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# **Solutions Manual**

To

**INTRODUCTORY QUANTUM OPTICS**

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# Chapter 1

## Introduction





# Chapter 2

## Field Quantization

### 2.1 problem 2.1

Eq. (2.5) has the form

$$E_x(z, t) = \sqrt{\frac{2\omega^2}{V\varepsilon_0}} q(t) \sin(kz), \quad (2.1.1)$$

and Eq. (2.2)

$$\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \quad (2.1.2)$$

Both equations lead to

$$-\partial_z B_y = \mu_0 \varepsilon_0 \sqrt{\frac{2\omega^2}{V\varepsilon_0}} \dot{q}(t) \sin(kz), \quad (2.1.3)$$

which itself leads to Eq. (2.6)

$$B_y(z, t) = \frac{\mu_0 \varepsilon_0}{k} \sqrt{\frac{2\omega^2}{V\varepsilon_0}} \dot{q}(t) \cos(kz). \quad (2.1.4)$$

### 2.2 problem 2.2

$$H = \frac{1}{2} \int dV \left[ \varepsilon_0 E_x^2(z, t) + \frac{1}{\mu_0} B_y^2(z, t) \right]. \quad (2.2.1)$$

From the previous problem

$$E_x(z, t) = \sqrt{\frac{2\omega^2}{V\varepsilon_0}} q(t) \sin(kz), \quad (2.2.2)$$

so

$$\varepsilon_0 E_x^2(z, t) = \frac{2\omega^2}{V} q^2(t) \sin^2(kz). \quad (2.2.3)$$

Also

$$B_y(z, t) = \frac{\mu_0 \varepsilon_0}{k} \sqrt{\frac{2\omega^2}{V\varepsilon_0}} \dot{q}(t) \cos(kz), \quad (2.2.4)$$

and

$$\frac{1}{\mu_0} B_y^2(z, t) = \frac{2}{V} p^2(t) \cos^2(kz), \quad (2.2.5)$$

where we have used that  $c^2 = (\mu_0 \varepsilon_0)^{-1}$ ,  $p(t) = \dot{q}(t)$ , and  $ck = \omega$ . Eq. 2.2.1 becomes then

$$H = \frac{1}{V} \int dV [\omega^2 q^2(t) \sin^2(kz) + p^2(t) \cos^2(kz)]. \quad (2.2.6)$$

Using these simple trigonometric identities  $\cos^2 x = \frac{1+\cos 2x}{2}$  and  $\sin^2 x = \frac{1-\cos 2x}{2}$ , we can simplify equation 2.2.6 further to:

$$H = \frac{1}{2V} \int dV [\omega^2 q^2(t)(1 + \cos 2kz) + p^2(t)(1 - \cos 2kz)]. \quad (2.2.7)$$

Because of the periodic boundaries both cosine terms drop out, also  $\frac{1}{V} \int dV = 1$  and we end up by

$$H = \frac{1}{2} (p^2 + \omega^2 q^2). \quad (2.2.8)$$

It is easy to see that this Hamiltonian has the form of a simple harmonic oscillator.

## 2.3 problem 2.3

Let  $f$  be a function defined as:

$$f(\lambda) = e^{i\lambda\hat{A}} \hat{B} e^{-i\lambda\hat{A}}. \quad (2.3.1)$$

If we expand  $f$  as

$$f(\lambda) = c_0 + c_1(i\lambda) + c_2 \frac{(i\lambda)^2}{2!} + \dots, \quad (2.3.2)$$

where

$$\begin{aligned} c_0 &= f(0) \\ c_1 &= f'(0) \\ c_2 &= f''(0) \dots \end{aligned}$$

Also

$$\begin{aligned} c_0 &= f(0) = \hat{B} \\ c_1 &= f'(0) = \left[ \hat{A} e^{i\lambda \hat{A}} \hat{B} e^{-i\lambda \hat{A}} - e^{i\lambda \hat{A}} \hat{B} \hat{A} e^{-i\lambda \hat{A}} \right] \Big|_{\lambda=0} = [\hat{A}, \hat{B}] \\ c_2 &= [\hat{B}, [\hat{A}, \hat{B}]]. \end{aligned}$$

The same way we can determine the other coefficients.

