $\lim_{h\to 0} \dfrac{f(a+h) - f(a)}{h}$ exists. Now, let $x = a + h$. Then, as h approaches 0, x approaches a, and

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}.$$

To prove that f is continuous at $x = a$, it must be shown that $\lim_{x\to a} f(x) = f(a)$. Because $f(a)$ is a constant, $\lim_{x\to a} f(a) = f(a)$ and

$$\lim_{x\to a} f(x) = f(a) \quad \text{is equivalent to} \quad \lim_{x\to a}[f(x) - f(a)] = 0.$$

Proceeding with this last limit, we find

$$\begin{aligned}\lim_{x\to a}[f(x)-f(a)] &= \lim_{x\to a}\left[\frac{f(x)-f(a)}{x-a}\cdot(x-a)\right] = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}\cdot\lim_{x\to a}(x-a)\\ &= \lim_{x\to a}\frac{f(x)-f(a)}{x-a}\cdot 0 = 0,\end{aligned}$$

where, in going from the first line to the second line, we have used the fact that $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ exists so the Limit of a Product property applies.

122. The rational function $\frac{x^2+x-2}{x-1}$ is continuous on the set of all real numbers except $x=1$, so f is continuous on the interval $(-\infty,1)$. The function f is also continuous on the intervals $(1,4)$ and $(4,\infty)$ because it is a polynomial on each of these intervals. At $x=1$,

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}\frac{x^2+x-2}{x-1} = \lim_{x\to 1^-}\frac{(x-1)(x+2)}{x-1} = \lim_{x\to 1^-}(x+2) = 3$$

and

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} B(x-C)^2 = B(1-C)^2.$$

Then, at $x=4$,

$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} B(x-C)^2 = B(4-C)^2$$

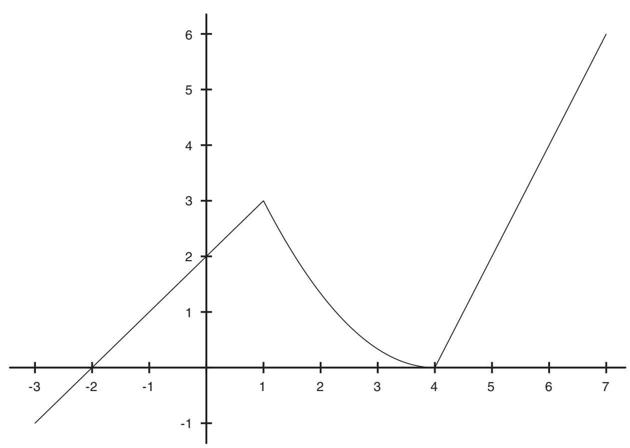
and

$$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+}(2x-8) = 0.$$

Therefore, for f to be continuous at $x=1$ and at $x=4$, the constants A, B, C, and D must satisfy the equations

$$A=3,\quad B(1-C)^2=3,\quad B(4-C)^2=0,\quad D=0.$$

The solution of these equations is $\boxed{A=3,\ B=\frac{1}{3},\ C=4,\text{ and } D=0}$. The figure below displays a graph of f with these values for the constants.



123. Let f be a function that is continuous on the closed interval $[0,1]$ and for which $0\le f(x)\le 1$ for all x in $[0,1]$. Define the function $g(x)=x-f(x)$. Because g is the difference between two functions that are continuous on $[0,1]$, g is also continuous on $[0,1]$. Now,

$$g(0)=0-f(0)\le 0 \quad\text{and}\quad g(1)=1-f(1)\ge 0.$$

If either $g(0)=0$ or $g(1)=0$, then either $f(0)=0$ or $f(1)=1$ and a c in $[0,1]$ has been found such that $f(c)=c$. Otherwise, $g(0)<0$ and $g(1)>0$, so that the Intermediate Value Theorem guarantees there exists a c in $(0,1)$ such that $g(c)=0$, or $f(c)=c$.

1.4 Limits and Continuity of Trigonometric, Exponential, and Logarithmic Functions

Concepts and Vocabulary

1. $\lim_{x\to 0} \sin x = \sin 0 = \boxed{0}$.

2. $\boxed{\text{False}}$. $\lim_{x\to 0} \dfrac{\cos x - 1}{x} = 0$.

3. The Squeeze Theorem states that if functions f, g, and h have the property $f(x) \le g(x) \le h(x)$ for all x in an open interval containing c, except possibly at c, and if $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L$, then $\lim_{x\to c} g(x) = \boxed{L}$.

4. $\boxed{\text{False}}$. $f(x) = \csc x$ is continuous for all real numbers except $x = k\pi$, where k is any integer.

Skill Building

5. Because $-x^2 + 1 \le g(x) \le x^2 + 1$ for all x in an open interval containing 0 and
$$\lim_{x\to 0}(-x^2+1) = 1 \quad \text{and} \quad \lim_{x\to 0}(x^2+1) = 1,$$
it follows from the Squeeze Theorem that $\boxed{\lim_{x\to 0} g(x) = 1}$.

6. Because $-(x-2)^2 - 3 \le g(x) \le (x-2)^2 - 3$ for all x in an open interval containing 2 and
$$\lim_{x\to 2}[-(x-2)^2 - 3] = -3 \quad \text{and} \quad \lim_{x\to 2}[(x-2)^2 - 3] = -3,$$
it follows from the Squeeze Theorem that $\boxed{\lim_{x\to 2} g(x) = -3}$.

7. Because $\cos x \le g(x) \le 1$ for all x in an open interval containing 0 and
$$\lim_{x\to 0}\cos x = 1 \quad \text{and} \quad \lim_{x\to 0} 1 = 1,$$
it follows from the Squeeze Theorem that $\boxed{\lim_{x\to 0} g(x) = 1}$.

8. Because $-x^2 + 1 \le g(x) \le \sec x$ for all x in an open interval containing 0 and
$$\lim_{x\to 0}(-x^2+1) = 1 \quad \text{and} \quad \lim_{x\to 0}\sec x = 1,$$
it follows from the Squeeze Theorem that $\boxed{\lim_{x\to 0} g(x) = 1}$.

9. $\lim_{x\to 0}(x^3 + \sin x) = 0^3 + \sin 0 = 0 + 0 = \boxed{0}$.

10. $\lim_{x\to 0}(x^2 - \cos x) = 0^2 - \cos 0 = 0 - 1 = \boxed{-1}$.

11. $\lim_{x\to \pi/3}(\cos x + \sin x) = \cos\dfrac{\pi}{3} + \sin\dfrac{\pi}{3} = \boxed{\dfrac{\sqrt{3}}{2} + \dfrac{1}{2}}$.

12. $\lim\limits_{x\to\pi/3}(\sin x - \cos x) = \sin\dfrac{\pi}{3} - \cos\dfrac{\pi}{3} = \boxed{\dfrac{\sqrt{3}}{2} - \dfrac{1}{2}}$.

13. $\lim\limits_{x\to 0}\dfrac{\cos x}{1+\sin x} = \dfrac{\cos 0}{1+\sin 0} = \dfrac{1}{1+0} = \boxed{1}$.

14. $\lim\limits_{x\to 0}\dfrac{\sin x}{1+\cos x} = \dfrac{\sin 0}{1+\cos 0} = \dfrac{0}{1+1} = \boxed{0}$.

15. $\lim\limits_{x\to 0}\dfrac{3}{1+e^x} = \dfrac{3}{1+e^0} = \dfrac{3}{1+1} = \boxed{\dfrac{3}{2}}$.

16. $\lim\limits_{x\to 0}\dfrac{e^x-1}{1+e^x} = \dfrac{e^0-1}{1+e^0} = \dfrac{1-1}{1+1} = \boxed{0}$.

17. $\lim\limits_{x\to 0}(e^x\sin x) = e^0\sin 0 = 1(0) = \boxed{0}$.

18. $\lim\limits_{x\to 0}(e^{-x}\tan x) = e^{-0}\tan 0 = 1(0) = \boxed{0}$.

19. $\lim\limits_{x\to 1}\ln\left(\dfrac{e^x}{x}\right) = \ln\left(\dfrac{e^1}{1}\right) = \ln e = \boxed{1}$.

20. $\lim\limits_{x\to 1}\ln\left(\dfrac{x}{e^x}\right) = \ln\left(\dfrac{1}{e^1}\right) = \ln e^{-1} = \boxed{-1}$.

21. $\lim\limits_{x\to 0}\dfrac{e^{2x}}{1+e^x} = \dfrac{e^{2(0)}}{1+e^0} = \dfrac{1}{1+1} = \boxed{\dfrac{1}{2}}$.

22. $\lim\limits_{x\to 0}\dfrac{1-e^x}{1-e^{2x}} = \lim\limits_{x\to 0}\dfrac{1-e^x}{(1-e^x)(1+e^x)} = \lim\limits_{x\to 0}\dfrac{1}{1+e^x} = \dfrac{1}{1+e^0} = \dfrac{1}{1+1} = \boxed{\dfrac{1}{2}}$.

23. $\lim\limits_{x\to 0}\dfrac{\sin(7x)}{x} = \lim\limits_{x\to 0}\dfrac{7\sin(7x)}{7x} = 7\lim\limits_{x\to 0}\dfrac{\sin(7x)}{7x} = 7(1) = \boxed{7}$.

24. $\lim\limits_{x\to 0}\dfrac{\sin\left(\frac{x}{3}\right)}{x} = \lim\limits_{x\to 0}\dfrac{\frac{1}{3}\sin\left(\frac{x}{3}\right)}{\frac{x}{3}} = \dfrac{1}{3}\lim\limits_{x\to 0}\dfrac{\sin\left(\frac{x}{3}\right)}{\frac{x}{3}} = \dfrac{1}{3}(1) = \boxed{\dfrac{1}{3}}$.

25. $\lim\limits_{\theta\to 0}\dfrac{\theta+3\sin\theta}{2\theta} = \lim\limits_{\theta\to 0}\dfrac{\theta}{2\theta} + \dfrac{3}{2}\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta} = \dfrac{1}{2} + \dfrac{3}{2} = \boxed{2}$.

26. $\lim\limits_{x\to 0}\dfrac{2x-5\sin(3x)}{x} = \lim\limits_{x\to 0}\dfrac{2x}{x} - 5\lim\limits_{x\to 0}\dfrac{\sin(3x)}{x} = \lim\limits_{x\to 0}2 - 15\lim\limits_{x\to 0}\dfrac{\sin(3x)}{3x} = 2 - 15 = \boxed{-13}$.

27. First note that

$$\lim_{\theta\to 0}\frac{\theta}{\sin\theta} = \lim_{\theta\to 0}\frac{1}{\frac{\sin\theta}{\theta}} = \frac{\lim_{\theta\to 0}1}{\lim_{\theta\to 0}\frac{\sin\theta}{\theta}} = \frac{1}{1} = 1.$$

Then

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta+\tan\theta} = \lim_{\theta\to 0}\frac{1}{\frac{\theta}{\sin\theta}+\sec\theta} = \frac{1}{1+1} = \boxed{\frac{1}{2}}.$$

28. $\lim\limits_{\theta\to 0}\dfrac{\tan\theta}{\theta} = \lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta\cdot\cos\theta} = \lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}\cdot\lim\limits_{\theta\to 0}\dfrac{1}{\cos\theta} = 1\cdot\dfrac{1}{1} = \boxed{1}$.

29. $\lim\limits_{\theta\to 0}\dfrac{5}{\theta\cdot\csc\theta} = 5\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta} = 5(1) = \boxed{5}$.

30. $\lim\limits_{\theta\to 0}\dfrac{\sin(3\theta)}{\sin(2\theta)} = \lim\limits_{\theta\to 0}\dfrac{\frac{\sin(3\theta)}{\theta}}{\frac{\sin(2\theta)}{\theta}} = \dfrac{3\lim\limits_{\theta\to 0}\dfrac{\sin(3\theta)}{3\theta}}{2\lim\limits_{\theta\to 0}\dfrac{\sin(2\theta)}{2\theta}} = \dfrac{3(1)}{2(1)} = \boxed{\dfrac{3}{2}}$.

31. $\lim\limits_{\theta\to 0}\dfrac{1-\cos^2\theta}{\theta}=\lim\limits_{\theta\to 0}\dfrac{\sin^2\theta}{\theta}=\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}\cdot\lim\limits_{\theta\to 0}\sin\theta=1(0)=\boxed{0}$.

32. $\lim\limits_{\theta\to 0}\dfrac{\cos(4\theta)-1}{2\theta}=\lim\limits_{\theta\to 0}\dfrac{2[\cos(4\theta)-1]}{4\theta}=2\lim\limits_{\theta\to 0}\dfrac{\cos(4\theta)-1}{4\theta}=2(0)=\boxed{0}$.

33. $\lim\limits_{\theta\to 0}(\theta\cdot\cot\theta)=\lim\limits_{\theta\to 0}\dfrac{\theta\cdot\cos\theta}{\sin\theta}=\dfrac{\lim\limits_{\theta\to 0}\cos\theta}{\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}}=\dfrac{1}{1}=\boxed{1}$.

34. Because
$$\sin\theta\left(\frac{\cot\theta-\csc\theta}{\theta}\right)=\frac{\cos\theta-1}{\theta},$$
it follows that
$$\lim_{\theta\to 0}\left[\sin\theta\left(\frac{\cot\theta-\csc\theta}{\theta}\right)\right]=\lim_{\theta\to 0}\frac{\cos\theta-1}{\theta}=\boxed{0}.$$

35. First, f is defined at $c=0$ with $f(0)=3$. Next,
$$\lim_{x\to 0^-}f(x)=\lim_{x\to 0^-}(3\cos x)=3(1)=3\quad\text{and}\quad\lim_{x\to 0^+}f(x)=\lim_{x\to 0^+}(x+3)=3.$$
Because the two one-sided limits as x approaches 0 are equal to 3, it follows that $\lim\limits_{x\to 0}f(x)$ exists and is equal to 3. Finally, $\lim\limits_{x\to 0}f(x)=f(0)$, so $\boxed{f\text{ is continuous at }c=0}$.

36. First, f is defined at $c=0$ with $f(0)=0$. Next,
$$\lim_{x\to 0^-}f(x)=\lim_{x\to 0^-}\cos x=\cos 0=1\quad\text{and}\quad\lim_{x\to 0^+}f(x)=\lim_{x\to 0^+}e^x=e^0=1.$$
Because the two one-sided limits as x approaches 0 are equal to 1, it follows that $\lim\limits_{x\to 0}f(x)$ exists and is equal to 1. However, $\lim\limits_{x\to 0}f(x)\neq f(0)$, so $\boxed{f\text{ is not continuous at }c=0}$.

37. First, f is defined at $c=\dfrac{\pi}{4}$ with $f\left(\dfrac{\pi}{4}\right)=\sin\dfrac{\pi}{4}=\dfrac{\sqrt{2}}{2}$. Next,
$$\lim_{x\to\pi/4^-}f(x)=\lim_{x\to\pi/4^-}\sin x=\sin\frac{\pi}{4}=\frac{\sqrt{2}}{2}\text{ and }\lim_{x\to\pi/4^+}f(x)=\lim_{x\to\pi/4^+}\cos x=\cos\frac{\pi}{4}=\frac{\sqrt{2}}{2}.$$
Because the two one-sided limits as x approaches $\pi/4$ are equal to $\dfrac{\sqrt{2}}{2}$, it follows that $\lim\limits_{x\to\pi/4}f(x)$ exists and is equal to $\dfrac{\sqrt{2}}{2}$. Finally, $\lim\limits_{x\to\pi/4}f(x)=f\left(\dfrac{\pi}{4}\right)$, so $\boxed{f\text{ is continuous at }c=\dfrac{\pi}{4}}$.

38. First, f is defined at $c=1$ with $f(1)=\ln 1=0$. Next,
$$\lim_{x\to 1^-}f(x)=\lim_{x\to 1^-}\tan^{-1}x=\tan^{-1}1=\frac{\pi}{4}$$
and
$$\lim_{x\to 1^+}f(x)=\lim_{x\to 1^+}\ln x=\ln 1=0.$$
Because the two one-sided limits as x approaches 0 are not equal, it follows that $\lim\limits_{x\to 1}f(x)$ does not exist. Therefore, $\boxed{f\text{ is not continuous at }c=1}$.

39. Let $g(x) = \sin x$ and $h(x) = \dfrac{x^2 - 4x}{x - 4}$. The trigonometric function g is continuous on the set of all real numbers, and the rational function h is continuous on the set $\{x|x \neq 4\}$. As the function f is the composition $g(h(x))$ and g is continuous at $h(x)$ for all x at which h is continuous, it follows that $\boxed{f \text{ is continuous on the set } \{x|x \neq 4\}}$.

40. Let $g(x) = \cos x$ and $h(x) = \dfrac{x^2 - 5x + 1}{2x}$. The trigonometric function g is continuous on the set of all real numbers, and the rational function h is continuous on the set $\{x|x \neq 0\}$. As the function f is the composition $g(h(x))$ and g is continuous at $h(x)$ for all x at which h is continuous, it follows that $\boxed{f \text{ is continuous on the set } \{x|x \neq 0\}}$.

41. The constant function 1 and the trigonometric function $\sin\theta$ are continuous on the set of all real numbers, so the sum of these functions, $1+\sin\theta$, is also continuous on the set of all real numbers. Now, $1 + \sin\theta = 0$ when $\sin\theta = -1$. This happens for $\theta = \frac{3\pi}{2} + 2k\pi$, where k is any integer. Because f is the quotient of the constant function 1 and the function $1+\sin\theta$, it follows that $\boxed{f \text{ is continuous on the set } \left\{x|x \neq \dfrac{3\pi}{2} + 2k\pi\right\}, \text{ where } k \text{ is an integer}}$.

42. The constant function 1 and the trigonometric function $\cos\theta$ are continuous on the set of all real numbers. The function $\cos^2\theta = \cos\theta \quad \cos\theta$, being the product of continuous functions, and the function $1+\cos^2\theta$, being the sum of continuous functions, are then also continuous on the set of all real numbers. Finally, because $1+\cos^2\theta$ is never equal to zero for any real number θ and f is the quotient of the constant function 1 and the function $1 + \cos^2\theta$, it follows that $\boxed{f \text{ is continuous on the set of all real numbers}}$.

43. Let $g(x) = \ln x$ and $h(x) = x - 3$. The logarithmic function g is continuous on the set $\{x|x > 0\}$, and the polynomial function h is continuous on the set of all real numbers. As f is the quotient of the functions g and h and the only value x for which $h(x) = 0$ is $x = 3$, it follows that the function $\boxed{f \text{ is continuous on the set } \{x|x > 0, x \neq 3\}}$.

44. Let $g(x) = \ln x$ and $h(x) = x^2 + 1$. The logarithmic function g is continuous on the set $\{x|x > 0\}$, and the polynomial function h is continuous on the set of all real numbers. As the function f is the composition $g(h(x))$ and g is continuous at $h(x)$ for all x because $x^2+1 \geq 1 > 0$ for any real number x, it follows that $\boxed{f \text{ is continuous on the set of all real numbers}}$.

45. Let $g(x) = e^{-x}$ and $h(x) = \sin x$. The exponential function g and the trigonometric function h are both continuous on the set of all real numbers. As f is the product of g and h, it follows that $\boxed{f \text{ is also continuous on the set of all real numbers}}$.

46. The exponential function e^x, the constant function 1, and the trigonometric function $\sin x$ are all continuous on the set of all real numbers. The function $\sin^2 x = \sin x \cdot \sin x$, being the product of continuous functions, and the function $1 + \sin^2 x$, being the sum of continuous functions, are then also continuous on the set of all real numbers. Finally, because $1+\sin^2 x$ is never equal to zero for any real number x and f is the quotient of the exponential function e^x and the function $1+\sin^2 x$, it follows that $\boxed{f \text{ is continuous on the set of all real numbers}}$.

Applications and Extensions

47. Start from the compound inequality

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1,$$

which holds for all $x \neq 0$. Multiplying by x^2, which is non-negative for all x, then yields

$$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2.$$

As

$$\lim_{x\to 0}(-x^2) = 0 \quad \text{and} \quad \lim_{x\to 0} x^2 = 0,$$

it follows from the Squeeze Theorem that

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = \boxed{0}.$$

48. Start from the compound inequality

$$-1 \leq \cos\left(\frac{1}{x}\right) \leq 1,$$

which holds for all $x \neq 0$. Then

$$0 \leq 1 - \cos\left(\frac{1}{x}\right) \leq 2$$

and

$$\left|1 - \cos\left(\frac{1}{x}\right)\right| \leq 2.$$

Multiplying by $|x|$, which is non-negative for all x, then yields

$$|x|\left|1 - \cos\left(\frac{1}{x}\right)\right| = \left|x\left(1 - \cos\left(\frac{1}{x}\right)\right)\right| \leq 2|x|,$$

or

$$-2|x| \leq x\left(1 - \cos\left(\frac{1}{x}\right)\right) \leq 2|x|.$$

As

$$\lim_{x\to 0}(-2|x|) = 0 \quad \text{and} \quad \lim_{x\to 0}(2|x|) = 0,$$

it follows from the Squeeze Theorem that

$$\lim_{x\to 0}\left[x\left(1 - \cos\left(\frac{1}{x}\right)\right)\right] = \boxed{0}.$$

49. Start from the compound inequality

$$-1 \leq \cos\left(\frac{1}{x}\right) \leq 1,$$

which holds for all $x \neq 0$. Then

$$0 \leq 1 - \cos\left(\frac{1}{x}\right) \leq 2.$$

Multiplying by x^2, which is non-negative for all x, then yields

$$0 \leq x^2\left(1 - \cos\left(\frac{1}{x}\right)\right) \leq 2x^2.$$

As

$$\lim_{x\to 0} 0 = 0 \quad \text{and} \quad \lim_{x\to 0}(2x^2) = 0,$$

it follows from the Squeeze Theorem that

$$\lim_{x\to 0}\left[x^2\left(1-\cos\left(\frac{1}{x}\right)\right)\right] = \boxed{0}.$$

50. First note that $x^3+3x^2 = x^2(x+3)$ is non-negative for $x \geq -3$. Thus, as x approaches 0, $\sqrt{x^3+3x^2}$ is defined. Now, consider the compound inequality

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1,$$

which holds for all $x \neq 0$. Multiplying by $\sqrt{x^3+3x^2}$, which is non-negative for all x, then yields

$$-\sqrt{x^3+3x^2} \leq \sqrt{x^3+3x^2}\sin\left(\frac{1}{x}\right) \leq \sqrt{x^3+3x^2}.$$

As

$$\lim_{x\to 0}(-\sqrt{x^3+3x^2}) = 0 \quad \text{and} \quad \lim_{x\to 0}\sqrt{x^3+3x^2} = 0,$$

it follows from the Squeeze Theorem that

$$\lim_{x\to 0}\sqrt{x^3+3x^2}\sin\left(\frac{1}{x}\right) = \boxed{0}.$$

51. $\displaystyle \lim_{x\to 0}\frac{\sin(ax)}{\sin(bx)} = \lim_{x\to 0}\frac{\frac{\sin(ax)}{x}}{\frac{\sin(bx)}{x}} = \lim_{x\to 0}\frac{\frac{a\sin(ax)}{ax}}{\frac{b\sin(bx)}{bx}} = \frac{a}{b}\,\frac{\lim\limits_{x\to 0}\frac{\sin(ax)}{ax}}{\lim\limits_{x\to 0}\frac{\sin(bx)}{bx}} = \frac{a}{b}\cdot\frac{1}{1} = \boxed{\frac{a}{b}}.$

52. $\displaystyle \lim_{x\to 0}\frac{\cos(ax)}{\cos(bx)} = \frac{\lim\limits_{x\to 0}\cos(ax)}{\lim\limits_{x\to 0}\cos(bx)} = \frac{1}{1} = \boxed{1}.$

53. $\displaystyle \lim_{x\to 0}\frac{\sin(ax)}{bx} = \lim_{x\to 0}\frac{a\sin(ax)}{abx} = \frac{a}{b}\lim_{x\to 0}\frac{\sin(ax)}{ax} = \frac{a}{b}\cdot 1 = \boxed{\frac{a}{b}}.$

54. $\displaystyle \lim_{x\to 0}\frac{1-\cos(ax)}{bx} = \lim_{x\to 0}\frac{a(1-\cos(ax))}{abx} = \frac{a}{b}\lim_{x\to 0}\frac{1-\cos(ax)}{ax} = \frac{a}{b}\cdot 0 = \boxed{0}.$

55.

$$\begin{aligned}\lim_{x\to 0}\frac{1-\cos x}{x^2} &= \lim_{x\to 0}\frac{1-\cos x}{x^2}\cdot\frac{1+\cos x}{1+\cos x} = \lim_{x\to 0}\frac{1-\cos^2 x}{x^2(1+\cos x)} = \lim_{x\to 0}\frac{\sin^2 x}{x^2(1+\cos x)} \\ &= \lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{1}{1+\cos x} = 1\cdot 1\cdot\frac{1}{1+1} = \frac{1}{2}.\end{aligned}$$

56. Let f be a function for which $0 \leq f(x) \leq 1$ for every x. Multiplying by x^2, which is non-negative for all x, then yields $0 \leq x^2 f(x) \leq x^2$ for every x. As

$$\lim_{x\to 0} 0 = 0 \quad \text{and} \quad \lim_{x\to 0} x^2 = 0,$$

it follows from the Squeeze Theorem that

$$\lim_{x\to 0}[x^2 f(x)] = 0.$$

57. Let f be a function for which $0 \le f(x) \le M$ for every x. Multiplying by x^2, which is non-negative for all x, then yields $0 \le x^2 f(x) \le Mx^2$ for every x. As

$$\lim_{x\to 0} 0 = 0 \quad \text{and} \quad \lim_{x\to 0}(Mx^2) = 0,$$

it follows from the Squeeze Theorem that

$$\lim_{x\to 0}[x^2 f(x)] = 0.$$

58. To make f continuous at 0, $f(0)$ should be defined equal to the value of $\lim_{x\to 0} f(x)$, provided this limit exists. Here,

$$\lim_{x\to 0}\frac{\sin(\pi x)}{x} = \lim_{x\to 0}\frac{\pi\sin(\pi x)}{\pi x} = \pi\lim_{x\to 0}\frac{\sin(\pi x)}{\pi x} = \pi\cdot 1 = \pi.$$

Thus, $\boxed{f(0) \text{ should be set equal to } \pi}$ to make f continuous at 0.

