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Solutions Manual

To

INTRODUCTORY QUANTUM OPTICS

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Field Quantization

2.1 problem 2.1

Eq. (2.5) has the form

$$E_x(z,t) = \sqrt{\frac{2\omega^2}{V\varepsilon_0}} q(t)\sin(kz), \tag{2.1.1}$$

and Eq. (2.2)

$$\nabla \times \mathbf{B} = \mu_0\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \tag{2.1.2}$$

Both equations lead to

$$-\partial_z B_y = \mu_0\varepsilon_0 \sqrt{\frac{2\omega^2}{V\varepsilon_0}} \dot{q}(t)\sin(kz), \tag{2.1.3}$$

which itself leads to Eq. (2.6)

$$B_y(z,t) = \frac{\mu_0\varepsilon_0}{k} \sqrt{\frac{2\omega^2}{V\varepsilon_0}} \dot{q}(t)\cos(kz). \tag{2.1.4}$$

2.2 problem 2.2

$$H = \frac{1}{2}\int dV \left[\varepsilon_0 E_x^2(z,t) + \frac{1}{\mu_0} B_y^2(z,t)\right]. \tag{2.2.1}$$

From the previous problem

$$E_x(z,t) = \sqrt{\frac{2\omega^2}{V\varepsilon_0}}q(t)\sin(kz), \tag{2.2.2}$$

so

$$\varepsilon_0 E_x^2(z,t) = \frac{2\omega^2}{V}q^2(t)\sin^2(kz). \tag{2.2.3}$$

Also

$$B_y(z,t) = \frac{\mu_0\varepsilon_0}{k}\sqrt{\frac{2\omega^2}{V\varepsilon_0}}\dot{q}(t)\cos(kz), \tag{2.2.4}$$

and

$$\frac{1}{\mu_0}B_y^2(z,t) = \frac{2}{V}p^2(t)\cos^2(kz), \tag{2.2.5}$$

where we have used that $c^2 = (\mu_0\varepsilon_0)^{-1}$, $p(t) = \dot{q}(t)$, and $ck = \omega$. Eq. 2.2.1 becomes then

$$H = \frac{1}{V}\int dV \left[\omega^2 q^2(t)\sin^2(kz) + p^2(t)\cos^2(kz)\right]. \tag{2.2.6}$$

Using these simple trigonometric identities $\cos^2 x = \frac{1+\cos 2x}{2}$ and $\sin^2 x = \frac{1-\cos 2x}{2}$, we can simplify equation 2.2.6 further to:

$$H = \frac{1}{2V}\int dV \left[\omega^2 q^2(t)(1+\cos 2kz) + p^2(t)(1-\cos 2kz)\right]. \tag{2.2.7}$$

Because of the periodic boundaries both cosine terms drop out, also $\frac{1}{V}\int dV = 1$ and we end up by

$$H = \frac{1}{2}\left(p^2 + w^2 q^2\right). \tag{2.2.8}$$

It is easy to see that this Hamiltonian has the form of a simple harmonic oscillator.

2.3 problem 2.3

Let f be a function defined as:

$$f(\lambda) = e^{i\lambda\hat{A}}\hat{B}e^{-i\lambda\hat{A}}. \tag{2.3.1}$$

If we expand f as

$$f(\lambda) = c_0 + c_1(i\lambda) + c_2\frac{(i\lambda)^2}{2!} + ..., \tag{2.3.2}$$

where

$$\begin{aligned} c_0 &= f(0) \\ c_1 &= f'(0) \\ c_2 &= f''(0)\cdots \end{aligned}$$

Also

$$\begin{aligned} c_0 &= f(0) = \hat{B} \\ c_1 &= f'(0) = \left[\hat{A}e^{i\lambda\hat{A}}\hat{B}e^{-i\lambda\hat{A}} - e^{i\lambda\hat{A}}\hat{B}\hat{A}e^{-i\lambda\hat{A}}\right]\Big|_{\lambda=0} = \left[\hat{A},\hat{B}\right] \\ c_2 &= \left[\hat{B},\left[\hat{A},\hat{B}\right]\right]. \end{aligned}$$

The same way we can determine the other coefficients.

