# Instructors' Solutions

*for*

# Mathematical Methods for Physics and Engineering (*third edition*)

K.F. Riley and M.P. Hobson

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# Introduction

The second edition of *Mathematical Methods for Physics and Engineering* carried more than twice as many exercises, based on its various chapters, as did the first. In the Preface we discussed the general question of how such exercises should be treated but, in the end, decided to provide hints and outline answers to all problems, as in the first edition. This decision was an uneasy one as, on the one hand, it did not allow the exercises to be set as totally unaided homework that could be used for assessment purposes but, on the other, it did not give a full explanation of how to tackle a problem when a student needed explicit guidance or a model answer.

In order to allow both of these educationally desirable goals to be achieved we have, in the third edition, completely changed the way this matter is handled. All of the exercises from the second edition, plus a number of additional ones testing the newly-added material, have been included in penultimate subsections of the appropriate, sometimes reorganised, chapters. Hints and outline answers are given, as previously, in the final subsections, but *only to the odd-numbered exercises*. This leaves all even-numbered exercises free to be set as unaided homework, as described below.

For the four hundred plus **odd-numbered** exercises, complete solutions are available, to both students and their teachers, in the form of a separate manual, K. F. Riley and M. P. Hobson, *Student Solutions Manual for Mathematical Methods for Physics and Engineering, 3rd edn.* (Cambridge: CUP, 2006). These full solutions are additional to the hints and outline answers given in the main text. For each exercise, the original question is reproduced and then followed by a fully-worked solution. For those exercises that make internal reference to the main text or to other (even-numbered) exercises not included in the manual, the questions have been reworded, usually by including additional information, so that the questions can stand alone.

The remaining four hundred or so **even-numbered** exercises have no hints or answers, outlined or detailed, available for general access. They can therefore be used by instructors as a basis for setting unaided homework. Full solutions to these exercises, in the same general format as those appearing in the manual (though they may contain cross-references to the main text or to other exercises), form the body of the material on this website.

In many cases, in the manual as well as here, the solution given is even fuller than one that might be expected of a good student who has understood the material. This is because we have aimed to make the solutions instructional as well as utilitarian. To this end, we have included comments that are intended to show how the plan for the solution is fomulated and have given the justifications for particular intermediate steps (something not always done, even by the best of students). We have also tried to write each individual substituted formula in the form that best indicates how it was obtained, before simplifying it at the next or a subsequent stage. Where several lines of algebraic manipulation or calculus are needed to obtain a final result they are normally included in full; this should enable the instructor to determine whether a student's incorrect answer is due to a misunderstanding of principles or to a technical error.

In all new publications, on paper or on a website, errors and typographical mistakes are virtually unavoidable and we would be grateful to any instructor who brings instances to our attention.

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# 1

# *Preliminary algebra*

*Polynomial equations*

> ***1.2*** *Determine how the number of real roots of the equation*
>
> $$g(x) = 4x^3 - 17x^2 + 10x + k = 0$$
>
> *depends upon k. Are there any cases for which the equation has exactly two distinct real roots?*

We first determine the positions of the turning points (if any) of $g(x)$ by equating its derivative $g'(x) = 12x^2 - 34x + 10$ to zero. The roots of $g'(x) = 0$ are given, either by factorising $g'(x)$, or by the standard formula,

$$\alpha_{1,2} = \frac{34 \pm \sqrt{1156 - 480}}{24},$$

as $\frac{5}{2}$ and $\frac{1}{3}$.

We now determine the values of $g(x)$ at these turning points; they are $g(\frac{5}{2}) = -\frac{75}{4} + k$ and $g(\frac{1}{3}) = \frac{43}{27} + k$. These will remain of opposite signs, as is required for three real roots, provided $k$ remains in the range $-\frac{43}{27} < k < \frac{75}{4}$. If $k$ is equal to one of these two extreme values, a graph of $g(x)$ just touches the $x$-axis and two of the roots become coincident, resulting in only two *distinct* real roots.

> ***1.4*** *Given that $x = 2$ is one root of*
>
> $$g(x) = 2x^4 + 4x^3 - 9x^2 - 11x - 6 = 0,$$
>
> *use factorisation to determine how many real roots it has.*

Given that $x = 2$ is one root of $g(x) = 0$, we write $g(x) = (x-2)h(x)$ or, more explicitly,

$$2x^4 + 4x^3 - 9x^2 - 11x - 6 = (x-2)(b_3x^3 + b_2x^2 + b_1x + b_0).$$

Equating the coefficients of successive (decreasing) powers of $x$, we obtain

$$b_3 = 2, \quad b_2 - 2b_3 = 4, \quad b_1 - 2b_2 = -9, \quad b_0 - 2b_1 = -11, \quad -2b_0 = -6.$$

These five equations have the consistent solution for the four unknowns $b_i$ of $b_3 = 2$, $b_2 = 8$, $b_1 = 7$ and $b_0 = 3$. Thus $h(x) = 2x^3 + 8x^2 + 7x + 3$.

Clearly, since all of its coefficients are positive, $h(x)$ can have no zeros for positive values of $x$. A few tests with negative integer values of $x$ (with the initial intention of making a rough sketch) reveal that $h(-3) = 0$, implying that $(x+3)$ is a factor of $h(x)$. We therefore write

$$2x^3 + 8x^2 + 7x + 3 = (x+3)(c_2x^2 + c_1x + c_0),$$

and, proceeding as previously, obtain $c_2 = 2$, $c_1 + 3c_2 = 8$, $c_0 + 3c_1 = 7$ and $3c_0 = 3$, with corresponding solution $c_2 = 2$, $c_1 = 2$ and $c_0 = 1$.

We now have that $g(x) = (x-2)(x+3)(2x^2 + 2x + 1)$. If we now try to determine the zeros of the quadratic term using the standard form (1.4) we find that, since $2^2 - (4 \times 2 \times 1)$, i.e. $-4$, is negative, its zeros are complex. In summary, the only real roots of $g(x) = 0$ are $x = 2$ and $x = -3$.

***1.6*** *Use the results of* (i) *equation (1.13),* (ii) *equation (1.12) and* (iii) *equation (1.14) to prove that if the roots of* $3x^3 - x^2 - 10x + 8 = 0$ *are* $\alpha_1, \alpha_2$ *and* $\alpha_3$ *then*

(a) $\alpha_1^{-1} + \alpha_2^{-1} + \alpha_3^{-1} = 5/4$,
(b) $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 61/9$,
(c) $\alpha_1^3 + \alpha_2^3 + \alpha_3^3 = -125/27$.
(d) *Convince yourself that eliminating (say)* $\alpha_2$ *and* $\alpha_3$ *from* (i)*,* (ii) *and* (iii) *does* not *give a simple explicit way of finding* $\alpha_1$.

If the roots of $3x^3 - x^2 - 10x + 8 = 0$ are $\alpha_1$, $\alpha_2$ and $\alpha_3$, then:

(i) from equation (1.13), $\alpha_1 + \alpha_2 + \alpha_3 = -\dfrac{-1}{3} = \dfrac{1}{3}$;

(ii) from equation (1.12), $\alpha_1\alpha_2\alpha_3 = (-1)^3\dfrac{8}{3} = -\dfrac{8}{3}$;

(iii) from equation (1.14), $\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1 = \dfrac{-10}{3} = -\dfrac{10}{3}$.

We now use these results in various combinations to obtain expressions for the given quantities:

(a) $\dfrac{1}{\alpha_1} + \dfrac{1}{\alpha_2} + \dfrac{1}{\alpha_3} = \dfrac{\alpha_2\alpha_3 + \alpha_1\alpha_3 + \alpha_2\alpha_1}{\alpha_1\alpha_2\alpha_3} = \dfrac{-(10/3)}{-(8/3)} = \dfrac{5}{4}$;

(b) $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = (\alpha_1 + \alpha_2 + \alpha_3)^2 - 2(\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1)$

$= \left(\frac{1}{3}\right)^2 - 2\left(-\frac{10}{3}\right) = \frac{61}{9}$;

(c) $\alpha_1^3 + \alpha_2^3 + \alpha_3^3 =$

$(\alpha_1 + \alpha_2 + \alpha_3)^3 - 3(\alpha_1 + \alpha_2 + \alpha_3)(\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1) + 3\alpha_1\alpha_2\alpha_3$

$= (\frac{1}{3})^3 - 3(\frac{1}{3})(-\frac{10}{3}) + 3(-\frac{8}{3}) = -\frac{125}{27}$.

(d) No answer is given as it cannot be done. All manipulation is complicated and, at best, leads back to the original equation. Unfortunately, the 'convincing' will have to come from frustration, rather than from a proof by contradiction!

*Trigonometric identities*

> ***1.8*** *The following exercises are based on the half-angle formulae.*
>
> (a) *Use the fact that* $\sin(\pi/6) = 1/2$ *to prove that* $\tan(\pi/12) = 2 - \sqrt{3}$.
> (b) *Use the result of* (a) *to show further that* $\tan(\pi/24) = q(2 - q)$, *where* $q^2 = 2 + \sqrt{3}$.

(a) Writing $\tan(\pi/12)$ as $t$ and using (1.32), we have

$$\frac{1}{2} = \sin\frac{\pi}{6} = \frac{2t}{1+t^2},$$

from which it follows that $t^2 - 4t + 1 = 0$.

The quadratic solution (1.6) then shows that $t = 2 \pm \sqrt{2^2 - 1} = 2 \pm \sqrt{3}$; there are two solutions because $\sin(5\pi/6)$ is also equal to $1/2$. To resolve the ambiguity, we note that, since $\pi/12 < \pi/4$ and $\tan(\pi/4) = 1$, we must have $t < 1$; hence, the negative sign is the appropriate choice.

