Problem P1.2-2: Conduction Through a Shape with Varying Cross-sectional Area

The temperature distribution for the shape shown in Figure P1.2-2 can be assumed to be 1-D in the coordinate *s*. The problem is at steady state and the area available for conduction changes with *s* according to an arbitrary function, $A(s)$. The temperatures of the two ends of the shape are specified; T_H at s_1 and T_C at s_2.

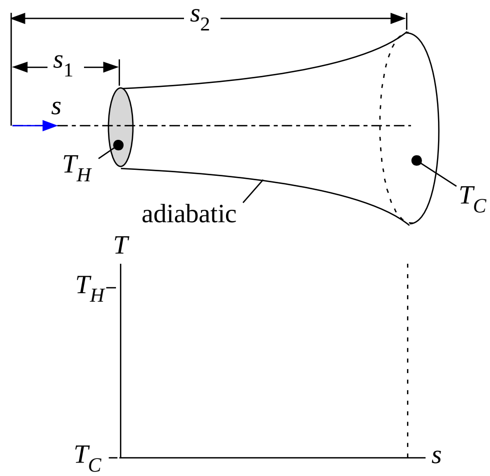

Figure P1.2-2: Conduction through a shape in which the cross-sectional area varies according to *A*(*s*).

a.) Sketch the temperature distribution through the shape on the axes below the figure.

The rate of conductive heat transfer ($\dot{q}$) at any position *s* is given by Fourier's law:

$$\dot{q} = -k\,A\frac{dT}{ds} \tag{1}$$

At steady state, the heat transfer rate must be constant with position and therefore the temperature gradient is inversely proportional to area:

$$\frac{dT}{ds} = -\frac{\dot{q}}{k\,A} \tag{2}$$

The temperature gradient will be steepest where the area is smallest, as shown in Figure 2.

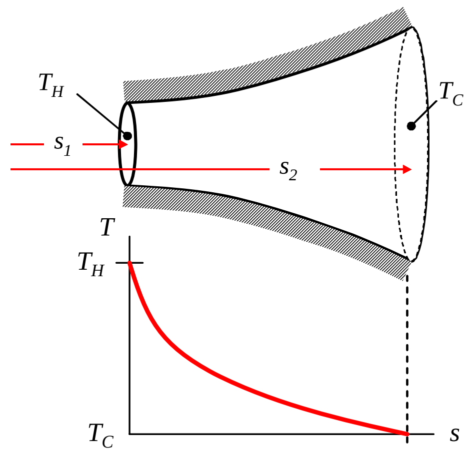

Figure 2: Temperature distribution.

b.) Derive the governing differential equation for the problem; the governing differential equation should include only temperature T and its derivatives with respect to s as well as the area and its derivatives with respect to s.

A differentially small control volume is defined, as shown in Figure 3.

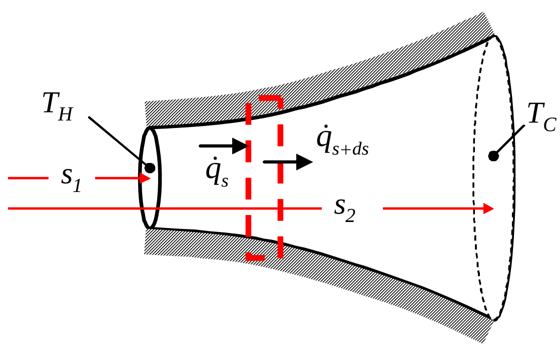

Figure 3: Differential control volume.

An energy balance on the control volume leads to:

$$\dot{q}_s = \dot{q}_{s+ds} \tag{3}$$

Expanding the s+ds term in Eq. (3) leads to:

$$\dot{q}_s = \dot{q}_s + \frac{d\dot{q}}{ds} ds \tag{4}$$

which can be simplified:

$$\frac{d\dot{q}}{ds} = 0 \tag{5}$$

Substituting in Fourier's law into Eq. (5) leads to:

$$\frac{d}{ds}\left[-k\,A\frac{dT}{ds}\right]=0 \tag{6}$$

You can divide through by -*k* to get the governing differential equation:

$$\boxed{\frac{d}{ds}\left[A\frac{dT}{ds}\right]=0} \text{ or } \frac{dA}{ds}\frac{dT}{ds}+A\frac{d^2T}{ds^2}=0 \tag{7}$$