2.4 problem 2.4

Let

$$f(x) = e^{\hat{A}x}e^{\hat{B}x} \tag{2.4.1}$$

$$\begin{aligned} \frac{df(x)}{dx} &= \hat{A}e^{\hat{A}x}e^{\hat{B}x} + e^{\hat{A}x}\hat{B}e^{\hat{B}x} \\ &= \left(\hat{A} + e^{\hat{A}x}\hat{B}e^{-\hat{A}x}\right)f(x) \end{aligned}$$

It is easy to prove that

$$\left[\hat{B},\hat{A}^n\right] = n\hat{A}^{n-1}\left[\hat{B},\hat{A}\right] \tag{2.4.2}$$

$$\begin{aligned}\left[\hat{B}, e^{-\hat{A}x}\right] &= \sum \left[\hat{B}, \frac{\left(-\hat{A}x\right)^n}{n!}\right] \\ &= \sum (-1)^n \frac{x^n}{n!} \left[\hat{B}, \hat{A}^n\right] \\ &= \sum (-1)^n \frac{x^n}{(n-1)!} \hat{A}^{n-1} \left[\hat{B}, \hat{A}\right] \\ &= -e^{-\hat{A}x} \left[\hat{B}, \hat{A}\right] x\end{aligned}$$

So

$$\hat{B}e^{-\hat{A}x} - e^{-\hat{A}x}\hat{B} = -e^{-\hat{A}x} \left[\hat{B}, \hat{A}\right] x$$

$$e^{-\hat{A}x}\hat{B}e^{\hat{A}x} = \hat{B} - e^{-\hat{A}x} \left[\hat{B}, \hat{A}\right] x \tag{2.4.3}$$

$$e^{\hat{A}x}\hat{B}e^{-\hat{A}x} = \hat{B} + e^{\hat{A}x} \left[\hat{A}, \hat{B}\right] x \tag{2.4.4}$$

Equation 4.1.1 becomes

$$\frac{df(x)}{dx} = \left(\hat{A} + \hat{B} + \left[\hat{A}, \hat{B}\right]\right) f(x). \tag{2.4.5}$$

Since $\left[\hat{A}, \hat{B}\right]$ commutes with $\hat{A}$ and $\hat{B}$, we can solve equation 2.4.5 as an ordinary equation. The solution is simply

$$f(x) = \exp\left[\left(\hat{A} + \hat{B}\right) x\right] \exp\left(\frac{1}{2}\left[\hat{A}, \hat{B}\right] x^2\right) \tag{2.4.6}$$

If we take $x = 1$ we will have

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{-\frac{1}{2}\left[\hat{A}, \hat{B}\right]} \tag{2.4.7}$$

2.5 problem 2.5

$$|\Psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|n\rangle + e^{i\varphi}|n+1\rangle\right). \tag{2.5.1}$$

$$
\begin{aligned}
|\Psi(t)\rangle &= e^{-i\frac{\hat{H}t}{\hbar}}|\Psi(0)\rangle \\
&= \frac{1}{\sqrt{2}}\left(e^{-i\frac{\hat{H}t}{\hbar}}|n\rangle + e^{-i\frac{\hat{H}t}{\hbar}}|n+1\rangle\right) \\
&= \frac{1}{\sqrt{2}}\left(e^{-in\omega t}|n\rangle + e^{i\varphi}e^{-i(n+1)\omega t}|n+1\rangle\right),
\end{aligned}
$$

where we have used $\frac{E}{\hbar} = \omega$

$$
\begin{aligned}
\hat{n}|\Psi(t)\rangle &= \hat{a}^\dagger\hat{a}|\Psi(t)\rangle \\
&= \frac{1}{\sqrt{2}}\left(e^{-in\omega t}n|n\rangle + e^{i\varphi}e^{-i(n+1)\omega t}(n+1)|n+1\rangle\right)
\end{aligned}
$$

$$
\begin{aligned}
\langle\hat{n}\rangle &= \langle\Psi(t)|\hat{n}|\Psi(t)\rangle \\
&= \frac{1}{2}(n+n+1) \\
&= n+\frac{1}{2}
\end{aligned}
$$

the same way

$$
\begin{aligned}
\left\langle\hat{n}^2\right\rangle &= \langle\Psi(t)|\hat{n}\hat{n}|\Psi(t)\rangle \\
&= \frac{1}{2}\left(n^2+(n+1)^2\right) \\
&= n^2+n+\frac{1}{2}
\end{aligned}
$$

$$
\begin{aligned}
\left\langle\ (\Delta\hat{n})^2\right\rangle &= \left\langle\hat{n}^2\right\rangle - \langle\hat{n}\rangle^2 \\
&= \frac{1}{4}
\end{aligned}
$$