59. The functions $\sin(\pi x)$ and $\dfrac{1}{x(1-x)}$ are both continuous on the open interval $(0,1)$; consequently, the function $f(x) = \dfrac{\sin(\pi x)}{x(1-x)}$ is also continuous on the open interval $(0,1)$. Now,

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\frac{\sin(\pi x)}{x(1-x)} = \lim_{x\to 0^+}\frac{\pi\sin(\pi x)}{\pi x(1-x)} = \pi\lim_{x\to 0^+}\frac{\sin(\pi x)}{\pi x}\cdot\lim_{x\to 0^+}\frac{1}{1-x} = \pi\cdot 1\cdot\frac{1}{1-0} = \pi,$$

and, using the identity $\sin\theta = \sin(\pi-\theta)$,

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}\frac{\sin(\pi x)}{x(1-x)} = \lim_{x\to 1^-}\frac{\pi\sin(\pi(1-x))}{\pi x(1-x)} = \pi\lim_{x\to 1^-}\frac{\sin(\pi(1-x))}{\pi(1-x)}\cdot\lim_{x\to 1^-}\frac{1}{x} = \pi\cdot 1\cdot\frac{1}{1} = \pi.$$

Thus, defining $\boxed{f(0) = f(1) = \pi}$, f will be continuous on the closed interval $[0,1]$.

60. Because

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{\sin x}{x} = 1 = f(0),$$

f is $\boxed{\text{continuous}}$ at 0.

61. Because

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{1-\cos x}{x} = 0 = f(0),$$

f is $\boxed{\text{continuous}}$ at 0.

62. Let n be a positive integer, and start from the inequality

$$\left|\sin\left(\frac{1}{x}\right)\right| \le 1,$$

which holds for all $x \ne 0$. Multiplying by $|x^n|$, which is non-negative for all x, then yields

$$|x^n|\cdot\left|\sin\left(\frac{1}{x}\right)\right| = \left|x^n\sin\left(\frac{1}{x}\right)\right| \le |x^n|,$$

or

$$-|x^n| \le x^n\sin\left(\frac{1}{x}\right) \le |x^n|.$$

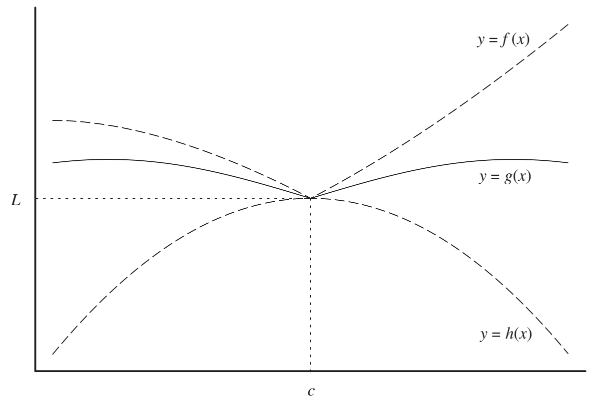
As
$$\lim_{x\to 0}(-|x^n|) = 0 \quad \text{and} \quad \lim_{x\to 0}|x^n| = 0,$$
it follows from the Squeeze Theorem that
$$\lim_{x\to 0}\left[x^n \sin\left(\frac{1}{x}\right)\right] = 0.$$

63. Let $0 < \theta < \frac{\pi}{2}$. Consider the diagram presented in the problem statement. The length of the segment $\overline{PB}$ is shorter than the length of the segment $\overline{PA}$ because $\overline{PA}$ is the hypotenuse of the right triangle PBA for which $\overline{PB}$ is one of the legs. Moreover, the length of the segment $\overline{PA}$ is shorter than the length of the arc PA because the shortest distance between any two distinct points is the length of the line segment connecting the two points. Thus, the segment joining the points P and B is shorter than the length of the arc AP along the circle. As the length of the segment joining P and B is $\sin\theta$ while the length of the arc AP is θ, it follows that $\sin\theta \le \theta$. Because θ is a first quadrant angle, it is also true that $\sin\theta \ge 0$. Thus,
$$0 \le \sin\theta \le \theta.$$
With
$$\lim_{\theta\to 0^+} 0 = 0 \quad \text{and} \quad \lim_{\theta\to 0^+} \theta = 0,$$
it follows from the Squeeze Theorem that
$$\lim_{\theta\to 0^+} \sin\theta = 0.$$
If, instead, $-\frac{\pi}{2} < \theta < 0$, then the length of the segment joining P and B is $-\sin\theta$, so that
$$0 \le -\sin\theta \le \theta \quad \text{or} \quad -\theta \le \sin\theta \le 0.$$
With
$$\lim_{\theta\to 0^-} (-\theta) = 0 \quad \text{and} \quad \lim_{\theta\to 0^-} 0 = 0,$$
it follows from the Squeeze Theorem that
$$\lim_{\theta\to 0^-} \sin\theta = 0.$$
Finally, because the two one-sided limits are equal,
$$\lim_{\theta\to 0} \sin\theta = 0.$$

64. From the Pythagorean identity $\cos^2\theta + \sin^2\theta = 1$ it follows that $\cos\theta = \pm\sqrt{1-\sin^2\theta}$. In the limit as θ approaches 0, θ will eventually lie in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, so that $\cos\theta$ will be positive; hence, $\cos\theta = \sqrt{1-\sin^2\theta}$. Therefore,
$$\lim_{\theta\to 0}\cos\theta = \lim_{\theta\to 0}\sqrt{1-\sin^2\theta} = \sqrt{1-0^2} = 1.$$

65. Answers will vary. The function $\cos\theta$ is continuous on the set of all real numbers, and the polynomial function $5x^3 + 2x^2 - 8x + 1$ is also continuous on the set of all real numbers. As $f(x) = \cos(5x^3 + 2x^2 - 8x + 1)$ is the composition of the two previously mentioned functions, it follows that f is continuous on the set of all real numbers.

66. Answers will vary. One possible response is the following. Suppose we wish to evaluate $\lim_{x\to c} g(x)$ for some function g. If two functions f and h can be found such that $f(x) \le g(x) \le h(x)$ for all x in a neighborhood of c, except possibly at c, and $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L$ for some real number L, then $\lim_{x\to c} g(x) = L$. In plain terms, the functions f and h squeeze the value of g toward L. A graph to illustrate the process is shown below.



Challenge Problems

67. To show that the sine function is continuous on its domain, it must be established that $\lim_{\theta\to c} \sin\theta = \sin c$ for any real number c. So let c be any real number. Then

$$\begin{aligned}\lim_{\theta\to c}\sin\theta &= \lim_{x\to 0}\sin(x+c) = \lim_{x\to 0}[\sin x\cos c + \cos x \sin c]\\ &= \cos c\lim_{x\to 0}\sin x + \sin c\lim_{x\to 0}\cos x = \cos c\cdot 0 + \sin c\cdot 1 = \sin c.\end{aligned}$$

Similarly, to show that the cosine function is continuous on its domain, it must be established that $\lim_{\theta\to c}\cos\theta = \cos c$ for any real number c. So let c be any real number. Then

$$\begin{aligned}\lim_{\theta\to c}\cos\theta &= \lim_{x\to 0}\cos(x+c) = \lim_{x\to 0}[\cos x\cos c - \sin x \sin c]\\ &= \cos c\lim_{x\to 0}\cos x - \sin c\lim_{x\to 0}\sin x = \cos c\cdot 1 - \sin c\cdot 0 = \cos c.\end{aligned}$$

68. $\lim_{x\to 0}\frac{\sin x^2}{x} = \lim_{x\to 0}\frac{x\sin x^2}{x^2} = \lim_{x\to 0} x\cdot\lim_{x\to 0}\frac{\sin x^2}{x^2} = 0(1) = \boxed{0}$.

69. As defined, $0 \le f(x) \le 1$ for all x, so that $|f(x)| \le 1$. Multiplying this last inequality by $|x|$, which is non-negative for all x, yields

$$|x|\cdot|f(x)| = |xf(x)| \le |x|, \quad \text{or} \quad -|x| \le xf(x) \le |x|.$$

Because

$$\lim_{x\to 0}(-|x|) = 0 \quad \text{and} \quad \lim_{x\to 0}|x| = 0,$$

it follows from the Squeeze Theorem that

$$\lim_{x\to 0}[xf(x)] = 0.$$

70. Using the diagram in the problem statement, let d denote the x-coordinate of the point D. Applying right angle trigonometry to the triangle ACD yields $d\tan\theta$ as the length of the segment $\overline{CD}$, while applying right angle trigonometry to the triangle BCD yields $(1-d)\tan(n\theta)$ as the length of the segment $\overline{CD}$. Thus,

$$d\tan\theta = (1-d)\tan(n\theta),$$

or

$$d = \frac{\tan(n\theta)}{\tan\theta + \tan(n\theta)} = \frac{\frac{\tan(n\theta)}{\tan\theta}}{1+\frac{\tan(n\theta)}{\tan\theta}}.$$

Now,

$$\lim_{\theta\to 0}\frac{\tan(n\theta)}{\tan\theta}=\lim_{\theta\to 0}\frac{\sin(n\theta)}{\sin\theta}\cdot\frac{\cos\theta}{\cos(n\theta)}=\lim_{\theta\to 0}\frac{\sin(n\theta)}{\sin\theta}\cdot\lim_{\theta\to 0}\frac{\cos\theta}{\cos(n\theta)}=n(1)=n,$$

using the results of Problems 51 and 52. Thus, as θ approaches 0, $d\to\dfrac{n}{n+1}$, and the limiting position of D is the point $\boxed{\left(\dfrac{n}{n+1},0\right)}$.

1.5 Infinite Limits; Limits at Infinity; Asymptotes

Concepts and Vocabulary

1. $\boxed{\text{False}}$. ∞ is **not** a number; ∞ is a symbol to represent the concept of becoming unbounded.

2. (a) $\lim\limits_{x\to 0^-}\dfrac{1}{x}=\boxed{-\infty}$.

   (b) $\lim\limits_{x\to 0^+}\dfrac{1}{x}=\boxed{\infty}$.

   (c) $\lim\limits_{x\to 0^+}\ln x=\boxed{-\infty}$.

3. $\boxed{\text{False}}$. The graph of a rational function **may have** a vertical asymptote at **a** number x at which the function is undefined. The graph will have a vertical asymptote provided that at least one of the one-sided limits at that number is infinite. If neither of the one-sided limits is infinite, the graph will have a hole.

4. If $\lim\limits_{x\to 4}f(x)=\infty$, then the line $x=4$ is a $\boxed{\textbf{vertical}}$ asymptote of the graph of f.

5. (a) $\lim\limits_{x\to\infty}\dfrac{1}{x}=\boxed{0}$.

   (b) $\lim\limits_{x\to\infty}\dfrac{1}{x^2}=\boxed{0}$.

   (c) $\lim\limits_{x\to\infty}\ln x=\boxed{\infty}$.

6. $\boxed{\text{False}}$. $\lim\limits_{x\to -\infty}5=5$.

7. (a) $\lim\limits_{x\to -\infty}e^x=\boxed{0}$.

   (b) $\lim\limits_{x\to\infty}e^x=\boxed{\infty}$.

   (c) $\lim\limits_{x\to\infty}e^{-x}=\boxed{0}$.

8. $\boxed{\text{True}}$. The graph of a function can have at most two horizontal asymptotes.

Skill Building

9. As x becomes unbounded in the positive direction, the graph of f approaches the line $y=2$. Thus, $\boxed{\lim\limits_{x\to\infty}f(x)=2}$.

10. As x becomes unbounded in the negative direction, the graph of f approaches the line $y=0$. Thus, $\boxed{\lim\limits_{x\to -\infty}f(x)=0}$.

11. As x approaches -1 from the left, the graph of f becomes unbounded in the positive direction. Thus, $\boxed{\lim_{x\to -1^-} f(x) = \infty}$.

12. As x approaches -1 from the right, the graph of f becomes unbounded in the positive direction. Thus, $\boxed{\lim_{x\to -1^+} f(x) = \infty}$.

13. As x approaches 3 from the left, the graph of f becomes unbounded in the positive direction. Thus, $\boxed{\lim_{x\to 3^-} f(x) = \infty}$.

14. As x approaches 3 from the right, the graph of f becomes unbounded in the positive direction. Thus, $\boxed{\lim_{x\to 3^+} f(x) = \infty}$.

15. The graph of f has two vertical asymptotes: $\boxed{x = -1 \text{ and } x = 3}$.

16. The graph of f has two horizontal asymptotes: $\boxed{y = 0 \text{ and } y = 2}$.

17. As x becomes unbounded in the positive direction, the graph of f approaches the line $y = -3$. Thus, $\boxed{\lim_{x\to\infty} f(x) = -3}$.

18. As x becomes unbounded in the negative direction, the graph of f approaches the line $y = 0$. Thus, $\boxed{\lim_{x\to -\infty} f(x) = 0}$.

19. As x approaches -3 from the left, the graph of f becomes unbounded in the positive direction. Thus, $\boxed{\lim_{x\to -3^-} f(x) = \infty}$.

20. As x approaches -3 from the right, the graph of f becomes unbounded in the negative direction. Thus, $\boxed{\lim_{x\to -3^+} f(x) = -\infty}$.

21. As x approaches 0 from the left, the graph of f approaches the origin. Thus, $\boxed{\lim_{x\to 0^-} f(x) = 0}$.

22. As x approaches 0 from the right, the graph of f becomes unbounded in the positive direction. Thus, $\boxed{\lim_{x\to 0^+} f(x) = \infty}$.

23. As x approaches 4 from the left, the graph of f becomes unbounded in the positive direction. Thus, $\boxed{\lim_{x\to 4^-} f(x) = \infty}$.

24. As x approaches 4 from the right, the graph of f becomes unbounded in the positive direction. Thus, $\boxed{\lim_{x\to 4^+} f(x) = \infty}$.

25. The graph of f has three vertical asymptotes: $\boxed{x = -3,\ x = 0, \text{ and } x = 4}$.

26. The graph of f has two horizontal asymptotes: $\boxed{y = 0 \text{ and } y = -3}$.

27. As x approaches 2 from the left, $3x$ approaches 6 and $x-2$ approaches 0 from the left. Therefore, the ratio $\frac{3x}{x-2}$ becomes unbounded in the negative direction, so

$$\lim_{x\to 2^-} \frac{3x}{x-2} = \boxed{-\infty}.$$

The values in the table below support this conclusion.

x	1.9	1.99	1.999	$\to 2$
$f(x) = \dfrac{3x}{x-2}$	-57	-597	-5997	$f(x)$ approaches $-\infty$

28. As x approaches -4 from the right, $2x+1$ approaches -7 and $x+4$ approaches 0 from the right. Therefore, the ratio $\frac{2x+1}{x+4}$ becomes unbounded in the negative direction, so

$$\lim_{x\to -4^+} \frac{2x+1}{x+4} = \boxed{-\infty}.$$

The values in the table below support this conclusion.

x	$-4 \leftarrow$	-3.999	-3.99	-3.9
$f(x) = \dfrac{2x+1}{x+4}$	$f(x)$ approaches $-\infty$	-6998	-698	-68

29. As x approaches 2 from the right, 5 approaches 5 and x^2-4 approaches 0 from the right. Therefore, the ratio $\frac{5}{x^2-4}$ becomes unbounded in the positive direction, so

$$\lim_{x\to 2^+} \frac{5}{x^2-4} = \boxed{\infty}.$$

The values in the table below, which have been rounded to two decimal places, support this conclusion.

x	$2 \leftarrow$	2.001	2.01	2.1
$f(x) = \dfrac{5}{x^2-4}$	$f(x)$ approaches ∞	1249.69	124.69	12.20

30. As x approaches 1 from the left, $2x$ approaches 2 and x^3-1 approaches 0 from the left. Therefore, the ratio $\frac{2x}{x^3-1}$ becomes unbounded in the negative direction, so

$$\lim_{x\to 1^-} \frac{2x}{x^3-1} = \boxed{-\infty}.$$

The values in the table below, which have been rounded to two decimal places, support this conclusion.

x	0.9	0.99	0.999	$\to 1$
$f(x) = \dfrac{2x}{x^3-1}$	-6.64	-66.66	-666.67	$f(x)$ approaches $-\infty$

31. As x approaches -1 from the right, $5x+3$ approaches -2 and $x(x+1)$ approaches 0 from the left. Therefore, the ratio $\frac{5x+3}{x(x+1)}$ becomes unbounded in the positive direction, so

$$\lim_{x\to -1^+} \frac{5x+3}{x(x+1)} = \boxed{\infty}.$$

The values in the table below, which have been rounded to two decimal places, support this conclusion.

x	$-1 \leftarrow$	-0.999	-0.99	-0.9
$f(x) = \dfrac{5x+3}{x(x+1)}$	$f(x)$ approaches ∞	1997.00	196.97	16.67

32. As x approaches 0 from the left, $5x+3$ approaches 3 and $5x(x-1)$ approaches 0 from the right. Therefore, the ratio $\frac{5x+3}{5x(x-1)}$ becomes unbounded in the positive direction, so

$$\lim_{x\to 0^-} \frac{5x+3}{5x(x-1)} = \boxed{\infty}.$$

The values in the table below, which have been rounded to two decimal places, support this conclusion.

x	-0.1	-0.01	-0.001	$\to 0$
$f(x) = \dfrac{5x+3}{5x(x-1)}$	4.55	58.42	598.40	$f(x)$ approaches ∞

33. As x approaches -3 from the left, 1 approaches 1 and x^2-9 approaches 0 from the right. Therefore, the ratio $\frac{1}{x^2-9}$ becomes unbounded in the positive direction, so

$$\lim_{x\to -3^-} \frac{1}{x^2-9} = \boxed{\infty}.$$

The values in the table below, which have been rounded to two decimal places, support this conclusion.

x	-3.1	-3.01	-3.001	$\to -3$
$f(x) = \dfrac{1}{x^2-9}$	1.64	16.64	166.64	$f(x)$ approaches ∞

34. As x approaches 2 from the right, x approaches 2 and x^2-4 approaches 0 from the right. Therefore, the ratio $\frac{x}{x^2-4}$ becomes unbounded in the positive direction, so

$$\lim_{x\to 2^+} \frac{x}{x^2-4} = \boxed{\infty}.$$

The values in the table below, which have been rounded to two decimal places, support this conclusion.

x	$2 \leftarrow$	2.001	2.01	2.1
$f(x) = \dfrac{x}{x^2-4}$	$f(x)$ approaches ∞	500.12	50.12	5.12

35. As x approaches 3, $1-x$ approaches -2 and $(3-x)^2$ approaches 0 from the right. Therefore, the ratio $\frac{1-x}{(3-x)^2}$ becomes unbounded in the negative direction, so

$$\lim_{x\to 3}\frac{1-x}{(3-x)^2} = \boxed{-\infty}.$$

The values in the table below support this conclusion.

x	2.9	2.99	2.999	$\to 3 \leftarrow$	3.001	3.01	3.1
$f(x) = \frac{1-x}{(3-x)^2}$	-190	-19900	-1999000	$f(x)$ approaches $-\infty$	-2001000	-20100	-210

36. As x approaches -1, $x+2$ approaches 1 and $(x+1)^2$ approaches 0 from the right. Therefore, the ratio $\frac{x+2}{(x+1)^2}$ becomes unbounded in the positive direction, so

$$\lim_{x\to -1}\frac{x+2}{(x+1)^2} = \boxed{\infty}.$$

The values in the table below support this conclusion.

x	-1.1	-1.01	-1.001	$\to -1 \leftarrow$	-0.999	-0.99	-0.9
$f(x) = \frac{x+2}{(x+1)^2}$	90	9900	999000	$f(x)$ approaches ∞	1001000	10100	110

37. As x approaches π from the left, $\cos x$ approaches -1 and $\sin x$ approaches 0 from the right. Therefore, the ratio $\frac{\cos x}{\sin x} = \cot x$ becomes unbounded in the negative direction, so

$$\lim_{x\to \pi^-}\cot x = \boxed{-\infty}.$$

The values in the table below, which have been rounded to two decimal places, support this conclusion.

x	$\pi - 0.1$	$\pi - 0.01$	$\pi - 0.001$	$\to \pi$
$f(x) = \cot x$	-9.97	-100.00	-1000.00	$f(x)$ approaches $-\infty$

38. As x approaches $-\pi/2$ from the left, $\sin x$ approaches -1 and $\cos x$ approaches 0 from the left. Therefore, the ratio $\frac{\sin x}{\cos x} = \tan x$ becomes unbounded in the positive direction, so

$$\lim_{x\to -\pi/2^-}\tan x = \boxed{\infty}.$$

The values in the table below, which have been rounded to two decimal places, support this conclusion.

x	$-\frac{\pi}{2} - 0.1$	$-\frac{\pi}{2} - 0.01$	$-\frac{\pi}{2} - 0.001$	$\to -\pi/2$
$f(x) = \tan x$	9.97	100.00	1000.00	$f(x)$ approaches ∞

39. As x approaches $\pi/2$ from the right, $2x$ approaches π from the right, so $\sin(2x)$ approaches 0 from the left. Therefore, the ratio $\frac{1}{\sin(2x)} = \csc(2x)$ becomes unbounded in the negative direction, so

$$\lim_{x\to \pi/2^+}\csc(2x) = \boxed{-\infty}.$$

The values in the table below, which have been rounded to two decimal places, support this conclusion.

x	$\frac{\pi}{2} \leftarrow$	$\frac{\pi}{2}+0.001$	$\frac{\pi}{2}+0.01$	$\frac{\pi}{2}+0.1$
$f(x)=\csc(2x)$	$f(x)$ approaches $-\infty$	-500.00	-50.00	-5.03

40. As x approaches $-\pi/2$ from the left, $\cos x$ approaches 0 from the left. Therefore, the ratio $\frac{1}{\cos x} = \sec x$ becomes unbounded in the negative direction, so

$$\lim_{x\to -\pi/2^-} \sec x = \boxed{-\infty}.$$

The values in the table below, which have been rounded to two decimal places, support this conclusion.

x	$-\frac{\pi}{2}-0.1$	$-\frac{\pi}{2}-0.01$	$-\frac{\pi}{2}-0.001$	$\rightarrow -\pi/2$
$f(x)=\sec x$	-10.02	-100.00	-1000.00	$f(x)$ approaches $-\infty$

41. As x approaches -1 from the right, $x+1$ approaches 0 from the right. Therefore, $\ln(x+1)$ becomes unbounded in the negative direction, so

$$\lim_{x\to -1^+} \ln(x+1) = \boxed{-\infty}.$$

The values in the table below, which have been rounded to two decimal places, support this conclusion.

x	$-1 \leftarrow$	$-1+10^{-6}$	$-1+10^{-4}$	$-1+10^{-2}$
$f(x)=\ln(x+1)$	$f(x)$ approaches $-\infty$	-13.82	-9.21	-4.61

42. As x approaches 1 from the right, $x-1$ approaches 0 from the right. Therefore, $\ln(x-1)$ becomes unbounded in the negative direction, so

$$\lim_{x\to 1^+} \ln(x-1) = \boxed{-\infty}.$$

The values in the table below, which have been rounded to two decimal places, support this conclusion.

