PROBLEM 1.5-1 (1-11 in text): Hay Temperature (revisited)

Reconsider Problem P1.3-8, but obtain a solution numerically using MATLAB. The description of the hay bale is provided in Problem P1.3-8. Prepare a model that can consider the effect of temperature on the volumetric generation. Increasing temperature tends to increase the rate of reaction and therefore increase the rate of generation of thermal energy; the volumetric rate of generation can be approximated by: $\dot{g}''' = a + bT$ where a = -1 W/m^3 and b = 0.01 W/m^3-K. Note that at T = 300 K, the generation is 2 W/m^3 but that the generation increases with temperature.

a.) Prepare a numerical model of the hay bale using EES. Plot the temperature as a function of position within the hay bale.

The input information is entered in EES and a function is used to define the volumetric generation:

```
$UnitSystem SI MASS RAD PA  K J
$TABSTOPS   0.2 0.4 0.6 0.8 3.5 in

function gen(T)
    "volumetric heat generation in wall"
    "Input - T, temperature [K]"
    "Output - gen, volumetric rate of heat generation [W/m^3]"

    a=-1 [W/m^3]                                    "coefficients in generation function"
    b=0.01 [W/m^3-K]
    gen=a+b*T

end

"Inputs"
L = 1 [m]                                           "per unit length of bale"
R_bale= 5 [ft]*convert(ft,m)                        "bale radius"
t_p=0.045 [inch]*convert(inch,m)                    "plastic thickness"
k_p=0.15 [W/m-K]                                    "plastic conductivity"
T_infinity=converttemp(C,K,20)                      "ambient temperature"
h=10 [W/m^2-K]                                      "heat transfer coefficient"
k=0.04 [W/m-K]                                      "hay conductivity"
```

Nodes are distributed uniformly throughout the computational domain (which consists only of the hay, not the plastic), the location of each node (r_i) is:

$$r_i = \frac{(i-1)}{(N-1)} R_{bale} \quad i = 1..N \tag{1}$$

where N is the number of nodes used for the simulation. The distance between adjacent nodes (Δr) is:

$$\Delta r = \frac{R_{bale}}{(N-1)} \tag{2}$$

```
"Setup grid"
N=50 [-]                                          "number of nodes"
duplicate i=1,N
    r[i]=(i-1)*R_bale/(N-1)                       "position of each node"
end
Deltar=R_bale/(N-1)                               "distance between adjacent nodes"
```

A control volume is defined around each node and an energy balance is written for each control volume. The control volume for an arbitrary, internal node (i.e., a node that is not placed on the edge or at the center of the hay) experiences conduction heat transfer passing through the internal surface ($\dot{q}_{LHS}$), conduction heat transfer passing through the external surface ($\dot{q}_{RHS}$), and heat generation within the control volume ($\dot{g}$). A steady-state energy balance for the control volume is shown in Figure 1:

$$\dot{q}_{LHS} + \dot{q}_{RHS} + \dot{g} = 0 \tag{3}$$

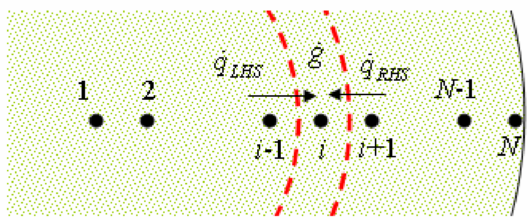

Figure 1: Internal node energy balance

Each of the terms in the energy balance in Eq. (3) must be modeled using a rate equation. Conduction through the inner surface is driven by the temperature difference between nodes *i*-1 and *i* through the material that lies between these nodes.

