Instructor Solutions Manual: *Modern General Relativity*

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This document gives the solutions for all problems at the ends of chapters for the first edition of *Modern General Relativity: Black Holes, Gravitational Waves, and Cosmology* by Mike Guidry (Cambridge University Press, 2019). Unless otherwise indicated, literature references, equation numbers, figure references, table references, and section numbers refer to the print version of that book.

1 Introduction

1.1 From Eq. (1.2), the value of γ is infinite if $v = c$, so there is no Lorentz transformation to an inertial frame corresponding to a rest frame for light.

1.2 Since $E = m\gamma$, for a 7 TeV proton,

$$\gamma = \frac{E}{m} = \frac{7 \times 10^{12}\,\text{eV}}{938.3 \times 10^{6}\,\text{eV}} = 7460.$$

Then from the definition of γ,

$$\frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}} = 0.999999991.$$

This is a speed that is only about 3 meters per second less than that of light.

1.3 This question is ambiguous, since it does not specify whether the curvature is that of the surface itself (which is called *intrinsic curvature*) or whether it is the apparent curvature of the surface seen embedded in a higher-dimensional euclidean space (which is called the *extrinsic curvature*). In general relativity the curvature of interest is usually intrinsic curvature. Then the sheet of paper can be laid out flat and is not curved, the cylinder is *also flat,* with no intrinsic curvature, because one can imagine cutting it longitudinally and rolling it out into a flat surface, but the sphere has finite intrinsic curvature because it cannot be cut and rolled out flat without distortion. The reason that the cylinder seems to be curved is because the 2D surface is being viewed embedded in 3D space, which gives a non-zero *extrinsic curvature,* but if attention is confined only to the 2D surface it has no *intrinsic curvature*. This is a rather qualitative discussion but in later chapters methods will be developed to quantify the amount of intrinsic curvature for a surface.

2 Coordinate Systems and Transformations

2.1 Utilizing Eq. (2.31) to integrate around the circumference of the circle,

$$C = \oint ds = \oint (dx^2 + dy^2)^{1/2} = 2\int_{-R}^{+R} dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2},$$

subject to the constraint $R^2 = x^2 + y^2$, where the factor of two and the limits are because x ranges from $-R$ to $+R$ over half a circle. The constraints yield $dy/dx = -(R^2 - x^2)^{-1/2}x$, which permits the integral to be written as

$$C = 2\int_{-R}^{R} dx \sqrt{\frac{R^2}{R^2 - x^2}}.$$

Introducing a new integration variable a through $a \equiv x/R$ then gives

$$C = 2R\int_{-1}^{+1} \frac{da}{\sqrt{1-a^2}} = 2\pi R,$$

since the integral is $\sin^{-1} a$. In plane polar coordinates the line element is given by Eq. (2.32) and proceeding as above the circumference is

$$\begin{aligned} C &= \oint ds = \oint (dr^2 + r^2 d\varphi^2)^{1/2} \\ &= \int_0^{2\pi} d\varphi \sqrt{r^2 + \left(\frac{dr}{d\varphi}\right)^2} = R\int_0^{2\pi} d\varphi = 2\pi R, \end{aligned}$$

where $r = R$ has been used, implying that $dr/d\varphi = 0$.

2.2 Under a Galilean transformation $\boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{v}t$ and $t' = t$ it is clear that the acceleration $\boldsymbol{a}$ and the separation vector $\boldsymbol{r} = \Delta\boldsymbol{x}$ between two masses are unchanged. Thus the second law $\boldsymbol{F} = m\boldsymbol{a}$ and the gravitational law $\boldsymbol{F} = Gm_1m_2\hat{\mathbf{r}}/r^2$ are invariant under Galilean transformations.

2.3 Our solution follows Example 1.2.1 of Foster and Nightingale [88]. The tangent and dual basis vectors, and the products for $g_{ij} = g_{ji} = \boldsymbol{e}_i \cdot \boldsymbol{e}_j$, were worked out in Example 2.3. The elements for $g^{ij} = g^{ji} = \boldsymbol{e}^i \cdot \boldsymbol{e}^j$ can be obtained in a similar fashion. For example,

$$g^{12} = g^{21} = (\tfrac{1}{2}\boldsymbol{i} + \tfrac{1}{2}\boldsymbol{j}) \cdot (\tfrac{1}{2}\boldsymbol{i} - \tfrac{1}{2}\boldsymbol{j}) = \tfrac{1}{4} - \tfrac{1}{4} = 0,$$

where the orthonormality of the cartesian basis vectors has been used. Summarizing the results,

$$g_{ij} = \begin{pmatrix} 4v^2+2 & 4uv & 2v \\ 4uv & 4u^2+2 & 2u \\ 2v & 2u & 1 \end{pmatrix} \qquad g^{ij} = \begin{pmatrix} \frac{1}{2} & 0 & -v \\ 0 & \frac{1}{2} & -u \\ -v & -u & 2u^2+2v^2+1 \end{pmatrix}$$

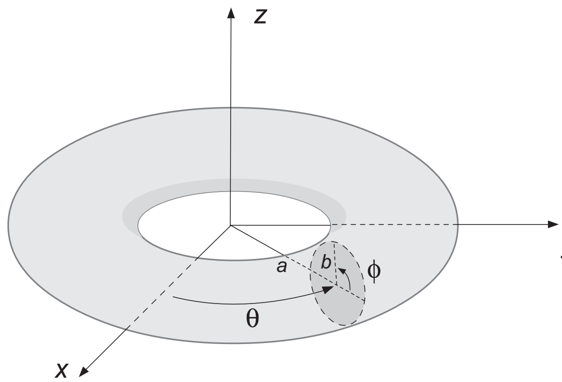

Fig. 2.1 Figure for Problem 2.5.