## 2.4 problem 2.4

Let

$$f(x) = e^{\hat{A}x} e^{\hat{B}x} \quad (2.4.1)$$

$$\begin{aligned} \frac{df(x)}{dx} &= \hat{A} e^{\hat{A}x} e^{\hat{B}x} + e^{\hat{A}x} \hat{B} e^{\hat{B}x} \\ &= \left( \hat{A} + e^{\hat{A}x} \hat{B} e^{-\hat{A}x} \right) f(x) \end{aligned}$$

It is easy to prove that

$$[\hat{B}, \hat{A}^n] = n \hat{A}^{n-1} [\hat{B}, \hat{A}] \quad (2.4.2)$$

$$\begin{aligned}
[\hat{B}, e^{-\hat{A}x}] &= \sum \left[ \hat{B}, \frac{(-\hat{A}x)^n}{n!} \right] \\
&= \sum (-1)^n \frac{x^n}{n!} [\hat{B}, \hat{A}^n] \\
&= \sum (-1)^n \frac{x^n}{(n-1)!} \hat{A}^{n-1} [\hat{B}, \hat{A}] \\
&= -e^{-\hat{A}x} [\hat{B}, \hat{A}] x
\end{aligned}$$

So

$$\hat{B}e^{-\hat{A}x} - e^{-\hat{A}x}\hat{B} = -e^{-\hat{A}x} [\hat{B}, \hat{A}] x$$

$$e^{-\hat{A}x}\hat{B}e^{\hat{A}x} = \hat{B} - e^{-\hat{A}x} [\hat{B}, \hat{A}] x \quad (2.4.3)$$

$$e^{\hat{A}x}\hat{B}e^{-\hat{A}x} = \hat{B} + e^{\hat{A}x} [\hat{A}, \hat{B}] x \quad (2.4.4)$$

Equation 4.1.1 becomes

$$\frac{df(x)}{dx} = \left( \hat{A} + \hat{B} + [\hat{A}, \hat{B}] \right) f(x). \quad (2.4.5)$$

Since  $[\hat{A}, \hat{B}]$  commutes with  $\hat{A}$  and  $\hat{B}$ , we can solve equation 2.4.5 as an ordinary equation. The solution is simply

$$f(x) = \exp \left[ (\hat{A} + \hat{B}) x \right] \exp \left( \frac{1}{2} [\hat{A}, \hat{B}] x^2 \right) \quad (2.4.6)$$

If we take  $x = 1$  we will have

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{-\frac{1}{2}[\hat{A}, \hat{B}]} \quad (2.4.7)$$

## 2.5 problem 2.5

$$|\Psi(0)\rangle = \frac{1}{\sqrt{2}} (|n\rangle + e^{i\varphi}|n+1\rangle). \quad (2.5.1)$$

$$\begin{aligned}
|\Psi(t)\rangle &= e^{-i\frac{\hat{H}t}{\hbar}}|\Psi(0)\rangle \\
&= \frac{1}{\sqrt{2}}\left(e^{-i\frac{\hat{H}t}{\hbar}}|n\rangle + e^{-i\frac{\hat{H}t}{\hbar}}|n+1\rangle\right) \\
&= \frac{1}{\sqrt{2}}\left(e^{-in\omega t}|n\rangle + e^{i\varphi}e^{-i(n+1)\omega t}|n+1\rangle\right),
\end{aligned}$$

where we have used  $\frac{E}{\hbar} = \omega$

$$\begin{aligned}
\hat{n}|\Psi(t)\rangle &= \hat{a}^\dagger\hat{a}|\Psi(t)\rangle \\
&= \frac{1}{\sqrt{2}}\left(e^{-in\omega t}n|n\rangle + e^{i\varphi}e^{-i(n+1)\omega t}(n+1)|n+1\rangle\right)
\end{aligned}$$

$$\begin{aligned}
\langle\hat{n}\rangle &= \langle\Psi(t)|\hat{n}|\Psi(t)\rangle \\
&= \frac{1}{2}(n+n+1) \\
&= n + \frac{1}{2}
\end{aligned}$$

the same way

$$\begin{aligned}
\langle\hat{n}^2\rangle &= \langle\Psi(t)|\hat{n}\hat{n}|\Psi(t)\rangle \\
&= \frac{1}{2}(n^2 + (n+1)^2) \\
&= n^2 + n + \frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
\langle(\Delta\hat{n})^2\rangle &= \langle\hat{n}^2\rangle - \langle\hat{n}\rangle^2 \\
&= \frac{1}{4}
\end{aligned}$$