(b) Writing $\tan(\pi/24)$ as $u$ and using (1.34) and the result of part (a), we have

$$2 - \sqrt{3} = \frac{2u}{1-u^2}.$$

Multiplying both sides by $q^2 = 2 + \sqrt{3}$, and then using $(2 + \sqrt{3})(2 - \sqrt{3}) = 1$, gives

$$1 - u^2 = 2q^2u.$$

This quadratic equation has the (positive) solution

$$\begin{aligned} u &= -q^2 + \sqrt{q^4 + 1} \\ &= -q^2 + \sqrt{4 + 4\sqrt{3} + 3 + 1} \\ &= -q^2 + 2\sqrt{2 + \sqrt{3}} \\ &= -q^2 + 2q = q(2 - q), \end{aligned}$$

as stated in the question.

> **1.10** *If* $s = \sin(\pi/8)$, *prove that*
>
> $$8s^4 - 8s^2 + 1 = 0,$$
>
> *and hence show that* $s = [(2 - \sqrt{2})/4]^{1/2}$.

With $s = \sin(\pi/8)$, using (1.29) gives

$$\sin\frac{\pi}{4} = 2s(1 - s^2)^{1/2}.$$

Squaring both sides, and then using $\sin(\pi/4) = 1/\sqrt{2}$, leads to

$$\frac{1}{2} = 4s^2(1 - s^2),$$

i.e. $8s^4 - 8s^2 + 1 = 0$. This is a quadratic equation in $u = s^2$, with solutions

$$s^2 = u = \frac{8 \pm \sqrt{64 - 32}}{16} = \frac{2 \pm \sqrt{2}}{4}.$$

Since $\pi/8 < \pi/4$ and $\sin(\pi/4) = 1/\sqrt{2} = \sqrt{2/4}$, it is clear that the minus sign is the appropriate one. Taking the square root of both sides then yields the stated answer.

## *Coordinate geometry*

> **1.12** *Obtain in the form (1.38), the equations that describe the following:*
>
> (a) *a circle of radius* 5 *with its centre at* $(1, -1)$;
> (b) *the line* $2x + 3y + 4 = 0$ *and the line orthogonal to it which passes through* $(1, 1)$;
> (c) *an ellipse of eccentricity* 0.6 *with centre* $(1, 1)$ *and its major axis of length* 10 *parallel to the* $y$*-axis.*

(a) Using (1.42) gives $(x-1)^2 + (y+1)^2 = 5^2$, i.e. $x^2 + y^2 - 2x + 2y - 23 = 0$.

(b) From (1.24), a line orthogonal to $2x + 3y + 4 = 0$ must have the form $3x - 2y + c = 0$, and, if it is to pass through $(1, 1)$, then $c = -1$. Expressed in the form (1.38), the pair of lines takes the form

$$0 = (2x + 3y + 4)(3x - 2y - 1) = 6x^2 - 6y^2 + 5xy + 10x - 11y - 4.$$

(c) As the major semi-axis has length 5 and the eccentricity is 0.6, the minor semi-axis has length $5[1 - (0.6)^2]^{1/2} = 4$. The equation of the ellipse is therefore

$$\frac{(x-1)^2}{4^2} + \frac{(y-1)^2}{5^2} = 1,$$

which can be written as $25x^2 + 16y^2 - 50x - 32y - 359 = 0$.

**1.14** *For the ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

*with eccentricity e, the two points $(-ae, 0)$ and $(ae, 0)$ are known as its foci. Show that the sum of the distances from* any *point on the ellipse to the foci is $2a$.*

[*The constancy of the sum of the distances from two fixed points can be used as an alternative defining property of an ellipse.*]

Let the sum of the distances be $s$. Then, for a point $(x, y)$ on the ellipse,

$$s = [\,(x + ae)^2 + y^2\,]^{1/2} + [\,(x - ae)^2 + y^2\,]^{1/2},$$

where the positive square roots are to be taken.

Now, $y^2 = b^2[1 - (x/a)^2]$, with $b^2 = a^2(1 - e^2)$. Thus, $y^2 = (1 - e^2)(a^2 - x^2)$ and

$$\begin{aligned} s &= (x^2 + 2aex + a^2e^2 + a^2 - a^2e^2 - x^2 + e^2x^2)^{1/2} \\ &\quad + (x^2 - 2aex + a^2e^2 + a^2 - a^2e^2 - x^2 + e^2x^2)^{1/2} \\ &= (a + ex) + (a - ex) = 2a. \end{aligned}$$

This result is independent of $x$ and hence holds for *any* point on the ellipse.

### *Partial fractions*

**1.16** *Express the following in partial fraction form:*

$$\text{(a)}\ \frac{2x^3 - 5x + 1}{x^2 - 2x - 8}, \qquad \text{(b)}\ \frac{x^2 + x - 1}{x^2 + x - 2}.$$

(a) For

$$f(x) = \frac{2x^3 - 5x + 1}{x^2 - 2x - 8},$$

we note that the degree of the numerator is higher than that of the denominator, and so we must first divide through by the latter. Write

$$2x^3 - 5x + 1 = (2x + s_0)(x^2 - 2x - 8) + (r_1 x + r_0).$$

Equating the coefficients of the powers of $x$: $0 = s_0 - 4$, $-5 = -16 - 2s_0 + r_1$, and $1 = -8s_0 + r_0$, giving $s_0 = 4$, $r_1 = 19$, and $r_0 = 33$. Thus,

$$f(x) = 2x + 4 + \frac{19x + 33}{x^2 - 2x - 8}.$$

The denominator in the final term factorises as $(x - 4)(x + 2)$, and so we write the term as

$$\frac{A}{x-4} + \frac{B}{x+2}.$$

Using the third method given in section 1.4:

$$A = \frac{19(4) + 33}{4 + 2} \quad \text{and} \quad B = \frac{19(-2) + 33}{-2 - 4}.$$

Thus,

$$f(x) = 2x + 4 + \frac{109}{6(x-4)} + \frac{5}{6(x+2)}.$$

(b) Since the highest powers of $x$ in the denominator and numerator are equal, the partial–fraction expansion takes the form

$$f(x) = \frac{x^2 + x - 1}{x^2 + x - 2} = 1 + \frac{1}{x^2 + x - 2} = 1 + \frac{A}{x+2} + \frac{B}{x-1}.$$

Using the same method as above, we have

$$A = \frac{1}{-2-1}; \quad B = \frac{1}{1+2}.$$

Thus,

$$f(x) = 1 - \frac{1}{3(x+2)} + \frac{1}{3(x-1)}.$$

***1.18*** *Resolve the following into partial fractions in such a way that $x$ does not appear in any numerator:*

(a) $\dfrac{2x^2 + x + 1}{(x-1)^2(x+3)}$, (b) $\dfrac{x^2 - 2}{x^3 + 8x^2 + 16x}$, (c) $\dfrac{x^3 - x - 1}{(x+3)^3(x+1)}$.

Since no factor $x$ may appear in a numerator, all repeated factors appearing in the denominator give rise to as many terms in the partial fraction expansion as the power to which that factor is raised in the denominator.

(a) The denominator is already factorised but contains the repeated factor $(x-1)^2$. Thus the expansion will contain a term of the form $(x-1)^{-1}$, as well as one of the form $(x-1)^{-2}$. So,

$$\frac{2x^2+x+1}{(x-1)^2(x+3)} = \frac{A}{x+3} + \frac{B}{(x-1)^2} + \frac{C}{x-1}.$$

We can evaluate $A$ and $B$ using the third method given in section 1.4:

$$A = \frac{2(-3)^2-3+1}{(-3-1)^2} = 1 \quad \text{and} \quad B = \frac{2(1)^2+1+1}{1+3} = 1.$$

We now evaluate $C$ by setting $x=0$ (say):

$$\frac{1}{(-1)^2 3} = \frac{1}{3} + \frac{1}{(-1)^2} + \frac{C}{-1},$$

giving $C=1$ and the full expansion as

$$\frac{2x^2+x+1}{(x-1)^2(x+3)} = \frac{1}{x+3} + \frac{1}{(x-1)^2} + \frac{1}{x-1}.$$

(b) Here the denominator needs factorising, but this is elementary,

$$\frac{x^2-2}{x^3+8x^2+16x} = \frac{x^2-2}{x(x+4)^2} = \frac{A}{x} + \frac{B}{(x+4)^2} + \frac{C}{x+4}.$$

Now, using the same method as in part (a):

$$A = \frac{0-2}{(0+4)^2} = -\frac{1}{8} \quad \text{and} \quad B = \frac{(-4)^2-2}{-4} = -\frac{7}{2}.$$

Setting $x=1$ (say) determines $C$ through

$$\frac{-1}{25} = -\frac{1}{8(1)} - \frac{7}{2(5)^2} + \frac{C}{5}.$$

Thus $C = 9/8$, and the full expression is

$$\frac{x^2-2}{x^3+8x^2+16x} = -\frac{1}{8x} - \frac{7}{2(x+4)^2} + \frac{9}{8(x+4)}.$$

(c)

$$\frac{x^3-x-1}{(x+3)^3(x+1)} = \frac{A}{x+1} + \frac{B}{(x+3)^3} + \frac{C}{(x+3)^2} + \frac{D}{x+3}.$$

As in parts (a) and (b), the third method in section 1.4 gives $A$ and $B$ as

$$A = \frac{(-1)^3-(-1)-1}{(-1+3)^3} = -\frac{1}{8} \quad \text{and} \quad B = \frac{(-3)^3-(-3)-1}{-3+1} = \frac{25}{2}.$$

Setting $x = 0$ requires that

$$\frac{-1}{27} = -\frac{1}{8} + \frac{25}{54} + \frac{C}{9} + \frac{D}{3} \qquad \text{i.e. } C + 3D = -\frac{27}{8}.$$

Setting $x = 1$ gives the additional requirement that

$$\frac{-1}{128} = -\frac{1}{16} + \frac{25}{128} + \frac{C}{16} + \frac{D}{4} \qquad \text{i.e. } C + 4D = -\frac{18}{8}.$$

Solving these two equations for $C$ and $D$ now yields $D = 9/8$ and $C = -54/8$. Thus,

$$\frac{x^3 - x - 1}{(x+3)^3(x+1)} = -\frac{1}{8(x+1)} + \frac{1}{8}\left[\frac{100}{(x+3)^3} - \frac{54}{(x+3)^2} + \frac{9}{x+3}\right].$$

If necessary, that the expansion is valid for all $x$ (and not just for 0 and 1) can be checked by writing all of its terms so as to have the common denominator $(x+3)^3(x+1)$.

