Problem 1.8-7: Bracket

A disk-shaped bracket connects a cylindrical heater to an outer shell, as shown in Figure P1.8-7(a).

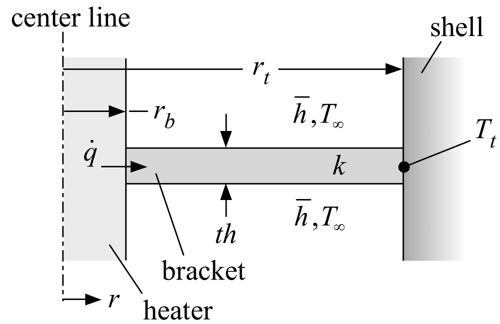

Figure P1.8-7(a): Disk-shaped bracket.

The thickness of the bracket is *th* and it is made of material with conductivity k. The bracket extends radially from r_b at the heater to r_t at the outer shell. The temperature of the bracket location where it intersects the shell (at $r = r_t$) is T_t. The heater provides a rate of heat transfer to the bracket, $\dot{q}$, at $r = r_b$. Both the upper and lower surfaces of the bracket are exposed to fluid at T_∞ with average heat transfer coefficient $\overline{h}$.

a.) What calculation would you do in order to justify treating the bracket as an extended surface (i.e., justify the assumption that temperature is only a function of r); provide an expression in terms of the symbols in the problem statement.

The appropriate Biot number is the ratio of the resistance to conduction across the thickness of the fin to the resistance to convection from the fin surface. The resistance to conduction across the thickness of the fin (i.e., in the x direction) is:

$$R_{cond,x} = \frac{th}{2\,k\,\pi\left(r_t^2 - r_b^2\right)} \tag{1}$$

The resistance to convection from the fin surface is:

$$R_{conv} = \frac{1}{\overline{h}\,\pi\left(r_t^2 - r_b^2\right)} \tag{2}$$

The Biot number is therefore:

$$Bi = \frac{R_{cond,x}}{R_{conv}} = \frac{th}{2\,k\,\pi\left(r_t^2 - r_b^2\right)}\frac{\overline{h}\,\pi\left(r_t^2 - r_b^2\right)}{1} = \boxed{\frac{th\,\overline{h}}{2\,k}} \tag{3}$$

For the remainder of the problem, assume that the bracket can be treated as an extended surface.

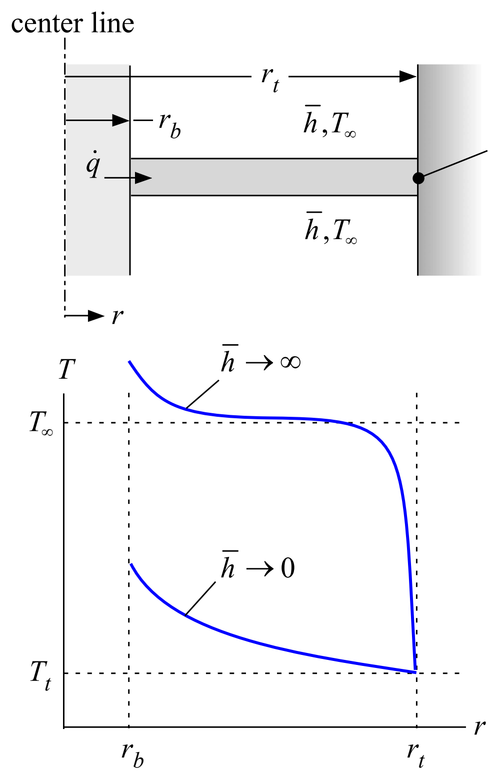

Figure P1.8-7(b): Qualitative sketch of the temperature distribution expected if $\bar{h} \rightarrow 0$ and $\bar{h} \rightarrow \infty$.

b.) On the axes in Figure P1.8-7(b), sketch the temperature distribution that you would expect if the heat transfer coefficient $\bar{h}$ is very low ($\bar{h} \rightarrow 0$). Note that the qualitative values of T_t and T_∞ are indicated in the plot - your sketch should be consistent with these values.

The sketch is shown in Figure P1.8-7(a) and has the following characteristics.

1. The slope at $r = r_b$ should be negative due to the heat transfer at the base.
2. The temperature at $r = r_t$ must be T_t.
3. The surfaces of the bracket are insulated if $\bar{h} \rightarrow 0$; therefore, the conduction heat transfer is constant in the r-direction. Because the area for conduction increases with r, the temperature gradient must decrease.

c.) On the axes in Figure P1.8-7(c), sketch the temperature distribution that you would expect if the heat transfer coefficient $\bar{h}$ is very high ($\bar{h} \rightarrow \infty$). Note that the qualitative values of T_t and T_∞ are indicated in the plot - your sketch should be consistent with these values.

The sketch is also shown in Figure P1.8-7(a) and has the following characteristics.

1. The slope at $r = r_b$ is negative due to the heat transfer at the base and is identical to the slope of the distribution from (b).
2. The temperature at $r = r_t$ must be T_t.
3. The temperature of the bracket otherwise tends to approach T_∞ due to the strong coupling between the surface and the surrounding fluid. The temperature must approach T_∞ from above at the base (because you are transferring heat from the bracket to the fluid) and from below at the tip (because you are transferring from the fluid to the bracket.

d.) What dimensionless number would you calculate in order to determine whether the actual temperature distribution is closer to your sketch from (b) or (c)? Provide an expression in terms of the symbols in the problem statement.

