MICROECONOMICS

Solutions Manual

Frank A. Cowell

STICERD and Department of Economics
London School of Economics

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Contents

Chapter 1

Introduction

This manual contains outline answers to the end-of-chapter exercises in *Microeconomics: Principles and Analysis* by Frank Cowell (Oxford University Press, second edition 2018). For convenience the (slightly edited) versions of the questions are reproduced here as well. Almost every question begins on a new page so that instructors can print off individual problems and outline answers for classroom use.

Some of the exercises are based on key contributions to the literature. For the bibliographic references consult the original question in the printed text.

Chapter 2

The Firm

Exercise 2.1 *Suppose that a unit of output q can be produced by any of the following combinations of inputs*

$$\mathbf{z}^1 = \begin{bmatrix} 0.2 \\ 0.5 \end{bmatrix}, \mathbf{z}^2 = \begin{bmatrix} 0.3 \\ 0.2 \end{bmatrix}, \mathbf{z}^3 = \begin{bmatrix} 0.5 \\ 0.1 \end{bmatrix}$$

1. *Construct the isoquant for $q = 1$.*

2. *Assuming constant returns to scale, construct the isoquant for $q = 2$.*

3. *If the technique $\mathbf{z}^4 = [0.25, 0.5]$ were also available would it be included in the isoquant for $q = 1$?*

Outline Answer

1. See Figure 2.1 for the simplest case. However, if other basic techniques are also available then an isoquant such as that in Figure 2.2 is consistent with the data in the question.

Figure 2.1: Isoquant-Simple Case

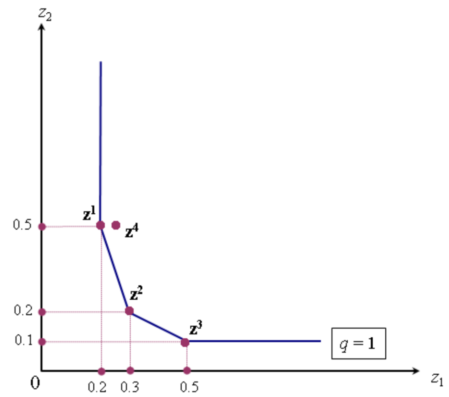


Figure 2.2: Isoquant-Alternative Case

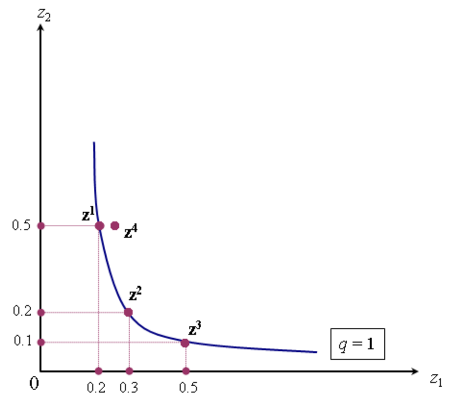
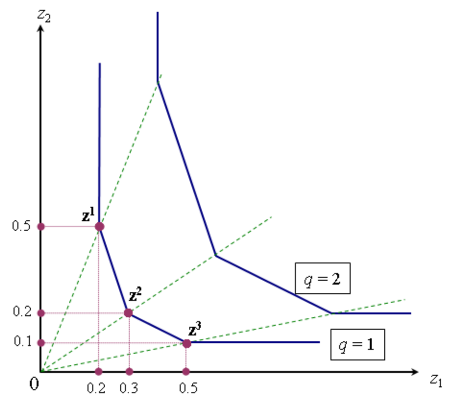


Figure 2.3: Isoquants under CRTS



2. See Figure 2.3. Draw the rays through the origin that pass through each of the corners of the isoquant for $q = 1$. Each corner of the isoquant for $q = 2$. lies twice as far out along the ray as the corner for the case $q = 1$.

3. Clearly $\mathbf{z}^4$ should not be included in the isoquant since $\mathbf{z}^4$ requires strictly more of either input to produce one unit of output than does $\mathbf{z}^2$ so that it cannot be efficient. This is true whatever the exact shape of the isoquant in – see Figures 2.1 and 2.2

Exercise 2.2 *An innovating firm produces a single output from two inputs.*

1. *If the production function is originally given by the single technique*

$$q \leq \min\left\{\frac{1}{3}z_1, z_2\right\}$$

draw the isoquants.

2. *The firm's research department develops a new technique*

$$q \leq \min\left\{z_1, \frac{1}{3}z_2\right\}.$$

Assuming that both techniques are now available to the firm, draw the isoquants.

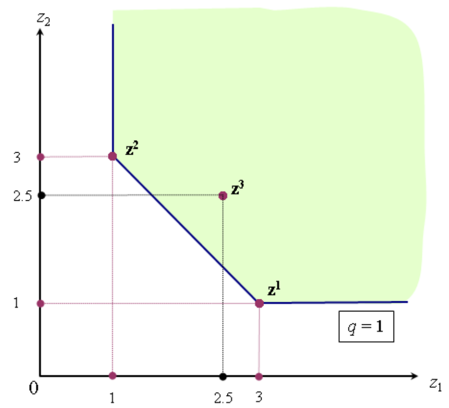
3. *The firm's research department now develops a third technique*

$$q \leq \min\left\{\frac{2}{5}z_1, \frac{2}{5}z_2\right\}.$$

Assuming that all three techniques are available to the firm, draw the isoquants.

Outline Answer

Figure 2.4: Input Requirement Set, Single Technique



1. For the case $q = 1$ see the shaded area in Figure 2.4 where $\mathbf{z}^1$ is the point $(3, 1)$.
2. See the shaded area in Figure 2.5 where $\mathbf{z}^2$ is the point $(1, 3)$.
3. The new technique would be represented, for $q = 1$, by the point $\mathbf{z}^3 = (2.5, 2.5)$ in Figure 2.5. But this is inefficient because one could do better by using a combination of $\mathbf{z}^1$ and $\mathbf{z}^2$. The input requirement set remains the same as in the previous case and $\mathbf{z}^3$ lies in its interior. See Figure 2.6.

Figure 2.5: Input Requirement Set, Multiple Techniques

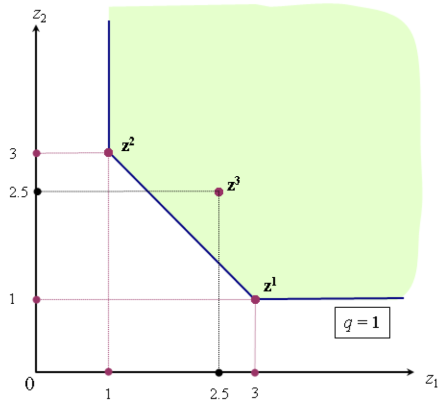
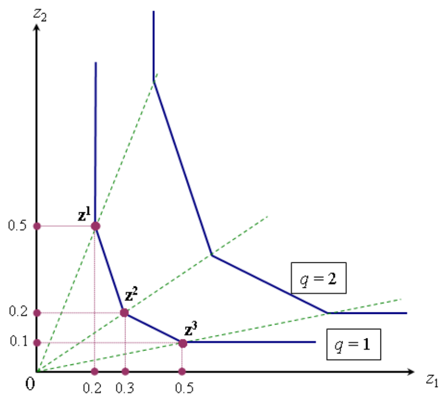


Figure 2.6: Isoquants in the case of CRTS.



