1.FUNDAMENTAL CONCEPTS

1.1. WHAT IS A GRAPH?

1.1.1. *Complete bipartite graphs and complete graphs.* The complete bipartite graph $K_{m,n}$ is a complete graph if and only if $m = n = 1$ or $\{m, n\} = \{1, 0\}$.

1.1.2. *Adjacency matrices and incidence matrices for a 3-vertex path.*

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$$

Adjacency matrices for a path and a cycle with six vertices.

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

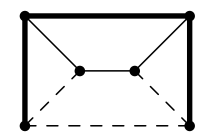
1.1.3. *Adjacency matrix for $K_{m,n}$.*

	m	n
m	0	1
n	1	0

1.1.4. *$G \cong H$ if and only if $\overline{G} \cong \overline{H}$.* If f is an isomorphism from G to H, then f is a vertex bijection preserving adjacency and nonadjacency, and hence f preserves non-adjacency and adjacency in $\overline{G}$ and is an isomorphism from $\overline{G}$ to $\overline{H}$. The same argument applies for the converse, since the complement of $\overline{G}$ is G.

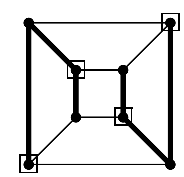
1.1.5. *If every vertex of a graph G has degree 2, then G is a cycle—FALSE.* Such a graph can be a disconnected graph with each component a cycle. (If infinite graphs are allowed, then the graph can be an infinite path.)

1.1.6. *The graph below decomposes into copies of P_4.*

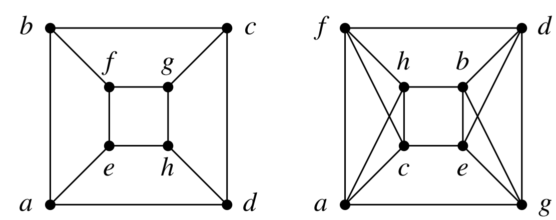


1.1.7. *A graph with more than six vertices of odd degree cannot be decomposed into three paths.* Every vertex of odd degree must be the endpoint of some path in a decomposition into paths. Three paths have only six endpoints.

1.1.8. *Decompositions of a graph.* The graph below decomposes into copies of $K_{1,3}$ with centers at the marked vertices. It decomposes into bold and solid copies of P_4 as shown.



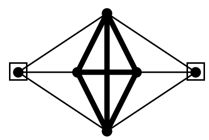
1.1.9. *A graph and its complement.* With vertices labeled as shown, two vertices are adjacent in the graph on the right if and only if they are not adjacent in the graph on the left.



1.1.10. *The complement of a simple disconnected graph must be connected—TRUE.* A disconnected graph G has vertices x, y that do not belong to a path. Hence$x$ and y are adjacent in $\overline{G}$. Furthermore, x and y have no common neighbor in G, since that would yield a path connecting them. Hence every vertex not in $\{x, y\}$ is adjacent in $\overline{G}$ to at least one of $\{x, y\}$. Hence every vertex can reach every other vertex in $\overline{G}$ using paths through $\{x, y\}$.

1.1.11. *Maximum clique and maximum independent set.* Since two vertices have degree 3 and there are only four other vertices, there is no clique of size 5. A complete subgraph with four vertices is shown in bold.

Since two vertices are adjacent to all others, an independent set containing either of them has only one vertex. Deleting them leaves P_4, in which the maximum size of an independent set is two, as marked.



1.1.12. *The Petersen graph.* The Petersen graph contains odd cycles, so it is not bipartite; for example, the vertices 12, 34, 51, 23, 45 form a 5-cycle.

The vertices 12, 13, 14, 15 form an independent set of size 4, since any two of these vertices have 1 as a common element and hence are nonadjacent. Visually, there is an independent set of size 4 marked on the drawing of the Petersen graph on the cover of the book. There are many ways to show that the graph has no larger independent set.

Proof 1. Two consecutive vertices on a cycle cannot both appear in an independent set, so every cycle contributes at most half its vertices. Since the vertex set is covered by two disjoint 5-cycles, every independent set has size at most 4.

Proof 2. Let ab be a vertex in an independent set S, where $a, b \in [5]$. We show that S has at most three additional vertices. The vertices not adjacent to ab are ac, bd, ae, bc, ad, be, and they form a cycle in that order. Hence at most half of them can be added to S.

1.1.13. *The graph with vertex set $\{0, 1\}^k$ and $x \leftrightarrow y$ when x and y differ in one place is bipartite.* The partite sets are determined by the parity of the number of 1's. Adjacent vertices have opposite parity. (This graph is the k-dimensional hypercube; see Section 1.3.)

1.1.14. *Cutting opposite corner squares from an eight by eight checkerboard leaves a subboard that cannot be partitioned into rectangles consisting of two adjacent unit squares.* 2-coloring the squares of a checkerboard so that adjacent squares have opposite colors shows that the graph having a vertex for each square and an edge for each pair of adjacent squares is bipartite. The squares at opposite corners have the same color; when these are deleted, there are 30 squares of that color and 32 of the other

color. Each pair of adjacent squares has one of each color, so the remaining squares cannot be partitioned into sets of this type.

Generalization: the two partite sets of a bipartite graph cannot be matched up using pairwise-disjoint edges if the two partite sets have unequal sizes.

1.1.15. *Common graphs in four families: $A = \{paths\}$, $B = \{cycles\}$, $C = \{complete\ graphs\}$, $D = \{bipartite\ graphs\}$.*

$A \cap B = \varnothing$: In a cycle, the numbers of vertices and edges are equal, but this is false for a path.

$A \cap C = \{K_1, K_2\}$: To be a path, a graph must contain no cycle.

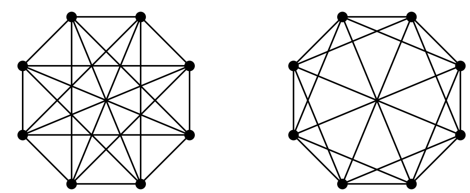
$A \cap D = A$: non-bipartite graphs have odd cycles.

$B \cap C = K_3$: Only when $n = 3$ does $\binom{n}{2} = n$.

$B \cap D = \{C_{2k}\colon k \geq 2\}$: odd cycles are not bipartite.

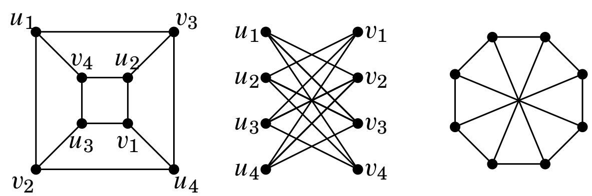
$C \cap D = \{K_1, K_2\}$: bipartite graphs cannot have triangles.

1.1.16. *The graphs below are not isomorphic.* The graph on the left has four cliques of size 4, and the graph on the right has only two. Alternatively, the complement of the graph on the left is disconnected (two 4-cycles), while the complement of the graph on the right is connected (one 8-cycle).



1.1.17. *There are exactly two isomorphism classes of 4-regular simple graphs with 7 vertices.* Simple graphs G and H are isomorphic if and only if their complements $\overline{G}$ and $\overline{H}$ are isomorphic, because an isomorphism $\phi\colon V(G) \to V(H)$ is also an isomorphism from $\overline{G}$ to $\overline{H}$, and vice versa. Hence it suffices to count the isomorphism classes of 2-regular simple graphs with 7 vertices. Every component of a finite 2-regular graph is a cycle. In a simple graph, each cycle has at least three vertices. Hence each class it determined by partitioning 7 into integers of size at least 3 to be the sizes of the cycles. The only two graphs that result are C_7 and $C_3 + C_4$ – a single cycle or two cycles of lengths three and four.

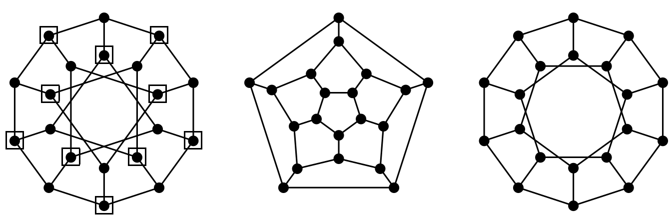
1.1.18. *Isomorphism.* Using the correspondence indicated below, the first two graphs are isomorphic; the graphs are bipartite, with $u_i \leftrightarrow v_j$ if and only if $i \neq j$. The third graph contains odd cycles and hence is not isomorphic to the others.



Visually, the first two graphs are Q_3 and the graph obtained by deleting four disjoint edges from $K_{4,4}$. In Q_3, each vertex is adjacent to the vertices whose names have opposite parity of the number of 1s, except for the complementary vertex. Hence Q_3 also has the structure of $K_{4,4}$ with four disjoint edges deleted; this proves isomorphism without specifying an explicit bijection.

1.1.19. *Isomorphism of graphs.* The rightmost two graphs below are isomorphic. The outside 10-cycle in the rightmost graph corresponds to the intermediate ring in the second graph. Pulling one of the inner 5-cycles of the rightmost graph out to the outside transforms the graph into the same drawing as the second graph.

The graph on the left is bipartite, as shown by marking one partite set. It cannot be isomorphic to the others, since they contain 5-cycles.



1.1.20. *Among the graphs below, the first (F) and third (H) are isomorphic, and the middle graph (G) is not isomorphic to either of these.*

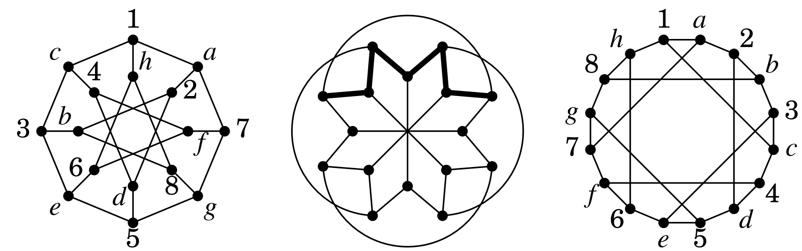
F and H are isomorphic. We exhibit an isomorphism (a bijection from $V(F)$ to $V(H)$ that preserves the adjacency relation). To do this, we name the vertices of F, write the name of each vertex of F on the corresponding vertex in H, and show that the names of the edges are the same in H and F. This proves that H is a way to redraw F. We have done this below using the first eight letters and the first eight natural numbers as names.

To prove quickly that the adjacency relation is preserved, observe that $1, a, 2, b, 3, c, 4, d, 5, e, 6, f, 7, g, 8, h$ is a cycle in both drawings, and the additional edges $1c, 2d, 3e, 4f, 5g, 6h, 7a, 8b$ are also the same in both drawings. Thus the two graphs have the same edges under this vertex correspondence. Equivalently, if we list the vertices in this specified order for

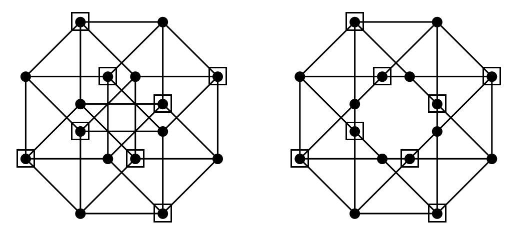
the two drawings, the two adjacency matrices are the same, but that is harder to verify.

G is not isomorphic to F or to H. In F and in H, the numbers form an independent set, as do the letters. Thus F and H are bipartite. The graph G cannot be bipartite, since it contains an odd cycle. The vertices above the horizontal axis of the picture induce a cycle of length 7.

It is also true that the middle graph contains a 4-cycle and the others do not, but it is harder to prove the absence of a 4-cycle than to prove the absence of an odd cycle.



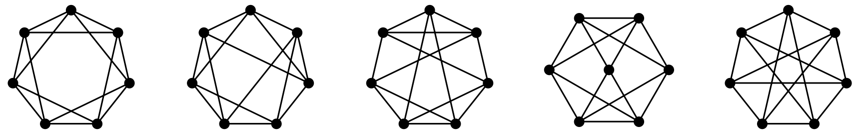
1.1.21. *Isomorphism.* Both graphs are bipartite, as shown below by marking one partite set. In the graph on the right, every vertex appears in eight 4-cycles. In the graph on the left, every vertex appears in only six 4-cycles (it is enough just to check one). Thus they are not isomorphic. Alternatively, for every vertex in the right graph there are five vertices having common neighbors with it, while in the left graph there are six such vertices.



1.1.22. *Isomorphism of explicit graphs.* Among the graphs below, $\{G_1, G_2, G_5\}$ are pairwise isomorphic. Also $G_3 \cong G_4$, and these are not isomorphic to any of the others. Thus there are exactly two isomorphism classes represented among these graphs.

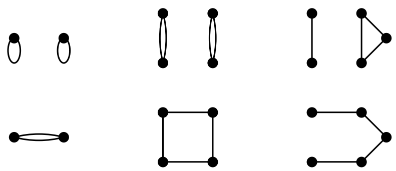
To prove these statements, one can present explicit bijections between vertex sets and verify that these preserve the adjacency relation (such as by displaying the adjacency matrix, for example). One can also make other structural arguments. For example, one can move the highest vertex in G_3 down into the middle of the picture to obtain G_4; from this one can list the desired bijection.

One can also recall that two graphs are isomorphic if and only if their complements are isomorphic. The complements of G_1, G_2, and G_5 are cycles of length 7, which are pairwise isomorphic. Each of $\overline{G}_3$ and $\overline{G}_4$ consists of two components that are cycles of lengths 3 and 4; these graphs are isomorphic to each other but not to a 7-cycle.



G_1 G_2 G_3 G_4 G_5

1.1.23. *Smallest pairs of nonisomorphic graphs with the same vertex degrees.* For multigraphs, loopless multigraphs, and simple graphs, the required numbers of vertices are 2,4,5; constructions for the upper bounds appear below. We must prove that these constructions are smallest.



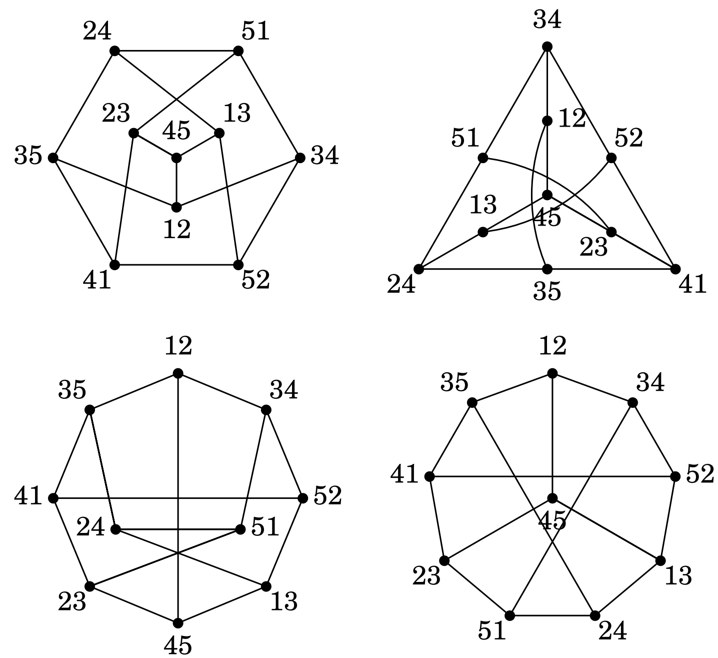
a) general b) loopless c) simple

a) With 1 vertex, every edge is a loop, and the isomorphism class is determined by the number of edges, which is determined by the vertex degree. Hence nonisomorphic graphs with the same vertex degrees have at least two vertices.

b) Every loopless graph is a graph, so the answer for loopless graphs is at least 2. The isomorphism class of a loopless graph with two vertices is determined by the number of copies of the edge, which is determined by the vertex degrees. The isomorphism class of a loopless graph with three vertices is determined by the edge multiplicities. Let the three vertex degrees be a, b, c, and let the multiplicities of the opposite edges be x, y, z, where Since $a = y + z$, $b = x + z$, and $c = x + y$, we can solve for the multiplicities in terms of the degrees by $x = (b+c-a)/2$, $y = (a+c-b)/2$, and $z = (a+b-c)/2$. Hence the multiplicities are determined by the degrees, and all loopless graphs with vertex degrees a, b, c are pairwise isomorphic. Hence nonisomorphic loopless graphs with the same vertex degrees have at least four vertices.

c) Since a simple graph is a loopless graph, the answer for simple graphs is at least 4. There are 11 isomorphism classes of simple graphs with four vertices. For each of 0,1,5, or 6 edges, there is only one isomorphism class. For 2 edges, there are two isomorphism classes, but they have different lists of vertex degrees (similarly for 4 edges). For 3 edges, the three isomorphism classes have degree lists 3111, 2220, and 2211, all different. Hence nonisomorphic simple graphs with the same vertex degrees must have at least five vertices.

1.1.24. *Isomorphisms for the Petersen graph.* Isomorphism is proved by giving an adjacency-preserving bijection between the vertex sets. For pictorial representations of graphs, this is equivalent to labeling the two graphs with the same vertex labels so that the adjacency relation is the same in both pictures; the labels correspond to a permutation of the rows and columns of the adjacency matrices to make them identical. The various drawings of the Petersen graph below illustrate its symmetries; the labelings indicate that these are all "the same" (unlabeled) graph. The number of isomorphisms from one graph to another is the same as the number of isomorphisms from the graph to itself.



1.1.25. *The Petersen graph has no cycle of length 7.* Suppose that the Petersen graph has a cycle C of length 7. Since any two vertices of C are connected by a path of length at most 3 on C, any additional edge with endpoints on C would create a cycle of length at most 4. Hence the third neighbor of each vertex on C is not on C.

Thus there are seven edges from $V(C)$ to the remaining three vertices. By the pigeonhole principle, one of the remaining vertices receives at least three of these edges. This vertex x not on C has three neighbors on C. For any three vertices on C, either two are adjacent or two have a common neighbor on C (again the pigeonhole principle applies). Using x, this completes a cycle of length at most 4. We have shown that the assumption of a 7-cycle leads to a contradiction.

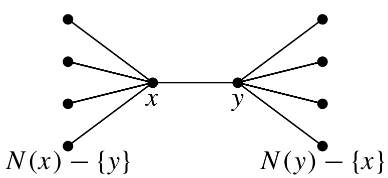
Alternative completion of proof. Let u be a vertex on C, and let v, w be the two vertices farthest from u on C. As argued earlier, no edges join vertices of C that are not consecutive on C. Thus u is not adjacent to v or w. Hence u, v have a common neighbor off C, as do u, w. Since u has only one neighbor off C, these two common neighbors are the same. The resulting vertex x is adjacent to all of u, v, w, and now x, v, w is a 3-cycle.

1.1.26. *A k-regular graph of girth four has at least $2k$ vertices, with equality only for $K_{k,k}$.* Let G be k-regular of girth four, and chose $xy \in E(G)$. Girth 4 implies that G is simple and that x and y have no common neighbors. Thus the neighborhoods $N(x)$ and $N(y)$ are disjoint sets of size k, which forces at least $2k$ vertices into G. Possibly there are others.

Note also that $N(x)$ and $N(y)$ are independent sets, since G has no triangle. If G has no vertices other than these, then the vertices in $N(x)$ can have neighbors only in $N(y)$. Since G is k-regular, every vertex of $N(x)$ must be adjacent to every vertex of $N(y)$. Thus G is isomorphic to $K_{k,k}$, with partite sets $N(x)$ and $N(y)$. In other words, there is only one such isomorphism class for each value of k.

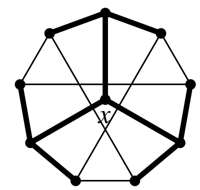
Comment. One can also start with a vertex x, choose y from among the k vertices in $N(x)$, and observe that $N(y)$ must have $k-1$ more vertices not in $N(x) \cup \{x\}$. The proof then proceeds as above.

(An alternative proof uses the methods of Section 1.3. A triangle-free simple graph with n vertices has at most $n^2/4$ edges. Since G is k-regular, this yields $n^2/4 \geq nk/2$, and hence $n \geq 2k$. Furthermore, equality holds in the edge bound only for $K_{n/2,n/2}$, so this is the only such graph with $2k$ vertices. (C. Pikscher))



1.1.27. *A simple graph of girth 5 in which every vertex has degree at least k has at least k^2+1 vertices, with equality achieveable when $k \in \{2, 3\}$.* Let G be k-regular of girth five. Let S be the set consisting of a vertex x and its neighbors. Since G has no cycle of length less than five, G is simple, and any two neighbors of x are nonadjacent and have no common neighbor other than x. Hence each $y \in S - \{x\}$ has at least $k-1$ neighbors that are not in S and not neighbors of any vertex in S. Hence G has at least $k(k-1)$ vertices outside S and at least $k+1$ vertices in S for at least k^2+1 altogether.

The 5-cycle achieves equality when $k=2$. For $k=3$, growing the graph symmetrically from x permits completing the graph by adding edges among the non-neighbors of x. The result is the Petersen graph. (Comment: For $k > 3$, it is known that girth 5 with minimum degree k and exactly k^2+1 vertices is impossible, except for $k=7$ and possibly for $k=57$.)



1.1.28. *The Odd Graph has girth 6.* The Odd Graph O_k is the disjointness graph of the set of k-element subsets of $[2k+1]$.

Vertices with a common neighbor correspond to k-sets with $k-1$ common elements. Thus they have exactly one common neighbor, and O_k has no 4-cycle. Two vertices at distance 2 from a single vertex have at least $k-2$ common neighbors. For $k > 2$, such vertices cannot be adjacent, and thus O_k has no 5-cycle when $k > 2$. To form a 6-cycle when $k \geq 2$, let $A = \{2, \ldots, k\}$, $B = \{k+2, \ldots, 2k\}$, $a = 1$, $b = k+1$, $c = 2k+1$. A 6-cycle is $A \cup \{a\}$, $B \cup \{b\}$, $A \cup \{c\}$, $B \cup \{a\}$, $A \cup \{b\}$, $B \cup \{c\}$.

The Odd Graph also is not bipartite. The successive elements $\{1, \ldots, k\}$, $\{k+1, \ldots, 2k\}$, $\{2k+1, 1, \ldots, k-1\}$, . . ., $\{k+2, \ldots, 2k+1\}$ form an odd cycle.

1.1.29. *Among any 6 people, there are 3 mutual acquaintances or 3 mutual strangers.* Let G be the graph of the acquaintance relation, and let x be one of the people. Since x has 5 potential neighbors, x has at least 3 neighbors or at least 3 nonneighbors. By symmetry (if we complement G, we still have to prove the same statement), we may assume that x has at least 3 neighbors. If any pair of these people are acquainted, then with x we have 3 mutual acquaintances, but if no pair of neighbors of x is acquainted, then the neighbors of x are three mutual strangers.

1.1.30. *The number of edges incident to v_i is the ith diagonal entry in MM^T and in A^2.* In both MM^T and A^2 this is the sum of the squares of the entries in the ith row. For MM^T, this follows immediately from the definition of matrix multiplication and transposition, but for A^2 this uses the graph-theoretic fact that $A = A^T$; i.e. column i is the same as row i. Because G is simple, the entries of the matrix are all 0 or 1, so the sum of the squares in a row equals the number of 1s in the row. In M, the 1s in a row denote incident edges; in A they denote vertex neighbors. In either case, the number of 1s is the degree of the vertex.

If $i \neq j$, then the entry in position (i, j) of A^2 is the number of common neighbors of v_i and v_j. The matrix multiplication puts into position (i, j) the "product" of row i and column j; that is $\sum_{k=1}^{n} a_{i,k}a_{k,j}$. When G is simple, entries in A are 1 or 0, depending on whether the corresponding vertices are adjacent. Hence $a_{i,k}a_{k,j} = 1$ if v_k is a common neighbor of v_i and v_j; otherwise, the contribution is 0. Thus the number of contributions of 1 to entry (i, j) is the number of common neighbos of v_i and v_j.

If $i \neq j$, then the entry in position (i, j) of MM^T is the number of edges joining v_i and v_j (0 or 1 when G has no multiple edges). The ith row of M has 1s corresponding to the edges incident to v_i. The jth column of M^T is the same as the jth row of M, which has 1s corresponding to the edges incident to v_j. Summing the products of corresponding entries will contribute 1 for each edge incident to both v_i and v_j; 0 otherwise.

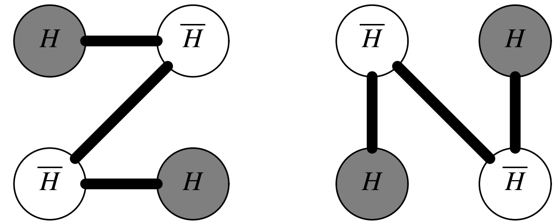
Comment. For graphs without loops, both arguments for (i, j) in general apply when $i = j$ to explain the diagonal entries.

1.1.31. *K_n decomposes into two isomorphic ("self-complementary") subgraphs if and only if n or $n-1$ is divisible by 4.*

a) The number of vertices in a self-complementary graph is congruent to 0 or 1 $(\mathrm{mod}\, 4)$. If G and $\overline{G}$ are isomorphic, then they have the same number of edges, but together they have $\binom{n}{2}$ edges (with none repeated), so the number of edges in each must be $n(n-1)/4$. Since this is an integer and the numbers n and $n-1$ are not both even, one of $\{n, n-1\}$ must be divisible by 4.

b) Construction of self-complementary graphs for all such n.

Proof 1 (explicit construction). We generalize the structure of the self-complementary graphs on 4 and 5 vertices, which are P_4 and C_5. For $n = 4k$, take four vertex sets of size k, say X_1, X_2, X_3, X_4, and join all vertices of X_i to those of X_{i+1}, for $i = 1, 2, 3$. To specify the rest of G, within these sets let X_1 and X_4 induce copies of a graph H with k vertices, and let X_2 and X_3 induce $\overline{H}$. (For example, H may be K_k.) In $\overline{G}$, both X_2 and X_3 induce H, while X_1 and X_4 induce $\overline{H}$, and the connections between sets are $X_2 \leftrightarrow X_4 \leftrightarrow X_1 \leftrightarrow X_3$. Thus relabeling the subsets defines an isomorphism between G and $\overline{G}$. (There are still other constructions for G.)



For $n = 4k+1$, we add a vertex x to the graph constructed above. Join x to the $2k$ vertices in X_1 and X_4 to form G. The isomorphism showing that $G - x$ is self-complementary also works for G (with x mapped to itself), since this isomorphism maps $N_G(x) = X_1 \cup X_4$ to $N_{\overline{G}}(x) = X_2 \cup X_3$.

Proof 2 (inductive construction). If G is self-complementary, then let H_1 be the graph obtained from G and P_4 by joining the two ends of P_4 to all vertices of G. Let H_2 be the graph obtained from G and P_4 by joining the two center vertices of P_4 to all vertices of G. Both H_1 and H_2 are self-complementary. Using this with $G = K_1$ produces the two self-complementary graphs of order 5, namely C_5 and the bull.