*Binomial expansion*

**1.20** *Use a binomial expansion to evaluate* $1/\sqrt{4.2}$ *to five places of decimals, and compare it with the accurate answer obtained using a calculator.*

To use the binomial expansion, we need to express the inverse square root in the form $(1+a)^{-1/2}$ with $|a| < 1$. We do this as follows.

$$\begin{aligned}\frac{1}{\sqrt{4.2}} &= \frac{1}{(4+0.2)^{1/2}} = \frac{1}{2(1+0.05)^{1/2}} \\ &= \frac{1}{2}\left[1 - \frac{1}{2}(0.05) + \frac{3}{8}(0.05)^2 - \frac{15}{48}(0.05)^3 + \cdots\right] \\ &= 0.487949218.\end{aligned}$$

This four-term sum and the accurate value differ by about $8 \times 10^{-7}$.

*Proof by induction and contradiction*

**1.22** *Prove by induction that*

$$1 + r + r^2 + \cdots + r^k + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}.$$

To prove that

$$\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r},$$

assume that the result is valid for $n = N$, and consider the corresponding sum for $n = N+1$, which is the original sum plus one additional term:

$$\begin{aligned}
\sum_{k=0}^{N+1} r^k &= \sum_{k=0}^{N} r^k + r^{N+1} \\
&= \frac{1-r^{N+1}}{1-r} + r^{N+1}, \quad \text{using the assumption,} \\
&= \frac{1-r^{N+1}+r^{N+1}-r^{N+2}}{1-r} \\
&= \frac{1-r^{N+2}}{1-r}.
\end{aligned}$$

This is the same form as in the assumption, except that $N$ has been replaced by $N+1$, and shows that the result is valid for $n = N+1$ if it is valid for $n = N$.

But, since $(1-r)/(1-r) = 1$, the result is trivially valid for $n = 0$. It therefore follows that it is valid for all $n$.

**1.24** *If a sequence of terms $u_n$ satisfies the recurrence relation $u_{n+1} = (1-x)u_n + nx$, with $u_1 = 0$, then show by induction that, for $n \geq 1$,*

$$u_n = \frac{1}{x}[nx - 1 + (1-x)^n].$$

Assume that the stated result is valid for $n = N$, and consider the expression for the next term in the sequence:

$$\begin{aligned}
u_{N+1} &= (1-x)u_N + Nx \\
&= \frac{1-x}{x}\left[Nx - 1 + (1-x)^N\right] + Nx, \quad \text{using the assumption,} \\
&= \frac{1}{x}\left[Nx - Nx^2 - 1 + x + (1-x)^{N+1} + Nx^2\right] \\
&= \frac{1}{x}\left[(N+1)x - 1 + (1-x)^{N+1}\right].
\end{aligned}$$

This has the same form as in the assumption, except that $N$ has been replaced by $N+1$, and shows that the result is valid for $n = N+1$ if it is valid for $n = N$.

The assumed result gives $u_1$ as $x^{-1}(x-1+1-x) = 0$ (i.e. as stated in the question), and so is valid for $n = 1$. It now follows, from the result proved earlier, that the given expression is valid for all $n \geq 1$.

**1.26** *The quantities $a_i$ in this exercise are all positive real numbers.*

(a) *Show that*

$$a_1 a_2 \le \left( \frac{a_1 + a_2}{2} \right)^2.$$

(b) *Hence, prove by induction on $m$ that*

$$a_1 a_2 \cdots a_p \le \left( \frac{a_1 + a_2 + \cdots + a_p}{p} \right)^p,$$

*where $p = 2^m$ with $m$ a positive integer. Note that each increase of $m$ by unity doubles the number of factors in the product.*

(a) Consider $(a_1 - a_2)^2$ which is always non-negative:

$$\begin{aligned}
(a_1 - a_2)^2 &\ge 0, \\
a_1^2 - 2a_1 a_2 + a_2^2 &\ge 0, \\
a_1^2 + 2a_1 a_2 + a_2^2 &\ge 4a_1 a_2, \\
(a_1 + a_2)^2 &\ge 4a_1 a_2, \\
\left( \frac{a_1 + a_2}{2} \right)^2 &\ge a_1 a_2.
\end{aligned}$$

(b) With $p = 2^m$, assume that

$$a_1 a_2 \cdots a_p \le \left( \frac{a_1 + a_2 + \cdots + a_p}{p} \right)^p$$

is valid for some $m = M$. Write $P = 2^M$, $P' = 2P$, $b_1 = a_1 + a_2 + \cdots + a_P$ and $b_2 = a_{P+1} + a_{P+2} + \cdots + a_{P'}$. Note that both $b_1$ and $b_2$ consist of $P$ terms.

Now consider the multiple product $u = a_1 a_2 \cdots a_P a_{P+1} a_{P+2} \cdots a_{P'}$.

$$\begin{aligned}
u &\le \left( \frac{a_1 + a_2 + \cdots + a_P}{P} \right)^P \left( \frac{a_{P+1} + a_{P+2} + \cdots + a_{P'}}{P} \right)^P \\
&= \left( \frac{b_1 b_2}{P^2} \right)^P,
\end{aligned}$$

where the assumed result has been applied twice, once to a set consisting of the first $P$ numbers, and then for a second time to the remaining set of $P$ numbers, $a_{P+1}, a_{P+2}, \ldots, a_{P'}$. We have also used the fact that, for positive real numbers, if $q \le r$ and $s \le t$ then $qs \le rt$.

But, from part (a),

$$b_1 b_2 \le \left( \frac{b_1 + b_2}{2} \right)^2.$$

Thus,

$$\begin{aligned} a_1 a_2 \cdots a_P a_{P+1} a_{P+2} \cdots a_{P'} &\le \left(\frac{1}{P^2}\right)^P \left(\frac{b_1+b_2}{2}\right)^{2P} \\ &= \frac{(b_1+b_2)^{P'}}{(2P)^{2P}} \\ &= \left(\frac{b_1+b_2}{P'}\right)^{P'}. \end{aligned}$$

This shows that the result is valid for $P' = 2^{M+1}$ if it is valid for $P = 2^M$. But for $m = M = 1$ the postulated inequality is simply result (a), which was shown directly. Thus the inequality holds for all positive integer values of $m$.

**1.28** *An arithmetic progression of integers $a_n$ is one in which $a_n = a_0 + nd$, where $a_0$ and $d$ are integers and $n$ takes successive values $0, 1, 2, \ldots$.*

(a) *Show that if any one term of the progression is the cube of an integer, then so are infinitely many others.*
(b) *Show that no cube of an integer can be expressed as $7n+5$ for some positive integer $n$.*

(a) We proceed by the method of contradiction. Suppose $d > 0$. Assume that there is a finite, but non-zero, number of natural cubes in the arithmetic progression. Then there must be a largest cube. Let it be $a_N = a_0 + Nd$, and write it as $a_N = a_0 + Nd = m^3$. Now consider $(m+d)^3$:

$$\begin{aligned} (m+d)^3 &= m^3 + 3dm^2 + 3d^2m + d^3 \\ &= a_0 + Nd + d(3m^2 + 3dm + d^2) \\ &= a_0 + dN_1, \end{aligned}$$

where $N_1 = N + 3m^2 + 3dm + d^2$ is necessarily an integer, since $N$, $m$ and $d$ all are. Further, $N_1 > N$. Thus $a_{N_1} = a_0 + N_1 d$ is also the cube of a natural number and is greater than $a_N$; this contradicts the assumption that it is possible to select a largest cube in the series and establishes the result that, if there is one such cube, then there are infinitely many of them. A similar argument (considering the smallest term in the series) can be carried through if $d < 0$.

We note that the result is also formally true in the case in which $d = 0$; if $a_0$ is a natural cube, then so is every term, since they are all equal to $a_0$.

(b) Again, we proceed by the method of contradiction. Suppose that $7N + 5 = m^3$

for some pair of positive integers $N$ and $m$. Consider the quantity

$$\begin{aligned}(m-7)^3 &= m^3 - 21m^2 + 147m - 343\\ &= 7N + 5 - 7(3m^2 - 21m + 49)\\ &= 7N_1 + 5,\end{aligned}$$

where $N_1 = N - 3m^2 + 21m - 49$ is an integer smaller than $N$. From this, it follows that if $m^3$ can be expressed in the form $7N+5$ then so can $(m-7)^3$, $(m-14)^3$, etc. Further, for some finite integer $p$, $(m-7p)$ must lie in the range $0 \le m - 7p \le 6$ and will have the property $(m-7p)^3 = 7N_p + 5$.

