Problem 1.3-5: Windings

Figure P1.3-5(a) illustrates a motor that is constructed using windings that surround laminated iron poles. You have been asked to estimate the maximum temperature that will occur within the windings. The windings and poles are both approximated as being cylindrical, as shown in Figure P1.3-5(b).

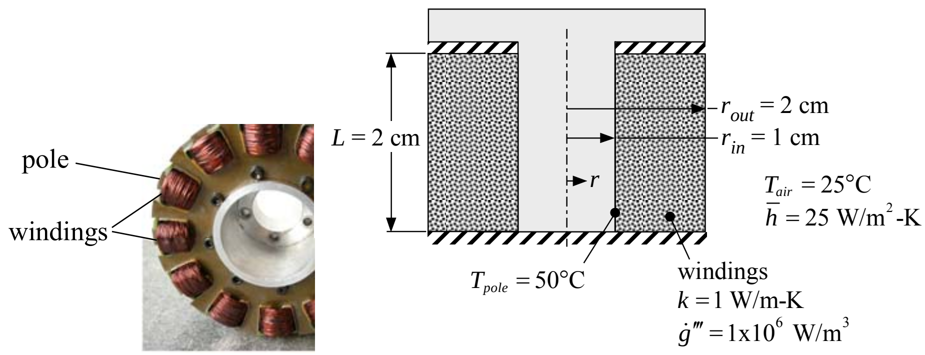

Figure P1.3-5: (a) Concentrated winding for a permanent magnet motor, (b) cylindrical model of the windings

The windings are a complicated composite formed from copper conductor, insulation and air that fills the gaps between adjacent wires. However, in most models, the windings are represented by a solid with equivalent properties that account for this underlying structure. You can therefore consider the windings in Figure P1.3-5(b) to be a solid. The electrical current in the windings causes an ohmic dissipation that can be modeled as a uniform volumetric generation rate of $\dot{g}'''$ =1x10^6 W/m^3. The conductivity of the windings is k = 1.0 W/m-K. The inner radius of the windings is r_{in} = 1.0 cm and the outer radius is r_{out} = 2.0 cm. The windings are L = 2.0 cm long and the upper and lower surfaces may be assumed to be insulated so that the temperature in the windings varies only in the radial direction. The stator pole is conductive and cooled externally; therefore, you can assume that the stator tooth has a uniform temperature of T_{pole} = 50°C. Neglect any contact resistance between the inner radius of the winding and the pole; therefore, the temperature of the windings at $r = r_{in}$ is T_{pole}. The outer radius of the windings is exposed to air at T_{air} = 20°C with a heat transfer coefficient of $\overline{h}$ = 25 W/m^2-K.

a.) Derive the governing differential equation for the temperature within the windings (i.e., the differential equation that is valid from $r_{in} < r < r_{out}$). You should end up with an ordinary differential equation for T in terms of the symbols provided in the problem statement. Clearly show your steps, which should include:
   1. define a differentially small control volume,
   2. do an energy balance on your control volume,
   3. expand the $r + dr$ terms in your energy balance and take the limit as $dr \rightarrow 0$,
   4. substitute rate equations into your energy balance.

A differential control volume is shown in Figure 3.

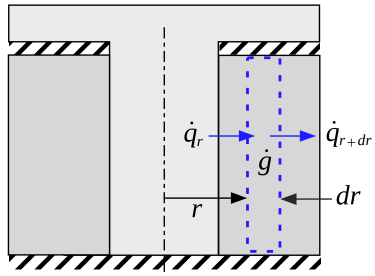

Figure 3: A differentially small control volume.