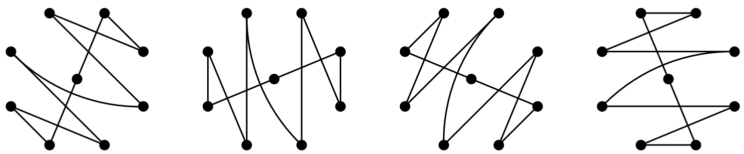
Self-complementary graphs with order divisible by 4 arise from repeated use of the above using $G = P_4$ as a starting point, and self-complementary graphs of order congruent to 1 modulo 4 arise from repeated use of the above using $G = K_1$ as a starting point. This construction produces many more self-complementary graphs than the explicit construction in Proof 1.

1.1.32. *$K_{m,n}$ decomposes into two isomorphic subgraphs if and only if m and n are not both odd.* The condition is necessary because the number of edges must be even. It is sufficient because $K_{m,n}$ decomposes into two copies of $K_{m,n/2}$ when n is even.

1.1.33. *Decomposition of complete graphs into cycles through all vertices.* View the vertex set of K_n as $\mathbb{Z}_n$, the values $0, \ldots, n-1$ in cyclic order. Since each vertex has degree $n-1$ and each cycle uses two edges at each vertex, the decomposition has $(n-1)/2$ cycles.

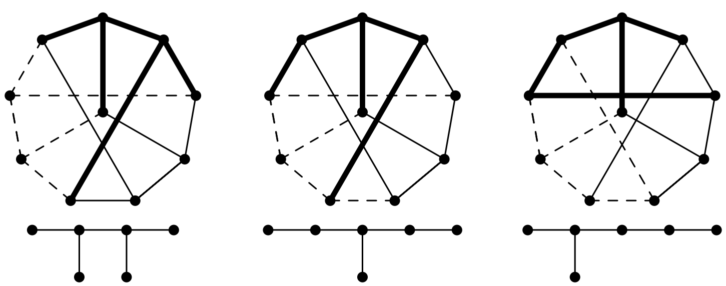
For $n = 5$ and $n = 7$, it suffices to use cycles formed by traversing the vertices with constant difference: $(0, 1, 2, 3, 4)$ and $(0, 2, 4, 1, 3)$ for $n = 5$ and $(0, 1, 2, 3, 4, 5, 6)$, $(0, 2, 4, 6, 1, 3, 5)$, and $(0, 3, 6, 2, 5, 1, 4)$ for $n = 7$.

This construction fails for $n = 9$ since the edges with difference 3 form three 3-cycles. The cyclically symmetric construction below treats the vertex set as $\mathbb{Z}_8$ together with one special vertex.



1.1.34. *Decomposition of the Petersen graph into copies of P_4.* Consider the drawing of the Petersen graph with an inner 5-cycle and outer 5-cycle. Each P_4 consists of one edge from each cycle and one cross edge joining them. Extend each cross edge e to a copy of P_4 by taking the edge on each of the two 5-cycles that goes in a clockwise direction from e. In this way, the edges on the outside 5-cycle are used in distinct copies of P_4, and the same holds for the edges on the inside 5-cycle.

Decomposition of the Petersen graph into three pairwise-isomorphic connected subgraphs. Three such decompositions are shown below. We restricted the search by seeking a decomposition that is unchanged by 120° rotations in a drawing of the Petersen graph with 3-fold rotational symmetry. The edges in this drawing form classes of size 3 that are unchanged under rotations of 120°; each subgraph in the decomposition uses exactly one edge from each class.



1.1.35. *K_n decomposes into three pairwise-isomorphic subgraphs if and only if $n+1$ is not divisible by 3.* The number of edges is $n(n-1)/2$. If $n+1$ is divisible by 3, then n and $n-1$ are not divisible by 3. Thus decomposition into three subgraphs of equal size is impossible in this case.

If $n+1$ is not divisible by 3, then $e(K_n)$ is divisible by 3, since n or $n-1$ is divisible by 3. We construct a decomposition into three subgraphs that are pairwise isomorphic (there are many such decompositions).

When n is a multiple of 3, we partition the vertex set into three subsets V_1, V_2, V_3 of equal size. Edges now have two types: within a set or joining two sets. Let the ith subgraph G_i consist of all the edges within V_i and all the edges joining the two other subsets. Each edge of K_n appears in exactly

one of these subgraphs, and each G_i is isomorphic to the disjoint union of $K_{n/3}$ and $K_{n/3,n/3}$.

When $n \equiv 1 \pmod 3$, consider one vertex w. Since $n-1$ is a multiple of 3, we can form the subgraphs G_i as above on the remaining $n-1$ vertices. Modify G_i to form H_i by joining w to every vertex of V_i. Each edge involving w has been added to exactly one of the three subgraphs. Each H_i is isomorphic to the disjoint union of $K_{1+(n-1)/3}$ and $K_{(n-1)/3,(n-1)/3}$.

1.1.36. *If K_n decomposes into triangles, then $n-1$ or $n-3$ is divisible by 6.* Such a decomposition requires that the degree of each vertex is even and the number of edges is divisible by 3. To have even degree, n must be odd. Also $n(n-1)/2$ is a multiple of 3, so 3 divides n or $n-1$. If 3 divides n and n is odd, then $n-3$ is divisible by 6. If 3 divides $n-1$ and n is odd, then $n-1$ is divisible by 6.

1.1.37. *A graph in which every vertex has degree 3 has no decomposition into paths with at least five vertices each.* Suppose that G has such a decomposition. Since every vertex has degree 3, each vertex is an endpoint of at least one of the paths and is an internal vertex on at most one of them. Since every path in the decomposition has two endpoints and has at least three internal vertices, we conclude that the number of paths in the decomposition is at least $n(G)/2$ and is at most $n(G)/3$, which is impossible.

Alternatively, let k be the number of paths. There are $2k$ endpoints of paths. On the other hand, since each internal vertex on a path in the decomposition must be an endpoint of some other path in the decomposition, there must be at least $3k$ endpoints of paths. The contradiction implies that there cannot be such a decomposition.

1.1.38. *A 3-regular graph G has a decomposition into claws if and only if G is bipartite.* When G is bipartite, we produce a decomposition into claws. We use all claws obtained by taking the three edges incident with a single vertex in the first partite set. Each claw uses all the edges incident to its central vertex. Since each edge has exactly one endpoint in the first partite set, each edge appears in exactly one of these claws.

When G has a decomposition into claws, we partition $V(G)$ into two independent sets. Let X be the set of centers of the claws in the decomposition. Since every vertex has degree 3, each claw in the decomposition uses all edges incident to its center. Since each edge is in at most one claw, this makes X an independent set. The remaining vertices also form an independent set, because every edge is in some claw in the decomposition, which means that one of its endpoints must be the center of that claw.

1.1.39. *Graphs that decompose K_6.*

Triangle—No. A graph decomposing into triangles must have even degree at each vertex. (This excludes all decompositions into cycles.)

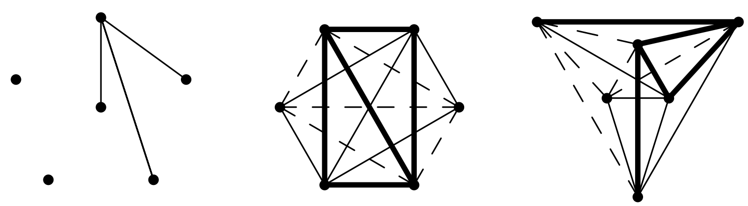
Paw, P_5—No. K_6 has 15 edges, but each paw or P_5 has four edges.

House, Bowtie, Dart–No. K_6 has 15 edges, but each house, bowtie, or dart has six edges.

Claw—Yes. Put five vertices $0, 1, 2, 3, 4$ on a circle and the other vertex z in the center. For $i \in \{0, 1, 2, 3, 4\}$, use a claw with edges from i to $i+1$, $i+2$, and z. Each edge appears in exactly one of these claws.

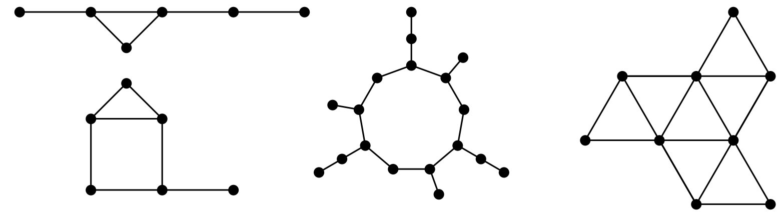
Kite—Yes. Put all six vertices on a circle. Each kite uses two opposite edges on the outside, one diagonal, and two opposite edges of "length" 2. Three rotations of the picture complete the decomposition.

Bull—Yes. The bull has five edges, so we need three bulls. Each bull uses degrees $3, 3, 2, 1, 1, 0$ at the six vertices. Each bull misses one vertex, and each vertex of K_6 has five incident edges, so three of the vertices will receive degrees $3, 2, 0$ from the three bulls, and the other three will receive degrees $3, 1, 1$. Thus we use vertices of two types, which leads us to position them on the inside and outside as on the right below. The bold, solid, and dashed bulls obtained by rotation complete the decomposition.



1.1.40. *Automorphisms of P_n, C_n, and K_n.* A path can be left alone or flipped, a cycle can be rotated or flipped, and a complete graph can be permuted arbtrarily. The numbers of automorphisms are $2, 2n, n!$, respectively. Correspondingly, the numbers of distinct labelings using vertex set $[n]$ are $n!/2$, $(n-1)!/2$, 1, respectively. For P_n, these formulas require $n > 1$.

1.1.41. *Graphs with one and three automorphisms.* The two graphs on the left have six vertices and only the identity automorphism. The two graphs on the right have three automorphisms.



1.1.42. *The set of automorphisms of a graph G satisfies the following:*

a) The composition of two automorphisms is an automorphism.

b) The identity permutation is an automorphism.

c) The inverse of an automorphism is also an automorphism.

d) Composition of automorphisms satisfies the associative property.

The first three properties are essentially the same as the transitive, reflexive, and symmetric properties for the isomorphism relation; see the discussion of these in the text. The fourth property holds because composition of functions always satisfies the associative property (see the discussion of composition in Appendix A).

1.1.43. *Every automorphism of the Petersen graph maps the 5-cycle (12,34,51,23,45) into a 5-cycle with vertices ab, cd, ea, bc, de by a permutation of* [5] *taking 1,2,3,4,5 to a, b, c, d, e, respectively.* Let σ denote the automorphism, and let the vertex ab be the image of the vertex 12 under σ. The image of 34 must be a pair disjoint from ab, so we may let $cd = \sigma(34)$. The third vertex must be disjoint from the second and share an element with the first. We may select a to be the common element in the first and third vertices. Similarly, we may select c to be the common element in the second and fourth vertices. Since nonadjacent vertices correspond to sets with a common element, the other element of the fourth vertex must be b, and the fifth vertex can't have a or b and must have d and e. Thus every 5-cycle must have this form and is the image of (12,34,51,23,45) under the specified permutation σ.

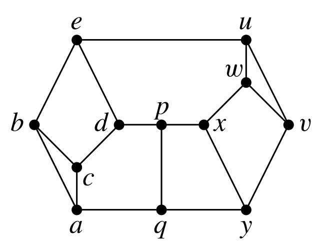
The Peterson graph has 120 automorphisms. Every permutation of [5] preserves the disjointness relation on 2-element subsets and therefore defines an automorphism of the Petersen graph. Thus there are at least 120 automorphism. To show that there are no others, consider an arbitrary automorphism σ. By the preceding paragraph, the 5-cycle C maps to some 5-cycle (ab, cd, ea, bc, de). This defines a permutation f mapping $1, 2, 3, 4, 5$ to a, b, c, d, e, respectively. It suffices to show that the other vertices must also have images under σ that are described by f.

The remaining vertices are pairs consisting of two nonconsecutive values modulo 5. By symmetry, it suffices to consider just one of them, say 24. The only vertex of C that 24 is adjacent to (disjoint from) is 51. Since

$\sigma(51) = ea$, and the only vertex not on (ab, cd, ea, bc, de) that is adjacent to ea is bd, we must have $\sigma(24) = bd$, as desired.

1.1.44. *For each pair of 3-edge paths $P = (u_0, u_1, u_2, u_3)$ and $Q = (v_0, v_1, v_2, v_3)$ in the Petersen graph, there is an automorphism of the Petersen graph that turns P into Q.* In the disjointness representation of the Petersen graph, suppose the pairs corresponding to the vertices of P are ab, cd, ef, g_h, respectively. Since consecutive pairs are disjoint and the edges are unordered pairs, we may write the pairs so that a, b, c, d, e are distinct, $f = a$, $g = b$, and $h = c$. Putting the vertex names of Q in the same format $AB, CD, EF, G \square H$, we chose the isomorphism generated by the permutation of [5] that turns a, b, c, d, e into A, B, C, D, E, respectively.

1.1.45. *A graph with 12 vertices in which every vertex has degree 3 and the only automorphism is the identity.*



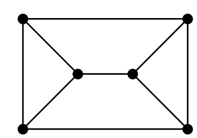
There are many ways to prove that an automorphism must fix all the vertices. The graph has only two triangles (abc and uvw). Now an automorphism must fix p, since is the only vertex having no neighbor on a triangle, and also e, since it is the only vertex with neighbors on both triangles. Now d is the unique common neighbor of p and e. The remaining vertices can be fixed iteratively in the same way, by finding each as the only unlabeled vertex with a specified neighborhood among the vertices already fixed. (This construction was provided by Luis Dissett, and the argument forbidding nontrivial automorphisms was shortened by Fred Galvin. Another such graph with three triangles was found by a student of Fred Galvin.)

1.1.46. *Vertex-transitivity and edge-transitivity.* The graph on the left in Exercise 1.1.21 is isomorphic to the 4-dimensional hypercube (see Section 1.3), which is vertex-transitive and edge-transitive via the permutation of coordinates. For the graph on the right, rotation and inside-out exchange takes care of vertex-transitivity. One further generating operation is needed to get edge-transitivity; the two bottom outside vertices can be switched with the two bottom inside vertices.

1.1.47. *Edge-transitive versus vertex-transitive. a) If G is obtained from K_n with $n \geq 4$ by replacing each edge of K_n with a path of two edges through a new vertex of degree 2, then G is edge-transitive but not vertex-transitive.* Every edge consists of an old vertex and a new vertex. The $n!$ permutations of old vertices yield automorphism. Let $x\&y$ denote the new vertex on the path replacing the old edge xy; note that $x\&y = y\&x$. The edge joining x and $x\&y$ is mapped to the edge joining u and $u\&v$ by any automorphism that maps x to u and y to v. The graph is not vertex-transitive, since $x\&y$ has degree 2, while x has degree $n - 1$.

b) If G is edge-transitive but not vertex-transitive and has no isolated vertices, then G is bipartite. Let uv be an arbitrary edge of G. Let S be the set of vertices to which u is mapped by automorphisms of G, and let T be the set of vertices to which v is mapped. Since G is edge-transitive and has no isolated vertex, $S \cup T = V(G)$. Since G is not vertex-transitive, $S \neq V(G)$. Together, these statements yield $S \cap T = \varnothing$, since the composition of two automorphisms is an automorphism. By edge-transitivity, every edge of G contains one vertex of S and one vertex of T. Since $S \cap T = \varnothing$, this implies that G is bipartite with vertex bipartition S, T.

c) The graph below is vertex-transitive but not edge-transitive. A composition of left-right reflections and vertical rotations can take each vertex to any other. The graph has some edges on triangles and some edges not on triangles, so it cannot be edge-transitive.



1.2. PATHS, CYCLES, AND TRAILS

1.2.1. *Statements about connection.*

a) Every disconnected graph has an isolated vertex—FALSE. A simple 4-vertex graph in which every vertex has degree 1 is disconnected and has no isolated vertex.

b) A graph is connected if and only if some vertex is connected to all other vertices—TRUE. A vertex is "connected to" another if they lie in a common path. If G is connected, then by definition each vertex is connected to every other. If some vertex x is connected to every other, then because a u, x-path and x, v-path together contain a u, v-path, every vertex is connected to every other, and G is connected.

c) The edge set of every closed trail can be partitioned into edge sets of cycles—TRUE. The vertices and edges of a closed trail form an even graph, and Proposition 1.2.27 applies.

d) If a maximal trail in a graph is not closed, then its endpoints have odd degree. If an endpoint v is different from the other endpoint, then the trail uses an odd number of edges incident to v. If v has even degree, then there remains an incident edge at v on which to extend the trail.

1.2.2. *Walks in K_4.*

a) K_4 has a walk that is not a trail; repeat an edge.

b) K_4 has a trail that is not closed and is not a path; traverse a triangle and then one additional edge.

c) The closed trails in K_4 that are not cycles are single vertices. A closed trail has even vertex degrees; in K_4 this requires degrees 2 or 0, which forbids connected nontrivial graphs that are not cycles. By convention, a single vertex forms a closed trail that is not a cycle.

1.2.3. *The non-coprimality graph with vertex set $\{1,\ldots,15\}$.* Vertices 1,11,13 are isolated. The remainder induce a single component. It has a spanning path 7,14,10,5,15,3,9,12,8,6,4,2. Thus there are four components, and the maximal path length is 11.

1.2.4. *Effect on the adjacency and incidence matrices of deleting a vertex or edge.* Assume that the graph has no loops.

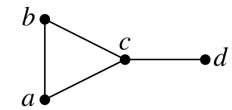
Consider the vertex ordering $v_1,\ldots,v_n$. Deleting edge v_iv_j simply deletes the corresponding column of the incidence matrix; in the adjacency matrix it reduces positions i,j and j,i by one.

Deleting a vertex v_i eliminates the ith row of the incidence matrix, and it also deletes the column for each edge incident to v_i. In the adjacency matrix, the ith row and ith column vanish, and there is no effect on the rest of the matrix.

1.2.5. *If v is a vertex in a connected graph G, then v has a neighbor in every component of $G-v$.* Since G is connected, the vertices in one component of $G-v$ must have paths in G to every other component of $G-v$, and a path can only leave a component of $G-v$ via v. Hence v has a neighbor in each component.

No cut-vertex has degree 1. If G is connected and $G-v$ has k components, then having a neighbor in each such component yields $d_G(v)\ge k$. If v is a cut-vertex, then $k\ge 2$, and hence $d_G(v)\ge 2$.

1.2.6. *The paw.* Maximal paths: acb, $abcd$, $bacd$ (two are maximum paths). Maximal cliques: abc, cd (one is a maximum clique). Maximal independent sets: c, bd, ad (two are maximum independent sets).



1.2.7. *A bipartite graph has a unique bipartition (except for interchanging the two partite sets) if and only if it is connected.* Let G be a bipartite graph. If u and v are vertices in distinct components, then there is a bipartition in which u and v are in the same partite set and another in which they are in opposite partite sets.

If G is connected, then from a fixed vertex u we can walk to all other vertices. A vertex v must be in the same partite set as u if there is a u,v-walk of even length, and it must be in the opposite set if there is a u,v-walk of odd length.

1.2.8. *The biclique $K_{m,n}$ is Eulerian if and only if m and n are both even or one of them is 0.* The graph is connected. It vertices have degrees m and n (if both are nonzero), which are all even if and only if m and n are both even. When m or n is 0, the graph has no edges and is Eulerian.

1.2.9. *The minimum number of trails that decompose the Petersen graph is 5.* The Petersen graph has exactly 10 vertices of odd degree, so it needs at least 5 trails, and Theorem 1.2.33 implies that five trails suffice.

The Petersen graph does have a decomposition into five paths. Given the drawing of the Petersen graph consisting of two disjoint 5-cycles and edges between them, form paths consisting of one edge from each cycle and one edge joining them.

1.2.10. *Statements about Eulerian graphs.*

a) Every Eulerian bipartite graph has an even number of edges—TRUE.

Proof 1. Every vertex has even degree. We can count the edges by summing the degrees of the vertices in one partite set; this counts every edge exactly once. Since the summands are all even, the total is also even.

Proof 2. Since every walk alternates between the partite sets, following an Eulerian circuit and ending at the initial vertex requires taking an even number of steps.

Proof 3. Every Eulerian graph has even vertex degrees and decomposes into cycles. In a bipartite graph, every cycle has even length. Hence the number of edges is a sum of even numbers.

b) Every Eulerian simple graph with an even number of vertices has an even number of edges—FALSE. The union of an even cycle and an odd cycle that share one vertex is an Eulerian graph with an even number of vertices and an odd number of edges.

1.2.11. *If G is an Eulerian graph with edges e, f that share a vertex, then G need not have an Eulerian circuit in which e, f appear consecutively.* If G consists of two edge-disjoint cycles sharing one common vertex v, then edges incident to v that belong to the same cycle cannot appear consecutively on an Eulerian circuit.

1.2.12. *Algorithm for Eulerian circuit.* We convert the proof by extremality to an iterative algorithm. Assume that G is a connected even graph. Initialize T to be a closed trail of length 0; a single vertex.

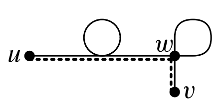
If T is not all of G, we traverse T to reach a vertex v on T that is incident to an edge e not in T. Beginning from v along e, traversing an arbitrary trail T' not using edges of T; eventually the trail cannot be extended. Since $G - E(T)$ is an even graph, this can only happen upon a return to the original vertex v, completing a closed trail. Splice T' into T by traversing T up to v, then following T', then the rest of T.

If this new trail includes all of $E(G)$, then it is an Eulerian circuit, and we stop. Otherwise, let this new trail be T and repeat the iterative step.

Since each successive trail is longer and G has finitely many edges, the procedure must terminate. It can only terminate when an Eulerian circuit has been found.

1.2.13. *Each u, v-walk contains a u, v-path.*

a) (induction). We use ordinary induction on the length l of the walk, proving the statement for all pairs of vertices. A u, v-walk of length 1 is a u, v-path of length 1; this provides the basis. For the induction step, suppose $l > 1$, and let W be a u, v-walk of length l; the induction hypothesis is that walks of length less than l contain paths linking their endpoints. If $u = v$, the desired path has length 0. If $u \neq v$, let wv be the last edge of W, and let W' be the u, w-walk obtained by deleting wv from W. Since W' has length $l - 1$, the induction hypothesis guarantees a u, w-path P in W'. If $w = v$, then P is the desired u, v-path. If $w \neq v$ and v is not on P, then we extend P by the edge wv to obtain a u, v-path. If $w \neq v$ and v is on P, then P contains a u, v-path. In each case, the edges of the u, v-path we construct all belong to W.



b) (extremality) Given a u, v-walk W, consider a shortest u, v-walk W' contained in W. If this is not a path, then it has a repeated vertex, and the portion between the instances of one vertex can be removed to obtain a shorter u, v-walk in W than W'.

1.2.14. *The union of the edge sets of distinct u, v-paths contains a cycle.*

Proof 1 (extremality). Let P and Q be distinct u, v-paths. Since a path in a simple graph is determined by its set of edges, we may assume (by symmetry) that P has an edge e not belonging to Q. Within the portion of P before P traverses e, let y be the last vertex that belongs to Q. Within the portion of P after P traverses e, let z be the first vertex that belongs to Q. The vertices y, z exist, because $u, v \in V(Q)$. The y, z-subpath of P combines with the y, z- or z, y-subpath of Q to form a cycle, since this subpath of Q contains no vertex of P between y and z.

Proof 2 (induction). We use induction on the sum l of the lengths of the two paths, for all vertex pairs simultaneously. If P and Q are u, v-paths, then $l \geq 2$. If $l = 2$, then we have distinct edges consisting of u and v, and together they form a cycle of length 2. For the induction step, suppose $l > 2$. If P and Q have no common internal vertices, then their union is a cycle. If P and Q have a common internal vertex w, then let P', P'' be the u, w-subpath of P and the w, v-subpath of P. Similarly define Q', Q''. Then P', Q' are u, w-paths with total length less than l. Similarly, P'', Q'' are w, v-paths with total length less than l. Since P, Q are distinct, we must have P', Q' distinct or P'', Q'' distinct. We can apply the induction hypothesis to the pair that is a pair of distinct paths joining the same endpoints. This pair contains the edges of a cycle, by the induction hypothesis, which in turn is contained in the union of P and Q.

The union of distinct u, v-walks need not contain a cycle. Let $G = P_3$, with vertices u, x, v in order. The distinct u, v-walks with vertex lists u, x, u, x, v and u, x, v, x, v do not contain a cycle in their union.

1.2.15. *If W is a nontrivial closed walk that does not contain a cycle, then some edge of W occurs twice in succession (once in each direction).*

Proof 1 (induction on the length l of W). We are given $l \geq 1$. A closed walk of length 1 is a loop, which is a cycle. Thus we may assume $l \geq 2$.

Basis step: $l = 2$. Since it contains no cycle, the walk must take a step and return immediately on the same edge.

Induction step: $l > 2$. If there is no vertex repetition other than first vertex = last vertex, then W traverses a cycle, which is forbidden. Hence there is some other vertex repetition. Let W' be the portion of W between the instances of such a repetition. The walk W' is a shorter closed walk than W and contains no cycle, since W has none. By the induction hypothesis, W' has an edge repeating twice in succession, and this repetition also appears in W.

Proof 2. Let w be the first repetition of a vertex along W, arriving from v on edge e. From the first occurrence of w to the visit to v is a w, v-walk, which is a cycle if $v = w$ or contains a nontrivial w, v-path P. This completes a cycle with e unless in fact P is the path of length 1 with edge e, in which case e repeats immediately in opposite directions.

1.2.16. *If edge e appears an odd number of times in a closed walk W, then W contains the edges of a cycle through e.*

Proof 1 (induction on the length of W, as in Lemma 1.2.7). The shortest closed walk has length 1. Basis step ($l = 1$): The edge e in a closed walk of length 1 is a loop and thus a cycle. Induction step ($l > 1$): If there is no vertex repetition, then W is a cycle. If there is a vertex repetition, choose two appearances of some vertex (other than the beginning and end of the walk). This splits the walk into two closed walks shorter than W. Since each step is in exactly one of these subwalks, one of them uses e an odd number of times. By the induction hypothesis, that subwalk contains the edges of a cycle through e, and this is contained in W.

Proof 2 (parity first, plus Lemma 1.2.6). Let x and y be the endpoints of e. As we traverse the walk, every trip through e is x, e, y or y, e, x. Since the number of trips is odd, the two types cannot alternate. Hence some two successive trips through e have the same direction. By symmetry, we may assume that this is $x, e, y, \ldots, x, e, y$.

The portion of the walk between these two trips through e is a y, x-walk that does not contain e. By Lemma 1.2.6, it contains a y, x-path (that does not contain e. Adding e to this path completes a cycle with e consisting of edges in W.

Proof 3 (contrapositive). If edge e in walk W does not lie on a cycle consisting of edges in W, then by our characterization of cut-edges, e is a cut-edge of the subgraph H consisting of the vertices and edges in W. This means that the walk can only return to e at the endpoint from which it most recently left e. This requires the traversals of e to alternate directions along e. Since a closed walk ends where it starts (that is, in the same component of $H - e$), the number of traversals of e by W must be even.