By direct multiplication the product of these two matrices is the unit matrix, verifying Eq. (2.26) explicitly for this case. Utilizing Eq. (2.29), the line element is

$$\begin{aligned} ds^2 &= g_{ij}du^i du^j \\ &= g_{uu}du^2 + 2g_{uv}dudv + 2g_{uw}dudw + g_{vv}dv^2 + 2g_{vw}dvdw + g_{ww}dw^2 \\ &= (4v^2+2)du^2 + 8uvdudv + 4vdudw + (4u^2+2)dv^2 + 4udvdw + dw^2 \end{aligned}$$

where $g_{ij} = g_{ji}$ has been used and no summation is implied by repeated indices.

2.4 Using the spherical coordinates

$$u^1 = r \qquad u^2 = \theta \qquad u^3 = \varphi$$

defined through Eq. (2.2) and the results of Example 2.2,

$$\boldsymbol{e}_1 \cdot \boldsymbol{e}_1 = 1 \qquad \boldsymbol{e}_2 \cdot \boldsymbol{e}_2 = r^2 \qquad \boldsymbol{e}_3 \cdot \boldsymbol{e}_3 = r^2 \sin^2\theta,$$

while all non-diagonal components vanish. Thus the metric tensor is

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}.$$

The corresponding line element is

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2,$$

where Eq. (2.29) has been used.

2.5 This solution is based on Problem 1.2 in Ref. [88]. From the parameterization $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ with

$$x = (a + b\cos\varphi)\cos\theta \qquad y = (a + b\cos\varphi)\sin\theta \qquad z = b\sin\varphi,$$

where the radius of the doughnut a and radius of the circle b are defined in Fig. 2.1 [this document], the tangent basis vectors are

$$\boldsymbol{e}_\theta = \frac{\partial \boldsymbol{r}}{\partial \theta} = -\sin\theta(a+b\cos\varphi)\,\boldsymbol{i} + (a+b\cos\varphi)\cos\theta\,\boldsymbol{j}$$
$$\boldsymbol{e}_\varphi = \frac{\partial \boldsymbol{r}}{\partial \varphi} = -(b\sin\varphi\cos\theta)\,\boldsymbol{i} - (b\sin\varphi\sin\theta)\,\boldsymbol{j} + (b\cos\varphi)\,\boldsymbol{k}.$$

The corresponding elements of the metric tensor $g_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{e}_j$ are

$$g_{\varphi\varphi} = b^2 \qquad g_{\varphi\theta} = g_{\theta\varphi} = 0 \qquad g_{\theta\theta} = (a+b\cos\varphi)^2.$$

2.6 The tangent basis vectors and metric tensor g_{ij} were given in Example 2.4. Since g^{ij} is the matrix inverse of g_{ij}, which is diagonal,

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \qquad \longrightarrow \qquad g^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1/r^2 \end{pmatrix}$$

Then the dual basis may be obtained by raising indices with the metric tensor: $\boldsymbol{e}^i = g^{ij}\boldsymbol{e}_j$, giving

$$\boldsymbol{e}^1 = g^{11}\boldsymbol{e}_1 + g^{12}\boldsymbol{e}_2 = \boldsymbol{e}_1 \qquad \boldsymbol{e}^2 = g^{21}\boldsymbol{e}_1 + g^{22}\boldsymbol{e}_2 = \frac{1}{r^2}\boldsymbol{e}_2$$

for the elements of the dual basis.

2.7 For a constant displacement d in the x direction

$$x' = x - d \qquad y' = y \qquad z' = z.$$

Since d is constant

$$dx' = dx \qquad dy' = dy \qquad dz' = dz$$

and therefore $ds'^2 = ds^2$. From Eq. (2.41), a rotation in the $x-y$ plane may be written

$$x' = x\cos\theta + y\sin\theta \qquad y' = -x\sin\theta + y\cos\theta \qquad z' = z,$$

which gives the transformed line element

$$\begin{aligned} ds'^2 &= (dx')^2 + (dy')^2 + (dz')^2 \\ &= (\cos\theta dx + \sin\theta dy)^2 + (-\sin\theta dx + \cos\theta dy)^2 + dz^2 \\ &= (\cos^2\theta + \sin^2\theta)dx^2 + (\cos^2\theta + \sin^2\theta)dy^2 + dz^2 \\ &= dx^2 + dy^2 + dz^2 \\ &= ds^2. \end{aligned}$$

Therefore the euclidean spatial line element is invariant under displacements by a constant amount and under rotations.

2.8 Taking the scalar products using Eqs. (2.8), (2.9), and (2.20) gives

$$\boldsymbol{e}^i \cdot \boldsymbol{V} = \boldsymbol{e}^i \cdot (V^j \boldsymbol{e}_j) = V^j \boldsymbol{e}^i \cdot \boldsymbol{e}_j = V^j \delta^i_j = V^i,$$
$$\boldsymbol{e}_i \cdot \boldsymbol{V} = \boldsymbol{e}_i \cdot (V_j \boldsymbol{e}^j) = V_j \boldsymbol{e}_i \cdot \boldsymbol{e}^j = V_j \delta^j_i = V_i,$$

which is Eq. (2.22).

2.9 Utilizing that the angle θ between the basis vectors is determined by $\cos\theta = \boldsymbol{e}_1 \cdot \boldsymbol{e}_2/|\boldsymbol{e}_1||\boldsymbol{e}_2|$, the area of the parallelogram is

$$\begin{aligned} dA &= |\boldsymbol{e}_1||\boldsymbol{e}_2| \sin\theta \, dx^1 dx^2 \\ &= |\boldsymbol{e}_1||\boldsymbol{e}_2|(1-\cos^2\theta)^{1/2} dx^1 dx^2 \\ &= \left(|\boldsymbol{e}_1|^2|\boldsymbol{e}_2|^2 - (\boldsymbol{e}_1 \cdot \boldsymbol{e}_2)^2\right)^{1/2} dx^1 dx^2. \end{aligned}$$

The components of the metric tensor g_{ij} are

$$\boldsymbol{e}_1 \cdot \boldsymbol{e}_2 = g_{12} = g_{21} \qquad |\boldsymbol{e}_1||\boldsymbol{e}_1| = \boldsymbol{e}_1 \cdot \boldsymbol{e}_1 = g_{11} \qquad |\boldsymbol{e}_2||\boldsymbol{e}_2| = \boldsymbol{e}_2 \cdot \boldsymbol{e}_2 = g_{22},$$

so the area of the parallelogram may be expressed as

$$dA = \left(g_{11}g_{22} - g_{12}^2\right)^{1/2} dx^1 dx^2 = \sqrt{\det g}\, dx^1 dx^2,$$

where $\det g$ is the determinant of the metric tensor. This is the 2D version of the invariant 4D volume element given in Eq. (3.48).

