Problem 1.5-2 (1-12 in text): Mass Flow Meter (re-visited)

Reconsider the mass flow meter that was investigated in Problem 1.3-9 (1-9 in text). Assume that the conductivity of the material that is used to make the test section is not actually constant as was assumed in Problem 1.3-9 (1-9 in text) but rather depends on temperature according to:

$$k = 10\frac{\text{W}}{\text{m-K}} + 0.035\left[\frac{\text{W}}{\text{m-K}^2}\right]\left(T - 300\left[\text{K}\right]\right)$$

a.) Develop a numerical model of the mass flow meter using MATLAB. Plot the temperature as a function of radial position for the conditions shown in Figure P1.3-9 (P1-9 in text) with the temperature-dependent conductivity.

The inputs are entered in a MATLAB function that requires as an input the number of nodes (N):

```
function[r,T_C]=P1p5_2(N)

    r_out=0.0254;   %outer radius of test section (m)
    r_in=0.01905;   %inner radius of test section (m)
    h_bar_out=10;   %external convection coefficient (W/m^2-K)
    T_infinity=293.2; %air temperature (K)
    T_f=291.2;      %fluid temperature (K)
    gv=1e7;         %rate of generation (W/m^3)
    m_dot=0.75;     %mass flow rate (kg/s)
    th_ins=0.00635; %thickness of the insulation (m)
    k_ins=1.5;      %insulation conductivity (W/m-K)
    L=0.0762;       %length of the test section (m)
    C=2500;         %constant for convection relationship
```

The convection coefficient on the internal surface is computed:

```
    h_bar_in=C*m_dot^0.8;   %internal convection coefficient
```

A function is defined that returns the conductivity of the material:

```
function[k]=k_t(T)
    %conductivity of the material
    %
    %Inputs:
    % T: temperature (K)
    %
    %Outputs:
    % k: conductivity (W/m-K)

    k=10+0.035*(T-300);
end
```

A uniform distribution of nodes is used, the radial location of each node (r_i) is:

$$r_i = r_{in} + \frac{(i-1)}{(N-1)}\left(r_{out} - r_{in}\right) \quad \text{for } \; i = 1..N \tag{1}$$

where N is the number of nodes. The radial distance between adjacent nodes (Δr) is:

$$\Delta r = \frac{\left(r_{out} - r_{in}\right)}{(N-1)} \tag{2}$$

```
DELTAr=(r_out-r_in)/(N-1);      %distance between adjacent nodes (m)
for i=1:N
    r(i)=r_in+(r_out-r_in)*(i-1)/(N-1); %position of each node (m)
end
```

The system of equations is placed in matrix format.

$$\underline{\underline{A}}\,\underline{X} = \underline{b} \tag{3}$$

The most logical technique for ordering the unknown temperatures in the vector $\underline{X}$ is:

$$X = \begin{bmatrix} X_1 = T_1 \\ X_2 = T_2 \\ \dots \\ X_N = T_N \end{bmatrix} \tag{4}$$

Equation (4) shows that the unknown temperature at node i (i.e., T_i) corresponds to element i of vector $\underline{X}$ (i.e., X_i). The most logical technique for placing the equations into the $\underline{\underline{A}}$ matrix is:

$$\underline{\underline{A}} = \begin{bmatrix} \text{row 1 = control volume 1 equation} \\ \text{row 2 = control volume 2 equation} \\ \dots \\ \text{row } N \text{ = control volume } N \text{ equation} \end{bmatrix} \tag{5}$$

In Eq. (5), the equation for control volume i is placed into row i.

An energy balance is carried out on a control volume surrounding each node. For node 1, placed at the inner surface (Figure P1.5-2-1):

$$\dot{q}_{conv,in} + \dot{q}_{outer} + \dot{g} = 0 \tag{6}$$

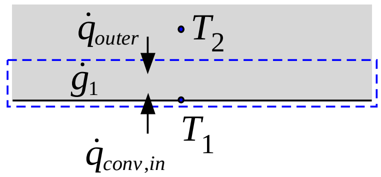

Figure P1.5-2-1: Control volume around node 1.