x	$1 \leftarrow$	$1+10^{-6}$	$1+10^{-4}$	$1+10^{-2}$
$f(x)=\ln(x-1)$	$f(x)$ approaches $-\infty$	-13.82	-9.21	-4.61

43. $$\lim_{x\to\infty} \frac{5}{x^2+4} = \lim_{x\to\infty} \frac{\frac{5}{x^2}}{\frac{x^2+4}{x^2}} = \lim_{x\to\infty} \frac{\frac{5}{x^2}}{1+\frac{4}{x^2}} = \frac{0}{1+0} = \boxed{0}.$$

44. $$\lim_{x\to-\infty} \frac{1}{x^2-9} = \lim_{x\to-\infty} \frac{\frac{1}{x^2}}{\frac{x^2-9}{x^2}} = \lim_{x\to-\infty} \frac{\frac{1}{x^2}}{1-\frac{9}{x^2}} = \frac{0}{1-0} = \boxed{0}.$$

45. $$\lim_{x\to\infty} \frac{2x+4}{5x} = \lim_{x\to\infty} \frac{\frac{2x+4}{5x}}{\frac{5x}{5x}} = \lim_{x\to\infty} \frac{\frac{2}{5}+\frac{4}{5x}}{1} = \frac{\frac{2}{5}+0}{1} = \boxed{\frac{2}{5}}.$$

46. $\lim\limits_{x\to\infty} \dfrac{x+1}{x} = \lim\limits_{x\to\infty} \dfrac{\dfrac{x+1}{x}}{\dfrac{x}{x}} = \lim\limits_{x\to\infty} \dfrac{1+\dfrac{1}{x}}{1} = \dfrac{1+0}{1} = \boxed{1}.$

47. $\lim\limits_{x\to\infty} \dfrac{x^{3/2}+2x-x^{1/2}}{5x-x^{1/2}} = \lim\limits_{x\to\infty} \dfrac{\dfrac{x^{3/2}+2x-x^{1/2}}{x}}{\dfrac{5x-x^{1/2}}{x}} = \lim\limits_{x\to\infty} \dfrac{x^{1/2}+2-\dfrac{1}{x^{1/2}}}{5-\dfrac{1}{x^{1/2}}}$

$= \dfrac{\lim\limits_{x\to\infty} x^{1/2}+2-0}{5-0} = \boxed{\infty}.$

48. $\lim\limits_{x\to\infty} \dfrac{6x^2+x^{3/4}}{2x^2+x^{3/2}-2x^{1/2}} = \lim\limits_{x\to\infty} \dfrac{\dfrac{6x^2+x^{3/4}}{x^2}}{\dfrac{2x^2+x^{3/2}-2x^{1/2}}{x^2}}$

$= \lim_{x\to\infty} \dfrac{6+\dfrac{1}{x^{5/4}}}{2+\dfrac{1}{x^{1/2}}-\dfrac{2}{x^{3/2}}} = \dfrac{6+0}{2+0-0} = \boxed{3}.$

49. $\lim\limits_{x\to-\infty} \dfrac{x^2+1}{x^3-1} = \lim\limits_{x\to-\infty} \dfrac{\dfrac{x^2+1}{x^3}}{\dfrac{x^3-1}{x^3}} = \lim\limits_{x\to-\infty} \dfrac{\dfrac{1}{x}+\dfrac{1}{x^3}}{1-\dfrac{1}{x^3}} = \dfrac{0+0}{1-0} = \boxed{0}.$

50. $\lim\limits_{x\to\infty} \dfrac{x^2-2x+1}{x^3+5x+4} = \lim\limits_{x\to\infty} \dfrac{\dfrac{x^2-2x+1}{x^3}}{\dfrac{x^3+5x+4}{x^3}} = \lim\limits_{x\to\infty} \dfrac{\dfrac{1}{x}-\dfrac{2}{x^2}+\dfrac{1}{x^3}}{1+\dfrac{5}{x^2}+\dfrac{4}{x^3}} = \dfrac{0-0+0}{1+0+0} = \boxed{0}.$

51.

$$\lim_{x\to\infty}\left[\frac{3x}{2x+5}-\frac{x^2+1}{4x^2+8}\right] = \lim_{x\to\infty}\frac{3x}{2x+5} - \lim_{x\to\infty}\frac{x^2+1}{4x^2+8} = \lim_{x\to\infty}\frac{\dfrac{3x}{2x}}{\dfrac{2x+5}{2x}} - \lim_{x\to\infty}\frac{\dfrac{x^2+1}{4x^2}}{\dfrac{4x^2+8}{4x^2}}$$

$$= \lim_{x\to\infty}\frac{\dfrac{3}{2}}{1+\dfrac{5}{2x}} - \lim_{x\to\infty}\frac{\dfrac{1}{4}+\dfrac{1}{4x^2}}{1+\dfrac{2}{x^2}} = \frac{\dfrac{3}{2}}{1+0} - \frac{\dfrac{1}{4}+0}{1+0} = \frac{3}{2}-\frac{1}{4} = \boxed{\frac{5}{4}}.$$

52.

$$\begin{aligned}
\lim_{x\to\infty}\left[\frac{1}{x^2+x+4}-\frac{x+1}{3x-1}\right] &= \lim_{x\to\infty}\frac{1}{x^2+x+4} - \lim_{x\to\infty}\frac{x+1}{3x-1} \\
&= \lim_{x\to\infty}\frac{\dfrac{1}{x^2}}{\dfrac{x^2+x+4}{x^2}} - \lim_{x\to\infty}\frac{\dfrac{x+1}{3x}}{\dfrac{3x-1}{3x}} \\
&= \lim_{x\to\infty}\frac{\dfrac{1}{x^2}}{1+\dfrac{1}{x}+\dfrac{4}{x^2}} - \lim_{x\to\infty}\frac{\dfrac{1}{3}+\dfrac{1}{3x}}{1-\dfrac{1}{3x}} \\
&= \frac{0}{1+0+0} - \frac{\dfrac{1}{3}+0}{1-0} = 0-\frac{1}{3} = \boxed{-\frac{1}{3}}.
\end{aligned}$$

53.

$$\lim_{x\to-\infty}\left[2e^x\left(\frac{5x+1}{3x}\right)\right] = \lim_{x\to-\infty}(2e^x)\cdot\lim_{x\to-\infty}\frac{5x+1}{3x} = \lim_{x\to-\infty}(2e^x)\cdot\left(\lim_{x\to-\infty}\frac{\dfrac{5x+1}{3x}}{\dfrac{3x}{3x}}\right)$$

$$= \lim_{x\to-\infty}(2e^x)\cdot\lim_{x\to-\infty}\frac{\dfrac{5}{3}+\dfrac{1}{3x}}{1} = 0\cdot\frac{\dfrac{5}{3}+0}{1} = \boxed{0}.$$

54.

$$\lim_{x\to-\infty}\left[e^x\left(\frac{x^2+x-3}{2x^3-x^2}\right)\right] = \lim_{x\to-\infty}e^x\cdot\lim_{x\to-\infty}\frac{x^2+x-3}{2x^3-x^2} = \lim_{x\to-\infty}e^x\cdot\left(\lim_{x\to-\infty}\frac{\dfrac{x^2+x-3}{2x^3}}{\dfrac{2x^3-x^2}{2x^3}}\right)$$

$$= \lim_{x\to-\infty}e^x\cdot\lim_{x\to-\infty}\frac{\dfrac{1}{2x}+\dfrac{1}{2x^2}-\dfrac{3}{2x^3}}{1-\dfrac{1}{2x}} = 0\cdot\frac{0+0-0}{1-0} = \boxed{0}.$$

55. $$\lim_{x\to\infty}\frac{\sqrt{x}+2}{3x-4} = \lim_{x\to\infty}\frac{\dfrac{\sqrt{x}+2}{3x}}{\dfrac{3x-4}{3x}} = \lim_{x\to\infty}\frac{\dfrac{1}{3\sqrt{x}}+\dfrac{2}{3x}}{1-\dfrac{4}{3x}} = \frac{0+0}{1-0} = \boxed{0}.$$

56. $$\lim_{x\to\infty}\frac{\sqrt{3x^3}+2}{x^2+6} = \lim_{x\to\infty}\frac{\dfrac{\sqrt{3x^3}+2}{x^2}}{\dfrac{x^2+6}{x^2}} = \lim_{x\to\infty}\frac{\dfrac{\sqrt{3}}{\sqrt{x}}+\dfrac{2}{x^2}}{1+\dfrac{6}{x^2}} = \frac{0+0}{1+0} = \boxed{0}.$$

57. Because

$$\lim_{x\to\infty}\frac{3x^2-1}{x^2+4} = \lim_{x\to\infty}\frac{\dfrac{3x^2-1}{x^2}}{\dfrac{x^2+4}{x^2}} = \lim_{x\to\infty}\frac{3-\dfrac{1}{x^2}}{1+\dfrac{4}{x^2}} = \frac{3-0}{1+0} = 3,$$

it follows that

$$\lim_{x\to\infty}\sqrt{\frac{3x^2-1}{x^2+4}} = \sqrt{\lim_{x\to\infty}\frac{3x^2-1}{x^2+4}} = \boxed{\sqrt{3}}.$$

58. Because

$$\lim_{x\to\infty}\frac{16x^3+2x+1}{2x^3+3x} = \lim_{x\to\infty}\frac{\dfrac{16x^3+2x+1}{2x^3}}{\dfrac{2x^3+3x}{2x^3}} = \lim_{x\to\infty}\frac{8+\dfrac{1}{x^2}+\dfrac{1}{2x^3}}{1+\dfrac{3}{2x^2}} = \frac{8+0+0}{1+0} = 8,$$

it follows that

$$\lim_{x\to\infty}\left(\frac{16x^3+2x+1}{2x^3+3x}\right)^{2/3} = \left(\lim_{x\to\infty}\frac{16x^3+2x+1}{2x^3+3x}\right)^{2/3} = 8^{2/3} = \boxed{4}.$$

59. $$\lim_{x\to-\infty}\frac{5x^3}{x^2+1} = \lim_{x\to-\infty}\frac{\dfrac{5x^3}{x^2}}{\dfrac{x^2+1}{x^2}} = \lim_{x\to-\infty}\frac{5x}{1+\dfrac{1}{x^2}} = \boxed{-\infty}.$$

60. $\lim\limits_{x\to-\infty} \dfrac{x^4}{x-2} = \lim\limits_{x\to-\infty} \dfrac{\dfrac{x^4}{x}}{\dfrac{x-2}{x}} = \lim\limits_{x\to-\infty} \dfrac{x^3}{1-\dfrac{2}{x}} = \boxed{-\infty}.$

61. The domain of $f(x) = 3 + \dfrac{1}{x}$ is the set $\{x|x \neq 0\}$. The one-sided limits as x approaches 0 are

$$\lim_{x\to0^-}\left(3+\frac{1}{x}\right) = -\infty \quad \text{and} \quad \lim_{x\to0^+}\left(3+\frac{1}{x}\right) = \infty,$$

so $\boxed{x = 0 \text{ is a vertical asymptote of the graph of } f}$. Because

$$\lim_{x\to-\infty}\left(3+\frac{1}{x}\right) = 3+0 = 3 \quad \text{and} \quad \lim_{x\to\infty}\left(3+\frac{1}{x}\right) = 3+0 = 3,$$

$\boxed{y = 3 \text{ is a horizontal asymptote of the graph of } f}$.

62. The domain of $f(x) = 2 - \dfrac{1}{x^2}$ is the set $\{x|x = 0\}$. The one-sided limits as x approaches 0 are

$$\lim_{x\to0^-}\left(2-\frac{1}{x^2}\right) = -\infty \quad \text{and} \quad \lim_{x\to0^+}\left(2-\frac{1}{x^2}\right) = -\infty,$$

so $\boxed{x = 0 \text{ is a vertical asymptote of the graph of } f}$. Because

$$\lim_{x\to-\infty}\left(2-\frac{1}{x^2}\right) = 2-0 = 2 \quad \text{and} \quad \lim_{x\to\infty}\left(2-\frac{1}{x^2}\right) = 2-0 = 2,$$

$\boxed{y = 2 \text{ is a horizontal asymptote of the graph of } f}$.

63. The domain of the function $f(x) = \dfrac{x^2}{x^2-1}$ is the set $\{x|x \neq \pm1\}$. The one-sided limits as x approaches -1 are

$$\lim_{x\to-1^-}\frac{x^2}{x^2-1} = \infty \quad \text{and} \quad \lim_{x\to-1^+}\frac{x^2}{x^2-1} = -\infty,$$

so $\boxed{x = -1 \text{ is a vertical asymptote of the graph of } f}$. The one-sided limits as x approaches 1 are

$$\lim_{x\to1^-}\frac{x^2}{x^2-1} = -\infty \quad \text{and} \quad \lim_{x\to1^+}\frac{x^2}{x^2-1} = \infty,$$

so $\boxed{x = 1 \text{ is also a vertical asymptote of the graph of } f}$. Because

$$\lim_{x\to-\infty}\frac{x^2}{x^2-1} = \lim_{x\to-\infty}\frac{\dfrac{x^2}{x^2}}{\dfrac{x^2-1}{x^2}} = \lim_{x\to-\infty}\frac{1}{1-\dfrac{1}{x^2}} = \frac{1}{1-0} = 1,$$

and

$$\lim_{x\to\infty}\frac{x^2}{x^2-1} = \lim_{x\to\infty}\frac{\dfrac{x^2}{x^2}}{\dfrac{x^2-1}{x^2}} = \lim_{x\to\infty}\frac{1}{1-\dfrac{1}{x^2}} = \frac{1}{1-0} = 1,$$

$\boxed{y = 1 \text{ is a horizontal asymptote of the graph of } f}$.

64. The domain of the function $f(x) = \dfrac{2x^2-1}{x^2-1}$ is the set $\{x | x \neq \pm 1\}$. The one-sided limits as x approaches -1 are

$$\lim_{x\to -1^-} \frac{2x^2-1}{x^2-1} = \infty \quad \text{and} \quad \lim_{x\to -1^+} \frac{2x^2-1}{x^2-1} = -\infty,$$

so $\boxed{x = -1 \text{ is a vertical asymptote of the graph of } f}$. The one-sided limits as x approaches 1 are

$$\lim_{x\to 1^-} \frac{2x^2-1}{x^2-1} = -\infty \quad \text{and} \quad \lim_{x\to 1^+} \frac{2x^2-1}{x^2-1} = \infty,$$

so $\boxed{x = 1 \text{ is also a vertical asymptote of the graph of } f}$. Because

$$\lim_{x\to -\infty} \frac{2x^2-1}{x^2-1} = \lim_{x\to -\infty} \frac{\frac{2x^2-1}{x^2}}{\frac{x^2-1}{x^2}} = \lim_{x\to -\infty} \frac{2-\frac{1}{x^2}}{1-\frac{1}{x^2}} = \frac{2-0}{1-0} = 2,$$

and

$$\lim_{x\to \infty} \frac{2x^2-1}{x^2-1} = \lim_{x\to \infty} \frac{\frac{2x^2-1}{x^2}}{\frac{x^2-1}{x^2}} = \lim_{x\to \infty} \frac{2-\frac{1}{x^2}}{1-\frac{1}{x^2}} = \frac{2-0}{1-0} = 2,$$

$\boxed{y = 2 \text{ is a horizontal asymptote of the graph of } f}$.

65. By completing the square, we find

$$2x^2 - x + 10 = 2\left(x^2 - \frac{1}{2}x + \frac{1}{16}\right) + 10 - \frac{1}{8} = 2\left(x - \frac{1}{4}\right)^2 + \frac{79}{8} \geq \frac{79}{8} > 0$$

for all x. Therefore, $\sqrt{2x^2-x+10}$ is defined for all x, and the domain of $f(x) = \dfrac{\sqrt{2x^2-x+10}}{2x-3}$ is the set $\{x | x \neq \frac{3}{2}\}$. The one-sided limits as x approaches $\frac{3}{2}$ are

$$\lim_{x\to 3/2^-} \frac{\sqrt{2x^2-x+10}}{2x-3} = -\infty \quad \text{and} \quad \lim_{x\to 3/2^+} \frac{\sqrt{2x^2-x+10}}{2x-3} = \infty,$$

so $\boxed{x = \frac{3}{2} \text{ is a vertical asymptote of the graph of } f}$. To examine the limits at infinity, note that

$$\frac{\sqrt{2x^2-x+10}}{2x-3} = \frac{|x|\sqrt{2-\frac{1}{x}+\frac{10}{x^2}}}{2x-3} = \frac{|x|}{x} \cdot \frac{\sqrt{2-\frac{1}{x}+\frac{10}{x^2}}}{2-\frac{3}{x}}.$$

Note that $|x|/x$ is -1 as x approaches $-\infty$ and $+1$ as x approaches ∞; therefore,

$$\lim_{x\to -\infty} R(x) = -\frac{\sqrt{2-0+0}}{2-0} = -\frac{\sqrt{2}}{2} \quad \text{and} \quad \lim_{x\to \infty} R(x) = \frac{\sqrt{2-0+0}}{2-0} = \frac{\sqrt{2}}{2},$$

so that $\boxed{y = -\frac{\sqrt{2}}{2} \text{ and } y = \frac{\sqrt{2}}{2} \text{ are horizontal asymptotes of the graph of } f}$.

66. The domain of the function $f(x) = \dfrac{\sqrt[3]{x^2+5x}}{x-6}$ is the set $\{x | x \neq 6\}$. The one-sided limits as x approaches 6 are

$$\lim_{x\to 6^-} \frac{\sqrt[3]{x^2+5x}}{x-6} = -\infty \quad \text{and} \quad \lim_{x\to 6^+} \frac{\sqrt[3]{x^2+5x}}{x-6} = \infty,$$

so $\boxed{x = 6 \text{ is a vertical asymptote of the graph of } f}$. Because

$$\lim_{x\to-\infty} \frac{\sqrt[3]{x^2+5x}}{x-6} = \lim_{x\to-\infty} \frac{\dfrac{\sqrt[3]{x^2+5x}}{x}}{\dfrac{x-6}{x}} = \lim_{x\to-\infty} \frac{\sqrt[3]{\dfrac{1}{x}+\dfrac{5}{x^2}}}{1-\dfrac{6}{x}} = \frac{\sqrt[3]{0+0}}{1-0} = 0,$$

and

$$\lim_{x\to\infty} \frac{\sqrt[3]{x^2+5x}}{x-6} = \lim_{x\to\infty} \frac{\dfrac{\sqrt[3]{x^2+5x}}{x}}{\dfrac{x-6}{x}} = \lim_{x\to\infty} \frac{\sqrt[3]{\dfrac{1}{x}+\dfrac{5}{x^2}}}{1-\dfrac{6}{x}} = \frac{\sqrt[3]{0+0}}{1-0} = 0,$$

$\boxed{y = 0 \text{ is a horizontal asymptote of the graph of } f}$.

67. (a) Factoring the denominator of R yields

$$2x^3 + 4x^2 = 2x^2(x+2),$$

so the domain of R is the set $\boxed{\{x | x \neq -2, x \neq 0\}}$.

(b) Because

$$\lim_{x\to-\infty} \frac{-2x^2+1}{2x^3+4x^2} = \lim_{x\to-\infty} \frac{\dfrac{-2x^2+1}{2x^3}}{\dfrac{2x^3+4x^2}{2x^3}} = \lim_{x\to-\infty} \frac{-\dfrac{1}{x}+\dfrac{1}{2x^3}}{1+\dfrac{2}{x}} = \frac{0+0}{1+0} = 0$$

and

$$\lim_{x\to\infty} \frac{-2x^2+1}{2x^3+4x^2} = \lim_{x\to\infty} \frac{\dfrac{-2x^2+1}{2x^3}}{\dfrac{2x^3+4x^2}{2x^3}} = \lim_{x\to\infty} \frac{-\dfrac{1}{x}+\dfrac{1}{2x^3}}{1+\dfrac{2}{x}} = \frac{0+0}{1+0} = 0,$$

$\boxed{y = 0 \text{ is a horizontal asymptote of the graph of } R}$.

(c) The one-sided limits as x approaches -2 are

$$\lim_{x\to-2^-} \frac{-2x^2+1}{2x^3+4x^2} = \infty \quad \text{and} \quad \lim_{x\to-2^+} \frac{-2x^2+1}{2x^3+4x^2} = -\infty,$$

so $\boxed{x = -2 \text{ is a vertical asymptote of the graph of } R}$. The one-sided limits as x approaches 0 are

$$\lim_{x\to 0^-} \frac{-2x^2+1}{2x^3+4x^2} = \infty \quad \text{and} \quad \lim_{x\to 0^+} \frac{-2x^2+1}{2x^3+4x^2} = \infty,$$

so $\boxed{x = 0 \text{ is also a vertical asymptote of the graph of } R}$.

(d) The only numbers where R is not defined are $x = -2$ and $x = 0$. Using the limits from part (c), it follows that as x approaches -2 from the left, the graph of R becomes unbounded in the positive direction, while as x approaches -2 from the right, the graph of R becomes unbounded in the negative direction. As x approaches 0 from either direction, the graph of R becomes unbounded in the positive direction.

68. (a) Factoring the denominator of R yields

$$x^4 - 1 = (x^2 - 1)(x^2 + 1) = (x - 1)(x + 1)(x^2 + 1),$$

so the domain of R is the set $\boxed{\{x|x \neq \pm 1\}}$.

(b) Because

$$\lim_{x\to-\infty} \frac{x^3}{x^4 - 1} = \lim_{x\to-\infty} \frac{\frac{x^3}{x^4}}{\frac{x^4 - 1}{x^4}} = \lim_{x\to-\infty} \frac{\frac{1}{x}}{1 - \frac{1}{x^4}} = \frac{0}{1 - 0} = 0$$

and

$$\lim_{x\to\infty} \frac{x^3}{x^4 - 1} = \lim_{x\to\infty} \frac{\frac{x^3}{x^4}}{\frac{x^4 - 1}{x^4}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1 - \frac{1}{x^4}} = \frac{0}{1 - 0} = 0,$$

$\boxed{y = 0 \text{ is a horizontal asymptote of the graph of } R}$.

(c) The one-sided limits as x approaches -1 are

$$\lim_{x\to-1^-} \frac{x^3}{x^4 - 1} = -\infty \quad \text{and} \quad \lim_{x\to-1^+} \frac{x^3}{x^4 - 1} = \infty,$$

so $\boxed{x = -1 \text{ is a vertical asymptote of the graph of } R}$. The one-sided limits as x approaches 1 are

$$\lim_{x\to1^-} \frac{x^3}{x^4 - 1} = -\infty \quad \text{and} \quad \lim_{x\to1^+} \frac{x^3}{x^4 - 1} = \infty,$$

so $\boxed{x = 1 \text{ is also a vertical asymptote of the graph of } R}$.

(d) The only numbers where R is not defined are $x = \pm 1$. Using the limits from part (c), it follows that as x approaches -1 from the left, the graph of R becomes unbounded in the negative direction, while as x approaches -1 from the right, the graph of R becomes unbounded in the positive direction. As x approaches 1 from the left, the graph of R becomes unbounded in the negative direction, while as x approaches 1 from the right, the graph of R becomes unbounded in the positive direction.

69. (a) Factoring the denominator of R yields

$$2x^2 - 7x + 6 = (2x - 3)(x - 2),$$

so the domain of R is the set $\boxed{\{x|x \neq \tfrac{3}{2}, x \neq 2\}}$.

(b) Because

$$\lim_{x\to-\infty} \frac{x^2 + 3x - 10}{2x^2 - 7x + 6} = \lim_{x\to-\infty} \frac{\frac{x^2 + 3x - 10}{2x^2}}{\frac{2x^2 - 7x + 6}{2x^2}} = \lim_{x\to-\infty} \frac{\frac{1}{2} + \frac{3}{2x} - \frac{5}{x^2}}{1 - \frac{7}{2x} + \frac{3}{x^2}} = \frac{\frac{1}{2} + 0 - 0}{1 - 0 + 0} = \frac{1}{2}$$

and

$$\lim_{x\to\infty}\frac{x^2+3x-10}{2x^2-7x+6}=\lim_{x\to\infty}\frac{\dfrac{x^2+3x-10}{2x^2}}{\dfrac{2x^2-7x+6}{2x^2}}=\lim_{x\to\infty}\frac{\dfrac{1}{2}+\dfrac{3}{2x}-\dfrac{5}{x^2}}{1-\dfrac{7}{2x}+\dfrac{3}{x^2}}=\frac{\dfrac{1}{2}+0-0}{1-0+0}=\frac{1}{2},$$

$\boxed{y=\frac{1}{2} \text{ is a horizontal asymptote of the graph of } R}$.

