Problem 1.8-4: Circular Fin

Figure P1.8-4 shows a typical fin design that is fabricated by attaching a thin washer to the outer radius of a tube. The inner and outer radii of the fin are r_{in} and r_{out}, respectively. The thickness of the fin is *th* and the fin material has conductivity, k. The fin is surrounded by fluid at T_∞ and the average heat transfer coefficient is $\overline{h}$. The base of the fin is maintained at T_b and the tip is adiabatic.

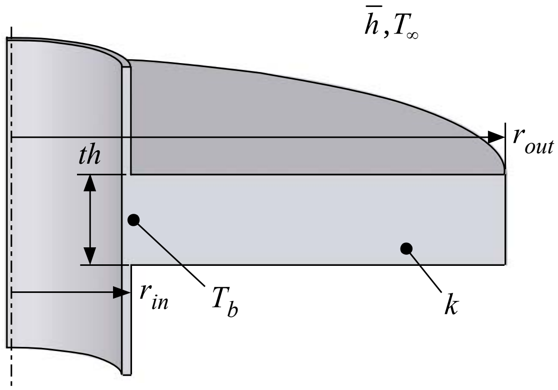

Figure P1.8-4: Circular fin.

Determine an analytical solution for the temperature distribution in the fin and the fin efficiency.

The differential control volume shown in Figure 2 can be used to derive the governing equation.

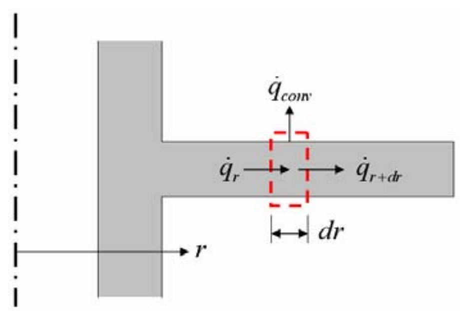

Figure 2: Differential control volume.

An energy balance for the control volume is:

$$\dot{q}_r = \dot{q}_{r+dr} + \dot{q}_{conv}$$

or

$$0 = \frac{d\dot{q}}{dr} dr + \dot{q}_{conv}$$

The conduction and convection terms are:

$$\dot{q} = -k\,2\pi\, r\, th\frac{dT}{dr}$$

$$\dot{q}_{conv} = 4\pi\, r\, dr\,\overline{h}\left(T - T_\infty\right)$$

Combining these equations leads to:

$$0 = \frac{d}{dr}\left[-k\,2\pi\, r\, th\frac{dT}{dr}\right]dr + 4\pi\, r\, dr\,\overline{h}\left(T - T_\infty\right) = 0$$

which can be simplified to:

$$\frac{d}{dr}\left[r\,\frac{dT}{dr}\right] - \frac{2\, r\,\overline{h}}{k\, th}T = -\frac{2\, r\,\overline{h}}{k\, th}T_\infty$$

The solution is divided into a homogeneous and particular component:

$$T = T_h + T_p$$

which leads to:

$$\underbrace{\frac{d}{dr}\left[r\,\frac{dT_h}{dr}\right] - \frac{2\, r\,\overline{h}}{k\, th}T_h}_{\text{=0 for homogeneous differential equation}} + \underbrace{\frac{d}{dr}\left[r\,\frac{dT_p}{dr}\right] - \frac{2\, r\,\overline{h}}{k\, th}T_p = -\frac{2\, r\,\overline{h}}{k\, th}T_\infty}_{\text{whatever is left is the particular differential equation}}$$