$$
\begin{aligned}
\hat{E}|\Psi(t)\rangle &= \mathcal{E}_0\sin(kz)\left(\hat{a}^\dagger+\hat{a}\right)|\Psi(t)\rangle \\
&= \frac{1}{\sqrt{2}}\mathcal{E}_0\sin(kz)\left(\hat{a}^\dagger+\hat{a}\right)\left(e^{-in\omega t}|n\rangle + e^{i\varphi}e^{-i(n+1)\omega t}|n+1\rangle\right) \\
&= \frac{1}{\sqrt{2}}\mathcal{E}_0\sin(kz)\Big[e^{-in\omega t}\left(\sqrt{n+1}|n+1\rangle + \sqrt{n}|n-1\rangle\right) \\
&\quad + e^{i\varphi}e^{-i(n+1)\omega t}\left(\sqrt{n+2}|n+2\rangle + \sqrt{n+1}|n\rangle\right)\Big]
\end{aligned}
$$

$$\begin{aligned}\langle\Psi(t)|\hat{E}|\Psi(t)\rangle &= \frac{1}{2}\mathcal{E}_0\sin(kz)\left(e^{i\omega t}\sqrt{n+1}+e^{i\varphi}e^{-i\omega t}\sqrt{n+1}\right)\\ &= \sqrt{n+1}\mathcal{E}_0\sin(kz)\cos(\varphi-\omega t)\end{aligned}$$

$$\begin{aligned}\left\langle\hat{E}^2\right\rangle &= \langle\Psi(t)|\hat{E}\hat{E}|\Psi(t)\rangle\\ &= 2(n+1)\mathcal{E}_0^2\sin^2(kz)\end{aligned}$$

$$\left\langle\left(\Delta\hat{E}\right)^2\right\rangle = (n+1)\mathcal{E}_0^2\sin^2(kz)\left[2-\cos^2(\varphi-\omega t)\right]$$

$$\begin{aligned}(\hat{a}^\dagger-\hat{a})|\Psi(t)\rangle &= \frac{1}{\sqrt{2}}\Big[e^{-in\omega t}\left(\sqrt{n+1}|n+1\rangle-\sqrt{n}|n-1\rangle\right)\\ &\quad + e^{i\varphi}e^{-i(n+1)\omega t}\left(\sqrt{n+2}|n+2\rangle-\sqrt{n+1}|n\rangle\right)\Big]\\ \langle(\hat{a}^\dagger-\hat{a})\rangle &= -i\sqrt{n+1}\sin(\varphi-\omega t)\end{aligned}$$

Finally we have the following quantities

$$\begin{aligned}\Delta n &= \frac{1}{2}\\ \Delta E &= \mathcal{E}_0|\sin(kz)|\sqrt{2(n+1)\left[2-\cos^2(\varphi-\omega t)\right]}\\ \left|\langle(\hat{a}^\dagger-\hat{a})\rangle\right| &= \sqrt{n+1}|\sin(\varphi-\omega t)|.\end{aligned}$$

Certainly the inequality in (2.49) holds true since

$$\sqrt{2\left(2-\cos^2(\varphi-\omega t)\right)} > |\sin(\varphi-\omega t)|.$$

2.6 problem 2.6

$$\begin{aligned}\hat{X}_1 &= \frac{1}{2}\left(\hat{a}+\hat{a}^\dagger\right)\\ \hat{X}_2 &= \frac{1}{2i}\left(\hat{a}-\hat{a}^\dagger\right)\\ \hat{X}_1^2 &= \frac{1}{4}\left(\hat{a}^{\dagger 2}+\hat{a}^2+2\hat{a}^\dagger\hat{a}+1\right)\\ \hat{X}_2^2 &= -\frac{1}{4}\left(\hat{a}^{\dagger 2}+\hat{a}^2-2\hat{a}^\dagger\hat{a}-1\right)\end{aligned}$$

$$|\Psi_{01}\rangle = \alpha|0\rangle + \beta|1\rangle$$

where $|\alpha|^2 + |\beta|^2 = 1$. So we can rewrite $\beta = \sqrt{1-|\alpha|^2}e^{i\phi}$ and $\alpha^2 = |\alpha|^2$ without any loss of generality.