Exercise 2.3 *A firm uses two inputs in the production of a single good. The input requirements per unit of output for a number of alternative techniques are given by the following table:*

Process	*1*	*2*	*3*	*4*	*5*	*6*
Input 1	*9*	*15*	*7*	*1*	*3*	*4*
Input 2	*4*	*2*	*6*	*10*	*9*	*7*

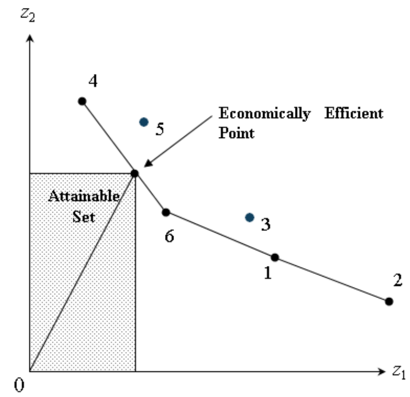
The firm has exactly 140 units of input 1 and 410 units of input 2 at its disposal.

1. *Discuss the concepts of technological and economic efficiency with reference to this example.*
2. *Describe the optimal production plan for the firm.*
3. *Would the firm prefer 10 extra units of input 1 or 20 extra units of input 2?*

Outline Answer

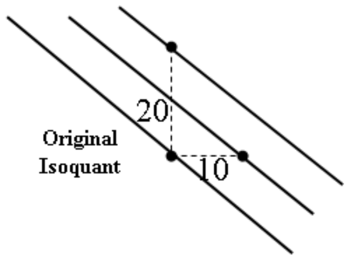
1. As illustrated in Figure 2.7 only processes 1,2,4 and 6 are technically efficient.
2. Given the resource constraint (see shaded area), the economically efficient input combination is a mixture of processes 4 and 6.

Figure 2.7: Economically Efficient Point



3. Note that in the neighbourhood of this efficient point MRTS=1. So, as illustrated in the enlarged diagram in Figure 2.8, 20 extra units of input 2 clearly enable more output to be produced than 10 extra units of input 1.

Figure 2.8: Effect of increase in input



Exercise 2.4 *Consider the following structure of the cost function:* $C(\mathbf{w},0) = 0$, $C_q(\mathbf{w},q) = \text{int}(q)$ *where* $\text{int}(x)$ *is the smallest integer greater than or equal to* x. *Sketch total, average and marginal cost curves.*

Outline Answer

From the question the cost function is given by

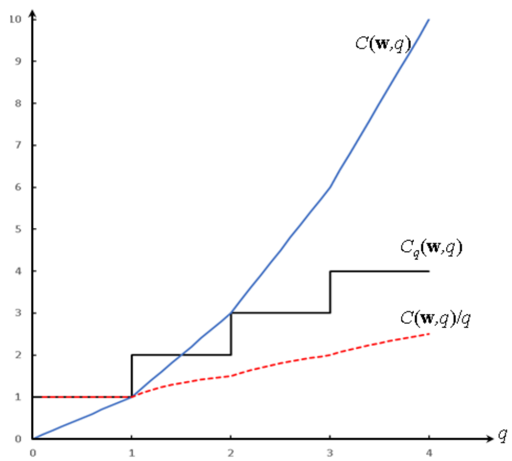
$$C(\mathbf{w},q) = kq - \frac{1}{2}k\,[k-1]\,, k-1 < q \leq k, k = 1, 2, 3...$$

so that average cost is

$$k + k\frac{1-k}{2q}, k-1 < q \leq k, k = 1, 2, 3...$$

– see Figure 2.9.

Figure 2.9: Step-wise Marginal Cost



Exercise 2.5 *Suppose a firm's production function has the* Cobb-Douglas *form*

$$q = z_1^{\alpha_1} z_2^{\alpha_2}$$

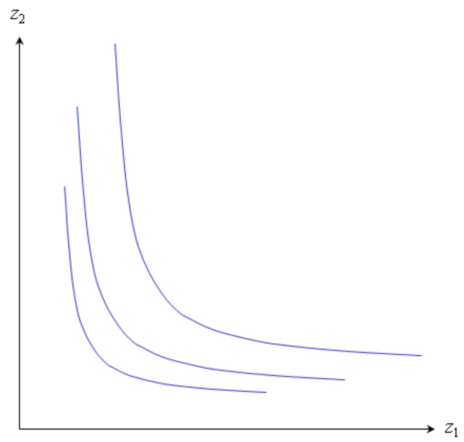
where z_1 and z_2 are inputs, q is output and α_1, α_2 are positive parameters.

1. *Draw the isoquants. Do they touch the axes?*
2. *What is the elasticity of substitution in this case?*
3. *Using the Lagrangian method find the cost-minimising values of the inputs and the cost function.*
4. *Under what circumstances will the production function exhibit (a) decreasing (b) constant (c) increasing returns to scale? Explain this using first the production function and then the cost function.*
5. *Find the conditional demand curve for input 1.*

Outline Answer

1. The isoquants are illustrated in Figure 2.10. They do not touch the axes.

Figure 2.10: Isoquants - Cobb Douglas



2. The elasticity of substitution is defined as

$$\sigma_{ij} := -\frac{\partial \log (z_j / z_i)}{\partial \log (\phi_j(\mathbf{z})/\phi_i(\mathbf{z}))}$$

which, in the two input case, becomes

$$\sigma = -\frac{\partial \log \left(\frac{z_1}{z_2}\right)}{\partial \log \left(\frac{\phi_1(\mathbf{z})}{\phi_2(\mathbf{z})}\right)} \tag{2.1}$$

In case 1 we have $\phi(\mathbf{z}) = z_1^{\alpha_1} z_2^{\alpha_2}$ and so, by differentiation, we find:

$$\frac{\phi_1(\mathbf{z})}{\phi_2(\mathbf{z})} = \frac{\alpha_1}{\alpha_2} / \frac{z_1}{z_2}$$

Taking logarithms we have

$$\log\left(\frac{z_1}{z_2}\right) = \log\left(\frac{\alpha_1}{\alpha_2}\right) - \log\left(\frac{\phi_1(\mathbf{z})}{\phi_2(\mathbf{z})}\right)$$

or

$$u = \log\left(\frac{\alpha_1}{\alpha_2}\right) - v$$

where $u := \log(z_1/z_2)$ and $v := \log(\phi_1/\phi_2)$. Differentiating u with respect to v we have

$$\frac{\partial u}{\partial v} = -1. \tag{2.2}$$

So, using the definitions of u and v in equation (2.2) we have

$$\sigma = -\frac{\partial u}{\partial v} = 1.$$

3. This is a *Cobb-Douglas production function.* This will yield a unique interior solution; the Lagrangian is:

$$\mathcal{L}(\mathbf{z}, \lambda) = w_1 z_1 + w_2 z_2 + \lambda\left[q - z_1^{\alpha_1} z_2^{\alpha_2}\right] , \tag{2.3}$$

and the first-order conditions are:

$$\frac{\partial \mathcal{L}(\mathbf{z}, \lambda)}{\partial z_1} = w_1 - \lambda \alpha_1 z_1^{\alpha_1 - 1} z_2^{\alpha_2} = 0 , \tag{2.4}$$

$$\frac{\partial \mathcal{L}(\mathbf{z}, \lambda)}{\partial z_2} = w_2 - \lambda \alpha_2 z_1^{\alpha_1} z_2^{\alpha_2 - 1} = 0 , \tag{2.5}$$

$$\frac{\partial \mathcal{L}(\mathbf{z}, \lambda)}{\partial \lambda} = q - z_1^{\alpha_1} z_2^{\alpha_2} = 0 . \tag{2.6}$$

Using these conditions and rearranging we can get an expression for minimized cost in terms of and q:

$$w_1 z_1 + w_2 z_2 = \lambda \alpha_1 z_1^{\alpha_1} z_2^{\alpha_2} + \lambda \alpha_2 z_1^{\alpha_1} z_2^{\alpha_2} = [\alpha_1 + \alpha_2] \lambda q.$$