$$\begin{aligned}
\hat{E}|\Psi(t)\rangle &= \mathcal{E}_0 \sin(kz) (\hat{a}^\dagger + \hat{a}) |\Psi(t)\rangle \\
&= \frac{1}{\sqrt{2}}\mathcal{E}_0 \sin(kz) (\hat{a}^\dagger + \hat{a}) \left(e^{-in\omega t}|n\rangle + e^{i\varphi}e^{-i(n+1)\omega t}|n+1\rangle\right) \\
&= \frac{1}{\sqrt{2}}\mathcal{E}_0 \sin(kz) \left[e^{-in\omega t} \left(\sqrt{n+1}|n+1\rangle + \sqrt{n}|n-1\rangle\right) \right. \\
&\quad \left. + e^{i\varphi}e^{-i(n+1)\omega t} \left(\sqrt{n+2}|n+2\rangle + \sqrt{n+1}|n\rangle\right)\right]
\end{aligned}$$

$$\begin{aligned}
\langle \Psi(t) | \hat{E} | \Psi(t) \rangle &= \frac{1}{2} \mathcal{E}_0 \sin(kz) \left( e^{i\omega t} \sqrt{n+1} + e^{i\varphi} e^{-i\omega t} \sqrt{n+1} \right) \\
&= \sqrt{n+1} \mathcal{E}_0 \sin(kz) \cos(\varphi - \omega t) \\
\langle \hat{E}^2 \rangle &= \langle \Psi(t) | \hat{E} \hat{E} | \Psi(t) \rangle \\
&= 2(n+1) \mathcal{E}_0^2 \sin^2(kz)
\end{aligned}$$

$$\left\langle (\Delta \hat{E})^2 \right\rangle = (n+1) \mathcal{E}_0^2 \sin^2(kz) [2 - \cos^2(\varphi - \omega t)]$$

$$\begin{aligned}
(\hat{a}^\dagger - \hat{a}) | \Psi(t) \rangle &= \frac{1}{\sqrt{2}} \left[ e^{-i\omega t} \left( \sqrt{n+1} |n+1\rangle - \sqrt{n} |n-1\rangle \right) \right. \\
&\quad \left. + e^{i\varphi} e^{-i(n+1)\omega t} \left( \sqrt{n+2} |n+2\rangle - \sqrt{n+1} |n\rangle \right) \right] \\
\langle (\hat{a}^\dagger - \hat{a}) \rangle &= -i\sqrt{n+1} \sin(\varphi - \omega t)
\end{aligned}$$

Finally we have the following quantities

$$\begin{aligned}
\Delta n &= \frac{1}{2} \\
\Delta E &= \mathcal{E}_0 |\sin(kz)| \sqrt{2(n+1) [2 - \cos^2(\varphi - \omega t)]} \\
|\langle (\hat{a}^\dagger - \hat{a}) \rangle| &= \sqrt{n+1} |\sin(\varphi - \omega t)|.
\end{aligned}$$

Certainly the inequality in (2.49) holds true since

$$\sqrt{2(2 - \cos^2(\varphi - \omega t))} > |\sin(\varphi - \omega t)|.$$

## 2.6 problem 2.6

$$\begin{aligned}
\hat{X}_1 &= \frac{1}{2} (\hat{a} + \hat{a}^\dagger) \\
\hat{X}_2 &= \frac{1}{2i} (\hat{a} - \hat{a}^\dagger) \\
\hat{X}_1^2 &= \frac{1}{4} (\hat{a}^{\dagger 2} + \hat{a}^2 + 2\hat{a}^\dagger \hat{a} + 1) \\
\hat{X}_2^2 &= -\frac{1}{4} (\hat{a}^{\dagger 2} + \hat{a}^2 - 2\hat{a}^\dagger \hat{a} - 1)
\end{aligned}$$

$$|\Psi_{01}\rangle = \alpha|0\rangle + \beta|1\rangle$$

where  $|\alpha|^2 + |\beta|^2 = 1$ . So we can rewrite  $\beta = \sqrt{1 - |\alpha|^2}e^{i\phi}$  and  $\alpha^2 = |\alpha|^2$  without any loss of generality.

