P1.1-1: Viscosity of a dilute gas

Momentum transfer occurs in a fluid due to interactions between molecules that results in a transfer of momentum. This process is characterized by viscosity, which relates the shear stress to a velocity gradient in the same way Fourier's Law relates heat flux to a temperature gradient. It is not surprising, then, that the viscosity and thermal conductivity of an ideal gas are analogous transport properties.

a.) Using reasoning similar to that provided in Section 1.1.2 for thermal conductivity, show that the viscosity of an ideal gas can be estimated according to $\mu \propto \sqrt{T\, MW} / \sigma^2$.

Consider momentum transfer through a fluid in which a velocity gradient has been established in the *x*-direction, as shown in Figure 1. We can evaluate the net rate of momentum transferred through a plane that is located at position *x*. The flux of molecules passing through the plane from left-to-right (i.e., in the positive *x*-direction) is proportional to the number density of the molecules (n_{ms}) and their mean velocity (v_{ms}). The molecules that are moving in the positive *x*-direction experienced their last interaction at $x - L_{ms}$ (on average), where L_{ms} is the distance between molecular interactions. The rate of momentum associated with these molecules per unit area is the product of the rate of molecules passing through the plane ($n_{ms}\, v_{ms}$) momentum and the momentum per molecule; the momentum per molecule is the product of the mass of the molecule (M) and its *x*-velocity at the point where it experienced its last collision, x-L_{ms} ($M\, u_{x-L_{ms}}$). Therefore, the rate of momentum passing through the plane from left-to-right ($\dot{M}''_{x+}$) is given approximately by:

$$\dot{M}''_{x+} \approx n_{ms}\, v_{ms}\, M\, u_{ms,x-L_{ms}} \tag{1}$$

Similarly, the momentum per unit area passing through the plane from right-to-left ($\dot{M}''_{x-}$) is given by:

$$\dot{M}''_{x+} \approx n_{ms}\, v_{ms}\, M\, u_{x+L_{ms}} \tag{2}$$

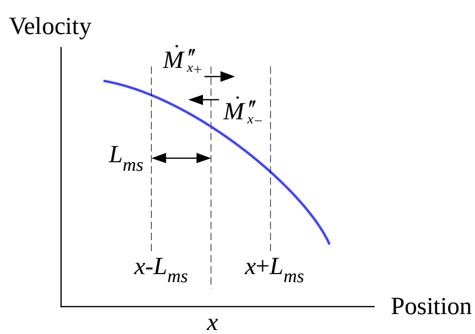

Figure 1: Momentum flows through a plane in a material.

The net rate of momentum flux passing through the plane per unit area in the positive *x*-direction ($\dot{M}''$) is the difference between $\dot{M}''_{x+}$ and $\dot{M}''_{x-}$,

$$\dot{M}'' \approx n_{ms}\, v_{ms}\, M \left(u_{x-L_{ms}} - u_{x+L_{ms}} \right) \tag{3}$$

which can be rearranged to yield:

$$\dot{M}'' \approx -2\, n_{ms}\, v_{ms}\, M\, L_{ms} \underbrace{\frac{\left(u_{x+L_{ms}} - u_{x-L_{ms}} \right)}{L_{ms}}}_{\frac{\partial u}{\partial x}} \approx -\underbrace{2\, n_{ms}\, v_{ms}\, M\, L_{ms}}_{\propto \mu} \frac{\partial u}{\partial x} \tag{4}$$

Comparing Eq. (4) with the definition of viscosity shows that the viscosity is proportional to the product of the number of molecules per unit volume, their average velocity, the mass of each molecule, and the mean distance between their interactions.

$$\mu \propto n_{ms}\, v_{ms}\, M\, L_{ms} \tag{5}$$

The mass of a molecule is the molecular weight, *MW*. As noted in Eq. (1-14), kinetic theory indicates that

$$v_{ms} \propto \sqrt{\frac{R_{univ}\, T}{MW}} \tag{6}$$

where R_{univ} is the universal gas constant and T is the absolute temperature. The distance between molecular interactions was derived in Eq. (1-17) is

$$L_{ms} = \frac{1}{n_{ms}\, \pi\, \sigma^2} \tag{7}$$

where σ is the equivalent radius of the molecule. Substituting Eqs. (6) and (7) into Eq. (5) shows that:

$$\boxed{\mu \propto \frac{1}{\sigma^2} \sqrt{T\, MW}} \tag{8}$$

which is identical to Eq. (1-18) for conductivity if the specific heat capacity is removed.