However, explicit calculation shows that, when expressed in the form $7n+q$, the cubes of the integers 0, 1, 2, $\cdots$, 6 have respective values of $q$ of 0, 1, 1, 6, 1, 6, 6; none of these is equal to 5. This contradicts the conclusion that followed from our initial supposition and subsequent argument. It was therefore wrong to assume that there is a natural cube that can be expressed in the form $7N+5$.

[ Note that it is not sufficient to carry out the above explicit calculations and then rely on the construct from part (a), as this does not guarantee to generate every cube. ]

### *Necessary and sufficient conditions*

***1.30*** *Prove that the equation $ax^2 + bx + c = 0$, in which $a$, $b$ and $c$ are real and $a > 0$, has two real distinct solutions IFF $b^2 > 4ac$.*

As is usual for IFF proofs, this answer will consist of two parts.

Firstly, assume that $b^2 > 4ac$. We can then write the equation as

$$\begin{aligned}a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) &= 0,\\ a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c &= 0,\\ a\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a} = \lambda^2.\end{aligned}$$

Since $b^2 > 4ac$ and $a > 0$, $\lambda$ is real, positive and non-zero. So, taking the square roots of both sides of the final equation gives

$$x = -\frac{b}{2a} \pm \frac{\lambda}{\sqrt{a}},$$

i.e. both roots are real and they are distinct; thus, the 'if' part of the proposition is established.

Now assume that both roots are real, $\alpha$ and $\beta$ say, with $\alpha \neq \beta$. Then,

$$a\alpha^2 + b\alpha + c = 0,$$
$$a\beta^2 + b\beta + c = 0.$$

Subtraction of the two equations gives

$$a(\alpha^2 - \beta^2) + b(\alpha - \beta) = 0 \quad \Rightarrow \quad b = -(\alpha + \beta)a, \text{ since } \alpha - \beta \neq 0.$$

Multiplying the first displayed equation by $\beta$ and the second by $\alpha$ and then subtracting, gives

$$a(\alpha^2\beta - \beta^2\alpha) + c(\beta - \alpha) = 0 \quad \Rightarrow \quad c = \alpha\beta a, \text{ since } \alpha - \beta \neq 0.$$

Now, recalling that $\alpha \neq \beta$ and that $a > 0$, consider the inequality

$$\begin{aligned} 0 < (\alpha - \beta)^2 &= \alpha^2 - 2\alpha\beta + \beta^2 \\ &= (\alpha + \beta)^2 - 4\alpha\beta \\ &= \frac{b^2}{a^2} - 4\frac{c}{a} = \frac{b^2 - 4ac}{a^2}. \end{aligned}$$

This inequality shows that $b^2$ is necessarily greater than $4ac$, and so establishes the 'only if' part of the proof.

***1.32*** *Given that at least one of a and b, and at least one of c and d, are non-zero, show that $ad = bc$ is both a necessary and sufficient condition for the equations*

$$ax + by = 0,$$
$$cx + dy = 0,$$

*to have a solution in which at least one of x and y is non-zero.*

First, suppose that $ad = bc$ with at least one of $a$ and $b$, and at least one of $c$ and $d$, non-zero. Assume, for definiteness, that $a$ and $c$ are non-zero; if this is not the case, then the following proof is modified in an obvious way by interchanging the roles of $a$ and $b$ and/or of $c$ and $d$, as necessary:

$$ax + by = 0 \Rightarrow x = -\frac{b}{a}y,$$
$$cx + dy = 0 \Rightarrow x = -\frac{d}{c}y.$$

Now

$$ad = bc \quad \Rightarrow \quad d = \frac{bc}{a} \quad \Rightarrow \quad \frac{d}{c} = \frac{b}{a},$$

where we have used, in turn, that $a \neq 0$ and $c \neq 0$. Thus the two solutions for $x$

in terms of $y$ are the same. Any non-zero value for $y$ may be chosen, but that for $x$ is then determined (and may be zero). This establishes that the condition is sufficient.

To show that it is a necessary condition, suppose that there is a non-trivial solution to the original equations and that, say, $x \neq 0$. Multiply the first equation by $d$ and the second by $b$ to obtain

$$dax + dby = 0,$$
$$bcx + bdy = 0.$$

Subtracting these equations gives $(ad - bc)x = 0$ and, since $x \neq 0$, it follows that $ad = bc$.

If $x = 0$ then $y \neq 0$, and multiplying the first of the original equations by $c$ and the second by $a$ leads to the same conclusion.

This completes the proof that the condition is both necessary and sufficient.

# 2

# *Preliminary calculus*

**2.2** *Find from first principles the first derivative of* $(x+3)^2$ *and compare your answer with that obtained using the chain rule.*

Using the definition of a derivative, we consider the difference between $(x+\Delta x+3)^2$ and $(x+3)^2$, and determine the following limit (if it exists):

$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{(x+\Delta x+3)^2 - (x+3)^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{[(x+3)^2 + 2(x+3)\Delta x + (\Delta x)^2] - (x+3)^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{(2(x+3)\Delta x + (\Delta x)^2}{\Delta x} \\ &= 2x+6. \end{aligned}$$

The limit does exist, and so the derivative is $2x+6$.
Rewriting the function as $f(x) = u^2$, where $u(x) = x+3$, and using the chain rule:

$$f'(x) = 2u \times \frac{du}{dx} = 2u \times 1 = 2u = 2x+6,$$

i.e. the same, as expected.

**2.4** *Find the first derivatives of*

(a) $x/(a+x)^2$, (b) $x/(1-x)^{1/2}$, (c) $\tan x$, *as* $\sin x/\cos x$,

(d) $(3x^2+2x+1)/(8x^2-4x+2)$.

In each case, using (2.13) for a quotient:

(a) $$f'(x) = \frac{[(a+x)^2 \times 1] - [x \times 2(a+x)]}{(a+x)^4} = \frac{a^2 - x^2}{(a+x)^4} = \frac{a-x}{(a+x)^3};$$

(b) $$f'(x) = \frac{[(1-x)^{1/2} \times 1] - [x \times -\frac{1}{2}(1-x)^{-1/2}]}{1-x} = \frac{1 - \frac{1}{2}x}{(1-x)^{3/2}};$$

(c) $$f'(x) = \frac{[\cos x \times \cos x] - [\sin x \times (-\sin x)]}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x;$$

(d) $$f'(x) = \frac{[(8x^2 - 4x + 2) \times (6x+2)] - [(3x^2 + 2x + 1) \times (16x - 4)]}{(8x^2 - 4x + 2)^2}$$
$$= \frac{x^3(48 - 48) + x^2(16 - 24 + 12 - 32) + \cdots}{(8x^2 - 4x + 2)^2}$$
$$\frac{\cdots + x(-8 + 12 + 8 - 16) + (4 + 4)}{(8x^2 - 4x + 2)^2}$$
$$= \frac{-28x^2 - 4x + 8}{(8x^2 - 4x + 2)^2} = \frac{-7x^2 - x + 2}{(4x^2 - 2x + 1)^2}.$$

**2.6** *Show that the function* $y(x) = \exp(-|x|)$ *defined as*

$$\begin{aligned} \exp\, x \quad &\text{for } x < 0, \\ 1 \quad &\text{for } x = 0, \\ \exp(-x) \quad &\text{for } x > 0, \end{aligned}$$

*is* not *differentiable at* $x = 0$. *Consider the limiting process for both* $\Delta x > 0$ *and* $\Delta x < 0$.

For $x > 0$, let $\Delta x = \eta$. Then,

$$y'(x > 0) = \lim_{\eta \to 0} \frac{e^{-0-\eta} - 1}{\eta}$$
$$= \lim_{\eta \to 0} \frac{1 - \eta + \frac{1}{2!}\eta^2 \cdots - 1}{\eta} = -1.$$

For $x < 0$, let $\Delta x = -\eta$. Then,

$$y'(x > 0) = \lim_{\eta \to 0} \frac{e^{0-\eta} - 1}{-\eta}$$
$$= \lim_{\eta \to 0} \frac{1 - \eta + \frac{1}{2!}\eta^2 \cdots - 1}{-\eta} = 1.$$

The two limits are not equal, and so $y(x)$ is not differentiable at $x = 0$.

**2.8** *If* $2y + \sin y + 5 = x^4 + 4x^3 + 2\pi$, *show that* $dy/dx = 16$ *when* $x = 1$.

For this equation neither $x$ nor $y$ can be made the subject of the equation, i.e neither can be written explicitly as a function of the other, and so we are forced to use implicit differentiation. Starting from

$$2y + \sin y + 5 = x^4 + 4x^3 + 2\pi$$

implicit differentiation, and the use of the chain rule when differentiating $\sin y$ with respect to $x$, gives

$$2\frac{dy}{dx} + \cos y\frac{dy}{dx} = 4x^3 + 12x^2.$$

When $x = 1$ the original equation reduces to $2y + \sin y = 2\pi$ with the obvious (and unique, as can be verified from a simple sketch) solution $y = \pi$. Thus, with $x = 1$ and $y = \pi$,

$$\left.\frac{dy}{dx}\right|_{x=1} = \frac{4 + 12}{2 + \cos\pi} = 16.$$

**2.10** *The function* $y(x)$ *is defined by* $y(x) = (1 + x^m)^n$.

(a) *Use the chain rule to show that the first derivative of* $y$ *is* $nmx^{m-1}(1+x^m)^{n-1}$.