3 Tensors and Covariance

3.1 For the three cases

$$T'^{\mu\nu} = V'^{\mu}V'^{\nu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}}\frac{\partial x'^{\nu}}{\partial x^{\beta}}V^{\alpha}V^{\beta} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}}\frac{\partial x'^{\nu}}{\partial x^{\beta}}T^{\alpha\beta}$$

$$T'_{\mu\nu} = V'_{\mu}V'_{\nu} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial x^{\beta}}{\partial x'^{\nu}}V_{\alpha}V_{\beta} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial x^{\beta}}{\partial x'^{\nu}}T_{\alpha\beta}$$

$$T'^{\nu}_{\mu} = V'_{\mu}V'^{\nu} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial x'^{\nu}}{\partial x^{\beta}}V_{\alpha}V^{\beta} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial x'^{\nu}}{\partial x^{\beta}}T_{\alpha}{}^{\beta}.$$

3.2 From Eqs. (3.50) and (3.51) with indices suitably relabeled

$$\begin{aligned}
A'_{\mu,\nu} - \Gamma'^{\lambda}_{\mu\nu}A'_{\lambda} &= A_{\alpha,\beta}\frac{\partial x^{\beta}}{\partial x'^{\nu}}\frac{\partial x^{\alpha}}{\partial x'^{\mu}} + A_{\alpha}\frac{\partial^2 x^{\alpha}}{\partial x'^{\nu}\partial x'^{\mu}} \\
&\quad - \left(\Gamma^{\kappa}_{\alpha\beta}\frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial x^{\beta}}{\partial x'^{\nu}}\frac{\partial x'^{\lambda}}{\partial x^{\kappa}} + \frac{\partial^2 x^{\alpha}}{\partial x'^{\mu}\partial x'^{\nu}}\frac{\partial x'^{\lambda}}{\partial x^{\alpha}}\right)\frac{\partial x^{\gamma}}{\partial x'^{\lambda}}A_{\gamma} \\
&= A_{\alpha,\beta}\frac{\partial x^{\beta}}{\partial x'^{\nu}}\frac{\partial x^{\alpha}}{\partial x'^{\mu}} + A_{\alpha}\frac{\partial^2 x^{\alpha}}{\partial x'^{\nu}\partial x'^{\mu}} \\
&\quad - \Gamma^{\kappa}_{\alpha\beta}\frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial x^{\beta}}{\partial x'^{\nu}}\frac{\partial x'^{\lambda}}{\partial x^{\kappa}}\frac{\partial x^{\gamma}}{\partial x'^{\lambda}}A_{\gamma} - \frac{\partial^2 x^{\alpha}}{\partial x'^{\mu}\partial x'^{\nu}}\frac{\partial x'^{\lambda}}{\partial x^{\alpha}}\frac{\partial x^{\gamma}}{\partial x'^{\lambda}}A_{\gamma} \\
&= A_{\alpha,\beta}\frac{\partial x^{\beta}}{\partial x'^{\nu}}\frac{\partial x^{\alpha}}{\partial x'^{\mu}} + A_{\alpha}\frac{\partial^2 x^{\alpha}}{\partial x'^{\nu}\partial x'^{\mu}} \\
&\quad - \Gamma^{\kappa}_{\alpha\beta}\frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial x^{\beta}}{\partial x'^{\nu}}A_{\kappa} - A_{\alpha}\frac{\partial^2 x^{\alpha}}{\partial x'^{\nu}\partial x'^{\mu}} \\
&= A_{\alpha,\beta}\frac{\partial x^{\beta}}{\partial x'^{\nu}}\frac{\partial x^{\alpha}}{\partial x'^{\mu}} - \Gamma^{\kappa}_{\alpha\beta}\frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial x^{\beta}}{\partial x'^{\nu}}A_{\kappa} \\
&= \left(A_{\alpha,\beta} - \Gamma^{\kappa}_{\alpha\beta}A_{\kappa}\right)\frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial x^{\beta}}{\partial x'^{\nu}},
\end{aligned}$$

which is Eq. (3.52).

3.3 (a) Since δ^{ν}_{μ} is a rank-2 tensor with the same components in all coordinate systems (see Section 3.8), under a coordinate transformation $g_{\mu\alpha}g^{\alpha\nu} = \delta^{\nu}_{\mu}$ becomes $g'_{\mu\alpha}g'^{\alpha\nu} = \delta^{\nu}_{\mu}$. Since $g_{\mu\nu}$ is a tensor, if we assume $g^{\mu\nu}$ is also a tensor then

$$g'_{\mu\alpha} = \frac{\partial x^{\kappa}}{\partial x'^{\mu}}\frac{\partial x^{\eta}}{\partial x'^{\alpha}}g_{\kappa\eta}. \qquad g'^{\alpha\nu} = \frac{\partial x'^{\alpha}}{\partial x^{\rho}}\frac{\partial x'^{\nu}}{\partial x^{\sigma}}g^{\rho\sigma}.$$

