Problem 1.8-6: Cryosurgical Probe

As an alternative to surgery, cancer tumors may be destroyed by placing cylindrically-shaped cryoprobes into the body, as shown in Figure P1.8-6. The probe surface is cooled causing the temperature of the surrounding tissue to drop to a lethal level, killing the tumor.

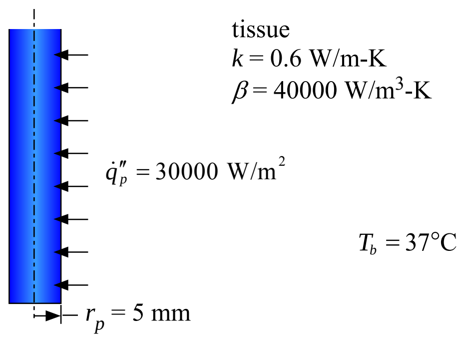

Figure 1.8-6: Cryosurgical probe.

The probe radius is r_p = 5 mm and the heat flux at the surface of the probe (leaving the tissue) is $\dot{q}''_p$ = 30000 W/m^2. The tissue has conductivity k = 0.6 W/m-K. The blood flow through the tissue results in a volumetric heating effect ($\dot{g}'''$) that is proportional to the difference between the local temperature and the blood temperature, T_b = 37ºC:

$$\dot{g}''' = \beta\left(T_b - T\right)$$

where β = 40000 W/m^3-K. The temperature of the tissue far from the probe is T_b. Assume that the temperature distribution is 1-D and steady-state.

a.) Develop an analytical model that can be used to predict the temperature distribution in the tissue. Implement your solution in EES and prepare a plot of the temperature distribution as a function of radius.

The differential control volume shown in Figure 2 can be used to derive the governing equation.

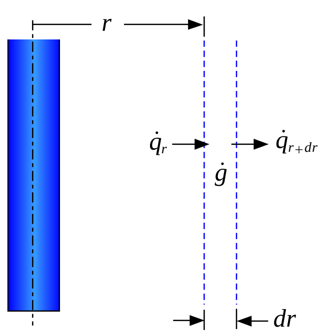

Figure 2: Differential control volume.

An energy balance for the control volume is:

$$\dot{g} + \dot{q}_r = \dot{q}_{r+dr} \tag{1}$$

or

$$\dot{g} = \frac{d\dot{q}}{dr} dr \tag{2}$$

The conduction and blood perfusion terms are:

$$\dot{q} = -k\, 2\pi\, r\, L \frac{dT}{dr} \tag{3}$$

$$\dot{g} = 2\pi\, r\, dr\, L\, \beta \left(T_b - T \right) \tag{4}$$

Combining these equations leads to:

$$2\pi\, r\, dr\, L\, \beta \left(T_b - T \right) = \frac{d}{dr}\left[-k\, 2\pi\, r\, L \frac{dT}{dr} \right] dr \tag{5}$$

which can be simplified to:

$$\frac{d}{dr}\left[r \frac{dT}{dr} \right] - r \frac{\beta}{k} T = -r \frac{\beta}{k} T_b \tag{6}$$

The solution is divided into a homogeneous and particular component:

$$T = T_h + T_p \tag{7}$$

which leads to:

$$\frac{d}{dr}\left[r \frac{d\left(T_h + T_p \right)}{dr} \right] - r \frac{\beta}{k}\left(T_h + T_p \right) = -r \frac{\beta}{k} T_b \tag{8}$$

or

$$\underbrace{\frac{d}{dr}\left[r \frac{dT_h}{dr} \right] - r \frac{\beta}{k} T_h}_{=0 \text{ for homogeneous ODE}} + \underbrace{\frac{d}{dr}\left[r \frac{dT_p}{dr} \right] - r \frac{\beta}{k} T_p = -r \frac{\beta}{k} T_b}_{\text{whatever is left is particular ODE}} \tag{9}$$

The solution to the particular differential equation:

$$\frac{d}{dr}\left[r \frac{dT_p}{dr} \right] - r \frac{\beta}{k} T_p = -r \frac{\beta}{k} T_b \tag{10}$$

is

$$T_p = T_b \tag{11}$$

The homogeneous differential equation is:

$$\frac{d}{dr}\left[r\frac{dT_h}{dr}\right] - r\frac{\beta}{k}T_h = 0 \tag{12}$$

Equation (12) is a form of Bessel's equation:

$$\frac{d}{dx}\left(x^p \frac{d\theta}{dx}\right) \pm c^2\, x^s\, \theta = 0 \tag{13}$$

where, by comparing Eqs. (12) and (13), p = 1, c = $\sqrt{\beta / k}$, and s = 1. Referring to the flow chart presented in Section 1.8, the value of s-p+2 is equal to 2 and therefore the solution parameters n and a must be computed:

$$n = \frac{1-1}{1-1+2} = 0 \tag{14}$$

$$a = \frac{2}{1-1+2} = 1 \tag{15}$$

The last term in Eq. (12) is negative; therefore the homogeneous solution is given by:

$$T_h = C_1\, x^{n/a}\, \mathrm{BesselI}\left(n, c\, a\, x^{1/a}\right) + C_2\, x^{n/a}\, \mathrm{BesselK}\left(n, c\, a\, x^{1/a}\right) \tag{16}$$

or, for this problem:

$$T_h = C_1\, \mathrm{BesselI}\left(0, r\sqrt{\frac{\beta}{k}}\right) + C_2\, \mathrm{BesselK}\left(0, r\sqrt{\frac{\beta}{k}}\right) \tag{17}$$

