Problem 1.8-8

Figure P1.8-8 illustrates a triangular fin with a circular cross-section.

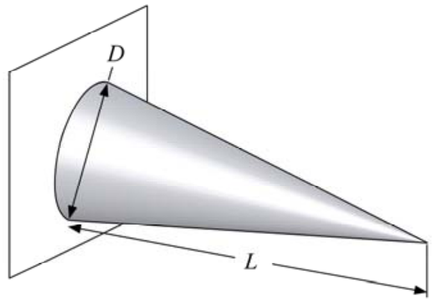


Figure P1.8-5: Fin on an evaporator.

The fin is surrounded by fluid at T_∞ with heat transfer coefficient $\bar{h}$. The base of the fin is at T_b and the fin conductivity is k.

a.) Derive the governing differential equation and the boundary conditions for the problem.

The x-coordinate is defined as starting from the tip of the fin and moving to the base. The cross-sectional area is therefore:

$$A_c = \pi \frac{D^2}{4} \frac{x^2}{L^2} \tag{1}$$

and the perimeter (assuming that $L >> D$) is:

$$per = \pi D \frac{x}{L} \tag{2}$$

A differential energy balance leads to:

$$\dot{q}_x = \dot{q}_{x+dx} + \dot{q}_{conv} \tag{3}$$

The rate of conduction and convection are:

$$\dot{q}_x = -k\, A_c \frac{dT}{dx} \tag{4}$$

$$\dot{q}_{conv} = \bar{h}\; per \left(T - T_\infty\right) dx \tag{5}$$

Substituting Eqs. (1), (2), (4), and (5) into Eq. (3) and expanding the x+dx term leads to:

$$0 = \frac{d}{dx}\left[-k\,\pi \frac{D^2}{4} \frac{x^2}{L^2} \frac{dT}{dx}\right] + \bar{h}\,\pi\, D \frac{x}{L}\left(T - T_\infty\right) \tag{6}$$

Simplifying:

$$\frac{d}{dx}\left[x^2 \frac{dT}{dx}\right] - m^2\, x\left(T - T_\infty\right) = 0 \tag{7}$$

where

$$m^2 = \frac{4\,\overline{h}\,L}{k\,D} \tag{8}$$

The boundary conditions are:

$$T_{x=0} \text{ must be bounded} \tag{9}$$

$$T_{x=L} = T_b \tag{10}$$

b.) Normalize the governing differential equation and the boundary conditions. This process should lead to the identification of a dimensionless fin parameter that governs the solution. Identify the physical significance of this parameter.

Dimensionless position and temperature difference are defined:

$$\tilde{x} = \frac{x}{L} \tag{11}$$

$$\tilde{\theta} = \frac{T - T_\infty}{T_b - T_\infty} \tag{12}$$

Substituting Eqs. (11) and (12) into Eq. (7) leads to:

$$\left(T_b - T_\infty\right)\frac{d}{d\tilde{x}}\left[\tilde{x}^2 \frac{d\tilde{\theta}}{d\tilde{x}}\right] - m^2\,L\,\tilde{x}\left(T_b - T_\infty\right)\tilde{\theta} = 0 \tag{13}$$

or

$$\frac{d}{d\tilde{x}}\left[\tilde{x}^2 \frac{d\tilde{\theta}}{d\tilde{x}}\right] - \left(m^2 L\right)\tilde{x}\,\tilde{\theta} = 0 \tag{14}$$

where

$$m^2\,L = \frac{4\,\overline{h}\,L^2}{k\,D} \tag{15}$$

is the dimensionless parameter identified by the process. The dimensionless parameter is nominally equal to the ratio of the resistance to conduction along the fin (R_{cond}) to the resistance to convection from the fin surface (R_{conv}):

$$R_{cond} \approx \frac{8\,L}{\pi\,D^2\,k} \tag{16}$$

$$R_{conv} \approx \frac{2}{\pi\,D\,L\,\overline{h}} \tag{17}$$

$$\frac{R_{cond}}{R_{conv}} = \frac{8\,L}{\pi\,D^2\,k}\frac{\pi\,D\,L\,\overline{h}}{2} = \frac{4\,L^2\,\overline{h}}{D\,k} = m^2\,L \tag{18}$$

The normalized boundary conditions are:

$$\tilde{\theta}_{\tilde{x}=0} \text{ must be bounded} \tag{19}$$

$$\tilde{\theta}_{\tilde{x}=1} = 1 \tag{20}$$

c.) Solve the differential equation subject to the boundary conditions.

