CHAPTER 3:
Preferences and Utility

These problems provide some practice in examining utility functions by looking at indifference curve maps and at a few functional forms. The primary focus is on illustrating the notion of quasi-concavity (a diminishing MRS) in various contexts. The concepts of the budget constraint and utility maximization are not used until the next chapter.

Comments on Problems

3.1 This problem requires students to graph indifference curves for a variety of functions, some of which are not quasi-concave.

3.2 Introduces the formal definition of quasi-concavity (from Chapter 2) to be applied to the functions studied graphically in Problem 3.1.

3.3 This problem shows that diminishing marginal utility is not required to obtain a diminishing *MRS*. All of the functions are monotonic transformations of one another, so this problem illustrates that diminishing *MRS* is preserved by monotonic transformations, but diminishing marginal utility is not.

3.4 This problem focuses on whether some simple utility functions exhibit convex indifference curves.

3.5 This problem is an exploration of the fixed-proportions utility function. The problem also shows how the goods in such problems can be treated as a composite commodity.

3.6 This problem asks students to use their imaginations to explain how advertising slogans might be captured in the form of a utility function

3.7 This problem shows how utility functions can be inferred from *MRS* segments. It is a very simple example of "integrability".

3.8 This problem offers some practice in deriving utility functions from indifference curve specifications.

Analytical Problems

3.9 **Initial endowments.** This problem shows how initial endowments can be treated in simple indifference curve analysis.

3.10 **Cobb–Douglas utility.** Provides some exercises with the Cobb–Douglas function including how to integrate subsistence levels of consumption into the functional form.

3.11 **Independent marginal utilities.** Shows how analysis can be simplified if the cross partials of the utility function are zero.

3.12 **CES utility.** Shows how distributional weights can be incorporated into the CES form introduced in the chapter without changing the basic conclusions about the function.

3.13 **The quasi-linear function.** The problem provides a brief introduction to the quasi-linear form which (in later chapters) will be used to illustrate a number of interesting outcomes.

3.14 **Preference relations.** This problem provides a very brief introduction to how preferences can be treated formally with set-theoretic concepts.

3.15 **The benefit function.** Introduces Luenberger's notion of reducing preferences to a cardinal number of replications of a basic bundle of goods..

Solutions

3.1 Here we calculate the *MRS* for each of these functions:

a. $MRS = \frac{f_x}{f_y} = \frac{3}{1}$. *MRS* is constant.

b. $MRS = \frac{f_x}{f_y} = \frac{0.5(y/x)^{0.5}}{0.5(y/x)^{-0.5}} = \frac{y}{x}$. Convex; *MRS* is diminishing.

c. $MRS = \frac{f_x}{f_y} = \frac{0.5x^{-0.5}}{1}$. *MRS* is diminishing.

d. $MRS = \frac{f_x}{f_y} = .5(x^2 - y^2)^{-0.5} \cdot \frac{2x}{.5(x^2 - y^2)^{-0.5}} \cdot 2y = \frac{x}{y}$. *MRS* is increasing.

e. $MRS = \frac{f_x}{f_y} = \frac{(y(x+y)-xy)/(x+y)^2}{(x(x+y)-xy)/(x+y)^2} = \frac{y^2}{x^2}$. Convex; *MRS* is diminishing.

3.2 Because all of the first-order partials are positive, we must only check the second-order partials.

a. $f_{11} = f_{22} = f_2 = 0$. Not strictly quasi-concave.

b. $f_{11}, f_{22} < 0, f_{12} > 0$. Strictly quasi-concave.

c. $f_{11} < 0, f_{22} = 0, f_{12} = 0$. Strictly quasi-concave.

d. Even if we only consider cases where $x \geq y$, both of the own second-order partials are ambiguous and therefore the function is not necessarily strictly quasi-concave.

AU: is 'own' desired?

e. $f_{11}, f_{22} < 0, f_{12} > 0$. Strictly quasi-concave.

3.3 a. $U_x = y, U_{xx} = 0, U_y = x, U_{yy} = 0, MRS = y/x$.

b. $U_x = 2xy^2, U_{xx} = 2y^2, U_y = 2x^2y, U_{yy} = 2x^2, MRS = y/x$.

c. $U_x = 1/x, U_{xx} = -1/x, U_y = 1/y, U_{yy} = -1/y^2, MRS = y/x$. This shows that monotonic transformations may affect diminishing marginal utility, but not the *MRS*.

3.4 a. In the range in which the same good is limiting, the indifference curve is linear. To see this, take the case in which both $x_1 \leq y_1$ and $x_2 \leq y_2$. Then $k = U(x_1, y_1) = x_1$ and $k = U(x_2, y_2) = x_2$, implying

$$U\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \frac{x_1 + x_2}{2} = \frac{k+k}{2} = k$$

as well.