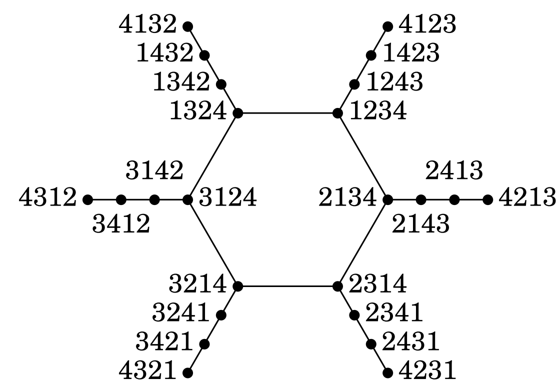
1.2.17. *The "adjacent-transposition graph" G_n on permutations of $[n]$ is connected.* Note that since every permutation of $[n]$ has $n - 1$ adjacent pairs that can be transposed, G_n is $(n-1)$-regular. Therefore, showing that G_n is connected shows that it is Eulerian if and only if n is odd.

Proof 1 (path to fixed vertex). We show that every permutation has a path to the identity permutation $I = 1, \ldots, n$. By the transitivity of the connection relation, this yields for all $u, v \in V(G)$ a u, v-path in G. To create a v, I-path, move element 1 to the front by adjacent interchanges, then move 2 forward to position 2, and so on. This builds a walk to I, which contains a path to I. (Actually, this builds a path.)

Proof 2 (direct u, v-path). Each vertex is a permutation of $[n]$. Let $u = a_1, \ldots, a_n$ and $v = b_1, \ldots, b_n$; we construct at u, v-path. The element

b_1 appears in u as some a_i; move it to the front by adjacent transpositions, beginning a walk from u. Next find b_2 among $a_2, \ldots, a_n$ and move it to position 2. Iterating this procedure brings the elements of v toward the front, in order, while following a walk. It reaches v when all positions have been "corrected". (Actually, the walk is a u, v-path.) Note that since we always bring the desired element forward, we never disturb the position of the elements that were already moved to their desired positions.

Proof 3 (induction on n). If $n = 1$, then $G_n \cong K_1$ and G is connected (we can also start with $n = 2$). For $n > 1$, assume that G_{n-1} is connected. In G_n, the subgraph H induced by the vertices having n at the end is isomorphic to G_{n-1}. Every vertex of G is connected to a vertex of H by a path formed by moving element n to the end, one step at a time. For $u, v \in V(G)$, we thus have a path from u to a vertex $u' \in V(H)$, a path from v to a vertex $v' \in V(H)$, and a u', v'-path in H that exists by the induction hypothesis. By the transitivity of the connection relation, there is a u, v-path in G. This completes the proof of the induction step. (The part of G_4 used in the induction step appears below.)



Proof 4 (induction on n). The basis is as in Proof 3. For $n > 1$, note that for each $i \in [n]$, the vertices with i at the end induce a copy H_i of G_{n-1}. By the induction hypothesis, each such subgraph is connected. Also, H_n has vertices with i in position $n - 1$ whenever $1 \le i \le n - 1$. We can interchange the last two positions to obtain a neighbor in H_i. Hence there is an edge from each H_i to H_n, and transitivity of the connection relation again completes the proof.

1.2.18. *For $k \ge 1$, there are two components in the graph G_k whose vertex set is the set of binary k-tuples and whose edge set consists of the pairs that differ in exactly two places.* Changing two coordinates changes the number of 1s in the name of the vertex by zero or by ± 2. Thus the parity of the number of 1s remains the same along every edge. This implies that G_k has at least two components, because there is no edge from an k-tuple with an even number of 1s to an k-tuple with an odd number of 1s.

To show that G_k has at most two components, there are several approaches. In each, we prove that any two vertices with the same parity lie on a path, where "parity" means parity of the number of 1s.

Proof 1. If u and v are vertices with the same parity, then they differ in an even number of places. This is true because each change of a bit in obtaining one label from the other switches the parity. Since they differ in an even number of places, we can change two places at a time to travel from u to v along a path in G_k.

Proof 2. We use induction on k. Basis step ($k = 1$): G_1 has two components, each an isolated vertex. Induction step ($k > 1$): when $k > 1$, G_k consists of two copies of G_{k-1} plus additional edges. The two copies are obtained by appending 0 to all the vertex names in G_{k-1} or appending 1 to them all. Within a copy, the edges don't change, since these vertices all agree in the new place. By the induction hypothesis, each subgraph has two components. The even piece in the 0-copy has $0\cdots000$, which is adjacent to $0\cdots011$ in the odd piece of the 1-copy. The odd piece in the 0-copy has $0\cdots010$, which is adjacent to $0\cdots001$ in the even piece of the 1-copy. Thus the four pieces reduce to (at most) two components in G_k.

1.2.19. *For $n, r, s \in \mathbb{N}$, the simple graph G with vertex set $\mathbb{Z}_n$ and edge set $\{ij\colon |j-i| \in \{r, s\}\}$ has $\gcd(n, r, s)$ components.* **Note**: The text gives the vertex set incorrectly. When $r = s = 2$ and n is odd, it is necessary to go up to $n \equiv 0$ to switch from odd to even.

Let $k = \gcd(n, r, s)$. Since k divides n, the congruence classes modulo n fall into congruence classes modulo k in a well-defined way. All neighbors of vertex i differ from i by a multiple of k. Thus all vertices in a component lie in the same congruence class modulo k, which makes at least k components.

To show that there are only k components, we show that all vertices with indices congruent to $i \pmod k$ lie in one component (for each i). It suffices to build a path from i to $i + k$. Let $l = \gcd(r, s)$, and let $a = r/l$ and $b = s/l$. Since there are integers (one positive and one negative) such that $pa + qb = 1$, moving p edges with difference $+r$ and q edges with difference $+s$ achieves a change of $+l$.

We thus have a path from i to $i + l$, for each i. Now, $k = \gcd(l, n)$. As above, there exist integers p', q' such that $p'(l/k) + q'(n/k) = 1$. Rewriting this as $p'l = k - q'n$ means that if we use p' of the paths that add l, then we will have moved from i to $i + k \pmod n$.

1.2.20. *If v is a cut-vertex of a simple graph G, then v is not a cut-vertex of $\overline{G}$.* Let $V_1, \ldots, V_k$ be the vertex sets of the components of $G - v$; note that $k \ge 2$. Then $\overline{G}$ contains the complete multipartite graph with partite sets $V_1, \ldots, V_k$. Since this includes all vertices of $\overline{G} - v$, the graph $\overline{G} - v$ is connected. Hence v is not a cut-vertex of $\overline{G}$.

1.2.21. *A self-complementary graph has a cut-vertex if and only if it has a vertex of degree 1.* If there is a vertex of degree 1, then its neighbor is a cut-vertex (K_2 is not self-complementary).

For the converse, let v be a cut-vertex in a self-complementary graph G. The graph $\overline{G - v}$ has a *spanning biclique*, meaning a complete bipartite subgraph that contains all its vertices. Since G is self-complementary, also G must have a vertex u such that $G - u$ has a spanning biclique.

Since each vertex of $G - v$ is nonadjacent to all vertices in the other components of $G - v$, a vertex other than u must be in the same partite set of the spanning biclique of $G - u$ as the vertices not in the same component as u in $G - v$. Hence only v can be in the other partite set, and v has degree at least $n - 2$. We conclude that v has degree at most 1 in $\overline{G}$, so G has a vertex of degree at most 1. Since a graph and its complement cannot both be disconnected, G has a vertex of degree 1.

1.2.22. *A graph is connected if and only if for every partition of its vertices into two nonempty sets, there is an edge with endpoints in both sets.*

Necessity. Let G be a connected graph. Given a partition of $V(G)$ into nonempty sets S, T, choose $u \in S$ and $v \in T$. Since G is connected, G has a u, v-path P. After its last vertex in S, P has an edge from S to T.

Sufficiency.

Proof 1 (contrapositive). We show that if G is not connected, then for some partition there is no edge across. In particular, if G is disconnected, then let H be a component of G. Since H is a maximal connected subgraph of G and the connection relation is transitive, there cannot be an edges with one endpoint in $V(H)$ and the other endpoint outside. Thus for the partition of $V(G)$ into $V(H)$ and $V(G) - V(H)$ there is no edge with endpoints in both sets.

Proof 2 (algorithmic approach). We grow a set of vertices that lie in the same equivalence class of the connection relation, eventually accumulating all vertices. Start with one vertex in S. While S does not include all vertices, there is an edge with endpoints $x \in S$ and $y \notin S$. Adding y to S produces a larger set within the same equivalence class, using the transitivity of the connection relation. This procedure ends only when there are no more vertices outside S, in which case all of G is in the same equivalence class, so G has only one component.

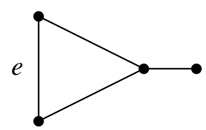
Proof 3 (extremality). Given a vertex $x \in V(G)$, let S be the set of all vertices that can be reached from x via paths. If $S \ne V(G)$, consider the partition into S and $V(G) - S$. By hypothesis, G has an edge with endpoints

$u \in S$ and $v \notin S$. Now there is an x, v-path formed by extending an x, u-path along the edge uv. This contradicts the choice of S, so in fact S is all of $V(G)$. Since there are paths from x to all other vertices, the transitivity of the connection relation implies that G is connected.

1.2.23. *a) If a connected simple graph G is not a complete graph, then every vertex of G belongs to some induced subgraph isomorphic to P_3.* Let v be a vertex of G. If the neighborhood of v is not a clique, then v has a pair x, y of nonadjacent neighbors; $\{x, v, y\}$ induces P_3. If the neighborhood of v is a clique, then since G is not complete there is some vertex y outside the set S consisting of v and its neighbors. Since G is connected, there is some edge between a neighbor w of v and a vertex x that is not a neighbor of v. Now the set $\{v, w, x\}$ induces P_3, since x is not a neighbor of v.

One can also use cases according to whether v is adjacent to all other vertices or not. The two cases are similar to those above.

b) When a connected simple graph G is not a complete graph, G may have edges that belong to no induced subgraph isomorphic to P_3. In the graph below, e lies in no such subgraph.



1.2.24. *If a simple graph with no isolated vertices has no induced subgraph with exactly two edges, then it is a complete graph.* Let G be such a graph. If G is disconnected, then edges from two components yield four vertices that induce a subgraph with two edges. If G is connected and not complete, then G has nonadjacent vertices x and y. Let Q be a shortest x, y-path; it has length at least 2. Any three successive vertices on Q induce P_3, with two edges.

Alternatively, one can use proof by contradiction. If G is not complete, then G has two nonadjacent vertices. Considering several cases (common neighbor or not, etc.) always yields an induced subgraph with two edges.

1.2.25. *Inductive proof that every graph G with no odd cycles is bipartite.*

Proof 1 (induction on $e(G)$). Basis step ($e(G) = 0$): Every graph with no edges is bipartite, using any two sets covering $V(G)$.

Induction step ($e(G) > 0$): Discarding an edge e introduces no odd cycles. Thus the induction hypothesis implies that $G - e$ is bipartite.

If e is a cut-edge, then combining bipartitions of the components of $G - e$ so that the endpoints of e are in opposite sets produces a bipartition of G. If e is not a cut-edge of G, then let u and v be its endpoints, and let X, Y be a bipartition of $G - e$. Adding e completes a cycle with a u, v-path in $G - e$; by hypothesis, this cycle has even length. This forces u and v to be in opposite sets in the bipartition X, Y. Hence the bipartition X, Y of $G - e$ is also a bipartition of G.

Proof 2 (induction on $n(G)$). Basis step ($n(G) = 1$): A graph with one vertex and no odd cycles has no loop and hence no edge and is bipartite.

Induction step ($n(G) > 1$): When we discard a vertex v, we introduce no odd cycles. Thus the induction hypothesis implies that $G - v$ is bipartite. Let $G_1, \ldots, G_k$ be the components of $G - v$; each has a bipartition. If v has neighbors u, w in both parts of the bipartition of G_i, then the edges uv and vw and a shortest u, w-path in G_i form a cycle of odd length. Hence we can specify the bipartition X_i, Y_i of G_i so that X_i contains all neighbors of v in G_i. We now have a bipartition of G by letting $X = \bigcup X_i$ and $Y = \{v\} \cup (\bigcup Y_i)$.

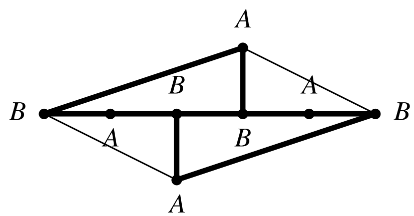
1.2.26. *A graph G is bipartite if and only if for every subgraph H of G, there is an independent set containing at least half of the vertices of H.* Every bipartite graph has a vertex partition into two independent sets, one of which must contain at least half the vertices (though it need not be a maximum independent set). Since every subgraph of a bipartite graph is bipartite, the argument applies to all subgraphs of a bipartite graph, and the condition is necessary.

For the converse, suppose that G is not bipartite. By the characterization of bipartite graphs, G contains an odd cycle H. This subgraph H has no independent set containing at least half its vertices, because every set consisting of at least half the vertices in an odd cycle must have two consecutive vertices on the cycle.

1.2.27. *The "transposition graph" on permutations of $[n]$ is bipartite.* The partite sets are determined by the parity of the number of pairs i, j such that $i < j$ and $a_i > a_j$ (these are called **inversions**). We claim that each transposition changes the parity of the number of inversions, and therefore each edge in the graph joins vertices with opposite parity. Thus the permutations with an even number of inversions form an independent set, as do those with an odd number of inversions. This is a bipartition, and thus the graph is bipartite.

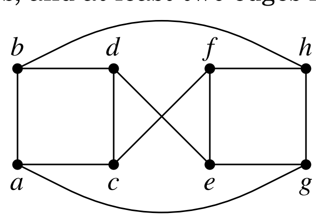
Consider the transposition that interchanges the elements in position r and position s, with $r < s$. No pairs involving elements that are before r or after s have their order changed. If $r < k < s$, then interchanging a_r and a_s changes the order of a_r and a_k, and also it changes the order of a_k and a_s. Thus for each such k the number of inversions changes twice and retains the same parity. This describes all changes in order except for the switch of a_r and a_s itself. Thus the total number of changes is odd, and the parity of the number of inversions changes.

1.2.28. *a) The graph below has a unique largest bipartite subgraph, obtained by deleting the central edge.* Deleting the central edge leaves a bipartite subgraph, since the indicated sets A and B are independent in that subgraph. If deleting one edge makes a graph bipartite, then that edge must belong to all odd cycles in the graph, since a bipartite subgraph has no odd cycles. The two odd cycles in bold have only the central edge in common, so no other edge belongs to all odd cycles.



b) In the graph below, the largest bipartite subgraph has 10 edges, and it is not unique. Deleting edges bh and ag yields an X,Y-bigraph with $X=\{b,c,e,h\}$ and $Y=\{a,d,f,g\}$. Another bipartite subgraph with 10 edges is obtained by deleting edges de and cf; the bipartition is $X=\{b,c,f,g\}$ and $Y=\{a,d,e,h\}$. (Although these two subgraphs are isomorphic, they are two subgraphs, just as the Petersen graph has ten claws, not one.)

It remains to show that we must delete at least two edges to obtain a bipartite subgraph. By the characterization of bipartite graphs, we must delete enough edges to break all odd cycles. We can do this with (at most) one edge if and only if all the odd cycles have a common edge. The 5-cycles (b,a,c,f,h) and (b,d,e,g,h) have only the edge bh in common. Therefore, if there is a single edge lying in all odd cycles, it must be bh. However, (a,c,f,h,g) is another 5-cycle that does not contain this. Therefore no edge lies in all odd cycles, and at least two edges must be deleted.



1.2.29. *A connected simple graph not having P_4 or C_3 as an induced subgraph is a biclique.* Choose a vertex x. Since G has no C_3, $N(x)$ is independent. Let $S=V(G)-N(X)-\{x\}$. Every $v\in S$ has a neighbor in $N(x)$; otherwise, a shortest v,x-path contains an induced P_4. If $v\in S$ is adjacent to w but not z in $N(x)$, then v,w,x,z is an induced P_4. Hence all of S is adjacent to all of $N(x)$. Now $S\cup\{x\}$ is an independent set, since G has no C_3. We have proved that G is a biclique with bipartition $N(x)$, $S\cup\{x\}$.

1.2.30. *Powers of the adjacency matrix.*

a) In a simple graph G, the (i,j)th entry in the kth power of the adjacency matrix $\mathbf{A}$ is the number of (v_i,v_j)-walks of length k in G. We use induction on k. When $k=1$, $a_{i,j}$ counts the edges (walks of length 1) from i to j. When $k>1$, every (v_i,v_j)-walk of length k has a unique vertex v_r reached one step before the end at v_j. By the induction hypothesis, the number of (v_i,v_r)-walks of length $k-1$ is entry (i,r) in $\mathbf{A}^{k-1}$, which we write as $a_{i,r}^{(k-1)}$. The number of (v_i,v_j)-paths of length k that arrive via v_r on the last step is $a_{i,r}^{(k-1)}a_{r,j}$, since $a_{r,j}$ is the number of edges from v_r to v_j that can complete the walk. Counting the (v_i,v_j)-walks of length k by which vertex appears one step before v_j yields $\sum_{r=1}^n a_{i,r}^{(k-1)}a_{r,j}$. By the definition of matrix multiplication, this is the (i,j)th entry in $\mathbf{A}^k$. (The proof allows loops and multiple edges and applies without change for digraphs. When loops are present, note that there is no choice of "direction" on a loop; a walk is a list of edge traversals).

b) A simple graph G with adjacency matrix A is bipartite if and only if, for each odd integer r, the diagonal entries of the matrix A^r are all 0. By part (a), $A_{i,i}^r$ counts the closed walks of length r beginning at v_i. If this is always 0, then G has no closed walks of odd length through any vertex; in particular, G has no odd cycle and is bipartite. Conversely, if G is bipartite, then G has no odd cycle and hence no closed odd walk, since every closed odd walk contains an odd cycle.

1.2.31. *K_n is the union of k bipartite graphs if and only if $n\le 2^k$ (without using induction).*

a) Construction when $n\le 2^k$. Given $n\le 2^k$, encode the vertices of K_n as distinct binary k-tuples. Let G_i be the complete bipartite subgraph with bipartition X_i,Y_i, where X_i is the set of vertices whose codes have 0 in position i, and Y_i is the set of vertices whose codes have 1 in position i. Since every two vertex codes differ in some position, $G_1\cup\cdots\cup G_k=K_n$.

b) Upper bound. Given that K_n is a union of bipartite graphs $G_1,\ldots,G_k$, we define a code for each vertex. For $1\le i\le k$, let X_i,Y_i be a bipartition of G_i. Assign vertex v the code $(a_1,\ldots,a_k)$, where $a_i=0$ if $v\in X_i$, and $a_i=1$ if $v\in Y_i$ or $v\notin X_i\cup Y_i$. Since every two vertices are adjacent and the edge joining them must be covered in the union, they lie in opposite partite sets in some G_i. Therefore the codes assigned to the vertices are distinct. Since the codes are binary k-tuples, there are at most 2^k of them, so $n\le 2^k$.

1.2.32. *"Every maximal trail in an even graph is an Eulerian circuit"—FALSE.* When an even graph has more than one component, each component has a maximal trail, and it will not be an Eulerian circuit unless the other components have no edges. The added hypothesis needed is that the graph is connected.

The proof of the corrected statement is essentially that of Theorem 1.2.32. If a maximal trail T is not an Eulerian circuit, then it is incident to a missing edge e, and a maximal trail in the even graph $G - E(T)$ that starts at e can be inserted to enlarge T, which contradicts the hypothesis that T is a maximal trail.

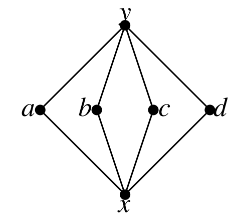
1.2.33. *The edges of a connected graph with $2k$ odd vertices can be partitioned into k trails if $k > 0$.* The assumption of connectedness is necessary, because the conclusion is not true for $G = H_1 + H_2$ when H_1 has some odd vertices and H_2 is Eulerian.

Proof 1 (induction on k). When $k = 1$, we add an edge between the two odd vertices, obtain an Eulerian circuit, and delete the added edge. When $k > 1$, let P be a path connecting two odd vertices. The graph $G' = G - E(P)$ has $2k - 2$ odd vertices, since deleting $E(P)$ changes degree parity only at the ends of P. The induction hypothesis applies to each component of G' that has odd vertices. Any component not having odd vertices has an Eulerian circuit that contains a vertex of P; we splice it into P to avoid having an additional trail. In total, we have used the desired number of trails to partition $E(G)$.

Proof 2 (induction on $e(G)$). If $e(G) = 1$, then $G = K_2$, and we have one trail. If G has an even vertex x adjacent to an odd vertex y, then $G' = G - xy$ has the same number of odd vertices as G. The trail decomposition of G' guaranteed by the induction hypothesis has one trail ending at x and no trail ending at y. Add xy to the trail ending at x to obtain the desired decomposition of G. If G has no even vertex adjacent to an odd vertex, then G is Eulerian or every vertex of G is odd. In this case, deleting an edge xy reduces k, and we can add xy as a trail of length one to the decomposition of $G - xy$ guaranteed by the induction hypothesis.

1.2.34. *The graph below has 6 equivalence classes of Eulerian circuits.* If two Eulerian circuits follow the same circular arrangement of edges, differing only in the starting edges or the direction, then we consider them equivalent. An equivalence class of circuits is characterized by the pairing of edges at each vertex corresponding to visits through that vertex.

A 2-valent vertex has exactly one such pairing; a 4-valent vertex has three possible pairings. The only restriction is that the pairings must yield a single closed trail. Given a pairing at one 4-valent vertex below, there is a forbidden pairing at the other, because it would produce two edge-disjoint 4-cycles instead of a single trail. The other two choices are okay. Thus the answer is $3 \cdot 2 = 6$.



Alternatively, think of making choices while following a circuit. Because each circuit uses each edge, and because the reversal of a circuit C is in the same class as C, we may follow a canonical representative of the class from a along ax. We now count the choices made to determine the circuit. After x we can follow one of 3 choices. This leads us through another neighbor of x to y. Now we cannot use the edge ya or the edge just used, so two choices remain. This determines the rest of the circuit. For each of the three ways to make the initial choice, there was a choice of two later, so there are $3 \cdot 2 = 6$ ways to specify distinct classes of circuits. (Distinct ways of making the choices yields a distinct pairing at some vertex.)

1.2.35. *Algorithm for Eulerian circuits.* Let G be a connected even graph. At each vertex partition the incident edges into pairs (each edge appears in a pair at each endpoint). Start along some edge. At each arrival at a vertex, there is an edge paired with the entering edge; use it to exit. This can end only by arriving at the initial vertex along the edge paired with the initial edge, and it must end since the graph is finite. At the point where the first edge would be repeated, stop; this completes a closed trail. Furthermore, there is no choice in assembling this trail, so every edge appears in exactly one such trail. Therefore, the pairing decomposes G into closed trails.

If there is more than one trail in the decomposition, then there are two trails with a common vertex, since G is connected. (A shortest path connecting vertices in two of the trails first leaves the first trail at some vertex v, and at v we have edges from two different trails.) Given edges from trails A and B at v, change the pairing by taking a pair in A and a pair in B and switching them to make two pairs that pair an edge of A with an edge of B. Now when A is followed from v, the return to A does not end the trail, but rather the trail continues and follows B before returning to the original edge. Thus changing the pairing at v combines these two trails into one trail and leaves the other trails unchanged.

We have shown that if the number of trails in the decomposition exceeds one, then we can obtain a decomposition with fewer trails be changing the pairing. Repeating the argument produces a decomposition using one closed trail. This trail is an Eulerian circuit.

1.2.36. *Alternative characterization of Eulerian graphs.*

a) If G is loopless and Eulerian and $G' = G - uv$, then G' has an odd number of u, v-trails that visit v only at the end.

Proof 1 (exhaustive counting and parity). Every extension of every trail from u in G' eventually reaches v, because a maximal trail ends only at a vertex of odd degree. We maintain a list of trails from u. The number of choices for the first edge is odd. For a trail T that has not yet reached v, there are an odd number of ways to extend T by one edge. We replace T in the list by these extensions. This changes the number of trails in the list by an even number. The process ends when all trails in the list end at v. Since the list always has odd size, the total number of these trails is odd.

Proof 2 (induction and stronger result). We prove that the same conclusion holds whenever u and v are the only vertices of odd degree in a graph H, regardless of whether they are adjacent. This is immediate if H has only the edge uv. For larger graphs, we show that there are an odd number of such trails starting with each edge e incident to u, so the sum is odd. If $e = uv$, then there is one such trail. Otherwise, when $e = uw$ with $w \neq v$, we apply the induction hypothesis to $H - e$, in which w and v are the only vertices of odd degree.

The number of non-paths in this list of trails is even. If T is such a trail that is not a path, then let w be the first instance of a vertex repetition on T. By traversing the edges between the first two occurrences of w in the opposite order, we obtain another trail T' in the list. For T', the first instance of a vertex repetition is again w, and thus $T'' = T$. This defines an involution under which the fixed points are the u, v-paths. The trails we wish to delete thus come in pairs, so there are an even number of them.

b) If v is a vertex of odd degree in a graph G, then some edge incident to v lies in an even number of cycles. Let $c(e)$ denote the number of cycles containing e. Summing $c(e)$ over edges incident to v counts each cycle through v exactly twice, so the sum is even. Since there are an odd number of terms in the sum, $c(e)$ must be even for some e incident to v.

c) A nontrivial connected graph is Eulerian if and only if every edge belongs to an odd number of cycles. Necessity: By part (a), the number of u, v-paths in $G - uv$ is odd. The cycles through uv in G correspond to the u, v-paths in $G - uv$, so the number of these cycles is odd.

Sufficiency: We observe the contrapositive. If G is not Eulerian, then G has a vertex v of odd degree. By part (b), some edge incident to v lies in an even number of cycles.

1.2.37. *The connection relation is transitive.* It suffices to show that if P is a u, v-path and P' is a v, w-path, then P and P' together contain a u, w-path. At least one vertex of P is in P', since both contain v. Let x be the first vertex of P that is in P'. Following P from u to x and then P' from x to w yields a u, w path, since no vertex of P before x belongs to P'.

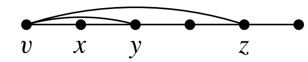
1.2.38. *Every n-vertex graph with at least n edges contains a cycle.*

Proof 1 (induction on n). A graph with one vertex that has an edge has a loop, which is a cycle. For the induction step, suppose that $n > 1$. If our graph G has a vertex v with degree at most 1, then $G - v$ has $n - 1$ vertices and at least $n - 1$ edges. By the induction hypothesis, $G - v$ contains a cycle, and this cycle appears also in G. If G has no vertex of degree at most 1, then every vertex of G has degree at least 2. Now Lemma 1.2.25 guarantees that G contains a cycle.

Proof 2 (use of cut-edges). If G has no cycle, then by Theorem 1.2.14 every edge is a cut-edge, and this remains true as edges are deleted. Deleting all the edges thus produces at least $n + 1$ components, which is impossible.

1.2.39. *If G is a loopless graph and $\delta(G) \geq 3$, then G has a cycle of even length.* An endpoint v of a maximal path P has at least three neighbors on P. Let x, y, z be three such neighbors of v in order on P. Consider three v, y-paths: the edge vy, the edge vx followed by the x, y-path in P, and the edge vz followed by the z, y-path in P.