(b) *The binomial expansion (see section 1.5) of* $(1 + z)^n$ *is*

$$(1 + z)^n = 1 + nz + \frac{n(n-1)}{2!}z^2 + \cdots + \frac{n(n-1)\cdots(n-r+1)}{r!}z^r + \cdots .$$

*Keeping only the terms of zeroth and first order in* $dx$, *apply this result twice to derive result* (a) *from first principles.*

(c) *Expand* $y$ *in a series of powers of* $x$ *before differentiating term by term. Show that the result is the series obtained by expanding the answer given for* $dy/dx$ *in part* (a).

(a) Writing $1 + x^m$ as $u$, $y(x) = u^n$, and so $dy/du = nu^{n-1}$, whilst $du/dx = mx^{m-1}$. Thus, from the chain rule,

$$\frac{dy}{dx} = nu^{n-1} \times mx^{m-1} = nmx^{m-1}(1 + x^m)^{n-1}.$$

(b) From the defining process for a derivative,

$$
\begin{aligned}
y'(x) &= \lim_{\Delta x \to 0} \frac{[1 + (x + \Delta x)^m]^n - (1 + x^m)^n}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{[1 + x^m(1 + \frac{\Delta x}{x})^m]^n - (1 + x^m)^n}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{[1 + x^m(1 + \frac{m\Delta x}{x} + \cdots)]^n - (1 + x^m)^n}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(1 + x^m + mx^{m-1}\Delta x + \cdots)^n - (1 + x^m)^n}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\left[(1 + x^m)\left(1 + \frac{mx^{m-1}\Delta x}{1+x^m} + \cdots\right)\right]^n - (1 + x^m)^n}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(1 + x^m)^n\left(1 + \frac{mnx^{m-1}\Delta x}{1+x^m} + \cdots\right) - (1 + x^m)^n}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{mn(1 + x^m)^{n-1}x^{m-1}\Delta x + \cdots}{\Delta x} \\
&= nmx^{m-1}(1 + x^m)^{n-1},
\end{aligned}
$$

i.e. the same as the result in part (a).

(c) Expanding in a power series before differentiating:

$$
\begin{aligned}
y(x) &= 1 + nx^m + \frac{n(n-1)}{2!}\,x^{2m} + \cdots \\
&\quad + \frac{n(n-1)\cdots(n-r+1)}{r!}\,x^{rm} + \cdots\,, \\
y'(x) &= m\,nx^{m-1} + \frac{2m\,n(n-1)}{2!}\,x^{2m-1} + \cdots \\
&\quad + \frac{rm\,n(n-1)\cdots(n-r+1)}{r!}\,x^{rm-1} + \cdots\,.
\end{aligned}
$$

Now, expanding the result given in part (a) gives

$$
\begin{aligned}
y'(x) &= nmx^{m-1}(1 + x^m)^{n-1} \\
&= nmx^{m-1}\left(1 + \cdots + \frac{(n-1)(n-2)\cdots(n-s)}{s!}\,x^{ms} + \cdots\right) \\
&= nmx^{m-1} + \cdots + \frac{mn(n-1)(n-2)\cdots(n-s)}{s!}\,x^{ms+m-1} + \cdots\,.
\end{aligned}
$$

This is the same as the previous expansion of $y'(x)$ if, in the general term, the index is moved by one, i.e. $s = r - 1$.

**2.12** *Find the positions and natures of the stationary points of the following functions:*

(a) $x^3 - 3x + 3$; (b) $x^3 - 3x^2 + 3x$; (c) $x^3 + 3x + 3$;
(d) $\sin ax$ *with* $a \neq 0$; (e) $x^5 + x^3$; (f) $x^5 - x^3$.

In each case, we need to determine the first and second derivatives of the function. The zeros of the 1st derivative give the positions of the stationary points, and the values of the 2nd derivatives at those points determine their natures.

(a) $$y = x^3 - 3x + 3; \quad y' = 3x^2 - 3; \quad y'' = 6x.$$

$y' = 0$ has roots at $x = \pm 1$, where the values of $y''$ are $\pm 6$. Therefore, there is a minimum at $x = 1$ and a maximum at $x = -1$.

(b) $$y = x^3 - 3x^2 + 3x; \quad y' = 3x^2 - 6x + 3; \quad y'' = 6x - 6.$$

$y' = 0$ has a double root at $x = 1$, where the value of $y''$ is 0. Therefore, there is a point of inflection at $x = 1$, but no other stationary points. At the point of inflection, the tangent to the curve $y = y(x)$ is horizontal.

(c) $$y = x^3 + 3x + 3; \quad y' = 3x^2 + 3; \quad y'' = 6x.$$

$y' = 0$ has no real roots, and so there are no stationary points.

(d) $$y = \sin ax; \quad y' = a \cos ax; \quad y'' = -a^2 \sin ax.$$

$y' = 0$ has roots at $x = (n + \frac{1}{2})\pi/a$ for integer $n$. The corresponding values of $y''$ are $\mp a^2$, depending on whether $n$ is even or odd. Therefore, there is a maximum for even $n$ and a minimum where $n$ is odd.

(e) $$y = x^5 + x^3; \quad y' = 5x^4 + 3x^2; \quad y'' = 20x^3 + 6x.$$

$y' = 0$ has, as its only real root, a double root at $x = 0$, where the value of $y''$ is 0. Thus, there is a (horizontal) point of inflection at $x = 0$, but no other stationary point.

(f) $$y = x^5 - x^3; \quad y' = 5x^4 - 3x^2; \quad y'' = 20x^3 - 6x.$$

$y' = 0$ has a double root at $x = 0$ and simple roots at $x = \pm(\frac{3}{5})^{1/2}$, where the respective values of $y''$ are 0 and $\pm 6(\frac{3}{5})^{1/2}$. Therefore, there is a point of inflection at $x = 0$, a maximum at $x = -(\frac{3}{5})^{1/2}$ and a minimum at $x = (\frac{3}{5})^{1/2}$.

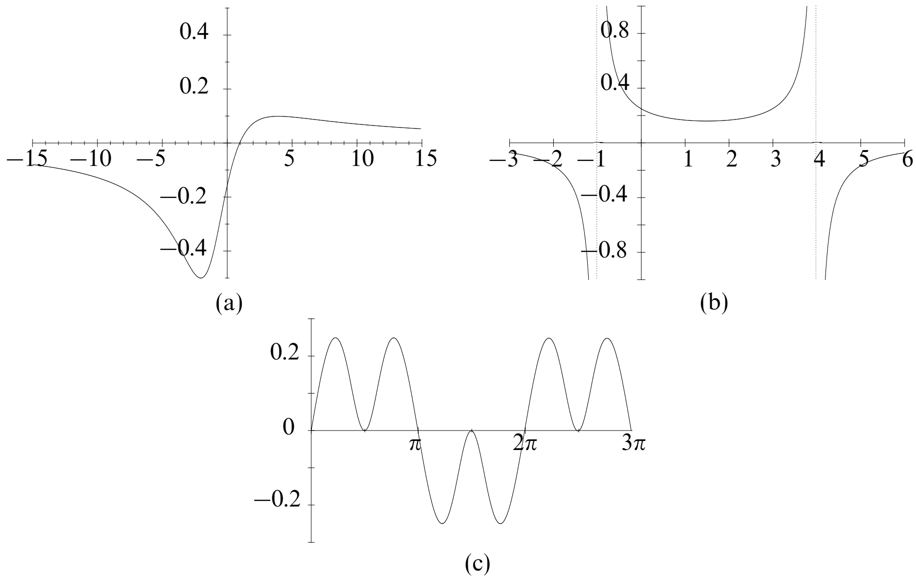


Figure 2.1 The solutions to exercise 2.14.

> ***2.14*** *By finding their stationary points and examining their general forms, determine the range of values that each of the following functions $y(x)$ can take. In each case make a sketch-graph incorporating the features you have identified.*
>
> (a) $y(x) = (x-1)/(x^2+2x+6)$.
> (b) $y(x) = 1/(4+3x-x^2)$.
> (c) $y(x) = (8\sin x)/(15+8\tan^2 x)$.

See figure 2.1 (a)–(c).

(a) Some simple points to calculate for

$$y = \frac{x-1}{x^2+2x+6}$$

are $y(0) = -\frac{1}{6}$, $y(1) = 0$ and $y(\pm\infty) = 0$, and, since the denominator has no real roots ($2^2 < 4 \times 1 \times 6$), there are no infinities. Its 1st derivative is

$$y' = \frac{-x^2+2x+8}{(x^2+2x+6)^2} = \frac{-(x+2)(x-4)}{(x^2+2x+6)^2}.$$

Thus there are turning points only at $x = -2$, with $y(-2) = -\frac{1}{2}$, and at $x = 4$, with $y(4) = \frac{1}{10}$. The former must be a minimum and the latter a maximum. The range in which $y(x)$ lies is $-\frac{1}{2} \le y \le \frac{1}{10}$.

(b) Some simple points to calculate for

$$y = \frac{1}{4 + 3x - x^2}.$$

are $y(0) = \frac{1}{4}$ and $y(\pm\infty) = 0$, approached from negative values. Since the denominator can be written as $(4-x)(1+x)$, the function has infinities at $x = -1$ and $x = 4$ and is positive in the range of $x$ between them.

The 1st derivative is

$$y' = \frac{2x - 3}{(4 + 3x - x^2)^2}.$$

Thus there is only one turning point; this is at $x = \frac{3}{2}$, with corresponding $y(\frac{3}{2}) = \frac{4}{25}$. Since $\frac{3}{2}$ lies in the range $-1 < x < 4$, at the ends of which the function $\to +\infty$, the stationary point must be a minimum. This sets a lower limit on the positive values of $y(x)$ and so the ranges in which it lies are $y < 0$ and $y \geq \frac{4}{25}$.