$$\dot{q}_{LHS} = \frac{k\,2\,\pi\left(r_i - \frac{\Delta r}{2}\right)L}{\Delta r}\left(T_{i-1} - T_i\right) \tag{4}$$

where *L* is the length of the bale (assumed to be 1 m, corresponding to doing the problem on a per unit length of bale basis). The conduction into the outer surface of the control volume is:

$$\dot{q}_{RHS} = \frac{k\,2\,\pi\left(r_i + \frac{\Delta r}{2}\right)L}{\Delta r}\left(T_{i+1} - T_i\right) \tag{5}$$

The generation is the product of the volume of the control volume and the volumetric generation rate, which is approximately:

$$\dot{g} = \dot{g}'''\left(T_i\right)\pi\,L\left[\left(r_i + \frac{\Delta r}{2}\right)^2 - \left(r_i - \frac{\Delta r}{2}\right)^2\right] \tag{6}$$

where $\dot{g}'''(T_i)$ is the volumetric rate of generation evaluated at the nodal temperature T_i. Substituting Eqs. (4) through (6) into Eq. (3) leads to:

$$\frac{k\,2\pi\left(r_i-\frac{\Delta r}{2}\right)L}{\Delta r}\left(T_{i-1}-T_i\right)+\frac{k\,2\pi\left(r_i+\frac{\Delta r}{2}\right)L}{\Delta r}\left(T_{i+1}-T_i\right)+\dot{g}'''\left(T_i\right)\pi\,L\left[\left(r_i+\frac{\Delta r}{2}\right)^2-\left(r_i-\frac{\Delta r}{2}\right)^2\right]=0 \quad (7)$$

$$\text{for } i=2...(N-1)$$

Figure 2 illustrates the control volume associated with the node that is placed on the outer surface of the hay (i.e., node N).

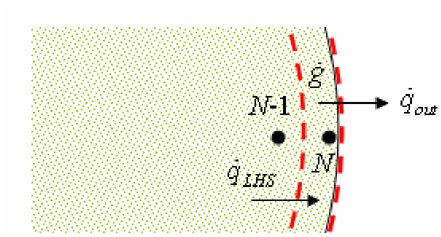

Figure 2: Control volume for node *N* located on hay outer surface

The energy balance for the control volume associated with node N is:

$$\dot{q}_{LHS}+\dot{g}=\dot{q}_{out} \quad (8)$$

where the conduction term is:

$$\dot{q}_{LHS}=\frac{k\,2\pi\left(r_N-\frac{\Delta r}{2}\right)L}{\Delta r}\left(T_{N-1}-T_N\right), \quad (9)$$

the generation term is:

$$\dot{g}=\dot{g}'''\left(T_N\right)\pi\,L\left[r_N^2-\left(r_N-\frac{\Delta r}{2}\right)^2\right], \quad (10)$$

Note that the volume in Eq. (10) is calculated differently from the volume in Eq. (6) because the control volume is half as wide radially. The heat transfer to the external air is:

$$\dot{q}_{out}=\frac{\left(T_N-T_\infty\right)}{R_p+R_{conv}} \quad (11)$$

where

$$R_p = \frac{th_p}{k_p\, 2\,\pi\, R_{bale}\, L} \tag{12}$$

and

$$R_{conv,out} = \frac{1}{h\, 2\,\pi\, R_{bale}\, L} \tag{13}$$

Substituting Eqs. (9) through (11) into Eq. (8) leads to:

$$\frac{k\, 2\,\pi \left(r_N - \dfrac{\Delta r}{2} \right) L}{\Delta r} \left(T_{N-1} - T_N \right) + \dot{g}''' \left(T_N \right) \pi\, L \left[r_N^2 - \left(r_N - \frac{\Delta r}{2} \right)^2 \right] = \frac{\left(T_N - T_\infty \right)}{R_p + R_{conv}} \tag{14}$$

A similar procedure applied to the control volume associated with node 1 leads to:

$$\frac{k\, 2\,\pi \left(r_1 + \dfrac{\Delta r}{2} \right) L}{\Delta r} \left(T_2 - T_1 \right) + \dot{g}''' \left(T_1 \right) \pi\, L \left(r_1 + \frac{\Delta r}{2} \right)^2 = 0 \tag{15}$$