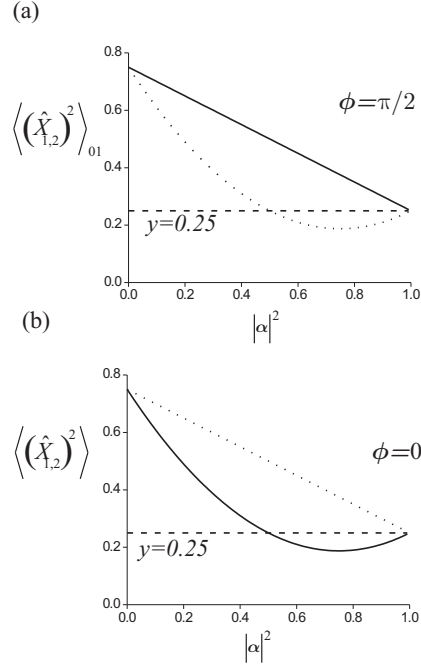
$$\begin{aligned}\langle \hat{X}_1 \rangle_{01} &= \frac{1}{2}(\alpha^*\beta + \alpha\beta^*) \\ \langle \hat{X}_2 \rangle_{01} &= \frac{1}{2i}(\alpha^*\beta - \alpha\beta^*)\end{aligned}$$

$$\begin{aligned}\langle \hat{a}^{\dagger 2} \rangle_{01} &= 0 \\ \langle \hat{a}^2 \rangle_{01} &= 0 \\ \langle \hat{a}^\dagger \hat{a} \rangle_{01} &= |\beta|^2\end{aligned}$$

$$\begin{aligned}\langle \hat{X}_1^2 \rangle_{01} &= \frac{1}{4}(2|\beta|^2 + 1) \\ \langle \hat{X}_2^2 \rangle_{01} &= \frac{1}{4}(2|\beta|^2 + 1)\end{aligned}$$

$$\begin{aligned}\left\langle \left( \Delta \hat{X}_1 \right)^2 \right\rangle_{01} &= \frac{1}{4} [2|\beta|^2 + 1 - (\alpha^*\beta)^2 - (\alpha\beta^*)^2 - 2|\alpha|^2|\beta|^2] \\ &= \frac{1}{4} [3 - 4|\alpha|^2 + 2|\alpha|^4 - 2|\alpha|^2(1 - |\alpha|^2)\cos(2\phi)] \\ \left\langle \left( \Delta \hat{X}_2 \right)^2 \right\rangle_{01} &= \frac{1}{4} [2|\beta|^2 + 1 + (\alpha^*\beta)^2 + (\alpha\beta^*)^2 - 2|\alpha|^2|\beta|^2] \\ &= \frac{1}{4} [3 - 4|\alpha|^2 + 2|\alpha|^4 + 2|\alpha|^2(1 - |\alpha|^2)\cos(2\phi)]\end{aligned}$$

In figures a and b below we plot  $\left\langle \left( \Delta \hat{X}_1 \right)^2 \right\rangle_{01}$  (solid line) for  $\phi = \pi/2$  and  $\left\langle \left( \Delta \hat{X}_2 \right)^2 \right\rangle_{01}$  (dotted line) for  $\phi = 0$ , respectively. Clearly the quadratures in hands go below the quadrature variances of the vacuum in more than one occasion.



$$|\Psi_{02}\rangle = \alpha|0\rangle + \beta|2\rangle. \quad (2.6.1)$$

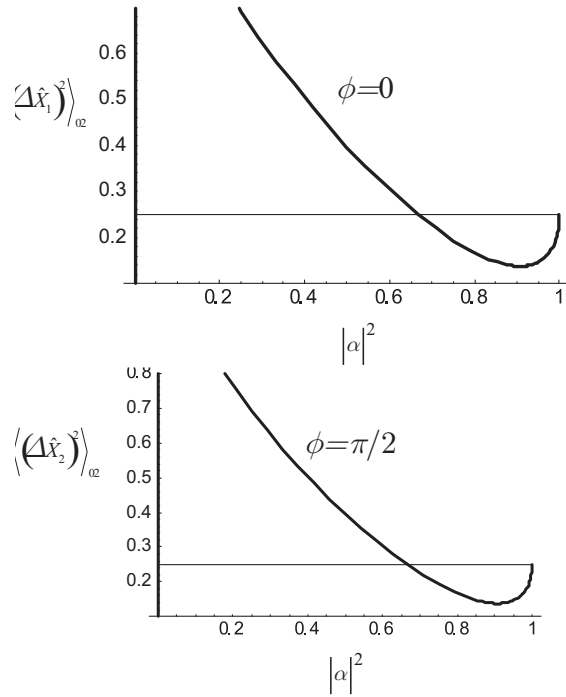
Again, where  $|\alpha|^2 + |\beta|^2 = 1$ . So we can rewrite  $\beta = \sqrt{1 - |\alpha|^2}e^{i\phi}$  and  $\alpha^2 = |\alpha|^2$  without any loss of generality.

