Problem 1.6-1 (1-13 in text): Temperature Sensor Error

A resistance temperature detector (RTD) utilizes a material that has a resistivity that is a strong function of temperature. The temperature of the RTD is inferred by measuring its electrical resistance. Figure P1.6-1 shows an RTD that is mounted at the end of a metal rod and inserted into a pipe in order to measure the temperature of a flowing liquid. The RTD is monitored by passing a known current through it and measuring the voltage across it. This process results in a constant amount of ohmic heating that may tend to cause the RTD temperature to rise relative to the temperature of the surrounding liquid; this effect is referred to as a self-heating error. Also, conduction from the wall of the pipe to the temperature sensor through the metal rod can also result in a temperature difference between the RTD and the liquid; this effect is referred to as a mounting error.

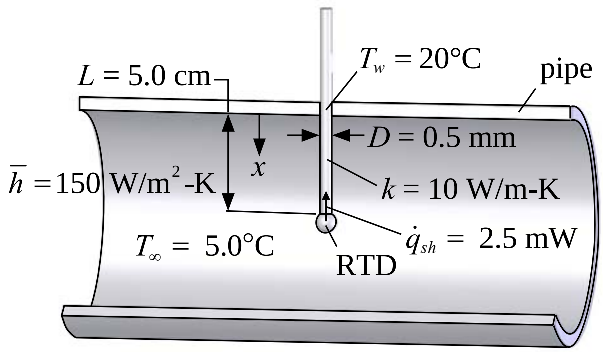


Figure P1.6-1: Temperature sensor mounted in a flowing liquid.

The thermal energy generation associated with ohmic heating is $\dot{q}_{sh}$ = 2.5 mW. All of this ohmic heating is assumed to be transferred from the RTD into the end of the rod at $x = L$. The rod has a thermal conductivity k = 10 W/m-K, diameter D = 0.5 mm, and length L = 5 cm. The end of the rod that is connected to the pipe wall (at x = 0) is maintained at a temperature of T_w = 20°C. The liquid is at a uniform temperature, T_∞ = 5°C and the heat transfer coefficient between the liquid and the rod is $\bar{h}$ = 150 W/m^2-K.

a.) Is it appropriate to treat the rod as an extended surface (i.e., can we assume that the temperature in the rod is a function only of x)? Justify your answer.

The input parameters are entered in EES.

```
$UnitSystem SI MASS RAD PA  K J
$TABSTOPS   0.2 0.4 0.6 0.8 3.5 in

"Inputs"
q_dot_sh=0.0025 [W]                 "self-heating power"
k=10 [W/m-K]                        "conductivity of mounting rod"
d=0.5 [mm]*convert(mm,m)            "diameter of mounting rod"
L=5.0 [cm]*convert(cm,m)            "length of mounting rod"
T_wall=convertTemp(C,K,20)          "temperature of wall"
T_f=convertTemp(C,K,5)              "temperature of liquid"
h=150 [W/m^2-K]                     “heat transfer coefficient”
```

The appropriate Biot number for this case is:

$$Bi = \frac{h\,d}{2\,k} \tag{1}$$

```
"Extended surface approximation"
Bi=h*d/(2*k)
```

The Biot number calculated by EES is 0.004 which is much less than 1.0 and therefore the extended surface approximation is justified.

b.) Develop an analytical model of the rod that will predict the temperature distribution in the rod and therefore the error in the temperature measurement; this error is the difference between the temperature at the tip of the rod and the liquid. You may find it easiest to use Maple for this process.

Figure 2 illustrates a differential control volume for the rod.

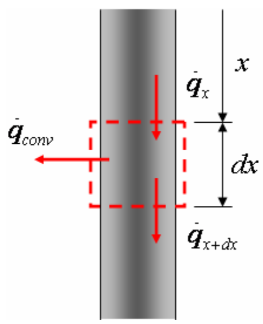


Figure 2: Differential control volume for the rod.