In the range in which the limiting goods differ, we can show the indifference curve is strictly convex. Take the case $k = x_1 < y_1$ and $k = y_2 < x_2$. Then $(x_1 + x_2)/2 > k$ and $(y_1 + y_2)/2 > k$, implying

$$U\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)>k.$$

Hence the indifference curve is convex.

b. Again, in the range in which the same good is maximum, the indifference curve can be shown to be linear. Consider a range in which different goods are maximum, specifically, $k=x_1>y_1$ and $k=y_2>x_2$. Then $(x_1+x_2)/2<k$ and $(y_1+y_2)/2<k$, implying

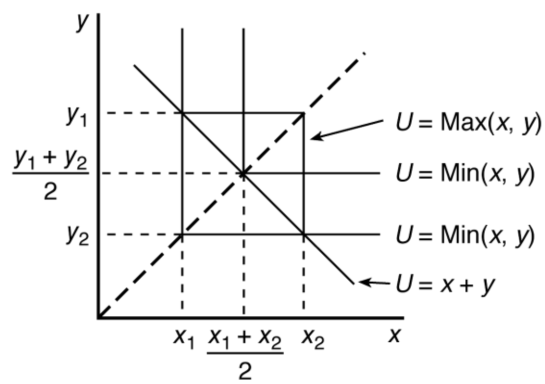
$$U\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)<k.$$

Hence the indifference curve is concave.

c. Here,

$$(x_1+y_1)=k=(x_2+y_2)=U\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right).$$

Hence the indifference curve is linear.



3.5 a. Since the four goods are perfect complements, $U(h,r,m,c)=\min(h,2r,m,0.5c)$.

b. A fully condimented hot dog.

c. £3.90

d. £5.15, an increase of 32%.

e. Price would increase only to £4.025, an increase of 3.2%.

f. Raise prices so that a fully condimented hot dog rises in price to £4.90. This would be equivalent to a lump-sum reduction in purchasing power.

3.6 For all the suggested utility functions, let x represent some other good and the good in question is represented by the appropriate letter(s):

a. $U(x,p) \geq U(x,b)$ for $p=b$.

b. Given $U(x,c)$, $U_{xc} = U_{cx} > 0$.

c. Given $U(x,kf)$ $U(x,1) < U(x,0) < U(x,kf>1)$.

d. $U(x, M\&B) > U(x, WB)$ for $M\&B = WB$.

e. $U(x, sab) < (x, sab_{responsible})$ for $sab > sab_{responsible}$.

3.7 a. $MRS = 1/3$ at both points. Since both the points lie on the same indifference curve (as the utility at both points is the same), this means that the slope of the indifference curve is constant (i.e., straight line). So the goods are perfect substitutes.

b. We know that for a Cobb–Douglas utility function,

$$MRS = \frac{\alpha}{\beta}\frac{y}{x}.$$

Using this formula, the values of MRS ($1/4$ at the first point and 2 at the second) and the values of x and y at the two points, we can construct a pair of equations in α and β that can be solved simultaneously. We get $\alpha = 2$ and $\beta = 1$; the utility function is of the form $U = x^2 y$.

c. Yes, there was a redundancy. We never used the information that the two points were on the same indifference curve. The MRS and the values of x and y at the points will suffice to find the utility function given the function is Cobb–Douglas. (or, alternatively, some other combination of the information in part b excluding one piece of information).

3.8 a. $U = x^\alpha y^\beta z^\delta$.

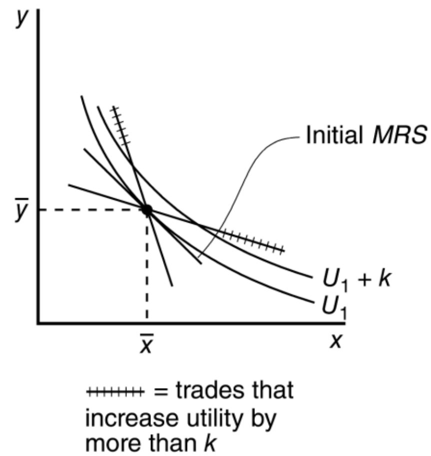
b. $U = x^2 + xy + y^2$.

c. $U = x^2 y + y^2 z + z^2 x$.

Analytical Problems

3.9 Initial endowments

a.



b. Any trading opportunities that differ from the *MRS* at $\bar{x}, \bar{y}$ will provide the opportunity to raise utility (see figure).

c. A preference for the initial endowment will require that trading opportunities raise utility substantially. This will be more likely if the trading opportunities are significantly different from the initial *MRS* (see figure).

3.10 Cobb–Douglas utility

a. $$MRS = \frac{\partial U/\partial x}{\partial U/\partial y} = \frac{\alpha x^{\alpha-1} y^{\beta}}{\beta x^{\alpha} y^{\beta-1}} = \frac{\alpha}{\beta} \cdot \frac{y}{x}.$$

This result does not depend on the sum $\alpha + \beta$ which, contrary to production theory, has no significance in consumer theory because utility is unique only up to a monotonic transformation.

b. The mathematics follow directly from part a. If $\alpha > \beta$, the individual values x relatively more highly. Hence, $MRS > 1$ for $x = y$.

c. The function is homothetic in $x - x_0$ and $y - y_0$, but not in x and y.

3.11 Independent marginal utilities

From Problem 3.2, $f_{12} = 0$ implies diminishing *MRS* providing $f_{11}, f_{22} < 0$. Conversely, the Cobb–Douglas has $f_{12} > 0$ and $f_{11}, f_{22} < 0$, but also has a diminishing *MRS* (see problem 3.8a).

