Problem 1.8-9

One side of a thin circular membrane is subjected to a flux of energy that varies according to:

$$\dot{q}'' = a\, r^2 \tag{1}$$

One side of the membrane is exposed to fluid at T_∞ with heat transfer coefficient $\overline{h}$. The outer edge of the membrane is held at T_∞. The radius of the membrane is r_o and the thickness is *th*. The conductivity of the membrane material is *k*. Assume that the temperature distribution in the membrane is only a function of radius.

a.) Derive the governing differential equation for the temperature in the membrane and the boundary conditions.

A differential energy balance leads to:

$$\dot{q}_r + \dot{q}''\, 2\,\pi\, r\, dr = \dot{q}_{r+dr} + 2\,\pi\, r\, dr\, \overline{h}\left(T - T_\infty\right) \tag{2}$$

Expanding the *r*+*dr* term leads to:

$$\dot{q}''\, 2\,\pi\, r\, dr = \frac{d\dot{q}}{dr} dr + 2\,\pi\, r\, dr\, \overline{h}\left(T - T_\infty\right) \tag{3}$$

The rate of conduction heat transfer is:

$$\dot{q} = -k\, 2\,\pi\, r\, th \frac{dT}{dr} \tag{4}$$

Substituting Eqs. (1) and (4) into Eq. (3) leads to:

$$a\, 2\,\pi\, r^3\, dr = \frac{d}{dr}\left(-k\, 2\,\pi\, r\, th \frac{dT}{dr}\right) dr + 2\,\pi\, r\, dr\, \overline{h}\left(T - T_\infty\right) \tag{5}$$

which can be simplified to:

$$a\, r^3 = \frac{d}{dr}\left(-k\, r\, th \frac{dT}{dr}\right) + r\, \overline{h}\left(T - T_\infty\right) \tag{6}$$

or

$$\frac{d}{dr}\left(r \frac{dT}{dr}\right) - \frac{\overline{h}}{k\, th} r\left(T - T_\infty\right) = -\frac{a}{k\, th} r^3 \tag{7}$$

The boundary conditions are:

$$T_{r=r_o} = T_\infty \tag{8}$$

$$T_{r=0} \text{ must be bounded} \tag{9}$$

b.) Define a dimensionless temperature difference and radius. Use them to non-dimensionalize the governing differential equation and boundary conditions from (a). This process should lead to the identification of another dimensionless parameter. Explain the significance of this parameter.

A dimensionless temperature is defined:

$$\tilde{\theta} = \frac{T - T_\infty}{\Delta T} \tag{10}$$

where ΔT is a normalizing temperature difference. The normalizing temperature difference is defined based on the temperature difference that would be induced if the entire rate of energy transfer from the heat flux were transferred convectively from the membrane. The total rate of heat transfer from the flux is:

$$\dot{q} = \int_0^{r_o} \underbrace{a\, r^2}_{\dot{q}''} 2\,\pi\, r\, dr = \frac{\pi\, a\, r_o^4}{2} \tag{11}$$

The reference temperature difference is therefore:

$$\Delta T = \frac{\dot{q}}{\pi\, r_o^2\, \overline{h}} = \frac{\pi\, a\, r_o^4}{2\,\pi\, r_o^2\, \overline{h}} = \frac{a\, r_o^2}{2\,\overline{h}} \tag{12}$$

which leads to:

$$\tilde{\theta} = \frac{2\,\overline{h}\left(T - T_\infty\right)}{a\, r_o^2} \tag{13}$$

A dimensionless radius is defined:

$$\tilde{r} = \frac{r}{r_o} \tag{14}$$

Substituting Eqs. (13) and (14) into Eq. (7) leads to:

$$\frac{a\, r_o}{2\,\overline{h}} \frac{d}{d\tilde{r}}\left(\tilde{r} \frac{d\tilde{\theta}}{d\tilde{r}}\right) - \frac{\overline{h}}{k\, th} \frac{a\, r_o^3}{2\,\overline{h}} \tilde{r}\, \tilde{\theta} = -\frac{a\, r_o^3}{k\, th} \tilde{r}^3 \tag{15}$$

or

$$\frac{d}{d\tilde{r}}\left(\tilde{r}\,\frac{d\tilde{\theta}}{d\tilde{r}}\right) - m^2\,\tilde{r}\,\tilde{\theta} = -2\,m^2\,\tilde{r}^3 \tag{16}$$

where

$$m^2 = \frac{\overline{h}\,r_o^2}{k\,th} \tag{17}$$

The parameter m^2 is, approximately, the ratio of the resistance to conduction along the membrane in the radial direction to the resistance to convection from the membrane surface.