The behavior of the bracket is governed by two resistances: the resistance to convection from the surface of the bracket, Eq. (2), and the resistance to conduction in the radial direction:

$$R_{cond,r} = \frac{\ln\left(\frac{r_t}{r_b}\right)}{2\,\pi\, th\, k} \tag{4}$$

The ratio of $R_{cond,r}$ to R_{conv} governs the behavior:

$$\frac{R_{cond,r}}{R_{conv}} = \frac{\ln\left(\frac{r_t}{r_b}\right)}{2\,\pi\, th\, k}\frac{\overline{h}\,\pi\left(r_t^2 - r_b^2\right)}{1} = \boxed{\ln\left(\frac{r_t}{r_b}\right)\frac{\overline{h}\left(r_t^2 - r_b^2\right)}{2\, th\, k}} \tag{5}$$

If $R_{cond,r}/R_{conv} <<1$, then the distribution will approach your answer from (b). If $R_{cond,r}/R_{conv} >>1$, then the distribution will approach your answer from (c).

e.) Derive the governing ordinary differential equation for the bracket.

An energy balance on a differential segment of the bracket is shown in Figure P1.8-7(c).

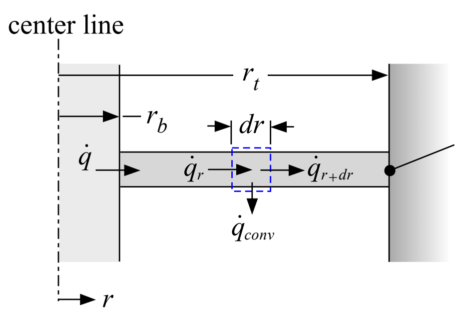

Figure P1.8-7(c): Energy balance on a differential segment of the bracket.

The energy balance in Figure P1.8-7(c) leads to:

$$\dot{q}_r = \dot{q}_{r+dr} + \dot{q}_{conv} \tag{6}$$

or

$$0 = \frac{d\dot{q}}{dr}dr + \dot{q}_{conv} \tag{7}$$

Substituting rate equations into Eq. (7) leads to:

$$0 = \frac{d}{dr}\left[-k\,2\,\pi\,r\,th\frac{dT}{dr}\right]dr + \overline{h}\,4\,\pi\,r\,dr\left(T - T_{\infty}\right) \tag{8}$$

or

$$\boxed{\frac{d}{dr}\left[r\frac{dT}{dr}\right] - \beta^{2}\,r\,T = -\beta^{2}\,r\,T_{\infty}} \tag{9}$$

where

$$\beta^{2} = \frac{2\,\overline{h}}{k\,th} \tag{10}$$

f.) What are the boundary conditions for the ordinary differential equation from (e)?

The boundary conditions are:

$$\boxed{-k\,2\,\pi\,r_{b}\,th\left.\frac{dT}{dr}\right|_{r=r_{b}} = \dot{q}} \tag{11}$$

and

$$\boxed{T_{r=r_{t}} = T_{t}} \tag{12}$$

g.) Obtain a solution to your ordinary differential equation that includes two undetermined constants.

The solution is split into a homogeneous and particular component:

$$T = T_{h} + T_{p} \tag{13}$$

which leads to:

$$\underbrace{\frac{d}{dr}\left[r\frac{dT_{h}}{dr}\right] - \beta^{2}\,r\,T_{h}}_{\text{homogeneous ODE}} + \underbrace{\frac{d}{dr}\left[r\frac{dT_{p}}{dr}\right] - \beta^{2}\,r\,T_{p} = -\beta^{2}\,r\,T_{\infty}}_{\text{particular ODE}} \tag{14}$$

The solution to the particular ordinary differential equation is:

$$T_p = T_\infty \tag{15}$$

Comparing the homogeneous ordinary differential equation to Bessel's equation and following the flow chart provided in Figure 1-54 of your notes leads to:

$$T_h = C_1 \,\text{BesselI}\left(0, \beta\, r\right) + C_2 \,\text{BesselK}\left(0, \beta\, r\right) \tag{16}$$

Substituting Eqs. (15) and (16) into Eq. (13) leads to:

$$\boxed{T = C_1 \,\text{BesselI}\left(0, \beta\, r\right) + C_2 \,\text{BesselK}\left(0, \beta\, r\right) + T_\infty} \tag{17}$$

h.) Write the two algebraic equation that could be solved to provide the undetermined constants (don't solve these equations).

Substituting Eq. (17) into Eq. (12) leads to:

$$\boxed{C_1 \,\text{BesselI}\left(0, \beta\, r_t\right) + C_2 \,\text{BesselK}\left(0, \beta\, r_t\right) + T_\infty = T_t} \tag{18}$$

Substituting Eq. (17) into Eq. (11) leads to:

$$\boxed{-k\, 2\, \pi\, r_b\, th\, \beta \left[C_1 \,\text{BesselI}\left(1, \beta\, r_b\right) - C_2 \,\text{BesselK}\left(1, \beta\, r_b\right)\right] = \dot{q}} \tag{19}$$