3.12 CES utility

a. $MRS = \dfrac{\partial U/\partial x}{\partial U/\partial y} = \dfrac{\alpha x^{\delta-1}}{\beta y^{\delta-1}} = \dfrac{\alpha}{\beta}\left(\dfrac{y}{x}\right)^{1-\delta}$, so this function is homothetic.

b. If $\delta = 1$, $MRS = \alpha/\beta$, a constant. If $\delta = 0$,

$$MRS = \frac{\alpha}{\beta}\cdot\frac{y}{x},$$

This agrees with Problem 3.10.

c. $\dfrac{\partial MRS}{\partial x} = (\delta - 1)\dfrac{\alpha}{\beta} y^{1-\delta} x^{\delta-2}$. This is negative if and only if $\delta < 1$.

d. Follows from part a. If $x = y$, $MRS = \alpha/\beta$.

e. With $\delta = .5$,

$$MRS(.9) = \frac{\alpha}{\beta}(.9)^{0.5} = .949\frac{\alpha}{\beta}$$
$$MRS(1.1) = \frac{\alpha}{\beta}(1.1)^{0.5} = 1.05\frac{\alpha}{\beta}.$$

With $\delta = -1$,

$$MRS(.9) = \frac{\alpha}{\beta}(.9)^{2} = .81\frac{\alpha}{\beta}$$
$$MRS(1.1) = \frac{\alpha}{\beta}(1.1)^{2} = 1.21\frac{\alpha}{\beta}.$$

Hence, the *MRS* changes more dramatically when $\delta = -1$ than when $\delta = .5$. The indifference curves are more sharply curved when δ is lower. When $\delta = -\infty$, the indifference curves are L-shaped, implying fixed proportions.

3.13 The quasi-linear function

a. $MRS = y$.

b. Check $f_{11}f_2^2 - 2f_{12}f_1f_2 + f_{22}f_1^2 < 0$. We have

$$f_x = f_1 = 1$$
$$f_y = f_2 = \frac{1}{y}$$
$$f_{11} = 0$$
$$f_{22} = -\frac{1}{y^2}$$
$$f_{12} = 0.$$

So,

$$f_{11}f_2^2 - 2f_{12}f_1f_2 + f_{22}f_1^2 = 0 + 0 - \frac{1}{y^2} = \frac{1}{y^2}.$$

For $y > 0$,, the value is negative.

c. $y = e^{C-x}$.

d. Since the marginal utility of x is a constant at 1, while that of y is decreasing as y increases (as it is of the form $1/y$), we would expect consumers to shift more towards x and away from y when they get to buy more goods to increase utility due to an income raise. This is because consumers will always try to maximize utility. They are better off with a higher marginal utility.

e. Refer to Example 3.4. This function is usually used to describe the consumption of one commodity with respect to all other commodities. So $\ln y$ could represent the singular commodity while x could represent all the other goods consumed.

3.14 Preference relations

All of the suggested preference relations are complete, transitive, and continuous.

a. **Summation**

Complete: Clearly all bundles are ranked by the sum of items contained.
Transitive: If bundle A has more items than B and B has more items than C, clearly A has more items than C.
Continuous: If bundle A contains more items than bundle B, then A is preferred to B and any bundle with slightly more items than B (but fewer than A) is also preferred to B.

b. **Lexicographic**

Complete: All bundles can be ranked in this ordered way.
Transitive: If bundle A is preferred to bundle B with ties being broken at the *i*th good and B is preferred to C with ties broken at the (*i*+*j*)th good, then A will be preferred to C because it will break the tie at the *i*th good also.
Continuous: Suppose bundle A is preferred to B with the tie break occurring at the *i*th good. Then there exists a bundle C with slightly more of this good than B but less than A which will be preferred to B.

c. **Bliss**

Complete: Clearly all bundles are ranked by the distance metric.
Transitive: The distance metric itself imposes a cardinal ranking which is clearly transitive.
Continuous: If bundle A is any positive distance from Bliss, there will exist another bundle slightly closer since any single good that is not at Bliss can be made closer to it.

3.15 The benefit function

a. $U^* = x_1^{\beta} y_1^{1-\beta} = \alpha^{\beta}\alpha^{1-\beta} = \alpha$. Hence $b(U^*) = U^*$.

b. In this case the benefit function cannot be computed because the Cobb–Douglas requires positive quantities of both goods to take a non-zero value.

c. In the graph below the benefit associated with any initial endowment is the length of the vector from the initial endowment to the utility target where the direction of the vector is given by the composition of the elementary bundle.

d. In the graph below two initial endowments are shown $(E_1 \text{ and } E_2)$. The benefit for each endowment is also shown by the vectors in the graph. The benefit is also shown for an initial endowment given by $(E_1 + E_2)/2$. By completing the parallelogram it is clear that the convexity of the indifference curve implies that

$$b(U^*, E_1) > b(U^*, E_1 + E_2/2) < b(U^*, E_2).$$

[Figure 3.15]