Then evaluating $g'_{\mu\alpha}g'^{\alpha\nu}$,

$$g'_{\mu\alpha}\frac{\partial x'^{\alpha}}{\partial x^{\rho}}\frac{\partial x'^{\nu}}{\partial x^{\sigma}}g^{\rho\sigma} = \frac{\partial x^{\kappa}}{\partial x'^{\mu}}\frac{\partial x^{\eta}}{\partial x'^{\alpha}}g_{\kappa\eta}\frac{\partial x'^{\alpha}}{\partial x^{\rho}}\frac{\partial x'^{\nu}}{\partial x^{\sigma}}g^{\rho\sigma} = \frac{\partial x^{\sigma}}{\partial x'^{\mu}}\frac{\partial x'^{\nu}}{\partial x^{\sigma}} = \delta^{\nu}_{\mu},$$

where we have used

$$\frac{\partial x^\eta}{\partial x'^\alpha}\frac{\partial x'^\alpha}{\partial x^\rho}=\delta^\eta_\rho \qquad g_{\kappa\rho}g^{\rho\sigma}=\delta^\sigma_\kappa .$$

Comparing the result

$$g'_{\mu\alpha}\frac{\partial x'^\alpha}{\partial x^\rho}\frac{\partial x'^\nu}{\partial x^\sigma}g^{\rho\sigma}=\delta^\nu_\mu$$

with $g'_{\mu\alpha}g'^{\alpha\nu}=\delta^\nu_\mu$ requires that

$$g'^{\alpha\nu}=\frac{\partial x'^\alpha}{\partial x^\rho}\frac{\partial x'^\nu}{\partial x^\sigma}g^{\rho\sigma}.$$

which is the transformation law for a rank-2 contravariant tensor. Note that this result is an example of the quotient theorem described in Problem 3.13. Since $g_{\mu\alpha}g^{\alpha\nu}=\delta^\nu_\mu$ and $g_{\mu\nu}$ and δ^ν_μ are known to be tensors, $g^{\mu\nu}$ must also be a tensor.

(b) From Eq. (3.44) an arbitrary rank-2 tensor can be decomposed into a symmetric and antisymmetric part,

$$g_{\mu\nu}=\tfrac{1}{2}(g_{\mu\nu}+g_{\nu\mu})+\tfrac{1}{2}(g_{\mu\nu}-g_{\nu\mu}).$$

Inserting this in the line element gives

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu \\ &= \tfrac{1}{2}(g_{\mu\nu}+g_{\nu\mu})dx^\mu dx^\nu+\tfrac{1}{2}(g_{\mu\nu}-g_{\nu\mu})dx^\mu dx^\nu \\ &= \left[g_{\mu\nu}+\tfrac{1}{2}(g_{\nu\mu}-g_{\nu\mu})\right]dx^\mu dx^\nu \\ &= g_{\mu\nu}dx^\mu dx^\nu . \end{aligned}$$

Thus only the symmetric part of $g_{\mu\nu}$ contributes to the line element.

3.4 Under the transformation $x\to x'$,

$$\begin{aligned} T'^{\mu}{}_{\nu} = g'_{\nu\alpha}T'^{\mu\alpha} &= \frac{\partial x^\alpha}{\partial x'^\nu}\frac{\partial x^\beta}{\partial x'^\alpha}g_{\alpha\beta}\frac{\partial x'^\mu}{\partial x^\gamma}\frac{\partial x'^\alpha}{\partial x^\sigma}T^{\gamma\delta} \\ &= g_{\alpha\delta}T^{\gamma\delta}\frac{\partial x^\alpha}{\partial x'^\nu}\frac{\partial x'^\mu}{\partial x^\gamma}=T^{\gamma}{}_{\alpha}\frac{\partial x^\alpha}{\partial x'^\nu}\frac{\partial x'^\mu}{\partial x^\gamma}, \end{aligned}$$

where in going from the first line to the second line

$$\frac{\partial x^\beta}{\partial x'^\alpha}\frac{\partial x'^\alpha}{\partial x^\delta}=\delta^\beta_\delta$$

has been used. This is a tensor transformation law so it is valid in all frames. Proceeding in similar fashion,

$$\begin{aligned} T'_{\mu\nu} &= g'_{\mu\alpha}g'_{\nu\beta}T'^{\alpha\beta}=\frac{\partial x^\varepsilon}{\partial x'^\mu}\frac{\partial x^\lambda}{\partial x'^\alpha}g_{\varepsilon\lambda}\frac{\partial x^\gamma}{\partial x'^\nu}\frac{\partial x^\delta}{\partial x'^\beta}g_{\gamma\delta}\frac{\partial x'^\alpha}{\partial x^\tau}\frac{\partial x'^\beta}{\partial x^\theta}T^{\tau\theta} \\ &= \frac{\partial x^\varepsilon}{\partial x'^\mu}\frac{\partial x^\gamma}{\partial x'^\nu}T_{\varepsilon\gamma}, \end{aligned}$$

where in the last step

$$\frac{\partial x^\lambda}{\partial x'^\alpha}\frac{\partial x'^\alpha}{\partial x^\tau}=\delta^\lambda_\tau \qquad \frac{\partial x^\delta}{\partial x'^\beta}\frac{\partial x'^\beta}{\partial x^\theta}=\delta^\delta_\theta \qquad g_{\varepsilon\lambda}g_{\gamma\delta}T^{\lambda\delta}=T_{\varepsilon\gamma}$$

have been used. This is a tensor transformation law so it is valid in all frames.