The nondimensional boundary conditions are:

$$\tilde{\theta}_{\tilde{r}=1} = 0 \tag{18}$$

$$\tilde{\theta}_{\tilde{r}=0} \text{ must be bounded} \tag{19}$$

c.) Solve the normalized problem from (b). Prepare a plot of the dimensionless temperature as a function of the dimensionless radius for various values of the dimensionless parameter identified in (b).

The solution is split into a homogeneous and nonhomogeneous component:

$$\tilde{\theta} = \tilde{\theta}_h + \tilde{\theta}_p \tag{20}$$

Equation (20) is substituted into Eq. (16), leading to:

$$\underbrace{\frac{d}{d\tilde{r}}\left(\tilde{r}\,\frac{d\tilde{\theta}_h}{d\tilde{r}}\right) - m^2\,\tilde{r}\,\tilde{\theta}_h}_{=0\text{ for homogeneous differential equation}} + \underbrace{\frac{d}{d\tilde{r}}\left(\tilde{r}\,\frac{d\tilde{\theta}_p}{d\tilde{r}}\right) - m^2\,\tilde{r}\,\tilde{\theta}_p = -2\,m^2\,\tilde{r}^3}_{\text{whatever is left is the particular differential equation}} \tag{21}$$

The solution to the particular differential equation is considered first.

$$\frac{d}{d\tilde{r}}\left(\tilde{r}\,\frac{d\tilde{\theta}_p}{d\tilde{r}}\right) - m^2\,\tilde{r}\,\tilde{\theta}_p = -2\,m^2\,\tilde{r}^3 \tag{22}$$

Based on the form of the left side, a second order polynomial is assumed for the particular solution:

$$\tilde{\theta}_p = a + b\,\tilde{r} + c\,\tilde{r}^2 \tag{23}$$

Substituting Eq. (23) into Eq. (22) leads to:

$$\frac{d}{d\tilde{r}}\left(\tilde{r}\left(b + 2\,c\,\tilde{r}\right)\right) - m^2\,\tilde{r}\left(a + b\,\tilde{r} + c\,\tilde{r}^2\right) = -2\,m^2\,\tilde{r}^3 \tag{24}$$

or

$$b + 4\,c\,\tilde{r} - m^2\,\tilde{r}\left(a + b\,\tilde{r} + c\,\tilde{r}^2\right) = -2\,m^2\,\tilde{r}^3 \tag{25}$$

Equating like coefficients of $\tilde{r}$ leads to:

$$b = 0 \tag{26}$$

$$4\,c - m^2\,a = 0 \tag{27}$$

$$-m^2\,b = 0 \tag{28}$$

$$-m^2\,c = -2\,m^2 \tag{29}$$

which leads to $a = 8/m^2$, $b = 0$, and $c = 2$. Therefore, the particular solution is:

$$\tilde{\theta}_p = \frac{8}{m^2} + 2\,\tilde{r}^2 \tag{30}$$

The solution to the homogeneous differential equation:

$$\frac{d}{d\tilde{r}}\left(\tilde{r}\,\frac{d\tilde{\theta}_h}{d\tilde{r}}\right) - m^2\,\tilde{r}\,\tilde{\theta}_h = 0 \tag{31}$$

is obtained using the flow chart:

$$\tilde{\theta}_h = C_1\,\mathrm{BesselI}\left(0, m\,\tilde{r}\right) + C_2\,\mathrm{BesselK}\left(0, m\,\tilde{r}\right) \tag{32}$$

The solution is:

$$\tilde{\theta} = C_1\,\mathrm{BesselI}\left(0, m\,\tilde{r}\right) + C_2\,\mathrm{BesselK}\left(0, m\,\tilde{r}\right) + \frac{8}{m^2} + 2\,\tilde{r}^2 \tag{33}$$