The energy balance on the control volume is:

$$\dot{q}_r + \dot{g} = \dot{q}_{r+dr} \tag{1}$$

or, after expanding the *r* + *dr* term:

$$\dot{q}_r + \dot{g} = \dot{q}_r + \frac{d\dot{q}_r}{dr} dr + \frac{d^2 \dot{q}_r}{dr^2} \frac{dr^2}{2!} + \frac{d^3 \dot{q}_r}{dr^3} \frac{dr^3}{3!} + \dots \tag{2}$$

Taking the limit as *dr* approaches zero leads to:

$$\dot{q}_r + \dot{g} = \dot{q}_r + \frac{d\dot{q}_r}{dr} dr \tag{3}$$

or

$$\dot{g} = \frac{d\dot{q}_r}{dr} dr \tag{4}$$

The parameter $\dot{g}$ is the rate of thermal energy generation within the control volume, which can be expressed as the product of the volume of the control volume and the volumetric rate of generation:

$$\dot{g} = \dot{g}''' \, 2\,\pi\, r\, L\, dr \tag{5}$$

and the conduction term is expressed using Fourier's law:

$$\dot{q} = -k\, 2\,\pi\, r\, L \frac{dT}{dr} \tag{6}$$

Substituting Eqs. (5) and (6) into Eq. (4) leads to:

$$\dot{g}''' \, 2\,\pi\, r\, L\, dr = \frac{d}{dr}\left[-k\, 2\,\pi\, r\, L \frac{dT}{dr}\right] dr \tag{7}$$

or

$$\boxed{\dot{g}''' r = -k \frac{d}{dr}\left[r \frac{dT}{dr} \right]} \tag{8}$$

b.) Specify the boundary conditions for your differential equation. This should be easy at the inner radius, where the temperature is specified, but you will need to carry out an interface energy balance at $r = r_{out}$.

The boundary condition at the inner radius is:

$$\boxed{T_{r=r_{in}} = T_{pole}} \tag{9}$$

An interface energy balance at the outer radius is shown in Figure 4.

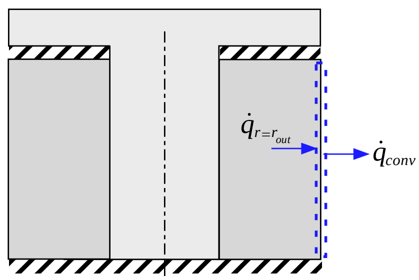

Figure 4: An interface energy balance at the outer radius.

The interface energy balance leads to:

$$\dot{q}_{r=r_{out}} = \dot{q}_{conv} \tag{10}$$

Substituting Fourier's law and Newton's law of cooling into Eq. (10) leads to:

$$-k\, 2\, \pi\, r_{out}\, L \left. \frac{dT}{dr} \right|_{r=r_{out}} = 2\, \pi\, r_{out}\, L\, h \left(T_{r=r_{out}} - T_{air} \right) \tag{11}$$

or

$$\boxed{-k \left. \frac{dT}{dr} \right|_{r=r_{out}} = h \left(T_{r=r_{out}} - T_{air} \right)} \tag{12}$$

c.) Solve the governing differential equation that you derived in part (a) by integrating twice. You should end up with a solution that involves two constants of integration, C_1 and C_2.

Equation (8) is rearranged:

$$d\left[r\frac{dT}{dr} \right] = -\frac{\dot{g}''' r}{k} dr \tag{13}$$

and integrated:

$$\int d\left[r\frac{dT}{dr} \right] = \int \left(-\frac{\dot{g}''' r}{k} \right) dr \tag{14}$$

to obtain:

$$r\frac{dT}{dr} = -\frac{\dot{g}''' r^2}{2k} + C_1 \tag{15}$$

so the temperature gradient is:

$$\frac{dT}{dr} = -\frac{\dot{g}''' r}{2k} + \frac{C_1}{r} \tag{16}$$

Equation (16) is again rearranged:

$$\int dT = \int \left(-\frac{\dot{g}''' r}{2k} + \frac{C_1}{r} \right) dr \tag{17}$$

to obtain:

$$\boxed{T = -\frac{\dot{g}''' r^2}{4k} + C_1 \ln(r) + C_2} \tag{18}$$

d.) Substitute your answer from part (c) into the boundary conditions specified in part (b) to obtain two equations for your two unknown constants of integration, C_1 and C_2.