3.5 (a) For example, consider a rank-4 tensor $T_\beta{}^{\mu\nu\alpha}$. Its transformation law is

$$T'^{\mu\nu\alpha}_{\beta} = \frac{\partial x'^\mu}{\partial x^\gamma}\frac{\partial x'^\nu}{\partial x^\delta}\frac{\partial x'^\alpha}{\partial x^\varepsilon}\frac{\partial x^\eta}{\partial x'^\beta}T_\eta{}^{\gamma\delta\varepsilon}.$$

Now set $\alpha=\beta$ for this tensor (implying a sum on this index). The resulting quantity must have two upper indices by the summation convention, so define it to be $T^{\mu\nu}$:

$$T^{\mu\nu} \equiv \delta^\beta_\alpha T_\beta{}^{\mu\nu\alpha} = T_\alpha{}^{\mu\nu\alpha}.$$

Is $T^{\mu\nu}$ a tensor? From the preceding equations, its transformation law is

$$\begin{aligned} T'^{\mu\nu} &\equiv T'^{\mu\nu\alpha}_{\alpha} = \delta^\beta_\alpha T'^{\mu\nu\alpha}_{\beta} \\ &= \delta^\beta_\alpha \frac{\partial x'^\mu}{\partial x^\gamma}\frac{\partial x'^\nu}{\partial x^\delta}\frac{\partial x'^\alpha}{\partial x^\varepsilon}\frac{\partial x^\eta}{\partial x'^\beta}T_\eta{}^{\gamma\delta\varepsilon} = \frac{\partial x'^\mu}{\partial x^\gamma}\frac{\partial x'^\nu}{\partial x^\delta}\frac{\partial x'^\alpha}{\partial x^\varepsilon}\frac{\partial x^\eta}{\partial x'^\alpha}T_\eta{}^{\gamma\delta\varepsilon} \\ &= \frac{\partial x'^\mu}{\partial x^\gamma}\frac{\partial x'^\nu}{\partial x^\delta}\delta^\eta_\varepsilon T_\eta{}^{\gamma\delta\varepsilon} = \frac{\partial x'^\mu}{\partial x^\gamma}\frac{\partial x'^\nu}{\partial x^\delta}T_\eta{}^{\gamma\delta\eta} = \frac{\partial x'^\mu}{\partial x^\gamma}\frac{\partial x'^\nu}{\partial x^\delta}T^{\gamma\delta}, \end{aligned}$$

which is the transformation law for a contravariant rank-2 tensor. Similar proofs can be carried out for tensors of any order. Thus, setting an upper and lower index equal on a rank-N tensor and summing yields a tensor of rank $N-2$.

(b) For example, consider the linear combination of two rank-2 tensors, $T_\mu{}^\nu = aA_\mu{}^\nu + bB_\mu{}^\nu$. The transformation law is

$$\begin{aligned} T'^{\nu}_{\mu} &= aA'^{\nu}_{\mu} + bB'^{\nu}_{\mu} = a\frac{\partial x'^\nu}{\partial x^\alpha}\frac{\partial x^\beta}{\partial x'^\mu}A_\beta{}^\alpha + b\frac{\partial x'^\nu}{\partial x^\alpha}\frac{\partial x^\beta}{\partial x'^\mu}B_\beta{}^\alpha \\ &= \frac{\partial x'^\nu}{\partial x^\alpha}\frac{\partial x^\beta}{\partial x'^\mu}\left(aA_\beta{}^\alpha + bB_\beta{}^\alpha\right) = \frac{\partial x'^\nu}{\partial x^\alpha}\frac{\partial x^\beta}{\partial x'^\mu}T_\beta{}^\alpha. \end{aligned}$$

A similar proof holds for any such linear combination of tensors.

3.6 The line element is $ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2$, so the non-zero components of the metric are

$$g_{00} = g_{tt} = -1 \quad g_{11} = g_{rr} = 1 \quad g_{22} = g_{\theta\theta} = r^2 \quad g_{33} = g_{\varphi\varphi} = r^2\sin^2\theta$$

and $\det g_{\mu\nu} = -r^4\sin^2\theta$. Then from Eq. (3.48) the invariant volume element is

$$dV = (-\det g_{\mu\nu})^{1/2} dr\, d\theta\, d\varphi = r^2 dr \sin\theta d\theta d\varphi,$$

which gives a volume

$$V = \int dV = \int_0^R r^2 dr \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\varphi = \tfrac{4}{3}\pi R^3,$$

as expected.

3.7 Since $A\cdot B = A_\mu B^\mu$ is a scalar it is unchanged by a coordinate transformation. Thus from the vector transformation law for B^μ

$$A'_\mu B'^\mu = A_\mu B^\mu = A_\nu \frac{\partial x^\nu}{\partial x'^\mu}B'^\mu \quad \longrightarrow \quad \left(A_\nu\frac{\partial x^\nu}{\partial x'^\mu} - A'_\mu\right)B'^\mu = 0.$$

But B'^{μ} is an arbitrary vector that does not generally vanish. Thus the quantity in parentheses must be equal to zero, implying that $A'_{\mu} = \left(\partial x^{\nu}/\partial x'^{\mu}\right) A_{\nu}$, which is the transformation law for a dual vector.

3.8 This problem is adapted from an example in Ref. [88]. From the transformation equations between spherical and cylindrical coordinates assuming $u = (r, \theta, \varphi)$ and $u' = (\rho, \varphi, z)$,

$$\begin{aligned} u'^1 &= \rho = r\sin\theta = u^1 \sin u^2 \\ u'^2 &= \varphi = u^3 \\ u'^3 &= z = r\cos\theta = u^1 \cos u^2 \end{aligned}$$

and the inverse transformations are

$$\begin{aligned} u^1 &= r = \sqrt{\rho^2 + z^2} = \sqrt{(u'^1)^2 + (u'^3)^2} \\ u^2 &= \theta = \tan^{-1}\left(\frac{\rho}{z}\right) = \tan^{-1}\left(\frac{u'^1}{u'^3}\right) \\ u^3 &= \varphi = u'^2. \end{aligned}$$

From these the partial derivative entries in the matrices U and $\hat{U}$ defined in Example 3.7 may be computed directly. For example,

$$\begin{aligned} U^1_2 &= \frac{\partial u'^1}{\partial u^2} = \frac{\partial}{\partial u^2}(u^1 \sin u^2) = u^1 \cos u^2 = r\cos\theta \\ \hat{U}^2_1 &= \frac{\partial u^2}{\partial u'^1} = \frac{\partial}{\partial u'^1}\left[\tan^{-1}\left(\frac{u'^1}{u'^3}\right)\right] = \frac{u'^3}{(u'^1)^2 + (u'^3)^2} = \frac{\cos\theta}{r}. \end{aligned}$$

Computing all the derivatives and assembling them gives

$$U = \begin{pmatrix} \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix} \qquad \hat{U} = \begin{pmatrix} \sin\theta & 0 & \cos\theta \\ \dfrac{\cos\theta}{r} & 0 & -\dfrac{\sin\theta}{r} \\ 0 & 1 & 0 \end{pmatrix},$$

and by explicit matrix multiplication, $\hat{U}U = I$.

