Solutions to Lectures on Quantum Mechanics

Steven Weinberg
The University of Texas at Austin
Prepared by Dr Joel Meyers.

Chapter 1 Problem Set Solutions

1. *Consider a non-relativistic particle of mass M in one dimension, confined in a potential that vanishes for $-a \leq x \leq a$, and becomes infinite at $x = \pm a$, so that the wave function must vanish at $x = \pm a$.*

   - *Find the energy values of states with definite energy, and the corresponding normalized wave functions.*
   - *Suppose that the particle is placed in a state with a wave function proportional to $a^2 - x^2$. If the energy of the particle is measured, what is the probability that the particle will be found in the state of lowest energy?*

   The states of definite energy are those which are solutions of the time-independent Schrödinger equation

$$-\frac{\hbar^2}{2M}\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) = E\psi(x).$$

   The potential for the infinite square well is given by

$$V(x) = \begin{cases} 0 & \text{for } |x| < a \\ \infty & \text{for } |x| \geq a. \end{cases}$$

   This potential requires that the wave function vanishes outside the well

$$\psi(x) = 0 \qquad \text{for } |x| \geq a,$$

   while inside the well the Schrödinger equation becomes

$$\frac{d^2}{dx^2}\psi(x) = -\frac{2ME}{\hbar^2}\psi(x) \qquad \text{for } |x| < a.$$

   The solution of this differential equation takes the form

$$\psi(x) = A\sin(kx) + B\cos(kx),$$

   where we have defined

$$k \equiv \frac{\sqrt{2ME}}{\hbar}.$$

   Continuity of the wave function requires that we impose the boundary conditions

$$\psi(x = \pm a) = 0,$$

   and so

$$0 = A\sin(ka) + B\cos(ka),$$
$$0 = -A\sin(ka) + B\cos(ka).$$

Adding these two equations gives

$$0 = 2B\cos(ka),$$

which requires that

$$B = 0 \text{ or } k = \frac{n\pi}{2a} \text{ where } n = 2, 4, 6, \ldots,$$

and subtracting the equations gives

$$0 = 2A\sin(ka),$$

which requires that

$$A = 0 \text{ or } k = \frac{n\pi}{2a} \text{ where } n = 1, 3, 5, \ldots.$$

In either case, the energy levels are given by

$$E_n = \frac{\hbar^2 k^2}{2M} = \frac{n^2\hbar^2\pi^2}{8Ma}.$$

Next, we need to normalize the wave functions in the sense of Eq. (1.5.4), such that

$$\int_{-\infty}^{\infty} |\psi(x)|^2\, dx = 1.$$

For n odd, we have

$$\int_{-a}^{a} \left| B\cos\left(\frac{n\pi x}{2a}\right)\right|^2 dx = |B|^2 \int_{-a}^{a} \cos^2\left(\frac{n\pi x}{2a}\right) dx = |B|^2 a = 1\,,$$

which requires that

$$B = \frac{1}{\sqrt{a}},$$

where we have fixed the arbitrary complex phase for convenience. For n even, the normalization condition takes the form

$$\int_{-a}^{a} \left| A\sin\left(\frac{n\pi x}{2a}\right)\right|^2 dx = |A|^2 \int_{-a}^{a} \sin^2\left(\frac{n\pi x}{2a}\right) dx = |A|^2 a = 1\,,$$

which requires that

$$A = \frac{1}{\sqrt{a}},$$

where we have again fixed the arbitrary complex phase for convenience. Summarizing our results, the normalized wave functions of states with definite energy are

$$\psi_n(x) = \begin{cases} \frac{1}{\sqrt{a}} \cos\left(\frac{n\pi x}{2a}\right) & \text{for } |x| < a \text{ and } n = 1, 3, 5, \ldots \\ \frac{1}{\sqrt{a}} \sin\left(\frac{n\pi x}{2a}\right) & \text{for } |x| < a \text{ and } n = 2, 4, 6, \ldots \\ 0 & \text{for } |x| \geq a \end{cases}$$

and the energy levels are

$$E_n = \frac{n^2\hbar^2\pi^2}{8Ma}.$$

Since these states are normalized and the energy level is determined by n, the argument below Eq. (1.4.26) proves that the states of definite energy are orthonormal

$$\int_{-\infty}^{\infty} \psi_m^*(x)\psi_n(x)\, dx = \delta_{mn}.$$

Let us define a state which vanishes for $|x| \geq a$, while for $|x| < a$ it is given by

$$\Phi(x, t = 0) = C\left(a^2 - x^2\right).$$

We must normalize this state in order to determine C.

