

Problem 1.5-2 (1-12 in text): Mass Flow Meter (re-visited)

Reconsider the mass flow meter that was investigated in Problem 1.3-9 (1-9 in text). Assume that the conductivity of the material that is used to make the test section is not actually constant as was assumed in Problem 1.3-9 (1-9 in text) but rather depends on temperature according to:

$$k = 10 \frac{\text{W}}{\text{m-K}} + 0.035 \left[\frac{\text{W}}{\text{m-K}^2} \right] (T - 300[\text{K}])$$

- a.) Develop a numerical model of the mass flow meter using MATLAB. Plot the temperature as a function of radial position for the conditions shown in Figure P1.3-9 (P1-9 in text) with the temperature-dependent conductivity.

The inputs are entered in a MATLAB function that requires as an input the number of nodes (N):

```
function[r,T_C]=P1p5_2(N)

    r_out=0.0254;    %outer radius of test section (m)
    r_in=0.01905;    %inner radius of test section (m)
    h_bar_out=10;     %external convection coefficient (W/m^2-K)
    T_infinity=293.2; %air temperature (K)
    T_f=291.2;        %fluid temperature (K)
    gv=1e7;           %rate of generation (W/m^3)
    m_dot=0.75;        %mass flow rate (kg/s)
    th_ins=0.00635;    %thickness of the insulation (m)
    k_ins=1.5;         %insulation conductivity (W/m-K)
    L=0.0762;         %length of the test section (m)
    C=2500;           %constant for convection relationship
```

The convection coefficient on the internal surface is computed:

```
h_bar_in=C*m_dot^0.8;    %internal convection coefficient
```

A function is defined that returns the conductivity of the material:

```
function[k]=k_t(T)
    %conductivity of the material
    %
    %Inputs:
    % T: temperature (K)
    %
    %Outputs:
    % k: conductivity (W/m-K)

    k=10+0.035*(T-300);
end
```

A uniform distribution of nodes is used, the radial location of each node (r_i) is:

$$r_i = r_{in} + \frac{(i-1)}{(N-1)}(r_{out} - r_{in}) \quad \text{for } i = 1..N \quad (1)$$

where N is the number of nodes. The radial distance between adjacent nodes (Δr) is:

$$\Delta r = \frac{(r_{out} - r_{in})}{(N-1)} \quad (2)$$

```
DELTA r=(r_out-r_in)/(N-1);           %distance between adjacent nodes (m)
for i=1:N
    r(i)=r_in+(r_out-r_in)*(i-1)/(N-1); %position of each node (m)
end
```

The system of equations is placed in matrix format.

$$\underline{\underline{A}} \underline{X} = \underline{b} \quad (3)$$

The most logical technique for ordering the unknown temperatures in the vector \underline{X} is:

$$\underline{X} = \begin{bmatrix} X_1 = T_1 \\ X_2 = T_2 \\ \dots \\ X_N = T_N \end{bmatrix} \quad (4)$$

Equation (4) shows that the unknown temperature at node i (i.e., T_i) corresponds to element i of vector \underline{X} (i.e., X_i). The most logical technique for placing the equations into the $\underline{\underline{A}}$ matrix is:

$$\underline{\underline{A}} = \begin{bmatrix} \text{row 1 = control volume 1 equation} \\ \text{row 2 = control volume 2 equation} \\ \dots \\ \text{row } N = \text{control volume } N \text{ equation} \end{bmatrix} \quad (5)$$

In Eq. (5), the equation for control volume i is placed into row i .

An energy balance is carried out on a control volume surrounding each node. For node 1, placed at the inner surface (Figure P1.5-2-1):

$$\dot{q}_{conv,in} + \dot{q}_{outer} + \dot{g} = 0 \quad (6)$$

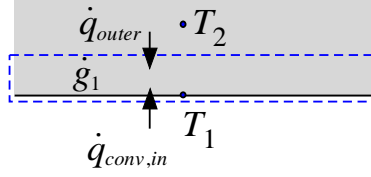


Figure P1.5-2-1: Control volume around node 1.