(c) The function

$$y = \frac{8 \sin x}{15 + 8 \tan^2 x}$$

is clearly periodic with period $2\pi$.

Since $\sin x$ and $\tan^2 x$ are both symmetric about $x = \frac{1}{2}\pi$, so is the function. Also, since $\sin x$ is antisymmetric about $x = \pi$ whilst $\tan^2 x$ is symmetric, the function is antisymmetric about $x = \pi$.

Some simple points to calculate are $y(n\pi) = 0$ for all integers $n$. Further, since $\tan(n + \frac{1}{2})\pi = \infty$, $y((n + \frac{1}{2})\pi) = 0$. As the denominator has no real roots there are no infinities.

Setting the derivative of $y(x) \equiv 8u(x)/v(x)$ equal to zero, i.e. writing $vu' = uv'$, and expressing all terms as powers of $\cos x$ gives (using $\tan^2 z = \sec^2 z - 1$ and $\sin^2 z = 1 - \cos^2 z$)

$$\begin{aligned}
(15 + 8\tan^2 x)\cos x &= 16 \sin x \tan x \sec^2 x, \\
15 + \frac{8}{\cos^2 x} - 8 &= \frac{16(1 - \cos^2 x)}{\cos^4 x}, \\
7\cos^4 x + 24\cos^2 x - 16 &= 0.
\end{aligned}$$

This quadratic equation for $\cos^2 x$ has roots of $\frac{4}{7}$ and $-4$. Only the first of these gives real values for $\cos x$ of $\pm\frac{2}{\sqrt{7}}$. The corresponding turning values of $y(x)$ are $\pm\frac{8}{7\sqrt{21}}$. The value of $y$ always lies between these two limits.

**2.16** *The curve* $4y^3 = a^2(x+3y)$ *can be parameterised as* $x = a\cos 3\theta$, $y = a\cos\theta$.

(a) *Obtain expressions for* $dy/dx$ (i) *by implicit differentiation and* (ii) *in parameterised form. Verify that they are equivalent.*
(b) *Show that the only point of inflection occurs at the origin. Is it a stationary point of inflection?*
(c) *Use the information gained in* (a) *and* (b) *to sketch the curve, paying particular attention to its shape near the points* $(-a, a/2)$ *and* $(a, -a/2)$ *and to its slope at the 'end points'* $(a, a)$ *and* $(-a, -a)$.

(a) (i) Differentiating the equation of the curve implicitly:

$$12y^2\frac{dy}{dx} = a^2 + 3a^2\frac{dy}{dx}, \quad \Rightarrow \quad \frac{dy}{dx} = \frac{a^2}{12y^2 - 3a^2}.$$

(ii) In parameterised form:

$$\frac{dy}{d\theta} = -a\sin\theta, \quad \frac{dx}{d\theta} = -3a\sin 3\theta, \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-a\sin\theta}{-3a\sin 3\theta}.$$

But, using the results from section 1.2, we have that

$$\begin{aligned}\sin 3\theta &= \sin(2\theta + \theta)\\ &= \sin 2\theta\cos\theta + \cos 2\theta\sin\theta\\ &= 2\sin\theta\cos^2\theta + (2\cos^2\theta - 1)\sin\theta\\ &= \sin\theta(4\cos^2\theta - 1),\end{aligned}$$

thus giving $dy/dx$ as

$$\frac{dy}{dx} = \frac{1}{12\cos^2\theta - 3} = \frac{a^2}{12a^2\cos^2\theta - 3a^2},$$

with $a\cos\theta = y$, i.e. as in (i).

(b) At a point of inflection $y'' = 0$. For the given function,

$$\frac{d^2y}{dx^2} = \frac{d}{dy}\left(\frac{dy}{dx}\right) \times \frac{dy}{dx} = -\frac{a^2}{(12y^2 - 3a^2)^2} \times 24y \times \frac{a^2}{12y^2 - 3a^2}.$$

This can only equal zero at $y = 0$, when $x = 0$ also. But, when $y = 0$ it follows from (a)(i) that $dy/dx = 1/(-3) = -\frac{1}{3}$. As this is non-zero the point of inflection is not a stationary point.

(c) See figure 2.2. Note in particular that the curve has vertical tangents when $y = \pm a/2$ and that $dy/dx = \frac{1}{9}$ at $y = \pm a$, i.e. the tangents at the end points of the 'S'-shaped curve are not horizontal.

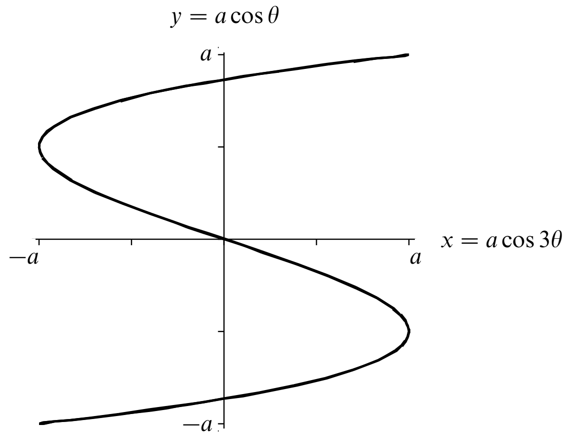


Figure 2.2 The parametric curve described in exercise 2.16.

***2.18** Show that the maximum curvature on the catenary $y(x) = a\cosh(x/a)$ is $1/a$. You will need some of the results about hyperbolic functions stated in subsection 3.7.6.*

The general expression for the curvature, $\rho^{-1}$, of the curve $y = y(x)$ is

$$\frac{1}{\rho} = \frac{y''}{(1 + y'^2)^{3/2}},$$

and so we begin by calculating the first two derivatives of $y$. Starting from $y = a\cosh(x/a)$, we obtain

$$y' = a\frac{1}{a}\sinh\frac{x}{a},$$
$$y'' = \frac{1}{a}\cosh\frac{x}{a}.$$

Therefore the curvature of the catenary at the point $(x, y)$ is given by

$$\frac{1}{\rho} = \frac{\frac{1}{a}\cosh\frac{x}{a}}{\left[1 + \sinh^2\frac{x}{a}\right]^{3/2}} = \frac{1}{a}\frac{\cosh\frac{x}{a}}{\cosh^3\frac{x}{a}} = \frac{a}{y^2}.$$

To obtain this result we have used the identity $\cosh^2 z = 1 + \sinh^2 z$. We see that the curvature is maximal when $y$ is minimal; this occurs when $x = 0$ and $y = a$. The maximum curvature is therefore $1/a$.

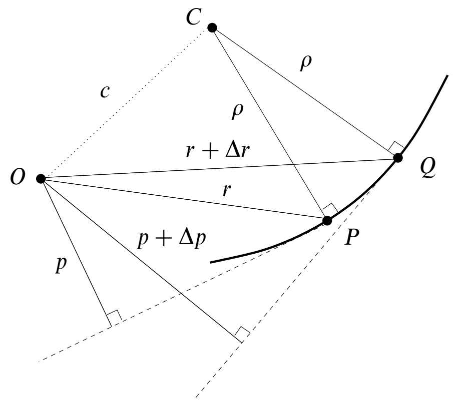


Figure 2.3 The coordinate system described in exercise 2.20.

***2.20** A two-dimensional coordinate system useful for orbit problems is the tangential-polar coordinate system (figure 2.3). In this system a curve is defined by r, the distance from a fixed point O to a general point P of the curve, and p, the perpendicular distance from O to the tangent to the curve at P. By proceeding as indicated below, show that the radius of curvature at P can be written in the form $\rho = r\,dr/dp$.*

*Consider two neighbouring points P and Q on the curve. The normals to the curve through those points meet at C, with (in the limit $Q \to P$) $CP = CQ = \rho$. Apply the cosine rule to triangles OPC and OQC to obtain two expressions for $c^2$, one in terms of r and p and the other in terms of $r + \Delta r$ and $p + \Delta p$. By equating them and letting $Q \to P$ deduce the stated result.*

We first note that cos $OPC$ is equal to the sine of the angle between $OP$ and the tangent at $P$, and that this in turn has the value $p/r$. Now, applying the cosine rule to the triangles $OCP$ and $OCQ$, we have

$$\begin{aligned} c^2 &= r^2 + \rho^2 - 2r\rho\cos OPC = r^2 + \rho^2 - 2\rho p \\ c^2 &= (r + \Delta r)^2 + \rho^2 - 2(r + \Delta r)\rho\cos OQC \\ &= (r + \Delta r)^2 + \rho^2 - 2\rho(p + \Delta p). \end{aligned}$$

Subtracting and rearranging then yields

$$\rho = \frac{r\Delta r + \frac{1}{2}(\Delta r)^2}{\Delta p},$$

or, in the limit $Q \to P$, that $\rho = r(dr/dp)$.

**2.22** *If $y = \exp(-x^2)$, show that $dy/dx = -2xy$ and hence, by applying Leibnitz' theorem, prove that for $n \geq 1$*

$$y^{(n+1)} + 2xy^{(n)} + 2ny^{(n-1)} = 0.$$

With $y(x) = \exp(-x^2)$,

$$\frac{dy}{dx} = -2x \exp(-x^2) = -2xy.$$

We now take the $n$th derivatives of both sides and use Leibnitz' theorem to find that of the product $xy$, noting that all derivatives of $x$ beyond the first are zero:

$$y^{(n+1)} = -2[\,(y^{(n)})(x) + n(y^{(n-1)})(1) + 0\,].$$

i.e.

$$y^{(n+1)} + 2xy^{(n)} + 2ny^{(n-1)} = 0,$$

as stated in the question.