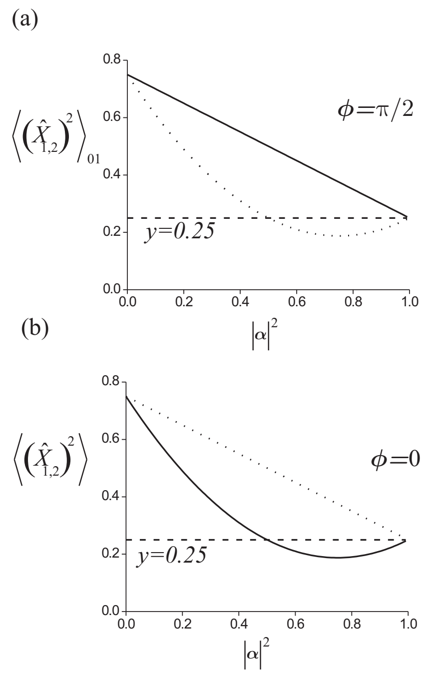
$$\left\langle \hat{X}_1 \right\rangle_{01} = \frac{1}{2}\left(\alpha^*\beta + \alpha\beta^*\right)$$
$$\left\langle \hat{X}_2 \right\rangle_{01} = \frac{1}{2i}\left(\alpha^*\beta - \alpha\beta^*\right)$$

$$\left\langle \hat{a}^{\dagger 2} \right\rangle_{01} = 0$$
$$\left\langle \hat{a}^{2} \right\rangle_{01} = 0$$
$$\langle \hat{a}^{\dagger}\hat{a} \rangle_{01} = |\beta|^2$$

$$\left\langle \hat{X}_1^2 \right\rangle_{01} = \frac{1}{4}\left(2|\beta|^2 + 1\right)$$
$$\left\langle \hat{X}_2^2 \right\rangle_{01} = \frac{1}{4}\left(2|\beta|^2 + 1\right)$$

$$\begin{aligned}
\left\langle \left(\Delta \hat{X}_1\right)^2 \right\rangle_{01} &= \frac{1}{4}\left[2|\beta|^2 + 1 - (\alpha^*\beta)^2 - (\alpha\beta^*)^2 - 2|\alpha|^2|\beta|^2\right] \\
&= \frac{1}{4}\left[3 - 4|\alpha|^2 + 2|\alpha|^4 - 2|\alpha|^2(1-|\alpha|^2)\cos(2\phi)\right] \\
\left\langle \left(\Delta \hat{X}_2\right)^2 \right\rangle_{01} &= \frac{1}{4}\left[2|\beta|^2 + 1 + (\alpha^*\beta)^2 + (\alpha\beta^*)^2 - 2|\alpha|^2|\beta|^2\right] \\
&= \frac{1}{4}\left[3 - 4|\alpha|^2 + 2|\alpha|^4 + 2|\alpha|^2(1-|\alpha|^2)\cos(2\phi)\right]
\end{aligned}$$

In figures a and b below we plot $\left\langle \left(\Delta \hat{X}_1\right)^2 \right\rangle_{01}$ (solid line) for $\phi = \pi/2$ and $\left\langle \left(\Delta \hat{X}_2\right)^2 \right\rangle_{01}$ (doted line) for $\phi = 0$, respectively. Clearly the quadratures in hands go below the quadrature variances of the vacuum in more than one occasion.



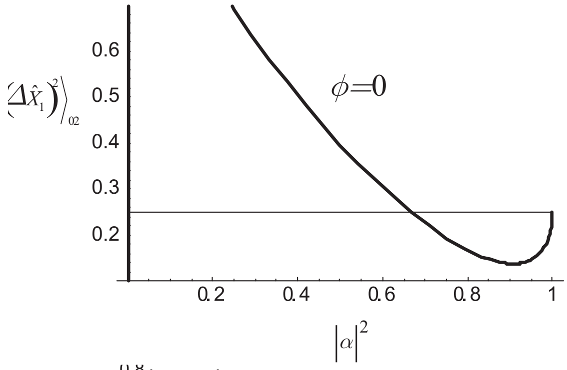
$$|\Psi_{02}\rangle = \alpha|0\rangle + \beta|2\rangle. \tag{2.6.1}$$

