Problem 1.9-1: A 4-Node Numerical Simulation of a Fuse

A fuse is a long, thin piece of metal that will heat up when current is passed through it. If a large amount of current is passed through the fuse, then the material will melt and this protects the electrical components downstream of the fuse. Figure P1.9-1 illustrates a fuse that is composed of a piece of metal with a square cross-section (a x a) that has length L. The conductivity of the fuse material is k. The fuse surface experiences convection with air at temperature T_a with heat transfer coefficient $\overline{h}$. Radiation from the surface can be neglected for this problem. The ohmic heating associated with the current passing through the fuse results in a uniform rate of volumetric thermal energy generation, $\dot{g}'''$. The two ends of the fuse (at x = 0 and x = L) are held at temperature T_b. You have been asked to generate a numerical model of the fuse. Figure P1.9-1 also shows a simple numerical model that includes only four nodes which are positioned uniformly along the length of the fuse.

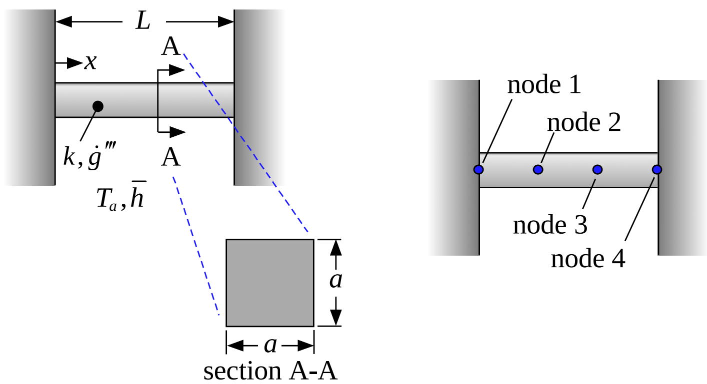

Figure P1.9-1: Fuse and a four node numerical model of the fuse.

a.) How would you determine if the extended surface approximation was appropriate for this problem?

The Biot number compares the resistance to conduction from the center to the edge of the fuse to the resistance to convection from the surface of the fuse; for this problem, the Biot number should be:

$$Bi = \frac{h\,a}{2\,k} \tag{1}$$

Any formula that is within a factor of 2 of Eq. (1) is fine.

For the remainder of this problem, assume that you can use the extended surface approximation.

b.) Derive a system of algebraic equations that can be solved in order to predict the temperatures at each of the four nodes in Figure P1.9-1 (T[1], T[2], T[3], and T[4]). Your equations should include only those symbols defined in the problem statement. Do not solve these equations.

An energy balance on a control volume around node 2 is shown in Figure 2.

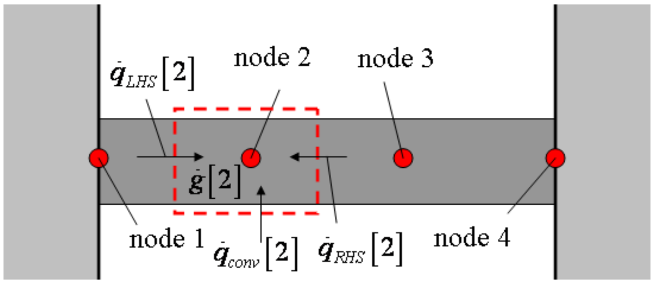

Figure 2: Energy balance on a control volume around node 2

The energy balance suggested by Figure 2 is:

$$\dot{g}[2] + \dot{q}_{LHS}[2] + \dot{q}_{RHS}[2] + \dot{q}_{conv}[2] = 0 \tag{2}$$

The rate equations for the terms in Eq. (2) are:

$$\dot{g}[2] = \frac{L\,a^2}{3}\,\dot{g}''' \tag{3}$$

$$\dot{q}_{LHS}[2] = \frac{3\,k\,a^2}{L}\left(T[1] - T[2]\right) \tag{4}$$

$$\dot{q}_{RHS}[2] = \frac{3\,k\,a^2}{L}\left(T[3] - T[2]\right) \tag{5}$$

$$\dot{q}_{conv}[2] = \frac{4\,a\,L\,\overline{h}}{3}\left(T_a - T[2]\right) \tag{6}$$