The rate equation for convection is:

$$\dot{q}_{conv,in} = \bar{h}_{in}\, 2\,\pi\, r_{in}\, L\left(T_f - T_1\right) \tag{7}$$

The rate equation for conduction is:

$$\dot{q}_{outer} = k_{T=(T_1+T_2)/2}\, 2\,\pi\left(r_{in} + \frac{\Delta r}{2}\right) L \frac{\left(T_2 - T_1\right)}{\Delta r} \tag{8}$$

The rate equation for generation is:

$$\dot{g} = 2\,\pi\, r_{in} \frac{\Delta r}{2} L\, \dot{g}''' \tag{9}$$

Substituting Eqs. (7) through (9) into Eq. (6) leads to:

$$\bar{h}_{in}\, 2\,\pi\, r_{in}\, L\left(T_f - T_1\right) + k_{T=(T_1+T_2)/2}\, 2\,\pi\left(r_{in} + \frac{\Delta r}{2}\right) L \frac{\left(T_2 - T_1\right)}{\Delta r} + \pi\, r_{in}\, \Delta r\, L\, \dot{g}''' = 0 \tag{10}$$

Equation (10) is rearranged to identify the coefficients that multiply each unknown temperature:

$$\begin{gathered} T_1\left[-\bar{h}_{in}\, 2\,\pi\, r_{in}\, L - k_{T=(T_1+T_2)/2}\, 2\,\pi\left(r_{in} + \frac{\Delta r}{2}\right)\frac{L}{\Delta r}\right] + T_2\left[k_{T=(T_1+T_2)/2}\, 2\,\pi\left(r_{in} + \frac{\Delta r}{2}\right)\frac{L}{\Delta r}\right] = \\ -\pi\, r_{in}\, \Delta r\, L\, \dot{g}''' - \bar{h}_{in}\, 2\,\pi\, r_{in}\, L\, T_f \end{gathered} \tag{11}$$

An energy balance on an internal node is shown in Figure P1.5-2-2:

$$\dot{q}_{inner} + \dot{q}_{outer} + \dot{g} = 0 \tag{12}$$

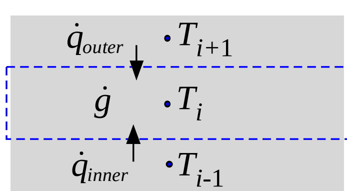

Figure P1.5-2-2: Control volume around internal node *i*.

The rate equations for conduction are:

$$\dot{q}_{outer} = k_{T=(T_i+T_{i+1})/2}\,2\,\pi\left(r_{in}+\frac{\Delta r}{2}\right)L\,\frac{\left(T_{i+1}-T_i\right)}{\Delta r} \tag{13}$$

$$\dot{q}_{inner} = k_{T=(T_i+T_{i-1})/2}\,2\,\pi\left(r_{in}-\frac{\Delta r}{2}\right)L\,\frac{\left(T_{i-1}-T_i\right)}{\Delta r} \tag{14}$$

The rate equation for generation is:

$$\dot{g} = 2\,\pi\,r_i\,\Delta r\,L\,\dot{g}''' \tag{15}$$

Substituting Eqs. (13) through (15) into Eq. (12) for all of the internal nodes leads to:

$$\begin{gathered} k_{T=(T_i+T_{i-1})/2}\,2\,\pi\left(r_{in}-\frac{\Delta r}{2}\right)L\,\frac{\left(T_{i-1}-T_i\right)}{\Delta r}+k_{T=(T_i+T_{i+1})/2}\,2\,\pi\left(r_{in}+\frac{\Delta r}{2}\right)L\,\frac{\left(T_{i+1}-T_i\right)}{\Delta r} \\ +2\,\pi\,r_i\,\Delta r\,L\,\dot{g}'''=0 \quad \text{for } i=2..\left(N-1\right) \end{gathered} \tag{16}$$

