Problem 1.8-1 (1-17 in text): Disk Brake

Figure P1.8-1 illustrates a disk brake for a rotating machine. The temperature distribution within the brake can be assumed to be a function of radius only. The brake is divided into two regions. In the outer region, from R_p = 3.0 cm to R_d = 4.0 cm, the stationary brake pads create frictional heating and the disk is not exposed to convection. The clamping pressure applied to the pads is P = 1.0 MPa and the coefficient of friction between the pad and the disk is μ = 0.15. You may assume that the pads are not conductive and therefore all of the frictional heating is conducted into the disk. The disk rotates at N = 3600 rev/min and is b = 5.0 mm thick. The conductivity of the disk is k = 75 W/m-K and you may assume that the outer rim of the disk is adiabatic.

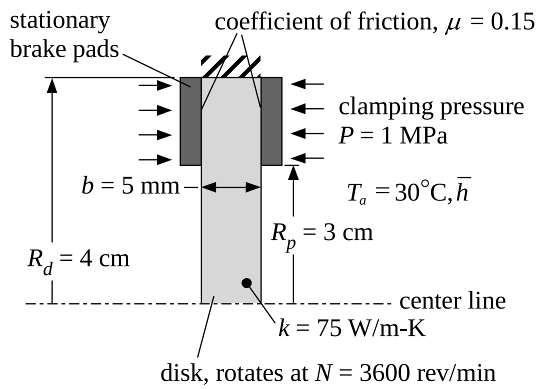

Figure P1.8-1: Disk brake.

In the inner region of the disk, from 0 to R_p, is exposed to air at T_a = 30°C. The heat transfer coefficient between the air and disk surface depends on the angular velocity of the disk, ω, according to:

$$\bar{h} = 20\left[\frac{\text{W}}{\text{m}^2\text{-K}}\right] + 1500\left[\frac{\text{W}}{\text{m}^2\text{-K}}\right]\left(\frac{\omega}{100\left[\text{rad/s}\right]}\right)^{1.25}$$

a.) Develop an analytical model of the temperature distribution in the disk brake; prepare a plot of the temperature as a function of radius for r = 0 to $r = R_d$.

The inputs are entered in EES and the heat transfer coefficient is computed according to Eq. **Error! Reference source not found.**.

```
$UnitSystem SI MASS RAD PA  K J
$TABSTOPS   0.2 0.4 0.6 0.8 3.5 in

b=5 [mm]*convert(mm,m)                          "thickness of disk"
N=3600 [rev/min]                                "rotational velocity of disk"
omega=N*convert(rev/min,rad/s)                  "angular velocity of disk"
mu=0.15 [-]                                     "coefficient of friction"
P=1 [MPa]*convert(MPa,Pa)                       "clamping pressure"
k=75 [W/m-K]                                    "conductivity"
Rd=4.0 [cm]*convert(cm,m)                       "outer radius of disk"
Rp=3.0 [cm]*convert(cm,m)                       "inner radius of pad"
```

```
Ta=converttemp(C,K,30)                                          "air temperature"
h=20[W/m^2-K]+1500 [W/m^2-K]*(omega/100 [rad/s])^1.25           "heat transfer coefficient"
```

In the outer region, region 1, the energy balance on a differential control volume is shown in Figure 2.

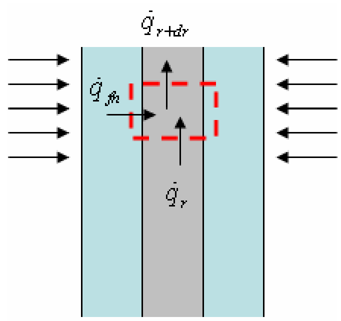

Figure 2: Differential energy balance in outer region, (region 1)

The energy balance suggested by Figure 2 is:

$$\dot{q}_r + \dot{q}_{fh} = \dot{q}_{r+dr} \tag{1}$$

where $\dot{q}_{fh}$ is the rate of thermal energy generated by frictional heating. After expanding the $r + dr$ term, Eq. (1) becomes:

$$\dot{q}_{fh} = \frac{d\dot{q}}{dr}dr \tag{2}$$

The rate equation for conduction is:

$$\dot{q} = -b\,2\,\pi\,r\,k\frac{dT_1}{dr} \tag{3}$$

where T_1 is the temperature in region 1. The force generated by the pad within the control volume is the product of the clamping pressure, the area of contact, and the coefficient of friction:

$$F = 4\,\pi\,r\,dr\,P\,\mu \tag{4}$$

Note that the factor of 4 in Eq. (4) is due to their being contact on both sides of the disk. The rate of frictional heating is the product of the force, the radius, and the angular velocity:

$$\dot{q}_{fh} = 4\,\pi\,r^2\,dr\,P\,\mu\,\omega \tag{5}$$

Substituting Eqs. (3) and (5) into Eq. (2) leads to:

$$4\pi r^2\, dr\, P\, \mu\, \omega = \frac{d}{dr}\left[-b\, 2\pi\, r\, k \frac{dT_1}{dr} \right] dr \tag{6}$$

which can be rearranged:

$$\frac{d}{dr}\left[r \frac{dT_1}{dr} \right] = -\frac{2\, P\, \mu\, \omega}{b\, k} r^2 \tag{7}$$

or

$$\frac{d}{dr}\left[r \frac{dT_1}{dr} \right] = -\beta\, r^2 \tag{8}$$

where

$$\beta = \frac{2\, P\, \mu\, \omega}{b\, k} \tag{9}$$

Equation (8) can be directly integrated:

$$\int d\left[r \frac{dT_1}{dr} \right] = -\beta \int r^2\, dr \tag{10}$$

to achieve:

$$r \frac{dT_1}{dr} = -\beta \frac{r^3}{3} + C_1 \tag{11}$$

Equation (11) can be directly integrated again:

$$\int dT_1 = \int \left(-\beta \frac{r^2}{3} + \frac{C_1}{r} \right) dr \tag{12}$$

to achieve:

$$T_1 = -\beta \frac{r^3}{9} + C_1 \ln\left(r\right) + C_2 \tag{13}$$

Equation (13) is the general solution for the temperature in region 1; the constants of integration will be selected in order to satisfy the boundary conditions.

In the inner region, region 2, the energy balance on a differential control volume is shown in Figure 3.

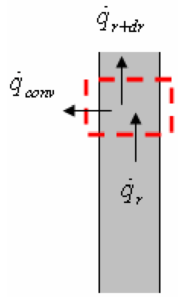

Figure 3: Differential energy balance in inner region, (region 2)

The energy balance suggested by Figure 2 is:

$$\dot{q}_r = \dot{q}_{r+dr} + \dot{q}_{conv} \tag{14}$$

After expanding the *r* + *dr* term, Eq. (14) becomes:

$$0 = \frac{d\dot{q}}{dr}dr + \dot{q}_{conv} \tag{15}$$

The rate equation for conduction remains the same:

$$\dot{q} = -b\,2\,\pi\,r\,k\frac{dT_2}{dr} \tag{16}$$

where T_2 is the temperature in region 2. The rate equation for convection is:

$$\dot{q}_{conv} = 4\,\pi\,r\,dr\,h\left(T_2 - T_a\right) \tag{17}$$

Substituting Eqs. (16) and (17) into Eq. (15) leads to:

$$\frac{d}{dr}\left[-b\,2\,\pi\,r\,k\frac{dT_2}{dr}\right]dr + 4\,\pi\,r\,dr\,h\left(T_2 - T_a\right) = 0 \tag{18}$$

or

$$\frac{d}{dr}\left[r\frac{dT_2}{dr}\right] - m^2\,r\,T_2 = -m^2\,r\,T_a \tag{19}$$

where

$$m = \sqrt{\frac{2h}{bk}} \tag{20}$$

The solution to Eq. (19) can be divided into its homogeneous (u_2) and particular (v_2) parts:

$$T_2 = u_2 + v_2 \tag{21}$$

The solution to the particular equation:

$$\frac{d}{dr}\left[r\frac{dv_2}{dr}\right] - m^2\, r\, v_2 = -m^2\, r\, T_a \tag{22}$$

is

$$v_2 = T_a \tag{23}$$

The homogeneous equation:

$$\frac{d}{dr}\left[r\frac{du_2}{dr}\right] - m^2\, r\, u_2 = 0 \tag{24}$$

is a form of Bessel's equation:

$$\frac{d}{dx}\left(x^p\,\frac{d\theta}{dx}\right) \pm c^2\, x^s\, \theta = 0 \tag{25}$$

where

$$x = r \tag{26}$$

$$\theta = u_2 \tag{27}$$

$$p = 1 \tag{28}$$

$$c = m \tag{29}$$

$$s = 1 \tag{30}$$

and the last term is negative. Following the flow chart provided in Section 1.8.4 of the book leads to $n = 0$, $a = 1$, and therefore the solution is:

$$u_2 = C_3\,\mathrm{BesselI}\left(0, m\,r\right) + C_4\,\mathrm{BesselK}\left(0, m\,r\right) \tag{31}$$