Substituting Eq. (18) into Eq. (9) leads to:

$$\boxed{-\frac{\dot{g}''' r_{in}^2}{4k} + C_1 \ln(r_{in}) + C_2 = T_{pole}} \tag{19}$$

Substituting Eqs. (18) and (16) into Eq. (12) leads to:

$$\boxed{-k\left(-\frac{\dot{g}''' r_{out}}{2k} + \frac{C_1}{r_{out}} \right) = h\left(-\frac{\dot{g}''' r_{out}^2}{4k} + C_1 \ln(r_{out}) + C_2 - T_{air} \right)} \tag{20}$$

e.) Implement your results from (c) and (d) in EES and prepare a plot of the temperature in the stator as a function of radius.

The inputs are entered in EES:

```
$UnitSystem SI MASS RAD PA  K J
$TABSTOPS   0.2 0.4 0.6 0.8 3.5 in

"Inputs"
k=1 [W/m-K]                              "winding conductivity"
gv=1e6 [W/m^3]                           "winding volumetric generation"
T_air=converttemp(C,K,25)                "air temperature"
h=25 [W/m^2-K]                           "heat transfer coefficient"
r_out=2.0 [cm]*convert(cm,m)             "outer radius of windings"
r_in=1.0 [cm]*convert(cm,m)              "inner radius of windings"
T_pole=converttemp(C,K,50)               "pole temperature"
L=2.0 [cm]*convert(cm,m)                 "length of windings"
```

The two algebraic equations for C_1 and C_2, Eqs. (19) and (20), are entered:

```
"boundary conditions"
-gv*r_in^2/(4*k)+C_1*ln(r_in)+C_2=T_pole                          "at r=r_in"
-k*(-gv*r_out/(2*k)+C_1/r_out)=h*(-gv*r_out^2/(4*k)+C_1*ln(r_out)+C_2-T_air)   "at r=r_out"
```

and the solution is obtained using Eq. (18).

```
"solution"
T=-gv*r^2/(4*k)+C_1*ln(r)+C_2
r_cm=r*convert(m,cm)
T_C=converttemp(K,C,T)
```

Figure 5 illustrates the temperature as a function of radial position within the windings.

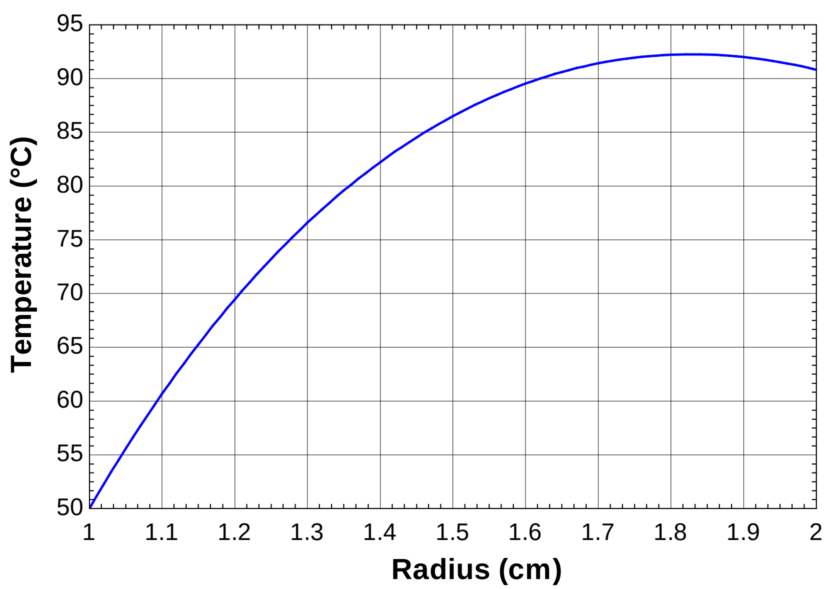

Figure 5: Temperature as a function of position within the windings.