Equation (16) is rearranged to identify the coefficients that multiply each unknown temperature:

$$\begin{gathered} T_i\left[-k_{T=(T_i+T_{i-1})/2}\,2\,\pi\left(r_{in}-\frac{\Delta r}{2}\right)\frac{L}{\Delta r}-k_{T=(T_i+T_{i+1})/2}\,2\,\pi\left(r_{in}+\frac{\Delta r}{2}\right)\frac{L}{\Delta r}\right] \\ +T_{i-1}\left[-k_{T=(T_i+T_{i-1})/2}\,2\,\pi\left(r_{in}-\frac{\Delta r}{2}\right)\frac{L}{\Delta r}\right]+T_{i+1}\left[-k_{T=(T_i+T_{i+1})/2}\,2\,\pi\left(r_{in}+\frac{\Delta r}{2}\right)\frac{L}{\Delta r}\right] \\ =-2\,\pi\,r_i\,\Delta r\,L\,\dot{g}''' \quad \text{for } i=2..\left(N-1\right) \end{gathered} \tag{17}$$

An energy balance on node *N* placed on the outer surface is shown in Figure P1.5-2-3:

$$\dot{q}_{inner}+\dot{q}_{air}+\dot{g}=0 \tag{18}$$

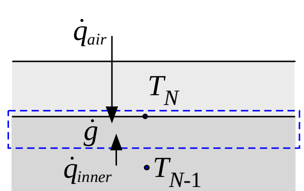

Figure P1.5-2-3: Control volume around internal node *N*.

The rate equation for the heat transfer with the air is:

$$\dot{q}_{air}=\frac{\left(T_\infty-T_N\right)}{\left(R_{ins}+R_{conv,out}\right)} \tag{19}$$

where

$$R_{ins} = \frac{\ln\left[\frac{\left(r_{out} + th_{ins}\right)}{r_{out}}\right]}{2\,\pi\,L\,k_{ins}} \tag{20}$$

$$R_{conv,out} = \frac{1}{2\,\pi\left(r_{out} + th_{ins}\right)L\,\overline{h}_{out}} \tag{21}$$

```
R_ins=log((r_out+th_ins)/r_out)/(2*pi*L*k_ins);
   %resistance to conduction through insulation
R_conv_out=1/(2*pi*(r_out+th_ins)*L*h_bar_out);
   %resistance to convection from the outside surface of the insulation
```

The rate equation for conduction is:

$$\dot{q}_{inner} = k_{T=\left(T_N+T_{N-1}\right)/2}\,2\,\pi\left(r_{out} - \frac{\Delta r}{2}\right)L\,\frac{\left(T_{N-1} - T_N\right)}{\Delta r} \tag{22}$$

The rate equation for generation is:

$$\dot{g} = 2\,\pi\,r_{out}\,\frac{\Delta r}{2}\,L\,\dot{g}''' \tag{23}$$

Substituting Eqs. (19), (22), and (23) into Eq. (18) leads to:

$$k_{T=\left(T_N+T_{N-1}\right)/2}\,2\,\pi\left(r_{out} - \frac{\Delta r}{2}\right)L\,\frac{\left(T_{N-1} - T_N\right)}{\Delta r} + \frac{\left(T_\infty - T_N\right)}{\left(R_{ins} + R_{conv,out}\right)} + 2\,\pi\,r_{out}\,\frac{\Delta r}{2}\,L\,\dot{g}''' = 0 \tag{24}$$

Equation (24) is rearranged to identify the coefficients that multiply each unknown temperature:

$$\begin{aligned} &T_N\left[-k_{T=\left(T_N+T_{N-1}\right)/2}\,2\,\pi\left(r_{out} - \frac{\Delta r}{2}\right)\frac{L}{\Delta r} - \frac{1}{\left(R_{ins} + R_{conv,out}\right)}\right] \\ &+T_{N-1}\left[-k_{T=\left(T_N+T_{N-1}\right)/2}\,2\,\pi\left(r_{out} - \frac{\Delta r}{2}\right)\frac{L}{\Delta r}\right] \\ &= -\pi\,r_{out}\,\Delta r\,L\,\dot{g}''' - \frac{T_\infty}{\left(R_{ins} + R_{conv,out}\right)} \end{aligned} \tag{25}$$