These paths share only their endpoints, so the union of any two is a cycle. By the pigeonhole principle, two of these paths have lengths with the same parity. The union of these two paths is an even cycle.



1.2.40. *If P and Q are two paths of maximum length in a connected graph G, then P and Q have a common vertex.* Let m be the common length of P and Q. Since G is connected, it has a shortest path R between $V(P)$ and $V(Q)$. Let l be the length of R. Let the endpoints of R be $r \in V(P)$ and $r' \in V(Q)$. The portion P' of P from r to the farther endpoint has length at least $m/2$. The portion Q' of Q from r to the farther endpoint has length at least $m/2$. Since R is a shortest path, R has no internal vertices in P or Q.

If P and Q are disjoint, then P' and Q' are disjoint, and the union of P', Q', and R is a path of length at least $m/2 + m/2 + l = m + l$. Since the maximum path length is m, we have $l = 0$. Thus $r = r'$, and P and Q have a common vertex.

The graph consisting of two edge-disjoint paths of length $2k$ sharing their midpoint is connected and hence shows that P and Q need not have a common edge.

1.2.41. *A connected graph with at least three vertices has two vertices x, y such that 1) $G - \{x, y\}$ is connected and 2) x, y are adjacent or have a common neighbor.* Let x be a endpoint of a longest path P in G, and let v be its neighbor on P. Note that P has at least three vertices. If $G - x - v$ is connected, let $y = v$. Otherwise, a component cut off from $P - x - v$ in $G - x - v$ has at most one vertex; call it w. The vertex w must be adjacent to v, since otherwise we could build a longer path. In this case, let $y = w$.

1.2.42. *A connected simple graph having no 4-vertex induced subgraph that is a path or a cycle has a vertex adjacent to every other vertex.* Consider a vertex x of maximum degree. If x has a nonneighbor y, let x, v, w be the begining of a shortest path to y (w may equal y). Since $d(v) \le d(x)$, some neighbor z of x is not adjacent to v. If $z \leftrightarrow w$, then $\{z, x, v, w\}$ induce C_4; otherwise, $\{z, x, v, w\}$ induce P_4. Thus x must have no nonneighbor.

1.2.43. *The edges of a connected simple graph with $2k$ edges can be partitioned into paths of length 2.* The assumption of connectedness is necessary, since the conclusion does not hold for a graph having components with an odd number of edges.

We use induction on $e(G)$; there is a single such path when $e(G) = 2$. For $e(G) > 2$, let $P = (x, y, z)$ be an arbitrary path of length two in G, and let $G' = G - \{xy, yz\}$. If we can partition $E(G)$ into smaller connected subgraphs of even size, then we can apply the induction hypothesis to each piece and combine the resulting decompositions. One way to do this is to partition $E(G')$ into connected subgraphs of even size and use P.

Hence we are finished unless G' has two components of odd size (G' cannot have more than three components, since an edge deletion increases the number of components by at most one). Each odd component contains at least one of $\{x, y, z\}$. Hence it is possible to add one of xy to one odd component and yz to the other odd component to obtain a partition of G into smaller connected subgraphs.

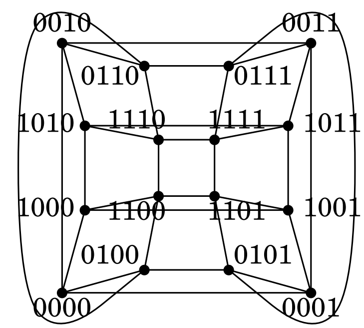
1.3. VERTEX DEGREES & COUNTING

1.3.1. *A graph having exactly two vertices of odd degree must contain a path from one to the other.* The degree of a vertex in a component of G is the same as its degree in G. If the vertices of odd degree are in different components, then those components are graphs with odd degree sum.

1.3.2. *In a class with nine students where each student sends valentine cards to three others, it is not possible that each student sends to and receives cards from the same people.* The sending of a valentine can be represented as a directed edge from the sender to the receiver. If each student sends to and receives cards from the same people, then the graph has $x \to y$ if and only if $y \to x$. Modeling each opposed pair of edges by a single unoriented edge yields a 3-regular graph with 9 vertices. This is impossible, since every graph has an even number of vertices of odd degree.

1.3.3. *If $d(u) + d(v) = n + k$ for an edge uv in a simple graph on n vertices, then uv belongs to at least k triangles.* This is the same as showing that u and v have at least k common neighbors. Let S be the neighbors of u and T the neighbors of v, and suppose $|S \cap T| = j$. Every vertex of G appears in S or T or none or both. Common neighbors are counted twice, so $n \ge |S| + |T| - j = n + k - j$. Hence $j \ge k$. (Almost every proof of this using induction or contradiction does not need it, and is essentially just this counting argument.)

1.3.4. *The graph below is isomorphic to Q_4.* It suffices to label the vertices with the names of the vertices in Q_4 so that vertices are adjacent if and only if their labels differ in exactly one place.



1.3.5. *The k-dimensional cube Q_k has $\binom{k}{2}2^k$ copies of P_3.*

Proof 1. To specify a particular subgraph isomorphic to P_3, the 3-vertex path, we can specify the middle vertex and its two neighbors. For each vertex of Q_k, there are $\binom{k}{2}$ ways to choose two distinct neighbors, since Q_k is a simple k-regular graph. Thus the total number of P_3's is $\binom{k}{2}2^k$.

Proof 2. We can alternatively choose the starting vertex and the next two. There are 2^k ways to pick the first vertex. For each vertex, there are k ways to pick a neighbor. For each way to pick these vertices, there are $k - 1$ ways to pick a third vertex completing P_3, since Q_k has no multiple edges. The product of these factors counts each P_3 twice, since we build it from each end. Thus the total number of them is $2^k k(k-1)/2$.

Q_k has $\binom{k}{2}2^{k-2}$ copies of C_4.

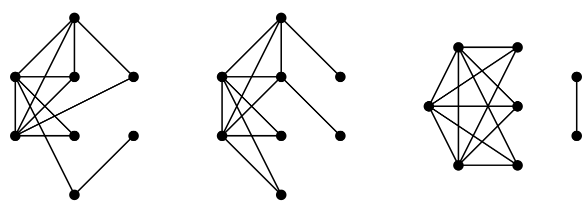
Proof 1 (direct counting). The vertices two apart on a 4-cycle must differ in two coordinates. Their two common neighbors each differ from each in exactly one of these coordinates. Hence the vertices of a 4-cycle must use all 2-tuples in two coordinates while keeping the remaining coordinates fixed. All such choices yield 4-cycles. There are $\binom{k}{2}$ ways to choose the two coordinates that vary and 2^{k-2} ways to set a fixed value in the remaining coordinates.

Proof 2 (prior result). Every 4-cycle contains four copies of P_3, and every P_3 contains two vertices at distance 2 in the cube and hence extends to exactly one 4-cycle. Hence the number of 4-cycles is one-fourth the number of copies of P_3.

1.3.6. *Counting components.* If G has k components and H has l components, then $G + H$ has $k + l$ components. The maximum degree of $G + H$ is $\max\{\Delta(G), \Delta(H)\}$.

1.3.7. *Largest bipartite subgraphs.* P_n is already bipartite. C_n loses one edge if n is odd, none if n is even. The largest bipartite subgraph of K_n is $K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$, which has $\lfloor n^2/4\rfloor$ edges.

1.3.8. *The lists (5,5,4,3,2,2,2,1), (5,5,4,4,2,2,1,1), and (5,5,5,3,2,2,1,1) are graphic, but (5,5,5,4,2,1,1,1) is not.* The answers can be obtained from the Havel-Hakimi test; a list is graphic if and only if the list obtained by deleting the largest element and deleting that many next-largest elements is graphic. Below are graphs realizing the first three lists, found by the Havel-Hakimi algorithm.



From the last list, we test $(4, 4, 3, 1, 0, 1, 1)$, reordered to $(4, 4, 3, 1, 1, 1, 0)$, then $(3, 2, 0, 0, 1, 0)$. This is not the degree list of a simple graph, since a vertex of degree 3 requires three other vertices with nonzero degree.

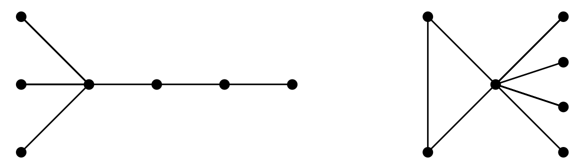
1.3.9. *In a league with two divisions of 13 teams each, no schedule has each team playing exactly nine games against teams in its own division and four games against teams in the other division.* If this were possible, then we could form a graph with the teams as vertices, making two vertices adjacent if those teams play a game in the schedule. We are asking for the subgraph induced by the 13 teams in a single division to be 9-regular. However, there is no regular graph of odd degree with an odd number of vertices, since for every graph the sum of the degrees is even.

1.3.10. *If l, m, n are nonnegative integers with $l + m = n \geq 1$, then there exists a connected simple n-vertex graph with l vertices of even degree and m*

vertices of odd degree if and only if m is even, except for $(l, m, n) = (2, 0, 2)$. Since every graph has an even number of vertices of odd degree, and the only simple connected graph with two vertices has both degrees odd, the condition is necessary.

To prove sufficiency, we construct such a graph G. If $m = 0$, let $G = C_l$ (except $G = K_1$ if $l = 1$). For $m > 0$, we can begin with $K_{1,m-1}$, which has m vertices of odd degree, and then add a path of length l beyond one of the leaves. (Illustration shows $l = 3$, $m = 4$.)

Alternatively, start with a cycle of length l, and add m vertices of degree one with a common neighbor on the cycle. That vertex of the cycle has even degree because m is even. Many other constructions also work. It is also possible to prove sufficiency by induction on n for $n \geq 3$, but this approach is longer and harder to get right than an explicit general construction.



1.3.11. *If C is a closed walk in a simple graph G, then the subgraph consisting of the edges appearing an odd number of times in C is an even graph.* Consider an arbitrary vertex $v \in V(G)$. Let S be the set of edges incident to v, and let $f(e)$ be the number of times an edge e is traversed by C. Each time C passes through v it enters and leaves. Therefore, $\sum_{e\in S} f(e)$ must be even, since it equals twice the number of times that C visits v. Hence there must an even number of odd contributions to the sum, which means there are an even number of edges incident to v that appear an odd number of times in C. Since we can start a closed walk at any of its vertices, this argument holds for every $v \in V(G)$.

1.3.12. *If every vertex of G has even degree, then G has no cut-edge.*

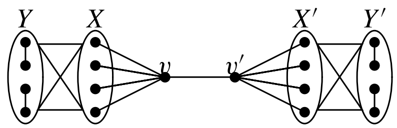
Proof 1 (contradiction). If G has a cut-edge, deleting it leaves two induced subgraphs whose degree sum is odd. This is impossible, since the degree sum in every graph is even.

Proof 2 (construction/extremality). For an edge uv, a maximal trail in $G - uv$ starting at u can only end at v, since whenever we reach a vertex we have use an odd number of edges there. Hence a maximal such trail is a (u, v)-trail. Every (u, v)-trail is a (u, v)-walk and contains a (u, v)-path. Hence there is still a (u, v)-path after deletion of uv, so uv is not a cut-edge.

Proof 3 (prior results). Let G be an even graph. By Proposition 1.2.27, G decomposes into cycles. By the meaning of "decomposition", every edge of G is in a cycle. By Theorem 1.2.14, every edge in a cycle is not a cut-edge. Hence every edge of G is not a cut-edge.

For $k \in \mathbb{N}$, some $(2k+1)$-regular simple graph has a cut-edge.

Construction 1. Let H, H' be copies of $K_{2k,2k}$ with partite sets X, Y for H and X', Y' for H'. Add an isolated edge vv' disjoint from these sets. To $H + H' + vv'$, add edges from v to all of X and from v' to all of X', and add k disjoint edges within Y and k disjoint edges within Y'. The resulting graph G_k is $(2k+1)$-regular with $8k+2$ vertices and has vv' as a cut-edge. Below we sketch G_2; the graph G_1 is the graph in Example 1.3.26.



Construction 2a (inductive). Let G_1 be the graph at the end of Example 1.3.26 (or in Construction 1). This graph is 3-regular with 10 vertices and cut-edge xy; note that $10 = 4 \cdot 1 + 6$. From a $(2k-1)$-regular graph G_{k-1} with $4k+2$ vertices such that $G_{k-1} - xy$ has two components of order $2k+1$, we form G_k. Add two vertices for each component of $G_{k-1} - xy$, adjacent to all the vertices of that component. This adds degree two to each old vertex, gives degree $2k+1$ to each new vertex, and leaves xy as a cut-edge. The result is a $(2k+1)$-regular graph G_k of order $4k+6$ with cut-edge xy.

Construction 2b (explicit). Form H_k from K_{2k+2} by removing k pairwise disjoint edges and adding one vertex that is adjacent to all vertices that lost an incident edge. Now H_k has $2k+2$ vertices of degree $2k+1$ and one of degree $2k$. Form G_k by taking two disjoint copies of H_k and adding an edge joining the vertices of degree $2k$. The graphs produced in Constructions 2a and 2b are identical.

1.3.13. *Meeting on a mountain range.* A **mountain range** is a polygonal curve from $(a, 0)$ to $(b, 0)$ in the upper half-plane; we start A and B at opposite endpoints. Let P be a highest peak; A and B will meet there. Let the segments from P to $(a, 0)$ be $x_1, \ldots, x_r$, and let the segments from P to $(b, 0)$ be $y_1, \ldots, y_s$. We define a graph to describe the positions; when A is on x_i and B is on y_j, the corresponding vertex is (i, j). We start at the vertex (r, s) and must reach $(1, 1)$. We introduce edges for the possible transitions. We can move from (i, j) to $(i, j+1)$ if the common endpoint of y_j and y_{j+1} has height between the heights of the endpoints of x_i. Similarly, (i, j) is adjacent to $(i+1, j)$ if the common endpoint of x_i and x_{i+1} has height between the heights of the endpoints of x_j. To avoid triviality, we may assume that $r + s > 2$.

We prove that (r, s) and $(1, 1)$ are the only vertices of odd degree in G. This suffices, because every graph has an even number of vertices of odd degree, which implies that (r, s) and $(1, 1)$ are in the same component, connected by a path.

The possible neighbors of (i, j) are the pairs obtained by changing i or j by 1. Let X and Y be the intervals of heights attained by x_i and y_j, and let $I = X \cap Y$. If the high end of I is the high end of exactly one of X and Y, then exactly one neighboring vertex can be reached by moving past the end of the corresponding segment. If it is the high end of both, then usually one or three neighboring vertices can be reached, the latter when both segments reach "peaks" at their high ends. However, if $(i, j) = (1, 1)$, then the high end of both segments is P and there is no neighbor of this type. Similarly, the low end of I generates one or three neighbors, except that when $(i, j) = (r, s)$ there is no neighbor of this type.

No neighbor of (i, j) is generated from both the low end and the high end of I. Since the contributions from the high and low end of I to the degree of (i, j) are both odd, each degree is even, except for (r, s) and $(1, 1)$, where exactly one of the contributions is odd.

1.3.14. *Every simple graph with at least two vertices has two vertices of equal degree.* The degree of a vertex in an n-vertex simple graph is in $\{0, \ldots, n-1\}$. These are n distinct values, so if no two are equal then all appear. However, a graph cannot have both an isolated vertex and a vertex adjacent to all others.

This does not hold for graphs allowing loops. In the 2-vertex graph with one loop edge and one non-loop edge, the vertex degrees are 1 and 3.

This does not hold for loopless graphs. In the 3-vertex loopless graph with pairs having multiplicity 0, 1, 2, the vertex degrees are 1, 3, 2.

1.3.15. *Smallest k-regular graphs.* A simple k-regular graph has at least $k+1$ vertices, so K_{k+1} is the smallest. This is the only isomorphism class of k-regular graphs with $k+1$ vertices. With $k+2$ vertices, the complement of a k-regular graph must be 1-regular. There is one such class when k is even ($(k+2)/2$ isolated edges), none when k is odd. (Two graphs are isomorphic if and only if their complements are isomorphic.)

With $k+3$ vertices, the complement is 2-regular. For $k \geq 3$, there are distinct choices for such a graph: a $(k+3)$-cycle or the disjoint union of a 3-cycle and a k-cycle. Since these two 2-regular graphs are nonisomorphic, their complements are nonisomorphic k-regular graphs with $k+3$ vertices.

1.3.16. *For $k \geq 2$ and $g \geq 2$, there exists a k-regular graph with girth g.* We use strong induction on g. For $g = 2$, take the graph consisting of two vertices and k edges joining them.

For the induction step, consider $g > 2$. Here we use induction on k. For $k = 2$, a cycle of length g suffices. For $k > 2$, the induction hypothesis provides a $(k-1)$-regular graph H with girth g. Since $\lceil g/2 \rceil < g$, the global induction hypothesis also provides a graph G with girth $\lceil g/2 \rceil$ that is $n(H)$-regular. Replace each vertex v in G with a copy of H; each vertex in the copy of H is made incident to one of the edges incident to v in G.

Each vertex in the resulting graph inherits $k-1$ incident edges from H and one from G, so the graph is k-regular. It has cycles of length g in copies of H. A cycle C in G is confined to a single copy of H or visits more than one such copy. In the first case, its length is at least g, since H has girth g. In the second case, the copies of H that C visits correspond to a cycle in G, so C visits at least $\lceil g/2 \rceil$ such copies. For each copy, C must enter on one edge and then move to another vertex before leaving, since the copy is entered by only one edge at each vertex. Hence the length of such a cycle is at least $2\lceil g/2 \rceil$.

1.3.17. *Deleting a vertex of maximum degree cannot increase the average degree, but deleting a vertex of minimum degree can reduce the average degree.* Deleting any vertex of a nontrivial regular graph reduces the average degree, which proves the second claim. For the first claim, suppose that G has n vertices and m edges, and let a and a' be the average degrees of G and $G-x$, respectively. Since $G-x$ has $m-d(x)$ edges and degree sum $2m-2d(x)$, we have $a' = \frac{na-2d(x)}{n-1} \le \frac{(n-2)a}{n-1} < a$ if $d(x) \ge a > 0$. Hence deleting a vertex of maximum degree in nontrivial graph reduces the average degree and cannot increase it.

1.3.18. *If $k \ge 2$, then a k-regular bipartite graph has no cut-edge.* Since components of k-regular graphs are k-regular, it suffices to consider a connected k-regular X,Y-bigraph. Let uv be a cut-edge, and let G and H be the components formed by deleting uv. Let $m = |V(G) \cap X|$ and $n = |V(G) \cap Y|$. By symmetry, we may assume that $u \in V(G) \cap Y$ and $v \in V(H) \cap X$.

We count the edges of G. The degree of each vertex of G in X is k, so G has mk edges. The degree of each vertex of G in Y is k except for $d_G(u) = k-1$, so G has $nk-1$ edges. Hence $m_k = nk-1$, which is impossible because one side is divisible by k and the other is not. The proof doesn't work if $k=1$, and the claim is false then.

If vertex degrees k and $k+1$ are allowed, then a cut-edge may exist. Consider the example of $2K_{k,k}$ plus one edge joining the two components.

1.3.19. *A claw-free simple graph with maximum degree at least 5 has a 4-cycle.* Consider five edges incident to a vertex v of maximum degree in such a graph G. Since G has no induced claw, the neighbors of v must induce at least three edges. Since these three edges have six endpoints among the five neighbors of v, two of them must be incident, say xy and yz. Adding the edges xv and zv to these two completes a 4-cycle.

There are arbitrarily large 4-regular claw-free graphs with no 4-cycles.

Consider a vertex v in such a graph G. Since v has degree 4 and is not the center of an induced claw and does not lie on a 4-cycle, the subgraph induced by v and its neighbors consists of two edge-disjoint triangles sharing v (a bowtie). Since this happens at each vertex, G consists of pairwise edge-disjoint triangles, with each vertex lying in two of them. Hence each triangle has three neighboring triangles. Furthermore, two triangles that neighbor a given triangle in this way cannot neighbor each other; that would create a 4-cycle in the graph.

Define a graph H with one vertex for each triangle in G; let vertices be adjacent in H if the corresponding triangles share a vertex in G. Now H is a 3-regular graph with no 3-cycles; a 3-cycle in H would yield a 4-cycle in G using two edges from one of the corresponding triangles. Also H must have no 4-cycles, because a 4-cycle in G could be built using one edge from each of the four triangles corresponding to the vertices of a 4-cycle in H. Note that $e(G) = 2n(G)$ and $n(H) = e(G)/3 = 2n(G)/3$.

On the other hand, given any 3-regular graph H with girth at least 5, reversing the construction yields G with the desired properties and $3n(H)/2$ vertices. Hence it suffices to show that there are arbitrarily large 3-regular graphs with girth at least 5. Disconnected such examples can be formed by taking many copies of the Petersen graph as components. The graph G is connected if and only if H is connected. Connected instances of H can be obtained from multiple copies of the Petersen graph by applying 2-switches (Definition 1.3.32).

Alternatively, arbitrarily large connected examples can be constructed by taking two odd cycles (say length $2m+1$) and joining the ith vertex on the first cycle to the $2i$th vertex (modulo $2m+1$) on the second cycle (this generalizes the Petersen graph). We have constructed a connected 3-regular graph. Since we add disjoint edges between the cycles, there is no triangle. A 4-cycle would have to alternate edges between the two odd cycles with one edge of each, but the neighbors of adjacent vertices on the first cycle are two apart on the second cycle.

1.3.20. *K_n has $(n-1)!/2$ cycles of length n, and $K_{n,n}$ has $n!(n-1)!/2$ cycles of length $2n$.* Each cycle in K_n is a listing of the vertices. These can be listed in $n!$ orders, but we obtain the same subgraph no matter where we start the cycle and no matter which direction we follow, so each cycle is listed $2n$ times. In $K_{n,n}$, we can list the vertices in order on a cycle (alternating between the partite sets), in $2(n!)^2$ ways, but by the same reasoning each cycle appears $(2n) \cdot 2$ times.

1.3.21. *$K_{m,n}$ has $6\binom{m}{3}\binom{n}{3}$ 6-cycles.* To extend an edge in $K_{m,n}$ to a 6-cycle, we choose two more vertices from each side to be visited in order as we follow the cycle. Hence each edge in $K_{n,n}$ appears in $(m-1)(n-1)(m-2)(n-2)$

6-cycles. Since each 6-cycle contains 6 edges, we conclude that $K_{n,n}$ has $mn(m-1)(n-1)(m-2)(n-2)/6$ 6-cycles.

Alternatively, each 6-cycle uses three vertices from each partite set, which we can choose in $\binom{m}{3}\binom{n}{3}$ ways. Each such choice of vertices induces a copy of $K_{3,3}$ with 9 edges. There are $3! = 6$ ways to pick three disjoint edges to be omitted by a 6-cycle, so each $K_{3,3}$ contains 6 6-cycles.

1.3.22. *Odd girth and minimum degree in nonbipartite triangle-free n-vertex graphs.* Let $k = \delta(G)$, and let l be the minimum length of an odd cycle in G. Let C be a cycle of length l in G.

a) Every vertex not in $V(C)$ has at most two neighbors in $V(C)$. It suffices to show that any two neighbors of such a vertex v on C must have distance 2 on C, since having three neighbors would then require $l = 6$.

Since G is triangle-free, v does not have consecutive neighbors on C. If v has neighbors x and y on C separated by distance more than 2 on C, then the detour through v can replace the x,y-path of even length on C to form a shorter odd cycle.

b) $n \ge kl/2$ (and thus $l \le 2n/k$). Since C is a shortest odd cycle, it has no chords (it is an induced cycle). Since $\delta(G) = k$, each vertex of C thus has at least $k-2$ edges to vertices outside C. However, each vertex outside C has at most two neighbors on C. Letting m be the number of edges from $V(C)$ to $V(G) - V(C)$, we thus have $l(k-2) \le m \le 2(n-l)$. Simplifying the inequality yields $n \ge kl/2$.

c) The inequality of part (b) is sharp when k is even. Form G from the cycle C_l by replacing each vertex of C_l with an independent set of size $k/2$ such that two vertices are adjacent if and only if the vertices they replaced were adjacent. Each vertex is now adjacent to the vertices arising from the two neighboring classes, so G is k-regular and has $lk/2$ vertices. Deleting the copies of any one vertex of C_l leaves a bipartite graph, since the partite sets can be labeled alternately around the classes arising from the rest of C_l. Hence every odd cycle uses a copy of each vertex of C_l and has length at least l, and taking one vertex from each class forms such a cycle.

1.3.23. *Equivalent definitions of the k-dimensional cube.* In the direct definition of Q_k, the vertices are the binary k-tuples, with edges consisting of pairs differing in one place. The inductive definition gives the same graph. For $k = 0$ both definitions specify K_1. For the induction step, suppose $k \ge 1$. The inductive definition uses two copies of Q_{k-1}, which by the induction hypothesis is the "1-place difference" graph of the binary $(k-1)$-tuples. If we append 0 to the $(k-1)$-tuples in one copy of Q_{k-1} and 1 to the $(k-1)$-tuples in the other copy, then within each set we still have edges between the labels differing in exactly one place. The inductive construction now adds edges consisting of corresponding vertices in the two copies. This is

also what the direction definition does, since k-tuples chosen from the two copies differ in the last position and therefore differ in exactly one position if and only if they are the same in all other positions.

$e(Q_k) = k2^{k-1}$. By the inductive definition, $e(Q_k) = 2e(Q_{k-1}) + 2^{k-1}$ for $k \ge 1$, with $e(Q_0) = 0$. Thus the inductive step for a proof of the formula is $e(Q_k) = 2(k-1)2^{k-2} + 2^{k-1} = k_2^{k-1}$.

1.3.24. *$K_{2,3}$ is the smallest simple bipartite graph that is not a subgraph of the k-dimensional cube for any k.* Suppose the vectors x, y, a, b, c are the vertices of a copy of $K_{2,3}$ in Q_k. Any one of a, b, c differs from x in exactly one coordinate and from y in another (it can't be the same coordinate, because then $x = y$). This implies that x and y differ in two coordinate i, j. Paths from x to y in two steps can be formed by changing i and then j or changing j and then i; these are the only ways. In a cube two vertices have at most two common neighbors. Hence $K_{2,3}$ is forbidden. Any bipartite graph with fewer vertices or edges is contained in $K_{2,3} - e$ or $K_{1,5}$, but $K_{2,3} - e$ is a subgraph of Q_3, and $K_{1,5}$ is a subgraph of Q_5, so $K_{2,3}$ is the smallest forbidden subgraph.

1.3.25. *Every cycle of length $2r$ in a hypercube belongs to a subcube of dimension at most r, uniquely if $r \le 3$.* Let C be a cycle of length $2r$ in Q_k; $V(C)$ is a collection of binary vectors of length k. Let S be the set of coordinates that change at some step while traversing the vectors in $V(C)$. In order to return to the first vector, each position must flip between 0 and 1 an even number of times. Thus traversing C changes each coordinate in S at least twice, but only one coordinate changes with each edge. Hence $2|S| \le 2r$, or $|S| \le r$. Outside the coordinates of S, the vectors of $V(C)$ all agree. Hence $V(C)$ is contained in a $|S|$-dimensional subcube.