$$\int_{-a}^{a} \left|C\left(a^2 - x^2\right)\right|^2 dx = |C|^2 \left[\frac{x^5}{5} - \frac{2a^2x^3}{3} + a^4 x\right]_{-a}^{a} = \frac{16a^5}{15}|C|^2 = 1,$$

and so we find (after fixing an arbitrary phase)

$$C = \sqrt{\frac{15}{16a^5}},$$

which then gives

$$\Phi(x, t = 0) = \sqrt{\frac{15}{16a^5}}\left(a^2 - x^2\right).$$

Since the states of definite energy that we found above form a complete orthonormal set on the interval $-a < x < a$, we can express Φ in terms of those states

$$\Phi(x, t = 0) = \sum_n c_n \psi_n(x),$$

and the probability of measuring the state Φ to have energy E_m, following Eq. (1.5.18), is given by

$$P\left(\Phi(x, t = 0) \to \psi_m(x)\right) = |c_m|^2 = \left|\int_{-a}^{a} \psi_m^*(x)\Phi(x, t = 0)\, dx\right|^2.$$

For the state of lowest energy this gives

$$\begin{aligned}
c_1 &= \int_{-a}^{a} \frac{1}{\sqrt{a}} \cos\left(\frac{\pi x}{2a}\right) \sqrt{\frac{15}{16a^5}} \left(a^2 - x^2\right) dx \\
&= \sqrt{\frac{15}{16}} \frac{1}{a^3} \left[\frac{2a^3}{\pi} \sin\left(\frac{\pi x}{2a}\right) - \frac{8xa^2}{\pi^2} \cos\left(\frac{\pi x}{2a}\right) - \frac{\frac{\pi^2 x^2}{4a^2} - 2}{\frac{\pi^3}{8a^3}} \sin\left(\frac{\pi x}{2a}\right) \right]_{-a}^{a} \\
&= \sqrt{\frac{15}{16}} \frac{1}{a^3} \left[\frac{4a^3}{\pi} - \frac{4a^3}{\pi} + \frac{32a^3}{\pi^3} \right] \\
&= \frac{8\sqrt{15}}{\pi^3},
\end{aligned}$$

and so the probability of finding Φ in the lowest energy state is

$$P\left(\Phi(x, t = 0) \to \psi_1(x)\right) = |c_1|^2 = \frac{960}{\pi^6} \approx 0.9986.$$

2. *Consider a non-relativistic particle of mass M in three dimensions, described by a Hamiltonian*

$$H = \frac{\mathbf{P}^2}{2M} + \frac{M\omega_0^2}{2}\mathbf{X}^2.$$

- *Find the energy values of states with definite energy, and the number of states for each energy.*
- *Find the rate at which a state of next-to-lowest energy decays by photon emission into the state of lowest energy.*

Hint: You can express the Hamiltonian as a sum of three Hamiltonians for one-dimensional oscillators, and use the results given in Section 1.4 for the energy levels and x-matrix elements for one-dimensional oscillators.

First, we will rewrite the Hamiltonian in components

$$\begin{aligned}
H &= \frac{\mathbf{p}^2}{2M} + \frac{M\omega_0^2}{2}\mathbf{x}^2 \\
&= \left(\frac{p_1^2}{2M} + \frac{M\omega_0^2}{2}x_1^2\right) + \left(\frac{p_2^2}{2M} + \frac{M\omega_0^2}{2}x_2^2\right) + \left(\frac{p_3^2}{2M} + \frac{M\omega_0^2}{2}x_3^2\right) \\
&= H_1 + H_2 + H_3,
\end{aligned}$$

where H_1, H_2, and H_3 are the one-dimensional harmonic oscillator Hamiltonians for x_1, x_2, and x_3, respectively. Now, we will assume a separable solution of the form

$$\psi(\mathbf{x}) = \psi_1(x_1)\psi_2(x_2)\psi_3(x_3),$$

where

$$\begin{aligned}
H_1\psi_1(x_1) &= E_{n_1}\psi_1(x_1) \\
H_2\psi_2(x_2) &= E_{n_2}\psi_2(x_2) \\
H_3\psi_3(x_3) &= E_{n_3}\psi_3(x_3) \\
H_1\psi_2(x_2) &= 0 \\
H_1\psi_3(x_3) &= 0 \\
&\vdots
\end{aligned}$$

and the E_n are given by the energies of the one-dimensional harmonic oscillator, Eq. (1.4.15)