We can then eliminate λ:

$$\left.\begin{array}{l} w_1 - \lambda \alpha_1 \frac{q}{z_1} = 0 \\ w_2 - \lambda \alpha_2 \frac{q}{z_2} = 0 \end{array}\right\}$$

which implies

$$\left.\begin{array}{l} z_1^* = \frac{\alpha_1}{w_1} \lambda q \\ z_2^* = \frac{\alpha_2}{w_2} \lambda q \end{array}\right\} . \tag{2.7}$$

Substituting the values of z_1^* and z_2^* back in the production function we have

$$\left[\frac{\alpha_1}{w_1} \lambda q\right]^{\alpha_1} \left[\frac{\alpha_2}{w_2} \lambda q\right]^{\alpha_2} = q$$

which implies

$$\lambda q = \left[q \left[\frac{w_1}{\alpha_1}\right]^{\alpha_1} \left[\frac{w_2}{\alpha_2}\right]^{\alpha_2} \right]^{\frac{1}{\alpha_1+\alpha_2}} \tag{2.8}$$

So, using (2.7) and (2.8), the corresponding cost function is

$$\begin{aligned} C(\mathbf{w}, q) &= w_1 z_1^* + w_2 z_2^* \\ &= [\alpha_1 + \alpha_2] \left[q \left[\frac{w_1}{\alpha_1}\right]^{\alpha_1} \left[\frac{w_2}{\alpha_2}\right]^{\alpha_2} \right]^{\frac{1}{\alpha_1+\alpha_2}}. \end{aligned}$$

4. Using the production functions we have, for any $t > 0$:

$$\phi(t\mathbf{z}) = [tz_1]^{\alpha_1} [tz_2]^{\alpha_2} = t^{\alpha_1+\alpha_2} \phi(\mathbf{z}).$$

Therefore we have DRTS/CRTS/IRTS according as $\alpha_1 + \alpha_2 \lesseqqgtr 1$. If we look at average cost as a function of q we find that AC is increasing/constant/decreasing in q according as $\alpha_1 + \alpha_2 \lesseqqgtr 1$.

5. Using (2.7) and (2.8) conditional demand functions are

$$H^1(\mathbf{w}, q) = \left[q \left[\frac{\alpha_1 w_2}{\alpha_2 w_1}\right]^{\alpha_2} \right]^{\frac{1}{\alpha_1+\alpha_2}}$$

$$H^2(\mathbf{w}, q) = \left[q \left[\frac{\alpha_2 w_1}{\alpha_1 w_2}\right]^{\alpha_1} \right]^{\frac{1}{\alpha_1+\alpha_2}}$$

and are smooth with respect to input prices.

Exercise 2.6 *Suppose a firm's production function has the* Leontief *form*

$$q = \min\left\{\frac{z_1}{\alpha_1}, \frac{z_2}{\alpha_2}\right\}$$

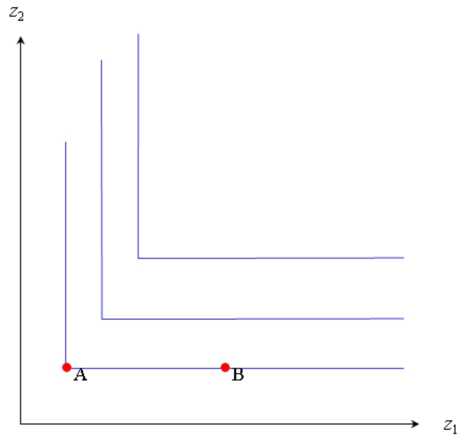
where the notation is the same as in Exercise 2.5.

1. *Draw the isoquants.*
2. *For a given level of output identify the cost-minimising input combination(s) on the diagram.*
3. *Hence write down the cost function in this case. Why would the Lagrangian method of Exercise 2.5 be inappropriate here?*
4. *What is the conditional input demand curve for input 1?*
5. *Repeat parts 1-4 for each of the two production functions*

$$\begin{aligned} q &= \alpha_1 z_1 + \alpha_2 z_2 \\ q &= \alpha_1 z_1^2 + \alpha_2 z_2^2 \end{aligned}$$

   Explain carefully how the solution to the cost-minimisation problem differs in these two cases.

Figure 2.11: Isoquants - Leontief



Outline Answer

1. The Isoquants are illustrated in Figure 2.11 – the so-called *Leontief* case,

Figure 2.12: Isoquants - Linear

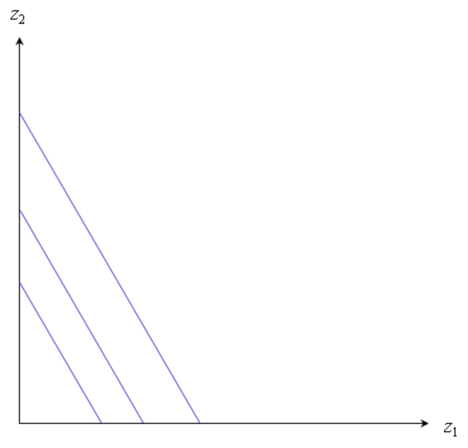
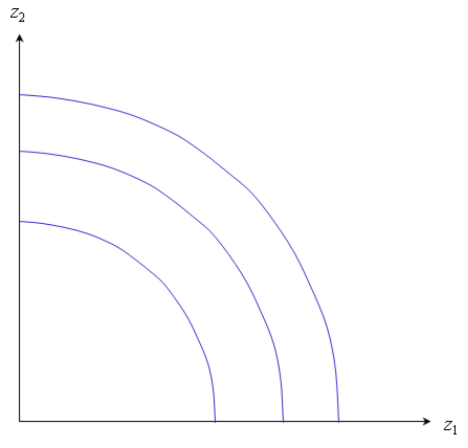


Figure 2.13: Isoquants - Nonconvex to origin



2. If all prices are positive, we have a unique cost-minimising solution at A: to see this, draw any straight line with positive finite slope through A and take this as an isocost line; if we considered any other point B on the isoquant through A then an isocost line through B (same slope as the one through A) must lie above the one you have just drawn.

3. The coordinates of the corner A are $(\alpha_1 q, \alpha_2 q)$ and, given $\mathbf{w}$, this immediately yields the minimised cost.

$$C(\mathbf{w}, q) = w_1\alpha_1 q + w_2\alpha_2 q.$$

   The methods in Exercise 2.5 since the Lagrangian is not differentiable at the corner.

4. Conditional demand is constant if all prices are positive

$$\begin{aligned} H^1(\mathbf{w}, q) &= \alpha_1 q \\ H^2(\mathbf{w}, q) &= \alpha_2 q. \end{aligned}$$

5. Given the linear case

$$q = \alpha_1 z_1 + \alpha_2 z_2$$

   - Isoquants are as in Figure 2.12.
   - It is obvious that the solution will be either at the corner $(q/\alpha_1, 0)$ if $w_1/w_2 < \alpha_1/\alpha_2$ or at the corner $(0, q/\alpha_2)$ if $w_1/w_2 > \alpha_1/\alpha_2$, or otherwise anywhere on the isoquant
   - This immediately shows us that minimised cost must be.

$$C(\mathbf{w}, q) = q \min\left\{\frac{w_1}{\alpha_1}, \frac{w_2}{\alpha_2}\right\}$$