As argued above, at most two coordinates vary among the vertices of a 4-cycle; at least two coordinates vary, because otherwise there are not enough vectors available to have four distinct vertices. By the same reasoning, exactly three three coordinates vary among the vertices of any 6-cycle; we cannot find six vertices in a 2-dimensional subcube. Thus the r-dimensional subcube containing a particular cycle is unique when $r \le 3$.

Some 8-cycles are contained in 3-dimensional subcubes, such as $000x$, $001x$, $011x$, $010x$, $110x$, $111x$, $101x$, $100x$, where x is a fixed vector of length $n-3$. Such an 8-cycle is contained in $n-3$ 4-dimensional subcubes, obtained by letting some position in x vary.

1.3.26. *A 3-dimensional cube contains 16 6-cycles, and the k-dimensional cube Q_k contains $16\binom{k}{3}2^{k-3}$ 6-cycles.* If we show that every 6-cycle appears in exactly one 3-dimensional subcube, then multiplying the number of 3-dimensional subcubes by the number of 6-cycles in each subcube counts each 6-cycle exactly once.

For any set S of vertices not contained in a 3-dimensional subcube, there must be four coordinates in the corresponding k-tuples that are not constant within S. A cycle through S makes changes in four coordinates. Completing the cycle requires returning to the original vertex, so any coordinate that changes must change back. Hence at least eight changes are needed, and each edge changes exactly one coordinate. The cycle has length at least 8; hence 6-cycles are contained in 3-dimensional subcubes.

Furthermore, there are only four vertices possible when $k-2$ coordinates are fixed, so every 6-cycle involves changes in three coordinates. Hence the only 3-dimensional subcube containing the 6-cycle is the one that varies in the same three coordinates as the 6-cycle.

By Example 1.3.8, there are $\binom{k}{3}2^{k-3}$ 3-dimensional subcubes, so it remains only to show that Q_3 has 16 cycles of length 6. We group them by the two omitted vertices. The two omitted vertices may differ in 1, 2, or 3 coordinates. If they differ in one place (they are adjacent), then deleting them leaves a 6-cycle plus one edge joining a pair of opposite vertices. Since Q_3 has 12 edges, there are 12 6-cycles of this type. Deleting two complementary vertices (differing in every coordinate) leaves only a 6-cycle. Since Q_3 has four such pairs, there are four such 6-cycles. The remaining pairs differ in two positions. Deleting such a pair leaves a 4-cycle plus two pendant edges, containing no 6-cycle. This considers all choices for the omitted vertices, so the number of 6-cycles in Q_3 is $12+4$.

1.3.27. *Properties of the "middle-levels" graph.* Let G be the subgraph of Q_{2k+1} induced by vertices in which the numbers of 1s and 0s differs by 1. These are the $(2k+1)$-tuples of weight k and weight $k+1$, where **weight** denotes the number of 1s.

Each vertex of weight k has $k+1$ neighbors of weight $k+1$, and each vertex of weight $k+1$ has $k+1$ neighbors of weight k. There are $\binom{2k+1}{k}$ vertices of each weight. Counting edges by the Degree-Sum Formula,

$$e(G) = (k+1)\frac{n(G)}{2} = (k+1)\binom{2k+1}{k+1} = (2k+1)\binom{2k}{k}.$$

The graph is bipartite and has no odd cycle. The 1s in two vertices of weight k must be covered by the 1s of any common neighbor of weight $k+1$. Since the union of distinct k-sets has size at least $k+1$, there can only be one common neighbor, and hence G has no 4-cycle. On the other hand, G does have a 6-cycle. Given any arbitary fixed vector of weight $k-1$ for the last $2k-2$ positions, we can form a cycle of length six by using 110, 100, 101, 001, 011, 010 successively in the first three positions.

1.3.28. *Alternative description of even-dimensional hypercubes.* The simple graph Q_k' has vertex set $\{0,1\}^k$, with $u \leftrightarrow v$ if and only if u and v *agree* in exactly one coordinate. Let the *odd vertices* be the vertices whose name has an odd number of 1s; the rest are *even vertices*.

When k is even, $Q_k' \cong Q_k$. To show this, rename all odd vertices by changing 1s into 0s and 0s into 1s. Since k is even, the resulting labels are still odd. Since k is even, every edge in Q_k' joins an even vertex to an odd vertex. Under the new naming, it joins the even vertex to an odd vertex that differs from it in one coordinate. Hence the adjacency relation becomes precisely the adjacency relation of Q_k.

When k is odd, $Q_k' \ncong Q_k$, because Q_k' contains an odd cycle and hence is not bipartite. Starting from one vertex, form a closed walk by successively following k edges where each coordinate is the coordinate of agreement along exactly one of these edges. Hence each coordinate changes exactly $k-1$ times and therefore ends with the value it had at the start. Thus this is a closed walk of odd length and contains an odd cycle.

1.3.29. *Automorphisms of Q_k.*

a) A subgraph H of Q_k is isomorphic to Q_l if and only if it is the subgraph induced by a set of vertices agreeing in some set of $k-l$ coordinates. Let f be an isomorphism from H to Q_l, and let v be the vertex mapped to the vertex $\mathbf{0}$ of Q_l whose coordinates are all 0. Let $u_1,\dots,u_l$ be the neighbors of v in H mapped to neighbors of $\mathbf{0}$ in Q_l by f. Each u_i differs from v in one coordinate; let S be the set of l coordinates where these vertices differ from v. It suffices to show that vertices of H differ from v only on the coordinates of S. This is immediate for $l \le 1$.

For $l \ge 2$, we prove that each vertex mapped by f to a vertex of Q_l having weight j differs from v in j positions of S, by induction on j. Let x be a vertex mapped to a vertex of weight j in Q_l. For $j \le 1$, we have already argued that x differs from v in j positions of S. For $j \ge 2$, let y and z be two neighbors of x whose images under f have weight $j-1$ in Q_l. By the induction hypothesis, y and z differ from v in j positions of S. Since $f(y)$ and $f(z)$ differ in two places, they have two common neighbors in Q_l, which are x and another vertex w. Since w has weight $j-2$, the induction hypothesis yields that w differs from v in $j-1$ positions of S. Since the images of x,y,z,w induce a 4-cycle in Q_l, also x,y,z,w induce a 4-cycle in H. The only 4-cycle in Q_k that contains all of y,z,w adds the vertex that differs from v in the $j-2$ positions of S where w differs, plus the two positions where y and z differ from w. This completes the proof that x has the desired property.

b) The k-dimensional cube Q_k has exactly $2^k k!$ automorphisms. (Part (a) is unnecessary.) Form automorphisms of Q_k by choosing a subset of the k coordinates in which to complement 0 and 1 and, independently, a permutation of the k coordinates. There are $2^k k!$ such automorphisms.

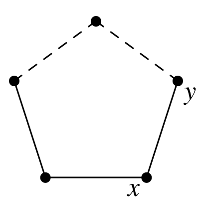
We prove that every automorphism has this form. Let $\mathbf{0}$ be the all-0 vertex. Let f be the inverse of an automorphism, and let v be the vertex mapped to $\mathbf{0}$ by f. The neighbors of v must be mapped to the neighbors of $\mathbf{0}$. If these choices completely determine f, then f complements the coordinates where v is nonzero, and the correspondence between the neighbors of $\mathbf{0}$ and the neighbors of v determines the permutation of the coordinates that expresses f as one of the maps listed above.

Suppose that x differs from v in coordinates $r_1, \ldots, r_j$. Let $u_1, \ldots, u_j$ be the neighbors of v differing from v in these coordinates. We prove that $f(x)$ is the k-tuple of weight j having 1 in the coordinates where $f(u_1), \ldots, f(u_j)$ have 1. We use induction on j.

For $j \le 1$, the claim follows by the definition of $u_1, \ldots, u_j$. For $j \ge 2$, let y and z be two neighbors of x that differ from v in $j-1$ coordinates. Let w be the common neighbor of y and z that differs from v in $j-2$ coordinates. By the induction hypothesis, $f(y)$ and $f(z)$ have weight $j-1$ (in the appropriate positions), and $f(w)$ has weight $j-1$. Since $f(x)$ must be the other common neighbor of $f(y)$ and $f(z)$, it has weight j, with 1s in the desired positions.

1.3.30. *The Petersen graph has twelve 5-cycles.* Let G be the Petersen graph. We show first that each edge of G appears in exactly four 5-cycles. For each edge $e = xy$ in G, there are two other edges incident to x and two others incident to y. Since G has no 3-cycles, we can thus extend xy at both ends to form a 4-vertex path in four ways. Since G has no 4-cycle, the endpoints of each such path are nonadjacent. By Proposition 1.1.38, there is exactly one vertex to add to such a path to complete a 5-cycle. Thus e is in exactly four 5-cycles.

When we sum this count over the 15 edges of G, we have counted 60 5-cycles. However, each 5-cycle has been counted five times—once for each of its edges. Thus the total number of 5-cycles in G is $60/5 = 12$.



1.3.31. *Combinatorial proofs with graphs.*

a) For $0 \le k \le n$, $\binom{n}{2} = \binom{k}{2} + k(n-k) + \binom{n-k}{2}$. Consider the complete graph K_n, which has $\binom{n}{2}$ edges. If we partition the vertices of K_n into a k-set and an $(n-k)$-set, then we can count the edges as those within one block of the partition and those choosing a vertex from each. Hence the total number of edges is $\binom{k}{2} + \binom{n-k}{2} + k(n-k)$.

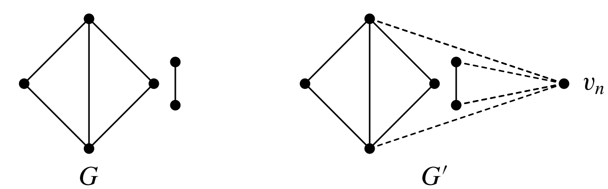
b) If $\sum n_i = n$, then $\sum \binom{n_i}{2} \le \binom{n}{2}$. Again consider the edges of K_n, and partition the vertices into sets with n_i being the size of the ith set. The left side of the inequality counts the edges in K_n having both ends in the same S_i, which is at most all of $E(K_n)$.

1.3.32. *For $n \ge 1$, there are $2^{\binom{n-1}{2}}$ simple even graphs with a fixed vertex set of size n.* Let A be the set of simple even graphs with vertex set $v_1, \ldots, v_n$. Since $2^{\binom{n-1}{2}}$ is the size of the set B of simple graphs with vertex set $v_1, \ldots, v_{n-1}$, we establish a bijection from A to B.

Given a graph in A, we obtain a graph in B by deleting v_n. To show that each graph in B arises exactly once, consider a graph $G \in B$. We form a new graph G' by adding a vertex v_n and making it adjacent to each vertex with odd degree in G, as illustrated below.

The vertices with odd degree in G have even degree in G'. Also, v_n itself has even degree because the number of vertices of odd degree in G is even. Thus $G' \in A$. Furthermore, G is the graph obtained from G' by deleting v_n, and every simple even graph in which deleting v_n yields G must have v_n adjacent to the same vertices as in G'.

Since there is a bijection from A to B, the two sets have the same size.



1.3.33. *Triangle-free graphs in which every two nonadjacent vertices have exactly two common neighbors.*

$n(G) = 1 + \binom{k+1}{2}$, where k is the degree of a vertex x in G. For every pair of neighbors of x, there is exactly one nonneighbor of x that they have as a common neighbor. Conversely, every nonneighbor of x has exactly one pair of neighbors of x in its neighborhood, because these are its common neighbors with x. This establishes a bijective correspondence between the pairs in $N(x)$ and the nonneighbors of x. Counting x, $N(x)$, and $\overline{N}(x)$, we have $n(G) = 1 + k + \binom{k}{2} = 1 + \binom{k+1}{2}$. Since this argument holds for every $x \in V(G)$, we conclude that G is k-regular.

Comment: Such graphs exist only for isolated values of k. Unique graphs exist for $k = 1, 2, 5$. Viewing the vertices as x, $N(x) = [k]$, and $\overline{N}(x) = \binom{[k]}{2}$, we have i adjacent to the pair $\{j, k\}$ if and only if $i \in \{j, k\}$. The lack of triangles guarantees that only disjoint pairs in $\binom{[k]}{2}$ can be adjacent, but each pair in $\binom{[k]}{2}$ must have exactly $k-2$ neighbors in $\binom{[k]}{2}$. For $k=5$, this implies that $\overline{N}(x)$ induces the 3-regular disjointness graph of $\binom{[5]}{2}$, which is the Petersen graph. Since the Petersen graph has girth 5 and diameter 2, each intersecting pair has exactly one common neighbor in $\overline{N}(x)$ in addition to its one common neighbor in $N(x)$, so this graph has the desired properties.

Numerical conditions eliminate $k \equiv 3 \pmod 4$, because G would be regular of odd degree with an odd number of vertices. There are stronger necessary conditions. After $k=5$, the next possibility is $k=10$, then 26, 37, 82, etc. A realization for $k=10$ is known to exist, but in general the set of realizable values is not known.

1.3.34. *If G is a kite-free simple n-vertex graph such that every pair of nonadjacent vertices has exactly two common neighbors, then G is regular.* Since nonadjacent vertices have common neighbors, G is connected. Hence it suffices to prove that adjacent vertices x and y have the same degree. To prove this, we establish a bijection from A to B, where $A = N(x) - N(y)$ and $B = N(y) - N(x)$.

Consider $u \in A$. Since $u \nleftrightarrow y$, there exists $v \in N(u) \cap N(y)$ with $v \neq x$. Since G is kite-free, $v \nleftrightarrow x$, so $v \in B$. Since x and v have common neighbors y and u, the vertex v cannot be generated in this way from another vertex of A. Hence we have defined an injection from A to B. Interchanging the roles of y and x yields an injection from B to A. Since these sets are finite, the injections are bijections, and $d(x) = d(y)$.

1.3.35. *If every induced k-vertex subgraph of a simple n-vertex graph G has the same number of edges, where $1 < k < n-1$, then G is a complete graph or an empty graph.*

a) If $l \geq k$ and G' is a graph on l vertices in which every induced k-vertex subgraph has m edges, then $e(G') = m\binom{l}{k}/\binom{l-2}{k-2}$. Counting the edges in all the k-vertex subgraphs of G' yields $m\binom{l}{k}$, but each edge appears in $\binom{l-2}{k-2}$ of these subgraphs, once for each k-set of vertices containing it. (Both sides of $\binom{l-2}{k-2}e(G') = m\binom{l}{k}$ count the ways to pick an edge of G' and a k-set of vertices in G' containing that edge. On the right, we pick the set first; on the left, we pick the edge first.)

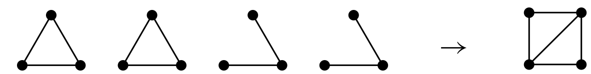
b) Under the stated conditions, $G = K_n$ or $G = \overline{K}_n$. Given vertices u and v, let A and B be the sets of edges incident to u and v, respectively. The set of edges with endpoints u and v is $A \cap B$. We compute

$$|A \cap B| = e(G) - \left|\overline{A \cap B}\right| = e(G) - \left|\overline{A} \cup \overline{B}\right| = e(G) - \left|\overline{A}\right| - \left|\overline{B}\right| + \left|\overline{A} \cap \overline{B}\right|.$$

In this formula, $\overline{A}$ and $\overline{B}$ are the edge sets of induced subgraphs of order $n-1$, and $\overline{A} \cap \overline{B}$ is the edge set of an induced subgraph of order $n-2$. By part (a), the sizes of these sets do not depend on the choice of u and v.

1.3.36. *The unique reconstruction of the graph with vertex-deleted subgraphs below is the kite.*

Proof 1. A vertex added to the first triangle may be joined to 0,1,2, or 3 of its vertices. We eliminate 0 and 1 because no vertex-deleted subgraph has an isolated vertex. We eliminate 3 because every vertex-deleted subgraph of K_4 is a triangle. Joining it to 2 yields the kite.



Proof 2. The graph G must have four vertices, and by Proposition 1.3.11 it has five edges. The only such simple graph is the kite.

1.3.37. *Retrieving a regular graph.* Suppose that H is a graph formed by deleting a vertex from a regular graph G. We have H, so we know $n(G) = n(H) + 1$, but we don't know the vertex degrees in G. If G is d-regular, then G has $dn(G)/2$ edges, and H has $dn(G)/2 - d$ edges. Thus $d = 2e(H)/(n(G)-2)$. Having determined d, we add one vertex w to H and add $d - d_H(v)$ edges from w to v for each $v \in V(H)$.

1.3.38. *A graph with at least 3 vertices is connected if and only if at least two of the subgraphs obtained by deleting one vertex are connected.* The endpoints of a maximal path are not cut-vertices. If G is connected, then the subgraphs obtained by deleted such vertices are connected, and there are at least of these.

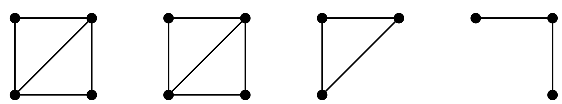
Conversely, suppose that at least two vertex-deleted subgraphs are connected. If $G - v$ is connected, then G is connected unless v is an isolated vertex. If v is an isolated vertex, then all the other subgraphs obtained by deleting one vertex are disconnected. Hence v cannot be isolated, and G is connected.

1.3.39. *Disconnected graphs are reconstructible.* First we show that G is connected if and only if it has at least two connected vertex-deleted subgraphs. Necessity holds, because the endpoints of a maximal path cannot be cut-vertices. If G is disconnected, then $G - v$ is disconnected unless v is an isolated vertex (degree 0) in G and $G - v$ is connected. This happens for at most one vertex in G.

After determining that G is disconnected, we obtain which disconnected graph it is from its vertex-deleted subgraphs. We aim to identify a connected graph M that is a component of G and a vds in the deck that arises by deleting a specified vertex u of M. Replacing $M - u$ by M in that subgraph will reconstruct G.

Among all components of all graphs in the deck, let M be one with maximum order. Since every component H of a potential reconstruction G appears as a component of some $G - v$, M cannot belong to any larger component of G. Hence M is a component of G. Let L be a fixed connected subgraph of M obtained by deleting a leaf u of some spanning tree of M. Then L is a component of $G - u$. We want to reconstruct G by substituting M for L in $G - u$; we must identify $G - u$. There may be several isomorphic copies of $G - u$.

As in the disconnected graph G shown above, M may appear as a component of every vds $G - v$. However, since M cannot be created by a vertex deletion, a vds with the fewest copies of M must arise by deleting a vertex of M. Among these, we seek a subgraph with the most copies of L as components, because in addition to occurrences of L as a component of G, we obtain an additional copy if and only if the deleted vertex of M can play the role of u. This identifies $G - u$, and we obtain G by replacing one of its components isomorphic to L with a component isomorphic to M.



1.3.40. *Largest graphs of specified types.*

a) Largest n-vertex simple graph with an independent set of size a.

Proof 1. Since there are no edges within the independent set, such a graph has at most $\binom{n}{2} - \binom{a}{2}$ edges, which equals $\binom{n-a}{2} + (n-a)a$. This bound is achieved by the graph consisting of a copy H of K_{n-a}, an independent set S of size a, and edges joining each vertex of H to each vertex of S.

Proof 2. Each vertex of an independent set of size a has degree at most $n-a$. Each other vertex has degree at most $n-1$. Thus $\sum d(v) \le a(n-a) + (n-a)(n-1)$. By the Degree-Sum Formula, $e(G) \le (n-a)(n-1+a)/2$. This formula equals those above and is achieved by the same graph, since this graph achieves the bound for each vertex degree.

b) The maximum size of an n-vertex simple graph with k components is $\binom{n-k+1}{2}$. The graph consisting of K_{n-k+1} plus $k-1$ isolated vertices has k components and $\binom{n-k+1}{2}$ edges. We prove that other n-vertex graphs with k components don't have maximum size. Let G be such a graph.

If G has a component that is not complete, then adding edges to make it complete does not change the number of components. Hence we may assume that every component is complete.

If G has components with r and s vertices, where $r \ge s > 1$, then we move one vertex from the s-clique to the r-clique. This deletes $s-1$ edges and creates r edges, all incident to the moved vertex. The other edges remain the same, so we gain $r - s + 1$ edges, which is positive.

Thus the number of edges is maximized only when every component is a complete graph and only one component has more than one vertex.

c) The maximum number of edges in a disconnected simple n-vertex graph is $\binom{n-1}{2}$, with equality only for $K_1 + K_{n-1}$.

Proof 1 (using part (b)). The maximum over graphs with k components is $\binom{n-k+1}{2}$, which decreases as k increases. For disconnected graphs, $k \ge 2$. We maximize the number of edges when $k = 2$, obtaining $\binom{n-1}{2}$.

Proof 2 (direct argument). Given a disconnected simple graph G, let S be the vertex set of one component of G, and let $t = |S|$. Since no edges join S and $\overline{S}$, $e(G) \le \binom{n}{2} - t(n-t)$. This bound is weakest when $t(n-t)$ is smallest, which for $1 \le t \le n-1$ happens when $t \in \{1, n-1\}$. Thus always $e(G) \le \binom{n}{2} - 1(n-1) = \binom{n-1}{2}$, and equality holds when $G = K_1 + K_{n-1}$.

Proof 3 (induction on n). When $n = 2$, the only simple graph with $e(G) > \binom{1}{2} = 1$ is K_2, which is connected. For $n > 2$, suppose $e(G) > \binom{n-1}{2}$. If $\Delta(G) = n-1$, then G is connected. Otherwise, we may select v with $d(v) \le n-2$. Then $e(G-v) > \binom{n-1}{2} - n + 2 = \binom{n-2}{2}$. By the induction hypothesis, $G - v$ is connected. Since $e(G) > \binom{n-1}{2}$ and G is simple, we have $d(v) > 0$, so there is an edge from v to $G - v$, and G is also connected.

Proof 4 (complementation). If G is disconnected, then $\overline{G}$ is connected, so $e(\overline{G}) \ge n-1$ and $e(G) \le \binom{n}{2} - (n-1) = \binom{n-1}{2}$. In fact, $\overline{G}$ must contain a spanning complete bipartite subgraph, which is as small as $n-1$ edges only when $\overline{G} = K_{1,n-1}$ and $G = K_1 + K_{n-1}$.

1.3.41. *Every n-vertex simple graph with maximum degree $\lceil n/2 \rceil$ and minimum degree $\lfloor n/2 \rfloor - 1$ is connected.* Let x be a vertex of maximum degree. It suffices to show that every vertex not adjacent to x has a common neighbor with x. Choose $y \notin N(x)$. We have $|N(x)| = \lceil n/2 \rceil$ and $|N(y)| \ge \lfloor n/2 \rfloor - 1$. Since $y \nleftrightarrow x$, we have $N(x), N(y) \subseteq V(G) - \{x, y\}$. Thus

$$|N(x) \cap N(y)| = |N(x)| + |N(y)| - |N(x) \cup N(y)| \ge \lceil n/2 \rceil + \lfloor n/2 \rfloor - 1 - (n-2) = 1.$$

1.3.42. *Strongly independent sets.* If S is an independent set with no common neighbors in a graph G, then the vertices of S have pairwise-disjoint closed neighborhoods of size at least $\delta(G) + 1$. Thus there are at most $\lfloor n(G)/(\delta(G)+1) \rfloor$ of them. Equality is achievable for the 3-dimensional cube using $S = \{000, 111\}$.

Equality is not achievable when $G = Q_4$, since with 16 vertices and minimum degree 4 it requires three parwise-disjoint closed neighborhoods of size 5. If $v \in S$, then no vertex differing from v in at most two places is in S. Also, at most one vertex differing from v in at least three places is in

S, since such vertices differ from each other in at most two places. Thus only two disjoint closed neighborhoods can be found in Q_4.

1.3.43. *Every simple graph has a vertex whose neighbors have average degree as large as the overall average degree.* Let $t(w)$ be the average degree of the neighbors of w. In the sum $\sum_{w\in V(G)} t(w) = \sum_{w\in V(G)} \sum_{y\in N(w)} d(y)/d(w)$, we have the terms $d(u)/d(v)$ and $d(v)/d(u)$ for each edge uv. Since $x/y + y/x \ge 2$ whenever x, y are positive real numbers (this is equivalent to $(x-y)^2 \ge 0$), each such contribution is at least 2. Hence $\sum t(w) \ge \sum_{uv\in E(G)} \frac{d(u)}{d(v)} + \frac{d(v)}{d(u)} \ge 2e(G)$. Hence the average of the neighborhood average degrees is at least the average degree, and the pigeonhole principle yields the desired vertex.

It is possible that every average neighborhood degree exceeds the average degree. Let G be the graph with $2n$ vertices formed by adding a matching between a complete graph and an independent set. Since G has $\binom{n}{2} + n$ edges and $2n$ vertices, G has average degree $(n+1)/2$. For each vertex of the n-clique, the neighborhood average degree is $n - 1 + 1/n$. For each leaf, the neighborhood average degree is n.

1.3.44. *Subgraphs with large minimum degree.* Let G be a loopless graph with average degree a.

a) If $x \in V(G)$, then $G' = G - x$ has average degree at least a if and only if $d(x) \le a/2$. Let a' be the average degree of G', and let n be the order of G. Deleting x reduces the degree sum by $2d(x)$, so $(n-1)a' = na - 2d(x)$. Hence $(n-1)(a'-a) = a - 2d(x)$. For $n > 1$, this implies that $a' \ge a$ if and only if $d(x) \le a/2$.

Alternative presentation. The average degree of G is $2e(G)/n(G)$. Since G' has $e(G) - d(x)$ edges, the average degree is at least a if and only if $\frac{2[e(G)-d(x)]}{n(G)-1} \ge a$. Since $e(G) = n(G)a/2$, we can rewrite this as $n(G)a - 2d(x) = 2e(G) - 2d(x) \ge an(G) - a$. By canceling $n(G)a$, we find that the original inequality is equivalent to $d(x) \le a/2$.

b) If $a > 0$, then G has a subgraph with minimum degree greater than $a/2$. Iteratively delete vertices with degree at most half the current average degree, until no such vertex exists. By part (a), the average degree never decreases. Since G is finite, the procedure must terminate. It ends only by finding a subgraph where every vertex has degree greater than $a/2$.

c) The result of part (b) is best possible. To prove that no fraction of a larger than $\frac{1}{2}a$ can be guaranteed, let G_n be an n-vertex tree. We have $a(G_n) = 2(n-1)/n = 2 - 2/n$, but subgraphs of G_n have minimum degree at most 1. Given $\beta > \frac{1}{2}$, we can choose n large enough so that $1 \le \beta a(G_n)$.

1.3.45. *Bipartite subgraphs of the Petersen graph.*

a) Every edge of the Petersen graph is in four 5-cycles. In every 5-cycle through an edge e, the edge e is the middle edge of a 4-vertex path. Such

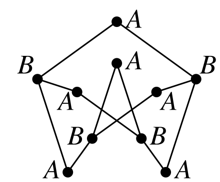
a path can be obtained in four ways, since each edge extends two ways at each endpoint. The neighbors at each endpoint of e are distinct and nonadjacent, since the girth is 5.