Equations (11), (17), and (25) are N equations in the N unknown temperatures. Because they are non-linear, they must be linearized and a successive substitution method used. A guess temperature distribution ($\hat{T}_i$) is assumed:

```
%initial guess for temperature distribution
for i=1:N
    Tg(i,1)=T_f;
end
```

The matrix $\underline{\underline{A}}$ is defined as a sparse matrix with at most $3N$ nonzero entries:

```
%initialize A and b
A=spalloc(N,N,3*N);
b=zeros(N,1);
```

The solution is placed within a while loop that terminates when the error between the solution and the guess is less than some tolerance:

```
err=999;        %initial value for error (K), must be larger than tol
tol=0.01;       %tolerance for convergence (K)
while(err>tol)
```

The equation for node 1, Eq. (11), is linearized by using the guess temperature distribution to compute the conductivity:

$$T_1 \underbrace{\left[-\bar{h}_{in}\, 2\,\pi\, r_{in}\, L - k_{T=\left(\hat{T}_1+\hat{T}_2\right)/2}\, 2\,\pi \left(r_{in} + \frac{\Delta r}{2} \right) \frac{L}{\Delta r} \right]}_{A_{1,1}} + T_2 \underbrace{\left[k_{T=\left(\hat{T}_1+\hat{T}_2\right)/2}\, 2\,\pi \left(r_{in} + \frac{\Delta r}{2} \right) \frac{L}{\Delta r} \right]}_{A_{1,2}} = \underbrace{-\pi\, r_{in}\, \Delta r\, L\, \dot{g}''' - \bar{h}_{in}\, 2\,\pi\, r_{in}\, L\, T_f}_{b_1} \tag{26}$$

```
        A(1,1)=-h_bar_in*2*pi*r_in*L-...
            k_t((Tg(1)+Tg(2))/2)*2*pi*(r_in+DELTAr/2)*L/DELTAr;
        A(1,2)=k_t((Tg(1)+Tg(2))/2)*2*pi*(r_in+DELTAr/2)*L/DELTAr;
        b(1)=-pi*r_in*DELTAr*L*gv-h_bar_in*2*pi*r_in*L*T_f;
```

The equations for the internal nodes, Eq. (17), is also linearized:

$$
\begin{aligned}
&T_i \underbrace{\left[-k_{T=(T_i+T_{i-1})/2}\, 2\pi \left(r_{in}-\frac{\Delta r}{2}\right)\frac{L}{\Delta r} - k_{T=(T_i+T_{i+1})/2}\, 2\pi \left(r_{in}+\frac{\Delta r}{2}\right)\frac{L}{\Delta r}\right]}_{A_{i,i}} \\
&+T_{i-1}\underbrace{\left[-k_{T=(T_i+T_{i-1})/2}\, 2\pi \left(r_{in}-\frac{\Delta r}{2}\right)\frac{L}{\Delta r}\right]}_{A_{i,i-1}} + T_{i+1}\underbrace{\left[-k_{T=(T_i+T_{i+1})/2}\, 2\pi \left(r_{in}+\frac{\Delta r}{2}\right)\frac{L}{\Delta r}\right]}_{A_{i,i+1}} \\
&= \underbrace{-2\pi\, r_i\, \Delta r\, L\, \dot{g}'''}_{b_i} \quad \text{for } i = 2..(N-1)
\end{aligned}
\tag{27}
$$

```
        for i=2:(N-1)
            A(i,i)=-k_t((Tg(i)+Tg(i-1))/2)*2*pi*(r(i)-DELTAr/2)*L/DELTAr...
                  -k_t((Tg(i)+Tg(i+1))/2)*2*pi*(r(i)+DELTAr/2)*L/DELTAr;
            A(i,i-1)=k_t((Tg(i)+Tg(i-1))/2)*2*pi*(r(i)-DELTAr/2)*L/DELTAr;
            A(i,i+1)=k_t((Tg(i)+Tg(i+1))/2)*2*pi*(r(i)+DELTAr/2)*L/DELTAr;
            b(i)=-2*pi*r(i)*DELTAr*L*gv;
        end
```