The energy balance suggested by Figure 2 is:

$$\dot{q}_x = \dot{q}_{x+dx} + \dot{q}_{conv} \tag{2}$$

or, expanding the $x+dx$ term:

$$\dot{q}_x = \dot{q}_x + \frac{d\dot{q}}{dx}dx + \dot{q}_{conv} \tag{3}$$

The rate equations for conduction and convection are:

$$\dot{q}_x = -k\,\pi\frac{d^2}{4}\frac{dT}{dx} \tag{4}$$

and

$$\dot{q}_{conv} = h\,\pi\,d\,dx\left(T - T_f\right) \tag{5}$$

Substituting Eqs. (4) and (5) into Eq. (3) leads to:

$$0 = \frac{d}{dx}\left[-k\,\pi\frac{d^2}{4}\frac{dT}{dx}\right]dx + h\,\pi\,d\,dx\left(T - T_f\right) \tag{6}$$

or

$$\frac{d^2T}{dx^2} - \frac{4\,h}{k\,d}\left(T - T_f\right) = 0 \tag{7}$$

which is a non-homogeneous 2nd order differential equation. The general solution to Eq. (7) can be found in your text as Eq. (3.66) or in the handout on Extended Surfaces as Eq. (6-22). The easiest thing to do is enter the differential equation into Maple and let it solve it for you:

```
> GDE:=diff(diff(T(x),x),x)-4*h*(T(x)-T_f)/(k*d)=0;
```

$$GDE := \left(\frac{d^2}{dx^2}\mathrm{T}(x)\right) - \frac{4\,h\,(\mathrm{T}(x) - T\_f)}{k\,d} = 0$$

```
> Ts:=dsolve(GDE);
```

$$Ts := \mathrm{T}(x) = \mathbf{e}^{\left(\frac{2\sqrt{h}\,x}{\sqrt{k}\,\sqrt{d}}\right)}\,\_C2 + \mathbf{e}^{\left(-\frac{2\sqrt{h}\,x}{\sqrt{k}\,\sqrt{d}}\right)}\,\_C1 + T\_f$$

The solution can be copied and pasted into EES (don't forget that you may need to change your output display to Maple Notation to facilitate the copying process depending on your version of Maple):

```
> Ts:=dsolve(GDE);
Ts := T(x) = exp(2*h^(1/2)*x/(k^(1/2)*d^(1/2)))*_C2+exp(-
2*h^(1/2)*x/(k^(1/2)*d^(1/2)))*_C1+T_f
```

which can be copied to EES:

```
Ts := T(x) = exp(2*h^(1/2)*x/(k^(1/2)*d^(1/2)))*_C2+exp(-2*h^(1/2)*x/(k^(1/2)*d^(1/2)))*_C1+T_f
        "solution copied from Maple"
```

The solution will need to be modified slightly so that it is compatible with EES (the _C1 must become C1, _C2 must be C2, Ts:= should be deleted and the T(x) must be just T:

```
T = exp(2*h^(1/2)*x/(k^(1/2)*d^(1/2)))*C2+exp(-2*h^(1/2)*x/(k^(1/2)*d^(1/2)))*C1+T_f
        "solution copied from Maple and modified"
```

Trying to solve now should give the message that you have 12 variables but only 9 equations – you need to specify C1, C2, and x to have a completely specified problem. Let's set x = 0:

```
x=0
```

and concentrate on determining symbolic expressions for the boundary conditions. The temperature at the pipe wall (x=0) is specified to be T_{wall}. Using Maple:

```
> rhs(eval(Ts,x=0))=T_wall;
_C2+_C1+T_f = T_wall
```

which can be pasted into EES (and modified):

```
C2+C1+T_f = T_wall      "wall boundary condition"
```

The boundary condition at the end of the rod with the sensor is associated with an energy balance on the interface:

$$k\,\pi\,\frac{d^2}{4}\left.\frac{dT}{dx}\right|_{x=L} = \dot{q}_{sh} \tag{8}$$

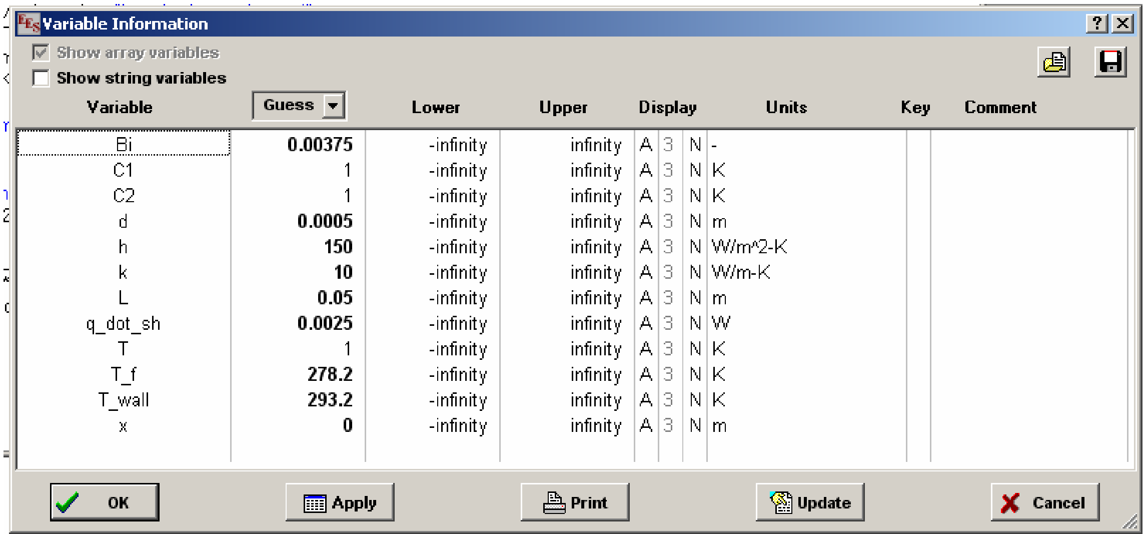
which can be evaluated symbolically in Maple:

```
> k*pi*d^2*rhs(eval(diff(Ts,x),x=L))/4=q_dot_sh;
1/4*k*pi*d^2*(2*h^(1/2)*exp(2*h^(1/2)*L/(k^(1/2)*d^(1/2)))*_C2/(k^(1/2)*d^(1/
2))-2*h^(1/2)*exp(-2*h^(1/2)*L/(k^(1/2)*d^(1/2)))*_C1/(k^(1/2)*d^(1/2))) =
q_dot_sh
```

Aren't you glad you don't have to do this by hand? The expression can be copied and pasted into EES to complete your solution:

```
1/4*k*pi*d^2*(2*h^(1/2)*exp(2*h^(1/2)*L/(k^(1/2)*d^(1/2)))*C2/(k^(1/2)*d^(1/2))-2*h^(1/2)*exp(-
2*h^(1/2)*L/(k^(1/2)*d^(1/2)))*C1/(k^(1/2)*d^(1/2))) = q_dot_sh "sensor boundary condition"
```

Check your units (Figure 3 shows the variable information window with the units set) to make sure that no errors were made.



Variable	Guess	Lower	Upper	Display			Units	Key	Comment
Bi	0.00375	-infinity	infinity	A	3	N	-		
C1	1	-infinity	infinity	A	3	N	K		
C2	1	-infinity	infinity	A	3	N	K		
d	0.0005	-infinity	infinity	A	3	N	m		
h	150	-infinity	infinity	A	3	N	W/m^2-K		
k	10	-infinity	infinity	A	3	N	W/m-K		
L	0.05	-infinity	infinity	A	3	N	m		
q_dot_sh	0.0025	-infinity	infinity	A	3	N	W		
T	1	-infinity	infinity	A	3	N	K		
T_f	278.2	-infinity	infinity	A	3	N	K		
T_wall	293.2	-infinity	infinity	A	3	N	K		
x	0	-infinity	infinity	A	3	N	m		

Figure 3: Variable Information window.

c.) Prepare a plot of the temperature as a function of position and compute the temperature error.

Comment out the specification that x=0 and prepare a parametric table that includes T and x. Alter x so that it varies from 0 to 0.05 and plot the result. You can convert the temperature to °C and position to cm for a better looking plot:

```
x_cm=x*convert(m,cm)
T_C=converttemp(K,C,T)
```

Figure 4 illustrates the temperature distribution; note that the temperature elevation at the tip with respect to the fluid is about 3.6 K and it represents the measurement error. For the conditions in the problem statement, it is clear that the measurement error is primarily due to the self-heating effect because the effect of the wall (the temperature elevation at the base) has died off after about 2.0 cm.