Since the endpoints of each such P_4 are nonadjacent, they have exactly one common neighbor. Thus each P_4 yields one 5-cycle, and each 5-cycle through e arises from such a P_4, so there are exactly four 5-cycles containing each edge.

b) The Petersen graph has twelve 5-cycles. Since there are 15 edges, summing the number of 5-cycles through each edge yields 60. Since each 5-cycle is counted five times in this total, the number of 5-cycles is 12.

c) The largest bipartite subgraph has twelve edges.

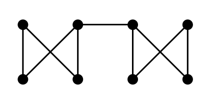
Proof 1 (breaking odd cycles). Each edge is in four 5-cycles, so we must delete at least 12/4 edges to break all 5-cycles. Hence we must delete at least three edges to have a bipartite subgraph. The illustration shows that deleting three is enough; the Petersen graph has a bipartite subgraph with 12 edges (see also the cover of the text).



Proof 2 (study of bipartite subgraphs). The Petersen graph G has an independent set of size 4, consisting of the vertices $\{ab, ac, ad, ae\}$ in the structural description. The 12 edges from these four vertices go to the other six vertices, so this is a bipartite subgraph with 12 edges.

Let X and Y be the partite sets of a bipartite subgraph H. If $|X| \le 4$, then $e(H) \le 12$, with equality only when X is an independent 4-set in G. Hence we need only consider the case $|X| = |Y| = 5$. To obtain $e(G) > 10$, some vertex $x \in X$ must have three neighbors in Y. The two nonneighbors of x in Y have common neighbors with x, and these must lie in $N(x)$, which is contained in Y. Hence $e(G[Y]) \ge 2$. Interchanging X and Y in the argument shows that also $e(G[X]) \ge 2$. Hence $e(H) \le 11$.

1.3.46. *When the algorithm of Theorem 1.4.2 is applied to a bipartite graph, it need not find the bipartite subgraph with the most edges.* For the bipartite graph below, the algorithm may reach the partition between the upper vertices and lower vertices.



This bipartite subgraph with eight edges has more than half of the edges at each vertex, and no further changes are made. However, the bipartite subgraph with the most edges is the full graph.

1.3.47. *Every nontrivial loopless graph G has a bipartite subgraph containing more than half its edges.* We use induction on $n(G)$. If $n(G) = 2$, then G consists of copies of a single edge and is bipartite. For $n(G) > 2$, choose $v \in V(G)$ that is not incident to all of $E(G)$ (at most two vertices can be incident to all of $E(G)$). Thus $e(G - v) > 0$. By the induction hypothesis, $G - v$ has a bipartite subgraph H containing more than $e(G)/2$ edges.

Let X, Y be a bipartition of H. If X contains at least half of $N_G(v)$, then add v to Y; otherwise add v to X. The augmented partition captures a bipartite subgraph of G having more than half of $E(G - v)$ and at least half of the remaining edges, so it has more than half of $E(G)$.

Comment. The statement can also be proved without induction. By Theorem 1.3.19, G has a bipartite subgraph H with at least $e(G)/2$ edges. By the proof of Theorem 1.3.19, equality holds only if $d_H(v) = d_G(v)/2$ for every $v \in V(G)$. Given an edge uv, each of u and v has exactly half its neighbors in its own partite set. Switching both to the opposite set will capture those edges while retaining the edge uv, so the new bipartite subgraph has more edges.

1.3.48. *No fraction of the edges larger than $1/2$ can be guaranteed for the largest bipartite subgraph.* If G_n is the complete graph K_{2n}, then $e(G_n) = \binom{2n}{2} = n(2n-1)$, and the largest bipartite subgraph is $K_{n,n}$, which has n^2 edges. Hence $\lim_{n\to\infty} f(G_n)/e(G_n) = \lim_{n\to\infty} \frac{n^2}{2n^2-n} = \frac{1}{2}$. For large enough n, the fraction of the edges in the largest bipartite subgraph is arbitrarily close to $1/2$. (In fact, in every graph the largest bipartite subgraph has **more** than half the edges.)

1.3.49. *Every loopless graph G has a spanning k-partite subgraph H such that $e(H) \ge (1 - 1/k)e(G)$.*

Proof 1 (local change). Begin with an arbitrary partition of $V(G)$ into k parts $V_1, \dots, V_k$, and consider the k-partite subgraph H containing all edges of G consisting of two vertices from distinct parts. Given a partition of $V(G)$, let $V(x)$ denote the part containing x. If in G some vertex x has more neighbors in V_j than in some other part, then shifting x to the other part increases the number of edges captured by the k-partite subgraph.

Since G has finitely many edges, this shifting process must terminate. It terminates when for each $x \in V(G)$ the number $|N(x) \cap V_i|$ is minimized by $V_i = V(x)$. Then $d_G(x) = \sum_i |N_G(x) \cap V_i| \ge k|N_G(x) \cap V(x)|$. We conclude that $|N_G(x) \cap V(x)| \le (1/k)d_G(x)$, and hence $d_H(x) \ge (1 - 1/k)d_G(x)$ for all $x \in V(G)$. By the degree-sum formula, $e(H) \ge (1 - 1/k)e(G)$.

Proof 2 (induction on n). We prove that when G is nontrivial, some such H has more than $(1 - 1/k)e(G)$ edges. This is true when $n = 2$. We procede by induction for $n > 2$. Choose a vertex $v \in V(G)$. By the induction hypothesis, $G - v$ has a spanning k-partite subgraph with more than $(1 - 1/k)e(G - v)$ edges. This subgraph partitions $V(G - v)$ into k partite sets. One of these sets contains at most $1/k$ neighbors of v. Add v to that set to obtain the desired k-partite subgraph H. Now $e(H) > (1 - 1/k)e(G - v) + (1 - 1/k)d_G(v) = (1 - 1/k)e(G)$.

1.3.50. For $n \ge 3$, the minimum number of edges in a connected n-vertex graph in which every edge belongs to a triangle is $\lceil 3(n-1)/2 \rceil$. To achieve the minimum, we need only consider simple graphs. Say that connected graphs with each edge in a triangle are *good* graphs. For $n = 3$, the only such graph is K_3, with three edges.

When n is odd, a construction with the claimed size consists of $(n-1)/2$ triangles sharing a common vertex. When n is even, add one vertex to the construction for $n - 1$ and make it adjacent to both endpoints of one edge.

For the lower bound, let G be a smallest n-vertex good graph. Since G has fewer than $3n/2$ edges (by the construction), G has a vertex v of degree 2. Let x and y be its neighbors. Since each edge belongs to a triangle, $x \leftrightarrow y$. If $n > 3$, then we form G' by deleting v and, if xy have no other neighbor, contracting xy. Every edge of G' belongs to a triangle that contained it in G. The change reduces the number of vertices by 1 or 2 and reduces the number of edges by at least $3/2$ times the reduction in the number of vertices. By the induction hypothesis, $e(G') \ge \lceil 3(n(G') - 1)/2 \rceil$, and hence the desired bound holds for G.

1.3.51. Let G be a simple n-vertex graph.

a) $e(G) = \frac{\sum_{v\in V(G)} e(G-v)}{n-2}$. If we count up all the edges in all the subgraphs obtained by deleting one vertex, then each edge of G is counted exactly $n-2$ times, because it shows up in the $n - 2$ subgraphs obtained by deleting a vertex other than its endpoints.

b) If $n \ge 4$ and G has more than $n^2/4$ edges, then G has a vertex whose deletion leaves a graph with more than $(n-1)^2/4$ edges. Since G has more than $n^2/4$ edges and $e(G)$ is an integer, we have $e(G) \ge (n^2 + 4)/4$ when n is even and $e(G) \ge (n^2 + 3)/4$ when n is odd (since $(2k + 1)^2 = 4k^2 + 4k + 1$, every square of an odd number is one more than a multiple of 4). Thus always we have $e(G) \ge (n^2 + 3)/4$.

By part (a), we have $\sum_{v\in V(G)} \frac{e(G-v)}{n-2} \ge (n^2 + 3)/4$. In the sum we have n terms. Since the largest number in a set is at least the average, there is a vertex v such that $\frac{e(G-v)}{n-2} \ge \frac{1}{n}\frac{n^2+3}{4}$. We rewrite this as

$$e(G - v) \ge \frac{(n^2 + 3)(n - 2)}{4n} = \frac{n^3 - 2n^2 + 3n - 6}{4n} = \frac{n^2 - 2n + 1}{4} + \frac{2n - 6}{4n}$$

When $n \ge 4$, the last term is positive, and we obtain the strict inequality $e(G-v) > (n-1)^2/4$.

c) Inductive proof that G contains a triangle if $e(G) > n^2/4$. We use induction on n. When $n \le 3$, they only simple graph with more than $n^2/4$ edges is when $n = 3$ and $G = K_3$, which indeed contains a triangle. For the induction step, consider $n \ge 4$, and let G be an n-vertex simple graph with more than $n^2/4$ vertices. By part (b), G has a subgraph $G - v$ with $n-1$ vertices and more than $(n-1)^2/4$ edges. By the induction hypothesis, $G - v$ therefore contains a triangle. This triangle appears also in G.

1.3.52. *$K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$ is the only n-vertex triangle-free graph of maximum size.* As in the proof of Mantel's result, let x be a vertex of maximum degree. Since $N(x)$ is an independent set, x and its non-neighbors capture all the edges, and we have $e(G) \le (n - \Delta(G))\Delta(G)$. If equality holds, then summing the degrees in $V(G) - N(x)$ counts each edge exactly once. This requires that $V(G) - N(x)$ also is an independent set, and hence G is bipartite. If G is bipartite and has $(n-\Delta(G))\Delta(G)$ edges, then $G = K_{(n-\Delta(G)),\Delta(G)}$. Hence $e(G)$ is maximized only by $K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$.

1.3.53. *The bridge club with 14 members (no game can be played if two of the four people table have previously been partners): If each member has played with four others and then six additional games have been played, then the arrival of a new member allows a game to be played.* We show that the new player yields a set of four people among which no two have been partners. This is true if and only if the previous games must leave three people (in the original 14) among which no two have been partners.

The graph of pairs who have NOT been partners initially is K_{14}. For each game played, two edges are lost from this graph. At the breakpoint in the session, each vertex has lost four incident edges, so 28 edges have been deleted. In the remaining six games, 12 more edges are deleted. Hence 40 edges have been deleted. Since $e(K_{14}) = 91$, there remain 51 edges for pairs that have not yet been partners.

By Mantel's Theorem (Theorem 1.3.23), the maximum number of edges in a simple 14-vertex graph with no triangle is $\lfloor 14^2/4 \rfloor$. Since $51 > 49$, the graph of remaining edges has a triangle. Thus, when the 15th person arrives, there will be four people of whom none have partnered each other.

1.3.54. *The minimum number of triangles $t(G)$ in an n-vertex graph G and its complement.*

a) $t(G) = \binom{n}{3} - (n-2)e + \sum_{v\in V(G)} \binom{d(v)}{2}$. Let $d_1, \ldots, d_n$ denote the vertex degrees. We prove that the right side of the formula assigns weight 1 to the vertex triples that induce a triangle in G or $\overline{G}$ and weight 0 to all other triples. Among these terms, $\binom{n}{3}$ counts all triples, $(n-2)e$ counts those determined by an edge of G and a vertex off that edge, and $\sum \binom{d_i}{2}$ counts 1 for each pair of incident edges. In the table below, we group these contributions by how many edges the corresponding triple induces in G.

$t(G)$	in G	$\binom{n}{3}$	$-(n-2)e$	$\sum\binom{d_i}{2}$
1	3 edges	1	-3	3
0	2 edges	1	-2	1
0	1 edge	1	-1	0
1	0 edges	1	-0	0

b) $t(G) \ge n(n-1)(n-5)/24$. Begin with the formula for $k_3(G) + k_3(\overline{G})$ from part (a). Using the convexity of quadratic functions, we get a lower bound for the sum on the right by replacing the vertex degrees by the average degree $2e/n$. The bound is $\binom{n}{3} - (n-2)e + n\binom{2e/n}{2}$, which reduces to $\binom{n}{3} - 2e(\binom{n}{2} - e)/n$. As a function of e, this is minimized when $e = \frac{1}{2}\binom{n}{2}$. This substitution and algebraic simplification produce $t(G) \ge n(n-1)(n-5)/24$.

Comment. The proof of part (b) uses two minimizations. These imply that equality can hold only for a regular graph ($d_i = 2e/n$ for all i) with $e = \frac{1}{2}\binom{n}{2}$. There is such a regular graph if and only if n is odd and $(n-1)/2$ is even. Thus we need $n = 4k + 1$ and G is $2k$-regular.

1.3.55. *Maximum size with no induced P_4. a) If G is a simple connected graph and $\overline{G}$ is disconnected, then $e(G) \le \Delta(G)^2$, with equality only for $K_{\Delta(G),\Delta(G)}$.* Since $\overline{G}$ is disconnected, $\Delta(G) \ge n(G)/2$, with equality only if $G = K_{\Delta(G),\Delta(G)}$. Thus $e(G) = \sum d_i/2 \le n(G)\Delta(G)/2 \le \Delta(G)^2$. As observed, equality when $\overline{G}$ is disconnected requires $G = K_{\Delta(G),\Delta(G)}$.

b) If G is a simple connected graph with maximum degree D and no induced subgraph isomorphic to P_4, then $e(G) \le D^2$. It suffices by part (a) to prove that $\overline{G}$ is disconnected when G is connected and P_4-free. We use induction on $n(G)$ for $n(G) \ge 2$; it is immediate when $n(G) = 2$. For the induction step, let v be a non-cut-vertex of G. The graph $G' = G - v$ is also P_4-free, so its complement is disconnected, by the induction hypothesis. Thus $V(G) - v$ has a vertex partition X, Y such that all of X is adjacent to all of Y in G. Since G is connected, v has a neighbor $z \in X \cup Y$; we may assume be symmetry that $z \in Y$. If $\overline{G}$ is connected, then $\overline{G}$ has a v, z-path. Let y be the vertex before z on this path; note that $y \in Y$. Also $\overline{G}$ connected requires $x \in X$ such that $vx \in E(\overline{G})$. Now $\{v, z, x, y\}$ induces P_4 in G.

1.3.56. *Inductive proof that for $\sum d_i$ even there is a multigraph with vertex degrees $d_1, \ldots, d_n$.*

Proof 1 (induction on $\sum d_i$). If $\sum d_i = 0$, then all d_i are 0, and the n-vertex graph with no edges has degree list d. For the induction step, suppose $\sum d_i > 0$. If only one d_i is nonzero, then it must be even, and the graph consisting of $n-1$ isolated vertices plus $d_i/2$ loops at one vertex has degree list d (multigraphs allow loops).

Otherwise, d has at least two nonzero entries, d_i and d_j. Replacing these with $d_i - 1$ and $d_j - 1$ yeilds a list d' with smaller even sum. By the induction hypothesis, some graph G' with degree list d'. Form G by adding an edge with endpoints u and v to G', where $d_{G'}(u) = d_i - 1$ and $d_{G'}(v) = d_j - 1$. Although u and v may already be adjacent in G', the resulting multigraph G has degree list d.

Proof 2 (induction on n). For $n = 1$, put $d_1/2$ loops at v_1. If d_n is even, put $d_n/2$ loops at v_n and apply the induction hypothesis. Otherwise, put an edge from v_n to some other vertex corresponding to positive d_i (which exists since $\sum d_i$ is even) and proceed as before.

1.3.57. *An n-tuple of nonnegative integers with largest entry k is graphic if the sum is even, $k < n$, and every entry is k or $k-1$.* Let $A(n)$ be the set of n-tuples satisfying these conditions. Let $B(n)$ be the set of graphic n-tuples. We prove by induction on n that n-tuples in $A(n)$ are also in $B(n)$. When $n = 1$, the only list in $A(n)$ is (0), and it is graphic.

For the induction step, let d be an n-tuple in $A(n)$, and let k be its largest element. Form d' from d by deleting a copy of k and subtracting 1 from k largest remaining elements. The operation is doable because $k < n$. To apply the induction hypothesis, we need to prove that $d' \in A(n-1)$. Since we delete an instance of k and subtract one from k other values, we reduce the sum by $2k$ to obtain d' from d, so d' does have even sum.

Let q be the number of copies of k in d. If $q > k+1$, then d' has ks and $(k-1)$s. If $q = k+1$, then d' has only $(k-1)$s. If $q < k+1$, then d' has $(k-1)$s and $(k-2)$s. Also, if $k = n-1$, then the first possibility cannot occur. Thus d' has length $n-1$, its largest value is less than $n-1$, and its largest and smallest values differ by at most 1. Thus $d' \in A(n-1)$, and we can apply the induction hypothesis to d'.

The induction hypothesis $(d' \in A(n-1)) \Rightarrow (d' \in B(n-1))$ tells us that d' is graphic. Now the Havel-Hakimi Theorem implies that d is graphic. (Actually, we use only the easy part of the HH Theorem, adding a vertex joined to vertices with desired degrees.)

1.3.58. *If d is a nonincreasing list of nonnegative integers, and d' is obtained by deleting d_k and subtracting 1 from the k largest other elements, then d is graphic if and only if d' is graphic.* The proof is like that of the Havel–Hakimi Theorem. Sufficiency is immediate. For necessity, let w be a vertex of degree d_k in a simple graph with degree sequence d. Alter G by 2-switches to obtain a graph in which w has the d_k highest-degree other vertices as neighbors. The argument to find a 2-switch increasing the number of desired neighbors of w is as in the proof of the Havel–Hakimi Theorem.

1.3.59. *The list $d = (d_1, \ldots, d_{2k})$ with $d_{2i} = d_{2i-1} = i$ for $1 \le i \le k$ is graphic.* This is the degree list for the bipartite graph with vertices $x_1, \ldots, x_k$ and $y_1, \ldots, y_k$ defined by $x_r \leftrightarrow y_s$ if and only if $r + s > k$. Since the neighborhood of x_r is $\{y_k, y_{k-1}, \ldots, y_{k-r+1}\}$, the degree of x_r is r. Thus the graph has two vertices of each degree from 1 to k.

1.3.60. *Necessary and sufficient conditions for a list d to be graphic when d consists of k copies of a and $n-k$ copies of b, with $a \ge b \ge 0$.* Since the degree sum must be even, the quantity $ka + (n-k)b$ must be even. In addition, the inequality $ka \le k(k-1) + (n-k)\min\{k, b\}$ must hold, since each vertex with degree b has at most $\min\{k, b\}$ incident edges whose other endpoint has degree a. We construct graphs with the desired degree sequence when these conditions hold. Note that the inequality implies $a \le n-1$.

Case 1: *$b \ge k$ and $a \ge n-k$.* Begin with $K_{k,n-k}$, having partite sets X of size k and Y of size $n-k$. If $k(a-n+k)$ and $(n-k)(b-k)$ are even, then add an $(a-n+k)$-regular graph on X and a $(b-k)$-regular graph on Y. To show that this is possible, note first that $0 \le a-n+k \le k-1$ and $0 \le b-k \le a-k \le n-k-1$. Also, when pq is even, a q-regular graph on p vertices in a circle can be constructed by making each vertex adjacent to the $\lfloor q/2 \rfloor$ nearest vertices in each direction and also to the opposite vertex if q is odd (since then p is even).

Note that $k(a-n+k)$ and $(n-k)(b-k)$ have the same parity, since their difference $ak - (n-k)b$ differs from the given even number $ka + (n-k)b$ by an even amount. If they are both odd, then we delete one edge from $K_{k,n-k}$, and now one vertex in the subgraph on X should have degree $a-n+k+1$ and one in the subgraph on Y should have degree $b-k+1$. When pq is odd, such a graph on vertices $v_0, \ldots, v_{p-1}$ in a circle (q-regular except for one vertex of degree $q+1$) can be constructed by making each vertex adjacent to the $(q-1)/2$ nearest vertices in each direction and then adding the edges $\{v_i v_{i+(p-1)/2}$: $0 \le i \le (p-1)/2$. Note that all vertices are incident to one of the added edges, except that $v_{(p-1)/2}$ is incident to two of them.

Case 2: $k-1 \le a < n-k$. Begin by placing a complete graph on a set S of k vertices. These vertices now have degree $k-1$ and will become the vertices of degree a, which is okay since $a \ge b$. Put a set T of $n-k$ additional vertices in a circle. For each vertex in S, add $a-k+1$ consecutive neighbors in T, starting the next set immediately after the previous set ends. Since $a \le n-1$, each vertex in S is assigned $a-k+1$ distinct neighbors in T. Since $k(a-k+1) \le (n-k)b$ and the edges are distributed nearly equally to vertices of T, there is room to add these edges.

For the subgraph induced by T, we need a graph with $n-k$ vertices and $[(n-k)b - k(a-k+1)]/2$ edges and degrees differing by at most 1. The desired number of edges is integral, since $ka + (n-k)b$ is even, and it is nonnegative, since $k(a-k+1) \le (n-k)b$. The largest degree needed is $\lceil (n-k)b - k(a-k+1) \rceil \, n-k$. This is at most b, which is less than $n-k$ since $b \le a < n-k$. The desired graph now exists by Exercise 1.3.57.

Case 3: $b < k$ *and* $a \ge n-k$. Put the set S of size k in a circle. For each vertex in the set T of size $n-k$, assign b consecutive neighbors in S; these are distinct since $b < k$. Since $a \ge n-k$, no vertex of S receives too many edges. On S we put an almost-regular graph with k vertices and $[ak - b(n-k)]/2$ edges. Again, this number of edges is integral, and in the case specified it is nonnegative. Existence of such a graph requires $a - b(n-k)/k \le k-1$, which is equivalent to the given inequality $k(a-k+1) \le (n-k)b$. Now again Exercise 1.3.57 provides the needed graph.

Case 4: $b < k$ *and* $a < \min\{k-1, n-k\}$. Since $a < n-k$, also $b < n-k$. Therefore, we can use the idea of Case 1 without the complete bipartite graph. Again take disjoint vertex sets X of size k and Y of size $n-k$. If ka and $(n-k)b$ are even, then we use an a-regular graph on X and a b-regular graph on Y. As observed before, these exist.

Note that ka and $(n-k)b$ have the same parity, since their sum is given to be even. If they are both odd, then we put $\min\{k, n-k\}$ disjoint edges with endpoints in both X and Y. We now complete the graph with a regular graph of even degree on one of these sets and an almost-regular graph guaranteed by Exercise 1.3.57 on the other.

1.3.61. *If G is a self-complementary n-vertex graph and n is odd, then G has a vertex of degree $(n-1)/2$.* Let $d_1, \ldots, d_n$ be the degree list of G in nonincreasing order. The degree list of $\overline{G}$ in nonincreasing order is $n-1-d_n, \ldots, n-1-d_1$. Since $G \cong \overline{G}$, the lists are the same. Since n is odd, the central elements in the list yield $d_{(n+1)/2} = n-1-d_{(n+1)/2}$, so $d_{(n+1)/2} = (n-1)/2$.

1.3.62. *When n is congruent to 0 or 1 modulo 4, there is an n-vertex simple graph G with $\frac{1}{2}\binom{n}{2}$ edges such that $\Delta(G) - \delta(G) \le 1$.* This is satisfied by the construction given in the answer to Exercise 1.1.31.

More generally, let G be any $2k$-regular simple graph with $4k+1$ vertices, where $n = 4k+1$. Such a graph can be constructed by placing $4k+1$ vertices around a circle and joining each vertex to the k closest vertices in each direction. By the Degree-Sum Formula, $e(G) = (4k+1)(2k)/2 = \frac{1}{2}\binom{n}{2}$.

For the case where $n = 4k$, delete one vertex from the graph constructed above to form G'. Now $e(G') = e(G) - 2k = (4k-1)(2k)/2 = \frac{1}{2}\binom{n}{2}$.

1.3.63. *The non-negative integers $d_1 \ge \cdots \ge d_n$ are the vertex degrees of a loopless graph if and only if $\sum d_i$ is even and $d_1 \le d_2 + \cdots + d_n$. Necessity.* If such a graph exists, then $\sum d_i$ counts two endpoints of each edge and must be even. Also, every edge incident to the vertex of largest degree has its other end counted among the degrees of the other vertices, so the inequality holds.

Sufficiency. Specify vertices $v_1, \ldots, v_n$ and construct a graph so that $d(v_i) = d_i$. Induction on n has problems: It is not enough to make d_n edges join v_1 and v_n degrees and apply the induction hypothesis to $(d_1 - d_n), d_2, \ldots, d_{n-1}$. Although $d_1 - d_n \le d_2 + \cdots + d_{n-1}$ holds, $d_1 - d_n$ may not be the largest of these numbers.

Proof 1 (induction on $\sum d_i$). The basis step is $\sum d_i = 0$, realized by an independent set. Suppose that $\sum d_i > 0$; we consider two cases. If $d_1 = \sum_{i=2}^n d_i$, then the desired graph consists of d_1 edges from v_1 to $v_2, \ldots, v_n$. If $d_1 < \sum_{i=2}^n d_i$, then the difference is at least 2, because the total degree sum is even. Also, at least two of the values after d_1 are nonzero, since d_1 is the largest. Thus we can subtract one from each of the last two nonzero values to obtain a list d' to which we can apply the induction hypothesis (it has even sum, and the largest value is at most the sum of the others. To the resulting G', we add one edge joining the two vertices whose degrees are the reduced values. (This can also be viewed as induction on $(\sum_{i=2}^n d_i) - d_1$.)

Proof 2 (induction on $\sum d_i$). Basis as above. Consider $\sum d_i > 0$. If $d_1 > d_2$, then we can subtract 1 from d_1 and from d_2 to obtain d' with smaller sum. Still $d_1 - 1$ is a largest value in d' and is bounded by the sum of the other values. If $d_1 = d_2$, then we subtract 1 from each of the two smallest values to form d'. If these are d_1 and d_2, then d' has the desired properties, and otherwise $\sum_{i=2}^n d_i$ exceeds d_1 by at least 2, and again d' has the desired properties. In each case, we can apply the induction hypothesis to d' and complete the proof as in Proof 1.

Proof 3 (local change). Every nonnegative integer sequence with even sum is realizable when loops and multiple edges are allowed. Given such a realization with a loop, we change it to reduce the number of loops without changing vertex degrees. Eliminating them all produces the desired realization. If we have loops at distinct vertices u and v, then we replace two loops with two copies of the edge uv. If we have loops only at v and have an edge xy between two vertices other than v, then we replace one loop and one copy of xy by edges vx and vy. Such an edge xy must exist because the sum of the degrees of the other vertices is as large as the degree of v.

1.3.64. *A simple graph with degree sequence $d_1 \le d_2 \le \cdots \le d_n$ is connected if $d_j \ge j$ for all j such that $j \le n-1-d_n$.* Let $V(G) = \{v_1, \ldots, v_n\}$, with $d(v_i) = d_i$, and let H be the component of G containing v_n; note that H has at least $1 + d_n$ vertices. If G is not connected, then G has another component H'. Let j be the number of vertices in H'. Since H has at least $1 + d_n$ vertices, we have $j \le n-1-d_n$. By the hypothesis, $d_j \ge j$. Since H' has j vertices, its maximum degree is at least d_j. Since $d_j \ge j$, there are at least $j+1$ vertices in H', which contradicts the definition of j. Hence G is in fact connected.