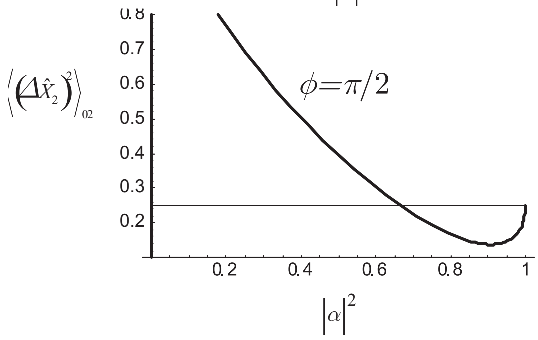
Again, where $|\alpha|^2 + |\beta|^2 = 1$. So we can rewrite $\beta = \sqrt{1-|\alpha|^2}e^{i\phi}$ and $\alpha^2 = |\alpha|^2$ without any loss of generality.

$$\begin{aligned}
\left\langle \hat{X}_1 \right\rangle_{02} &= 0 = \left\langle \hat{X}_2 \right\rangle_{02} \\
\left\langle \left(\Delta \hat{X}_1\right)^2 \right\rangle_{02} &= \left\langle \hat{X}_1^2 \right\rangle_{02} \\
&= \frac{1}{4}\left(|\alpha + \sqrt{2}\beta|^2 + 3|\beta|^2\right) \\
&= \frac{1}{4}\left[5 - 4|\alpha|^2 + 2\sqrt{2|\alpha|^2(1-|\alpha|^2)}\cos\phi\right] \\
\left\langle \left(\Delta \hat{X}_2\right)^2 \right\rangle_{02} &= \left\langle \hat{X}_2^2 \right\rangle_{02} \\
&= \frac{1}{4}\left(|\alpha - \sqrt{2}\beta|^2 + 3|\beta|^2\right) \\
&= \frac{1}{4}\left[5 - 4|\alpha|^2 - 2\sqrt{2|\alpha|^2(1-|\alpha|^2)}\cos\phi\right]
\end{aligned}$$

In figures c and d below we plot $\left\langle \left(\Delta \hat{X}_1\right)^2 \right\rangle_{02}$ for $\phi = 0$ and $\left\langle \left(\Delta \hat{X}_2\right)^2 \right\rangle_{02}$

for $\phi = \pi/2$, respectively. Clearly the quadratures in hands go below the quadrature variances of the vacuum in more than one occasion.





2.7 Problem 2.7

$$|\Psi'\rangle = \mathcal{N}\hat{a}\,|\Psi\rangle$$

$$\begin{aligned}|\mathcal{N}|^2 &= \langle\hat{n}\rangle \\ &= \bar{n} \\ \mathcal{N} &= \frac{1}{\sqrt{\bar{n}}}\end{aligned}$$

$$|\Psi'\rangle = \frac{1}{\sqrt{\bar{n}}}\hat{a}\,|\Psi\rangle$$

$$\begin{aligned}
\overline{n'} &= \langle \Psi' | \hat{n} | \Psi' \rangle \\
&= \frac{1}{\bar{n}} \langle \Psi | \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} | \Psi \rangle \\
&= \frac{1}{\bar{n}} \left(\langle \Psi | \hat{n}^2 | \Psi \rangle - \langle \Psi | \hat{n} | \Psi \rangle \right) \\
&= \frac{\langle \Psi | \hat{n}^2 | \Psi \rangle}{\bar{n}} - 1 \\
&= \frac{\langle \hat{n}^2 \rangle}{\langle \hat{n} \rangle} - 1.
\end{aligned}$$

Notice that $\overline{n'} \neq \overline{n} - 1$ in general, but for the number state $|n\rangle$, and only of this state we have $\overline{n'} = \overline{n} - 1$.

2.8 Problem 2.8

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |10\rangle) \tag{2.8.1}$$

The average photon number, $\bar{n}$, of this state is

$$\bar{n} = \langle \Psi | \hat{a}^\dagger \hat{a} | \Psi \rangle, \tag{2.8.2}$$

which can be easily calculated to be

$$\bar{n} = \frac{1}{2}(0 + 10) = 5. \tag{2.8.3}$$

If we assume that a single photon is absorbed, our normalized state will become

$$|\Psi\rangle = |9\rangle, \tag{2.8.4}$$

then the average photon becomes

$$\bar{n} = 9. \tag{2.8.5}$$

2.9 Problem 2.9

$$\begin{aligned}
\mathbf{E}(\mathbf{r}, t) &= i \sum_{\mathbf{k},s} \omega_k \mathbf{e}_{\mathbf{k}s} \left[A_{\mathbf{k}s} e^{i(\mathbf{k}\cdot\mathbf{r} - \omega_k t)} - A^*_{\mathbf{k}s} e^{-i(\mathbf{k}\cdot\mathbf{r} - \omega_k t)} \right] \\
\mathbf{B}(\mathbf{r}, t) &= \frac{i}{c} \sum_{\mathbf{k},s} \omega_k \left(\kappa \times \mathbf{e}_{\mathbf{k}s} \right) \left[A_{\mathbf{k}s} e^{i(\mathbf{k}\cdot\mathbf{r} - \omega_k t)} - A^*_{\mathbf{k}s} e^{-i(\mathbf{k}\cdot\mathbf{r} - \omega_k t)} \right]
\end{aligned}$$