The same solution can be identified in Maple:

```
> restart;
```

```
> ODE:=diff(r*diff(q(r),r),r)-m^2*r*q(r)=-2*m^2*r^3;
```

$$ODE := \left(\frac{d}{dr}\,\mathrm{q}(r)\right) + r\left(\frac{d^2}{dr^2}\,\mathrm{q}(r)\right) - m^2\, r\, \mathrm{q}(r) = -2\, m^2\, r^3$$

```
> qs:=dsolve(ODE);
```

$$qs := \mathrm{q}(r) = \mathrm{BesselI}(0, m\, r)\ \_C2 + \mathrm{BesselK}(0, m\, r)\ \_C1 + \frac{8 + 2\, m^2\, r^2}{m^2}$$

The boundary condition:

$$\tilde{\theta}_{\tilde{r}=0} \text{ must be bounded} \tag{34}$$

is satisfied by evaluating the limits of the two Bessel functions:

```
> limit(BesselI(0,r),r=0);
```

$$1$$

```
> limit(BesselK(0,r),r=0);
```

$$\infty$$

which means that C_2 in Eq. (33) must be zero:

$$\tilde{\theta} = C_1\, \mathrm{BesselI}(0, m\,\tilde{r}) + \frac{8}{m^2} + 2\,\tilde{r}^2 \tag{35}$$

The boundary condition:

$$\tilde{\theta}_{\tilde{r}=1} = 0 \tag{36}$$

is enforced:

$$C_1\, \mathrm{BesselI}(0, m) + \frac{8}{m^2} + 2 = 0 \tag{37}$$

which leads to:

$$C_1 = -\frac{\left(\frac{8}{m^2} + 2\right)}{\mathrm{BesselI}(0, m)} \tag{38}$$

Substituting Eq. (38) into Eq. (35) leads to:

$$\tilde{\theta} = -\frac{\left(\frac{8}{m^2}+2\right)}{\text{BesselI}(0,m)}\text{BesselI}(0,m\,\tilde{r}) + \frac{8}{m^2} + 2\,\tilde{r}^2 \tag{39}$$

Equation (39) is programmed in EES:

```
"P1.8-9"
$UnitSystem SI MASS RAD PA  K J
$TABSTOPS   0.2 0.4 0.6 0.8 3.5 in

"Inputs"
m=1 [-]                                   "dimensionless parameter - ratio of conduction to convection"
theta_hat=-(8/m^2+2)*BesselI(0,m*r_hat)/BesselI(0,m)+8/m^2+2*r_hat^2 "solution"
```

and used to generate Figure 1, which shows the dimensionless temperature as a function of dimensionless radius.

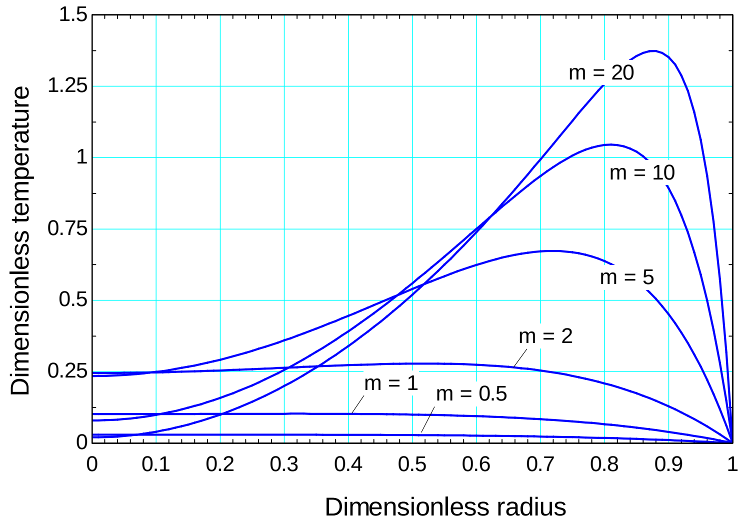

Figure 1: Dimensionless temperature as a function of dimensionless radius for various values of *m*.