Problem 1.2-21

Figure P1.2-21 illustrates a plane wall made of a material with a temperature-dependent conductivity. The conductivity of the material is given by:

$$k = b\,T \tag{1}$$

where b = 1 W/m^2-K^2 and T is the temperature in K.

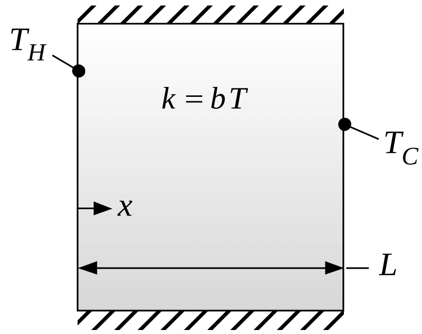

Figure P1.2-21: Plane wall with temperature-dependent conductivity.

The thickness of the wall is L = 1 m. The left side of the wall (at x= 0) is maintained at T_H = 500 K and the right side (at x= L) is kept at T_C = 50 K. The problem is steady-state and 1-D.

a.) Sketch the temperature distribution in the wall (i.e., sketch T as a function of x). Make sure that you get the qualitative features of your sketch right.

The temperatures at x = 0 and x = L are specified. The temperature variation from $0 < x < L$ will not be linear. The rate of heat transfer will be constant with x for this problem. Fourier's law governs the rate of conduction:

$$\dot{q} = -k\,A_c \frac{dT}{dx} \tag{2}$$

According to Eq. (1), in regions where the temperature is high, the conductivity will be high; therefore, the temperature gradient will be small. In regions where the temperature is low, the conductivity will be low and the temperature gradient higher. Figure 2 reflects these characteristics.

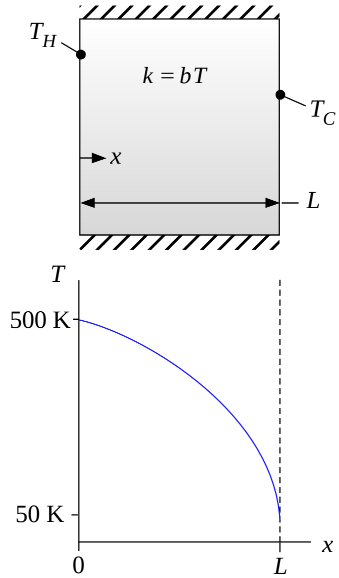

Figure 2: Sketch of temperature distribution.

b.) Derive the ordinary differential equation that governs this problem.

The first step towards developing an analytical solution for this, or any, problem involves the definition of a differential control volume. The control volume must encompass material at a uniform temperature; therefore, in this case it must be differentially small in the *x*-direction (i.e., it has width *dx*, see Figure 3) but can extend across the entire cross-sectional area of the wall as there are no temperature gradients in the *y*- or *z*-directions.

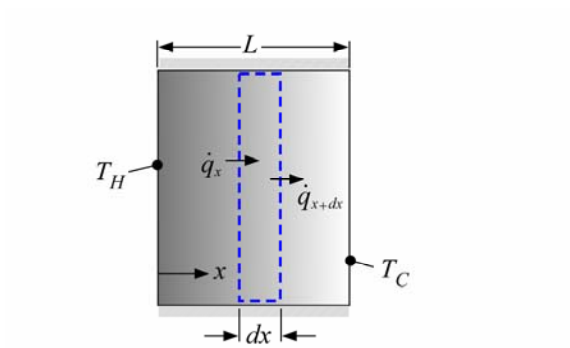

Figure 3: **Differential control volume.**

Next, the energy transfers across the control surfaces must be defined as well as any thermal energy generation or storage terms. For the steady-state, 1-D case considered here, there are only two energy transfers, corresponding to the rate of conduction heat transfer into the left side (i.e., at position x, $\dot{q}_x$) and out of the right side (i.e., at position $x+dx$, $\dot{q}_{x+dx}$) of the control volume. A steady-state energy balance for the differential control volume is therefore:

$$\dot{q}_x = \dot{q}_{x+dx} \tag{3}$$

A Taylor series expansion of the term at *x*+*dx* leads to:

$$\dot{q}_{x+dx} = \dot{q}_x + \frac{d\dot{q}}{dx}dx + \frac{d^2\dot{q}}{dx^2}\frac{dx^2}{2!} + \frac{d^3\dot{q}}{dx^3}\frac{dx^3}{3!} + \ldots \tag{4}$$