$$\begin{aligned}
\nabla \cdot \mathbf{E}(\mathbf{r},t) &= i\nabla \cdot \left(\sum_{\mathbf{k},s} \omega_k \mathbf{e_{ks}} \left[A_{\mathbf{k}s} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} - A^*_{\mathbf{k}s} e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} \right] \right) \\
&= i \sum_{\mathbf{k},s} \omega_k \mathbf{e_{ks}} \cdot \left[A_{\mathbf{k}s} \nabla \left(e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} \right) - A^*_{\mathbf{k}s} \nabla \left(e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} \right) \right] \\
&= i \sum_{\mathbf{k},s} \omega_k \mathbf{e_{ks}} \cdot \left[i\mathbf{k} A_{\mathbf{k}s} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} + i\mathbf{k} A^*_{\mathbf{k}s} e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} \right] \\
&= -\sum_{\mathbf{k},s} \omega_k \mathbf{e_{ks}} \cdot \mathbf{k} \left[A_{\mathbf{k}s} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} + A^*_{\mathbf{k}s} e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} \right] \\
&= 0
\end{aligned}$$

where we have used the vector identity

$$\nabla \cdot (f\mathbf{A}) = f\,(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)\,, \tag{2.9.1}$$

and

$$\mathbf{e_{ks}} \cdot \mathbf{k} = 0. \tag{2.9.2}$$

$$\begin{aligned}
\nabla \cdot \mathbf{B}(\mathbf{r},t) &= \frac{i}{c}\nabla \cdot \left(\sum_{\mathbf{k},s} \omega_k \left(\kappa \times \mathbf{e_{ks}} \right) \left[A_{\mathbf{k}s} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} - A^*_{\mathbf{k}s} e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} \right] \right) \\
&= \frac{i}{c} \sum_{\mathbf{k},s} \omega_k \left(\kappa \times \mathbf{e_{ks}} \right) \cdot \left[A_{\mathbf{k}s} \nabla \left(e^{i(\mathbf{kr}-\omega_k t)} \right) - A^*_{\mathbf{k}s} \nabla \left(e^{-i(\mathbf{kr}-\omega_k t)} \right) \right] \\
&= -\frac{1}{c} \sum_{\mathbf{k},s} \omega_k \left(\kappa \times \mathbf{e_{ks}} \right) \cdot \mathbf{k} \left[A_{\mathbf{k}s} e^{i(\mathbf{kr}-\omega_k t)} + A^*_{\mathbf{k}s} e^{-i(\mathbf{kr}-\omega_k t)} \right] \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\nabla \times \mathbf{E}(\mathbf{r}, t) &= i\nabla \times \left(\sum_{\mathbf{k},s} \omega_k \mathbf{e}_{\mathbf{k}s} \left[A_{\mathbf{k}s} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} - A^*_{\mathbf{k}s} e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} \right] \right) \\
&= i \sum_{\mathbf{k},s} \omega_k \mathbf{e}_{\mathbf{k}s} \times \left[A_{\mathbf{k}s} \nabla \left(e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} \right) - A^*_{\mathbf{k}s} \nabla \left(e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} \right) \right] \\
&= i \sum_{\mathbf{k},s} \omega_k \mathbf{e}_{\mathbf{k}s} \times \left[i\mathbf{k} A_{\mathbf{k}s} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} + i\mathbf{k} A^*_{\mathbf{k}s} e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} \right] \\
&= -\sum_{\mathbf{k},s} \omega_k \mathbf{e}_{\mathbf{k}s} \times \mathbf{k} \left[A_{\mathbf{k}s} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} + A^*_{\mathbf{k}s} e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} \right] \\
&= -\sum_{\mathbf{k},s} \frac{\omega_k^2}{c} \mathbf{e}_{\mathbf{k}s} \times \kappa \left[A_{\mathbf{k}s} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} + A^*_{\mathbf{k}s} e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} \right] \\
&= \sum_{\mathbf{k},s} \frac{\omega_k^2}{c} \kappa \times \mathbf{e}_{\mathbf{k}s} \left[A_{\mathbf{k}s} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} + A^*_{\mathbf{k}s} e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} \right]
\end{aligned}$$