   - So conditional demand can be multivalued:

$$H^1(\mathbf{w}, q) = \begin{cases} \frac{q}{\alpha_1} & \text{if } \frac{w_1}{w_2} < \frac{\alpha_1}{\alpha_2} \\ z_1^* \in \left[0, \frac{q}{\alpha_1}\right] & \text{if } \frac{w_1}{w_2} = \frac{\alpha_1}{\alpha_2} \\ 0 & \text{if } \frac{w_1}{w_2} > \frac{\alpha_1}{\alpha_2} \end{cases}$$

$$H^2(\mathbf{w}, q) = \frac{q - \alpha_1 H^1(\mathbf{w}, q)}{\alpha_2}$$

   - Case 3 is a test to see if you are awake: the isoquants are not convex to the origin – see Figure 2.13. An experiment with a straight-edge to simulate an isocost line will show that it is almost like case 2 – the solution will be either at the corner $(\sqrt{q/\alpha_1}, 0)$ if $w_1/w_2 < \sqrt{\alpha_1/\alpha_2}$ or at the corner $(0, \sqrt{q/\alpha_2})$ if $w_1/w_2 > \sqrt{\alpha_1/\alpha_2}$ (but nowhere else). So the cost function is :

$$C(\mathbf{w}, q) = \min\left\{w_1\sqrt{\frac{q}{\alpha_1}}, w_2\sqrt{\frac{q}{\alpha_2}}\right\}.$$

The conditional demand function is similar to, but slightly different from, the previous case:

$$H^1(\mathbf{w},q) = \begin{cases} \sqrt{\frac{q}{\alpha_1}} & \text{if } \frac{w_1}{w_2} < \sqrt{\frac{\alpha_1}{\alpha_2}} \\ z_1^* \in \left\{0, \sqrt{\frac{q}{\alpha_1}}\right\} & \text{if } \frac{w_1}{w_2} = \sqrt{\frac{\alpha_1}{\alpha_2}} \\ 0 & \text{if } \frac{w_1}{w_2} > \sqrt{\frac{\alpha_1}{\alpha_2}} \end{cases}$$

$$H^2(\mathbf{w},q) = \sqrt{\frac{q - \alpha_1 \left[H^1(\mathbf{w},q)\right]^2}{\alpha_2}}$$

Note the discontinuity exactly at $w_1/w_2 = \sqrt{\alpha_1/\alpha_2}$

Exercise 2.7 *Assume the production function*

$$\phi(\mathbf{z}) = \left[\alpha_1 z_1^\beta + \alpha_2 z_2^\beta\right]^{\frac{1}{\beta}}$$

where z_i is the quantity of input i and $\alpha_i \geq 0$, $-\infty < \beta \leq 1$ are parameters. This is an example of the CES (Constant Elasticity of Substitution) production function.

1. *Show that the elasticity of substitution is $\frac{1}{1-\beta}$.*
2. *Explain what happens to the form of the production function and the elasticity of substitution in each of the following three cases: $\beta \to -\infty$, $\beta \to 0$, $\beta \to 1$.*
3. *Relate your answer to the answers to Exercises 2.5 and 2.6.*

Outline Answer

1. Differentiating the production function

$$\phi(\mathbf{z}) := \left[\alpha_1 z_1^\beta + \alpha_2 z_2^\beta\right]^{\frac{1}{\beta}}$$

it is clear that the marginal product of input i is

$$\phi_i(\mathbf{z}) := \left[\alpha_1 z_1^\beta + \alpha_2 z_2^\beta\right]^{\frac{1}{\beta}-1} \alpha_i z_i^{\beta-1} \tag{2.9}$$

Therefore the MRTS is

$$\frac{\phi_1(\mathbf{z})}{\phi_2(\mathbf{z})} = \frac{\alpha_1}{\alpha_2}\left[\frac{z_1}{z_2}\right]^{\beta-1} \tag{2.10}$$

which implies

$$\log\left(\frac{z_1}{z_2}\right) = \frac{1}{1-\beta}\log\frac{\alpha_1}{\alpha_2} - \frac{1}{1-\beta}\log\left(\frac{\phi_1(\mathbf{z})}{\phi_2(\mathbf{z})}\right).$$

Therefore

$$\sigma = -\frac{\partial \log\left(\frac{z_1}{z_2}\right)}{\partial \log\left(\frac{\phi_1(\mathbf{z})}{\phi_2(\mathbf{z})}\right)} = \frac{1}{1-\beta}$$

2. Clearly $\beta \to -\infty$ yields $\sigma = 0$ ($\phi(\mathbf{z}) = \min\{\alpha_1 z_1, \alpha_2 z_2\}$), $\beta \to 0$ yields $\sigma = 1$ ($\phi(\mathbf{z}) = z_1^{\alpha_1} z_2^{\alpha_2}$), $\beta \to 1$ yields $\sigma = \infty$ ($\phi(\mathbf{z}) = \alpha_1 z_1 + \alpha_2 z_2$).
3. The case $\beta \to -\infty$ corresponds to that in part 1 of Exercise 2.6; $\beta \to 0$ corresponds to that in Exercise 2.5; $\beta \to 1$ corresponds to that in part 5 of Exercise 2.6.

Exercise 2.8 *For the CES function in Exercise 2.7 find $H^1(\mathbf{w}, q)$, the conditional demand for good 1, for the case where $\beta \neq 0, 1$. Verify that it is decreasing in w_1 and homogeneous of degree 0 in (w_1, w_2).*

Outline Answer

From the minimization of the following Lagrangian

$$\mathcal{L}(\mathbf{z}, \lambda; \mathbf{w}, q) := \sum_{i=1}^{m} w_i z_i + \lambda \left[q - \phi(\mathbf{z})\right]$$

we obtain

$$\lambda^* \alpha_1 \left[z_1^*\right]^{\beta-1} q^{1-\beta} = w_1 \tag{2.11}$$
$$\lambda^* \alpha_2 \left[z_2^*\right]^{\beta-1} q^{1-\beta} = w_2 \tag{2.12}$$

On rearranging:

$$\frac{w_1}{\alpha_1} \frac{1}{\lambda^* q^{1-\beta}} = \left[z_1^*\right]^{\beta-1}$$
$$\frac{w_2}{\alpha_2} \frac{1}{\lambda^* q^{1-\beta}} = \left[z_2^*\right]^{\beta-1}$$

Using the production function we get

$$\alpha_1 \left[\frac{w_1}{\alpha_1} \frac{1}{\lambda^* q^{1-\beta}}\right]^{\frac{\beta}{\beta-1}} + \alpha_2 \left[\frac{w_2}{\alpha_2} \frac{1}{\lambda^* q^{1-\beta}}\right]^{\frac{\beta}{\beta-1}} = q^\beta$$

Rearranging we find

$$\lambda^* q^{1-\beta} = \left[\alpha_1^{-\frac{1}{\beta-1}} \left[w_1\right]^{\frac{\beta}{\beta-1}} + \alpha_2^{-\frac{1}{\beta-1}} \left[w_2\right]^{\frac{\beta}{\beta-1}}\right]^{\frac{\beta-1}{\beta}} q^{1-\beta}$$

Substituting this into (2.11) we get:

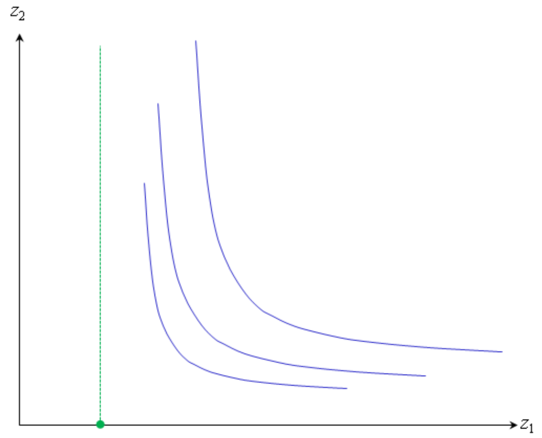
$$w_1 = \alpha_1 \left[z_1^*\right]^{\beta-1} \left[\alpha_1^{-\frac{1}{\beta-1}} \left[w_1\right]^{\frac{\beta}{\beta-1}} + \alpha_2^{-\frac{1}{\beta-1}} \left[w_2\right]^{\frac{\beta}{\beta-1}}\right]^{\frac{\beta-1}{\beta}} q^{1-\beta}$$

Rearranging this we have:

$$z_1^* = \left[\alpha_1 + \alpha_2 \left[\frac{\alpha_1}{\alpha_2} \frac{w_2}{w_1}\right]^{\frac{\beta}{\beta-1}}\right]^{\frac{-1}{\beta}} q$$

Clearly z_1^* is decreasing in w_1 if $\beta < 1$. Furthermore, rescaling w_1 and w_2 by some positive constant will leave z_1^* unchanged.