$$\begin{aligned} \langle \hat{X}_1 \rangle_{02} &= 0 = \langle \hat{X}_2 \rangle_{02} \\ \langle (\Delta \hat{X}_1)^2 \rangle_{02} &= \langle \hat{X}_1^2 \rangle_{02} \\ &= \frac{1}{4} (|\alpha + \sqrt{2}\beta|^2 + 3|\beta|^2) \\ &= \frac{1}{4} [5 - 4|\alpha|^2 + 2\sqrt{2}|\alpha|^2(1 - |\alpha|^2) \cos \phi] \\ \langle (\Delta \hat{X}_2)^2 \rangle_{02} &= \langle \hat{X}_2^2 \rangle_{02} \\ &= \frac{1}{4} (|\alpha - \sqrt{2}\beta|^2 + 3|\beta|^2) \\ &= \frac{1}{4} [5 - 4|\alpha|^2 - 2\sqrt{2}|\alpha|^2(1 - |\alpha|^2) \cos \phi] \end{aligned}$$

In figures c and d below we plot  $\langle (\Delta \hat{X}_1)^2 \rangle_{02}$  for  $\phi = 0$  and  $\langle (\Delta \hat{X}_2)^2 \rangle_{02}$



for  $\phi = \pi/2$ , respectively. Clearly the quadratures in hands go below the quadrature variances of the vacuum in more than one occasion.



## 2.7 Problem 2.7

$$|\Psi'\rangle = \mathcal{N} \hat{a} |\Psi\rangle$$

$$|\mathcal{N}|^2 = \langle \hat{n} \rangle$$

$$= \bar{n}$$

$$\mathcal{N} = \frac{1}{\sqrt{\bar{n}}}$$

$$|\Psi'\rangle = \frac{1}{\sqrt{\bar{n}}} \hat{a} |\Psi\rangle$$

$$\begin{aligned}
\bar{n}' &= \langle \Psi' | \hat{n} | \Psi' \rangle \\
&= \frac{1}{\bar{n}} \langle \Psi | \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} | \Psi \rangle \\
&= \frac{1}{\bar{n}} (\langle \Psi | \hat{n}^2 | \Psi \rangle - \langle \Psi | \hat{n} | \Psi \rangle) \\
&= \frac{\langle \Psi | \hat{n}^2 | \Psi \rangle}{\bar{n}} - 1 \\
&= \frac{\langle \hat{n}^2 \rangle}{\langle \hat{n} \rangle} - 1.
\end{aligned}$$

Notice that  $\bar{n}' \neq \bar{n} - 1$  in general, but for the number state  $|n\rangle$ , and only of this state we have  $\bar{n}' = \bar{n} - 1$ .

## 2.8 Problem 2.8

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |10\rangle) \quad (2.8.1)$$

The average photon number,  $\bar{n}$ , of this state is

$$\bar{n} = \langle \Psi | \hat{a}^\dagger \hat{a} | \Psi \rangle, \quad (2.8.2)$$

which can be easily calculated to be

$$\bar{n} = \frac{1}{2}(0 + 10) = 5. \quad (2.8.3)$$

If we assume that a single photon is absorbed, our normalized state will become

$$|\Psi\rangle = |9\rangle, \quad (2.8.4)$$

then the average photon becomes

$$\bar{n} = 9. \quad (2.8.5)$$

## 2.9 Problem 2.9

$$\begin{aligned}
\mathbf{E}(\mathbf{r}, t) &= i \sum_{\mathbf{k}, s} \omega_k \mathbf{e}_{\mathbf{k}s} [A_{\mathbf{k}s} e^{i(\mathbf{k}\cdot\mathbf{r} - \omega_k t)} - A_{\mathbf{k}s}^* e^{-i(\mathbf{k}\cdot\mathbf{r} - \omega_k t)}] \\
\mathbf{B}(\mathbf{r}, t) &= \frac{i}{c} \sum_{\mathbf{k}, s} \omega_k (\boldsymbol{\kappa} \times \mathbf{e}_{\mathbf{k}s}) [A_{\mathbf{k}s} e^{i(\mathbf{k}\cdot\mathbf{r} - \omega_k t)} - A_{\mathbf{k}s}^* e^{-i(\mathbf{k}\cdot\mathbf{r} - \omega_k t)}]
\end{aligned}$$