The rate equation for convection is:

$$\dot{q}_{conv,in} = \bar{h}_{in} 2\pi r_{in} L (T_f - T_1) \quad (7)$$

The rate equation for conduction is:

$$\dot{q}_{outer} = k_{T=(T_1+T_2)/2} 2\pi \left(r_{in} + \frac{\Delta r}{2} \right) L \frac{(T_2 - T_1)}{\Delta r} \quad (8)$$

The rate equation for generation is:

$$\dot{g} = 2\pi r_{in} \frac{\Delta r}{2} L \dot{g}''' \quad (9)$$

Substituting Eqs. (7) through (9) into Eq. (6) leads to:

$$\bar{h}_{in} 2\pi r_{in} L (T_f - T_1) + k_{T=(T_1+T_2)/2} 2\pi \left(r_{in} + \frac{\Delta r}{2} \right) L \frac{(T_2 - T_1)}{\Delta r} + \pi r_{in} \Delta r L \dot{g}''' = 0 \quad (10)$$

Equation (10) is rearranged to identify the coefficients that multiply each unknown temperature:

$$T_1 \left[-\bar{h}_{in} 2\pi r_{in} L - k_{T=(T_1+T_2)/2} 2\pi \left(r_{in} + \frac{\Delta r}{2} \right) \frac{L}{\Delta r} \right] + T_2 \left[k_{T=(T_1+T_2)/2} 2\pi \left(r_{in} + \frac{\Delta r}{2} \right) \frac{L}{\Delta r} \right] = -\pi r_{in} \Delta r L \dot{g}''' - \bar{h}_{in} 2\pi r_{in} L T_f \quad (11)$$

An energy balance on an internal node is shown in Figure P1.5-2-2:

$$\dot{q}_{inner} + \dot{q}_{outer} + \dot{g} = 0 \quad (12)$$

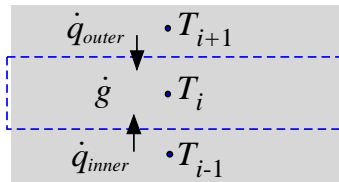


Figure P1.5-2-2: Control volume around internal node i .

The rate equations for conduction are:

$$\dot{q}_{outer} = k_{T=(T_i+T_{i+1})/2} 2\pi \left(r_{in} + \frac{\Delta r}{2} \right) L \frac{(T_{i+1} - T_i)}{\Delta r} \quad (13)$$

$$\dot{q}_{inner} = k_{T=(T_i+T_{i-1})/2} 2\pi \left(r_{in} - \frac{\Delta r}{2} \right) L \frac{(T_{i-1} - T_i)}{\Delta r} \quad (14)$$

The rate equation for generation is:

$$\dot{g} = 2\pi r_i \Delta r L \dot{g}''' \quad (15)$$

Substituting Eqs. (13) through (15) into Eq. (12) for all of the internal nodes leads to:

$$k_{T=(T_i+T_{i-1})/2} 2\pi \left(r_{in} - \frac{\Delta r}{2} \right) L \frac{(T_{i-1} - T_i)}{\Delta r} + k_{T=(T_i+T_{i+1})/2} 2\pi \left(r_{in} + \frac{\Delta r}{2} \right) L \frac{(T_{i+1} - T_i)}{\Delta r} + 2\pi r_i \Delta r L \dot{g}''' = 0 \text{ for } i = 2..(N-1) \quad (16)$$