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega_0,$$

where $n = 0, 1, 2, \ldots$. We therefore find for the three-dimensional harmonic oscillator

$$\begin{aligned}
H\psi(\mathbf{x}) &= E_N\psi(\mathbf{x}) \\
&= H_1\psi_1(x_1)\psi_2(x_2)\psi_3(x_3) + \psi_1(x_1)H_2\psi_2(x_2)\psi_3(x_3) \\
&\quad + \psi_1(x_1)\psi_2(x_2)H_3\psi_3(x_3) \\
&= E_{n_1}\psi_1(x_1)\psi_2(x_2)\psi_3(x_3) + \psi_1(x_1)E_{n_2}\psi_2(x_2)\psi_3(x_3) \\
&\quad + \psi_1(x_1)\psi_2(x_2)E_{n_3}\psi_3(x_3),
\end{aligned}$$

and so

$$E_N = E_{n_1} + E_{n_2} + E_{n_3} = \left(n_1 + n_2 + n_3 + \frac{3}{2}\right)\hbar\omega_0,$$

where n_1, n_2, and n_3 are each non-negative integers. If we define

$$N = n_1 + n_2 + n_3,$$

then the energy levels for states of definite energy are

$$E_N = \left(N + \frac{3}{2}\right)\hbar\omega_0,$$

where $N = 0, 1, 2, \ldots$.
Now we must count the number of states with each energy. For a definite value of N, the integer n_1 can take values $0, 1, \ldots, N$, then n_2 will take values $0, 1, \ldots, N - n_1$ (which represents $N - n_1 + 1$ possibilities), and n_3 is fixed to be $N - n_1 - n_2$. Then for each energy E_N there is a degeneracy

$$g_N = \sum_{n_1=0}^{N}(N - n_1 + 1) = N(N+1) - \frac{1}{2}N(N+1) + N + 1 = \frac{1}{2}(N+1)(N+2).$$

The rate of spontaneous emission of photons carrying energy $\hbar\omega_{nm} = E_m - E_n$ is given by Eq. (1.4.5) to be

$$A_m^n = \frac{4e^2\omega_{nm}^3}{3c^3\hbar} \left|[\mathbf{x}]_{nm}\right|^2 .$$

In three dimensions, we have $E_1 = \frac{5}{2}\hbar\omega_0$ and $E_0 = \frac{3}{2}\hbar\omega_0$, so $\omega_{01} = \omega_0$. The relevant matrix elements for the one-dimensional harmonic oscillator are given by Eq. (1.4.15) to be

$$[x]_{n+1,n}^* = [x]_{n,n+1} = e^{-i\omega_0 t}\sqrt{\frac{(n+1)\hbar}{2m_e\omega_0}}.$$

Since the wave function for the three-dimensional harmonic oscillator is just a product of three one-dimensional harmonic oscillator wave functions, we can also take these matrix elements to represent the matrix elements of a single spatial component x_1, x_2, or x_3

$$[x_i]_{mn} = [x]_{mn}.$$

The first excited state has n_1, n_2, or n_3 equal to 1 with the others equal to zero, and so the rate of spontaneous emission from the first excited state to the ground state for the three-dimensional harmonic oscillator is given by

$$A_1^0 = \frac{4e^2\omega_{01}^3}{3c^3\hbar}\left(\frac{\hbar}{2m_e\omega_0}\right) = \frac{2e^2\omega_0^2}{3c^3 m_e}.$$

3. *Suppose the photon had three polarization states rather than two. What difference would that make in the relations between Einstein's A and B coefficients?*
For black-body radiation in a cubical box with side L, the frequency of a normal mode is given by Eq. (1.1.2) as $\nu = |\mathbf{n}|c/L$. The number of normal modes $N(\nu)d\nu$ in a range of frequencies between ν and $\nu + d\nu$ is three times the volume of a spherical shell in frequency space (the factor of three here comes from the assumed three polarization states of the photon)

$$N(\nu)\,d\nu = 3 \times 4\pi|\mathbf{n}|^2\,d|\mathbf{n}| = 12\pi\left(\frac{L}{c}\right)^3 \nu^2\,d\nu.$$

Assuming that the energies of the light quanta are integer multiples of $h\nu$, the mean energy is

$$\bar{E} = \frac{\sum_n \exp\left(\frac{-nh\nu}{k_B T}\right) nh\nu}{\sum_n \exp\left(\frac{-nh\nu}{k_B T}\right)} = \frac{h\nu}{\exp\left(\frac{h\nu}{k_B T}\right) - 1}.$$