$$\begin{aligned}
\nabla \cdot \mathbf{E}(\mathbf{r}, t) &= i \nabla \cdot \left( \sum_{\mathbf{k}, s} \omega_k \mathbf{e}_{\mathbf{k}s} [A_{\mathbf{k}s} e^{i(\mathbf{k} \cdot \mathbf{r} - \omega_k t)} - A_{\mathbf{k}s}^* e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega_k t)}] \right) \\
&= i \sum_{\mathbf{k}, s} \omega_k \mathbf{e}_{\mathbf{k}s} \cdot [A_{\mathbf{k}s} \nabla (e^{i(\mathbf{k} \cdot \mathbf{r} - \omega_k t)}) - A_{\mathbf{k}s}^* \nabla (e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega_k t)})] \\
&= i \sum_{\mathbf{k}, s} \omega_k \mathbf{e}_{\mathbf{k}s} \cdot [i \mathbf{k} A_{\mathbf{k}s} e^{i(\mathbf{k} \cdot \mathbf{r} - \omega_k t)} + i \mathbf{k} A_{\mathbf{k}s}^* e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega_k t)}] \\
&= - \sum_{\mathbf{k}, s} \omega_k \mathbf{e}_{\mathbf{k}s} \cdot \mathbf{k} [A_{\mathbf{k}s} e^{i(\mathbf{k} \cdot \mathbf{r} - \omega_k t)} + A_{\mathbf{k}s}^* e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega_k t)}] \\
&= 0
\end{aligned}$$

where we have used the vector identity

$$\nabla \cdot (f \mathbf{A}) = f (\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f), \quad (2.9.1)$$

and

$$\mathbf{e}_{\mathbf{k}s} \cdot \mathbf{k} = 0. \quad (2.9.2)$$

$$\begin{aligned}
\nabla \cdot \mathbf{B}(\mathbf{r}, t) &= \frac{i}{c} \nabla \cdot \left( \sum_{\mathbf{k}, s} \omega_k (\boldsymbol{\kappa} \times \mathbf{e}_{\mathbf{k}s}) [A_{\mathbf{k}s} e^{i(\mathbf{k} \cdot \mathbf{r} - \omega_k t)} - A_{\mathbf{k}s}^* e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega_k t)}] \right) \\
&= \frac{i}{c} \sum_{\mathbf{k}, s} \omega_k (\boldsymbol{\kappa} \times \mathbf{e}_{\mathbf{k}s}) \cdot [A_{\mathbf{k}s} \nabla (e^{i(\mathbf{k} \cdot \mathbf{r} - \omega_k t)}) - A_{\mathbf{k}s}^* \nabla (e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega_k t)})] \\
&= -\frac{1}{c} \sum_{\mathbf{k}, s} \omega_k (\boldsymbol{\kappa} \times \mathbf{e}_{\mathbf{k}s}) \cdot \mathbf{k} [A_{\mathbf{k}s} e^{i(\mathbf{k} \cdot \mathbf{r} - \omega_k t)} + A_{\mathbf{k}s}^* e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega_k t)}] \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\nabla \times \mathbf{E}(\mathbf{r}, t) &= i\nabla \times \left( \sum_{\mathbf{k}, s} \omega_k \mathbf{e}_{\mathbf{k}s} [A_{\mathbf{k}s} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} - A_{\mathbf{k}s}^* e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)}] \right) \\
&= i \sum_{\mathbf{k}, s} \omega_k \mathbf{e}_{\mathbf{k}s} \times [A_{\mathbf{k}s} \nabla (e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)}) - A_{\mathbf{k}s}^* \nabla (e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)})] \\
&= i \sum_{\mathbf{k}, s} \omega_k \mathbf{e}_{\mathbf{k}s} \times [i\mathbf{k} A_{\mathbf{k}s} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} + i\mathbf{k} A_{\mathbf{k}s}^* e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)}] \\
&= - \sum_{\mathbf{k}, s} \omega_k \mathbf{e}_{\mathbf{k}s} \times \mathbf{k} [A_{\mathbf{k}s} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} + A_{\mathbf{k}s}^* e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)}] \\
&= - \sum_{\mathbf{k}, s} \frac{\omega_k^2}{c} \mathbf{e}_{\mathbf{k}s} \times \boldsymbol{\kappa} [A_{\mathbf{k}s} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} + A_{\mathbf{k}s}^* e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)}] \\
&= \sum_{\mathbf{k}, s} \frac{\omega_k^2}{c} \boldsymbol{\kappa} \times \mathbf{e}_{\mathbf{k}s} [A_{\mathbf{k}s} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} + A_{\mathbf{k}s}^* e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)}]
\end{aligned}$$

where we have used the vector identity

$$\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) + \mathbf{A} \times (\nabla f), \quad (2.9.3)$$