Figure 2.14: Isoquants Shifted Cobb Douglas



Exercise 2.9 *A firm's production function is given by:*

$$\phi(\mathbf{z}) = \begin{cases} [z_1 - a]^b z_2^b & \textit{if } z_1 > a,\ z_2 > 0 \\ 0 & \textit{otherwise} \end{cases}$$

where z_1, z_2 are quantities of two inputs, and $a, b \geq 0$ are parameters.

1. *Sketch the isoquants for this production function.*
2. *Find the firm's cost function. Sketch the average and marginal cost curves. Does the production function exhibit increasing or decreasing returns to scale?*
3. *Find the firm's conditional demand for the two inputs, given input prices. How would a decrease in a change these demands?*
4. *If the firm is selling in a competitive market find the firm's supply function. How would a decrease in a affect the supply of output?*

Outline Answer

1. See Figure 2.14.
2. The problem

$$\min w_1 z_1 + w_2 z_2 \text{ subject to } \ q \leq \phi(\mathbf{z}), z_1 > a$$

is equivalent to

$$\min w_1 z_1 + w_2 z_2 \text{ subject to } \frac{1}{b} \log q \leq \log(z_1 - a) + \log(z_2).$$

The Lagrangian for the transformed problem is

$$w_1 z_1 + w_2 z_2 + \lambda \left[\frac{1}{b} \log q - \log(z_1 - a) - \log(z_2) \right].$$

The first-order conditions for an interior optimum are

$$w_1 - \frac{\lambda^*}{z_1^* - a} = 0 \quad (2.13)$$

$$w_2 - \frac{\lambda^*}{z_2^*} = 0 \quad (2.14)$$

$$\log(z_1^* - a) + \log(z_2^*) = \frac{1}{b}\log q \quad (2.15)$$

Substituting from (2.13) and (2.14) into (2.15) we get

$$\log\left(\frac{\lambda^*}{w_1}\right) + \log\left(\frac{\lambda^*}{w_2}\right) = \frac{1}{b}\log q,$$

$$\frac{1}{b}\log q + \log(w_1 w_2) = 2\log\lambda^*,$$

so that

$$\lambda^* = q^{\frac{1}{2b}}\sqrt{w_1 w_2}. \quad (2.16)$$

Using (2.13), (2.14) and (2.16) we get

$$w_1 z_1^* + w_2 z_2^* = w_1 a + 2\lambda^*$$

$$= w_1 a + 2q^{\frac{1}{2b}}\sqrt{w_1 w_2}. \quad (2.17)$$

This is the cost function $C(\mathbf{w}, q)$. From (2.17) we immediately get average and marginal cost respectively as

$$\frac{C(\mathbf{w}, q)}{q} = \frac{w_1 a}{q} + 2q^k\sqrt{w_1 w_2} \quad (2.18)$$

$$\frac{\partial C(\mathbf{w}, q)}{\partial q} = \frac{1}{b}q^k\sqrt{w_1 w_2} \quad (2.19)$$

where $k := \frac{1}{2b} - 1$. Differentiating (2.18) we have

$$\frac{\partial}{\partial q}\left(\frac{C(\mathbf{w}, q)}{q}\right) = -\frac{w_1 a}{q^2} + 2kq^{k-1}\sqrt{w_1 w_2}$$

Clearly AC must be everywhere falling if $k \leq 0$ ($b \geq 0.5$); in this case we can be sure that there is increasing returns to scale; Figure 2.15 illustrates three possibilities for the values $w_1 = w_2 = a = 1$.

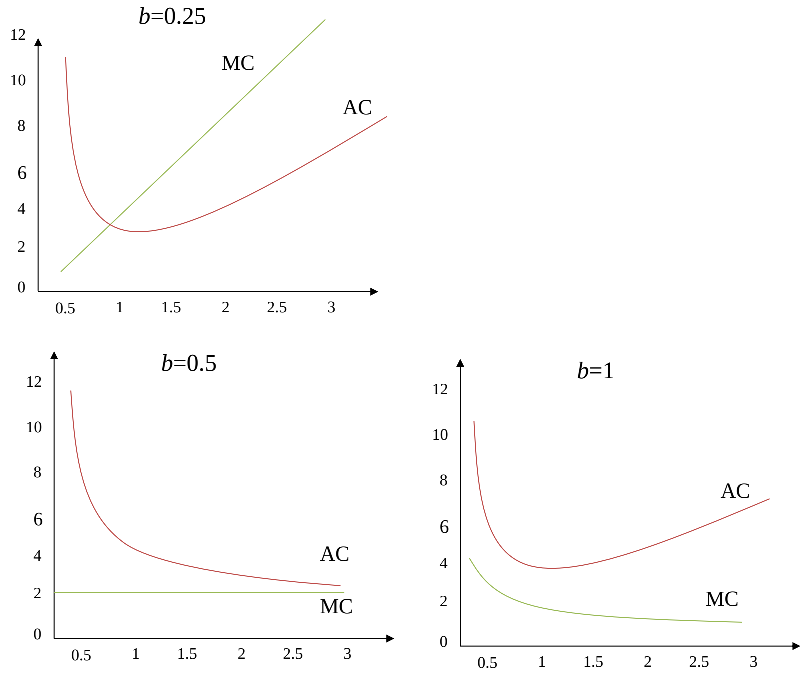
3. The conditional demand function is found by differentiating the cost function. So we have

$$H^1(\mathbf{w}, q) = \frac{\partial C(\mathbf{w}, q)}{\partial w_1} = a + q^{\frac{1}{2b}}\sqrt{\frac{w_2}{w_1}} \quad (2.20)$$

$$H^2(\mathbf{w}, q) = \frac{\partial C(\mathbf{w}, q)}{\partial w_2} = q^{\frac{1}{2b}}\sqrt{\frac{w_1}{w_2}} \quad (2.21)$$

If a decreases this reduces the conditional demand for good 1 but not that for good 2.

Figure 2.15: Average Costs and Marginal Costs in three cases



4. The supply curve for the competitive firm is determined by two conditions (a) $p =$MC if output is positive and (b) p must cover AC. The competitive firm will supply a positive amount of output if there is some q such that MC$\geq$AC which, using (2.18) and (2.19) in this case requires

$$kq^{k+1} \geq \frac{1}{2}a\sqrt{\frac{w_1}{w_2}}. \tag{2.22}$$

If $k > 0$ ($b < 0.5$) then there will be some $\underline{q}$ sufficiently large such that condition (2.22) will be satisfied for $q \geq \underline{q}$. Equating price to MC we have

$$p = \frac{1}{b}q^k\sqrt{w_1 w_2}$$

So the supply curve is

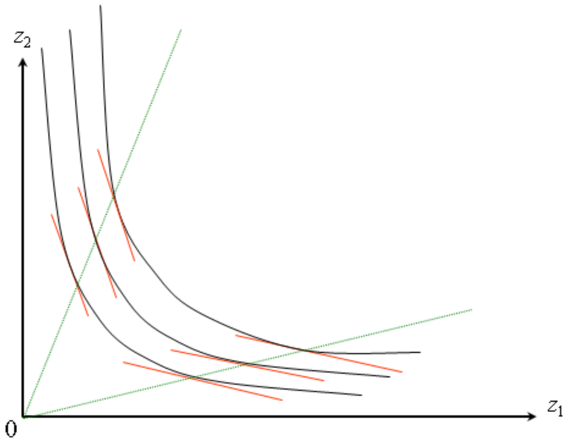
$$q = \begin{cases} \left[\frac{bp}{\sqrt{w_1 w_2}}\right]^{\frac{1}{k}} & \text{if } q \geq \underline{q} \\ 0 & \text{otherwise} \end{cases}$$

If a were to decrease then $\underline{q}$ would decrease but the slope of the supply curve would remain unchanged in the region $q \geq \underline{q}$.