Then the energy density in radiation between ν and $\nu + d\nu$ is given by

$$\rho(\nu)\,d\nu = \frac{\bar{E}N(\nu)\,d\nu}{L^3} = \frac{12\pi h}{c^3}\frac{\nu^3\,d\nu}{\exp\left(\frac{h\nu}{k_B T}\right) - 1}.$$

Now assume that we have black-body radiation in equilibrium with atoms at a temperature T. The transition rate for atoms to go from state m to state n must equal the rate for the transition from state n to state m, so

$$N_m\left[A_m^n + B_m^n \rho(\nu_{nm}, T)\right] = N_n B_n^m \rho(\nu_{nm}, T).$$

Using the Boltzmann distribution for the atoms gives

$$\frac{N_m}{N_n} = \exp\left(-\frac{(E_m - E_n)}{k_B T}\right) = \exp\left(\frac{-h\nu_{nm}}{k_B T}\right).$$

We can then rearrange the condition of equilibrium to give

$$A_m^n = \frac{12\pi h}{c^3}\frac{\nu^3\,d\nu}{\exp\left(\frac{h\nu}{k_B T}\right) - 1}\left[\exp\left(\frac{-h\nu_{nm}}{k_B T}\right)B_n^m - B_m^n\right].$$

Requiring that the Einstein coefficients A and B are temperature independent then gives

$$B_m^n = B_n^m,$$

and

$$A_m^n = \left(\frac{12\pi h\nu_{nm}^3}{c^3}\right)B_m^n.$$

This conclusion gives a value for A which is larger than the usual expression Eq. (1.2.16) by a factor of 3/2.

4. *Show that the solution $\psi(\mathbf{x}, t)$ of the time-dependent Schrödinger equation for a particle in a real potential has the property that $\partial|\psi|^2/\partial t$ is the divergence of a three-vector.*
We wish to calculate

$$\frac{\partial}{\partial t}|\psi(x, t)|^2,$$

for some $\psi(x, t)$ which is a solution of the time-dependent Schrödinger equation, which reads

$$i\hbar\frac{\partial}{\partial t}\psi(x, t) = \frac{-\hbar^2}{2M}\nabla^2\psi(x, t) + V(x, t)\psi(x, t).$$

Taking the complex conjugate of the Schrödinger equation gives

$$-i\hbar\frac{\partial}{\partial t}\psi^*(x, t) = \frac{-\hbar^2}{2M}\nabla^2\psi^*(x, t) + V(x, t)\psi^*(x, t).$$

Recall that $|\psi(x,t)|^2$ can be rewritten as $\psi(x,t)\psi^*(x,t)$, and so we have

$$\frac{\partial}{\partial t}|\psi(x,t)|^2 = \psi^*(x,t)\frac{\partial}{\partial t}\psi(x,t) + \psi(x,t)\frac{\partial}{\partial t}\psi^*(x,t).$$

Using the Schrödinger equation to replace the time derivatives (now dropping space and time arguments), this becomes

$$\begin{aligned}
\frac{\partial}{\partial t}|\psi|^2 &= \psi^*\left(\frac{i\hbar}{2M}\nabla^2\psi - \frac{i}{\hbar}V\psi\right) + \psi\left(\frac{-i\hbar}{2M}\nabla^2\psi^* + \frac{i}{\hbar}V\psi^*\right) \\
&= \frac{i\hbar}{2M}\psi^*\nabla^2\psi - \frac{i\hbar}{2M}\psi\nabla^2\psi^* - \frac{i}{\hbar}V|\psi|^2 + \frac{i}{\hbar}|\psi|^2 \\
&= \frac{i\hbar}{2M}\left(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*\right) \\
&= \frac{i\hbar}{2M}\nabla\cdot\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right).
\end{aligned}$$

We see that we can identify a three-vector

$$\mathbf{j} \equiv \frac{-i\hbar}{2M}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right),$$

such that

$$\frac{\partial}{\partial t}|\psi|^2 = -\nabla\cdot\mathbf{j}.$$

Notice that this is a continuity equation which implies that probability is conserved in quantum mechanics. The time rate of change of probability density $|\psi|^2$ in some infinitesimal volume is equal to the rate at which the probability current $\mathbf{j}$ flows into the same infinitesimal volume.