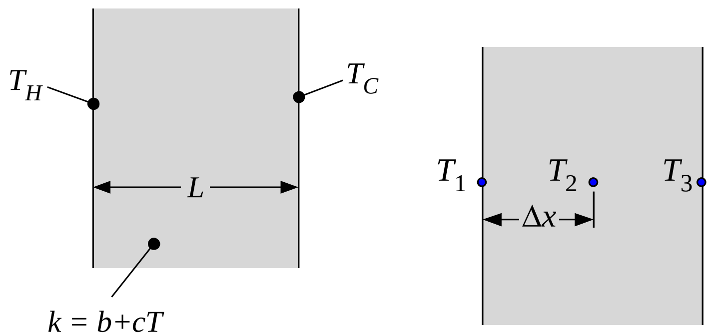
Problem 1.4-1: A 3-Node Numerical Solution

Figure P1.4.1(a) illustrates a plane wall with thickness L and cross-sectional area A that has a specified temperature T_H on the left side (at x = 0) and a specified temperature T_C on the right side (at x = L). There is no volumetric generation in the wall. However, the conductivity of the wall material is a function of temperature such that: $k = b + c\,T$ where a and b are constants. You would like to model the wall using a finite difference solution; a model with only 3 nodes is shown in Figure P1.4-1(b).



(a) **(b)**

Figure P1.4-1: (a) A plane wall and (b) a numerical model with 3 nodes.

The distance between adjacent nodes for the 3 node solution is: $\Delta x = L/2$.

a.) Write down the system of equations that could be solved in order to obtain the temperatures at the three nodes. Your equations should include the temperature of the nodes (T_1, T_2, and T_3) and the other parameters listed in the problem statement: T_H, T_C, Δx, A, b, and c.

The equations for T_1 and T_3 are easy, their temperatures are specified:

$$\boxed{T[1] = T_H} \tag{1}$$

$$\boxed{T[3] = T_C} \tag{2}$$

Figure 2 illustrates the control volume for the 2[nd] node.

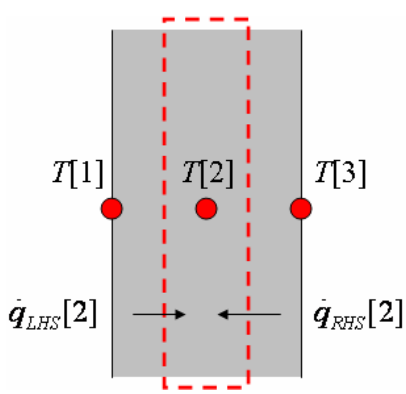


Figure 2: Control volume for node 2.

An energy balance for the control volume shown in Fig. 3 leads to:

$$\dot{q}_{RHS}[2] + \dot{q}_{LHS}[2] = 0 \tag{3}$$

The energy transfer rates must be approximated according to:

$$\dot{q}_{RHS}[2] = \frac{\left(T[3] - T[2]\right)}{\Delta x} A\left[b + c\left(\frac{T[3] + T[2]}{2}\right)\right] \tag{4}$$

and

$$\dot{q}_{LHS}[2] = \frac{\left(T[1] - T[2]\right)}{\Delta x} A\left[b + c\left(\frac{T[1] + T[2]}{2}\right)\right] \tag{5}$$

Notice that the temperature differences agree with the sign convention used in Figure 3 and that the conductivity is evaluated at the temperature of the interface. Substituting Eqs. (4) and (5) into Eq. (3) leads to:

$$\boxed{\frac{\left(T[3] - T[2]\right)}{\Delta x} A\left[b + c\left(\frac{T[3] + T[2]}{2}\right)\right] + \frac{\left(T[1] - T[2]\right)}{\Delta x} A\left[b + c\left(\frac{T[1] + T[2]}{2}\right)\right] = 0} \tag{6}$$

Equations (1), (2), and (6) together represent a system of three equations in the three unknown temperatures.