Exercise 2.10 *For any homothetic production function show that the cost function must be expressible in the form*

$$C(\mathbf{w}, q) = a(\mathbf{w})\, b(q).$$

Figure 2.16: Homothetic Expansion Path



Outline Answer

From the definition of homotheticity, the isoquants must look like Figure 2.16; interpreting the tangents as isocost lines it is clear from the figure that the expansion paths are rays through the origin. So, if $H^i(\mathbf{w}, q)$ is the demand for input i conditional on output q, the optimal input ratio

$$\frac{H^i(\mathbf{w}, q)}{H^j(\mathbf{w}, q)}$$

must be independent of q and so we must have

$$\frac{H^i(\mathbf{w}, q)}{H^i(\mathbf{w}, q')} = \frac{H^j(\mathbf{w}, q)}{H^j(\mathbf{w}, q')}$$

for any q, q'. For this to true it is clear that the ratio $H^i(\mathbf{w}, q)/H^i(\mathbf{w}, q')$ must be independent of $\mathbf{w}$. Setting $q' = 1$ we therefore have

$$\frac{H^1(\mathbf{w}, q)}{H^1(\mathbf{w}, 1)} = \frac{H^2(\mathbf{w}, q)}{H^2(\mathbf{w}, 1)} = ... = \frac{H^m(\mathbf{w}, q)}{H^m(\mathbf{w}, 1)} = b(q)$$

and so

$$H^i(\mathbf{w}, q) = b(q) H^i(\mathbf{w}, 1).$$

Therefore the minimized cost is given by

$$\begin{aligned} C(\mathbf{w}, q) &= \sum_{i=1}^{m} w_i H^i(\mathbf{w}, q) \\ &= \sum_{i=1}^{m} w_i b(q) H^i(\mathbf{w}, 1) \\ &= b(q) \sum_{i=1}^{m} w_i H^i(\mathbf{w}, 1) \\ &= a(\mathbf{w}) b(q) \end{aligned}$$

where $a(\mathbf{w}) = \sum_{i=1}^{m} w_i H^i(\mathbf{w}, 1)$.

Exercise 2.11 *Consider the production function*

$$q = \left[\alpha_1 z_1^{-1} + \alpha_2 z_2^{-1} + \alpha_3 z_3^{-1}\right]^{-1}$$

1. *Find the long-run cost function and sketch the long-run and short-run marginal and average cost curves and comment on their form.*
2. *Suppose input 3 is fixed in the short run. Repeat the analysis for the short-run case.*
3. *What is the elasticity of supply in the short and the long run?*

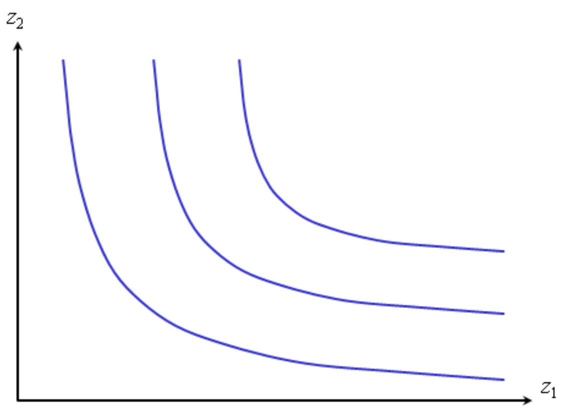
Outline Answer

1. In the long run all inputs are freely variable, including input 3. The production function is clearly homogeneous of degree 1 in all inputs – i.e. in the long run we have constant returns to scale. But CRTS implies constant average cost. So

$$\text{LRMC} = \text{LRAC} = \text{constant}$$

Their graphs will be an identical straight line.

Figure 2.17: Isoquants- Do not touch the axes



2. In the short run $z_3 = \bar{z}_3$ so input 3 represents a fixed cost. We can write the problem as the following Lagrangian

$$\hat{\mathcal{L}}(\mathbf{z}, \hat{\lambda}) = w_1 z_1 + w_2 z_2 + \hat{\lambda}\left[q - \left[\alpha_1 z_1^{-1} + \alpha_2 z_2^{-1} + \alpha_3 \bar{z}_3^{-1}\right]^{-1}\right]; \quad (2.23)$$

or, using a transformation of the constraint to make the manipulation easier, we can use the Lagrangian

$$\mathcal{L}(\mathbf{z}, \lambda) = w_1 z_1 + w_2 z_2 + \lambda\left[\alpha_1 z_1^{-1} + \alpha_2 z_2^{-1} - k\right] \quad (2.24)$$

where λ is the Lagrange multiplier for the transformed constraint and

$$k := q^{-1} - \alpha_3 \bar{z}_3^{-1}. \quad (2.25)$$

Note that the isoquant is

$$z_2 = \frac{\alpha_2}{k - \alpha_1 z_1^{-1}}.$$

From the Figure 2.17 it is clear that the isoquants do not touch the axes and so we will have an interior solution. The first-order conditions are

$$w_i - \lambda \alpha_i z_i^{-2} = 0, \ i = 1, 2 \tag{2.26}$$

which imply

$$z_i = \sqrt{\frac{\lambda \alpha_i}{w_i}}, \ i = 1, 2 \tag{2.27}$$

To find the conditional demand function we need to solve for λ. Using the production function and equations (2.25), (2.27) we get

$$k = \sum_{j=1}^{2} \alpha_j \left[\frac{\lambda \alpha_j}{w_j}\right]^{-1/2} \tag{2.28}$$

from which we find

$$\sqrt{\lambda} = \frac{b}{k} \tag{2.29}$$

where

$$b := \sqrt{\alpha_1 w_1} + \sqrt{\alpha_2 w_2}.$$

Substituting from (2.29) into (2.27) we get minimised cost as

$$\begin{aligned} \tilde{C}(\mathbf{w}, q; \bar{z}_3) &= \sum_{i=1}^{2} w_i z_i^* + w_3 \bar{z}_3 && (2.30) \\ &= \frac{b^2}{k} + w_3 \bar{z}_3 && (2.31) \\ &= \frac{qb^2}{1 - \alpha_3 \bar{z}_3^{-1} q} + w_3 \bar{z}_3. && (2.32) \end{aligned}$$

Marginal cost is

$$\frac{b^2}{\left[1 - \alpha_3 \bar{z}_3^{-1} q\right]^2} \tag{2.33}$$

and average cost is

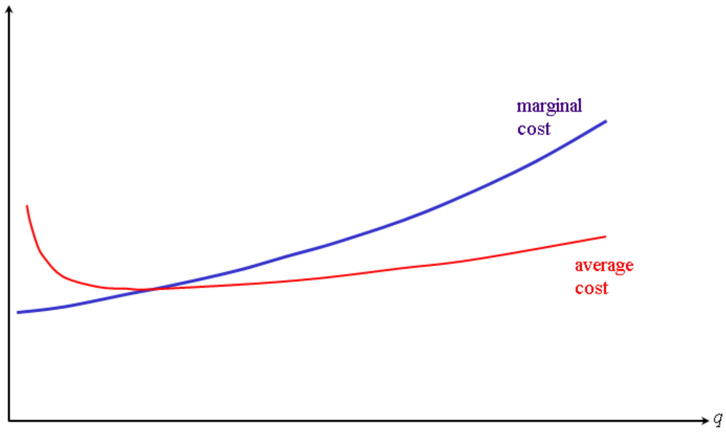
$$\frac{b^2}{1 - \alpha_3 \bar{z}_3^{-1} q} + \frac{w_3 \bar{z}_3}{q}. \tag{2.34}$$