(c) The one-sided limits as x approaches $\frac{3}{2}$ are

$$\lim_{x\to 3/2^-}\frac{x^2+3x-10}{2x^2-7x+6}=\lim_{x\to 3/2^-}\frac{(x-2)(x+5)}{(2x-3)(x-2)}=-\infty$$

and

$$\lim_{x\to 3/2^+}\frac{x^2+3x-10}{2x^2-7x+6}=\lim_{x\to 3/2^+}\frac{(x-2)(x+5)}{(2x-3)(x-2)}=\infty,$$

so $\boxed{x=\frac{3}{2} \text{ is a vertical asymptote of the graph of } R}$. Because

$$\lim_{x\to 2}\frac{x^2+3x-10}{2x^2-7x+6}=\lim_{x\to 2}\frac{(x-2)(x+5)}{(2x-3)(x-2)}=\lim_{x\to 2}\frac{x+5}{2x-3}=7,$$

$x=2$ is not a vertical asymptote of the graph of R. Rather, the graph of R has a hole at the point $(2,7)$.

(d) The only numbers where R is not defined are $x=\frac{3}{2}$ and $x=2$. Using the limits from part (c), it follows that as x approaches $\frac{3}{2}$ from the left, the graph of R becomes unbounded in the negative direction, while as x approaches $\frac{3}{2}$ from the right, the graph of R becomes unbounded in the positive direction. The graph of R has a hole at the point $(2,7)$ and approaches that point as x approaches 2 from either direction.

70. (a) The domain of R is the set $\boxed{\{x|x\neq -3\}}$.

(b) Because

$$\lim_{x\to-\infty}\frac{x(x-1)^2}{(x+3)^3}=\lim_{x\to-\infty}\frac{\dfrac{x(x-1)^2}{x^3}}{\dfrac{(x+3)^3}{x^3}}=\lim_{x\to-\infty}\frac{\left(1-\dfrac{1}{x}\right)^2}{\left(1+\dfrac{3}{x}\right)^3}=\frac{(1-0)^2}{(1+0)^3}=1$$

and

$$\lim_{x\to\infty}\frac{x(x-1)^2}{(x+3)^3}=\lim_{x\to\infty}\frac{\dfrac{x(x-1)^2}{x^3}}{\dfrac{(x+3)^3}{x^3}}=\lim_{x\to\infty}\frac{\left(1-\dfrac{1}{x}\right)^2}{\left(1+\dfrac{3}{x}\right)^3}=\frac{(1-0)^2}{(1+0)^3}=1,$$

$\boxed{y=1 \text{ is a horizontal asymptote of the graph of } R}$.

(c) The one-sided limits as x approaches -3 are

$$\lim_{x\to-3^-}\frac{x(x-1)^2}{(x+3)^3}=\infty \quad\text{and}\quad \lim_{x\to-3^+}\frac{x(x-1)^2}{(x+3)^3}=-\infty,$$

so $\boxed{x=-3 \text{ is a vertical asymptote of the graph of } R}$.

(d) The only number where R is not defined is $x = -3$. Using the limits from part (c), it follows that as x approaches -3 from the left, the graph of R becomes unbounded in the positive direction, while as x approaches -3 from the right, the graph of R becomes unbounded in the negative direction.

71. (a) Factoring the denominator of R yields

$$x - x^2 = x(1-x),$$

so the domain of R is the set $\boxed{\{x|x \neq 0, x \neq 1\}}$.

(b) Because

$$\lim_{x\to-\infty} \frac{x^3-1}{x-x^2} = \lim_{x\to-\infty} \frac{\dfrac{x^3-1}{-x^2}}{\dfrac{x-x^2}{-x^2}} = \lim_{x\to-\infty} \frac{-x+\dfrac{1}{x^2}}{-\dfrac{1}{x}+1} = \infty$$

and

$$\lim_{x\to\infty} \frac{x^3-1}{x-x^2} = \lim_{x\to\infty} \frac{\dfrac{x^3-1}{-x^2}}{\dfrac{x-x^2}{-x^2}} = \lim_{x\to\infty} \frac{-x+\dfrac{1}{x^2}}{-\dfrac{1}{x}+1} = -\infty,$$

$\boxed{\text{the graph of } R \text{ has no horizontal asymptotes}}$.

(c) The one-sided limits as x approaches 0 are

$$\lim_{x\to0^-} \frac{x^3-1}{x-x^2} = \lim_{x\to0^-} \frac{(x-1)(x^2+x+1)}{x(1-x)} = \infty$$

and

$$\lim_{x\to0^+} \frac{x^3-1}{x-x^2} = \lim_{x\to0^+} \frac{(x-1)(x^2+x+1)}{x(1-x)} = -\infty,$$

so $\boxed{x = 0 \text{ is a vertical asymptote of the graph of } R}$. As x approaches 1,

$$\begin{aligned}\lim_{x\to1} \frac{x^3-1}{x-x^2} &= \lim_{x\to1} \frac{(x-1)(x^2+x+1)}{x(1-x)} \\ &= -\lim_{x\to1} \frac{x^2+x+1}{x} = -\frac{1+1+1}{1} = -3;\end{aligned}$$

therefore, $x = 1$ is not a vertical asymptote of the graph of R. Rather, the graph of R has a hole at the point $(1, -3)$.

(d) The only numbers where R is not defined are $x = 0$ and $x = 1$. Using the limits from part (c), it follows that as x approaches 0 from the left, the graph of R becomes unbounded in the positive direction, while as x approaches 0 from the right, the graph of R becomes unbounded in the negative direction. The graph of R has a hole at the point $(1, -3)$ and approaches that point as x approaches 1 from either direction.

72. (a) Factoring the denominator of R yields

$$x^3 - 1 = (x-1)(x^2+x+1),$$

so the domain of R is the set $\boxed{\{x|x \neq 1\}}$.

(b) Because

$$\lim_{x\to-\infty}\frac{4x^5}{x^3-1}=\lim_{x\to-\infty}\frac{\dfrac{4x^5}{x^3}}{\dfrac{x^3-1}{x^3}}=\lim_{x\to-\infty}\frac{4x^2}{1-\dfrac{1}{x^3}}=\infty$$

and

$$\lim_{x\to\infty}\frac{4x^5}{x^3-1}=\lim_{x\to\infty}\frac{\dfrac{4x^5}{x^3}}{\dfrac{x^3-1}{x^3}}=\lim_{x\to\infty}\frac{4x^2}{1-\dfrac{1}{x^3}}=\infty,$$

$\boxed{\text{the graph of } R \text{ has no horizontal asymptotes}}$.

(c) The one-sided limits as x approaches 1 are

$$\lim_{x\to1^-}\frac{4x^5}{x^3-1}=-\infty \quad\text{and}\quad \lim_{x\to1^+}\frac{4x^5}{x^3-1}=\infty,$$

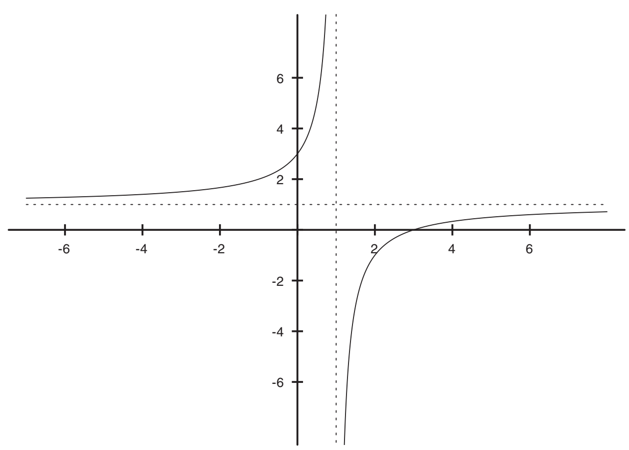
so $\boxed{x=1 \text{ is a vertical asymptote of the graph of } R}$.

(d) The only number where R is not defined is $x=1$. Using the limits from part (c), it follows that as x approaches 1 from the left, the graph of R becomes unbounded in the negative direction, while as x approaches 1 from the right, the graph of R becomes unbounded in the positive direction.

Applications and Extensions

73. (a) Answers will vary. The figure below displays the graph of a function f with the properties

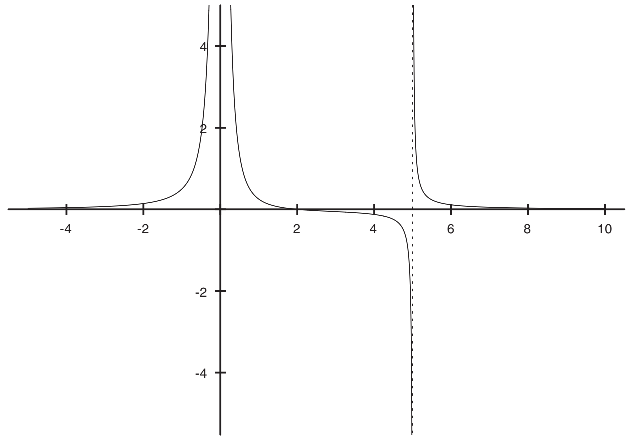
$$f(3)=0,\quad \lim_{x\to\infty}f(x)=1,\quad \lim_{x\to-\infty}f(x)=1,$$
$$\lim_{x\to1^-}f(x)=\infty,\quad\text{and}\quad \lim_{x\to1^+}f(x)=-\infty.$$



(b) Answers will vary. The function shown above is $\boxed{f(x)=\dfrac{x-3}{x-1}}$.

74. (a) Answers will vary. The figure below displays the graph of a function f with the properties

$$f(2)=0,\quad \lim_{x\to\infty}f(x)=0,\quad \lim_{x\to-\infty}f(x)=0,\quad \lim_{x\to0}f(x)=\infty,$$
$$\lim_{x\to5^-}f(x)=-\infty,\quad\text{and}\quad \lim_{x\to5^+}f(x)=\infty.$$



(b) Answers will vary. The function shown above is $\boxed{f(x) = \dfrac{x-2}{x^2(x-5)}}$.

75. (a) Because $k < 0$,

$$\lim_{t\to\infty} u(t) = \lim_{t\to\infty} [(u_0 - T)e^{kt} + T] = (u_0 - T) \lim_{t\to\infty} e^{kt} + \lim_{t\to\infty} T = (u_0 - T)(0) + T = \boxed{T}.$$

As time increases, one would expect the object to lose heat to its surroundings until the temperature of the object had decreased to the temperature of the surroundings, so this limit value is $\boxed{\text{in line with expectations}}$.

(b)

$$\begin{aligned}\lim_{t\to 0^+} u(t) &= \lim_{t\to 0^+} [(u_0 - T)e^{kt} + T] = (u_0 - T) \lim_{t\to 0^+} e^{kt} + \lim_{t\to 0^+} T \\ &= (u_0 - T)e^{k(0)} + T = (u_0 - T) + T = \boxed{u_0}.\end{aligned}$$

This value is again $\boxed{\text{in line with expectations}}$ because as t approaches 0 from the right, the temperature should approach the temperature of the object when it was placed into the lower temperature medium.

76. (a) To remove 45% of the pollutants, $p = 45$, and

$$C(45) = \frac{70,000(45)}{100 - 45} \approx 57,272.73.$$

The cost to remove 45% of the pollutants is therefore approximately $\boxed{\$57,272.73}$.

(b) To remove 90% of the pollutants, $p = 90$, and

$$C(90) = \frac{70,000(90)}{100 - 90} = 630,000.$$

The cost to remove 90% of the pollutants is therefore $\boxed{\$630,000}$.

(c) $\displaystyle\lim_{p\to 100^-} C(p) = \lim_{p\to 100^-} \frac{70,000p}{100 - p} = \boxed{\infty}$.

(d) As the percentage of the pollutants removed from the air increases toward 100, the cost of removing those pollutants increases without bound.

77. $\lim_{x\to 100^-} C(x) = \lim_{x\to 100^-} \dfrac{5x}{100-x} = \boxed{\infty}$. As the percentage of the pollutants removed from the lake increases toward 100, the cost of removing those pollutants increases without bound.

78. (a) The projected size of the colony after 1 year (365 days) is

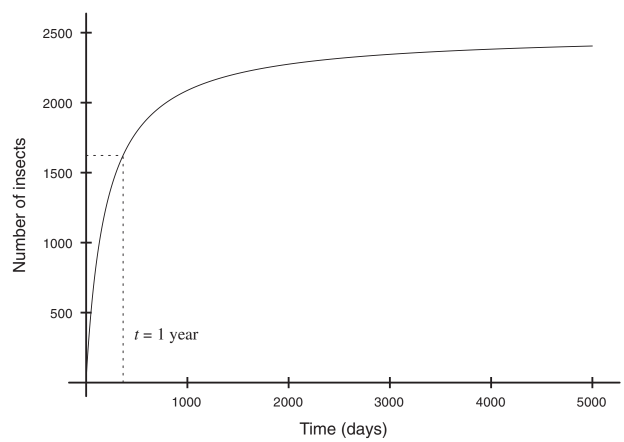
$$P(365) = \frac{50(1+0.5\cdot 365)}{2+0.01\cdot 365} = \frac{9175}{5.65} \approx 1623.89,$$

or $\boxed{1624 \text{ insects}}$.

(b) The largest population that the protected area can sustain is

$$\begin{aligned} \lim_{t\to\infty} P(t) &= \lim_{t\to\infty} \frac{50(1+0.5t)}{2+0.01t} = \lim_{t\to\infty} \frac{50(1+0.5t)}{2+0.01t}\cdot\frac{\frac{1}{t}}{\frac{1}{t}} \\ &= \lim_{t\to\infty} \frac{50\left(\frac{1}{t}+0.5\right)}{\frac{2}{t}+0.01} = \frac{50(0+0.5)}{0+0.01} = \boxed{2500 \text{ insects}}. \end{aligned}$$

(c) The figure below displays the graph of the population P as a function of time t.

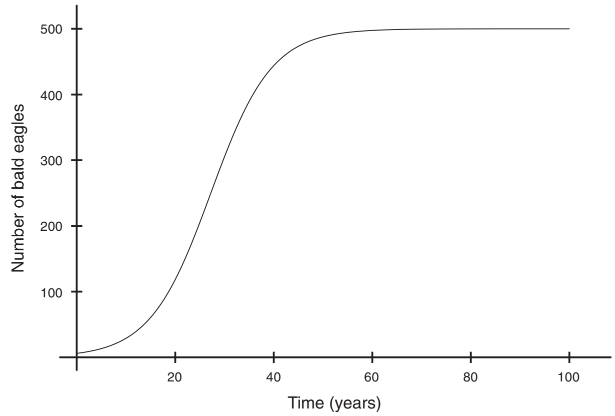


(d) Within the first year, the population increases rapidly from the initial size of 25 insects to over 1600 insects. After that the population grows more slowly, eventually leveling off at 2500 insects, which is $\boxed{\text{consistent with the result from part (b)}}$.

79. (a) If the model is correct, the environment can sustain

$$\begin{aligned} \lim_{t\to\infty} P(t) &= \lim_{t\to\infty} \frac{500}{1+82.3e^{-0.162t}} = \frac{\lim_{t\to\infty} 500}{\lim_{t\to\infty}\left[1+82.3e^{-0.162t}\right]} \\ &= \frac{500}{1+82.3\lim_{t\to\infty} e^{-0.162t}} = \frac{500}{1+82.3(0)} = \boxed{500 \text{ bald eagles}}. \end{aligned}$$

(b) The figure below displays the graph of the population P as a function of time t.



(c) Answers will vary. The number of bald eagles increases slowly for roughly the first ten years, after which the rate of population growth accelerates rapidly. This phase of growth continues until roughly year 35, after which the rate of growth slows dramatically. Eventually, the number of bald eagles levels off at 500, which is $\boxed{\text{consistent with the result from part (a)}}$.

80. (a) The limiting speed of the hailstone is

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty}\left[\frac{mg}{k}(1-e^{-kt/m})\right] = \frac{mg}{k}\left(1-\lim_{t\to\infty} e^{-kt/m}\right) = \frac{mg}{k}(1-0) = \frac{mg}{k}.$$

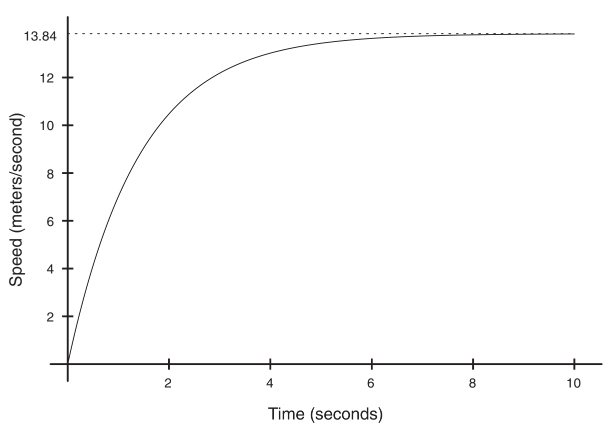
With $m = 4.8\times 10^{-4}$ kg, $g = 9.8$ m/s^2, and $k = 3.4\times 10^{-4}$ kg/s, the limiting speed is

$$\frac{4.8\times 10^{-4}\ \text{kg}\cdot 9.8\ \text{m/s}^2}{3.4\times 10^{-4}\ \text{kg/s}} \approx \boxed{13.84\ \text{m/s}}.$$

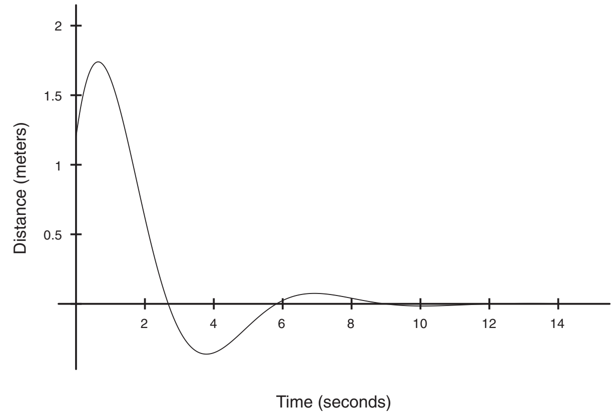
In miles per hour, this is

$$13.84\ \text{m/s}\cdot\frac{1\ \text{mi/h}}{0.447\ \text{m/s}} \approx \boxed{30.96\ \text{mi/h}}.$$

(b) The figure below displays a graph of $v(t)$. The speed appears to approach 13.84 m/s, $\boxed{\text{consistent with the result from part (a)}}$.



81. (a) The figure below displays the graph of $y = x(t)$. The graph suggests that $\boxed{\lim_{t\to\infty} x(t) = 0}$.



(b) Note that $-1 \le \cos t \le 1$ and $-1 \le \sin t \le 1$ for all t. Thus,

$$-1.2e^{-t/2} \le 1.2e^{-t/2}\cos t \le 1.2e^{-t/2}$$

and

$$-2.4e^{-t/2} \le 2.4e^{-t/2}\sin t \le 2.4e^{-t/2}.$$

Because $\lim_{t\to\infty} e^{-t/2} = 0$, it follows that

$$\lim_{t\to\infty}(-1.2e^{-t/2}) = \lim_{t\to\infty}(1.2e^{-t/2}) = \lim_{t\to\infty}(-2.4e^{-t/2}) = \lim_{t\to\infty}(2.4e^{-t/2}) = 0,$$

so the Squeeze Theorem guarantees

$$\lim_{t\to\infty}(1.2e^{-t/2}\cos t) = 0 \quad \text{and} \quad \lim_{t\to\infty}(2.4e^{-t/2}\sin t) = 0.$$

Therefore,

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty}(1.2e^{-t/2}\cos t) + \lim_{t\to\infty}(2.4e^{-t/2}\sin t) = 0 + 0 = \boxed{0}.$$

(c) Answers will vary. The answer from part (b) $\boxed{\text{is supported}}$ by the graph in part (a).

82. (a) First determine the value of the rate constant k. With $C(0) = 2.5$ ppm, $C(t) = 2.5e^{kt}$. Given that the amount of free chlorine after 24 hours is 2.2 ppm, it follows that

$$2.2 = 2.5e^{24k} \quad \text{or} \quad e^{24k} = \frac{2.2}{2.5},$$

so that

$$k = \frac{1}{24}\ln\left(\frac{2.2}{2.5}\right) \approx -0.005326 \text{ hours}^{-1}.$$

Thus, after 72 hours,

$$C(72) \approx 2.5e^{-0.005326(72)} \approx 1.7,$$

so the amount of free chlorine is $\boxed{\text{approximately 1.7 ppm}}$.

(b) The approximate time when the amount of free chlorine reaches 1.0 ppm is the solution of the equation

$$1.0 = 2.5e^{-0.005326t},$$

which is

$$t = \frac{1}{-0.005326} \ln\left(\frac{1.0}{2.5}\right) \approx 172 \text{ hours.}$$

Therefore, Ben can go $\boxed{\text{roughly 172 hours (a little more than one week)}}$ before he must shock the pool again.

(c) $\lim_{t\to\infty} C(t) = \lim_{t\to\infty} (2.5e^{-0.005326t}) = \boxed{0}$ ppm.

(d) In the long run, all of the free chlorine in the pool will decompose.

83. (a) First determine the value of the rate constant k. With $A(0) = 0.40$ moles, $A(t) = 0.40e^{kt}$. Given that the amount of sucrose present after 30 minutes is 0.36 moles, it follows that

$$0.36 = 0.40e^{30k} \quad \text{or} \quad e^{30k} = \frac{0.36}{0.40},$$

so that

$$k = \frac{1}{30} \ln\left(\frac{0.36}{0.40}\right) \approx -0.003512 \text{ minutes}^{-1}.$$

Thus, after 2 hours (120 minutes),

$$A(120) \approx 0.40e^{-0.003512(120)} \approx 0.26,$$

so the amount of sucrose present is $\boxed{\text{approximately 0.26 moles}}$.

(b) The approximate time when the amount of sucrose remaining will be 0.10 moles is the solution of the equation

$$0.10 = 0.40e^{-0.003512t},$$

which is

$$t = \frac{1}{-0.003512} \ln\left(\frac{0.10}{0.40}\right) \approx \boxed{395 \text{ minutes}}.$$

(c) $\lim_{t\to\infty} A(t) = \lim_{t\to\infty} (0.40e^{-0.003512t}) = \boxed{0}$ moles.

(d) In the long run, all of the sucrose will decompose into glucose and fructose.

84. (a) Solving the thin film equation for q yields

$$\frac{1}{q} = \frac{1}{f} - \frac{1}{p} = \frac{p-f}{pf} \quad \text{or} \quad q = \frac{pf}{p-f}.$$

Note that division by $p-f$ is permitted here because the problem statement indicates that $p > f$, so that $p - f$ will never be equal to zero. Thus,

$$\lim_{p\to f^+} q = \lim_{p\to f^+} \frac{pf}{p-f} = \infty.$$

Therefore, the distance q of the image from the lens is $\boxed{\text{not continuous}}$ as the distance of the object approaches the focal length of the lens.

(b) A camera cannot focus on an object placed close to its focal length because the distance of the image from the lens becomes unbounded.

85. If $c = 0$, then

$$\lim_{x\to\infty} \frac{ax^3+b}{cx^4+d} = \lim_{x\to\infty} \frac{ax^3+b}{cx^4+d} \cdot \frac{\frac{1}{x^4}}{\frac{1}{x^4}} = \lim_{x\to\infty} \frac{\frac{a}{x}+\frac{b}{x^4}}{c+\frac{d}{x^4}} = \frac{0+0}{c+0} = 0,$$

so that $y = 0$ would be a horizontal asymptote of the graph of f. Thus, for the graph of f to have no horizontal asymptotes, c must be zero. It then follows that d cannot be zero, otherwise the denominator of f would be zero. Now, with $c = 0$ and $d \neq 0$, the function f reduces to a polynomial, and therefore does not have vertical asymptotes. Next, note that if $a = 0$, then $f(x) = \frac{b}{d}$, so that

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{b}{d} = \frac{b}{d},$$

and $y = \frac{b}{d}$ would be a horizontal asymptote of the graph of f. Thus, a cannot be zero. Finally, for the graph of f to have no horizontal or vertical asymptotes, we must have $\boxed{a \neq 0,\ c = 0,\ d \neq 0, \text{ and } b \text{ can be any real number}}$.

86. If $c \neq 0$, then

$$\lim_{x\to\infty} \frac{ax+b}{cx+d} = \lim_{x\to\infty} \frac{ax+b}{cx+d} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{a+\frac{b}{x}}{c+\frac{d}{x}} = \frac{a+0}{c+0} = \frac{a}{c},$$

so that $y = \frac{a}{c}$ would be a horizontal asymptote of the graph of f. Thus, for the graph of f to have no horizontal asymptotes, c must be zero. It then follows that d cannot be zero, otherwise the denominator of f would be zero. Now, with $c = 0$ and $d \neq 0$, the function f reduces to a polynomial, and therefore does not have vertical asymptotes. Next, note that if $a = 0$, then $f(x) = \frac{b}{d}$, so that

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{b}{d} = \frac{b}{d},$$

and $y = \frac{b}{d}$ would be a horizontal asymptote of the graph of f. Thus, a cannot be zero. Finally, for the graph of f to have no horizontal or vertical asymptotes, we must have $\boxed{a \neq 0,\ c = 0,\ d \neq 0, \text{ and } b \text{ can be any real number}}$.