The solution to the particular differential equation:

$$\frac{d}{dr}\left[r\,\frac{dT_p}{dr}\right] - \frac{2\, r\,\overline{h}}{k\, th}T_p = -\frac{2\, r\,\overline{h}}{k\, th}T_\infty$$

is

$$T_p = T_\infty$$

The homogeneous differential equation is:

$$\frac{d}{dr}\left[r\,\frac{dT_h}{dr}\right] - m^2\, r\, T_h = 0 \tag{1}$$

where *m* is the fin parameter, defined as:

$$m = \sqrt{\frac{2\,h}{k\,b}}$$

Equation (1) is a form of Bessel's equation:

$$\frac{d}{dx}\left(x^{p}\,\frac{d\theta}{dx}\right) \pm c^{2}\,x^{s}\,\theta = 0 \tag{2}$$

where (by comparing Eqs. (1) and (2)), p= 1, c = m, and s = 1. Referring to the flow chart presented in Section 1.8.4, the value of s-p+2 is equal to 2 and therefore the solution parameters n and a must be computed:

$$n = \frac{1-1}{1-1+2} = 0$$

$$a = \frac{2}{1-1+2} = 1$$

The last term in Eq. (1) is negative and therefore the solution to Eq. (1) is given by:

$$T_h = C_1\,x^{n/a}\,\mathrm{BesselI}\left(n, c\,a\,x^{1/a}\right) + C_2\,x^{n/a}\,\mathrm{BesselK}\left(n, c\,a\,x^{1/a}\right)$$

or

$$T_h = C_1\,\mathrm{BesselI}\left(0, m\,r\right) + C_2\,\mathrm{BesselK}\left(0, m\,r\right)$$

The solution to the governing differential equation is:

$$T = C_1\,\mathrm{BesselI}\left(0, m\,r\right) + C_2\,\mathrm{BesselK}\left(0, m\,r\right) + T_\infty \tag{3}$$

Note that Maple would provide this information as well:

```
> restart;
> ODE:=diff(r*diff(T(r),r),r)-m^2*r*T(r)=-m^2*r*T_infinity;
```

$$ODE := \left(\frac{d}{dr}\,\mathrm{T}(r)\right) + r\left(\frac{d^2}{dr^2}\,\mathrm{T}(r)\right) - m^2\,r\,\mathrm{T}(r) = -m^2\,r\,T\_infinity$$

```
> Ts:=dsolve(ODE);
```

$$Ts := \mathrm{T}(r) = \mathrm{BesselI}(0,\, m\,r)\,\_C2 + \mathrm{BesselK}(0,\, m\,r)\,\_C1 + T\_infinity$$

The boundary conditions must be used to obtain the constants C_1 and C_2. The base temperature is specified:

$$T_{r=r_{in}} = T_b$$

or:

$$C_1 \, \mathrm{BesselI}\left(0, m\, r\right) + C_2 \, \mathrm{BesselK}\left(0, m\, r\right) + T_\infty = T_b \tag{4}$$

The tip of the fin is adiabatic:

$$-k \, 2\, \pi \, r_{in} \left.\frac{dT}{dr}\right|_{r=r_{out}} = 0$$

or

$$C_1 \frac{d}{dr}\left[\mathrm{BesselI}\left(0, m\, r\right)\right]_{r=r_{out}} + C_2 \frac{d}{dr}\left[\mathrm{BesselK}\left(0, m\, r\right)\right]_{r=r_{out}} = 0$$

Using the rules for differentiating Bessel functions presented in Section 1.8.4 leads to:

$$C_1 \, m \, \mathrm{Bessel\_I}\left(1, m\, r_{out}\right) - C_2 \, m \, \mathrm{Bessel\_K}\left(1, m\, r_{out}\right) = 0 \tag{5}$$

The boundary condition equations, Eqs. (4) and (5), can be obtained using Maple:

```
> BC1:=rhs(eval(Ts,r=r_in))=T_b;
```

$$BC1 := \mathrm{BesselI}(0,\, m\ r\_in)\ \_C2 + \mathrm{BesselK}(0,\, m\ r\_in)\ \_C1 + T\_infinity = T\_b$$

```
> BC2:=rhs(eval(diff(Ts,r),r=r_out))=0;
```

$$BC2 := \mathrm{BesselI}(1,\, m\ r\_out)\ m\ \_C2 - \mathrm{BesselK}(1,\, m\ r\_out)\ m\ \_C1 = 0$$

These equations can be copied into EES in order to obtain the solution for arbitrary conditions:

```
"Boundary conditions"
theta_b = C_1*BesselI(0, m*r_tube)+C_2*BesselK(0, m*r_tube)
 0 = C_1*BesselI(1, m*r_fin)*m-C_2*BesselK(1, m*r_fin)*m

"Temperature distribution"
theta = C_1*BesselI(0, m*r)+C_2*BesselK(0, m*r)
```

Given arbitrary values of the variables T_b, T_infinity, m, r_in, and r_out, the EES code above will provide the temperature distribution.