The general solution for the temperature distribution in region 2 is therefore:

$$T_2 = C_3 \,\mathrm{BesselI}\left(0, m\,r\right) + C_4 \,\mathrm{BesselK}\left(0, m\,r\right) + T_a \tag{32}$$

Note that this could be obtained directly from Maple by entering Eq. (19):

```
> restart;
> ODE:=diff(r*diff(T2(r),r),r)-m^2*r*T2(r)=-m^2*r*Ta;
```

$$ODE := \left(\frac{d}{dr}\mathrm{T2}(r)\right) + r\left(\frac{d^2}{dr^2}\mathrm{T2}(r)\right) - m^2\,r\,\mathrm{T2}(r) = -m^2\,r\,Ta$$

```
> T2s:=dsolve(ODE);
```

$$T2s := \mathrm{T2}(r) = \mathrm{BesselI}(0, m\,r)\,\_C2 + \mathrm{BesselK}(0, m\,r)\,\_C1 + Ta$$

The constants C_1 through C_4 in Eqs. (13) and (32) are obtained by applying the correct boundary conditions. At $r = 0$, the temperature must remain finite. The figures provided in Section 1.8.4 of the book or the limit capability in Maple show that BesselK(0,m r) will become infinite as r approaches zero:

```
> limit(BesselI(0,m*r),r=0);
```

$$1$$

```
> limit(BesselK(0,m*r),r=0);
```

$$\infty$$

therefore:

$$C_4 = 0 \tag{33}$$

The temperature and temperature gradient at the interface between the regions must be continuous:

$$T_{2,r=R_p} = T_{1,r=R_p} \tag{34}$$

and

$$\left.\frac{dT_2}{dr}\right|_{r=R_p} = \left.\frac{dT_1}{dr}\right|_{r=R_p} \tag{35}$$

The temperature gradient at the outer rim must be zero:

$$\left.\frac{dT_1}{dr}\right|_{r=R_p} = 0 \tag{36}$$

Substituting Eqs. (13) and (32) into Eqs. (33) through (36) leads to:

$$C_3\,\mathrm{BesselI}\left(0, m\,R_p\right) + T_a = -\beta\,\frac{R_p^3}{9} + C_1 \ln\left(R_p\right) + C_2 \tag{37}$$

$$C_3\, m\,\mathrm{BesselI}\left(1, m\,R_p\right) = -\beta\,\frac{R_p^2}{3} + \frac{C_1}{R_p} \tag{38}$$

$$-\beta\,\frac{R_d^2}{3} + \frac{C_1}{R_d} = 0 \tag{39}$$

Equations (37) through (39) are 3 equations for the unknown constants and can be solved in EES.

```
beta=2*mu*P*omega/(k*b)                                    "generation parameter"
m=sqrt(2*h/(k*b))                                          "fin parameter"
BesselI(0,m*Rp)*C_3+Ta=-1/9*beta*Rp^3+C_1*ln(Rp)+C_2       "equality of temperature at r=Rp"
BesselI(1,m*Rp)*m*C_3=-1/3*beta*Rp^2+1/Rp*C_1
     "equality of temperature gradient at r=Rp"
-1/3*beta*Rd^2+1/Rd*C_1=0                                  "zero temperature gradient at r=Rd"
```

The general solutions are entered in EES:

```
T2 = BesselI(0,m*r2)*C_3+Ta                                "solution in region 2"
T1 = -1/9*beta*r1^3+C_1*ln(r1)+C_2                         "solution in region 1"
```

A dimensionless radius, the variable rbar, is defined in order to allow a Parametric Table to be generated where the variable r1 can be easily altered from R_p to R_d and the r2 can be easily altered from 0 to R_p:

```
r1=Rp+(Rd-Rp)*rbar
r2=rbar*Rp
```

Figure 4 illustrates the temperature distribution in the disk.