87. (a) Let n be an even positive integer. Then, as x approaches c, $(x-c)^n$ approaches 0 from the right and $\dfrac{1}{(x-c)^n}$ becomes unbounded in the positive direction; that is, $\displaystyle\lim_{x\to c} \frac{1}{(x-c)^n} = \infty$. Answers will vary, but one example is $\displaystyle\lim_{x\to 0} \frac{1}{x^2} = \infty$.

(b) Let n be an odd positive integer. Then, as x approaches c from the left, $(x-c)^n$ approaches 0 from the left and $\dfrac{1}{(x-c)^n}$ becomes unbounded in the negative direction; that is, $\displaystyle\lim_{x\to c^-} \frac{1}{(x-c)^n} = -\infty$. Answers will vary, but one example is $\displaystyle\lim_{x\to 4^-} \frac{1}{x-4} = -\infty$.

(c) Let n be an odd positive integer. Then, as x approaches c from the right, $(x-c)^n$ approaches 0 from the right and $\dfrac{1}{(x-c)^n}$ becomes unbounded in the positive direction; that is, $\displaystyle\lim_{x\to c^+} \frac{1}{(x-c)^n} = \infty$. Answers will vary, but one example is $\displaystyle\lim_{x\to -2^+} \frac{1}{(x+2)^3} = \infty$.

88. Let R be a rational function whose numerator and denominator have no common zeros, and let c be a point of discontinuity of R. Then c must be a zero of the denominator of

R. Because the numerator and denominator of R have no common zeros, c is not a zero of the numerator, so that, as x approaches c, the numerator of R approaches a non-zero number, while the denominator approaches 0. It follows that the value of R must become unbounded as x approaches c and that the graph of R must have a vertical asymptote at $x = c$.

89. Let p be a polynomial function of degree 1 or higher. Because all polynomial functions are defined for all real numbers, the graph of p will have no vertical asymptotes. Moreover, because p contains at least one term of the form ax^k where $a \neq 0$ and k is an integer greater than or equal to 1 (as otherwise p would not be a polynomial function of degree 1 or higher), $p(x)$ will become unbounded as x becomes unbounded in either direction. Thus, the graph of f will also have no horizontal asymptotes.

90. Let P and Q be polynomial functions of degree m and n, respectively. In particular, suppose

$$P(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_0$$

and

$$Q(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_0,$$

where the a_j ($j = 0, 1, 2, \ldots, m$) and the b_k ($k = 0, 1, 2, \ldots, n$) are real numbers with $a_m \neq 0$ and $b_n \neq 0$.

(a) Suppose $m > n$. Multiplying the numerator and denominator of $\dfrac{P(x)}{Q(x)}$ by $\dfrac{1}{x^n}$ then yields an expression of the form

$$\frac{a_m x^{m-n} + a_{m-1}x^{m-n-1} + \cdots + \dfrac{a_0}{x^n}}{b_n + \dfrac{b_{n-1}}{x} + \cdots + \dfrac{b_0}{x^n}}.$$

Thus,

$$\boxed{\lim_{x\to\infty} \frac{P(x)}{Q(x)} = \begin{cases} \infty, & \text{if } a_m > 0 \\ -\infty, & \text{if } a_m < 0. \end{cases}}$$

(b) Suppose $m = n$. Multiplying the numerator and denominator of $\dfrac{P(x)}{Q(x)}$ by $\dfrac{1}{x^n}$ then yields an expression of the form

$$\frac{a_m + \dfrac{a_{m-1}}{x} + \cdots + \dfrac{a_0}{x^n}}{b_n + \dfrac{b_{n-1}}{x} + \cdots + \dfrac{b_0}{x^n}}.$$

Thus,

$$\lim_{x\to\infty} \frac{P(x)}{Q(x)} = \frac{a_m + 0 + \cdots + 0}{b_n + 0 + \cdots + 0} = \boxed{\frac{a_m}{b_n}}.$$

In other words, when $m = n$, the limit is the ratio of the leading coefficients of the two polynomial functions.

(c) Suppose $m < n$. Multiplying the numerator and denominator of $\dfrac{P(x)}{Q(x)}$ by $\dfrac{1}{x^n}$ then yields an expression of the form

$$\frac{\dfrac{a_m}{x^{n-m}} + \dfrac{a_{m-1}}{x^{n-m+1}} + \cdots + \dfrac{a_0}{x^n}}{b_n + \dfrac{b_{n-1}}{x} + \cdots + \dfrac{b_0}{x^n}}.$$

Thus,

$$\lim_{x\to\infty} \frac{P(x)}{Q(x)} = \frac{0 + 0 + \cdots + 0}{b_n + 0 + \cdots + 0} = \boxed{0}.$$

91. (a) The table of values below, which have been rounded to six decimal places, suggests

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x \approx \boxed{2.718282}.$$

x	100	10,000	1,000,000	100,000,000	$\to\infty$
$f(x)=\left(1+\frac{1}{x}\right)^x$	2.704814	2.718146	2.718280	2.718282	$f(x)$ approaches 2.718282

(b) Using the computer algebra system *Mathematica*,

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \boxed{e\approx 2.718281828}.$$

(c) Answers will vary. One possible response is that the results from parts (a) and (b) agree to five decimal places. Though it is possible to achieve accuracy to as many decimal places as desired by calculating $\left(1+\frac{1}{x}\right)^x$ for larger and larger x, it is impossible to determine every digit of this limit. The number e, like the number π, has a nonrepeating, nonterminating decimal expansion.

Challenge Problems

92. We may prove the result using the standard properties of limits at infinity and the fact that $\lim\limits_{x\to\pm\infty}\frac{1}{x}=0$. For any real number k and $p>0$ such that x^p is defined, we have the following:

$$\lim_{x\to\pm\infty}\frac{k}{x^p}=k\left(\lim_{x\to\pm\infty}\frac{1}{x^p}\right)=k\left(\lim_{x\to\pm\infty}\left(\frac{1}{x}\right)^p\right)=k\left(\lim_{x\to\pm\infty}\frac{1}{x}\right)^p=k(0)^p=0.$$

93. (a) As v approaches c from the left, $\frac{v^2}{c^2}$ approaches 1 and $1-\frac{v^2}{c^2}$ approaches 0. Therefore

$$\lim_{v\to c^-}K_{\text{gen}}(v)=mc^2\lim_{v\to c^-}\left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}-1\right)=\boxed{\infty}.$$

(b) Because it is not possible to have infinite kinetic energy, the result from part (a) suggests that it is not possible to reach the speed of light.

94. In $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x$, the exponent is the variable x. The property $\lim\limits_{x\to\infty}[f(x)]^n=\left[\lim\limits_{x\to\infty}f(x)\right]^n$ requires the exponent to be a constant, independent of the variable x.

1.6 The ϵ-δ Definition of a Limit

Concepts and Vocabulary

1. $\boxed{\text{False}}$. The limit of a function as x approaches c **does not depend** on the value of the function at c.

2. $\boxed{\text{True}}$. In the ϵ-δ definition of a limit, we require $0<|x-c|$ to ensure that $x\neq c$.

3. $\boxed{\text{True}}$. In an ϵ-δ proof of a limit, the size of δ usually depends on the size of ϵ.

4. $\boxed{\text{False}}$. When proving $\lim\limits_{x\to c} f(x) = L$ using the ϵ-δ definition, you try to find a connection between $|f(x) - L|$ and $|x - c|$.

5. $\boxed{\text{True}}$. Given any $\epsilon > 0$, suppose there is a $\delta > 0$, so that whenever $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$. Then $\lim\limits_{x\to c} f(x) = L$.

6. $\boxed{\text{False}}$. A function f has a limit L at infinity, if for any given $\epsilon > 0$, there is a positive number M so that whenever $x > M$, $|f(x) - L| < \epsilon$.

Skill Building

7. Here, $f(x) = 2x$, $c = 1$, and $L = 2$. To make

$$|f(x) - L| = |2x - 2| = 2|x - 1| < 0.01$$

requires

$$|x - 1| < \frac{0.01}{2} = 0.005.$$

Thus, the largest δ that "works" for $\epsilon = 0.01$ is $\boxed{\delta = 0.005}$.

8. Here, $f(x) = -3x$, $c = 2$, and $L = -6$. To make

$$|f(x) - L| = |-3x - (-6)| = |-3x + 6| = |-3|\ |x - 2| = 3|x - 2| < 0.01$$

requires

$$|x - 2| < \frac{0.01}{3} = \frac{1}{300}.$$

Thus, the largest δ that "works" for $\epsilon = 0.01$ is $\boxed{\delta = \frac{1}{300}}$.

9. Here $f(x) = 6x - 1$, $c = 2$, and $L = 11$. To make

$$|f(x) - L| = |(6x - 1) - 11| = |6x - 12| = 6|x - 2| < \frac{1}{2}$$

requires

$$|x - 2| < \frac{1}{12}.$$

Thus, the largest δ that "works" for $\epsilon = \frac{1}{2}$ is $\boxed{\delta = \frac{1}{12}}$.

10. Here $f(x) = 2 - 3x$, $c = -3$, and $L = 11$. To make

$$|f(x) - L| = |(2 - 3x) - 11| = |-3x - 9| = |-3|\ |x + 3| = 3|x + 3| < \frac{1}{3}$$

requires

$$|x + 3| < \frac{1}{9}.$$

Thus, the largest δ that "works" for $\epsilon = \frac{1}{3}$ is $\boxed{\delta = \frac{1}{9}}$.

11. Here $f(x) = -\frac{1}{2}x + 5$, $c = 2$, and $L = 4$. To make

$$|f(x) - L| = \left|\left(-\frac{1}{2}x + 5\right) - 4\right| = \left|-\frac{1}{2}x + 1\right| = \left|-\frac{1}{2}\right| |x - 2| = \frac{1}{2}|x - 2| < 0.01$$

requires

$$|x - 2| < 2(0.01) = 0.02.$$

Thus, the largest δ that "works" for $\epsilon = 0.01$ is $\boxed{\delta = 0.02}$.

12. Here $f(x) = 3x + \frac{1}{2}$, $c = \frac{5}{6}$, and $L = 3$. To make

$$|f(x) - L| = \left|\left(3x + \frac{1}{2}\right) - 3\right| = \left|3x - \frac{5}{2}\right| = 3\left|x - \frac{5}{6}\right| < 0.3$$

requires

$$\left|x - \frac{5}{6}\right| < \frac{0.3}{3} = 0.1.$$

Thus, the largest δ that "works" for $\epsilon = 0.3$ is $\boxed{\delta = 0.1}$.

13. To make

$$|(4x - 1) - 11| = |4x - 12| = 4|x - 3| < \epsilon$$

requires

$$|x - 3| < \frac{\epsilon}{4}.$$

Thus, the largest δ that "works" for an arbitrary ϵ is $\delta = \frac{\epsilon}{4}$.

(a) For $\epsilon = 0.1$, we can choose any $\boxed{\delta \le \frac{0.1}{4} = 0.025}$.

(b) For $\epsilon = 0.01$, we can choose any $\boxed{\delta \le \frac{0.01}{4} = 0.0025}$.

(c) For $\epsilon = 0.001$, we can choose any $\boxed{\delta \le \frac{0.001}{4} = 0.00025}$.

(d) For arbitrary $\epsilon > 0$, we can choose any $\boxed{\delta \le \frac{\epsilon}{4}}$.

14. To make

$$|(2 - 5x) - 12| = |-5x - 10| = |-5|\,|x + 2| = 5|x + 2| < \epsilon$$

requires

$$|x + 2| < \frac{\epsilon}{5}.$$

Thus, the largest δ that "works" for an arbitrary ϵ is $\delta = \frac{\epsilon}{5}$.

(a) For $\epsilon = 0.2$, we can choose any $\boxed{\delta \le \frac{0.2}{5} = 0.04}$.

(b) For $\epsilon = 0.02$, we can choose any $\boxed{\delta \le \frac{0.02}{5} = 0.004}$.

(c) For $\epsilon = 0.002$, we can choose any $\boxed{\delta \le \frac{0.002}{5} = 0.0004}$.

(d) For arbitrary $\epsilon > 0$, we can choose any $\boxed{\delta \le \dfrac{\epsilon}{5}}$.

15. The inequality $0 < |x+3|$ guarantees that x cannot be equal to -3. Now, for $x \neq -3$,

$$\frac{x^2-9}{x+3} = \frac{(x+3)(x-3)}{x+3} = x - 3.$$

Therefore, for $x \neq -3$,

$$\left|\frac{x^2-9}{x+3} - (-6)\right| = |(x-3)+6| = |x+3|.$$

To make this less than ϵ, requires $|x+3| < \epsilon$, so the largest δ that "works" for an arbitrary ϵ is $\delta = \epsilon$.

(a) For $\epsilon = 0.1$, we can choose any $\boxed{\delta \le 0.1}$.

(b) For $\epsilon = 0.01$, we can choose any $\boxed{\delta \le 0.01}$.

(c) For arbitrary $\epsilon > 0$, we can choose any $\boxed{\delta \le \epsilon}$.

16. The inequality $0 < |x-2|$ guarantees that x cannot be equal to 2. Now, for $x = 2$,

$$\frac{x^2-4}{x-2} = \frac{(x+2)(x-2)}{x-2} = x + 2.$$

Therefore, for $x \neq -2$,

$$\left|\frac{x^2-4}{x-2} - 4\right| = |(x+2)-4| = |x-2|.$$

To make this less than ϵ, requires $|x-2| < \epsilon$, so the largest δ that "works" for an arbitrary ϵ is $\delta = \epsilon$.

(a) For $\epsilon = 0.1$, we can choose any $\boxed{\delta \le 0.1}$.

(b) For $\epsilon = 0.01$, we can choose any $\boxed{\delta \le 0.01}$.

(c) For arbitrary $\epsilon > 0$, we can choose any $\boxed{\delta \le \epsilon}$.

17. Given any $\epsilon > 0$, we must find a number $\delta > 0$ so that whenever $0 < |x-c| < \delta$, then $|f(x) - L| < \epsilon$. Here $f(x) = 3x$, $c = 2$, and $L = 6$. To make

$$|f(x) - L| = |3x - 6| = 3|x-2| < \epsilon$$

requires $|x-2| < \epsilon/3$. Thus, the largest δ that "works" for an arbitrary ϵ is $\delta = \dfrac{\epsilon}{3}$. The ϵ-δ proof may be written as follows:

$\boxed{\text{Given any } \epsilon > 0, \text{ we can choose } \delta = \epsilon/3}$. Whenever $0 < |x-2| < \delta$, then

$$|f(x) - L| = |3x-6| = 3|x-2| < 3\delta = 3 \cdot \frac{\epsilon}{3} = \epsilon.$$

Therefore, $\lim\limits_{x\to 2}(3x) = 6$.

18. Given any $\epsilon > 0$, we must find a number $\delta > 0$ so that whenever $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$. Here $f(x) = 4x$, $c = 3$, and $L = 12$. To make

$$|f(x) - L| = |4x - 12| = 4|x - 3| < \epsilon$$

requires $|x - 3| < \epsilon/4$. Thus, the largest δ that "works" for an arbitrary ϵ is $\delta = \dfrac{\epsilon}{4}$. The ϵ-δ proof may be written as follows:

$\boxed{\text{Given any } \epsilon > 0, \text{ we can choose } \delta = \epsilon/4}$. Whenever $0 < |x - 3| < \delta$, then

$$|f(x) - L| = |4x - 12| = 4|x - 3| < 4\delta = 4 \cdot \frac{\epsilon}{4} = \epsilon.$$

Therefore, $\lim\limits_{x \to 3} (4x) = 12$.

19. Given any $\epsilon > 0$, we must find a number $\delta > 0$ so that whenever $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$. Here $f(x) = 2x + 5$, $c = 0$, and $L = 5$. To make

$$|f(x) - L| = |(2x + 5) - 5| = 2|x| < \epsilon$$

requires $|x| = |x - 0| < \epsilon/2$. Thus, the largest δ that "works" for an arbitrary ϵ is $\delta = \dfrac{\epsilon}{2}$. The ϵ-δ proof may be written as follows:

$\boxed{\text{Given any } \epsilon > 0, \text{ we can choose } \delta = \epsilon/2}$. Whenever $0 < |x - 0| = |x| < \delta$, then

$$|f(x) - L| = |(2x + 5) - 5| = 2|x| < 2\delta = 2 \cdot \frac{\epsilon}{2} = \epsilon.$$

Therefore, $\lim\limits_{x \to 0} (2x + 5) = 5$.

20. Given any $\epsilon > 0$, we must find a number $\delta > 0$ so that whenever $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$. Here $f(x) = 2 - 3x$, $c = -1$, and $L = 5$. To make

$$|f(x) - L| = |(2 - 3x) - 5| = |-3x - 3| = |-3|\ |x + 1| = 3|x + 1| < \epsilon$$

requires $|x + 1| < \epsilon/3$. Thus, the largest δ that "works" for an arbitrary ϵ is $\delta = \dfrac{\epsilon}{3}$. The ϵ-δ proof may be written as follows:

$\boxed{\text{Given any } \epsilon > 0, \text{ we can choose } \delta = \epsilon/3}$. Whenever $0 < |x - (-1)| = |x + 1| < \delta$, then

$$|f(x) - L| = |(2 - 3x) - 5| = |-3x - 3| = 3|x + 1| < 3\delta = 3 \cdot \frac{\epsilon}{3} = \epsilon.$$

Therefore, $\lim\limits_{x \to -1} (2 - 3x) = 5$.

21. Given any $\epsilon > 0$, we must find a number $\delta > 0$ so that whenever $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$. Here $f(x) = -5x + 2$, $c = -3$, and $L = 17$. To make

$$|f(x) - L| = |(-5x + 2) - 17| = |-5x - 15| = |-5|\ |x + 3| = 5|x + 3| < \epsilon$$

requires $|x + 3| < \epsilon/5$. Thus, the largest δ that "works" for an arbitrary ϵ is $\delta = \dfrac{\epsilon}{5}$. The ϵ-δ proof may be written as follows:

$\boxed{\text{Given any } \epsilon > 0, \text{ we can choose } \delta = \epsilon/5}$. Whenever $0 < |x - (-3)| = |x + 3| < \delta$, then

$$|f(x) - L| = |(-5x + 2) - 17| = |-5x - 15| = 5|x + 3| < 5\delta = 5 \cdot \frac{\epsilon}{5} = \epsilon.$$

Therefore, $\lim\limits_{x \to -3} (-5x + 2) = 17$.

22. Given any $\epsilon > 0$, we must find a number $\delta > 0$ so that whenever $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$. Here $f(x) = 2x - 3$, $c = 2$, and $L = 1$. To make

$$|f(x) - L| = |(2x - 3) - 1| = |2x - 4| = 2|x - 2| < \epsilon$$

requires $|x - 2| < \epsilon/2$. Thus, the largest δ that "works" for an arbitrary ϵ is $\delta = \dfrac{\epsilon}{2}$. The ϵ-δ proof may be written as follows:

$\boxed{\text{Given any } \epsilon > 0, \text{ we can choose } \delta = \epsilon/2}$. Whenever $0 < |x - 2| < \delta$, then

$$|f(x) - L| = |(2x - 3) - 1| = 2|x - 2| < 2\delta = 2 \cdot \frac{\epsilon}{2} = \epsilon.$$

Therefore, $\lim\limits_{x \to 2} (2x - 3) = 1$.

23. Given any $\epsilon > 0$, we must find a number $\delta > 0$ so that whenever $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$. Here $f(x) = x^2 - 2x$, $c = 2$, and $L = 0$. Then

$$|f(x) - L| = |(x^2 - 2x) - 0| = |x(x - 2)| = |x| \cdot |x - 2|.$$

The factor $|x - 2|$ will be smaller than δ, but what about the factor $|x|$? If, in addition to any other restrictions placed on δ, we require $\delta \leq 1$, then $|x - 2| < \delta$ guarantees that $|x - 2| < 1$. Removing the absolute value from this last inequality yields $-1 < x - 2 < 1$, or $1 < x < 3$. Thus, $|x| < 3$ and

$$|f(x) - L| = |x| \cdot |x - 2| < 3|x - 2| < 3\delta.$$

To make this less than ϵ, we can choose $\delta \leq \epsilon/3$. Combining all of this information, the ϵ-δ proof may be written as follows:

$\boxed{\text{Given any } \epsilon > 0, \text{ we can choose } \delta = \min\left\{1, \frac{\epsilon}{3}\right\}}$. Whenever $0 < |x - 2| < \delta$, it follows that $|x| < 3$, and then

$$|f(x) - L| = |(x^2 - 2x) - 0| = |x| \cdot |x - 2| < 3|x - 2| < 3\delta \leq 3 \cdot \frac{\epsilon}{3} = \epsilon.$$

Therefore, $\lim\limits_{x \to 2} (x^2 - 2x) = 0$.

24. Given any $\epsilon > 0$, we must find a number $\delta > 0$ so that whenever $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$. Here $f(x) = x^2 + 3x$, $c = 0$, and $L = 0$. Then

$$|f(x) - L| = |(x^2 + 3x) - 0| = |x(x + 3)| = |x| \cdot |x + 3|.$$

The factor $|x|$ will be smaller than δ, but what about the factor $|x + 3|$? If, in addition to any other restrictions placed on δ, we require $\delta \leq 1$, then $|x| < \delta$ guarantees that $|x| < 1$, or $-1 < x < 1$. Thus, $2 < x + 3 < 4$, so that $|x + 3| < 4$ and

$$|f(x) - L| = |x| \cdot |x + 3| < 4|x| < 4\delta.$$

To make this less than ϵ, we can choose $\delta \leq \epsilon/4$. Combining all of this information, the ϵ-δ proof may be written as follows:

$\boxed{\text{Given any } \epsilon > 0, \text{ we can choose } \delta = \min\left\{1, \frac{\epsilon}{4}\right\}}$. Whenever $0 < |x| < \delta$, it follows that $|x + 3| < 4$, and then

$$|f(x) - L| = |(x^2 + 3x) - 0| = |x| \cdot |x + 3| < 4|x| < 4\delta \leq 4 \cdot \frac{\epsilon}{4} = \epsilon.$$

Therefore, $\lim\limits_{x \to 0} (x^2 + 3x) = 0$.

25. Given any $\epsilon > 0$, we must find a number $\delta > 0$ so that whenever $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$. Here $f(x) = \dfrac{1+2x}{3-x}$, $c = 1$, and $L = \dfrac{3}{2}$. Then

$$|f(x) - L| = \left|\frac{1+2x}{3-x} - \frac{3}{2}\right| = \left|\frac{7x-7}{2(3-x)}\right| = \frac{7}{2}\frac{|x-1|}{|3-x|}.$$

The factor $|x-1|$ will be smaller than δ, but what about the factor $|3-x|$? If, in addition to any other restrictions placed on δ, we require $\delta \le 1$, then $|x-1| < \delta$ guarantees that $|x-1| < 1$. Removing the absolute value from this last inequality yields $-1 < x-1 < 1$, or $0 < x < 2$. Thus, $-2 < -x < 0$ and $1 < 3-x < 3$. Because $3-x > 1 > 0$, $3-x = |3-x|$, so that $1 < |3-x| < 3$. Therefore, $\dfrac{1}{3} < \dfrac{1}{|3-x|} < 1$ and

$$|f(x) - L| = \frac{7}{2}\frac{|x-1|}{|3-x|} < \frac{7}{2}|x-1| < \frac{7}{2}\delta.$$

To make this less than ϵ, we can choose $\delta \le 2\epsilon/7$. Combining all of this information, the ϵ-δ proof may be written as follows:

Given any $\epsilon > 0$, we can choose $\delta = \min\left\{1, \dfrac{2\epsilon}{7}\right\}$. Whenever $0 < |x-1| < \delta$, it follows that $\dfrac{1}{|3-x|} < 1$, and then

$$|f(x) - L| = \left|\frac{1+2x}{3-x} - \frac{3}{2}\right| = \left|\frac{7x-7}{2(3-x)}\right| = \frac{7}{2}\frac{|x-1|}{|3-x|} < \frac{7}{2}|x-1| < \frac{7}{2}\delta \le \frac{7}{2}\cdot\frac{2\epsilon}{7} = \epsilon.$$

Therefore, $\displaystyle\lim_{x\to 1}\frac{1+2x}{3-x} = \frac{3}{2}$.