The analytical solution proceeds by taking the limit as *dx* goes to zero so that the higher order terms in Eq. (4) can be neglected:

$$\dot{q}_{x+dx} = \dot{q}_x + \frac{d\dot{q}}{dx}dx \tag{5}$$

Substituting Eq. (5) into Eq. (3) leads to:

$$\dot{q}_x = \dot{q}_x + \frac{d\dot{q}}{dx}dx \tag{6}$$

or

$$\frac{d\dot{q}}{dx} = 0 \tag{7}$$

Equation (7) indicates that the rate of conduction heat transfer is not a function of *x*. For the problem in Figure 1, there are no sources or sinks of energy and no energy storage within the wall; therefore, there is no reason for the rate of heat transfer to vary with position. The final step in the derivation of the governing equation is to substitute appropriate rate equations that relate energy transfer rates to temperatures. The rate equation for conduction is Fourier's law:

$$\dot{q} = -k\, A_c \frac{dT}{dx} \tag{8}$$

Substituting Eq. (8) into Eq. (7) leads to:

$$\frac{d}{dx}\left[-k\, A_c \frac{dT}{dx}\right] = 0 \tag{9}$$

The area is constant and can be divided out of Eq. (9). The thermal conductivity is given by Eq. (1):

$$\frac{d}{dx}\left[b\,T \frac{dT}{dx}\right] = 0 \tag{10}$$

c.) What are the boundary conditions for this problem?

The boundary conditions are:

$$T_{x=0} = T_H \tag{11}$$

and

$$T_{x=L} = T_C \tag{12}$$

d.) Solve the governing differential equation from (b) - you should end up with a solution that involves two unknown constants of integration.

Equation (10) is separated:

$$d\left[b\,T\,\frac{dT}{dx}\right] = 0 \tag{13}$$

and integrated:

$$\int d\left[b\,T\,\frac{dT}{dx}\right] = \int 0 \tag{14}$$

which leads to:

$$b\,T\,\frac{dT}{dx} = C_1 \tag{15}$$

where C_1 is a constant of integration. Equation (15) is separated:

$$T\,dT = \frac{C_1}{b}\,dx \tag{16}$$

and integrated:

$$\int T\,dT = \frac{C_1}{b}\int dx \tag{17}$$

which leads to:

$$\frac{T^2}{2} = \frac{C_1}{b}\,x + C_2 \tag{18}$$

e.) Use the boundary conditions from (c) with the solution from (d) in order to obtain two equations in the two unknown constants.

Equation (18) is substituted into Eqs. (11) and (12):

$$\frac{T_H^2}{2} = C_2 \tag{19}$$

$$\frac{T_C^2}{2} = \frac{C_1}{b} L + C_2 \tag{20}$$

f.) Type the inputs for the problem and the equations from (e) into EES in order to evaluate the undetermined constants.

The inputs are entered in EES:

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$UnitSystem SI MASS RAD PA K J
$Tabstops 0.2 0.4 0.6 3.5 in

"Inputs"
T_H=500 [K]                    "temperature at left side of wall"
T_C=50 [K]                     "temperature at right side of wall"
b=1 [W/m-K^2]                  "coefficient for conductivity function"
L=1 [m]                        "thickness of wall"
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and Eqs. (19) and (20) are entered in EES:

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T_H^2/2=C_2                    "boundary condition at x=0"
T_C^2/2=C_1*L/b+C_2            "boundary condition at x=L"
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which leads to C_1 = -123750 W/m^2 and C_2 = 125000 K^2.

g.) Prepare a plot of the temperature as a function of position in the wall using EES.

The solution, Eq. (18), is entered in EES.

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T^2/2=C_1*x/b+C_2              "solution"
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and a plot is prepared using a parametric table that contains the variables x and T. The result is shown in Figure 4, which is qualitatively similar to the sketch in Figure 2.

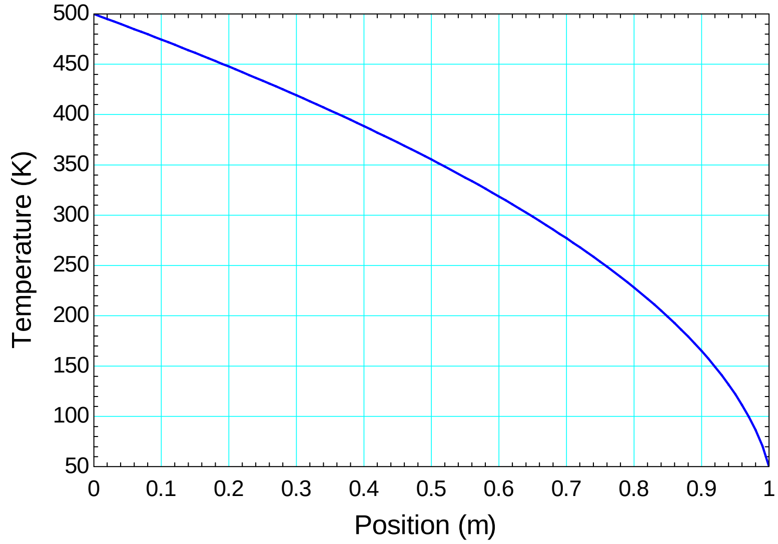

Figure 4: Temperature as a function of position.