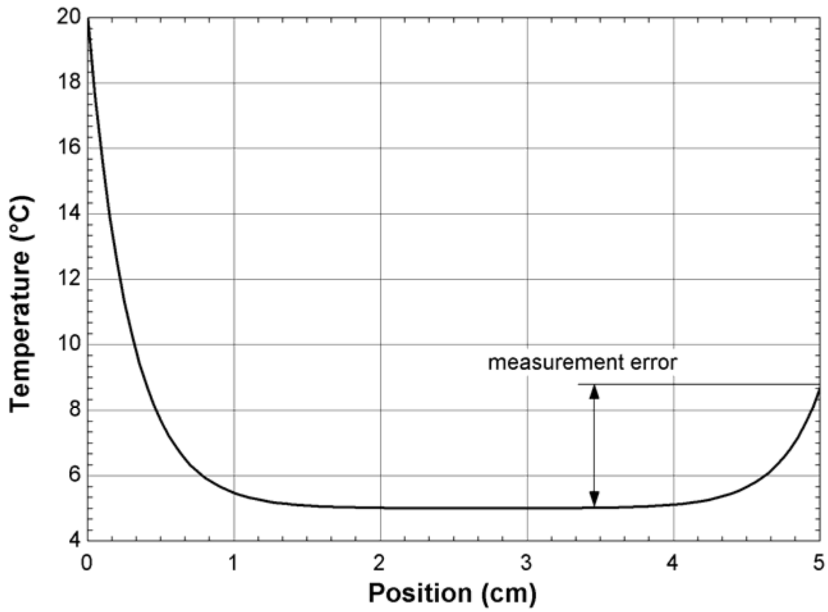


Figure 4: Temperature distribution in the mounting rod.

d.) Investigate the effect of thermal conductivity on the temperature measurement error. Identify the optimal thermal conductivity and explain why an optimal thermal conductivity exists.

The temperature measurement error can be calculated from your solution by setting $x = L$:

```
"Part d - temperature measurement error"
x=L
errT=T-T_f
```

Figure 5 illustrates the temperature measurement error as a function of the thermal conductivity of the rod material. Figure 5 shows that the optimal thermal conductivity, corresponding to the minimum measurement error, is around 100 W/m-K. Below the optimal value, the self-heating error dominates as the local temperature rise at the tip of the rod is large. Above the optimal value, the conduction from the wall dominates. The inset figures show the temperature distribution for high and low thermal conductivity in order to illustrate these different behaviors.

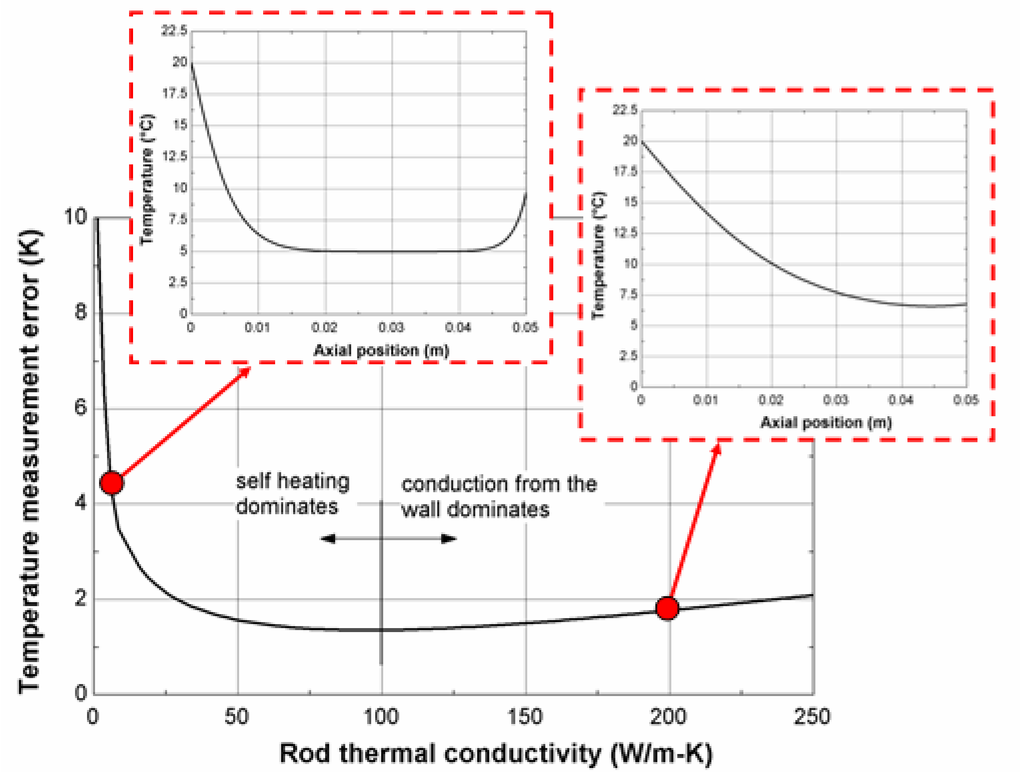


Figure 5: Temperature measurement error as a function of rod thermal conductivity. The inset figures show the temperature distribution at low conductivity and high conductivity.