26. Given any $\epsilon > 0$, we must find a number $\delta > 0$ so that whenever $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$. Here $f(x) = \dfrac{2x}{4+x}$, $c = 2$, and $L = \dfrac{2}{3}$. Then

$$|f(x) - L| = \left|\frac{2x}{4+x} - \frac{2}{3}\right| = \left|\frac{4x-8}{3(4+x)}\right| = \frac{4}{3}\frac{|x-2|}{|4+x|}.$$

The factor $|x-2|$ will be smaller than δ, but what about the factor $|4+x|$? If, in addition to any other restrictions placed on δ, we require $\delta \le 1$, then $|x-2| < \delta$ guarantees that $|x-2| < 1$. Removing the absolutely value from this last inequality yields $-1 < x-2 < 1$, or $1 < x < 3$. Thus, $5 < 4+x < 7$. Because $4+x > 5 > 0$, $4+x = |4+x|$, so that $5 < |4+x| < 7$. Therefore, $\dfrac{1}{7} < \dfrac{1}{|4+x|} < \dfrac{1}{5}$ and

$$|f(x) - L| = \frac{4}{3}\frac{|x-2|}{|4+x|} < \frac{4}{15}|x-2| < \frac{4}{15}\delta.$$

To make this less than ϵ, we can choose $\delta \le 15\epsilon/4$. Combining all of this information, the ϵ-δ proof may be written as follows:

Given any $\epsilon > 0$, we can choose $\delta = \min\left\{1, \dfrac{15\epsilon}{4}\right\}$. Whenever $0 < |x-2| < \delta$, it follows that $\dfrac{1}{|4+x|} < \dfrac{1}{5}$, and then

$$|f(x) - L| = \left|\frac{2x}{4+x} - \frac{2}{3}\right| = \left|\frac{4x-8}{3(4+x)}\right| = \frac{4}{3}\frac{|x-2|}{|4+x|} < \frac{4}{15}|x-2| < \frac{4}{15}\delta \le \frac{4}{15}\cdot\frac{15\epsilon}{4} = \epsilon.$$

Therefore, $\displaystyle\lim_{x\to 2}\frac{2x}{4+x} = \frac{2}{3}$.

27. Given any $\epsilon > 0$, we must find a number $\delta > 0$ so that whenever $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$. Here $f(x) = \sqrt[3]{x}$, $c = 0$, and $L = 0$. To make

$$|f(x) - L| = |\sqrt[3]{x} - 0| = \sqrt[3]{|x|} < \epsilon$$

requires $|x| = |x - 0| < \epsilon^3$. Thus, the largest δ that "works" for an arbitrary ϵ is $\delta = \epsilon^3$. The ϵ-δ proof may be written as follows:

$\boxed{\text{Given any } \epsilon > 0, \text{ we can choose } \delta = \epsilon^3}$. Whenever $0 < |x - 0| = |x| < \delta$, then

$$|f(x) - L| = |\sqrt[3]{x} - 0| = \sqrt[3]{|x|} < \sqrt[3]{\delta} = \sqrt[3]{\epsilon^3} = \epsilon.$$

Therefore, $\lim_{x \to 0} \sqrt[3]{x} = 0$.

28. Given any $\epsilon > 0$, we must find a number $\delta > 0$ so that whenever $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$. Here $f(x) = \sqrt{2 - x}$, $c = 1$, and $L = 1$. Then

$$|f(x) - L| = |\sqrt{2 - x} - 1| \cdot \frac{|\sqrt{2 - x} + 1|}{|\sqrt{2 - x} + 1|} = \frac{|(2 - x) - 1|}{|\sqrt{2 - x} + 1|} = \frac{|1 - x|}{|\sqrt{2 - x} + 1|} = \frac{|x - 1|}{|\sqrt{2 - x} + 1|}.$$

The factor $|x - 1|$ will be smaller than δ, but what about the factor $|\sqrt{2 - x} + 1|$. Because $\sqrt{2 - x} \geq 0$, $\sqrt{2 - x} + 1 \geq 1$. Thus,

$$0 < \frac{1}{\sqrt{2 - x} + 1} \leq 1 \quad \text{so that} \quad \frac{1}{|\sqrt{2 - x} + 1|} \leq 1,$$

and

$$|f(x) - L| = \frac{|x - 1|}{|\sqrt{2 - x} + 1|} \leq |x - 1|.$$

To make this less than ϵ requires $|x - 1| < \epsilon$. Therefore, the largest δ that "works" for an arbitrary ϵ is $\delta = \epsilon$. Combining all of this information, the ϵ-δ proof may be written as follows:

$\boxed{\text{Given any } \epsilon > 0, \text{ we can choose } \delta = \epsilon}$. Whenever $0 < |x - 1| < \delta$, then

$$\begin{aligned} |f(x) - L| &= |\sqrt{2 - x} - 1| \cdot \frac{|\sqrt{2 - x} + 1|}{|\sqrt{2 - x} + 1|} = \frac{|(2 - x) - 1|}{|\sqrt{2 - x} + 1|} \\ &= \frac{|1 - x|}{|\sqrt{2 - x} + 1|} = \frac{|x - 1|}{|\sqrt{2 - x} + 1|} \leq |x - 1| < \delta = \epsilon. \end{aligned}$$

Therefore, $\lim_{x \to 1} \sqrt{2 - x} = 1$.

29. Given any $\epsilon > 0$, we must find a number $\delta > 0$ so that whenever $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$. Here $f(x) = x^2$, $c = -1$, and $L = 1$. Then

$$|f(x) - L| = |x^2 - 1| = |(x - 1)(x + 1)| = |x - 1| \cdot |x + 1|.$$

The factor $|x + 1|$ will be smaller than δ, but what about the factor $|x - 1|$? If, in addition to any other restrictions placed on δ, we require $\delta \leq 1$, then $|x + 1| < \delta$ guarantees that $|x + 1| < 1$. Removing the absolute value from this last inequality yields $-1 < x + 1 < 1$, or $-2 < x < 0$. Thus, $-3 < x - 1 < -1$, so that $|x - 1| < 3$ and

$$|f(x) - L| = |x - 1| \cdot |x + 1| < 3|x + 1| < 3\delta.$$

To make this less than ϵ, we can choose $\delta \leq \epsilon/3$. Combining all of this information, the ϵ-δ proof may be written as follows:

$\boxed{\text{Given any } \epsilon > 0, \text{ we can choose } \delta = \min\left\{1, \frac{\epsilon}{3}\right\}}$. Whenever $0 < |x+1| < \delta$, it follows that $|x-1| < 3$, and then

$$|f(x) - L| = |x^2 - 1| = |x-1| \cdot |x+1| < 3|x+1| < 3\delta \leq 3 \cdot \frac{\epsilon}{3} = \epsilon.$$

Therefore, $\lim_{x \to -1} x^2 = 1$.

30. Given any $\epsilon > 0$, we must find a number $\delta > 0$ so that whenever $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$. Here $f(x) = x^3$, $c = 2$, and $L = 8$. Then

$$|f(x) - L| = |x^3 - 8| = |(x-2)(x^2 + 2x + 4)| = |x^2 + 2x + 4| \cdot |x - 2|.$$

The factor $|x - 2|$ will be smaller than δ, but what about the factor $|x^2 + 2x + 4|$? If, in addition to any other restrictions placed on δ, we require $\delta \leq 1$, then $|x-2| < \delta$ guarantees that $|x - 2| < 1$. Removing the absolute value from this last inequality yields $-1 < x - 2 < 1$, or $1 < x < 3$. Thus, $|x| < 3$, so that $|x^2 + 2x + 4| \leq |x|^2 + 2|x| + 4 < 19$ and

$$|f(x) - L| = |x^2 + 2x + 4| \cdot |x - 2| < 19|x - 2| < 19\delta.$$

To make this less than ϵ, we can choose $\delta \leq \epsilon/19$. Combining all of this information, the ϵ-δ proof may be written as follows:

$\boxed{\text{Given any } \epsilon > 0, \text{ we can choose } \delta = \min\left\{1, \frac{\epsilon}{19}\right\}}$. Whenever $0 < |x-2| < \delta$, it follows that
$|x^2 + 2x + 4| < 19$, and then

$$\begin{aligned} |f(x) - L| &= |x^3 - 8| = |(x-2)(x^2+2x+4)| = |x^2 + 2x + 4| \cdot |x-2| \\ &< 19|x-2| < 19\delta \leq 19 \cdot \frac{\epsilon}{19} = \epsilon. \end{aligned}$$

Therefore, $\lim_{x \to 2} x^3 = 8$.

31. Given any $\epsilon > 0$, we must find a number $\delta > 0$ so that whenever $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$. Here $f(x) = \frac{1}{x}$, $c = 3$, and $L = \frac{1}{3}$. Then

$$|f(x) - L| = \left|\frac{1}{x} - \frac{1}{3}\right| = \frac{|3 - x|}{3|x|} = \frac{|x-3|}{3|x|}.$$

The factor $|x - 3|$ will be smaller than δ, but what about the factor $|x|$? If, in addition to any other restrictions placed on δ, we require $\delta \leq 1$, then $|x - 3| < \delta$ guarantees that $|x - 3| < 1$. Removing the absolute value from this last inequality yields $-1 < x - 3 < 1$, or $2 < x < 4$. Because $x > 2 > 0$, $x = |x|$ so that $2 < |x-2| < 4$. Therefore, $\frac{1}{4} < \frac{1}{|x|} < \frac{1}{2}$, and

$$|f(x) - L| = \frac{|x-3|}{3|x|} < \frac{|x-2|}{6} < \frac{\delta}{6}.$$

To make this less than ϵ, we can choose $\delta \leq 6\epsilon$. Combining all of this information, the ϵ-δ proof may be written as follows:

$\boxed{\text{Given any } \epsilon > 0, \text{ we can choose } \delta = \min\{1, 6\epsilon\}}$. Whenever $0 < |x-3| < \delta$, it follows that $\frac{1}{|x|} < \frac{1}{2}$, and then

$$|f(x) - L| = \left|\frac{1}{x} - \frac{1}{3}\right| = \frac{|3-x|}{3|x|} = \frac{|x-3|}{3|x|} < \frac{|x-3|}{6} < \frac{\delta}{6} \leq \frac{6\epsilon}{6} = \epsilon.$$

Therefore, $\lim_{x \to 3} \frac{1}{x} = \frac{1}{3}$.

32. Given any $\epsilon > 0$, we must find a number $\delta > 0$ so that whenever $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$. Here $f(x) = \dfrac{1}{x^2}$, $c = 2$, and $L = \dfrac{1}{4}$. Then

$$|f(x) - L| = \left|\frac{1}{x^2} - \frac{1}{4}\right| = \frac{|4 - x^2|}{4x^2} = \frac{|x-2| \cdot |x+2|}{4x^2}.$$

The factor $|x - 2|$ will be smaller than δ, but what about the factors $|x + 2|$ and x^2? If, in addition to any other restrictions placed on δ, we require $\delta \le 1$, then $|x - 2| < \delta$ guarantees that $|x - 2| < 1$. Removing the absolute value from this last inequality yields $-1 < x - 2 < 1$, or $1 < x < 3$. Thus, $\dfrac{1}{x^2} < 1$ and $3 < x + 2 < 5$, so that $|x + 2| < 5$ and

$$|f(x) - L| = \frac{|x-2| \cdot |x+2|}{4x^2} < \frac{5}{4}|x - 2| < \frac{5}{4}\delta.$$

To make this less than ϵ, we can choose $\delta \le 4\epsilon/5$. Combining all of this information, the ϵ-δ proof may be written as follows:

Given any $\epsilon > 0$, we can choose $\delta = \min\left\{1, \dfrac{4\epsilon}{5}\right\}$. Whenever $0 < |x - 2| < \delta$, it follows that $\dfrac{1}{x^2} < 1$ and $|x + 2| < 5$. Then

$$|f(x) - L| = \left|\frac{1}{x^2} - \frac{1}{4}\right| = \frac{|4 - x^2|}{4x^2} = \frac{|x-2| \cdot |x+2|}{4x^2} < \frac{5}{4}|x - 2| < \frac{5}{4}\delta \le \frac{5}{4} \cdot \frac{4\epsilon}{5} = \epsilon.$$

Therefore, $\displaystyle\lim_{x \to 2} \frac{1}{x^2} = \frac{1}{4}$.

33. Negating the ϵ-δ definition of a limit yields the statement: $\displaystyle\lim_{x \to c} f(x) \ne L$ provided there is an $\epsilon > 0$ such that for any $\delta > 0$, there is an x satisfying $0 < |x - c| < \delta$ but $|f(x) - L| \ge \epsilon$. To establish that $\displaystyle\lim_{x \to 3}(3x - 1) \ne 12$, let $\epsilon = 1$ and $\delta > 0$. From the set of x values satisfying $0 < |x - 3| < \delta$ select any x for which $-\delta < x - 3 < 0$, or $3 - \delta < x < 3$. Then

$$-4 - 3\delta < (3x - 1) - 12 < -4 \quad \text{or} \quad |(3x - 1) - 12| > 4 > 1 = \epsilon.$$

Therefore, $\displaystyle\lim_{x \to 3}(3x - 1) \ne 12$.

34. Negating the ϵ-δ definition of a limit yields the statement: $\displaystyle\lim_{x \to c} f(x) \ne L$ provided there is an $\epsilon > 0$ such that for any $\delta > 0$, there is an x satisfying $0 < |x - c| < \delta$ but $|f(x) - L| \ge \epsilon$. To establish that $\displaystyle\lim_{x \to -2}(4x) \ne -7$, let $\epsilon = \frac{1}{2}$ and $\delta > 0$. From the set of x values satisfying $0 < |x + 2| < \delta$ select any x for which $-\delta < x + 2 < 0$, or $-2 - \delta < x < -2$. Then

$$-1 - 4\delta < 4x + 7 < -1 \quad \text{or} \quad |4x + 7| > 1 > \frac{1}{2} = \epsilon.$$

Therefore, $\displaystyle\lim_{x \to -2}(4x) \ne -7$.

Applications and Extensions

35. Note that

$$\left|\frac{1}{x^2 + 9} - \frac{1}{18}\right| = \left|\frac{18 - (x^2 + 9)}{18(x^2 + 9)}\right| = \left|\frac{9 - x^2}{18(x^2 + 9)}\right| = \frac{|3 - x| \cdot |3 + x|}{18(x^2 + 9)} = \frac{|x + 3|}{18(x^2 + 9)}|x - 3|.$$

If $2 < x < 4$, then

$$13 < x^2 + 9 < 25 \quad \text{so that} \quad \frac{1}{x^2+9} < \frac{1}{13}$$

and

$$5 < x + 3 < 7 \quad \text{so that} \quad |x+3| < 7;$$

therefore,

$$\left|\frac{1}{x^2+9} - \frac{1}{18}\right| = \frac{|x+3|}{18(x^2+9)}|x-3| < \frac{7}{18(13)}|x-3| = \frac{7}{234}|x-3| < \frac{7}{234}\delta.$$

To make this less than ϵ, we can choose $\delta \le 234\epsilon/7$. It is important to note that another restriction on δ is implicit in the analysis that has just been performed. If $2 < x < 4$, then $-1 < x - 3 < 1$, and $|x-3| < 1$. Thus, δ must also be less than or equal to 1. Combining all of this information, the ϵ-δ proof may be written as follows:

$\boxed{\text{Given any } \epsilon > 0, \text{ we can choose } \delta = \min\left\{1, \frac{234}{7}\epsilon\right\}}$. Whenever $0 < |x-3| < \delta$, it follows that $\frac{1}{x^2+9} < \frac{1}{13}$ and $|x+3| < 7$. Then

$$\left|\frac{1}{x^2+9} - \frac{1}{18}\right| = \frac{|x+3|}{18(x^2+9)}|x-3| < \frac{7}{234}|x-3| < \frac{7}{234}\delta \le \frac{7}{234}\cdot\frac{234}{7}\epsilon = \epsilon.$$

Therefore, $\lim_{x\to 3} \frac{1}{x^2+9} = \frac{1}{18}$.

36. Note that

$$|(2+x)^2 - 4| = |4 + 4x + x^2 - 4| = |x(4+x)| = |x| \cdot |x+4|.$$

If $-1 < x < 1$, then $3 < x + 4 < 5$ and $|x+4| < 5$. Thus,

$$|(2+x)^2 - 4| = |x| \cdot |x+4| < 5|x| < 5\delta.$$

To make this less than ϵ, we can choose $\delta \le \epsilon/5$. It is important to note that another restriction on δ is implicit in the analysis that has just been performed. If $-1 < x < 1$, then $|x| < 1$. Thus, δ must also be less than or equal to 1. Combining all of this information, the ϵ-δ proof may be written as follows:

$\boxed{\text{Given any } \epsilon > 0, \text{ we can choose } \delta = \min\left\{1, \frac{\epsilon}{5}\right\}}$. Whenever $0 < |x| < \delta$, it follows that $|x+4| < 5$, and then

$$|(2+x)^2 - 4| = |4 + 4x + x^2 - 4| = |x(4+x)| = |x| \cdot |x+4| < 5|x| < 5\delta \le 5 \cdot \frac{\epsilon}{5} = \epsilon.$$

Therefore, $\lim_{x\to 0} (2+x)^2 = 4$.

37. Note that

$$\left|\frac{1}{x^2+9} - \frac{1}{13}\right| = \left|\frac{13 - (x^2+9)}{13(x^2+9)}\right| = \left|\frac{4 - x^2}{13(x^2+9)}\right| = \frac{|2-x| \cdot |2+x|}{13(x^2+9)} = \frac{|x+2|}{13(x^2+9)}|x-2|.$$

If $1 < x < 3$, then

$$10 < x^2 + 9 < 18 \quad \text{so that} \quad \frac{1}{x^2+9} < \frac{1}{10}$$

and

$$3 < x + 2 < 5 \quad \text{so that} \quad |x+2| < 5.$$

Thus,

$$\left|\frac{1}{x^2+9}-\frac{1}{13}\right| = \frac{|x+2|}{13(x^2+2)}|x-2| < \frac{5}{13(10)}|x-2| = \frac{1}{26}|x-2| < \frac{1}{26}\delta.$$

To make this less than ϵ, we can choose $\delta \le 26\epsilon$. It is important to note that another restriction on δ is implicit in the analysis that has just been performed. If $1 < x < 3$, then $-1 < x-2 < 1$ and $|x-2| < 1$. Thus, δ must also be less than or equal to 1. Combining all of this information, the ϵ-δ proof may be written as follows:

$\boxed{\text{Given any } \epsilon > 0, \text{ we can choose } \delta = \min\{1, 26\epsilon\}}$. Whenever $0 < |x-2| < \delta$, it follows that
$\dfrac{1}{x^2+9} < \dfrac{1}{10}$ and $|x+2| < 5$. Then

$$\left|\frac{1}{x^2+9}-\frac{1}{13}\right| = \frac{|x+2|}{13(x^2+9)}|x-2| < \frac{1}{26}|x-2| < \frac{1}{26}\delta \le \frac{1}{26}\cdot 26\epsilon = \epsilon.$$

Therefore, $\displaystyle\lim_{x\to 2}\frac{1}{x^2+9} = \frac{1}{13}$.

38. Negating the ϵ-δ definition of a limit yields the statement: $\displaystyle\lim_{x\to c} f(x) = L$ provided there is an $\epsilon > 0$ such that for any $\delta > 0$, there is an x satisfying $0 < |x-c| < \delta$ but $|f(x)-L| \ge \epsilon$. To establish that $\displaystyle\lim_{x\to 1} x^2 = 1.31$, let $\epsilon = 0.1$ and $\delta > 0$. If $\delta < 1$, from the set of x values satisfying $0 < |x-1| < \delta$ select any x for which $-\delta < x-1 < 0$, or $1-\delta < x < 1$, and note that $1-\delta > 0$. On the other hand, if $\delta \ge 1$, then from the set of x values satisfying $0 < |x-1| < \delta$ select any x for which $-1 < x-1 < 0$, or $0 < x < 1$. Then, for any $\delta > 0$,

$$x^2 < 1 \quad \text{so that} \quad |x^2 - 1.31| > 0.31 > 0.1 = \epsilon.$$

Therefore, $\displaystyle\lim_{x\to 1} x^2 = 1.31$.

39. $\boxed{\text{Given any } \epsilon > 0, \text{ we can choose } \delta = \dfrac{\epsilon}{1+|m|}}$. The absolute value of m appears in the formula for δ to insure that δ will be positive. Moreover, 1 has been added to the denominator to allow for the possibility that m could be zero. Whenever $0 < |x-c| < \delta$, then

$$|(mx+b)-(mc+b)| = |m|\ |x-c| < |m|\cdot\delta = \frac{|m|}{1+|m|}\epsilon < \epsilon.$$

Therefore, $\displaystyle\lim_{x\to c}(mx+b) = mc+b$.

40. Recall that

$$|x^2-4| = |(x-2)(x+2)| = |x-2|\cdot|x+2|.$$

If $|x-2| < \dfrac{1}{3}$, then $\dfrac{5}{3} < x < \dfrac{7}{3}$, so that

$$\frac{11}{3} < x+2 < \frac{13}{3} \quad \text{and} \quad |x+2| < \frac{13}{3}.$$

Thus,

$$|x^2-4| = |x-2|\cdot|x+2| < \frac{13}{3}|x-2| < \frac{13}{3}\delta.$$

To make this less than ϵ, we can choose $\delta \le \dfrac{3\epsilon}{13}$. Combining the two restrictions placed on δ yields
$\delta \le \min\left\{\dfrac{1}{3}, \dfrac{3\epsilon}{13}\right\}$.

41. For $x \neq 3$, to guarantee that $|(2x-1)-5| < 0.1$ requires

$$|2x-6| = 2|x-3| < 0.1 \quad \text{or} \quad |x-3| < 0.05.$$

Thus, $\boxed{x \text{ must be within } 0.05 \text{ of } 3}$ to guarantee that $2x-1$ differs from 5 by less than 0.1.

42. For $x \neq 0$, to guarantee that $|3^x - 1| < 0.1$ requires

$$-0.1 < 3^x - 1 < 0.1 \quad \text{or} \quad 0.9 < 3^x < 1.1.$$

This last compound inequality yields

$$-0.0959 \approx \frac{\ln 0.9}{\ln 3} < x < \frac{\ln 1.1}{\ln 3} \approx 0.0868.$$

Thus, $\boxed{x \text{ must be within approximately } 0.087 \text{ of } 0}$ to guarantee that 3^x differs from 1 by less than 0.1.

43. Suppose $\lim_{x\to c} f(x) = L$ where $L < 0$, and let $\epsilon = \dfrac{|L|}{2} > 0$. Then there is a $\delta > 0$ such that, whenever $0 < |x-c| < \delta$,

$$|f(x) - L| < \frac{|L|}{2} \quad \text{or} \quad -\frac{|L|}{2} < f(x) - L < \frac{|L|}{2}.$$

Because $L < 0$, $|L| = -L$, so the last compound inequality becomes

$$\frac{L}{2} < f(x) - L < -\frac{L}{2} \quad \text{or} \quad \frac{3L}{2} < f(x) < \frac{L}{2} < 0.$$

Thus, everywhere in the open interval $|x-c| < \delta$, except possibly at c, $f(x) < 0$.

44. $\boxed{\text{Given any } \epsilon > 0, \text{ we can choose } N = -\dfrac{1}{\epsilon}}$. Whenever $x < N$, then

$$\left|\frac{1}{x} - 0\right| = -\frac{1}{x} < -\frac{1}{N} = \epsilon.$$

Therefore, $\lim_{x\to-\infty} \dfrac{1}{x} = 0$.

45. $\boxed{\text{Given any } \epsilon > 0, \text{ we can choose } M = \dfrac{1}{\epsilon^2}}$. Whenever $x > M$, then

$$\left|-\frac{1}{\sqrt{x}} - 0\right| = \frac{1}{\sqrt{x}} < \frac{1}{\sqrt{M}} = \epsilon.$$

Therefore, $\lim_{x\to\infty} \left(-\dfrac{1}{\sqrt{x}}\right) = 0$.

46. To make

$$\left|\frac{1}{x^2} - 0\right| = \frac{1}{x^2} < 0.1$$

requires

$$x^2 > 10 \quad \text{or} \quad |x| > \sqrt{10}.$$

As this limit is for x approaching $-\infty$, take $\boxed{N = -\sqrt{10}}$.