Let $\underline{q}$ be the value of q for which MC=AC in (2.33) and (2.34) – at the minimum of AC in Figure 2.18 – and let $\underline{p}$ be the corresponding minimum value of AC. Then, using p =MC in (2.33) for $p \geq \underline{p}$ the short-run supply curve is given by

$$q = \frac{\bar{z}_3}{\alpha_3} \left[1 - \frac{b}{\sqrt{p}}\right]$$

For $p < \underline{p}$ the firm will produce 0 if $w_3 \bar{z}_3$ represents a fixed cost but not a "sunk cost" – i.e. if input 3 can just be disposed of and the cost recovered, should the firm decide to cease production.

Figure 2.18: Short run Marginal cost and Average Cost



3. Differentiating the last line in the previous formula we get

$$\frac{d\ln q}{d\ln p} = \frac{p}{q}\frac{dq}{dp} = \frac{1}{2}\frac{1}{\sqrt{p/b}-1} > 0$$

Note that the elasticity decreases with b. In the long run the supply curve coincides with the MC,AC curves and so has infinite elasticity.

Exercise 2.12 *A competitive firm's output q is determined by*

$$q = z_1^{\alpha_1} z_2^{\alpha_2} ... z_m^{\alpha_m}$$

where z_i is its usage of input i and $\alpha_i > 0$ is a parameter $i = 1, 2, ..., m$. Assume that in the short run only k of the m inputs are variable.

1. *Find the long-run average and marginal cost functions for this firm. Under what conditions will marginal cost rise with output?*
2. *Find the short-run marginal cost function.*
3. *Find the firm's short-run elasticity of supply. What would happen to this elasticity if k were reduced?*

Outline Answer

Write the production function in the equivalent form:

$$\log q = \sum_{i=1}^{m} \alpha_i \log z_i \tag{2.35}$$

The isoquant for the case $m = 2$ would take the form

$$z_2 = \left[q z_1^{-\alpha_1} \right]^{\frac{1}{\alpha_2}} \tag{2.36}$$

which does not touch the axis for finite (z_1, z_2).

1. The cost-minimisation problem can be represented as minimising the Lagrangian

$$\sum_{i=1}^{m} w_i z_i + \lambda \left[\log q - \sum_{i=1}^{m} \alpha_i \log z_i \right] \tag{2.37}$$

where w_i is the given price of input i, and λ is the Lagrange multiplier for the modified production constraint. Given that the isoquant does not touch the axis we must have an interior solution: first-order conditions are

$$w_i - \lambda \alpha_i z_i^{-1} = 0, \ i = 1, 2, .., m \tag{2.38}$$

which imply

$$z_i = \frac{\lambda \alpha_i}{w_i}, \ i = 1, 2, .., m \tag{2.39}$$

Now solve for λ. Using (2.35) and (2.39) we get

$$z_i^{\alpha_i} = \left[\frac{\lambda \alpha_i}{w_i} \right]^{\alpha_i}, \ i = 1, 2, .., m \tag{2.40}$$

$$q = \prod_{i=1}^{m} z_i^{\alpha_i} = \left[\frac{\lambda}{A} \right]^{\gamma} \prod_{i=1}^{m} w_i^{-\alpha_i} \tag{2.41}$$

where $\gamma := \sum_{j=1}^{m} \alpha_j$ and $A := \left[\prod_{i=1}^{m} \alpha_i^{\alpha_i}\right]^{-1/\gamma}$ are constants, from which we find

$$\begin{aligned} \lambda &= A\left[\frac{q}{\prod_{i=1}^{m} w_i^{-\alpha_i}}\right]^{1/\gamma} \\ &= A\left[q w_1^{\alpha_1} w_2^{\alpha_2} ... w_m^{\alpha_m}\right]^{1/\gamma}. \end{aligned} \tag{2.42}$$

Substituting from (2.42) into (2.39) we get the conditional demand function:

$$H^i(\mathbf{w}, q) = z_i^* = \frac{\alpha_i}{w_i} A\left[q w_1^{\alpha_1} w_2^{\alpha_2} ... w_m^{\alpha_m}\right]^{1/\gamma} \tag{2.43}$$

and minimised cost is

$$C(\mathbf{w}, q) = \sum_{i=1}^{m} w_i z_i^* = \gamma A\left[q w_1^{\alpha_1} w_2^{\alpha_2} ... w_m^{\alpha_m}\right]^{1/\gamma} \tag{2.44}$$

$$= \gamma B q^{1/\gamma} \tag{2.45}$$

where $B := A\left[w_1^{\alpha_1} w_2^{\alpha_2} ... w_m^{\alpha_m}\right]^{1/\gamma}$. It is clear from (2.45) that cost is increasing in q and increasing in w_i if $\alpha_i > 0$ (it is always nondecreasing in w_i). Differentiating (2.45) with respect to q marginal cost is

$$C_q(\mathbf{w}, q) = B q^{\frac{1-\gamma}{\gamma}} \tag{2.46}$$

Clearly marginal cost falls/stays constant/rises with q as $\gamma \gtreqless 1$.

2. In the short run inputs $1, ..., k$ ($k \leq m$) remain variable and the remaining inputs are fixed. In the short-run the production function can be written as

$$\log q = \sum_{i=1}^{k} \alpha_i \log z_i + \log \theta_k \tag{2.47}$$

where

$$\theta_k := \exp\left(\sum_{i=k+1}^{m} \alpha_i \log \bar{z}_i\right) \tag{2.48}$$

and $\bar{z}_i$ is the arbitrary value at which input i is fixed. The general form of the Lagrangian (2.37) remains unchanged, but with q replaced by q/θ_k and m replaced by k. So the first-order conditions and their corollaries (2.38)-(2.42) are essentially as before, but γ and A are replaced by

$$\gamma_k := \sum_{j=1}^{k} \alpha_j \tag{2.49}$$

and $A_k := \left[\prod_{i=1}^{k} \alpha_i^{\alpha_i}\right]^{-1/\gamma_k}$. Hence short-run conditional demand is

$$\tilde{H}^i(\mathbf{w}, q; \bar{z}_{k+1}, ..., \bar{z}_m) = \frac{\alpha_i}{w_i} A_k\left[\frac{q}{\theta_k} w_1^{\alpha_1} w_2^{\alpha_2} ... w_k^{\alpha_k}\right]^{1/\gamma_k} \tag{2.50}$$

and minimised cost in the short run is

$$\begin{aligned}\tilde{C}(\mathbf{w}, q; \bar{z}_{k+1}, ..., \bar{z}_m) &= \sum_{i=1}^{k} w_i z_i^* + c_k \\ &= \gamma_k A_k \left[\frac{q}{\theta_k} w_1^{\alpha_1} w_2^{\alpha_2} ... w_k^{\alpha_k}\right]^{1/\gamma_k} + c_k \quad (2.51) \\ &= \gamma_k B_k q^{1/\gamma_k} + c_k \quad (2.52)\end{aligned}$$

where

$$c_k := \sum_{i=k+1}^{m} w_i \bar{z}_i \quad (2.53)$$

is the fixed-cost component in the short run and $B_k := A_k \left[w_1^{\alpha_1} w_2^{\alpha_2} ... w_k^{\alpha_k} / \theta_k\right]^{1/\gamma_k}$. Differentiating (2.52) we find that short-run marginal cost is

$$\tilde{C}_q(\mathbf{w}, q; \bar{z}_{k+1}, ..., \bar{z}_m) = B_k q^{\frac{1-\gamma_k}{\gamma_k}}$$