**2.24** *Determine what can be learned from applying Rolle's theorem to the following functions $f(x)$:* (a)$e^x$; (b)$x^2+6x$; (c)$2x^2+3x+1$; (d)$2x^2+3x+2$; (e)$2x^3-21x^2+60x+k$. (f)*If $k = -45$ in* (e), *show that $x = 3$ is one root of $f(x) = 0$, find the other roots, and verify that the conclusions from* (e) *are satisfied.*

(a) Since the derivative of $f(x) = e^x$ is $f'(x) = e^x$, Rolle's theorem states that between any two consecutive roots of $f(x) = e^x = 0$ there must be a root of $f'(x) = e^x = 0$, i.e. another root of the same equation. This is clearly a contradiction and it is wrong to suppose that there is more than one root of $e^x = 0$. In fact, there are no finite roots of the equation and the only zero of $e^x$ lies formally at $x = -\infty$.

(b) Since $f(x) = x(x+6)$, it has zeros at $x = -6$ and $x = 0$. Therefore the (only) root of $f'(x) = 2x + 6 = 0$ must lie between these values; it clearly does, as $-6 < -3 < 0$.

(c) With $f(x) = 2x^2 + 3x + 1$ and hence $f'(x) = 4x + 3$, any roots of $f(x) = 0$ (actually $-1$ and $-\frac{1}{2}$) must lie on either side of the root of $f'(x) = 0$, i.e. $x = -\frac{3}{4}$. They clearly do.

(d) This is as in (c), but there are no real roots. However, it can be more generally stated that if there are two values of $x$ that give $2x^2 + 3x + k$ equal values then they lie one on each side of $x = -\frac{3}{4}$.

(e) With $f(x) = 2x^3 - 21x^2 + 60x + k$,

$$f'(x) = 6x^2 - 42x + 60 = 6(x-5)(x-2)$$

and $f'(x) = 0$ has roots 2 and 5. Therefore, if $f(x) = 0$ has three real roots $\alpha_i$ with $\alpha_1 < \alpha_2 < \alpha_3$, then $\alpha_1 < 2 < \alpha_2 < 5 < \alpha_3$.

(f) When $k = -45$, $f(3) = 54 - 189 + 180 - 45 = 0$ and so $x = 3$ is a root of $f(x) = 0$ and $(x-3)$ is a factor of $f(x)$. Writing $f(x) = 2x^3 - 21x^2 + 60x - 45$ as $(x-3)(a_2x^2 + a_1x + a_0)$ and equating coefficients gives $a_2 = 2$, $a_1 = -15$ and $a_0 = 15$. The other two roots are therefore

$$\frac{15 \pm \sqrt{225 - 120}}{4} = \frac{1}{4}(15 \pm \sqrt{105}) = 1.19 \text{ or } 6.31.$$

Result (e) is verified in this case since $1.19 < 2 < 3 < 5 < 6.31$.

**2.26** *Use the mean value theorem to establish bounds*

(a) *for* $-\ln(1-y)$, *by considering* $\ln x$ *in the range* $0 < 1 - y < x < 1$,
(b) *for* $e^y - 1$, *by considering* $e^x - 1$ *in the range* $0 < x < y$.

(a) The mean value theorem applied to $\ln x$ within limits $1 - y$ and 1 gives

$$\frac{\ln(1) - \ln(1-y)}{1 - (1-y)} = \frac{d}{dx}(\ln x) = \frac{1}{x} \qquad (*)$$

for some $x$ in the range $1 - y < x < 1$. Now, since $1 - y < x < 1$ it follows that

$$\frac{1}{1-y} > \frac{1}{x} > 1,$$

$$\Rightarrow \quad \frac{1}{1-y} > \frac{-\ln(1-y)}{y} > 1,$$

$$\Rightarrow \quad \frac{y}{1-y} > -\ln(1-y) > y.$$

The second line was obtained by substitution from $(*)$.

(b) The mean value theorem applied to $e^x - 1$ within limits 0 and $y$ gives

$$\frac{e^y - 1 - 0}{y - 0} = e^x \quad \text{for some } x \text{ in the range } 0 < x < y.$$

Now, since $0 < x < y$ it follows that

$$1 < e^x < e^y,$$

$$\Rightarrow \quad 1 < \frac{e^y - 1}{y} < e^y,$$

$$\Rightarrow \quad y < e^y - 1 < ye^y.$$

Again, the second line was obtained by substitution for $x$ from the mean value theorem result.

**2.28** *Use Rolle's theorem to deduce that if the equation $f(x) = 0$ has a repeated root $x_1$ then $x_1$ is also a root of the equation $f'(x) = 0$.*

(a) *Apply this result to the 'standard' quadratic equation $ax^2 + bx + c = 0$, to show that a necessary condition for equal roots is $b^2 = 4ac$.*
(b) *Find all the roots of $f(x) = x^3 + 4x^2 - 3x - 18 = 0$, given that one of them is a repeated root.*
(c) *The equation $f(x) = x^4 + 4x^3 + 7x^2 + 6x + 2 = 0$ has a repeated integer root. How many real roots does it have altogether?*

If two roots of $f(x) = 0$ are $x_1$ and $x_2$, i.e. $f(x_1) = f(x_2) = 0$, then it follows from Rolle's theorem that there is some $x_3$ in the range $x_1 \leq x_3 \leq x_2$ for which $f'(x_3) = 0$. Now let $x_2 \to x_1$ to form the repeated root; $x_3$ must also tend to the limit $x_1$, i.e. $x_1$ is a root of $f'(x) = 0$ as well as of $f(x) = 0$.

(a) A quadratic equation $f(x) = ax^2 + bx + c = 0$ only has two roots and so if they are equal the common root $\alpha$ must also be a root of $f'(x) = 2ax + b = 0$, i.e. $\alpha = -b/2a$. Thus

$$a\frac{b^2}{4a^2} + b\frac{-b}{2a} + c = 0.$$

It then follows that $c - (b^2/4a) = 0$ and that $b^2 = 4ac$.

(b) With $f(x) = x^3 + 4x^2 - 3x - 18$, the repeated root must satisfy

$$f'(x) = 3x^2 + 8x - 3 = (3x - 1)(x + 3) = 0 \quad \text{i.e. } x = \frac{1}{3} \text{ or } x = -3.$$

Trying the two possibilities: $f(\frac{1}{3}) \neq 0$ but $f(-3) = -27 + 36 + 9 - 18 = 0$. Thus $f(x)$ must factorise as $(x + 3)^2(x - b)$, and comparing the constant terms in the two expressions for $f(x)$ immediately gives $b = 2$. Hence, $x = 2$ is the third root.

(c) Here $f(x) = x^4 + 4x^3 + 7x^2 + 6x + 2$. As previously, we examine $f'(x) = 0$, i.e. $f'(x) = 4x^3 + 12x^2 + 14x + 6 = 0$. This has to have an integer solution and, by inspection, this is $x = -1$. We can therefore factorise $f(x)$ as the product $(x + 1)^2(a_2x^2 + a_1x + a_0)$. Comparison of the coefficients gives immediately that $a_2 = 1$ and $a_0 = 2$. From the coefficients of $x^3$ we have $2a_2 + a_1 = 4$; hence $a_1 = 2$. Thus $f(x)$ can be written

$$f(x) = (x + 1)^2(x^2 + 2x + 2) = (x + 1)^2[(x + 1)^2 + 1].$$

The second factor, containing only positive terms, can have no real zeros and hence $f(x) = 0$ has only two real roots (coincident at $x = -1$).

**2.30** *Find the following indefinite integrals:*

(a) $\int (4 + x^2)^{-1}\, dx$; (b) $\int (8 + 2x - x^2)^{-1/2}\, dx$ for $2 \le x \le 4$;
(c) $\int (1 + \sin\theta)^{-1}\, d\theta$; (d) $\int (x\sqrt{1 - x})^{-1}\, dx$ for $0 < x \le 1$.

We make reference to the 12 standard forms given in subsection 2.2.3 and, where relevant, select the appropriate model.

(a) Using model 9,

$$\int \frac{1}{4 + x^2}\, dx = \frac{1}{2} \tan^{-1} \frac{x}{2} + c.$$

(b) We rearrange the integrand in the form of model 12:

$$\int \frac{1}{\sqrt{8 + 2x - x^2}}\, dx = \int \frac{1}{\sqrt{8 + 1 - (x - 1)^2}}\, dx = \sin^{-1} \frac{x - 1}{3} + c.$$

(c) See equation (2.35) and the subsequent text.

$$\begin{aligned}
\int \frac{1}{1 + \sin\theta}\, d\theta &= \int \frac{1}{1 + \dfrac{2t}{1 + t^2}} \, \frac{2}{1 + t^2}\, dt \\
&= \int \frac{2}{(1 + t)^2}\, dt \\
&= -\frac{2}{1 + t} + c \\
&= -\frac{2}{1 + \tan\dfrac{\theta}{2}} + c.
\end{aligned}$$

(d) To remove the square root, set $u^2 = 1 - x$; then $2u\, du = -dx$ and

$$\begin{aligned}
\int \frac{1}{x\sqrt{1 - x}}\, dx &= \int \frac{1}{(1 - u^2)u} \times -2u\, du \\
&= \int \frac{-2}{1 - u^2}\, du \\
&= \int \left( \frac{-1}{1 - u} + \frac{-1}{1 + u} \right) du \\
&= \ln(1 - u) - \ln(1 + u) + c \\
&= \ln \frac{1 - \sqrt{1 - x}}{1 + \sqrt{1 - x}} + c.
\end{aligned}$$