3.9 From Eqs. (3.45) and (3.46),

$$\begin{aligned} T_{[\alpha\beta](\gamma\delta)} &= \frac{1}{2}\left(T_{\alpha\beta(\gamma\delta)} - T_{\beta\alpha(\gamma\delta)}\right) \\ &= \frac{1}{2}\left(\tfrac{1}{2}(T_{\alpha\beta\gamma\delta} + T_{\alpha\beta\delta\gamma}) - \tfrac{1}{2}(T_{\beta\alpha\gamma\delta} + T_{\beta\alpha\delta\gamma})\right) \\ &= \frac{1}{4}\left(T_{\alpha\beta\gamma\delta} + T_{\alpha\beta\delta\gamma} - T_{\beta\alpha\gamma\delta} - T_{\beta\alpha\delta\gamma}\right). \end{aligned}$$

3.10 (a) Use the symmetry properties and relabeling of dummy indices to write,

$$\begin{aligned} A^{\mu\nu}B_{\mu\nu} &= -A^{\nu\mu}B_{\mu\nu} && (A^{\mu\nu} \text{ is antisymmetic}) \\ &= -A^{\nu\mu}B_{\nu\mu} && (B_{\mu\nu} \text{ is symmetric}) \\ &= -A^{\mu\nu}B_{\mu\nu} && (\text{Interchange dummy indices } \mu \leftrightarrow \nu). \end{aligned}$$

But $A^{\mu\nu}B_{\mu\nu} = -A^{\mu\nu}B_{\mu\nu}$ can be true only if $A^{\mu\nu}B_{\mu\nu} = 0$.
(b) For example, if $A^{\mu\nu}$ is symmetric, $A^{\mu\nu} = A^{\nu\mu}$, then

$$A'^{\mu\nu} = \frac{\partial x'^{\mu}}{\partial x^{\gamma}}\frac{\partial x'^{\nu}}{\partial x^{\delta}}A^{\gamma\delta} = \frac{\partial x'^{\mu}}{\partial x^{\gamma}}\frac{\partial x'^{\nu}}{\partial x^{\delta}}A^{\delta\gamma} = A'^{\nu\mu},$$

with an analogous proof if $A^{\mu\nu} = -A^{\nu\mu}$.

3.11 Contracting δ^{μ}_{ν} with the components V^{ν} of an arbitrary vector gives

$$\delta^{\mu}_{\nu}V^{\nu} = V^{\mu} = g^{\mu\alpha}V_{\alpha} = g^{\mu\alpha}g_{\alpha\nu}V^{\nu}.$$

But V is arbitrary so $g^{\mu\alpha}g_{\alpha\nu} = \delta^{\mu}_{\nu}$.

3.12 Multiply both sides of $T_{\mu\nu} = U_{\mu\nu}$ by $\partial x^{\mu}/\partial x'^{\alpha}$ and $\partial x^{\nu}/\partial x'^{\beta}$ and take the implied sums to give

$$\frac{\partial x^{\mu}}{\partial x'^{\alpha}}\frac{\partial x^{\nu}}{\partial x'^{\beta}}T_{\mu\nu} = \frac{\partial x^{\mu}}{\partial x'^{\alpha}}\frac{\partial x^{\nu}}{\partial x'^{\beta}}U_{\mu\nu}.$$

But from Eq. (3.36) this is just $T'_{\mu\nu} = U'_{\mu\nu}$.

3.13 In the scalar product expression $A\cdot B = g_{\mu\nu}A^{\mu}B^{\nu}$ of Eq. (3.43) the left side is a scalar and A and B on the right side are vectors. Since the quantities $g_{\mu\nu}$ contracted with tensors on the right side yield a tensor on the left side, by the quotient theorem $g_{\mu\nu}$ must define the components of a type $(0,2)$ tensor.

3.14 This solution is adapted from Example 1.8.1 in Ref. [88]. For an arbitrary contravariant vector V^{γ} the transformation law given in the problem is

$$T^{\alpha}{}_{\beta\gamma}V^{\gamma} = \frac{\partial x'^{\alpha}}{\partial x^{\delta}}\frac{\partial x^{\varepsilon}}{\partial x'^{\beta}}T^{\delta}{}_{\varepsilon\varphi}V^{\varphi},$$

indicating that $T^{\alpha}{}_{\beta\gamma}V^{\gamma}$ transforms as a $(1,1)$ tensor. By the quotient theorem then $T^{\alpha}{}_{\beta\gamma}$ must be a $(1,2)$ tensor. The proof follows from inserting $V^{\gamma} = (\partial x'^{\gamma}/\partial x^{\varphi})V^{\varphi}$ on the left side of the above equation and rearranging to give

$$\left(T^{\alpha}{}_{\beta\gamma}\frac{\partial x'^{\gamma}}{\partial x^{\varphi}} - \frac{\partial x'^{\alpha}}{\partial x^{\delta}}\frac{\partial x^{\varepsilon}}{\partial x'^{\beta}}T^{\delta}{}_{\varepsilon\varphi}\right)V^{\varphi} = 0.$$