Equation (16) is rearranged to identify the coefficients that multiply each unknown temperature:

$$\begin{aligned} & T_i \left[-k_{T=(T_i+T_{i-1})/2} 2\pi \left(r_{in} - \frac{\Delta r}{2} \right) \frac{L}{\Delta r} - k_{T=(T_i+T_{i+1})/2} 2\pi \left(r_{in} + \frac{\Delta r}{2} \right) \frac{L}{\Delta r} \right] \\ & + T_{i-1} \left[-k_{T=(T_i+T_{i-1})/2} 2\pi \left(r_{in} - \frac{\Delta r}{2} \right) \frac{L}{\Delta r} \right] + T_{i+1} \left[-k_{T=(T_i+T_{i+1})/2} 2\pi \left(r_{in} + \frac{\Delta r}{2} \right) \frac{L}{\Delta r} \right] \\ & = -2\pi r_i \Delta r L \dot{g}''' \text{ for } i = 2..(N-1) \end{aligned} \quad (17)$$

An energy balance on node N placed on the outer surface is shown in Figure P1.5-2-3:

$$\dot{q}_{inner} + \dot{q}_{air} + \dot{g} = 0 \quad (18)$$

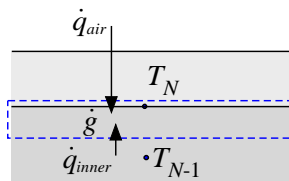


Figure P1.5-2-3: Control volume around internal node N .

The rate equation for the heat transfer with the air is:

$$\dot{q}_{air} = \frac{(T_{\infty} - T_N)}{(R_{ins} + R_{conv,out})} \quad (19)$$

where

$$R_{ins} = \frac{\ln \left[\frac{(r_{out} + th_{ins})}{r_{out}} \right]}{2 \pi L k_{ins}} \quad (20)$$

$$R_{conv,out} = \frac{1}{2 \pi (r_{out} + th_{ins}) L \bar{h}_{out}} \quad (21)$$

```
R_ins=log((r_out+th_ins)/r_out)/(2*pi*L*k_ins);
%resistance to conduction through insulation
R_conv_out=1/(2*pi*(r_out+th_ins)*L*h_bar_out);
%resistance to convection from the outside surface of the insulation
```

The rate equation for conduction is:

$$\dot{q}_{inner} = k_{T=(T_N+T_{N-1})/2} 2 \pi \left(r_{out} - \frac{\Delta r}{2} \right) L \frac{(T_{N-1} - T_N)}{\Delta r} \quad (22)$$

The rate equation for generation is:

$$\dot{g} = 2 \pi r_{out} \frac{\Delta r}{2} L \dot{g}''' \quad (23)$$

Substituting Eqs. (19), (22), and (23) into Eq. (18) leads to:

$$k_{T=(T_N+T_{N-1})/2} 2 \pi \left(r_{out} - \frac{\Delta r}{2} \right) L \frac{(T_{N-1} - T_N)}{\Delta r} + \frac{(T_{\infty} - T_N)}{(R_{ins} + R_{conv,out})} + 2 \pi r_{out} \frac{\Delta r}{2} L \dot{g}''' = 0 \quad (24)$$

Equation (24) is rearranged to identify the coefficients that multiply each unknown temperature:

$$\begin{aligned} T_N & \left[-k_{T=(T_N+T_{N-1})/2} 2 \pi \left(r_{out} - \frac{\Delta r}{2} \right) \frac{L}{\Delta r} - \frac{1}{(R_{ins} + R_{conv,out})} \right] \\ & + T_{N-1} \left[-k_{T=(T_N+T_{N-1})/2} 2 \pi \left(r_{out} - \frac{\Delta r}{2} \right) \frac{L}{\Delta r} \right] \\ & = -\pi r_{out} \Delta r L \dot{g}''' - \frac{T_{\infty}}{(R_{ins} + R_{conv,out})} \end{aligned} \quad (25)$$

Equations (11), (17), and (25) are N equations in the N unknown temperatures. Because they are non-linear, they must be linearized and a successive substitution method used. A guess temperature distribution (\hat{T}_i) is assumed:

```
%initial guess for temperature distribution
for i=1:N
    Tg(i,1)=T_f;
end
```