**2.32** *Express $x^2(ax+b)^{-1}$ as the sum of powers of $x$ and another integrable term, and hence evaluate*

$$\int_0^{b/a} \frac{x^2}{ax+b}\, dx.$$

We need to write the numerator in such a way that every term in it that involves $x$ contains a factor $ax+b$. Therefore, write $x^2$ as

$$x^2 = \frac{x}{a}(ax+b) - \frac{b}{a^2}(ax+b) + \frac{b^2}{a^2}.$$

Then,

$$\begin{aligned}\int_0^{b/a} \frac{x^2}{ax+b}\, dx &= \int_0^{b/a} \left( \frac{x}{a} - \frac{b}{a^2} + \frac{b^2}{a^2(ax+b)} \right) dx \\ &= \left[ \frac{x^2}{2a} - \frac{bx}{a^2} + \frac{b^2}{a^3}\ln(ax+b) \right]_0^{b/a} \\ &= \frac{b^2}{a^3}\left( \ln 2 - \frac{1}{2} \right).\end{aligned}$$

An alternative approach, consistent with the wording of the question, is to use the binomial theorem to write the integrand as

$$\frac{x^2}{ax+b} = \frac{x^2}{b}\left(1 + \frac{ax}{b}\right)^{-1} = \frac{x^2}{b}\sum_{n=0}^{\infty}\left(-\frac{ax}{b}\right)^n.$$

Then the integral is

$$\begin{aligned}\int_0^{b/a} \frac{x^2}{ax+b}\, dx &= \frac{1}{b}\int_0^{b/a} \sum_{n=0}^{\infty} (-1)^n \left(\frac{a}{b}\right)^n x^{n+2}\, dx \\ &= \frac{1}{b}\sum_{n=0}^{\infty} (-1)^n \left(\frac{a}{b}\right)^n \frac{1}{n+3}\left(\frac{b}{a}\right)^{n+3} \\ &= \frac{b^2}{a^3}\sum_{n=0}^{\infty} \frac{(-1)^n}{n+3}.\end{aligned}$$

That these two solutions are the same can be seen by writing $\ln 2 - \frac{1}{2}$ as

$$\begin{aligned}\ln 2 - \frac{1}{2} &= \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots \right) - \frac{1}{2} \\ &= \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+3}.\end{aligned}$$

**2.34** *Use logarithmic integration to find the indefinite integrals $J$ of the following:*

(a) $\sin 2x/(1+4\sin^2 x)$;
(b) $e^x/(e^x - e^{-x})$;
(c) $(1 + x\ln x)/(x\ln x)$;
(d) $[x(x^n + a^n)]^{-1}$.

To use logarithmic integration each integrand needs to be arranged as a fraction that has the derivative of the denominator appearing in the numerator.

(a) Either by noting that $\sin 2x = 2\sin x\cos x$ and so is proportional to the derivative of $\sin^2 x$ or by recognising that $\sin^2 x$ can be written in terms of $\cos 2x$ and constants and that $\sin 2x$ is then its derivative, we have

$$J = \int \frac{\sin 2x}{1+4\sin^2 x}\,dx$$
$$= \int \frac{2\sin x\cos x}{1+4\sin^2 x}\,dx = \frac{1}{4}\ln(1+4\sin^2 x) + c,$$

or

$$J = \int \frac{\sin 2x}{1+2(1-\cos 2x)}\,dx = \frac{1}{4}\ln(3-2\cos 2x) + c.$$

These two answers are equivalent since $3 - 2\cos 2x = 3 - 2(1 - 2\sin^2 x) = 1 + 4\sin^2 x$.

(b) This is straightforward if it is noticed that multiplying both numerator and denominator by $e^x$ produces the required form:

$$J = \int \frac{e^x}{e^x - e^{-x}}\,dx = \int \frac{e^{2x}}{e^{2x}-1}\,dx = \frac{1}{2}\ln(e^{2x}-1) + c.$$

An alternative, but longer, method is to write the numerator as $\cosh x + \sinh x$ and the denominator as $2\sinh x$. This leads to $J = \frac{1}{2}(x + \ln\sinh x)$, which can be re-written as

$$J = \tfrac{1}{2}(\ln e^x + \ln\sinh x) = \tfrac{1}{2}\ln(e^x\sinh x) = \tfrac{1}{2}\ln(e^{2x}-1) + \tfrac{1}{2}\ln\tfrac{1}{2}.$$

The $\frac{1}{2}\ln\frac{1}{2}$ forms part of $c$.

(c) Here we must first divide the numerator by the denominator to produce two separate terms, and then twice apply the result that $1/z$ is the derivative of $\ln z$:

$$J = \int \frac{1+x\ln x}{x\ln x}\,dx = \int \left(\frac{1}{x\ln x} + 1\right)dx = \ln(\ln x) + x + c.$$

(d) To put the integrand in a form suitable for logaritmic integration, we must first multiply both numerator and denominator by $nx^{n-1}$ and then use partial fractions so that each denominator contains $x$ only in the form $x^m$, of which $mx^{m-1}$ is the derivative.

$$\begin{aligned}
J &= \int \frac{dx}{x(x^n + a^n)} = \int \frac{nx^{n-1}}{nx^n(x^n + a^n)}\,dx \\
&= \frac{1}{na^n} \int \left( \frac{nx^{n-1}}{x^n} - \frac{nx^{n-1}}{x^n + a^n} \right) dx \\
&= \frac{1}{na^n} \left[ n \ln x - \ln(x^n + a^n) \right] + c \\
&= \frac{1}{na^n} \ln \left( \frac{x^n}{x^n + a^n} \right) + c.
\end{aligned}$$

**2.36** *Find the indefinite integrals $J$ of the following functions involving sinusoids:*

(a) $\cos^5 x - \cos^3 x$;
(b) $(1 - \cos x)/(1 + \cos x)$;
(c) $\cos x \sin x/(1 + \cos x)$;
(d) $\sec^2 x/(1 - \tan^2 x)$.

(a) As the integrand contains only odd powers of $\cos x$, take $\cos x$ out as a common factor and express the remainder in terms of $\sin x$, of which $\cos x$ is the derivative:

$$\begin{aligned}
\cos^5 x - \cos^3 x &= \left[ (1 - \sin^2 x)^2 - (1 - \sin^2 x) \right] \cos x \\
&= (\sin^4 x - \sin^2 x) \cos x.
\end{aligned}$$

Hence,

$$J = \int (\sin^4 x - \sin^2 x) \cos x \, dx = \frac{1}{5} \sin^5 x - \frac{1}{3} \sin^3 x + c.$$

A more formal way of expressing this approach is to say 'set $\sin x = u$ with $\cos x \, dx = du$.'

(b) This integral can be found either by writing the numerator and denominator in terms of sinusoidal functions of $x/2$ or by making the substitution $t = \tan(x/2)$. Using first the half-angle identities, we have

$$\begin{aligned}
J &= \int \frac{1 - \cos x}{1 + \cos x}\,dx = \int \frac{2 \sin^2 \frac{x}{2}}{2 \cos^2 \frac{x}{2}} \\
&= \int \tan^2 \frac{x}{2}\,dx = \int \left( \sec^2 \frac{x}{2} - 1 \right) dx = 2 \tan \frac{x}{2} - x + c.
\end{aligned}$$

The second approach (see subsection 2.2.7) is

$$\begin{aligned}
J &= \int \frac{1 - \dfrac{1-t^2}{1+t^2}}{1 + \dfrac{1-t^2}{1+t^2}} \frac{2\,dt}{1+t^2} \\
&= \int \frac{2t^2}{1+t^2}\,dt \\
&= \int 2\,dt - \int \frac{2}{1+t^2}\,dt \\
&= 2t - 2\tan^{-1} t + c = 2\tan\frac{x}{2} - x + c.
\end{aligned}$$

(c) This integrand, containing only sinusoidal functions, can be converted to an algebraic one by writing $t = \tan(x/2)$ and expressing the functions appearing in the integrand in terms of it,

$$\begin{aligned}
\frac{\cos x \sin x}{1 + \cos x}\,dx &= \frac{\dfrac{1-t^2}{1+t^2}\,\dfrac{2t}{1+t^2}\,\dfrac{2}{1+t^2}}{1 + \dfrac{1-t^2}{1+t^2}}\,dt \\
&= \frac{2t(1-t^2)}{(1+t^2)^2}\,dt \\
&= 2t\left[\frac{A}{(1+t^2)^2} + \frac{B}{1+t^2}\right]\,dt,
\end{aligned}$$

with $A + B(1+t^2) = 1 - t^2$, implying that $B = -1$ and $A = 2$.
And so, recalling that $1 + t^2 = \sec^2(x/2) = 1/[\cos^2(x/2)]$,

$$\begin{aligned}
J &= \int \left(\frac{4t}{(1+t^2)^2} - \frac{2t}{1+t^2}\right)\,dt \\
&= -\frac{2}{1+t^2} - \ln(1+t^2) + c \\
&= -2\cos^2\frac{x}{2} + \ln(\cos^2\frac{x}{2}) + c.
\end{aligned}$$

(d) We can either set $\tan x = u$ or show that the integrand is $\sec 2x$ and then use the result of exercise 2.35; here we use the latter method.

$$\int \frac{\sec^2 x}{1 - \tan^2 x}\,dx = \int \frac{1}{\cos^2 x - \sin^2 x}\,dx = \int \sec 2x\,dx.$$

It then follows from the earlier result that $J = \frac{1}{2}\ln(\sec 2x + \tan 2x) + c$. This can also be written as $\frac{1}{2}\ln[(1 + \tan x)/(1 - \tan x)] + c$.