Equations (7), (14), and (15) represent *N* equations in an equal number of unknowns; the solution of these equations provides the numerical solution.

```
"Internal control volumes"
duplicate i=2,(N-1)
     k*2*pi*(r[i]-Deltar/2)*L*(T[i-1]-T[i])/Deltar+k*2*pi*(r[i]+Deltar/2)*L*(T[i+1]-
T[i])/Deltar+gen(T[i])*pi*L*((r[i]+Deltar/2)^2-(r[i]-Deltar/2)^2)=0
end

"node N"
R_p=t_p/(k_p*2*pi*R_bale*L)                                   "conduction resistance of plastic"
R_conv=1/(h*2*pi*R_bale*L)                                    "convection resistance"
k*2*pi*(r[N]-Deltar/2)*L*(T[N-1]-T[N])/Deltar+gen(T[N])*pi*L*(r[N]^2-(r[N]-Deltar/2)^2)=(T[N]-
T_infinity)/(R_p+R_conv)

"node 1"
k*2*pi*(r[1]+Deltar/2)*L*(T[2]-T[1])/Deltar+gen(T[1])*pi*L*(r[1]+Deltar/2)^2=0
```

If the EES program is solved then the temperature distribution will be placed in the Arrays window. The temperature as a function of position is shown in Figure 3.

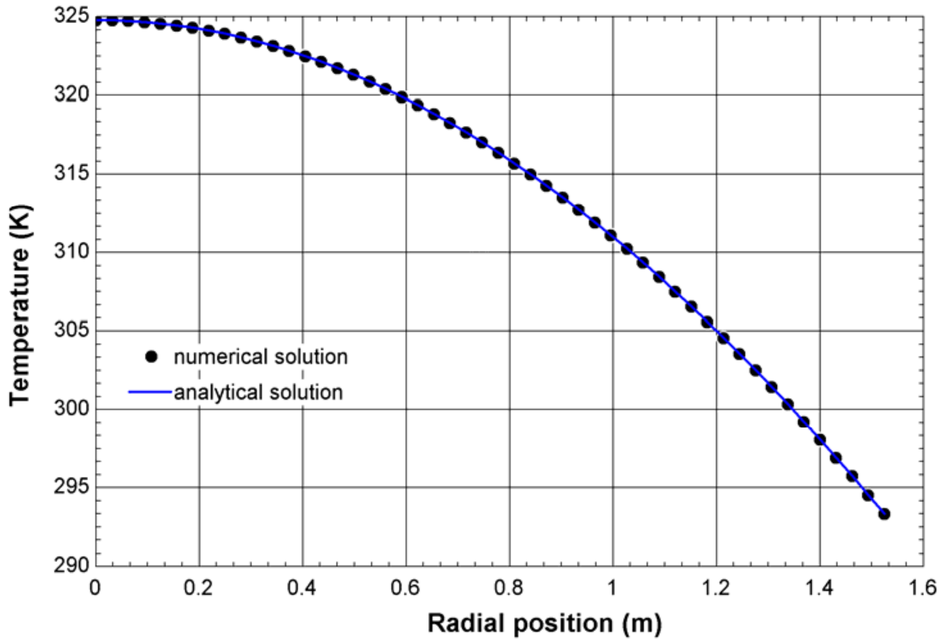

Figure 3: Temperature as a function of position within the bale

b.) Show that your model has numerically converged; that is, show some aspect of your solution as a function of the number of nodes in your solution and discuss an appropriate number of nodes to use.

The maximum temperature (i.e., the temperature at the center of the bale) is shown in Figure 4 as a function of the number of nodes. The model is numerically converged after approximately N = 20.