The matrix $\underline{\underline{A}}$ is defined as a sparse matrix with at most $3N$ nonzero entries:

```
%initialize A and b
A=spalloc(N,N,3*N);
b=zeros(N,1);
```

The solution is placed within a while loop that terminates when the error between the solution and the guess is less than some tolerance:

```
err=999;          %initial value for error (K), must be larger than tol
tol=0.01;         %tolerance for convergence (K)
while(err>tol)
```

The equation for node 1, Eq. (11), is linearized by using the guess temperature distribution to compute the conductivity:

$$T_1 \left[\underbrace{-\bar{h}_{in} 2\pi r_{in} L - k_{T=(\hat{T}_1+\hat{T}_2)/2} 2\pi \left(r_{in} + \frac{\Delta r}{2} \right) \frac{L}{\Delta r}}_{A_{1,1}} \right] + T_2 \left[\underbrace{k_{T=(\hat{T}_1+\hat{T}_2)/2} 2\pi \left(r_{in} + \frac{\Delta r}{2} \right) \frac{L}{\Delta r}}_{A_{1,2}} \right] = \underbrace{-\pi r_{in} \Delta r L \dot{g}''' - \bar{h}_{in} 2\pi r_{in} L T_f}_{b_1} \quad (26)$$

```
A(1,1)=-h_bar_in*2*pi*r_in*L-...
    k_t((Tg(1)+Tg(2))/2)*2*pi*(r_in+DELTA_r/2)*L/DELTA_r;
A(1,2)=k_t((Tg(1)+Tg(2))/2)*2*pi*(r_in+DELTA_r/2)*L/DELTA_r;
b(1)=-pi*r_in*DELTA_r*L*gv-h_bar_in*2*pi*r_in*L*T_f;
```

The equations for the internal nodes, Eq. (17), is also linearized:

$$\begin{aligned}
& T_i \underbrace{\left[-k_{T=(T_i+T_{i-1})/2} 2\pi \left(r_{in} - \frac{\Delta r}{2} \right) \frac{L}{\Delta r} - k_{T=(T_i+T_{i+1})/2} 2\pi \left(r_{in} + \frac{\Delta r}{2} \right) \frac{L}{\Delta r} \right]}_{A_{i,j}} \\
& + T_{i-1} \underbrace{\left[-k_{T=(T_i+T_{i-1})/2} 2\pi \left(r_{in} - \frac{\Delta r}{2} \right) \frac{L}{\Delta r} \right]}_{A_{i,j-1}} + T_{i+1} \underbrace{\left[-k_{T=(T_i+T_{i+1})/2} 2\pi \left(r_{in} + \frac{\Delta r}{2} \right) \frac{L}{\Delta r} \right]}_{A_{i,j+1}} \\
& = \underbrace{-2\pi r_i \Delta r L \dot{g}'''}_{b_i} \text{ for } i = 2..(N-1)
\end{aligned} \tag{27}$$

```

for i=2:(N-1)
    A(i,i)=-k_t((Tg(i)+Tg(i-1))/2)*2*pi*(r(i)-DELTA r/2)*L/DELTA r...
        -k_t((Tg(i)+Tg(i+1))/2)*2*pi*(r(i)+DELTA r/2)*L/DELTA r;
    A(i,i-1)=k_t((Tg(i)+Tg(i-1))/2)*2*pi*(r(i)-DELTA r/2)*L/DELTA r;
    A(i,i+1)=k_t((Tg(i)+Tg(i+1))/2)*2*pi*(r(i)+DELTA r/2)*L/DELTA r;
    b(i)=-2*pi*r(i)*DELTA r*L*gv;
end