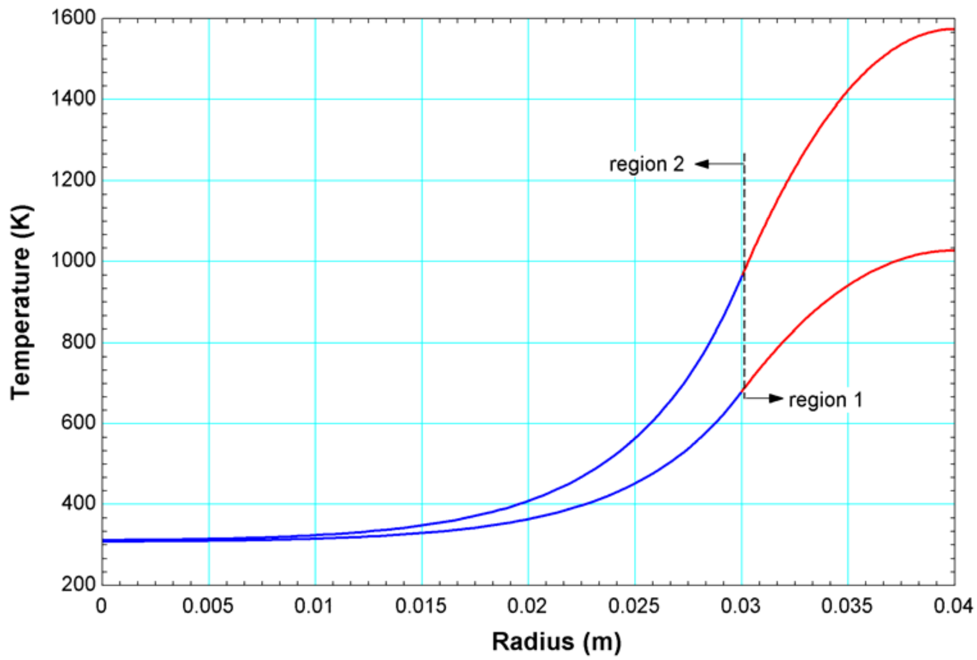

Figure 4: Temperature distribution in the disk

b.) If the disk material can withstand a maximum safe operating temperature of 750°C then what is the maximum allowable clamping pressure that can be applied? Plot the temperature distribution in the disk at this clamping pressure. What is the braking torque that results?

The maximum operating temperature is obtained at $r = R_d$ (see Figure 4). The clamping pressure that results in T_1 at the outer rim reaching the maximum allowable temperature can be determined by commenting out the originally specified clamping pressure and specifying this temperature:

```
{P=1 [MPa]*convert(MPa,Pa)}                          "clamping pressure"
T_max_allowed=converttemp(C,K,750)                   "maximum allowable temperature"
rbar=1.0
T1=T_max_allowed
```

which leads to a clamping pressure of P = 0.57 MPa. The temperature distribution for this clamping pressure is shown in Figure 4. The torque applied by the pads (Tq) is obtained from the integral:

$$Tq = \int_{R_p}^{R_d} 4\pi r^2 \mu P \, dr \tag{40}$$

or

$$Tq = \frac{4}{3}\pi \mu P \left[R_d^3 - R_p^3 \right] \tag{41}$$

which leads to Tq = 13.2 N-m.

c.) Assume that you can control the clamping pressure so that as the machine slows down the maximum temperature is always kept at the maximum allowable temperature, 750°C. Plot the torque as a function of rotational speed for 100 rev/min to 3600 rev/min.

A parametric table is created that includes the variables N and Tq,; N is varied from 100 rev/min to 3600 rev/min. The results are shown in Figure 5. Notice that it is possible to dramatically improve the performance of the brake if you can adjust the clamping pressure with speed.

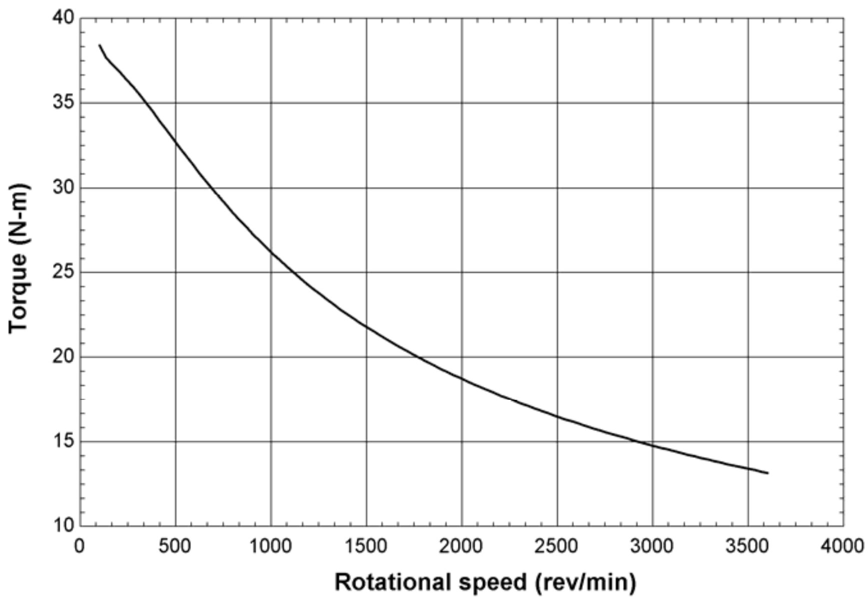

Figure 5: Clamping pressure and torque as a function of rotational velocity.