This must be valid for any V^{φ} so choose $V_{\varphi} = \delta^{\varphi}_{\lambda}$ such that the quantity inside the parentheses is required to vanish, giving

$$T^{\alpha}{}_{\beta\gamma}\frac{\partial x'^{\gamma}}{\partial x^{\lambda}} = \frac{\partial x'^{\alpha}}{\partial x^{\delta}}\frac{\partial x^{\varepsilon}}{\partial x'^{\beta}}T^{\delta}{}_{\varepsilon\lambda}.$$

Multiply both sides of this expression by $\partial x^\lambda / \partial x'^\mu$ to give

$$T^{\alpha}{}_{\beta\gamma} \frac{\partial x'^\gamma}{\partial x^\lambda} \frac{\partial x^\lambda}{\partial x'^\mu} = \frac{\partial x'^\alpha}{\partial x^\delta} \frac{\partial x^\varepsilon}{\partial x'^\beta} \frac{\partial x^\lambda}{\partial x'^\mu} T^{\delta}{}_{\varepsilon\lambda}.$$

But on the left side

$$\frac{\partial x'^\gamma}{\partial x^\lambda} \frac{\partial x^\lambda}{\partial x'^\mu} = \delta^\gamma_\mu,$$

giving finally

$$T^{\alpha}{}_{\beta\mu} = \frac{\partial x'^\alpha}{\partial x^\delta} \frac{\partial x^\varepsilon}{\partial x'^\beta} \frac{\partial x^\lambda}{\partial x'^\mu} T^{\delta}{}_{\varepsilon\lambda},$$

which is the transformation law obeyed by a $(1,2)$ tensor.

3.15 (a) One may write

$$\delta^\nu_\mu \frac{\partial x'^\alpha}{\partial x^\nu} \frac{\partial x^\mu}{\partial x'^\beta} = \frac{\partial x'^\alpha}{\partial x^\mu} \frac{\partial x^\mu}{\partial x'^\beta} = \frac{\partial x'^\alpha}{\partial x'^\beta} = \delta'^\alpha_\beta$$

which is the transformation law for a mixed, rank-2 tensor.

(b) In some coordinate system let $K^\nu_\mu = \delta^\nu_\mu = \text{diag}\,(1,1,1,1)$. Then under an arbitrary coordinate transformation,

$$K'^\nu_\mu = \frac{\partial x'^\nu}{\partial x^\alpha} \frac{\partial x^\beta}{\partial x'^\mu} K^\alpha_\beta = \frac{\partial x'^\nu}{\partial x^\alpha} \frac{\partial x^\beta}{\partial x'^\mu} \delta^\alpha_\beta = \frac{\partial x'^\nu}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial x'^\mu} = \delta^\nu_\mu.$$

Thus $K^\nu_\mu = \delta^\nu_\mu$ is a $(1,1)$ tensor that has the same components (those of the unit matrix) in any coordinate system.

3.16 This is a particular example of a scalar product, so it must transform as a scalar. Explicitly,

$$\begin{aligned}
ds'^2 &= g_{\alpha\beta} \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} \frac{\partial x'^\mu}{\partial x^\gamma} ds^\gamma \frac{\partial x'^\nu}{\partial x^\delta} ds^\delta \\
&= g_{\alpha\beta} ds^\gamma ds^\delta \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x'^\mu}{\partial x^\gamma} \frac{\partial x^\beta}{\partial x'^\nu} \frac{\partial x'^\nu}{\partial x^\delta} \\
&= g_{\alpha\beta} ds^\gamma ds^\delta \frac{\partial x^\alpha}{\partial x^\gamma} \frac{\partial x^\beta}{\partial x^\delta} = g_{\alpha\beta} ds^\gamma ds^\delta \delta^\alpha_\gamma \delta^\beta_\delta \\
&= g_{\alpha\beta} ds^\alpha ds^\beta = ds^2
\end{aligned}$$

where Eq. (3.35) has been used. The squared line element (3.39) is clearly a scalar invariant and so it has the same value in all coordinate systems.

3.17 By the usual rank-2 tensor transformation law,

$$T'^{\mu\nu}(x') = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} T^{\alpha\beta}(x).$$

Upon differentiating Eq. (3.66),

$$\frac{\partial x'^\mu}{\partial x^\alpha} = \delta^\mu_\alpha + (\delta u)\, \partial_\alpha X^\mu(x),$$

which may be substituted into the first equation to give

$$\begin{aligned} T'^{\mu\nu}(x') &= \left(\delta^\mu_\alpha + (\delta u)\partial_\alpha X^\mu\right)\left(\delta^\nu_\beta + (\delta u)\partial_\beta X^\nu\right) T^{\alpha\beta}(x) \\ &= \left(\delta^\mu_\alpha \delta^\nu_\beta + \delta^\mu_\alpha(\delta u)\partial_\beta X^\nu + \delta^\nu_\beta(\delta u)\partial_\alpha X^\mu + \mathscr{O}\left(\delta u^2\right)\right) T^{\alpha\beta}(x) \\ &= T^{\mu\nu}(x) + \left[\partial_\beta X^\nu T^{\mu\beta} + \partial_\alpha X^\mu T^{\alpha\nu}\right]\delta u, \end{aligned}$$

where only terms first-order in δu have been retained.

3.18 The transformation law for dual vectors is given by Eq. (3.29). Using the expansion (3.66) to evaluate the partial derivative gives

$$A'_\mu(x') = \frac{\partial x^\alpha}{\partial x'^\mu} A_\alpha(x) = \left(\delta^\alpha_\mu - \frac{\partial X^\alpha}{\partial x'^\mu}\right)\delta u A_\alpha(x) = A_\mu(x) - \partial_\mu X^\alpha (\delta u) A_\alpha(x).$$

By analogy with Eq. (3.68) the Lie derivative is then

$$\mathscr{L}_X A_\mu \equiv \lim_{\delta u \to 0}\left(\frac{A_\mu(x') - A'_\mu(x')}{\delta u}\right) = X^\alpha \partial_\alpha A_\mu + A_\alpha \partial_\mu X^\alpha,$$

where a Taylor expansion as in Eq. (3.69) was used to evaluate $A_\mu(x')$.