and

$$\mathbf{k} = \frac{\omega_k}{c} \boldsymbol{\kappa}. \quad (2.9.4)$$

$$\begin{aligned}
\frac{\partial \mathbf{B}}{\partial t} &= \frac{i}{c} \sum_{\mathbf{k}, s} \omega_k (\boldsymbol{\kappa} \times \mathbf{e}_{\mathbf{k}s}) \left[ A_{\mathbf{k}s} \frac{\partial e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)}}{\partial t} - A_{\mathbf{k}s}^* \frac{\partial e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)}}{\partial t} \right] \\
&= \frac{1}{c} \sum_{\mathbf{k}, s} \omega_k^2 (\boldsymbol{\kappa} \times \mathbf{e}_{\mathbf{k}s}) [A_{\mathbf{k}s} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} + A_{\mathbf{k}s}^* e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)}] \\
&= -\nabla \times \mathbf{E}
\end{aligned}$$

$$\begin{aligned}
\nabla \times \mathbf{B}(\mathbf{r}, t) &= \frac{i}{c} \nabla \times \left( \sum_{\mathbf{k}, s} \omega_k (\boldsymbol{\kappa} \times \mathbf{e}_{\mathbf{k}s}) [A_{\mathbf{k}s} e^{i(\mathbf{k} \cdot \mathbf{r} - \omega_k t)} - A_{\mathbf{k}s}^* e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega_k t)}] \right) \\
&= \frac{i}{c} \sum_{\mathbf{k}, s} \omega_k (\boldsymbol{\kappa} \times \mathbf{e}_{\mathbf{k}s}) \times [A_{\mathbf{k}s} \nabla (e^{i(\mathbf{k} \cdot \mathbf{r} - \omega_k t)}) - A_{\mathbf{k}s}^* \nabla (e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega_k t)})] \\
&= \frac{i}{c} \sum_{\mathbf{k}, s} \omega_k (\boldsymbol{\kappa} \times \mathbf{e}_{\mathbf{k}s}) \times [i\mathbf{k} A_{\mathbf{k}s} e^{i(\mathbf{k} \cdot \mathbf{r} - \omega_k t)} + i\mathbf{k} A_{\mathbf{k}s}^* e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega_k t)}] \\
&= -\frac{1}{c} \sum_{\mathbf{k}, s} \omega_k (\boldsymbol{\kappa} \times \mathbf{e}_{\mathbf{k}s}) \times \mathbf{k} [A_{\mathbf{k}s} e^{i(\mathbf{k} \cdot \mathbf{r} - \omega_k t)} + A_{\mathbf{k}s}^* e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega_k t)}] \\
&= -\sum_{\mathbf{k}, s} \frac{\omega_k^2}{c^2} \mathbf{e}_{\mathbf{k}s} [A_{\mathbf{k}s} e^{i(\mathbf{k} \cdot \mathbf{r} - \omega_k t)} + A_{\mathbf{k}s}^* e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega_k t)}] \\
&= \mu_0 \epsilon_0 \sum_{\mathbf{k}, s} \omega_k^2 \mathbf{e}_{\mathbf{k}s} [A_{\mathbf{k}s} e^{i(\mathbf{k} \cdot \mathbf{r} - \omega_k t)} + A_{\mathbf{k}s}^* e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega_k t)}] \\
&= \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}
\end{aligned}$$

## 2.10 Problem 2.10

For thermal light

$$P_n = \frac{\bar{n}^n}{(1 + \bar{n})^{n+1}} \quad (2.10.1)$$

$$\begin{aligned}
\sum_n n(n-1)\dots(n-r+1)P_n &= \sum_n n(n-1)\dots(n-r+1) \frac{\bar{n}^n}{(1+\bar{n})^{n+1}} \\
&= \frac{1}{\bar{n}} \left( \frac{\bar{n}}{1+\bar{n}} \right)^{r+1} \sum_n n(n-1)\dots(n-r+1) \left( \frac{\bar{n}}{1+\bar{n}} \right)^{n-r}
\end{aligned}$$

To simplify the last expression, let's define  $x = \frac{\bar{n}}{1+\bar{n}}$ , for which  $x < 1$ ,

$$\begin{aligned} \sum_n n(n-1)\dots(n-r+1)P_n &= \frac{1}{\bar{n}}x^{r+1} \sum_n n(n-1)\dots(n-r+1)x^{n-r} \\ &= \frac{1}{\bar{n}}x^{r+1} \frac{\partial^r}{\partial x^r} \sum_n x^n \\ &= \frac{1}{\bar{n}}x^{r+1} \frac{\partial^r}{\partial x^r} \frac{1}{1-x} \\ &= \frac{1}{\bar{n}}x^{r+1} r! \frac{1}{(1-x)^{r+1}} \end{aligned}$$

$$\langle \hat{n}(\hat{n}-1)(\hat{n}-1)\dots(\hat{n}-r+1) \rangle = r! \bar{n}^r \quad (2.10.2)$$