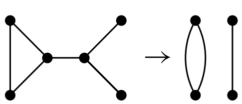
1.3.65. *If $D = \{a_i\}$ is a set of distinct positive integers, with $0 < a_1 < \cdots < a_k$, then there is a simple graph on $a_k + 1$ vertices whose set of vertex degrees (repetition allowed) is D.*

Proof 1 (inductive construction). We use induction on k. For $k = 1$, use K_{a_1+1}. For $k = 2$, use the join $K_{a_1} \vee \overline{K}_{a_2-a_1+1}$. That is, G consists of a clique Q with a_1 vertices, an independent set S with $a_2 - a_1 + 1$ vertices, and all edges from Q to S. The vertices of S have degree a_1, and those of Q have degree a_2.

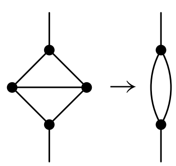
For $k \ge 2$, take a clique Q with a_1 vertices and an independent set S with $a_k - a_{k-1}$ vertices. Each vertex of S has neighborhood Q, and each vertex of Q is adjacent to all other vertices. Other vertices have a_1 neighbors in Q and none in S, so the degree set of $G - Q - S$ should be $\{a_2 - a_1, \ldots, a_{k-1} - a_1\}$. By the induction hypthesis, there is a simple graph H with $a_{k-1} - a_1 + 1$ vertices having this degree set (the degree set is smaller by 2). Using H for $G - Q - S$ completes G as desired.

Proof 2 (induction and complementation). Again use induction on k, using K_{a_1+1} when $k = 1$. For $k > 1$ and $0 < a_1 \cdots < a_k$, the complement of the desired graph with $a_1 + 1$ vertices has degree set $\{a_k - a_1, \ldots, a_k - a_{k-1}, 0\}$. By the induction hypothesis, there is a graph of order $a_k - a_1 + 1$ with degree set $\{a_k - a_1, \cdots, a_k - a_{k-1}\}$. Add a_1 isolated vertices and take the complement to obtain the desired graph G.

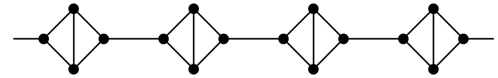
1.3.66. *Construction of cubic graphs not obtainable by expansion alone.* A simple cubic graph G that cannot be obtained from a smaller cubic graph by the expansion operation is the same as a cubic graph on which no erasure can be performed, since any erasure yielding a smaller H from G could be inverted by an expansion to obtain G from H. An edge cannot be erased by this operation if and only if one of the subsequent contractions produces a multiple edge. This happens if the other edges incident to the edge being erased belong to a triangle, or in one other case, as indicated below.



and

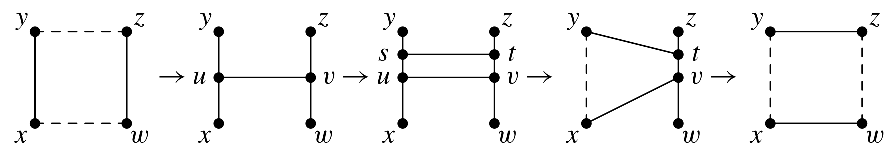


Finally, we need only provide a simple cubic graph with $4k$ vertices where every edge is non-erasable in one of these two ways. To do this place copies of $G_1, \ldots, G_k$ of $K_4 - e$ (the unique 4-vertex graph with 5 edges) around in a ring, and for each consecutive pair G_i, G_{i+1} add an edge joining a pair of vertices with degree two in the subgraphs, as indicated below, where the wraparound edge has been cut.



1.3.67. *Construction of 3-regular simple graphs*

a) A 2-switch can be performed by performing a sequence of expansions and erasures. We achieve a 2-switch using two expansions and then two erasures as shown below. If the 2-switch deletes xy and zw and introduces xw and yz, then the first expansion places new vertices u and v on xy and zw, the second introduces s and t on the resulting edges ux and vz, the first erasure deletes su and its vertices, and the second erasure deletes tv and its vertices. The resulting vertices are the same as in the original graph, the erasures were legal because they created only edges that were not present originally, and we have deleted xy and zw and introduced xw and yz.



b) Every 3-regular simple graph can be obtained from K_4 by a sequence of expansions and erasures. Erasure is allowed only if no multiple edges result. Suppose H is the desired 3-regular graph. Every 3-regular graph has an even number of vertices, at least four. Any expansion of a 3-regular graph is a 3-regular graph with two more vertices. Hence successive expansions from K_4 produce a 3-regular graph G with $n(H)$ vertices. Since G and H have the same vertex degrees, there is a sequence of 2-switches from G to H. Since every 2-switch can be produced by a sequence of expansions and erasures, we can construct a sequence of expansions and erasures from K_4 to H by going through G.

1.3.68. *If G and H are X, Y-bigraphs, then $d_G(v) = d_H(v)$ for all $v \in X \cup Y$ if and only if there is a sequence of 2-switches that transforms G into H without ever changing the bipartition.* The condition is sufficient, since 2-switches do not change vertex degrees. For necessity, assume that $d_G(v) = d_H(v)$ for all v. We build a sequence of 2-switches transforming G to H.

Proof 1 (induction on $|X|$). If $|X| = 1$, then already $G = H$, so we may assume that $|X| > 1$. Choose $x \in X$ and let $k = d(x)$. Let S be a selection of k vertices of highest degree in Y. If $N(x) \ne S$, choose $y \in S$ and $y' \in Y - S$ so that $x \nleftrightarrow y$ and $x \leftrightarrow y'$. Since $d(y) \ge d(y')$, there exists $x' \in X$ so that $y \leftrightarrow x'$ and $y' \nleftrightarrow x'$. Switching $xy', x'y$ for $xy, x'y'$ increases

$|N(x) \cap S|$ with the same bipartition. Iterating this reaches $N(x) = S$; let G' be the resulting graph.

Doing the same in H yields graphs G' from G and H' from H such that $N_{G'}(x) = N_{H'}(x)$. Deleting x and applying the induction hypothesis to the graphs $G^* = G' - x$ and $H^* = H' - x$ completes the construction of the desired sequence of 2-switches.

Proof 2 (induction on number of discrepancies). Let F be the X, Y-bigraph whose edges are those belonging to exactly one of G and H. Let $d = e(F)$. Since G and H have identical vertex degrees, each vertex of F has the same number of incident edges from $E(G) - E(H)$ and $E(H) - E(G)$. When $d > 0$, F therefore has a cycle alternating between $E(G)$ and $E(H)$ (when we enter a vertex on an edge of one type, we can exit on the other type, we can't continue forever, and all cycles have even length).

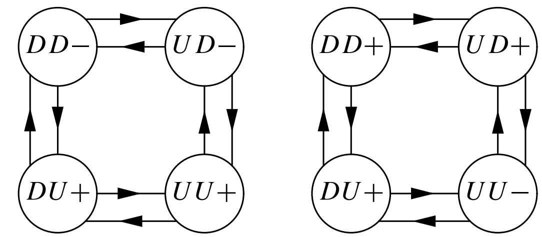
Let C be a shortest alternating cycle in F, with first $xy \in E(G) - E(H)$ and then $yx' \in E(H) - E(G)$ and $x'y' \in E(G) - E(H)$. We consider a 2-switch involving $\{x, y, x', y'\}$. If $y'x \in E(H) - E(G)$, then the 2-switch in G reduces d by 4. If $y'x \in E(G) - E(H)$, then we would have a shorter cycle in F. If $y'x \notin E(G) \cup E(H)$, then we perform the 2-switch in G; if $y'x \in E(G) \cup E(H)$, then we perform the 2-switch in H. Each of these last two cases yields a new pair of graphs with d reduced by 2, and the induction hypothesis applies to this pair to provide the rest of the exchanges.

1.4. DIRECTED GRAPHS

1.4.1. *Digraphs in the real world.* Many digraphs based on temporal order have no cycles. For example, given a set of football games, we can put an edge from game x to game y if game x ends before game y begins. The relation "is a parent of" also works.

Asymmetric digraphs without cycles often arise from tournaments. Each team plays every other team, and there is an edge for each game from the winner to the loser. The result can be without cycles, but usually cycles exist. Another example is the relation "has sent a letter to".

1.4.2. *If the first switch becomes disconnected from the wiring in the lightswitch system of Application 1.4.4, then the digraph for the resulting system is that below.*



1.4.3. *Every u, v-walk in a digraph contains a u, v-path.* The shortest u, v-walk contained in a u, v-walk W is a u, v-path, since the shortest walk has no vertex repetition.

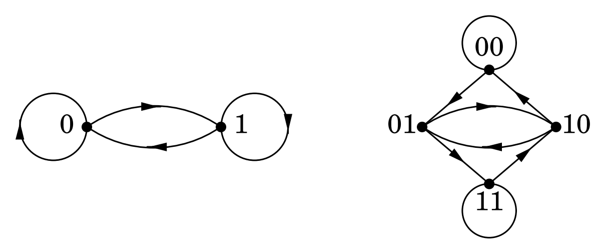
1.4.4. *Every closed walk of odd length in a digraph contains the edges of an odd cycle.* The proof follows that of the corresponding statement for graphs in Lemma 1.2.15, given that the definitions of walk and cycle require the head of each edge to be the tail of the next edge.

We use induction on the length l of a closed odd walk W. Basis step: $l = 1$. A closed walk of length 1 traverses a cycle of length 1.

Induction step: $l > 1$. Assume the claim for closed odd walks shorter than W. If W has no repeated vertex (other than first = last), then W itself forms a cycle of odd length. If vertex v is repeated in W, then we view W as starting at v and break W into two v, v-walks. Since W has odd length, one of these is odd and the other is even. The odd one is shorter than W. By the induction hypothesis, it contains an odd cycle, and this cycle appears in order in W.

1.4.5. *A finite directed graph contains a (directed) cycle if every vertex is the tail of at least one edge (has positive outdegree).* (The same conclusion holds if every vertex is the head of at least one edge.) Let G be such a graph, let P be a maximal (directed) path in G, and let x be the final vertex of P. Since x has at least one edge going out, there is an edge xy. Since P cannot be extended, y must belong to P. Now xy completes a cycle with the y, x-subpath of P.

1.4.6. *The De Bruijn graphs D_2 and D_3.*



1.4.7. *In an orientation of a simple graph with 10 vertices, the vertices can have distinct outdegrees.* Take the orientation of the complete graph with vertices $0, \ldots, 9$ by orienting the edge ij from i to j if $i > j$. In this digraph, the outdegree of vertex i is i.

1.4.8. *There is an n-vertex tournament with $d^+(v) = d^-(v)$ for every vertex v if and only if n is odd.* If n is even, then $d^+(v) + d^-(v) = n - 1$ is odd, so the summands can't be equal integers. For odd n, we construct such a tournament.

Proof 1 (explicit construction). Place the n vertices equally spaced around a circle, and direct the edges from v to the $(n-1)/2$ vertices that follow v in the clockwise direction. After doing this for each vertex, the $(n-1)/2$ nearest vertices in the counterclockwise direction from v have edges directed to v, and each edge has been oriented.

Proof 2 (inductive construction). When $n = 1$, the 1-vertex tournament satisfies the degree condition. For $k > 1$, suppose that T is a tournament with $2k - 1$ vertices that satisfies the condition. Partition $V(T)$ into sets A and B with $|A| = k$ and $|B| = k - 1$. Add two vertices x and y. Add all edges from x to A, from A to y, from y to B, and from B to x. Each vertex in $V(T)$ now has one predecessor and one successor in $\{x, y\}$. We have $d^+(x) = k$, $d^-(x) = k - 1$, $d^+(y) = k - 1$, $d^-(y) = k$. Complete the construction of T' by adding the edge yx. Now T' is a tournament with $2k + 1$ vertices that satisfies the degree condition.

Proof 3 (Eulerian graphs). When n is odd, K_n is a connected even graph and hence is Eulerian. Orienting edges of K_n in the forward direction while following an Eulerian circuit yields the desired tournament.

1.4.9. *For each n, there is an n-vertex digraph in which the vertices have distinct indegrees and distinct outdegrees.* Using vertices $v_1, \ldots, v_n$, let the edges be $\{v_i v_j : 1 \le i < j \le n\}$. Now $d^-(v_i) = i - 1$ and $d^+(v_i) = n - i$. Thus the indegrees are distinct, and the outdegrees are distinct.

1.4.10. *A digraph is strongly connected if and only if for each partition of the vertex set into nonempty sets S and T, there is an edge from S to T.* Given that D is strong, choose $x \in S$ and $y \in T$. Since D has an x, y-path, the path must leave S and enter T and do so along some edge.

Conversely, if there is such an edge for every partition, let S be the set of all vertices reachable from vertex x. If $S \ne V(D)$, then the hypothesis yield an edge leaving S, which adds a vertex to S. Since x was arbitrary, each vertex is reachable from every other, and D is strongly connected.

1.4.11. *In every digraph, some strong component has no entering edges, and some strong component has no exiting edges.*

Proof 1 (using cycles). Given a digraph D, form a digraph D^* with one vertex for each strong component of D. Let the strong components of D be $X_1, \ldots, X_k$, with corresponding vertices $x_1, \ldots, x_k$ in D^*. Put an edge from x_i to x_j in D^* if in D there is an edge from some vertex of X_i to some vertex of X_j. The problem is to show that D^* has a vertex with indegree 0 and a vertex with outdegree 0.

If such vertices do not exist, then D^* has a cycle (by Lemma 1.4.23). If D^* has a cycle, then the union of the strong components of D corresponding to the vertices of the cycle is a strongly connected subgraph of D containing all those components. This is a contradiction, because they were maximal strong subgraphs.

Proof 2 (extremality). For a vertex v in D, let $R(v)$ be the set of vertices reachable from v. Let u be a vertex minimizing $|R(u)|$. If $v \in R(u)$, then $R(v) \subseteq R(u)$, so $R(v) = R(u)$. Since $u \in R(u)$, also u is reachable from v. Thus $R(u)$ induces a strong subdigraph. By the definition of $R(u)$, no edges leave it, so it is a strong component. Applying the same argument to the reverse digraph yields a strong component with no entering edge.

1.4.12. *In a digraph the connection relation is an equivalence relation, and its equivalence classes are the vertex sets of the strong components.* We are defining x to be connected to y if the digraph has both an x, y-path and a y, x-path. The reflexive property holds using paths of length 0. The symmetric property holds by the definition.

For transitivity, consider an x, y-path P_1 and a y, z-path P_2. Let w be the first vertex of P_1 that belongs to P_2. Following P_1 from x to w and P_2 from w to z yields an x, z-path, by the choice of w. Applying this to obtain paths in both directions shows that the connection relation is transitive.

Since a strong component is a strongly connected subdigraph, its pairs of vertices satisfy the connection relation. Hence every strong component is contained in an equivalence class of the connection relation. In order to show that every equivalence class is contained in a strong component, we show that when x is connected to y, there is an x, y-path using only vertices of the equivalence class.

Let P be an x, y-path, and let Q be a y, x-path. The concatenation of Q with P is a closed walk in the digraph; let S be its vertex set. By following the walk, we find a u, v-walk for all $u, v \in S$. Such a walk contains a u, v-path. The same argument yields a v, u-path in the walk. Hence all pairs of vertices on it satisfy the connection relation, and we have found an x, y-path (and y, x-path) witin the equivalence class. Hence the subdigraph induced by the equivalence class is strongly connected.

1.4.13. *Strong components.*

a) Two maximal strongly connected subgraphs of a directed graph share no vertices. If strong components D_1, D_2 of D share a vertex v, then for all

$x \in V(D_1)$ and $y \in V(D_2)$, the union of an x,v-path in D_1 and a v,y-path in D_2 contains an x,y-path in D. Similarly, D has a y,x-path. Thus $D_1 \cup D_2$ is strongly connected.

b) The digraph D^* obtained by contracting the strong components of a digraph D is acyclic (D^* has a vertex v_i for each strong component D_i, with $v_i \to v_j$ if and only if $i \neq j$ and D has an edge from D_i to D_j). If D^* has a cycle with vertices $d_0, \ldots, d_{l-1}$, then D has strong components $D_0, \ldots, D_{l-1}$ such that D has an edge $u_i v_{i+1}$ from D_i to D_{i+1}, for each i (modulo l). If $x \in D_i$ and $y \in D_j$, this means that D contains an x,y-walk consisting of the concatenation of paths with successive endpoints $x, u_i, v_{i+1}, u_{i+1}, v_{i+2}, \ldots, u_{j-1}, v_j, y$. This walk contains an x,y-path. Since x,y were chosen arbitrarily from $D_0 \cup \cdots \cup D_{l-1}$, we conclude that $D_0 \cup \cdots \cup D_{l-1}$ is strongly connected, which contradicts $D_0, \ldots, D_{l-1}$ being maximal strongly connected subgraphs.

1.4.14. *If G is an n-vertex digraph with no cycles, then the vertices of G can be ordered as $v_1, \ldots, v_n$ so that if $v_i v_j \in E(G)$, then $i < j$.* If G has no cycles, then some vertex v has outdegree 0. Put v last in the ordering. Now $G - v$ also has no cycles, and we proceed iteratively. When we choose v_j, it has no successors among $v_1, \ldots, v_{j-1}$, so the desired condition on the edges holds.

1.4.15. *In the simple digraph with vertex set $\{(i,j) \in \mathbb{Z}^2 \colon 0 \le i \le m$ and $0 \le n\}$ and an edge from (i,j) to (i',j') if and only if (i',j') is obtained from (i,j) by adding 1 to one coordinate, there are $\binom{m+n}{n}$ paths from $(0,0)$ to (m,n).* Traversing each edge adds one to each coordinate, so every such path has $m+n$ edges. We can record such a path as a 0, 1-list, recording 0 when we follow an edge that increases the first coordinate, 1 when we follow an edge that increases the second coordinate. Each list with m 0s and n 1s records a unique path. Since there are $\binom{m+n}{n}$ ways to form such a list by choosing positions for the 1s, the bijection implies that the number of paths is $\binom{m+n}{n}$.

1.4.16. *Fermat's Little Theorem.* Let $\mathbb{Z}_n$ denote the set of congruence classes of integers modulo a PRIME NUMBER n (the first printing of the second edition omitted this!). Multiplication by a positive integer a that is not a multiple of n defines a permutation of $\mathbb{Z}_n$, since $ai \equiv aj \pmod n$ yields $a(j-i) \equiv 0 \pmod n$, which requires n to divide $j-i$ when a and n are relatively prime. The functional digraph consists of pairwise disjoint cycles.

a) If G is the functional digraph with vertex set $\mathbb{Z}_n$ for the permutation defined by multiplication by a, then all cycles in G (except the loop on n) have length $l-1$, where l is the least natural number such that $a^l \equiv a \pmod n$. This is the length of the cycle containing the element 1. Traversing a cycle of length k (not the cycle consisting of n) yields $xa^k \equiv x \pmod n$, or $x(a^k - 1) \equiv 0 \pmod n$, for some x not divisible by n. Since n is prime, this requires $a^k \equiv 1 \pmod n$, and hence $k \ge l-1$. On the other hand $xa^{l-1} = x$, and hence $k \le l-1$.

b) $a^{n-1} \equiv 1 \pmod n$. Since all nontrivial cycles have the same length, $l-1$ divides $n-1$. Let $m = (n-1)/(l-1)$. Now $a^{n-1} = a^{(l-1)m} = (a^{l-1})^m \equiv 1^m \equiv 1 \pmod n$.

1.4.17. *A (directed) odd cycle is a digraph with no kernel.* Let S be a kernel in an odd cycle C. Every vertex must be in S or have a successor in S. Since S is an independent set, exactly one of these two conditions holds at each vertex. Hence we must alternate between vertices in S and vertices not in S as we follow the C. We cannot alternate two conditions as we follow an odd cycle, so there is no kernel.

A digraph having an odd cycle as an induced subgraph and having a kernel. To an odd cycle, add one new vertex as a successor of each vertex on the cycle. The new vertex forms a kernel by itself.

1.4.18. *An acyclic digraph D has a unique kernel.*

Proof 1 (parity of cycles). By Theorem 1.4.16, a digraph with no odd cycles has at least one kernel. We show that a digraph with no even cycles has at most one kernel, by proving the contrapositive. If K and L are distinct kernels (each induces no edges), then every vertex of $K - L$ has a successor in $L - K$, and every vertex of $L - K$ has a successor in $K - L$.

Proof 2 (induction on $n(D)$). In a digraph with one vertex and no cycle, the vertex is a kernel. When $n(D) > 1$, the absence of cycles guarantees a vertex with outdegree 0 (Lemma 1.4.23). Such a vertex lies in every kernel, since it has no successor. Let $S' = \{v \in V(D) \colon d^+(v) = 0\}$. Note that S' induces no edges. Let D' be the subdigraph obtained from D by deleting S' and all vertices having successors in S'. The digraph D' has no cycles; by the induction hypothesis, D' has a unique kernel S''.

Let $S = S' \cup S''$. Since there are no edges from $V(D')$ to S', the set S is a kernel in D. Furthermore, S is the only kernel. We have argued that all of S' is present in every kernel, and independence of the kernel implies that no other vertex outside $V(D')$ is present. The lack of edges from $V(D')$ to S' implies that the remainder of the kernel must be a kernel in D', and there is only one such set.

1.4.19. *A digraph is Eulerian if and only if $d^+(v) = d^-(v)$ for every vertex v and the underlying graph has at most one nontrivial component.*

Necessity. Each passage through a vertex by a circuit uses an entering edge and an exiting edge; this applies also to the "last" and "first" edges of the circuit. Also, two edges can be in the same trail only when they lie in the same component of the underlying graph.

Sufficiency. We use induction on the number of edges, m. Basis step: When $m = 0$, a closed trail consisting of one vertex contains all the edges.

Induction step: Consider $m > 0$. With equal indegree and outdegree, each vertex in the nontrivial component of the underlying graph of our digraph G has outdegree at least 1 in G. By Lemma 1.2.25, G has a cycle C. Let G' be the digraph obtained from G by deleting $E(C)$.

Since C has 1 entering and 1 departing edge at each vertex, G' also has equal indegree and outdegree at each vertex. Each component of the underlying graph H' of G' is the underlying graph of some subgraph of G'. Since G' has fewer than m edges, the induction hypothesis yields an Eulerian circuit of each such subgraph of G'.

To form an Eulerian circuit of G, we traverse C, but when a component of H' is entered for the first time we detour along an Eulerian circuit of the corresponding subgraph of G', ending where the detour began. When we complete the traversal of C, we have an Eulerian circuit of G.

1.4.20. *A digraph is Eulerian if and only if indegree equals outdegree at every vertex and the underlying graph has at most one nontrivial component.* The conditions are necessary, since each passage through a vertex uses one entering edge and one departing edge.

For sufficiency, suppose that G is a digraph satisfying the conditions. We prove first that every non-extendible trail in G is closed. Let T be a non-extendible trail starting at u. Each time T passes through a vertex v other than u, it uses one entering edge and one departing edge. Thus upon each arrival at v, T has used one more edge entering v than departing v. Since $d^+(v) = d^-(v)$, there remains an edge on which T can continue. Hence a non-extendible trail can only end at v and must be closed.

We now show that a trail of maximal length in G must be an Eulerian circuit. Let T be a trail of maximum length; T must also be non-extendible, and hence T is closed. Suppose that T omits some edge e of G. Since the underlying graph of G has only one nontrivial component, it has a shortest path from e to the vertex set of T. Hence some edge e' not in T is incident to some vertex v of T. It may enter or leave v.

Since T is closed, there is a trail T' that starts and ends at v and uses the same edges as T. We now extend T' along e' (forward or backward depending on whether e leaves or enters v) to obtain a longer trail than T. This contradicts the choice of T, and hence T traverses all edges of G.

1.4.21. *A digraph has an Eulerian trail if and only if the underlying graph has only one nontrivial component and $d^-(v) = d^+(v)$ for all vertices or for all but two vertices, in which case in-degree and out-degree differ by one for the other two vertices.* Sufficiency: since the total number of heads equals the total number of tails, the vertices out of balance consist of x with an extra head and y with an extra tail. Add the directed edge xy and apply the characterization above for Eulerian digraphs.

1.4.22. *If D is a digraph with $d^-(v) = d^+(v)$ for every vertex v, except that $d^+(x) - d^-(x) = k = d^-(y) - d^+(y)$, then D contains k pairwise edge-disjoint x, y-paths.* Form a digraph D' by adding k edges from y to x. Since indegree equals outdegree for every vertex of D', the "component" of D' containing x and y is Eulerian. Deleting the added edges from an Eulerian circuit cuts it at k places; the resulting k directed trails are x, y-trails in the digraph D. As proved in Chapter 1, the edge set of every x, y-trail contains an x, y-path; the proof in Chapter 1 applies to both graphs and digraphs.

1.4.23. *Every graph G has an orientation such that $|d^+(v) - d^-(v)| \le 1$ for all v.*

Proof 1 (Eulerian circuits). Add edges to pair up vertices of odd degree (if any exist). Each component of this supergraph G' is Eulerian. Orient G' by following an Eulerian circuit in each component, orienting each edge forward as the circuit is traversed. The circuit leaves each vertex the same number of times as it enters, so the resulting orientation has equal indegree and outdegree at each vertex.

Deleting the edges of $E(G') - E(G)$ now yields the desired orientation of G, because at most one edge was added at each vertex to pair the vertices of odd degree. Deleting at most one incident edge at v produces difference at most one between $d^+(v)$ and $d^-(v)$.

Proof 2 (induction on $e(G)$). If $e(G) = 0$, then the claim holds. For $e(G) > 0$, if G has a cycle H, then orient H consistently, with no imbalance anywhere. If G has no cycle, then find a maximal path H and orient it consistently. This creates imbalance of 1 at the endpoints and 0 elsewhere. The endpoints have degree 1, so no further imbalance occurs there. In both cases, delete $E(H)$ and apply the induction hypothesis to complete the orientation.

1.4.24. *Not every graph has an orientation such that for every vertex subset, the numbers of edges entering and leaving differ by at most one.* Let G be a graph with at least four vertices such that every vertex degree is odd. Let D be an orientation of G. In D, no vertex of G has the same number of vertices entering and leaving. Let $S = \{v \in V : d^+(v) > d^-(v)\}$. Since each edge within S contributes the same amount to $\sum_{v\in S} d^+(v)$ and $\sum_{v\in S} d^-(v)$, there are $\sum_{v\in S} d^+(v) - \sum_{v\in S} d^-(v)$ more edges leaving S than entering. The difference is at least $|S|$. Similarly, for $\overline{S}$ the absolute difference is at least $|\overline{S}|$, so always some set has difference at least $n(G)/2$.

1.4.25. *Orientations and P_3-decomposition. a) Every connected graph has an orientation having at most one vertex with odd outdegree.*

Proof 1 (local change). Given an orientation of G with vertices x and y having odd outdegree, find an x, y-path P in the underlying graph and flip the orientation of every edge on P. This does not change the parity of the outdegree for any internal vertex of P, but it changes the parity of the outdegree for the endpoints, which previously had odd outdegree. Hence this operation reduces the number of vertices of odd outdegree by 2. We can apply this operation whenever at least two vertices have odd outdegree, so we can reduce the number of vertices with odd outdegree to 0 or 1.