It is convenient to solve for the two constants explicitly and substitute them into the temperature distribution; we can let Maple accomplish this process and avoid the algebra. The first step is to solve the two boundary conditions equations simultaneously to obtain the unknown constants; this is done using the solve command in Maple where the first argument is the set of equations (BC1 and BC2) and the second are the arguments to be solved for (_C1 and _C2):

```
> constants:=solve({BC1,BC2},{_C1,_C2});
```

$$constants := \{\_C2 = \frac{\text{BesselK}(1, m\ r\_out)\ (-T\_infinity + T\_b)}{\text{BesselK}(1, m\ r\_out)\ \text{BesselI}(0, m\ r\_in) + \text{BesselK}(0, m\ r\_in)\ \text{BesselI}(}$$

$$, \_C1 = \frac{\text{BesselI}(1, m\ r\_out)\ (-T\_infinity + T\_b)}{\text{BesselK}(1, m\ r\_out)\ \text{BesselI}(0, m\ r\_in) + \text{BesselK}(0, m\ r\_in)\ \text{BesselI}(1, m\ r\_}$$

$$\}$$

These equations for the constants can be substituted into the solution using the eval command, where the first argument is the base expression and the second contains the sub-expressions that must be substituted into the base expression:

```
> Ts:=eval(Ts,constants);
```

$$Ts := \mathrm{T}(r) = \frac{\text{BesselI}(0, m\ r)\ \text{BesselK}(1, m\ r\_out)\ (-T\_infinity + T\_b)}{\text{BesselK}(1, m\ r\_out)\ \text{BesselI}(0, m\ r\_in) + \text{BesselK}(0, m\ r\_in)\ \text{BesselI}(1, m\ r\_}$$

$$+ \frac{\text{BesselK}(0, m\ r)\ \text{BesselI}(1, m\ r\_out)\ (-T\_infinity + T\_b)}{\text{BesselK}(1, m\ r\_out)\ \text{BesselI}(0, m\ r\_in) + \text{BesselK}(0, m\ r\_in)\ \text{BesselI}(1, m\ r\_out)}$$

$$+ T\_infinity$$

which can be copied into EES in place of the 3 original equations:

```
"Explicit solution"
T = BesselI(0,m*r)*BesselK(1,m*r_out)*(-T_infinity+T_b)/(BesselK(1,m*r_out)*BesselI(0,m*r_in)+&
    BesselK(0,m*r_in)*BesselI(1,m*r_out))+BesselK(0,m*r)*BesselI(1,m*r_out)*(-T_infinity+T_b)&
        /(BesselK(1,m*r_out)*BesselI(0,m*r_in)+BesselK(0,m*r_in)*BesselI(1,m*r_out))+T_infinity
```

So the temperature distribution through the circular fin is given by:

$$T = T_{\infty} + \left(T_b - T_{\infty}\right) \frac{\left[\text{BesselK}\left(1, m\, r_{out}\right)\text{BesselI}\left(0, m\, r\right) + \text{BesselI}\left(1, m\, r_{out}\right)\text{BesselK}\left(0, m\, r\right)\right]}{\left[\text{BesselI}\left(1, m\, r_{out}\right)\text{BesselK}\left(0, m\, r_{in}\right) + \text{BesselI}\left(0, m\, r_{in}\right)\text{BesselK}\left(1, m\, r_{out}\right)\right]} \quad (6)$$