Substituting Eqs. (11) and (17) into Eq. (7) leads to:

$$T = C_1\, \mathrm{BesselI}\left(0, r\sqrt{\frac{\beta}{k}}\right) + C_2\, \mathrm{BesselK}\left(0, r\sqrt{\frac{\beta}{k}}\right) + T_b \tag{18}$$

Note that Maple could be used to identify this solution as well; it is necessary to specify that the parameters β and k are positive so that Maple identifies the solution in terms of modified Bessel functions (as opposed to Bessel functions with complex arguments):

```
> restart;
> assume(beta>0);
> assume(k>0);
> ODE:=diff(r*diff(T(r),r),r)-r*beta*T(r)/k=-r*beta*T_b/k;
```

$$ODE := \left(\frac{d}{dr}\mathrm{T}(r)\right) + r\left(\frac{d^2}{dr^2}\mathrm{T}(r)\right) - \frac{r\,\beta\sim\,\mathrm{T}(r)}{k\sim} = -\frac{r\,\beta\sim\,T\_b}{k\sim}$$

```
> Ts:=dsolve(ODE);
```

$$Ts := \mathrm{T}(r) = \mathrm{BesselI}\left(0, \frac{\sqrt{\beta\sim}\,r}{\sqrt{k\sim}}\right)\_C2 + \mathrm{BesselK}\left(0, \frac{\sqrt{\beta\sim}\,r}{\sqrt{k\sim}}\right)\_C1 + T\_b$$

The boundary conditions must be used to obtain the constants C_1 and C_2. As radius approaches infinity, the body temperature is recovered:

$$T_{r\to\infty} = T_b \tag{19}$$

Substituting Eq. (18) into Eq. (19) leads to:

$$T_{r\to\infty} = C_1\,\mathrm{BesselI}\left(0, \infty\sqrt{\frac{\beta}{k}}\right) + C_2\,\mathrm{BesselK}\left(0, \infty\sqrt{\frac{\beta}{k}}\right) + T_b = T_b \tag{20}$$

or

$$C_1\underbrace{\mathrm{BesselI}\left(0, \infty\sqrt{\frac{\beta}{k}}\right)}_{\infty} + C_2\underbrace{\mathrm{BesselK}\left(0, \infty\sqrt{\frac{\beta}{k}}\right)}_{0} = 0 \tag{21}$$

The 0^{th} order modified Bessel function of the 1^{st} kind, BesselI(0,x), approaches ∞ as $x \to \infty$ and 0^{th} order modified Bessel function of the 2^{nd} kind, BesselK(0,x), approaches 0 as $x \to \infty$:

```
> limit(BesselI(0,x),x=infinity);
```

$$\infty$$

```
> limit(BesselK(0,x),x=infinity);
```

$$0$$

Therefore, C_1 must be zero and Eq. (18) becomes:

$$T = C_2\,\mathrm{BesselK}\left(0, r\sqrt{\frac{\beta}{k}}\right) + T_b \tag{22}$$

The heat flux into the probe (i.e., in the negative r-direction) at $r = r_p$ is specified, providing the additional boundary condition:

$$k \left. \frac{dT}{dr} \right|_{r=r_p} = \dot{q}''_p \tag{23}$$

Substituting Eq. (22) into Eq. (23) leads to:

$$k \frac{d}{dr}\left[C_2 \, \text{BesselK}\left(0, r\sqrt{\frac{\beta}{k}}\right) + T_b \right]_{r=r_p} = \dot{q}''_p \tag{24}$$

Using the rules for differentiating Bessel functions presented in Section 1.8.4 leads to:

$$-k\, C_2 \sqrt{\frac{\beta}{k}} \text{BesselK}\left(1, r_p\sqrt{\frac{\beta}{k}}\right) = \dot{q}''_p \tag{25}$$

which leads to:

$$C_2 = -\frac{\dot{q}''_p}{\sqrt{\beta\, k} \; \text{BesselK}\left(1, r_p\sqrt{\frac{\beta}{k}}\right)} \tag{26}$$

Substituting Eq. (26) into Eq. (22) leads to:

$$T = T_b - \frac{\dot{q}''_p}{\sqrt{\beta\, k}} \frac{\text{BesselK}\left(0, r\sqrt{\frac{\beta}{k}}\right)}{\text{BesselK}\left(1, r_p\sqrt{\frac{\beta}{k}}\right)} \tag{27}$$