A similar process for node 3 leads to:

$$\dot{g}[3] + \dot{q}_{LHS}[3] + \dot{q}_{RHS}[3] + \dot{q}_{conv}[3] = 0 \tag{7}$$

where:

$$\dot{g}[3] = \frac{L\,a^2}{3}\,\dot{g}''' \tag{8}$$

$$\dot{q}_{LHS}[3] = \frac{3\,k\,a^2}{L}\left(T[2] - T[3]\right) \tag{9}$$

$$\dot{q}_{RHS}[3] = \frac{3\,k\,a^2}{L}\left(T[4] - T[3]\right) \tag{10}$$

$$\dot{q}_{conv}[3] = \frac{4\,a\,L\,\bar{h}}{3}\left(T_a - T[3]\right) \tag{11}$$

The temperatures of the edge nodes are specified:

$$T[1] = T_b \tag{12}$$

$$T[4] = T_b \tag{13}$$

Equations (2) through (13) can be solved (using, for example, EES) to provide the temperature at each node.

c.) How would you determine the amount of heat transferred from the fuse to the wall at $x = 0$ using your solution from (b)?

In order to determine the amount of heat transferred to the wall at $x= 0$, it is necessary to do an energy balance on the control volume that surrounds node 1.

$$\dot{g}[1] + \dot{q}_{RHS}[1] + \dot{q}_{conv}[1] = \dot{q}_{wall} \tag{14}$$

where

$$\dot{g}[1] = \frac{L\,a^2}{6}\,\dot{g}''' \tag{15}$$

$$\dot{q}_{RHS}[1] = \frac{3\,k\,a^2}{L}\left(T[2] - T[1]\right) \tag{16}$$

$$\dot{q}_{conv}[1] = \frac{2\,a\,L\,\bar{h}}{3}\left(T_a - T[1]\right) \tag{17}$$

d.) **Derive** the differential equation and boundary conditions that you would need in order to solve this problem analytically. Show your steps clearly.

A differential control volume is shown in Figure 3.

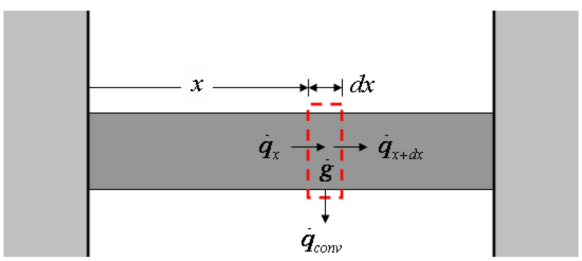

Figure 3: Differential control volume.

The energy balance suggested by Figure 3 is:

$$\dot{q}_x + \dot{g} = \dot{q}_{x+dx} + \dot{q}_{conv} \tag{18}$$

Expanding the *x*+*dx* term leads to:

$$\dot{q}_x + \dot{g} = \dot{q}_x + \frac{d\dot{q}}{dx}dx + \dot{q}_{conv} \tag{19}$$

or

$$\dot{g} = \frac{d\dot{q}}{dx}dx + \dot{q}_{conv} \tag{20}$$

The rate equations are:

$$\dot{g} = a^2\,\dot{g}'''\,dx \tag{21}$$

$$\dot{q} = -k\,a^2\,\frac{dT}{dx} \tag{22}$$

$$\dot{q}_{conv} = 4\,\overline{h}\,a\,dx\left(T - T_a\right) \tag{23}$$

Substituting Eqs. (21) through (23) into Eq. (20) leads to:

$$a^2\,\dot{g}'''\,dx = \frac{d}{dx}\left[-k\,a^2\,\frac{dT}{dx}\right]dx + 4\,\overline{h}\,a\,dx\left(T - T_a\right) \tag{24}$$

or

$$\frac{d^2T}{dx^2} - \frac{4\,\overline{h}}{k\,a}\left(T - T_a\right) = -\frac{\dot{g}'''}{k} \tag{25}$$

The boundary conditions are:

$$T_{x=0} = T_b \tag{26}$$

$$T_{x=L} = T_b \tag{27}$$