3. Using the "Marginal cost=price"condition we find

$$B_k q^{\frac{1-\gamma_k}{\gamma_k}} = p \quad (2.54)$$

where p is the price of output so that, rearranging (2.54) the supply function is

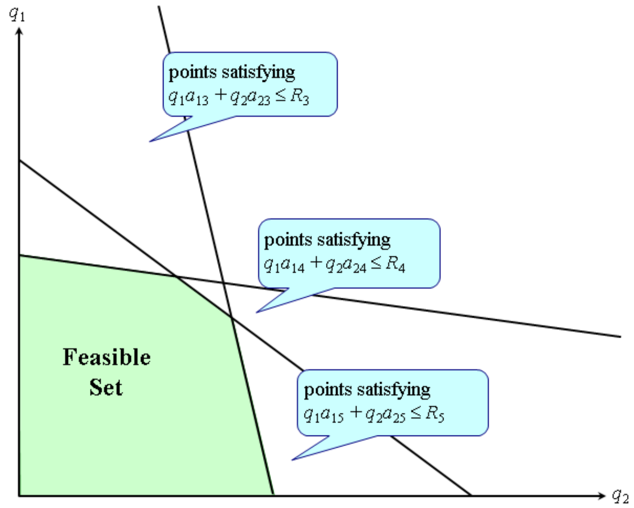
$$q = S(\mathbf{w}, p; \bar{z}_{k+1}, ..., \bar{z}_m) = \left[\frac{p}{B_k}\right]^{\frac{\gamma_k}{1-\gamma_k}} \quad (2.55)$$

wherever MC≥AC. The elasticity of (2.55) is given by

$$\frac{\partial \log S(\mathbf{w}, p; \bar{z}_{k+1}, ..., \bar{z}_m)}{\partial \log p} = \frac{\gamma_k}{1-\gamma_k} > 0 \quad (2.56)$$

It is clear from (2.49) that $\gamma_k > \gamma_{k-1} > \gamma_{k-2}...$ and so the positive supply elasticity in (2.56) must fall as k falls.

Figure 2.19: Feasible Set



Exercise 2.13 *A firm produces goods 1 and 2 using goods 3,...,5 as inputs. The production of one unit of good i ($i = 1, 2$) requires at least a_{ij} units of good j, ($j = 3, 4, 5$).*

1. *Assuming constant returns to scale, how much of resource j will be needed to produce q_1 units of commodity 1?*

2. *For given values of q_3, q_4, q_5 sketch the set of technologically feasible outputs of goods 1 and 2.*

Outline Answer

1. To produce q_1 units of commodity 1 $a_{1j}q_1$ units of resource j will be needed.
$$q_1a_{1i} + q_2a_{2i} \leq R_i.$$

2. The feasibility constraint for resource j is therefore going to be
$$q_1a_{1j} + q_2a_{2j} \leq R_j.$$
Taking into account all three resources, the feasible set is given as in Figure 2.19

Exercise 2.14 *An agricultural producer raises sheep to produce wool (good 1) and meat (good 2). There is a choice of four breeds (A, B, C, D) that can be used to stock the farm; each breed can be considered as a separate input to the production process. The yield of wool and of meat per 1000 sheep (in arbitrary units) for each breed is given in Table 2.1.*

	A	**B**	**C**	**D**
wool	20	65	85	90
meat	70	50	20	10

Table 2.1: Yield per 1000 sheep for breeds A,...,D

1. *On a diagram show the production possibilities if the producer stocks exactly 1000 sheep using just one breed from the set {A,B,C,D}.*
2. *Using this diagram show the production possibilities if the producer's 1000 sheep are a mixture of breeds A and B. Do the same for a mixture of breeds B and C; and again for a mixture of breeds C and D. Hence draw the (wool, meat) transformation curve for 1000 sheep. What would be the transformation curve for 2000 sheep?*
3. *What is the MRT of meat into wool if a combination of breeds A and B are used? What is the MRT if a combination of breeds B and C are used?And if breeds C and D are used?*
4. *Why will the producer not find it necessary to use more than two breeds?*
5. *A new breed E becomes available that has a (wool, meat) yield per 1000 sheep of (50,50). Explain why the producer would never be interested in stocking breed E if breeds A,...,D are still available and why the transformation curve remains unaffected.*
6. *Another new breed F becomes available that has a (wool, meat) yield per 1000 sheep of (74,46). Explain how this will alter the transformation curve.*

Outline Answer

1. See Figure 2.20.
2. See Figure 2.20.
3. The MRT if A and B are used is $\dfrac{70-50}{20-65} = -\dfrac{4}{9}$. If B and C are used it is going to be $\dfrac{20-50}{85-65} = -\dfrac{3}{2}$.
4. In general for m inputs and n outputs if $m > n$ then $m - n$ inputs are redundant.
5. As we can observe in Figure 2.20, by using breed E the producer cannot move the frontier (the transformation curve) outwards.

Figure 2.20: The wool and meat tradeoff

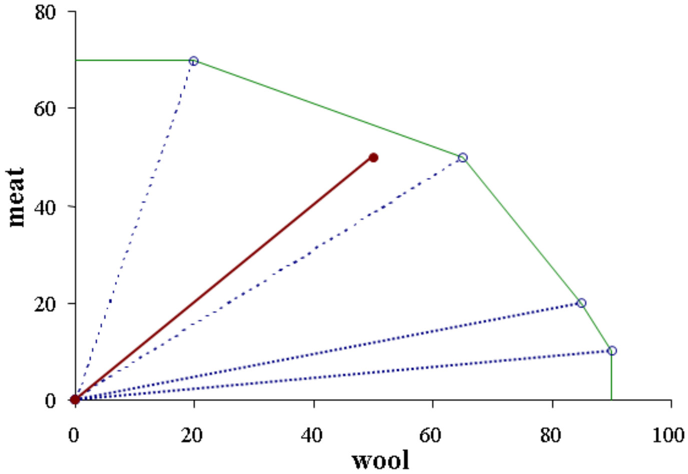
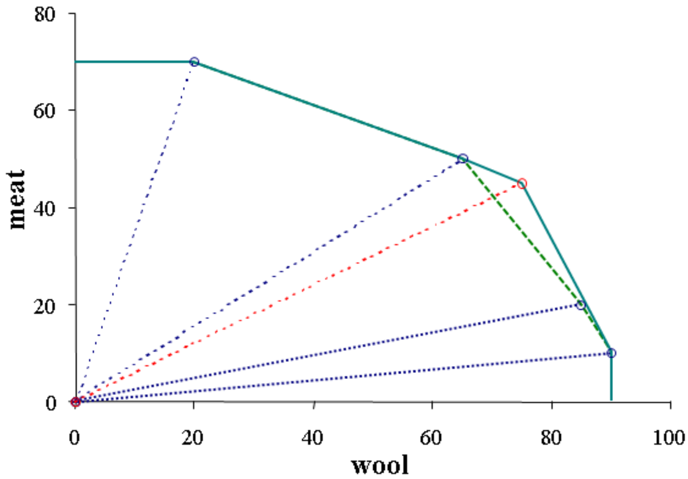


Figure 2.21: New Breed



6. As we can observe in Figure 2.21 now the technological frontier has moved outwards: one of the former techniques is no longer on the frontier.