The equation for node N, Eq. (25), is linearized:

$$
\begin{aligned}
&T_N \underbrace{\left[-k_{T=(T_N+T_{N-1})/2}\, 2\pi \left(r_{out}-\frac{\Delta r}{2}\right)\frac{L}{\Delta r} - \frac{1}{\left(R_{ins}+R_{conv,out}\right)}\right]}_{A_{N,N}} \\
&+T_{N-1}\underbrace{\left[-k_{T=(T_N+T_{N-1})/2}\, 2\pi \left(r_{out}-\frac{\Delta r}{2}\right)\frac{L}{\Delta r}\right]}_{A_{N,N-1}} \\
&= \underbrace{-\pi\, r_{out}\, \Delta r\, L\, \dot{g}''' - \frac{T_\infty}{\left(R_{ins}+R_{conv,out}\right)}}_{b_N}
\end{aligned}
\tag{28}
$$

```
        A(N,N)=-k_t((Tg(N)+Tg(N-1))/2)*2*pi*(r_in-DELTAr/2)*L/DELTAr-...
                  1/(R_ins+R_conv_out);
        A(N,N-1)=k_t((Tg(N)+Tg(N-1))/2)*2*pi*(r_in-DELTAr/2)*L/DELTAr;
        b(N)=-pi*r_out*DELTAr*L*gv-T_infinity/(R_ins+R_conv_out);
```

The solution is obtained:

```
        X=A\b;
        T=X;
```

and used to compute the error between the assumed and calculated solutions is obtained:

$$
err = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(T_i - \hat{T}_i\right)^2} \tag{29}
$$

```
        err=sqrt(sum((T-Tg).^2)/N) %compute rms error
```

The calculated solution becomes the guess value for the next iteration:

```
        Tg=T;   %reset guess values used to setup A and b
    end
```

The solution is converted to degrees Celsius:

```
    T_C=T-273.2;    %convert to C
end
```

The solution is illustrated in Figure P1.5-2-4.

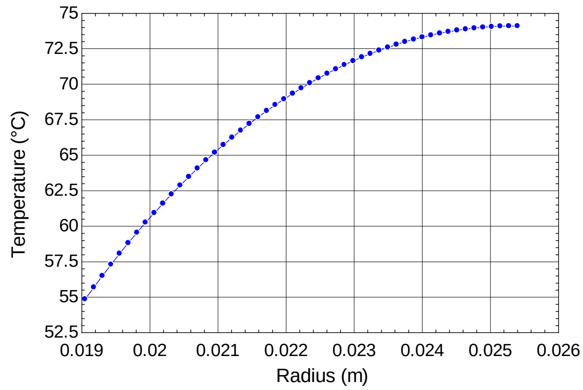

Figure P1.5-2-4: Temperature as a function of radius.

b.) Verify that your numerical solution limits to the analytical solution from Problem 1.3-9 (1-9 in text) in the limit that the conductivity is constant.

The conductivity function is modified temporarily so that it returns a constant value:

```
function[k]=k_t(T)
    %conductivity of the material
    %
    %Inputs:
    % T: temperature (K)
    %
    %Outputs:
    % k: conductivity (W/m-K)

    k=10;%+0.035*(T-300);
end
```

Figure P1.5-2-5 illustrates the temperature distribution predicted by the numerical and analytical solutions in the limit that *k* is constant.

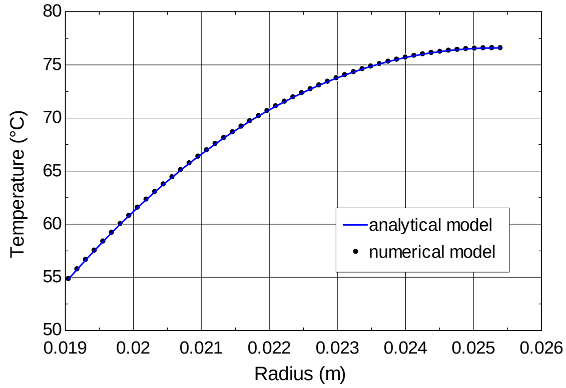

Figure P1.5-2-5: Temperature as a function of radius predicted by the analytical and numerical models in the limit that *k* is constant.