47. First, consider $L > 0$. Take $\epsilon = L/2$, let $\delta > 0$, and choose $x = -\delta/2$. Then $0 < |x - 0| = |x| < \delta$ but
$$\left|\frac{1}{x} - L\right| = \left|-\frac{2}{\delta} - L\right| = \left|\frac{2}{\delta} + L\right| > L > \frac{L}{2} = \epsilon,$$
where the inequality $L > L/2$ follows because $L > 0$. Therefore, $\lim\limits_{x \to 0} \frac{1}{x}$ cannot equal L. Next, consider $L < 0$. Take $\epsilon = -L/2$, let $\delta > 0$, and choose $x = \delta/2$. Then $0 < |x - 0| = |x| < \delta$ but
$$\left|\frac{1}{x} - L\right| = \left|\frac{2}{\delta} + (-L)\right| > -L > -\frac{L}{2} = \epsilon,$$
where the inequality $-L > -L/2$ follows because $-L > 0$. Therefore, $\lim\limits_{x \to 0} \frac{1}{x}$ cannot equal L. Finally, consider $L = 0$. Take $\epsilon = 1$, let $\delta > 0$, and choose $|x| < \min\{1, \delta\}$. Then $0 < |x - 0| = |x| < \delta$ but
$$\left|\frac{1}{x} - L\right| = \left|\frac{1}{x}\right| > 1 = \epsilon.$$
Therefore, $\lim\limits_{x \to 0} \frac{1}{x}$ cannot equal 0. In summary, there is no number L such that $\lim\limits_{x \to 0} \frac{1}{x} = L$.

48. The strict inequality on the left ($0 < |x - c|$) is to remove the function value at $x = c$ from consideration; the strict inequality on the right ($|x - c| < \delta$) is to create an **open** interval containing $x = c$.

49. A limit is supposed to capture the behavior of the value of a function as the value of the independent variable approaches a target value. We do not want the limit to depend on the value of the function at that location or even to depend on whether the function is defined at that location. Including the phrase *except possibly at c* imposes these restrictions.

50. In the ϵ-δ definition of a limit, ϵ measures the "closeness" of the function value $f(x)$ to the value of the limit L, while δ measures the "closeness" of the value of the independent variable x to the target value c. For example, notice the placement of ϵ and δ in the proof that $\lim\limits_{x \to 3}(2x - 1) = 5$:

    Given any $\epsilon > 0$, take $\delta = \epsilon/2$. Then, whenever $0 < |x - 3| < \delta$,
$$|(2x - 1) - 5| = |2x - 6| = 2|x - 3| < 2 \cdot \delta = 2 \cdot \frac{\epsilon}{2} = \epsilon.$$
    Therefore, $\lim\limits_{x \to 3}(2x - 1) = 5$.

51. First consider $\lim\limits_{x \to 0} f(x)$. Any open interval containing 0 will contain both rational numbers and irrational numbers. As the rational numbers approach 0, the function value x^2 will approach 0; as the irrational numbers approach 0, the function value will also approach 0. This suggests that $\boxed{\lim\limits_{x \to 0} f(x) = 0}$. This argument can be made precise using $\delta = \sqrt{\epsilon}$ within the ϵ-δ definition.

    Next, consider $\lim\limits_{x \to 1} f(x)$. Any open interval containing 1 will contain both rational numbers and irrational numbers. As the rational numbers approach 1, the function value x^2 will approach 1; as the irrational numbers approach 1, however, the function value will approach 0. This suggests that $\boxed{\lim\limits_{x \to 1} f(x) \text{ does not exist}}$. This argument can be made precise as follows: Suppose the limit does exist and is equal to L. Take $\epsilon = 1/4$. There would then be a δ such that whenever $0 < |x - 1| < \delta$, the function value would be within $1/4$ of L. Considering the rational and irrational values of x separately would require that L be simultaneously within $1/4$ of 1 and 0, which is impossible.

52. Any open interval containing 0 will contain both rational numbers and irrational numbers. As the rational numbers approach 0, the function value x^2 will approach 0; as the irrational numbers approach 0, the function value $\tan x$ will also approach 0. This suggests that $\boxed{\lim_{x\to 0} f(x) = 0}$.

Challenge Problems

53. Note that

$$|f(x) - L| = |4x^3 + 3x^2 - 24x + 22 - 5| = |4x^3 + 3x^2 - 24x + 17| = |4x^2 + 7x - 17| \cdot |x - 1|.$$

The factor $|x - 1|$ will be smaller than δ, but what about the factor $|4x^2 + 7x - 17|$? If, in addition to any other restrictions placed on δ, we require $\delta \le 1$, then $|x-1| < \delta$ guarantees that $|x-1| < 1$, or $0 < x < 2$. Thus, $|x| < 2$ so that $|4x^2+7x-17| \le 4|x|^2+7|x|+17 < 47$ and

$$|f(x) - L| = |4x^2 + 7x - 17| \cdot |x - 1| < 47|x - 1| < 47\delta.$$

To make this less than ϵ, we can choose $\delta \le \epsilon/47$. Combining all of this information, the ϵ-δ proof may be written as follows:

$\boxed{\text{Given any } \epsilon > 0, \text{ we can choose } \delta = \min\left\{1, \frac{\epsilon}{47}\right\}}$. Whenever $0 < |x - 1| < \delta$, it follows that
$|4x^2 + 7x - 17| < 19$, and then

$$\begin{aligned} |f(x) - L| &= |4x^3 + 3x^2 - 24x + 22 - 5| = |4x^3 + 3x^2 - 24x + 17| \\ &= |4x^2 + 7x - 17| \cdot |x - 1| < 47|x - 1| < 47\delta \le 47 \cdot \frac{\epsilon}{47} = \epsilon. \end{aligned}$$

Therefore, $\lim_{x\to 1}(4x^3 + 3x^2 - 24x + 22) = 5$.

54. Suppose $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$. Given any $\epsilon > 0$, there then exists a $\delta_1 > 0$ such that

$$|f(x) - L| < \frac{\epsilon}{2} \quad \text{whenever} \quad 0 < |x - c| < \delta_1,$$

and a $\delta_2 > 0$ such that

$$|g(x) - M| < \frac{\epsilon}{2} \quad \text{whenever} \quad 0 < |x - c| < \delta_2.$$

Choose $\delta = \min\{\delta_1, \delta_2\}$. Whenever $0 < |x - c| < \delta$, then

$$\begin{aligned} |(f(x) + g(x)) - (L + M)| &= |(f(x) - L) + (g(x) - M)| \\ &\le |f(x) - L| + |g(x) - M| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Therefore, $\lim_{x\to c}[f(x) + g(x)] = L + M$.

55. First note that

$$\sqrt{5 + 4x^2} > \sqrt{4x^2} \quad \text{so that} \quad \frac{1}{\sqrt{5 + 4x^2}} < \frac{1}{\sqrt{4x^2}}$$

for all x. For $x > 2$, $2 - x < 0$ so that upon multiplying the latter inequality above by the negative expression $2 - x$,

$$\frac{2 - x}{\sqrt{5 + 4x^2}} > \frac{2 - x}{\sqrt{4x^2}} = \frac{2 - x}{2x} = \frac{1}{x} - \frac{1}{2} > -\frac{1}{2}.$$

Thus, to be within $\epsilon = 0.01$ of $L = -1/2$, we need

$$\frac{2 - x}{\sqrt{5 + 4x^2}} < -0.49 \quad \text{or} \quad \frac{x - 2}{\sqrt{5 + 4x^2}} > 0.49.$$

Squaring both sides of this last inequality and gathering terms yields

$$0.0396x^2 - 4x + 2.7995 > 0.$$

The roots of the quadratic function are

$$x = \frac{4 \pm \sqrt{16 - 4(0.0396)(2.7995)}}{2(0.0396)} \approx 100.3, 0.7,$$

so

$$\frac{2-x}{\sqrt{5+4x^2}} < -0.49$$

for approximately $x < 0.7$ and $x > 100.3$. In the ϵ-δ definition of a limit at infinity, we can therefore take $\boxed{M = 101}$.

56. To prove that the linear function $f(x) = ax+b$ is continuous everywhere, it must be shown that $\lim_{x\to c} f(x) = ac + b$ for any real number c. Now, $\boxed{\text{given any } \epsilon > 0\text{, we can choose } \delta = \frac{\epsilon}{1+|a|}}$. The absolute value of a appears in the formula for δ to ensure that δ will be positive. Moreover, 1 has been added to the denominator to allow for the possibility that a could be zero. Whenever $0 < |x-c| < \delta$, then

$$|(ax+b) - (ac+b)| = |a|\ |x-c| < |a| \cdot \delta = \frac{|a|}{1+|a|}\epsilon < \epsilon.$$

Therefore, $\lim_{x\to c} f(x) = ac+b$, and the linear function $f(x) = ax+b$ is continuous everywhere.

57. Start by proving continuity at $x = 0$. Because 0 is a rational number, $f(0) = 0$. Now, given any $\epsilon > 0$, choose $\delta = \epsilon$. Whenever $0 < |x-c| < \delta$, then

$$|f(x) - 0| = |f(x)| = |x| < \delta = \epsilon$$

for every rational number x and

$$|f(x) - 0| = |f(x)| = 0 < \delta = \epsilon$$

for every irrational number x. Therefore, $\lim_{x\to 0} f(x) = 0 = f(0)$ and f is continuous at $x = 0$.

Now, let c be any non-zero real number. We will proceed with a proof by contradiction. Toward this end, suppose that f is continuous at $x = c$. Then the limit as x approaches c must exist and be equal to $f(c)$. Based on the definition of f, $\lim_{x\to c} f(x)$ must therefore be equal to either 0 (if c is a rational number) or c (if c is an irrational number). We will now show that neither of these are the limit. Take $\epsilon = |c|/2$, and choose $\delta > 0$. For any irrational number x such that $0 < |x-c| < \delta$,

$$|f(x) - 0| = |f(x)| = |c| > \frac{|c|}{2} = \epsilon,$$

so $\lim_{x\to c} f(x) \neq 0$. Moreover, for every rational number x such that $0 < |x-c| < \delta$,

$$|f(x) - c| = |c| > \frac{|c|}{2} = \epsilon,$$

so $\lim_{x\to c} f(x) \neq c$. Thus, $\lim_{x\to c} f(x)$ does not exist, and f is not continuous at $x = c$.

58. Let $f(x) = x^3$. Then

$$|f(x) - f(c)| = |x^3 - c^3| = |(x-c)(x^2 + cx + c^2)| = |x - c| \cdot |x^2 + cx + c^2|.$$

With both x and c in the open interval $(0, 2)$, $|x| < 2$ and $|c| < 2$. Therefore

$$|x^2 + cx + c^2| \le |x|^2 + |c|\ |x| + |c|^2 < 2^2 + 2^2 + 2^2 = 12,$$

and

$$|f(x) - f(c)| = |x - c| \cdot |x^2 + cx + c^2| < 12|x - c|.$$

Thus, $\boxed{K = 12}$ is a Lipschitz constant for $f(x) = x^3$ on $(0, 2)$.

Chapter 1 Review Exercises

1. The values in the table below, which have been rounded for display purposes, suggest that the value of $f(x) = \dfrac{1 - \cos x}{1 + \cos x}$ can be made "as close as we please" to 0 by choosing x "sufficiently close" to 0. It therefore appears that

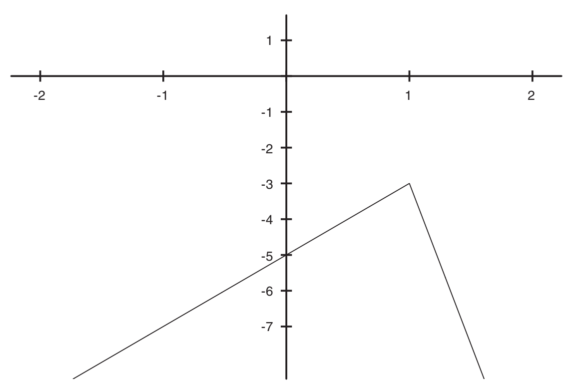
$$\boxed{\lim_{x \to 0} \frac{1 - \cos x}{1 + \cos x} = 0}.$$

x	-0.1	-0.01	-0.001	$\to 0 \leftarrow$	0.001	0.01	0.1
$f(x) = \dfrac{1 - \cos x}{1 + \cos x}$	0.002504	0.000025	0.00000025	$f(x)$ approaches 0	0.00000025	0.000025	0.002504

2. The figure below displays a graph of f. Using the graph,

$$\lim_{x \to 1^-} f(x) = -3 \quad \text{and} \quad \lim_{x \to 1^+} f(x) = -3.$$

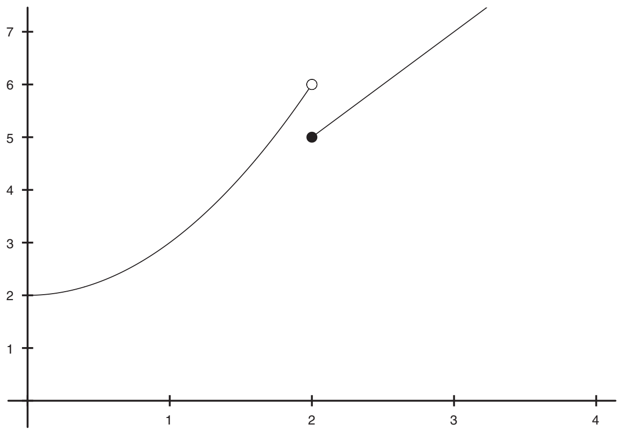
Because the two one-sided limits as x approaches 1 are equal, $\lim_{x \to 1} f(x)$ exists. Moreover, because the two one-sided limits are equal to -3, $\boxed{\lim_{x \to 1} f(x) = -3}$.



3. The figure below displays a graph of f. Using the graph,

$$\lim_{x \to 2^-} f(x) = 6 \quad \text{and} \quad \lim_{x \to 2^+} f(x) = 5.$$

Because the two one-sided limits as x approaches 2 are not equal, $\boxed{\lim_{x \to 2} f(x) \text{ does not exist}}$.



4. The statement can be written as $\lim_{x\to 3} f(x) = 5$.

5. Let $f(x) = \dfrac{3}{x}$. Then

$$\begin{aligned}\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{\frac{3}{x+h}-\frac{3}{x}}{h}\cdot\frac{x(x+h)}{x(x+h)} = \lim_{h\to 0}\frac{3x-3(x+h)}{hx(x+h)} = \lim_{h\to 0}\frac{-3h}{hx(x+h)}\\ &= -3\lim_{h\to 0}\frac{1}{x(x+h)} = -3\cdot\frac{1}{x^2} = \boxed{-\frac{3}{x^2}}.\end{aligned}$$

6. Let $f(x) = 3x^2 + 2x$. Then

$$\begin{aligned}\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{3(x+h)^2+2(x+h)-(3x^2+2x)}{h}\\ &= \lim_{h\to 0}\frac{3x^2+6xh+3h^2+2x+2h-3x^2-2x}{h}\\ &= \lim_{h\to 0}\frac{6xh+3h^2+2h}{h} = \lim_{h\to 0}(6x+3h+2) = \boxed{6x+2}.\end{aligned}$$

7. Because $1+\sin x \le f(x) \le |x|+1$ for all x in the open interval $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ containing 0 and

$$\lim_{x\to 0}(1+\sin x) = 1+\sin 0 = 1 \quad \text{and} \quad \lim_{x\to 0}(|x|+1) = |0|+1 = 1,$$

it follows from the Squeeze Theorem that

$$\lim_{x\to 0} f(x) = \boxed{1}.$$

8. $\displaystyle\lim_{x\to 2}\left(2x-\frac{1}{x}\right) = 2(2)-\frac{1}{2} = \boxed{\frac{7}{2}}$.

9. $\displaystyle\lim_{x\to \pi}(x\cos x) = \pi\cos\pi = \boxed{-\pi}$.

10. $\displaystyle\lim_{x\to -1}(x^3+3x^2-x-1) = (-1)^3+3(-1)^2-(-1)-1 = \boxed{2}$.

11. $\displaystyle\lim_{x\to 0}\sqrt[3]{x(x+2)^3} = \sqrt[3]{0(2)^3} = \boxed{0}$.

12. $\lim_{x\to 0}[(2x+3)(x^5+5x)] = (0+3)(0^5+0) = \boxed{0}$.

13. $\lim_{x\to 3} \frac{x^3-27}{x-3} = \lim_{x\to 3} \frac{(x-3)(x^2+3x+9)}{x-3} = \lim_{x\to 3}(x^2+3x+9) = 3^2+3(3)+9 = \boxed{27}$.

14. $\lim_{x\to 3} \left(\frac{x^2}{x-3} - \frac{3x}{x-3}\right) = \lim_{x\to 3} \frac{x^2-3x}{x-3} = \lim_{x\to 3} \frac{x(x-3)}{x-3} = \lim_{x\to 3} x = \boxed{3}$.

15. $\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2}(x+2) = 2+2 = \boxed{4}$.

16. $\lim_{x\to -1} \frac{x^2+3x+2}{x^2+4x+3} = \lim_{x\to -1} \frac{(x+1)(x+2)}{(x+1)(x+3)} = \lim_{x\to -1} \frac{x+2}{x+3} = \frac{-1+2}{-1+3} = \boxed{\frac{1}{2}}$.

17. $\lim_{x\to -2} \frac{x^3+5x^2+6x}{x^2+x-2} = \lim_{x\to -2} \frac{x(x+2)(x+3)}{(x+2)(x-1)} = \lim_{x\to -2} \frac{x(x+3)}{x-1} = \frac{-2(-2+3)}{-2-1} = \boxed{\frac{2}{3}}$.

18. $\lim_{x\to 1} \left(x^2-3x+\frac{1}{x}\right)^{15} = \left(1^2-3(1)+\frac{1}{1}\right)^{15} = (-1)^{15} = \boxed{-1}$.

19.

$$\begin{aligned}
\lim_{x\to 2} \frac{3-\sqrt{x^2+5}}{x^2-4} &= \lim_{x\to 2} \frac{3-\sqrt{x^2+5}}{x^2-4} \cdot \frac{3+\sqrt{x^2+5}}{3+\sqrt{x^2+5}} = \lim_{x\to 2} \frac{9-(x^2+5)}{(x^2-4)(3+\sqrt{x^2+5})} \\
&= \lim_{x\to 2} \frac{4-x^2}{(x^2-4)(3+\sqrt{x^2+5})} = -\lim_{x\to 2} \frac{1}{3+\sqrt{x^2+5}} \\
&= -\frac{1}{3+\sqrt{2^2+5}} = \boxed{-\frac{1}{6}}.
\end{aligned}$$

20. Note that

$$\frac{1}{(2+x)^2} - \frac{1}{4} = \frac{4-(2+x)^2}{4(2+x)^2} = \frac{4-4-4x-x^2}{4(2+x)^2} = -\frac{x(x+4)}{4(2+x)^2}.$$

Therefore,

$$\begin{aligned}
\lim_{x\to 0} \left\{\frac{1}{x}\left[\frac{1}{(2+x)^2} - \frac{1}{4}\right]\right\} &= -\lim_{x\to 0}\left[\frac{1}{x}\cdot\frac{x(x+4)}{4(2+x)^2}\right] \\
&= -\lim_{x\to 0} \frac{x+4}{4(2+x)^2} = -\frac{0+4}{4(2+0)^2} = \boxed{-\frac{1}{4}}.
\end{aligned}$$

21.

$$\begin{aligned}
\lim_{x\to 0} \frac{(x+3)^2-9}{x} &= \lim_{x\to 0} \frac{x^2+6x+9-9}{x} \\
&= \lim_{x\to 0} \frac{x^2+6x}{x} = \lim_{x\to 0} \frac{x(x+6)}{x} = \lim_{x\to 0}(x+6) = 0+6 = \boxed{6}.
\end{aligned}$$

22. $\lim_{x\to 1}[(x^3-3x^2+3x-1)(x+1)^2] = (1^3-3(1)^2+3(1)-1)(1+1)^2 = 0(4) = \boxed{0}$.

23. $\lim_{x\to -2^+} \frac{x^2+5x+6}{x+2} = \lim_{x\to -2^+} \frac{(x+2)(x+3)}{x+2} = \lim_{x\to -2^+}(x+3) = -2+3 = \boxed{1}$.

24. As x approaches 5 from the right, $x-5>0$, so that

$$\lim_{x\to 5^+}\frac{|x-5|}{x-5}=\lim_{x\to 5^+}\frac{x-5}{x-5}=\lim_{x\to 5^+}1=\boxed{1}.$$

25. As x approaches 1 from the left, $x-1<0$, so that

$$\lim_{x\to 1^-}\frac{|x-1|}{x-1}=\lim_{x\to 1^-}\frac{-(x-1)}{x-1}=\lim_{x\to 1^-}-1=\boxed{-1}.$$

26. As x approaches $3/2$ from the right, $2x$ approaches 3 from the right, so that

$$\lim_{x\to 3/2^+}\lfloor 2x\rfloor=\boxed{3}.$$

27. $\displaystyle\lim_{x\to 4^-}\frac{x^2-16}{x-4}=\lim_{x\to 4^-}\frac{(x-4)(x+4)}{x-4}=\lim_{x\to 4^-}(x+4)=4+4=\boxed{8}.$

28. As x approaches 1 from the right, $x-1>0$, so that

$$\lim_{x\to 1^+}\sqrt{x-1}=\sqrt{1-1}=\boxed{0}.$$

29. Because

$$\lim_{x\to 2^-}f(x)=\lim_{x\to 2^-}(2x+3)=2(2)+3=\boxed{7}$$

and

$$\lim_{x\to 2^+}f(x)=\lim_{x\to 2^+}(9-x)=9-2=\boxed{7}$$

are equal, $\lim_{x\to 2}f(x)$ exists. Moreover, because both one-sided limits are equal to 7, $\boxed{\lim_{x\to 2}f(x)=7}$.

30. Because

$$\lim_{x\to 3^-}f(x)=\lim_{x\to 3^-}(3x+1)=3(3)+1=\boxed{10}$$

and

$$\lim_{x\to 3^+}f(x)=\lim_{x\to 3^+}(4x-2)=4(3)-2=\boxed{10}$$

are equal, $\lim_{x\to 3}f(x)$ exists. Moreover, because both one-sided limits are equal to 10, $\boxed{\lim_{x\to 3}f(x)=10}$.

31. The function f is defined at $c=1$ with $f(1)=5$. Because

$$\lim_{x\to 1^-}f(x)=\lim_{x\to 1^-}(5x-2)=5(1)-2=3$$

and

$$\lim_{x\to 1^+}f(x)=\lim_{x\to 1^+}(2x+1)=2(1)+1=3$$

are equal, $\lim_{x\to 1}f(x)$ exists and is equal to 3. However $f(1)=5=3=\lim_{x\to 1}f(x)$, so $\boxed{f\text{ is not continuous at }c=1}$.

32. The function f is defined at $c = -1$ with $f(-1) = 2$. Because

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} x^2 = (-1)^2 = 1$$

and

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} (-3x - 2) = -3(-1) - 2 = 1$$

are equal, $\lim_{x \to -1} f(x)$ exists and is equal to 1. However $f(-1) = 2 \neq 1 = \lim_{x \to -1} f(x)$, so $\boxed{f \text{ is not continuous at } c = -1}$.

33. The function f is defined at $c = 0$ with $f(0) = 4$. Because

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (4 - 3x^2) = 4 - 3(0)^2 = 4$$

and

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \sqrt{16 - x^2} = \sqrt{16 - 0^2} = 4$$

are equal, $\lim_{x \to 0} f(x)$ exists and is equal to 4. Finally, $f(0) = \lim_{x \to 0} f(x)$, so $\boxed{f \text{ is continuous at } c = 0}$.

34. The function f is defined at $c = 4$ with $f(4) = \sqrt{4 + 4} = 2\sqrt{2}$. Because

$$\lim_{x \to 4^-} f(x) = \lim_{x \to 4^-} \sqrt{4 + x} = \sqrt{4 + 4} = 2\sqrt{2}$$

and

$$\lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} \sqrt{\frac{x^2 - 16}{x - 4}} = \lim_{x \to 4^+} \sqrt{\frac{(x - 4)(x + 4)}{x - 4}} = \lim_{x \to 4^+} \sqrt{x + 4} = \sqrt{4 + 4} = 2\sqrt{2}$$

are equal, $\lim_{x \to 4} f(x)$ exists and is equal to $2\sqrt{2}$. Finally, $f(4) = \lim_{x \to 4} f(x)$, so $\boxed{f \text{ is continuous at } c = 4}$.

35. The function f is defined at $c = 1/2$ with $f(1/2) = \lfloor 1 \rfloor = 1$. As x approaches $1/2$ from the left, $2x$ approaches 1 from the left, so that

$$\lim_{x \to 1/2^-} f(x) = \lim_{x \to 1/2^-} \lfloor 2x \rfloor = 0.$$

On the other hand, as x approaches $1/2$ from the right, $2x$ approaches 1 from the right, so that

$$\lim_{x \to 1/2^+} f(x) = \lim_{x \to 1/2^+} \lfloor 2x \rfloor = 1.$$

Because the two one-sided limits as x approaches $1/2$ are not equal $\lim_{x \to 1/2} f(x)$ does not exist. Therefore, $\boxed{f \text{ is not continuous at } c = 1/2}$.