The differential equation:

$$\frac{d}{d\tilde{x}}\left[\tilde{x}^2\,\frac{d\tilde{\theta}}{d\tilde{x}}\right] - \left(m^2 L\right)\tilde{x}\,\tilde{\theta} = 0 \tag{21}$$

is homogeneous and a form of the Bessel's equation:

$$\frac{d}{dx}\left(x^p\,\frac{d\theta}{dx}\right) \pm c^2\,x^s\,\theta = 0 \tag{22}$$

where $p = 2$, $c = m\sqrt{L}$, and $s = 1$. The parameter $s - p + 2$ is 1; therefore the solution parameters are:

$$n = \frac{1-2}{1-2+2} = -1 \tag{23}$$

$$a = \frac{2}{1-2+2} = 2 \tag{24}$$

$$\frac{n}{a}=\frac{1-2}{2}=-\frac{1}{2} \tag{25}$$

The solution is therefore:

$$\tilde{\theta}=C_1\frac{\text{BesselI}\left(-1,2m\sqrt{L}\,\tilde{x}^{1/2}\right)}{\sqrt{\tilde{x}}}+C_2\frac{\text{BesselK}\left(-1,2m\sqrt{L}\,\tilde{x}^{1/2}\right)}{\sqrt{\tilde{x}}} \tag{26}$$

The same solution can be identified in Maple:

```
> restart;
> assume(m2L,positive);
> ODE:=diff(x^2*diff(q(x),x),x)-m2L*x*q(x)=0;
```

$$ODE := 2\,x\left(\frac{d}{dx}\,\mathrm{q}(x)\right)+x^2\left(\frac{d^2}{dx^2}\,\mathrm{q}(x)\right)-m2L\sim x\,\mathrm{q}(x)=0$$

```
> qs:=dsolve(ODE);
```

$$qs := \mathrm{q}(x)=\frac{\_C1\ \text{BesselI}(1,\,2\sqrt{m2L\sim}\,\sqrt{x})}{\sqrt{x}}+\frac{\_C2\ \text{BesselK}(1,\,2\sqrt{m2L\sim}\,\sqrt{x})}{\sqrt{x}}$$

Note that $\text{BesselI}(-1,x)$ is equal to $\text{BesselI}(1,x)$ and so Maple has identified the same solution that we found manually. The boundary condition:

$$\tilde{\theta}_{\tilde{x}=0}\ \text{must be bounded} \tag{27}$$

requires that C_2 in Eq. (26) must be zero. To see this, take the limit of the second term as $\tilde{x}$ goes to zero.

```
> limit(BesselK(-1,x)/x,x=0);
```

$$\infty$$

Therefore:

$$\tilde{\theta}=C_1\frac{\text{BesselI}\left(-1,2m\sqrt{L}\,\tilde{x}^{1/2}\right)}{\sqrt{\tilde{x}}} \tag{28}$$

The second boundary condition:

$$\tilde{\theta}_{\tilde{x}=1}=1 \tag{29}$$

leads to:

$$1 = C_1\, \text{BesselI}\left(-1, 2\, m \sqrt{L}\right) \tag{30}$$

Substituting Eq. (30) into Eq. (28) leads to:

$$\tilde{\theta} = \frac{\text{BesselI}\left(-1, 2\, m \sqrt{L}\, \tilde{x}^{1/2}\right)}{\text{BesselI}\left(-1, 2\, m \sqrt{L}\right) \sqrt{\tilde{x}}} \tag{31}$$

d.) Prepare a plot of dimensionless temperature as a function of dimensionless position for various values of the remaining dimensionless parameter, identified in (b).

The solution is programmed in EES:

```
"P1.8-8"
$UnitSystem SI MASS RAD PA  K J
$TABSTOPS   0.2 0.4 0.6 0.8 3.5 in

msqrtL=10 [-]                                                   "fin parameter"
theta_bar=BesselI(-1,2*msqrtL*sqrt(x_hat))/(BesselI(-1,2*msqrtL)*sqrt(x_hat)) "temperature solution"
```

and used to generate Figure 2.

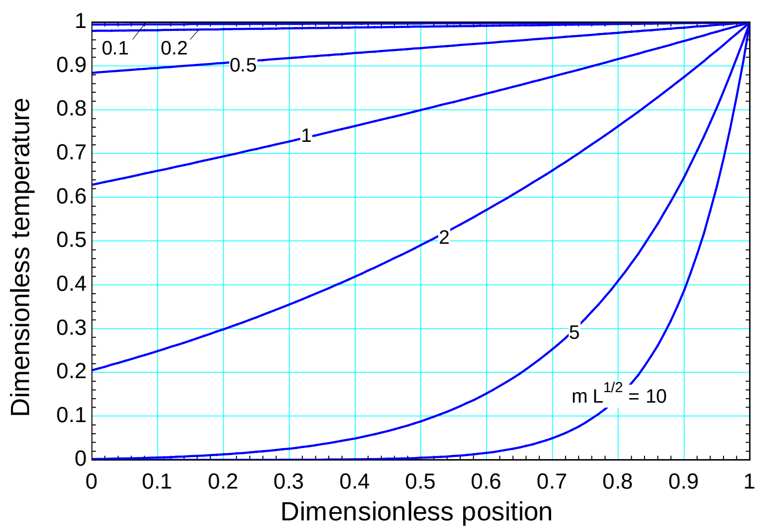


Figure 2: Dimensionless temperature as a function of dimensionless position for various values of the fin parameter.

e.) The fin efficiency is defined as the ratio of the heat transfer into the base of the fin to the heat transfer that would occur if the entire fin were isothermal and at the base temperature (i.e., if the fin material were infinitely conductive). Develop an equation for the fin efficiency and plot the fin efficiency as a function of the dimensionless fin parameter identified in (b).