where we have used the vector identity

$$\nabla \times (f\mathbf{A}) = f\left(\nabla \times \mathbf{A}\right) + \mathbf{A} \times \left(\nabla f\right), \tag{2.9.3}$$

and

$$\mathbf{k} = \frac{\omega_k}{c}\kappa. \tag{2.9.4}$$

$$\begin{aligned}
\frac{\partial \mathbf{B}}{\partial t} &= \frac{i}{c} \sum_{\mathbf{k},s} \omega_k \left(\kappa \times \mathbf{e}_{\mathbf{k}s}\right) \left[A_{\mathbf{k}s} \frac{\partial e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)}}{\partial t} - A^*_{\mathbf{k}s} \frac{\partial e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)}}{\partial t} \right] \\
&= \frac{1}{c} \sum_{\mathbf{k},s} \omega_k^2 \left(\kappa \times \mathbf{e}_{\mathbf{k}s}\right) \left[A_{\mathbf{k}s} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} + A^*_{\mathbf{k}s} e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} \right] \\
&= -\nabla \times \mathbf{E}
\end{aligned}$$

$$\begin{aligned}
\nabla \times \mathbf{B}(\mathbf{r},t) &= \frac{i}{c}\nabla \times \left(\sum_{\mathbf{k},s} \omega_k \left(\kappa \times \mathbf{e}_{\mathbf{k}s}\right) \left[A_{\mathbf{k}s} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} - A^*_{\mathbf{k}s} e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} \right] \right) \\
&= \frac{i}{c} \sum_{\mathbf{k},s} \omega_k \left(\kappa \times \mathbf{e}_{\mathbf{k}s}\right) \times \left[A_{\mathbf{k}s} \nabla \left(e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} \right) - A^*_{\mathbf{k}s} \nabla \left(e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} \right) \right] \\
&= \frac{i}{c} \sum_{\mathbf{k},s} \omega_k \left(\kappa \times \mathbf{e}_{\mathbf{k}s}\right) \times \left[i\mathbf{k} A_{\mathbf{k}s} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} + i\mathbf{k} A^*_{\mathbf{k}s} e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} \right] \\
&= -\frac{1}{c} \sum_{\mathbf{k},s} \omega_k \left(\kappa \times \mathbf{e}_{\mathbf{k}s}\right) \times \mathbf{k} \left[A_{\mathbf{k}s} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} + A^*_{\mathbf{k}s} e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} \right] \\
&= -\sum_{\mathbf{k},s} \frac{\omega_k^2}{c^2} \mathbf{e}_{\mathbf{k}s} \left[A_{\mathbf{k}s} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} + A^*_{\mathbf{k}s} e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} \right] \\
&= \mu_0 \epsilon_0 \sum_{\mathbf{k},s} \omega_k^2 \mathbf{e}_{\mathbf{k}s} \left[A_{\mathbf{k}s} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} + A^*_{\mathbf{k}s} e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} \right] \\
&= \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}
\end{aligned}$$

2.10 Problem 2.10

For thermal light

$$P_n = \frac{\bar{n}^n}{(1+\bar{n})^{n+1}} \tag{2.10.1}$$

$$\begin{aligned}
\sum_n n(n-1)...(n-r+1)P_n &= \sum_n n(n-1)...(n-r+1)\frac{\bar{n}^n}{(1+\bar{n})^{n+1}} \\
&= \frac{1}{\bar{n}}\left(\frac{\bar{n}}{1+\bar{n}}\right)^{r+1} \sum_n n(n-1)...(n-r+1)\left(\frac{\bar{n}}{1+\bar{n}}\right)^{n-r}
\end{aligned}$$

To simplify the last expression, let's define $x = \frac{\bar{n}}{1+\bar{n}}$, for which $x < 1$,

$$\begin{aligned}
\sum_n n(n-1)...(n-r+1)P_n &= \frac{1}{\bar{n}} x^{r+1} \sum_n n(n-1)...(n-r+1)x^{n-r} \\
&= \frac{1}{\bar{n}} x^{r+1} \frac{\partial^r}{\partial x^r} \sum x^n \\
&= \frac{1}{\bar{n}} x^{r+1} \frac{\partial^r}{\partial x^r} \frac{1}{1-x} \\
&= \frac{1}{\bar{n}} x^{r+1} r! \frac{1}{(1-x)^{r+1}} \\
\langle \hat{n}(\hat{n}-1)(\hat{n}-1)\cdots(\hat{n}-r+1)\rangle &= r!\bar{n}^r \qquad (2.10.2)
\end{aligned}$$