The heat transfer rate to the base of the fin, $\dot{q}_{fin}$, is obtained by applying Fourier's law to evaluate the conduction heat transfer rate at the base of the fin:

$$\dot{q}_{fin} = -k\, 2\, \pi\, r_{in}\, th \left.\frac{dT}{dr}\right|_{r=r_{in}} \quad (7)$$

Substituting Eq. (6) into Eq. (7) leads to:

$$\dot{q}_{fin} = -k\,2\,\pi\,r_{in}\,th\left(T_b - T_\infty\right)\left\{\begin{array}{l}\dfrac{\text{BesselK}\left(1, m\,r_{out}\right)\dfrac{d}{dr}\left[\text{BesselI}\left(0, m\,r\right)\right]_{r=r_{in}}}{\text{BesselI}\left(1, m\,r_{out}\right)\text{BesselK}\left(0, m\,r_{in}\right) + \text{BesselI}\left(0, m\,r_{in}\right)\text{BesselK}\left(1, m\,r_{out}\right)} + \\ \dfrac{\text{BesselI}\left(1, m\,r_{out}\right)\dfrac{d}{dr}\left[\text{BesselK}\left(0, m\,r\right)\right]_{r=r_{in}}}{\text{BesselI}\left(1, m\,r_{out}\right)\text{BesselK}\left(0, m\,r_{in}\right) + \text{BesselI}\left(0, m\,r_{in}\right)\text{BesselK}\left(1, m\,r_{out}\right)}\end{array}\right\}$$

Using the rules for differentiating Bessel functions, presented in Section 1.8.4, to evaluate the derivatives leads to:

$$\dot{q}_{fin} = -k\,2\,\pi\,r_{in}\,th\left(T_b - T_\infty\right)m\,\frac{\left[\text{BesselK}\left(1, m\,r_{out}\right)\text{BesselI}\left(1, m\,r_{in}\right) - \text{BesselI}\left(1, m\,r_{out}\right)\text{BesselK}\left(1, m\,r_{in}\right)\right]}{\left[\text{BesselI}\left(1, m\,r_{out}\right)\text{BesselK}\left(0, m\,r_{in}\right) + \text{BesselI}\left(0, m\,r_{in}\right)\text{BesselK}\left(1, m\,r_{out}\right)\right]} \quad (8)$$

Maple achieves the same result:

```
> q_dot_fin:=-k*2*pi*r_in*th*rhs(eval(diff(Ts,r),r=r_in));
q_dot_fin := - 2 k π r_in th (-BesselI( 1, m r_in ) m BesselK( 1, m r_out ) T_infinity
      + BesselI( 1, m r_in ) m BesselK( 1, m r_out ) T_b
      + BesselK( 1, m r_in ) m BesselI( 1, m r_out ) T_infinity
      - BesselK( 1, m r_in ) m BesselI( 1, m r_out ) T_b)/(
     BesselK( 1, m r_out ) BesselI( 0, m r_in )
      + BesselK( 0, m r_in ) BesselI( 1, m r_out ))
```

which can be cut and pasted directly into EES:

```
"Fin heat transfer rate"
q_dot_fin=-2*k*pi*r_in*th*(-BesselI(1,m*r_in)*m*BesselK(1,m*r_out)*T_infinity+&
    BesselI(1,m*r_in)*m*BesselK(1,m*r_out)*T_b+BesselK(1,m*r_in)*m*BesselI(1,m*r_out)&
    *T_infinity-BesselK(1,m*r_in)*m*BesselI(1,m*r_out)*T_b)/(BesselK(1,m*r_out)*&
        BesselI(0,m*r_in)+BesselK(0,m*r_in)*BesselI(1,m*r_out))
```

Finally, the fin efficiency (η_{fin}) is the ratio of the heat transfer rate to the heat transfer rate from an isothermal fin at the base temperature:

$$\eta_{fin} = \frac{\dot{q}_{fin}}{2\,\pi\left(r_{out}^2 - r_{in}^2\right)\overline{h}\left(T_b - T_\infty\right)}$$