The fin heat transfer rate is:

$$\dot{q}_{fin} = k\frac{\pi D^2}{4}\left(\frac{dT}{dx}\right)_{x=L} \tag{32}$$

or

$$\dot{q}_{fin} = k\frac{\pi D^2 \left(T_b - T_\infty\right)}{4L}\left(\frac{d\tilde{\theta}}{d\tilde{x}}\right)_{\tilde{x}=1} \tag{33}$$

The fin efficiency is therefore:

$$\eta_{fin} = \frac{2\dot{q}_{fin}}{\pi D L \bar{h}\left(T_b - T_\infty\right)} = k\frac{2\pi D^2\left(T_b - T_\infty\right)}{4\pi D L \bar{h}\left(T_b - T_\infty\right)L}\left(\frac{d\tilde{\theta}}{d\tilde{x}}\right)_{\tilde{x}=1} \tag{34}$$

which can be simplified to:

$$\eta_{fin} = \frac{2}{2}\frac{k D}{2\bar{h} L^2}\left(\frac{d\tilde{\theta}}{d\tilde{x}}\right)_{\tilde{x}=1} = 2m^2 L\left(\frac{d\tilde{\theta}}{d\tilde{x}}\right)_{\tilde{x}=1} \tag{35}$$

Substituting Eq. (31) into Eq. (35) leads to:

$$\eta_{fin} = \frac{2}{\mathrm{BesselI}\left(-1, 2m\sqrt{L}\right)m^2 L}\frac{d}{d\tilde{x}}\left(\frac{\mathrm{BesselI}\left(-1, 2m\sqrt{L}\,\tilde{x}^{1/2}\right)}{\sqrt{\tilde{x}}}\right)_{\tilde{x}=1} \tag{36}$$

The derivative in Eq. (36) is evaluated in Maple:

```
> restart;
> eval(diff(BesselI(-1,2*msqrtL*sqrt(x))/sqrt(x),x),x=1);
```

$$\left(\mathrm{BesselI}(0, 2\,msqrtL) - \frac{1}{2}\frac{\mathrm{BesselI}(1, 2\,msqrtL)}{msqrtL}\right) msqrtL - \frac{1}{2}\mathrm{BesselI}(1, 2\,msqrtL)$$

```
> simplify(%);
```

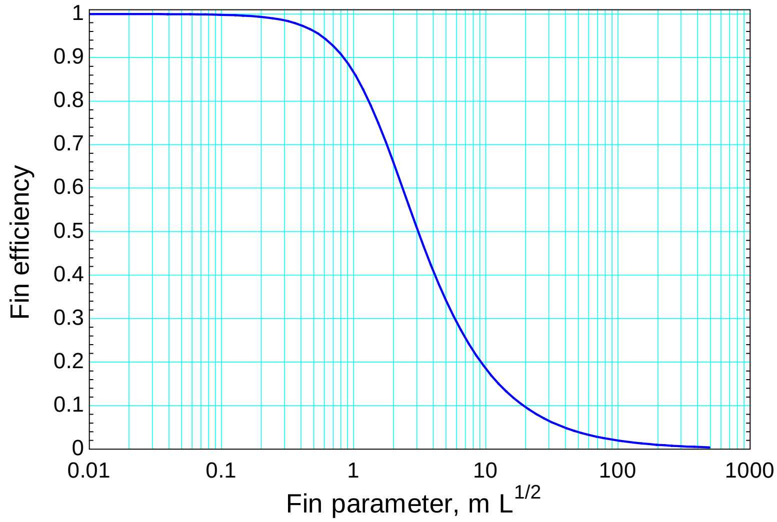
$$\mathrm{BesselI}(0, 2\,msqrtL)\,msqrtL - \mathrm{BesselI}(1, 2\,msqrtL)$$

Therefore, the fin efficiency is:

$$\eta_{fin} = \frac{2\left[\mathrm{BesselI}\left(0, 2m\sqrt{L}\right)m\sqrt{L} - \mathrm{BesselI}\left(1, 2m\sqrt{L}\right)\right]}{\mathrm{BesselI}\left(-1, 2m\sqrt{L}\right)m^2 L} \tag{37}$$

```
eta_fin=2*(BesselI(0,2*msqrtL)*msqrtL-BesselI(1,2*msqrtL))/(msqrtL^2*BesselI(-1,2*msqrtL))
          "fin efficiency"
```

Figure 3 illustrates the fin efficiency as a function of the fin parameter $m\sqrt{L}$.



Fin efficiency as a function of the fin parameter.