2.11 Problem 2.11

$$\begin{aligned}
\left[\hat{C}, \hat{S}\right] &= -\frac{i}{4}\left[\hat{E} + \hat{E}^\dagger, \hat{E} - \hat{E}^\dagger\right] \\
&= \frac{i}{2}\left[\hat{E}, \hat{E}^\dagger\right] \\
&= \frac{i}{2}\left(\hat{E}\hat{E}^\dagger - \hat{E}^\dagger\hat{E}\right) \\
&= \frac{i}{2}\left(1 - 1 + |0\rangle\langle 0|\right) \\
&= \frac{i}{2}|0\rangle\langle 0|
\end{aligned}$$

$$\langle m| \left[\hat{C}, \hat{S}\right] |n\rangle = \frac{i}{2}\delta_{m,0}\delta_{n,0}.$$

Obviously, only the diagonal matrix elements are nonzero.

2.12 Problem 2.12

Using equation (2.229) for

$$\hat{\rho} = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right) \qquad (2.12.1)$$

we have

$$\begin{aligned}\mathcal{P}(\varphi) &= \frac{1}{2\pi}\langle\varphi|\hat{\rho}|\varphi\rangle \\ &= \frac{1}{2\pi}\sum_n\sum_{n'}\langle n'|e^{-in'\varphi}\hat{\rho}e^{in\varphi}|n\rangle \\ &= \frac{1}{2\pi}\left(1+e^{i\varphi}e^{-i\varphi}\right) \\ &= \frac{1}{\pi}.\end{aligned}$$

This is similar to a thermal state. On the other hand using equation (2.227) for $|\psi\rangle = \frac{1}{2}(|0\rangle + e^{i\theta}|1\rangle)$ we have

$$\begin{aligned}\mathcal{P}(\phi) &= \frac{1}{2\pi}|\langle\phi|\psi\rangle|^2 \\ &= \frac{1}{2\pi}\left[1+\cos(\phi-\theta)\right].\end{aligned}$$

As expected, it is different than a statistical mixture state, the one for the pure state exhibiting a phase dependence.

2.13 Problem 2.13

$$\hat{\rho}_{th} = \sum_{n=0}^{\infty} P_n|n\rangle\langle n| \tag{2.13.1}$$

$$\begin{aligned}\mathcal{P}(\varphi) &= \frac{1}{2\pi}\langle\varphi|\hat{\rho}|\varphi\rangle \\ &= \frac{1}{2\pi}\sum_n\sum_{n'}\langle n'|e^{-in'\varphi}\hat{\rho}e^{in\varphi}|n\rangle \\ &= \frac{1}{2\pi}\sum_n\sum_{n'}\sum_k P_k\langle n'|e^{-in'\varphi}|k\rangle\langle k|e^{in\varphi}|n\rangle \\ &= \frac{1}{2\pi}\sum_k P_k \\ &= \frac{1}{2\pi}\end{aligned}$$

Chapter 3

Coherent States

3.1 Problem 3.1

Let assume that the eigenvector of the creation operator $\hat{a}^\dagger$ exists. So we can write

$$\hat{a}^\dagger|\beta\rangle = \beta|\beta\rangle. \tag{3.1.1}$$

Now let's write $|\beta\rangle$ as a superposition of the number states, namely

$$|\beta\rangle = \sum_{n=0}^{\infty} c_n|n\rangle \tag{3.1.2}$$

Now let's plug the last expression in equation 3.1.1:

$$\hat{a}^\dagger|\beta\rangle = \sum_{n=0}^{\infty} c_n\sqrt{n+1}|n+1\rangle \tag{3.1.3}$$

$$= \beta\sum_{n=0}^{\infty} c_n|n\rangle. \tag{3.1.4}$$

From the last express we deduce that

$$c_0 = 0, \tag{3.1.5}$$

$$c_{n+1} = \frac{1}{\beta}c_n\sqrt{n+1}, \tag{3.1.6}$$

which means all c_n's $= 0$.