3.19 Let $A_{\mu\nu} = U_u V_\nu$. Then by the Leibniz rule,

$$\begin{aligned} \mathscr{L}_X A_{\mu\nu} &= \mathscr{L}_X(U_\mu V_\nu) = (\mathscr{L}_X U_\mu)V_\nu + U_\mu(\mathscr{L}_X V_\nu) \\ &= X^\alpha\left[(\partial_\alpha U_\mu)V_\nu + U_\mu(\partial_\alpha V_\nu)\right] + U_\alpha(\partial_\mu X^\alpha)V_\nu + U_\mu V_\alpha(\partial_\nu X^\alpha) \\ &= X^\alpha \partial_\alpha A_{\mu\nu} + A_{\alpha\nu}\partial_\mu X^\alpha + A_{\mu\alpha}\partial_\nu X^\alpha, \end{aligned}$$

where in the second line Eq. (3.73) was used and in the third line $A_{\mu\nu} = U_u V_\nu$ and

$$\partial_\alpha A_{\mu\nu} = \partial_\alpha(U_\mu V_\nu) = U_\mu(\partial_\alpha V^\nu) + (\partial_\alpha U_\mu)V_\nu$$

were used.

3.20 Let $C = [A,B] = AB - BA$ and operate on an arbitrary function f,

$$\begin{aligned} Cf &= [A,B]f = ABf - BAf \\ &= A^\nu \partial_\nu(B^\mu \partial_\mu f) - B^\nu \partial_\nu(A^\mu \partial_\mu f) \\ &= A^\nu \partial_\nu B^\mu \partial_\mu f + A^\nu B^\mu \partial_\nu \partial_\mu f - B^\nu \partial_\nu A^\mu \partial_\mu f - B^\nu A^\mu \partial_\nu \partial_\mu f \\ &= (A^\nu \partial_\nu B^\mu)\partial_\mu f - (B^\nu \partial_\nu A^\mu)\partial_\mu f, \end{aligned}$$

where in the second line the vectors were expanded in the basis ∂_ν and the third line results from taking the partial derivative of the product. Since the function f is arbitrary, this implies the operator relation

$$C = [A,B] = (A^\nu \partial_\nu B^\mu - B^\nu \partial_\nu A^\mu)\partial_\mu,$$

and since ∂_μ is a vector basis, C is a vector with components

$$C^\mu = [A,B]^\mu = A^\nu \partial_\nu B^\mu - B^\nu \partial_\nu A^\mu,$$

which defines the *Lie bracket* $[A,B] = -[B,A]$ for the vectors A and B. Comparison with

Eq. (3.72) indicates that the Lie bracket is equivalent to a Lie derivative of a vector field: $[A,B]^\mu = \mathscr{L}_A B^\mu$. The Lie derivative of a tensor then may be viewed as a generalization of the Lie bracket for vectors.

3.21 (a) From Eqs. (3.15)–(3.17) and Example 3.4,

$$V(e^\mu) = V^\nu e_\nu(e^\mu) = \delta^\mu_\nu V^\nu = V^\mu$$
$$\omega(e_\mu) = \omega_\nu e^\nu(e_\mu) = \omega_\nu \delta^\nu_\mu = \omega_\mu,$$

which is Eq. (3.19).

(b) For vectors $V = V^\alpha e_\alpha$, by the chain rule under a coordinate transformation $x^\mu \to x'^\mu$ the basis vectors transform as

$$e_\alpha \to e'_\alpha = \frac{\partial x^\nu}{\partial x'^\alpha} e_\nu.$$

Thus, to keep V invariant under $x^\mu \to x'^\mu$ its components must transform as

$$V'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} V^\nu,$$

which is equivalent to (3.31), since then

$$\begin{aligned} V \to V' = V'^\mu e'_\mu &= \frac{\partial x'^\mu}{\partial x^\nu} V^\nu \frac{\partial x^\alpha}{\partial x'^\mu} e_\alpha \\ &= \frac{\partial x'^\mu}{\partial x^\nu} \frac{\partial x^\alpha}{\partial x'^\mu} V^\nu e_\alpha \\ &= \delta^\alpha_\nu V^\nu e_\alpha \\ &= V^\alpha e_\alpha = V. \end{aligned}$$

3.22 The first two examples are trivial. Since two successive partial derivative operations commute,

$$\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right] = 0 \qquad \left[\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}\right] = 0$$

and obviously these are coordinate bases. But for the third example

$$\begin{aligned} [\hat{e}_1, \hat{e}_2,] = \left[\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \theta}\right] &= \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial \theta}\right) - \frac{1}{r}\frac{\partial}{\partial \theta}\frac{\partial}{\partial r} \\ &= -\frac{1}{r^2}\frac{\partial}{\partial \theta} + \frac{1}{r}\frac{\partial^2}{\partial r \partial \theta} - \frac{1}{r}\frac{\partial^2}{\partial r \partial \theta} \\ &= -\frac{1}{r^2}\frac{\partial}{\partial \theta} = -\frac{\hat{e}_2}{r} \neq 0. \end{aligned}$$

Thus $\hat{e}_1$ and $\hat{e}_2$ do not commute and they define a non-coordinate basis.

3.23 The Lie derivative for a vector is given by Eq. (3.72). Replacing the partial derivatives with covariant derivatives in this expression gives

$$\begin{aligned} \mathscr{L}_X A^\mu = X^\alpha \partial_\alpha A^\mu - A^\alpha \partial_\alpha X^\mu &\longrightarrow X^\alpha \nabla_\alpha A^\mu - A^\alpha \nabla_\alpha X^\mu \\ &= X^\alpha \left(\partial_\alpha A^\mu + \Gamma^\mu_{\beta\alpha} A^\beta\right) - A^\alpha \left(\partial_\alpha X^\mu + \Gamma^\mu_{\beta\alpha} X^\beta\right). \end{aligned}$$