Substituting Eq. (8) into the definition of the fin efficiency leads to:

$$\eta_{fin} = \frac{2\, r_{in}}{m\left(r_{out}^2 - r_{in}^2\right)} \frac{\left[\text{BesselI}\left(1, m\, r_{out}\right)\text{BesselK}\left(1, m\, r_{in}\right)\text{-BesselK}\left(1, m\, r_{out}\right)\text{BesselI}\left(1, m\, r_{in}\right)\right]}{\left[\text{BesselI}\left(1, m\, r_{out}\right)\text{BesselK}\left(0, m\, r_{in}\right) + \text{BesselI}\left(0, m\, r_{in}\right)\text{BesselK}\left(1, m\, r_{out}\right)\right]}$$

which can be expressed as a function of the tube-to-fin radius ratio (r_{in}/r_{out}) and the product of the fin parameter and the fin radius ($m\, r_{out}$).

$$\eta_{fin} = \frac{2\left(\frac{r_{in}}{r_{out}}\right)}{m\, r_{out}\left(1-\left(\frac{r_{in}}{r_{out}}\right)^2\right)} \frac{\left[\text{BesselI}\left(1, m\, r_{out}\right)\text{BesselK}\left(1, m\, r_{out}\frac{r_{in}}{r_{out}}\right)\text{-BesselK}\left(1, m\, r_{out}\right)\text{BesselI}\left(1, m\, r_{out}\frac{r_{in}}{r_{out}}\right)\right]}{\left[\text{BesselI}\left(1, m\, r_{out}\right)\text{BesselK}\left(0, m\, r_{out}\frac{r_{in}}{r_{out}}\right) + \text{BesselI}\left(0, m\, r_{out}\frac{r_{in}}{r_{out}}\right)\text{BesselK}\left(1, m\, r_{out}\right)\right]}$$

The fin efficiency for a circular fin is shown in Figure 3 as a function of $m\, r_{out}$ for various values of r_{in}/r_{out}. Note that the fin radius can be corrected approximately to account for convection from the tip by adding the half-thickness of the fin; as previously discussed in Section 1.6.5, this correction is small and rarely worth considering.

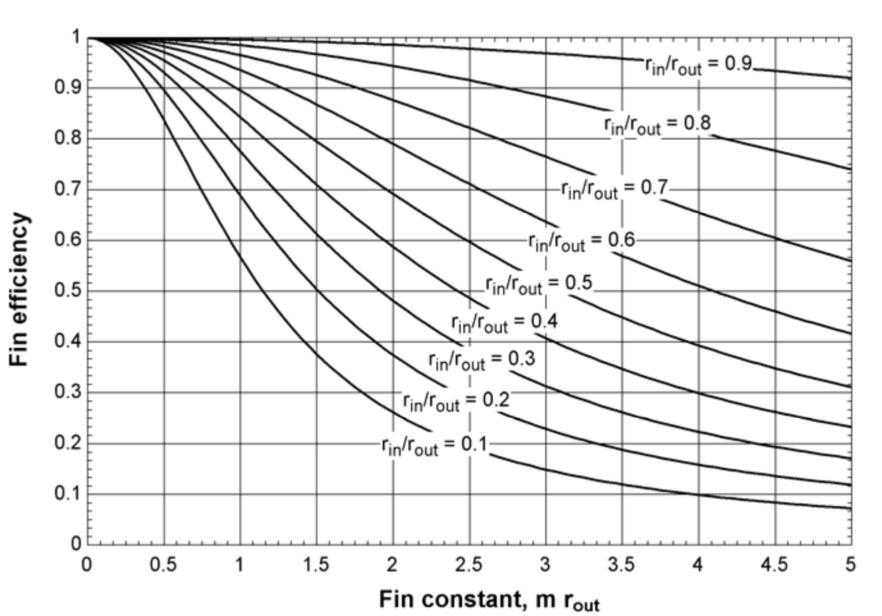

Figure 3: Fin efficiency of a circular fin as a function of $m\, r_{out}$ for various values of r_{in}/r_{out}.