36. The function f is defined at $c = 5$ with $f(5) = |5 - 5| = 0$. Because

$$\lim_{x \to 5^-} f(x) = \lim_{x \to 5^-} |x - 5| = \lim_{x \to 5^-} [-(x - 5)] = -(5 - 5) = 0$$

and

$$\lim_{x \to 5^+} f(x) = \lim_{x \to 5^+} |x - 5| = \lim_{x \to 5^+} (x - 5) = 5 - 5 = 0$$

are equal, $\lim_{x \to 5} f(x)$ exists and is equal to 0. Finally, $f(5) = \lim_{x \to 5} f(x)$, so $\boxed{f \text{ is continuous at } c = 5}$.

37. Let $f(x) = 2x^2 - 5x$.

   (a) The average rate of change of f from 1 to x is

$$\frac{f(x)-f(1)}{x-1} = \frac{2x^2-5x-(-3)}{x-1} = \frac{2x^2-5x+3}{x-1} = \frac{(x-1)(2x-3)}{x-1} = \boxed{2x-3},$$

   for $x \neq 1$.

   (b) Using the result from part (a),

$$\lim_{x\to 1} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1}(2x-3) = 2(1) - 3 = \boxed{-1}.$$

38. The function

$$f(x) = \begin{cases} -1, & \text{if } -1 \le x \le 0 \\ 1, & \text{if } 0 < x \le 1 \end{cases}$$

satisfies the indicated conditions: f is continuous on the interval $[-1, 1]$ except at 0, $f(-1) = -1 < 0$, $f(1) = 1 > 0$, and f has no zeroes. This $\boxed{\text{does not contradict the Intermediate Value Theorem}}$ because the function is not continuous on the closed interval $[-1, 1]$.

39. Let $g(x) = x$ and $h(x) = x^3 - 27$. The polynomial functions g and h are both continuous on the set of all real numbers. Because f is the quotient of g and h, f is continuous on the set of all real numbers except those for which $h(x) = 0$. The only real solution to the equation

$$h(x) = x^3 - 27 = (x-3)(x^2+3x+9) = 0$$

is $x = 3$, so $\boxed{f \text{ is continuous on the set } \{x|x \neq 3\}}$.

40. Let $g(x) = x^2 - 3$ and $h(x) = x^2 + 5x + 6$. The polynomial functions g and h are both continuous on the set of all real numbers. Because f is the quotient of g and h, f is continuous on the set of all real numbers except those for which $h(x) = 0$. The solutions to the equation $h(x) = x^2 + 5x + 6 = (x+3)(x+2) = 0$ are $x = -3$ and $x = -2$, so $\boxed{f \text{ is continuous on the set } \{x|x \neq -3, x \neq -2\}}$.

41. Let $g(x) = 2x + 1$ and $h(x) = x^3 + 4x^2 + 4x$. The polynomial functions g and h are both continuous on the set of all real numbers. Because f is the quotient of g and h, f is continuous on the set of all real numbers except those for which $h(x) = 0$. The solutions to the equation $h(x) = x^3 + 4x^2 + 4x = x(x+2)^2 = 0$ are $x = -2$ and $x = 0$, so $\boxed{f \text{ is continuous on the set } \{x|x \neq -2, x \neq 0\}}$.

42. Let $g(x) = \sqrt{x}$ and $h(x) = x - 1$. The function g is continuous on the set $\{x|x \ge 0\}$, and the polynomial function h is the continuous on the set of all real numbers. Moreover, the solution of the inequality $h(x) \ge 0$ is the set $\{x|x \ge 1\}$. As f is the composition $g(h(x))$, the function f is continuous at c provided h is continuous at c and g is continuous at $h(c)$; thus, $\boxed{f \text{ is continuous on the set } \{x|x \ge 1\}}$.

43. Let $g(x) = 2^x$ and $h(x) = -x$. The exponential function g and the polynomial function h are both continuous on the set of all real numbers. As f is the composition $g(h(x))$, the function f is continuous at c provided h is continuous at c and g is continuous at $h(c)$; $\boxed{f \text{ is continuous on the set of all real numbers}}$.

44. Let $f(x) = 2x^3 + 3x^2 - 23x - 42$. This polynomial function is continuous on the set of all real numbers, so it is continuous on the closed interval $[3, 4]$. Because

$$f(3) = 2(3)^3 + 3(3)^2 - 23(3) - 42 = -30 < 0$$

and

$$f(4) = 2(4)^3 + 3(4)^2 - 23(4) - 42 = 42 > 0,$$

the Intermediate Value Theorem guarantees there is a number c in the interval $(3,4)$ such that $f(c) = 0$. Thus, the equation $2x^3 + 3x^2 - 23x - 42 = 0$ $\boxed{\text{does have a solution in the interval } (3,4)}$.

45. The polynomial function $f(x) = 8x^4 - 2x^2 + 5x - 1$ is continuous for all real numbers, so it is continuous on the closed interval $[0,1]$. Because $f(0) = -1 < 0$ and $f(1) = 8 - 2 + 5 - 1 = 10 > 0$, the Intermediate Value Theorem guarantees that f must have a zero on the interval $(0,1)$. To approximate this zero, subdivide the interval $[0,1]$ into 10 subintervals, each of length 0.1, and evaluate f at each endpoint. The results are shown in the first two columns of the table below. Because $f(0.2) = -0.0672 < 0$ and $f(0.3) = 0.3848 > 0$, the Intermediate Value Theorem guarantees the zero lies in the interval $(0.2, 0.3)$. Repeating the process by subdividing the interval $[0.2, 0.3]$ into 10 subintervals of length 0.01 yields the results in the middle two columns of the table, where the function values have been rounded to five decimal places for display purposes. The zero has now been bracketed in the interval $(0.21, 0.22)$. Repeating the subdivision process once more, the results in the last two columns of the table are produced, again with the function values rounded to five decimal places. Examining the function values in the last column, it follows that the zero of the function f is $\boxed{0.215}$, correct to three decimal places.

$[0,1]$		$[0.2, 0.3]$		$[0.21, 0.22]$	
x	$f(x)$	x	$f(x)$	x	$f(x)$
0.0	−1.0000	0.20	−0.06720	0.210	−0.02264
0.1	−0.5192	0.21	−0.02264	0.211	−0.01819
0.2	−0.0672	0.22	0.02194	0.212	−0.01373
0.3	0.3848	0.23	0.06659	0.213	−0.00927
0.4	0.8848	0.24	0.11134	0.214	−0.00481
0.5	1.5000	0.25	0.15625	0.215	−0.00036
0.6	2.3168	0.26	0.20136	0.216	0.00410
0.7	3.4408	0.27	0.24672	0.217	0.00856
0.8	4.9968	0.28	0.29237	0.218	0.01302
0.9	7.1288	0.29	0.33838	0.219	0.01748
1.0	10.0000	0.30	0.38480	0.220	0.02194

46. The polynomial function $f(x) = 3x^3 - 10x + 9$ is continuous for all real numbers, so it is continuous on the closed interval $[-3,-2]$. Because $f(-3) = 3(-3)^3 - 10(-3) + 9 = -42 < 0$ and $f(-2) = 3(-2)^3 - 10(-2) + 9 = 5 > 0$, the Intermediate Value Theorem guarantees that f must have a zero on the interval $(-3,-2)$. To approximate this zero, subdivide the interval $[-3,-2]$ into 10 subintervals, each of length 0.1, and evaluate f at each endpoint. The results are shown in the first two columns of the table below. Because $f(-2.2) = -0.944 < 0$ and $f(-2.1) = 2.217 > 0$, the Intermediate Value Theorem guarantees the zero lies in the interval $(-2.2, -2.1)$. Repeating the process by subdividing the interval $[-2.2, -2.1]$ into 10 subintervals of length 0.01 yields the results in the middle two columns of the table, where the function values have been rounded to five decimal places for display purposes. The zero has now been bracketed in the interval $(-2.18, -2.17)$. Repeating the subdivision process once more, the results in the last two columns of the table are produced, again with the function values rounded to five decimal places. Examining the function values in the last column, it follows that the zero of the function f is $\boxed{-2.171}$, correct to three decimal places.

$[-3,-2]$		$[-2.2,-2.1]$		$[-2.18,-2.17]$	
x	$f(x)$	x	$f(x)$	x	$f(x)$
-3.0	-42.000	-2.20	-0.94400	-2.180	-0.28070
-2.9	-35.167	-2.19	-0.61038	-2.179	-0.24794
-2.8	-28.856	-2.18	-0.28070	-2.178	-0.21523
-2.7	-23.049	-2.17	0.04506	-2.177	-0.18256
-2.6	-17.728	-2.16	0.36691	-2.176	-0.14992
-2.5	-12.875	-2.15	0.68488	-2.175	-0.11733
-2.4	-8.472	-2.14	0.99897	-2.174	-0.08477
-2.3	-4.501	-2.13	1.30921	-2.173	-0.05226
-2.2	-0.944	-2.12	1.61562	-2.172	-0.01978
-2.1	2.217	-2.11	1.91821	-2.171	0.01266
-2.0	5.000	-2.10	2.21700	-2.170	0.04506

47. For $x > 0$, $|x| = x$, so

$$\lim_{x\to 0^+} \frac{|x|}{x}(1-x) = \lim_{x\to 0+} \frac{x}{x}(1-x) = \lim_{x\to 0^+}(1-x) = 1 - 0 = \boxed{1}.$$

For $x < 0$, $|x| = -x$, so

$$\lim_{x\to 0^-} \frac{|x|}{x}(1-x) = \lim_{x\to 0-} \frac{-x}{x}(1-x) = \lim_{x\to 0^-}(x-1) = 0 - 1 = \boxed{-1}.$$

Because the two one-sided limits as x approaches 0 are not equal, $\boxed{\lim_{x\to 0} \frac{|x|}{x}(1-x) \text{ does not exist}}$.

48. $\lim_{x\to 2}\left(\frac{x^2}{x-2} - \frac{2x}{x-2}\right) = \lim_{x\to 2} \frac{x^2-2x}{x-2} = \lim_{x\to 2} \frac{x(x-2)}{x-2} = \lim_{x\to 2} x = \boxed{2}$.
Individually, the functions

$$\frac{x^2}{x-2} \quad \text{and} \quad \frac{2x}{x-2}$$

become unbounded as x approaches 2, so that neither of the individual limits exists. Because the individual limits do not exist, the Limit of a Sum property does not apply, and

$$\boxed{\lim_{x\to 2}\left(\frac{x^2}{x-2} - \frac{2x}{x-2}\right) \quad \text{is not given by} \quad \lim_{x\to 2}\frac{x^2}{x-2} - \lim_{x\to 2}\frac{2x}{x-2}}.$$

49. Let $f(x) = \sqrt{x}$. Then

$$\begin{aligned}\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} = \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} \\ &= \lim_{h\to 0} \frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{\sqrt{x}+\sqrt{x}} = \boxed{\frac{1}{2\sqrt{x}}}.\end{aligned}$$

50. To make

$$|(2x+1)-7| = |2x-6| = 2|x-3| < 0.01$$

requires that $|x-3| < 0.005$. Thus, the largest δ that "works" for $\epsilon = 0.01$ is $\boxed{\delta = 0.005}$.

51. $\lim_{x\to 0} \cos(\tan x) = \cos(\tan 0) = \cos 0 = \boxed{1}$.

52. $\lim_{x\to 0} \frac{\sin \frac{x}{4}}{x} = \lim_{x\to 0} \frac{\frac{1}{4}\sin\frac{x}{4}}{\frac{x}{4}} = \frac{1}{4}\lim_{x\to 0}\frac{\sin\frac{x}{4}}{\frac{x}{4}} = \frac{1}{4}\cdot 1 = \boxed{\frac{1}{4}}$.

53.

$$\begin{aligned}\lim_{x\to 0}\frac{\tan(3x)}{\tan(4x)} &= \lim_{x\to 0}\frac{\sin(3x)}{\sin(4x)}\cdot\frac{\cos(4x)}{\cos(3x)} = \lim_{x\to 0}\frac{\frac{3\sin(3x)}{3x}}{\frac{4\sin(4x)}{4x}}\cdot\lim_{x\to 0}\frac{\cos(4x)}{\cos(3x)}\\ &= \frac{3}{4}\frac{\lim\limits_{x\to 0}\dfrac{\sin(3x)}{3x}}{\lim\limits_{x\to 0}\dfrac{\sin(4x)}{4x}}\cdot\frac{\cos 0}{\cos 0} = \frac{3}{4}\cdot\frac{1}{1}\cdot\frac{1}{1} = \boxed{\frac{3}{4}}.\end{aligned}$$

54. $\displaystyle\lim_{x\to 0}\frac{\cos\frac{x}{3}-1}{x} = \lim_{x\to 0}\frac{\frac{1}{3}\left(\cos\frac{x}{3}-1\right)}{\frac{x}{3}} = \frac{1}{3}\lim_{x\to 0}\frac{\cos\frac{x}{3}-1}{\frac{x}{3}} = \frac{1}{3}\cdot 0 = \boxed{0}.$

55. $\displaystyle\lim_{x\to 0}\left(\frac{\cos x-1}{x}\right)^{10} = \left(\lim_{x\to 0}\frac{\cos x-1}{x}\right)^{10} = 0^{10} = \boxed{0}.$

56.

$$\begin{aligned}\lim_{x\to 0}\frac{e^{4x}-1}{e^x-1} &= \lim_{x\to 0}\frac{(e^{2x}-1)(e^{2x}+1)}{e^x-1} = \lim_{x\to 0}\frac{(e^x-1)(e^x+1)(e^{2x}+1)}{e^x-1}\\ &= \lim_{x\to 0}[(e^x+1)(e^{2x}+1)] = (e^0+1)(e^0+1) = \boxed{4}.\end{aligned}$$

57. As x approaches $\pi/2$ from the right, $\sin x$ approaches 1 and $\cos x$ approaches 0 from the left. Thus, $\tan x = \dfrac{\sin x}{\cos x}$ becomes unbounded in the negative direction, so that

$$\lim_{x\to\pi/2^+}\tan x = \boxed{-\infty}.$$

58. As x approaches -3, $2+x$ approaches -1 and $(x+3)^2$ approaches 0 from the right. Thus, $\dfrac{2+x}{(x+3)^2}$ becomes unbounded in the negative direction, so that

$$\lim_{x\to -3}\frac{2+x}{(x+3)^2} = \boxed{-\infty}.$$

59. $\displaystyle\lim_{x\to\infty}\frac{3x^3-2x+1}{x^3-8} = \lim_{x\to\infty}\frac{\frac{3x^3-2x+1}{x^3}}{\frac{x^3-8}{x^3}} = \lim_{x\to\infty}\frac{3-\frac{2}{x^2}+\frac{1}{x^3}}{1-\frac{8}{x^3}} = \frac{3-0+0}{1-0} = \boxed{3}.$

60. $\displaystyle\lim_{x\to\infty}\frac{3x^4+x}{2x^2} = \lim_{x\to\infty}\left(\frac{3}{2}x^2+\frac{1}{2x}\right) = \boxed{\infty}.$

61. The domain of the rational function $f(x) = \dfrac{4x-2}{x+3}$ is the set $\{x|x\neq -3\}$. The one-sided limits as x approaches -3 are

$$\lim_{x\to -3^-}\frac{4x-2}{x+3} = \infty \quad\text{and}\quad \lim_{x\to -3^+}\frac{4x-2}{x+3} = -\infty,$$

so $\boxed{x=-3 \text{ is a vertical asymptote of the graph of } f}$. Moreover, because

$$\lim_{x\to -\infty}\frac{4x-2}{x+3} = \lim_{x\to -\infty}\frac{\frac{4x-2}{x}}{\frac{x+3}{x}} = \lim_{x\to -\infty}\frac{4-\frac{2}{x}}{1+\frac{3}{x}} = \frac{4-0}{1+0} = 4$$

and

$$\lim_{x\to\infty}\frac{4x-2}{x+3}=\lim_{x\to\infty}\frac{\frac{4x-2}{x}}{\frac{x+3}{x}}=\lim_{x\to\infty}\frac{4-\frac{2}{x}}{1+\frac{3}{x}}=\frac{4-0}{1+0}=4,$$

$\boxed{y=4 \text{ is a horizontal asymptote of the graph of } f}$.

62. The domain of the rational function $f(x)=\dfrac{2x}{x^2-4}$ is the set $\{x|x\neq\pm2\}$. The one-sided limits as x approaches -2 are

$$\lim_{x\to-2^-}\frac{2x}{x^2-4}=-\infty \quad\text{and}\quad \lim_{x\to-2^+}\frac{2x}{x^2-4}=\infty,$$

so $\boxed{x=-2 \text{ is a vertical asymptote of the graph of } f}$. The one-sided limits as x approaches 2 are

$$\lim_{x\to2^-}\frac{2x}{x^2-4}=-\infty \quad\text{and}\quad \lim_{x\to2^+}\frac{2x}{x^2-4}=\infty,$$

so $\boxed{x=2 \text{ is also a vertical asymptote of the graph of } f}$. Moreover, because

$$\lim_{x\to-\infty}\frac{2x}{x^2-4}=\lim_{x\to-\infty}\frac{\frac{2x}{x^2}}{\frac{x^2-4}{x^2}}=\lim_{x\to-\infty}\frac{\frac{2}{x}}{1-\frac{4}{x^2}}=\frac{0}{1-0}=0$$

and

$$\lim_{x\to\infty}\frac{2x}{x^2-4}=\lim_{x\to\infty}\frac{\frac{2x}{x^2}}{\frac{x^2-4}{x^2}}=\lim_{x\to\infty}\frac{\frac{2}{x}}{1-\frac{4}{x^2}}=\frac{0}{1-0}=0,$$

$\boxed{y=0 \text{ is a horizontal asymptote of the graph of } f}$.

63. The function f is defined at 0 with $f(0)=1/2$. Because

$$\begin{aligned}\lim_{x\to0}f(x)&=\lim_{x\to0}\frac{\tan x}{2x}=\frac{1}{2}\lim_{x\to0}\frac{\sin x}{x}\cdot\frac{1}{\cos x}\\ &=\frac{1}{2}\lim_{x\to0}\frac{\sin x}{x}\cdot\lim_{x\to0}\frac{1}{\cos x}=\frac{1}{2}\cdot1\cdot\frac{1}{1}=\frac{1}{2},\end{aligned}$$

it follows that $\lim_{x\to0}f(x)$ exists and $f(0)=\lim_{x\to0}f(x)$. Therefore, f is $\boxed{\text{continuous}}$ at 0.

64. The function f is defined at 0 with $f(0)=1$. Because

$$\lim_{x\to0}f(x)=\lim_{x\to0}\frac{\sin(3x)}{x}=\lim_{x\to0}\frac{3\sin(3x)}{3x}=3\lim_{x\to0}\frac{\sin(3x)}{3x}=3\cdot1=3,$$

it follows that $\lim_{x\to0}f(x)$ exists but $f(0)\neq\lim_{x\to0}f(x)$. Therefore, f is $\boxed{\text{not continuous}}$ at 0.

65. Note that

$$\cos\left(\pi x+\frac{\pi}{2}\right)=\cos(\pi x)\cos\frac{\pi}{2}-\sin(\pi x)\sin\frac{\pi}{2}=\cos(\pi x)\cdot0-\sin(\pi x)\cdot1=-\sin(\pi x).$$

Thus,

$$\lim_{x\to0}f(x)=-\lim_{x\to0}\frac{\sin(\pi x)}{x}=-\lim_{x\to0}\frac{\pi\sin(\pi x)}{\pi x}=-\pi\lim_{x\to0}\frac{\sin(\pi x)}{\pi x}=-\pi\cdot1=-\pi.$$

To make f continuous at 0, define $\boxed{f(0)=-\pi}$.

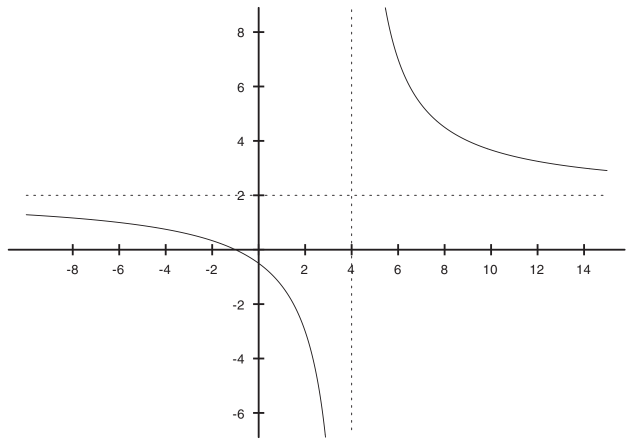
66. Take $\epsilon = 1$, let $\delta > 0$, and choose any x satisfying $0 < |x+3| < \delta$. Then

$$|(x^2-9)-(-18)| = |x^2+9| = x^2+9 \geq 9 > 1 = \epsilon,$$

so that $\lim_{x\to -3}(x^2-9) \neq -18$.

67. (a) Answers will vary. The figure below displays the graph of a function f with the properties

$$f(-1)=0, \quad \lim_{x\to\infty} f(x) = 2, \quad \lim_{x\to-\infty} f(x) = 2,$$
$$\lim_{x\to 4^-} f(x) = -\infty, \quad \text{and} \quad \lim_{x\to 4^+} f(x) = \infty.$$



(b) Answers will vary. The function shown above is $\boxed{f(x) = \dfrac{2x+2}{x-4}}$.

68. (a) Let $R(x) = \dfrac{2x^2-5x+2}{5x^2-x-2}$. Factoring the numerator yields

$$2x^2-5x+2 = (2x-1)(x-2).$$

Applying the quadratic formula to the equation $5x^2-x-2=0$ yields

$$x = \frac{1 \pm \sqrt{(-1)^2-4(5)(-2)}}{2(5)} = \frac{1\pm\sqrt{41}}{10}.$$

Therefore, the domain of R is the set $\boxed{\left\{x \middle| x \neq \dfrac{1\pm\sqrt{41}}{10}\right\}}$, the $\boxed{x\text{-intercepts are } \tfrac{1}{2} \text{ and } 2}$, and the $\boxed{y\text{-intercept is } R(0) = -1}$.

(b) The one-sided limits as x approaches $\dfrac{1-\sqrt{41}}{10} \approx -0.54$ are

$$\boxed{\lim_{x\to \frac{1-\sqrt{41}}{10}^-} R(x) = \infty \quad \text{and} \quad \lim_{x\to \frac{1-\sqrt{41}}{10}^+} R(x) = -\infty}.$$

Thus, the graph of R becomes unbounded in the positive direction as x approaches $\frac{1-\sqrt{41}}{10}$ from the left and becomes unbounded in the negative direction as x approaches $\frac{1-\sqrt{41}}{10}$ from the right. The one-sided limits as x approaches $\frac{1+\sqrt{41}}{10} \approx 0.74$ are

$$\boxed{\lim_{x \to \frac{1+\sqrt{41}}{10}^-} R(x) = \infty \quad \text{and} \quad \lim_{x \to \frac{1+\sqrt{41}}{10}^+} R(x) = -\infty}.$$

Thus, the graph of R becomes unbounded in the positive direction as x approaches $\frac{1+\sqrt{41}}{10}$ from the left and becomes unbounded in the negative direction as x approaches $\frac{1+\sqrt{41}}{10}$ from the right.

(c) Based on the limits from part (b), $\boxed{x = \frac{1-\sqrt{41}}{10} \text{ and } x = \frac{1+\sqrt{41}}{10} \text{ are both vertical asymptotes}}$ of the graph of R. Because

$$\lim_{x\to-\infty} \frac{2x^2-5x+2}{5x^2-x-2} = \lim_{x\to-\infty} \frac{\frac{2x^2-5x+2}{5x^2}}{\frac{5x^2-x-2}{5x^2}} = \lim_{x\to-\infty} \frac{\frac{2}{5}-\frac{1}{x}+\frac{2}{2x^2}}{1-\frac{1}{5x}-\frac{2}{5x^2}} = \frac{\frac{2}{5}-0+0}{1-0-0} = \frac{2}{5}$$

and

$$\lim_{x\to\infty} \frac{2x^2-5x+2}{5x^2-x-2} = \lim_{x\to\infty} \frac{\frac{2x^2-5x+2}{5x^2}}{\frac{5x^2-x-2}{5x^2}} = \lim_{x\to\infty} \frac{\frac{2}{5}-\frac{1}{x}+\frac{2}{2x^2}}{1-\frac{1}{5x}-\frac{2}{5x^2}} = \frac{\frac{2}{5}-0+0}{1-0-0} = \frac{2}{5},$$

$\boxed{y = \frac{2}{5} \text{ is a horizontal asymptote}}$ of the graph of R.

69. Because $1 - x^2 \le f(x) \le \cos x$ for all x in the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ containing 0 and

$$\lim_{x\to 0}(1-x^2) = 1 - 0 = 1 \quad \text{and} \quad \lim_{x\to 0} \cos x = \cos 0 = 1,$$

it follows from the Squeeze Theorem that

$$\lim_{x\to 0} f(x) = 1.$$