## 2.11 Problem 2.11

$$\begin{aligned} [\hat{C}, \hat{S}] &= -\frac{i}{4} [\hat{E} + \hat{E}^\dagger, \hat{E} - \hat{E}^\dagger] \\ &= \frac{i}{2} [\hat{E}, \hat{E}^\dagger] \\ &= \frac{i}{2} (\hat{E}\hat{E}^\dagger - \hat{E}^\dagger\hat{E}) \\ &= \frac{i}{2} (1 - 1 + |0\rangle\langle 0|) \\ &= \frac{i}{2} |0\rangle\langle 0| \end{aligned}$$

$$\langle m | [\hat{C}, \hat{S}] | n \rangle = \frac{i}{2} \delta_{m,0} \delta_{n,0}.$$

Obviously, only the diagonal matrix elements are nonzero.

## 2.12 Problem 2.12

Using equation (2.229) for

$$\hat{\rho} = \frac{1}{2} (|0\rangle\langle 0| + |1\rangle\langle 1|) \quad (2.12.1)$$

we have

$$\begin{aligned}
 \mathcal{P}(\varphi) &= \frac{1}{2\pi} \langle \varphi | \hat{\rho} | \varphi \rangle \\
 &= \frac{1}{2\pi} \sum_n \sum_{n'} \langle n' | e^{-in'\varphi} \hat{\rho} e^{in\varphi} | n \rangle \\
 &= \frac{1}{2\pi} (1 + e^{i\varphi} e^{-i\varphi}) \\
 &= \frac{1}{\pi}.
 \end{aligned}$$

This is similar to a thermal state. On the other hand using equation (2.227) for  $|\psi\rangle = \frac{1}{2}(|0\rangle + e^{i\theta}|1\rangle)$  we have

$$\begin{aligned}
 \mathcal{P}(\phi) &= \frac{1}{2\pi} |\langle \phi | \psi \rangle|^2 \\
 &= \frac{1}{2\pi} [1 + \cos(\phi - \theta)].
 \end{aligned}$$

As expected, it is different than a statistical mixture state, the one for the pure state exhibiting a phase dependence.

## 2.13 Problem 2.13

$$\hat{\rho}_{th} = \sum_{n=0}^{\infty} P_n |n\rangle \langle n| \tag{2.13.1}$$

$$\begin{aligned}
 \mathcal{P}(\varphi) &= \frac{1}{2\pi} \langle \varphi | \hat{\rho} | \varphi \rangle \\
 &= \frac{1}{2\pi} \sum_n \sum_{n'} \langle n' | e^{-in'\varphi} \hat{\rho} e^{in\varphi} | n \rangle \\
 &= \frac{1}{2\pi} \sum_n \sum_{n'} \sum_k P_k \langle n' | e^{-in'\varphi} | k \rangle \langle k | e^{in\varphi} | n \rangle \\
 &= \frac{1}{2\pi} \sum_k P_k \\
 &= \frac{1}{2\pi}
 \end{aligned}$$





# Chapter 3

## Coherent States

### 3.1 Problem 3.1

Let assume that the eigenvector of the creation operator  $\hat{a}^\dagger$  exists. So we can write

$$\hat{a}^\dagger|\beta\rangle = \beta|\beta\rangle. \quad (3.1.1)$$

Now let's write  $|\beta\rangle$  as a superposition of the number states, namely

$$|\beta\rangle = \sum_{n=0}^{\infty} c_n |n\rangle \quad (3.1.2)$$

Now let's plug the last expression in equation 3.1.1:

$$\hat{a}^\dagger|\beta\rangle = \sum_{n=0}^{\infty} c_n \sqrt{n+1} |n+1\rangle \quad (3.1.3)$$

$$= \beta \sum_{n=0}^{\infty} c_n |n\rangle. \quad (3.1.4)$$

From the last express we deduce that

$$c_0 = 0, \quad (3.1.5)$$

$$c_{n+1} = \frac{1}{\beta} c_n \sqrt{n+1}, \quad (3.1.6)$$

which means all  $c_n$ 's = 0.