The solution can also be identified using Maple. Substitute $C_2 = 0$ into the previously obtained solution:

```
> Ts:=subs(_C2=0,Ts);
```

$$Ts := \mathrm{T}(r) = \text{BesselK}\left(0, \frac{\sqrt{\beta\sim}\; r}{\sqrt{k\sim}}\right) \_C1 + T\_b$$

Obtain an equation for the boundary condition associated with Eq. (23):

```
> BC:=k*rhs(eval(diff(Ts,r),r=r_p))=qf_p;
```

$$BC := -\sqrt{k\sim}\ \mathrm{BesselK}\left(1, \frac{\sqrt{\beta\sim}\ r\_p}{\sqrt{k\sim}}\right)\sqrt{\beta\sim}\ \_C1 = qf\_p$$

Substitute the solution to the boundary condition equation into the general solution:

```
> subs(_C1=solve(BC,_C1),Ts);
```

$$\mathrm{T}(r) = -\frac{\mathrm{BesselK}\left(0, \frac{\sqrt{\beta\sim}\ r}{\sqrt{k\sim}}\right) qf\_p}{\sqrt{k\sim}\ \mathrm{BesselK}\left(1, \frac{\sqrt{\beta\sim}\ r\_p}{\sqrt{k\sim}}\right)\sqrt{\beta\sim}} + T\_b$$

The solution is implemented in EES. The inputs are entered:

```
$UnitSystem SI MASS RAD PA  K J
$TABSTOPS   0.2 0.4 0.6 0.8 3.5 in

"Inputs"
r_p_mm=5 [mm]                                   "probe radius, in mm"
r_p=r_p_mm*convert(mm,m)                        "probe radius"
q``_p=30000 [W/m^2]                             "probe heat flux"
k=0.6 [W/m-K]                                   "tissue conductivity"
beta=40000 [W/m^3-K]                            "blood perfusion effect"
T_b=converttemp(C,K,37[C])                      "blood temperature"
```

Equation (27) is implemented in EES; the radius and temperature are converted to mm and ºC, respectively.

```
T=T_b-q``_p*BesselK(0,r*sqrt(beta/k))/(sqrt(beta*k)*BesselK(1,r_p*sqrt(beta/k)))    "solution"
r_mm=r*convert(m,mm)                                                                "radius"
T_C=converttemp(K,C,T)                                                              "in C"
```

Figure 3 illustrates the temperature as a function of radius.

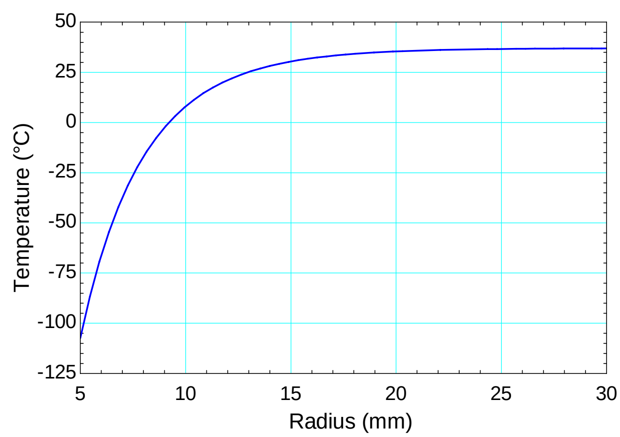

Figure 3: Temperature in tissue as a function of radius.

b.) The lethal temperature for cell death is T_{lethal} = -30ºC. Plot the radius of the cryolesion (i.e., the kill radius - all tissue inside of this radius is dead) as a function of the heat flux provided by the cryoprobe.

The temperature is set to the lethal temperature. The variable r must be constrained to be positive in the Variable Information Window to avoid convergence errors. The kill radius as a function of heat flux is shown in Figure 4.

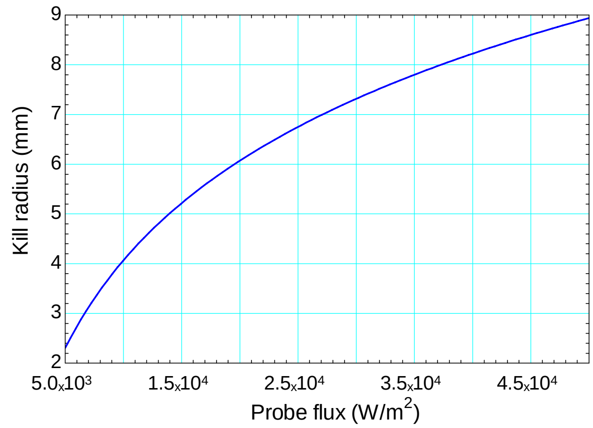

Figure 4: Kill radius as a function of the cryoprobe heat flux.