Proof 2 (application of Eulerian circuits). Suppose that G has $2k$ vertices of odd degree. Add edges that pair these vertices to form an Eulerian supergraph G'. Follow an Eulerian circuit of G', starting from u along $uv \in E(G)$, producing an orientation of G as follows. Orient uv out from u; now u has odd outdegree and all other vertices have even outdegree. Subsequently, when the circuit traverses an edge $xy \in E(G)$, orient it so that x has even outdegree among the edges oriented so far. At each stage, the only vertex that can have odd outdegree among edges of G is the current vertex. The orientation chosen for the edges not in $E(G)$ is unimportant.

b) A simple connected graph with an even number of edges can be decomposed into paths with two edges. Since the sum of the outdegrees is the number of edges, the parity of the number of vertices with odd outdegree is the same as the parity of the number of edges. Hence part (a) implies that a connected graph with an even number of edges has an orientation in which every vertex has even outdegree. At each vertex, pair up exiting edges arbitrarily. Since G is simple, this decomposes G into copies of P_3.

1.4.26. *De Bruijn cycle for binary words of length 4, avoiding 0101 and 1010.* Make a vertex for each of the 8 sequences of length 3 from the alphabet $S = \{0, 1\}$. Put an edge from sequence a to sequence b, with label $\alpha \in S$, if b is obtained from a by dropping the first letter of a and appending α to the end. Traveling this edge from a corresponds to having α in sequence after a. We want our digraph to have 14 edges corresponding to the desired 14 words, and we want an Eulerian circuit through them to generate the cyclic arrangement of labels. The difference between this digraph and the De Bruijn digraph in Application 1.4.25 is omitting the two edges joining 010 and 101. The resulting digraph still has indegree = outdegree at every vertex, so it is Eulerian. One arrangement of labels generated by an Eulerian circuit is 00001001101111.

1.4.27. *De Bruijn cycle for any alphabet and length.* When A is an alphabet of size k, there exists a cyclic arrangement of k^l characters chosen from A such that the k^l strings of length l in the sequence are all distinct.

Idea: The indegree and outdegree is k at each vertex of the digraph constructed in the matter analogous to that for $k = 2$. Thus the digraph is Eulerian, and recording the edge labels along an Eulerian circuit yields the desired sequence. Below we repeat the details.

Define a digraph $D_{k,l}$ whose vertices are the $(l-1)$-tuples with elements in A. Place an edge from a to b if the last $n-2$ entries of a agree with the first $n-2$ entries of b. Label the edge with the last entry of b. For each vertex a, there are k ways to append a element of A to lengthen its name, and hence there are k edges leaving each vertex.

Similarly, there are k choices for a character deleted from the front of a predecessor's name to obtain name b, so each vertex has indegree k. Also, we can reach $b = (b_1, \ldots, b_{n-1})$ from any vertex by successively following the edges labeled $b_1, \ldots, b_{n-1}$. Since $D_{k,l}$ is strongly connected and has indegree equal to outdegree at every vertex, the characterization of Eulerian digraphs implies that $D_{k,l}$ is Eulerian.

Let C be an Eulerian circuit of $D_{k,l}$. When we are at the vertex with name $a = (a_1, \ldots, a_{n-1})$ while traversing C, the most recent edge had label a_{n-1}, because the label on an edge entering a vertex agrees with the last digit of the sequence at the vertex. Since we delete the front and shift the rest to obtain the rest of the label at the head, the successive earlier labels (looking backward) must have been $a_{n-2}, \ldots, a_1$ in order. If C next traverses an edge with label a_n, then the subsequence consisting of the n most recent edge labels at that time is $a_1, \ldots, a_n$.

Since the k^{l-1} vertex labels are distinct, and the edges leaving each vertex have distinct labels, and we traverse each edge from each vertex exactly once along C, the k^l strings of length l in the circular arrangement given by the edge labels along C are distinct.

1.4.28. *De Bruijn cycle for length 4 without the constant words.* Make a vertex for each of the m^3 sequences of length 3 from the alphabet S. Put an edge from sequence a to sequence b, with label $\alpha \in S$, if b is obtained from a by dropping the first letter and appending α to the end. Since there are m ways to append a letter, the out-degree of each vertex is m. For each sequence, there are m possible letters that could have been deleted to reach it, so the in-degree of each vertex is m.

Deleting the loops at the m constant vertices (aaa, bbb, etc.) reduces the indegree and outdegree at those vertices by 1, so the resulting digraph has equal indegree and outdegree at every vertex. Also the underlying graph is connected, since vertex abc can be reach from any other vertex by following the edge labeled a, then b, then c.

Thus an Eulerian circuit exists. Recording the edge labels while following an Eulerian circuit yields the desired arrangement. The 4-digit strings obtained are those formed by the 3-digit name of a vertex plus the label on an exiting edge. These $m^4 - m$ strings are distinct and avoid the constant words, since the loops were deleted from the digraph.

Alternative proof. If we know (from Exercise 1.4.27, for example) that there exists a De Bruijn cycle including the constant words, then we can simply delete one letter from each string of four consecutive identical letters, without using graph theory.

1.4.29. *A strong orientation of a graph that has an odd cycle also has an odd (directed) cycle.* Suppose that D is a strong orientation of a graph G that has an odd cycle $v_1, \ldots, v_{2k+1}$. Since D is strongly connected, for each i there is a v_i, v_{i+1}-path in D. If for some i every such path has even length, then the edge between v_i and v_{i+1} points from v_{i+1} to v_i, since the other orientation would be a v_i, v_{i+1}-path of length 1 (odd). In this case, we have an odd cycle through v_i and v_{i+1}. Otherwise, we have a path of odd length from each v_i to v_{i+1}. Combining these gives a closed trail of odd length. In a digraph as well as in a graph (by the same proof), a closed odd trail contains the edges of an odd cycle.

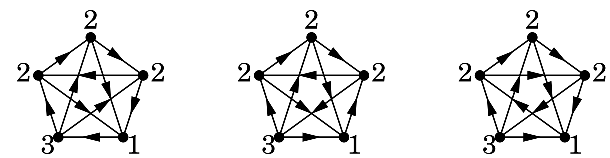
1.4.30. *The maximum length of a shortest spanning closed walk in a strongly-connected n-vertex digraph is $\lfloor (n+1)^2/4 \rfloor$ if $n \ge 3$.* For the lower bound, let G consist of a u, v-path P of $n-l$ vertices, plus l vertices with edges from v and to u. When leaving a vertex not on P, P must be reached and traversed before the next vertex off P. Hence G requires $l(n-l+1)$ steps to walk through every vertex, maximized by setting $l = \lfloor (n+1)/2 \rfloor$. The length of the walk is then $\lfloor (n+1)^2/4 \rfloor$.

For any strongly-connected n-vertex digraph G, we obtain a spanning closed walk of length at most $\lfloor (n+1)^2/4 \rfloor$. Let m be the maximum length of a path in G; from each vertex to every other, there is a path of length at most m. Begin with a path P of length m; this visits $m+1$ vertices. Next use paths to reach each of the remaining vertices in turn, followed by a path returning to the beginning of P. In this closed walk, $1+(n-m-1)+1$ paths have been followed, each of length at most m. The total length is at most $m(n+1-m)$, which is bounded by $\lfloor (n+1)^2/4 \rfloor$.

1.4.31. *The smallest nonisomorphic pair of tournaments with the same score sequences have five vertices.*

At least five vertices are needed. The score sequence (outdegrees) of an n-vertex tournament can have only one 0 or $n-1$. Nonisomorphic tournaments with such a vertex must continue to be nonisomorphic when that vertex is deleted. Hence a smallest nonisomorphic pair has no vertex with score 0 or $n-1$. The only such score sequences with fewer than 5 vertices are 111 and 2211. The first is realized only by the 3-cycle. For 2211, name the low-degree vertices as v_1 and v_2 such that $v_1 \leftarrow v_2$, and name the high-degree vertices as v_3 and v_4 such that $v_3 \leftarrow v_4$. The only way to complete a tournament with this score sequence is now $N^+(v_1) = \{v_4\}$, $N^+(v_2) = \{v_1\}$, $N^+(v_3) = \{v_1, v_2\}$, and $N^+(v_4) = \{v_2, v_3\}$.

Five vertices suffice, by construction. On five vertices, the sequences to consider are 33211, 32221, and 22222. There is only one isomorphic class with score sequence 22222, but there are more for the other two sequences. In fact, there are 3 nonisomorphic tournaments with score sequence 32221. They may be characterized as follows: (1) the bottom player beats the top player, and the three middle players induce a cyclic subtournament; (2) the top player beats the bottom player, and the three middle players induce a cyclic subtournament; (3) the top player beats the bottom player, and the three middle players induce a transitive subtournament.



Five vertices suffice, by counting. Each score sequence sums to 10 and has maximum outdegree at most 4; also there is at most one 4 and at most one 0. The possibilities are thus 43210, 43111, 42220, 42211, 33310, 33220, 33211, 32221, 22222. There are 2^{10} tournaments on five vertices; we show that they cannot fit into nine isomorphism classes. The isomorphism class consisting of a 5-cycle plus edges from each vertex to the vertex two later along the cycle occurs 4! times; once for each cyclic ordering of the vertices. Each of the other isomorphism classes occurs at most 5! times. Hence the nine isomorphism classes contain at most $24 + 8 \cdot 120$ of the 2^{10} tournaments. Since $1024 > 984$, there must be at least 10 isomorphism classes among the nine score sequences.

1.4.32. *Characterization of bigraphic sequences.* With $p = p_1, \ldots, p_m$ and $q = q_1, \ldots, q_n$, the pair (p, q) is **bigraphic** if there is a simple bipartite graph in which $p_1, \ldots, p_m$ are the degrees for one partite set and $q_1, \ldots, q_n$ are the degrees for the other.

If p has positive sum, then (p, q) is bigraphic if and only if (p', q') is bigraphic, where (p', q') is obtained from (p, q) by deleting the largest element Δ from p and subtracting 1 from each of the Δ largest elements of q. We follow the method of Theorem 1.3.31. Sufficiency of the condition follows by adding one vertex to a realization of the smaller pair.

For necessity, choose indices in a realization G so that $p_1 \ge \cdots \ge p_m$, $q_1 \ge \cdots \ge q_n$, $d(x_i) = p_i$, and $d(y_j) = q_j$. We produce a realization in which x_1 is adjacent to $y_1, \ldots, y_{p_1}$. If $y_j \not\leftrightarrow x_1$ for some $j \le p_1$, then $y_k \leftrightarrow x_1$ for some $k > p_1$. Since $q_j \ge q_k$, there exists x_i with $i > 1$ such that $x_i \in N(y_j) - N(y_k)$. We perform the 2-switch to replace $\{x_1 y_k, x_i y_j\}$ with $\{x_1 y_j, x_i y_k\}$. This reduces the number of missing neighbors, so we can obtain the desired realization. (Comment: the statement also holds when $m = 1$.)

1.4.33. *Bipartite 2-switch and 0,1-matrices with fixed row and column sums.* With a simple X,Y-bigraph G, we associate a 0,1-matrix $B(G)$ with rows indexed by X and columns indexed by Y. The matrix has a 1 in position i,j if and only if $x_i \leftrightarrow y_j$. Applying a 2-switch to G that exchanges $xy, x'y'$ for $xy', x'y$ (preserving the bipartition) affects $B(G)$ by interchanging the 0's and 1's in the 2 by 2 permutation submatrix induced by rows x,x' and columns y,y'. Hence there is a sequence of 2-switches transforming G to H without changing the bipartition if and only if there is a sequence of switches on 2 by 2 permutation submatrices that transforms $B(G)$ to $B(H)$.

Furthermore, G and H have the same bipartition and same vertex degrees if and only if $B(G)$ and $B(H)$ have the same row sums and the same column sums. Therefore, in the language of bipartite graphs the statement about matrices becomes "all bipartite graphs with the same bipartition and vertex degrees can be reached from each other using 2-switches preserving the bipartition." We prove either statement by induction. We use the phrasing of bipartite graphs.

Proof 1 (induction on m). If $m=1$, then already $G=H$. For $m>1$, let G be an X,Y-bigraph. Let x be a vertex of maximum degree in X, with $d(x)=k$. Let S be a set of k vertices of highest degree in Y. Using bipartition-preserving 2-switches, we transform G so that $N(x)=S$. If $N(x)\neq S$, we choose $y\in S$ and $y'\in Y-S$ so that $x \not\leftrightarrow y$ and $x\leftrightarrow y'$. Since $d(y)\geq d(y')$, we have $x'\in X$ so that $y\leftrightarrow x'$ and $y'\not\leftrightarrow x'$. Switching $xy', x'y$ for $xy, x'y'$ increases $|N(x)\cap S|$. Iterating this reaches $N(x)=S$. We can do the same thing in H to reach graphs G' from G and H' from H such that $N_{G'}(x)=N_{H'}(x)$. Now we can delete x and apply the induction hypothesis to the graphs $G^*=G'-x$ and $H^*=H'-x$ to complete the construction of the desired sequence of 2-switches.

Proof 2 (induction on number of discrepancies). Let F be the bipartite graph with the same bipartition as G and H consisting of edges belonging to exactly one of G and H. Let $d=e(F)$. Orient F by directing each edge of $G-E(H)$ from X to Y and each edge of $H-e(G)$ from Y to X. Since G,H have identical vertex degrees, in-degree equals outdegree at each vertex of F. If $d>0$, this implies that F contains a cycle. There is a 2-switch in G that introduces two edges of $E(G)-E(H)$ and reduces d by 4 if and only if F has a 4-cycle. Otherwise, Let C be a shortest cycle in F, and let x,y,x',y' be consecutive vertices on C. We have $xy\in E(G)-E(H)$, $x'y\in E(H)-E(G)$, and $x'y'\in E(G)-E(H)$. We also have $xy'\notin E(G)$, else we could replace these three edges of C by xy' to obtain a shorter cycle in F. We can now perform the 2-switch in G that replaces $xy, x'y'$ with $xy', x'y$. This reduces d by at least 2.

1.4.34. *If G and H are two tournaments on a vertex set V, then $d_G^+(v) =$*

$d_H^+(v)$ for all $v\in V$ if and only if G can be turned into H by a sequence of direction-reversals on cycles of length 3. Reversal of a 3-cycle changes no outdegree, so the condition is sufficient.

For necessity, let F be the subgraph of G consisting of edges oriented the opposite way in H. Since $d_G^+(v)=d_H^+(v)$ and $d_G^-(v)=d_H^-(v)$ for all v, every vertex has the same indegree and outdegree in F. Let x be a vertex of maximum degree in F, and let $S=N_F^+(x)$ and $T=N_F^-(x)$.

An edge from S to T in G completes a 3-cycle with x whose reversal in G reduces the number of pairs on which G and H disagree. An edge from T to S in H completes a 3-cycle with x whose reversal in H reduces the number of disagreements. If neither of these possibilities occurs, then G orients every edge of $S\times T$ from T to S, and H orients every such edge from S to T. Also F has edges from T to x. This gives every vertex of T higher outdegree than x in F, contradicting the choice of x.

1.4.35. *$p_1\leq\cdots\leq p_n$ is the sequence of outdegrees of a tournament if and only if $\sum_{i=1}^k p_i \geq \binom{k}{2}$ and $\sum_{i=1}^n p_i = \binom{n}{2}$.* *Necessity*. A tournament has $\binom{n}{2}$ edges in total, and any k vertices have out-degree-sum at least $\binom{k}{2}$ within the subtournament they induce.

Sufficiency. Given a sequence p satisfying the conditions, let $q_k=\sum_{i=1}^k p_k$ and $e_k=q_k-\binom{k}{2}$. We prove sufficiency by induction on $\sum e_k$. The only sequence p with $\sum e_k=0$ is $0,1,\ldots,n-1$; this is realized by the *transitive* tournament T_n having $v_k\to v_j$ if and only if $k>j$. If $\sum e_k>0$, let r be the least k with $e_k>0$, and let s be the least index above r with $e_k=0$, which exists since $e_n=0$. We have $q_{s-1}>\binom{s-1}{2}$, $q_s=\binom{s}{2}$, and $q_{s+1}\geq\binom{s+1}{2}$. This yields $p_{s+1}\geq s$ and $p_s<s-1$, or $p_{s+1}-p_s\geq 2$. Similarly, if $r=1$ we have $p_1\geq 1$, and if $r>1$ we have $p_r-p_{r-1}\geq 2$.

Hence we can subtract one from p_r and add one to p_s to obtain a new sequence p' that is non-decreasing, satisfies the conditions, and reduces $\sum e_k$ by $s-r$. By the induction hypothesis, there is a tournament with score sequence p'. If $v_s\to v_r$ in this tournament, we can reverse this edge to obtain the score sequence p. If not, then the fact that $p'_s\geq p'_r$ implies there is another vertex u such that $v_s\to u$ and $u\to v_r$; obtain the desired tournament by reversing these two edges.

1.4.36. Let T be a tournament having no vertex with indegree 0.

a) If x is a king in T, then T has another king in $N^-(x)$. The subdigraph induced by the vertices of $N^-(x)$ is also a tournament; call it T'. Since every tournament has a king, T' has a king. Let y be a king in T'. Since x is a successor of y and every vertex of $N^+(x)$ is a successor of x, every vertex of $V(T)-V(T')$ is reachable from y by a path in T of length at most T. Hence y is also a king in the original tournament T.

b) T has at least three kings. Since T is a tournament, it has some king, x. By part (a), T has another king y in $N^-(x)$. By part (a) again, T has another king z in $N^-(y)$. Since $y \to x$, we have $x \notin N^-(y)$, and hence $z \neq x$. Thus x, y, z are three distinct kings in T.

c) For $n \geq 3$, an n-vertex tournament T with no source and only three kings. Let $S = \{x, y, z\}$ be a set of three vertices in $V(T)$. Let the subtournament on S be a 3-cycle. For all edges joining S and $V(T) - S$, let the endpoint in S be the tail. Place any tournament on $V(T) - S$. Now x, y, z are kings, but no vertex outside S is a king, because no edge enters S.

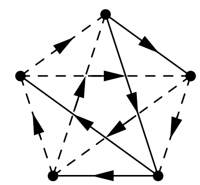
1.4.37. *Algorithm to find a king in a tournament T: Select $x \in V(T)$. If x has indegree 0, call it a king and stop. Otherwise, delete $\{x\} \cup N^+(x)$ from T to form T', and call the output from T' a king in T.* We prove the claims by induction on the number of vertices. The algorithm terminates, because it either stops by selecting a source (indegree 0) or moves to a smaller tournament. By the induction hypothesis, it terminates on the smaller tournament. Thus in each case it terminates and declares a king.

We prove by induction on the number of vertices that the vertex declared a king is a king. When there is only one vertex, it is a king. Suppose that $n(T) > 1$. If the initial vertex x is declared a king immediately, then it has outdegree $n - 1$ and is a king. Otherwise, the algorithm deletes x and its successors and runs on the tournament T' induced by the set of predecessors (in-neighbors) of x.

By the induction hypothesis, the vertex z that the algorithm selects as king in T' is a king in T', reaching each vertex of T' in at most two steps. It suffices to show that z is also a king in the full tournament. Since T' contains only predecessors of x, $z \to x$. Also, z reaches all successors of x in two steps through x. Thus z also reaches all discarded vertices in at most two steps and is a king in T.

1.4.38. *Tournaments with all players kings. a) If n is odd, then there is an tournament with n vertices such that every player is a king.*

Proof 1 (explicit construction). Place the players around a circle. Let each player defeat the $(n - 1)/2$ players closest to it in the clockwise direction, and lose to the $(n - 1)/2$ players closest to it in the counterclockwise direction. Since every pair of players is separated by fewer players around one side of the circle than the other, this gives a well-defined orientation to each edge. All players have exactly $(n - 1)/2$ wins. Thus every outdegree is the maximum outdegree, and we have proved that every vertex of maximum outdegree in a tournament is a king. It is also easy to construct explicit paths. Each player beats the next $(n - 1)/2$ players. The remaining $(n - 1)/2$ players all lose to the last of these first $(n - 1)/2$ players. The construction is illustrated below for five players.



Proof 2 (induction on n). For $n = 3$, every vertex in the 3-cycle is a king. For $n \geq 3$, given a tournament on vertex set S of size n in which every vertex is a king, we add two new vertices x, y. We orient $S \to x \to y \to S$. Every vertex of S reaches x in one step and y in two; x reaches y in one step and each vertex of S in two. Every vertex is a king. (The resulting tournaments are not regular.) Note: Since there is no such tournament when $n = 4$, one must also give an explicit construction for $n = 6$ to include in the basis. The next proof avoids this necessity.

Proof 3 (induction on n). For $n = 3$, we have the cyclic tournament. For $n = 5$, we have the cyclically symmetric tournament in which each vertex beats the two vertices that follow it on the circle. For $n > 5$, let T be an $(n-1)$-vertex tournament in which every vertex is a king, as guaranteed by the induction hypothesis. Add a new vertex x.

If n is odd, then partition $V(T)$ into pairs. For each pair, let a and b be the tail and head of the edge joining them, and add the edges xa and bx.

If n is even, then among any four vertices of $V(T)$ we can find a triple $\{u, v, w\}$ that induces a non-cyclic tournament. Pick one such triple, and partition the remaining vertices of $V(T)$ into pairs. Treat the edges joining x to these pairs as in the other case. Letting u be the vertex of the special triple with edges to the two other vertices, add edges xu, vx, and wx.

b) There is no tournament with four players in which every player is a king. Suppose G is such a tournament. A player with no wins cannot be a king. If some vertex has no losses, then no **other** vertex can be a king. Hence every player of G has 1 or 2 wins. Since the total wins must equal the total losses, there must be two players with 1 win and two players with 2 wins. Suppose x, y are the players with 1 win; by symmetry, suppose x beats y. Since x has no other win and y has exactly one win, the fourth player is not reached in two steps from x, and x is not a king.

1.4.39. *Every loopless digraph D has a vertex subset S such that $D[S]$ has no edges but every vertex is reachable from S by a path of length at most 2.*

Proof 1 (induction). The claim holds when $n(D) = 1$ and when there is a vertex with edges to all others. Otherwise, consider an arbitrary vertex x, and let $D' = D - x - N^+(x)$. Let S' be the subset of $V(D')$ guaranteed by the induction hypothesis. Observe that $S' \cap N^+(x) = \varnothing$. If $yx \in E(D)$ for some $y \in S'$, then $x \cup N^+(x)$ is reachable from y within two steps, and S' is the desired set S. Otherwise, the set $S = S' \cup \{x\}$ works.

Proof 2 (construction). Index the vertices as $v_1, \ldots, v_n$. Process the list in increasing order; when a vertex v_i is reached that has not been deleted, delete all successors of v_i with higher indices. Next process the list in decreasing order; when a vertex v_i is reached that has not been deleted (in either pass), delete all successors of v_i with lower indices.

The set S of vertices that are not deleted in either pass is independent. Every vertex deleted in the second pass has a predecessor in S. Every vertex deleted in the first pass can be reached from S directly or from a vertex deleted in the second pass, giving it a path of length at most two from S. Hence S has the desired properties.

Proof 3 (kernels). By looking at the reverse digraph, it suffices to show that every loopless digraph D has an independent set S that can be reached by a path of length at most 2 from each vertex outside S. Given a vertex ordering $v_1, \ldots, v_n$, decompose D into two acyclic spanning subgraphs G and H consisting of the edges that are forward and backwards in the ordering, respectively. All subgraphs of G and H are acyclic, and hence by Theorem 1.4.16 they have kernels. Let S be a kernel of the subgraph of G induced by a kernel T of H. Every vertex not in T has a successor in T, and every vertex in $T - S$ has a successor in S, so every vertex not in S has a path of length at most 2 to S. (Comment: The set S produced in this way is the same set produced in the reverse digraph by Proof 2. This proof is attributed to S. Thomasse on p. 163 of J. A. Bondy, Short proofs of classical theorems, *J. Graph Theory* 44 (2003), 159–165.)

1.4.40. *The largest unipathic subgraphs of the transitive tournament have* $\lfloor n^2/4 \rfloor$ *edges.* If a subgraph of T_n contains all three edges of any 3-vertex induced subtournament, then it contains two paths from the least-indexed of these vertices to the highest. Hence a unipathic subgraph must have as its underlying graph a triangle-free subgraph of K_n. By Mantel's Theorem, the maximum number of edges in such a subgraph is $\lfloor n^2/4 \rfloor$, achieved only by the complete equibipartite graph.

This leaves the problem of finding unipathic orientations of $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ in T_n. Suppose G is such a subgraph, with partite sets X, Y. If there are four vertices, say $i < j < k < l$, that alternate from the two partite sets of G or have i, l in one set and j, k in the other, then the oriented bipartite subgraph induced by X, Y as partite sets has two i, l-paths. Hence when $n \ge 4$ all the vertices of X must precede all the vertices of Y, or vice versa. To obtain $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$, we will have all edges ij such that $i \le \lfloor n/2 \rfloor$ and $j > \lfloor n/2 \rfloor$, or all edges such that $i \le \lceil n/2 \rceil$ and $j > \lceil n/2 \rceil$. Hence for $n \ge 4$ there are two extremal subgraphs when n is odd and only one when n is

even. (There is only one when $n = 1$, and there are three when $n = 3$.)

1.4.41. *Given any listing of the vertices of a tournament, every sequence of switchings of consecutive vertices that induce a reverse edge leads to a list with no reverse edges in at most* $\binom{n}{2}$ *steps.* Under this algorithm, each switch changes the order of only one pair. Furthermore, the order of two elements in the list can change only when they are consecutive and induce a reverse edge. Hence each pair is interchanged at most once, and the algorithm terminates after at most $\binom{n}{2}$ steps with a spanning path.

1.4.42. *Every ordering of the vertices of a tournament that minimizes the sum of lengths of the feedback edges puts the vertices in nonincreasing order of outdegree.* For the ordering $v_1, \ldots, v_n$, the sum is the sum of $j - i$ over edges $v_j v_i$ such that $j > i$. Consider the interchange of v_i and v_{i+1}. If some vertex is a successor of both or predecessor of both, then the contribution to the sum from the edges involving it remains unchanged. If $x \in N^+(v_i) - N^+(v_{i+1})$, then the switch increases the contribution from these edges by 1. If $x \in N^+(v_{i+1}) - N^+(v_i)$, then the switch decreases the contribution from these edges by 1. If $v_i \to v_{i+1}$, then the switch increases the cost by 1, otherwise it decreases. Hence the net change in the sum of the lengths of feedback edges is $d^+(v_i) - d^+(v_{i+1})$.

This implies that if the ordering has any vertex followed by a vertex with larger outdegree, then the sum can be decreased. Hence minimizing the sum puts the vertices in nonincreasing order of outdegree. Furthermore, permuting the vertices of a given outdegree among themselves does not change the sum of the lengths of feedback edges, so every ordering in nonincreasing order of outdegree minimizes the sum.