```

The equation for node N , Eq. (25), is linearized:

$$\begin{aligned}
& T_N \underbrace{\left[-k_{T=(T_N+T_{N-1})/2} 2\pi \left(r_{out} - \frac{\Delta r}{2} \right) \frac{L}{\Delta r} - \frac{1}{(R_{ins} + R_{conv,out})} \right]}_{A_{N,N}} \\
& + T_{N-1} \underbrace{\left[-k_{T=(T_N+T_{N-1})/2} 2\pi \left(r_{out} - \frac{\Delta r}{2} \right) \frac{L}{\Delta r} \right]}_{A_{N,N-1}} \\
& = \underbrace{-\pi r_{out} \Delta r L \dot{g}'''}_{b_N} - \frac{T_\infty}{(R_{ins} + R_{conv,out})}
\end{aligned} \tag{28}$$

```

A(N,N)=-k_t((Tg(N)+Tg(N-1))/2)*2*pi*(r_in-DELTA r/2)*L/DELTA r-...
    1/(R_ins+R_conv_out);
A(N,N-1)=k_t((Tg(N)+Tg(N-1))/2)*2*pi*(r_in-DELTA r/2)*L/DELTA r;
b(N)=-pi*r_out*DELTA r*L*gv-T_infinity/(R_ins+R_conv_out);

```

The solution is obtained:

```

X=A\b;
T=X;

```

and used to compute the error between the assumed and calculated solutions is obtained:

$$err = \sqrt{\frac{1}{N} \sum_{i=1}^N (T_i - \hat{T}_i)^2} \tag{29}$$

```
err=sqrt(sum((T-Tg).^2)/N) %compute rms error
```

The calculated solution becomes the guess value for the next iteration:

```
Tg=T; %reset guess values used to setup A and b
end
```

The solution is converted to degrees Celsius:

```
T_C=T-273.2; %convert to C
end
```

The solution is illustrated in Figure P1.5-2-4.

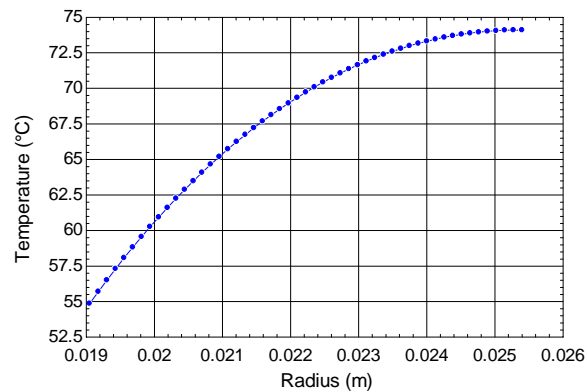


Figure P1.5-2-4: Temperature as a function of radius.

b.) Verify that your numerical solution limits to the analytical solution from Problem 1.3-9 (1-9 in text) in the limit that the conductivity is constant.

The conductivity function is modified temporarily so that it returns a constant value:

```
function[k]=k_t(T)
%conductivity of the material
%
%Inputs:
% T: temperature (K)
%
%Outputs:
% k: conductivity (W/m-K)

k=10;%+0.035*(T-300);
end
```

Figure P1.5-2-5 illustrates the temperature distribution predicted by the numerical and analytical solutions in the limit that k is constant.

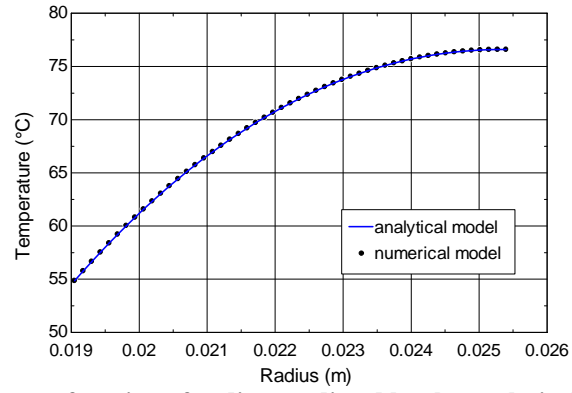


Figure P1.5-2-5: Temperature as a function of radius predicted by the analytical and numerical models in the limit that k is constant.