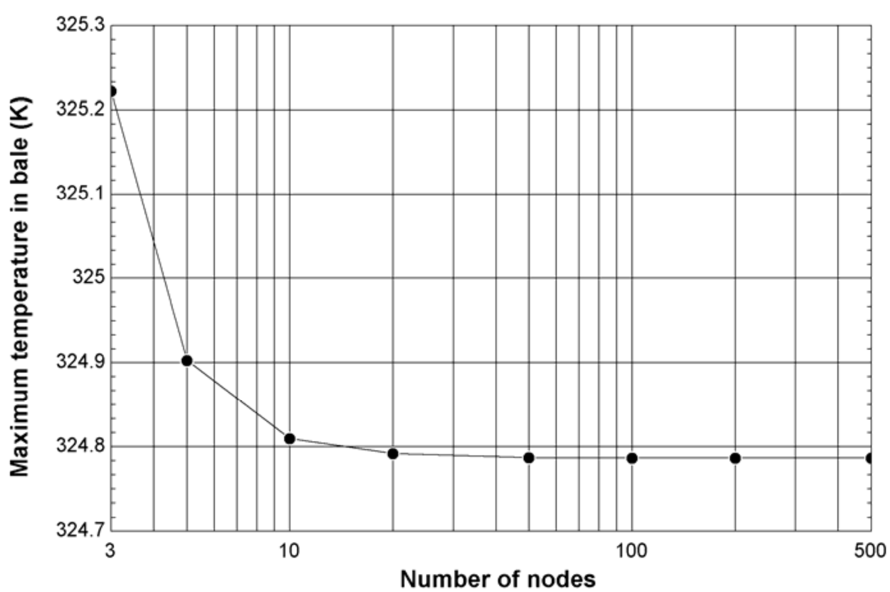

Figure 4: Predicted maximum temperature as a function of the number of nodes

c.) Verify your numerical model by comparing your answer to an analytical solution in some, appropriate limit. The result of this step should be a plot which shows the temperature as a function of radius predicted by both your numerical solution and the analytical solution and demonstrates that they agree.

The analytical solution derived in the problem 1.3-8 is used to compute the temperature at each nodal position:

```
"Analytical solution from Problem 1.3-8"
a=-1 [W/m^3]                                        "coefficients for volumetric generation function"
b=0.01 [W/m^3-K]
dTdr_Rbale = -BesselJ(1,(b/k)^(1/2)*R_bale)*(b/k)^(1/2)*C_2  "symbolic expressions from Maple"
T_Rbale = BesselJ(0,(b/k)^(1/2)*R_bale)*C_2-1/b*a
-k*2*pi*R_bale*L*dTdr_Rbale=(T_Rbale-T_infinity)/(R_p+R_conv)"interface energy balance"
duplicate i=1,N
    T_an[i]=BesselJ(0,sqrt(b/k)*r[i])*C_2-a/b
end
```

Figure 3 illustrates the analytical solution overlaid on the numerical solution and demonstrates agreement.

d.) Prepare a numerical model of the hay bale using MATLAB. Plot the temperature as a function of position within the hay bale.

A new m-file is opened and formatted as a function with a single input (the number of nodes) and two outputs (vectors containing the radial position and temperature at each node).

```
function[r,T]=P1p5_1(N)

L = 1;              %per unit length of bale (m)
R_bale= 1.524;      %bale radius (m)
t_p=0.00114;        %plastic thickness (m)
k_p=0.15;           %plastic conductivity (W/m-K)
T_infinity=293.2;   %ambient temperature (K)
h=10;               %heat transfer coefficient (W/m^2-K)
k=0.04;             %hay conductivity (W/m-K)

end
```

A function is defined that returns the volumetric rate of generation as a function of temperature; the function is placed at the bottom of the same m-file so that it is accessible locally to P1p5_1.

```
function[gv]=gen(T)

%coefficients of function
a=-1;               %(W/m^3)
b=0.01;             %(W/m^3-K)
gv=a+b*T;

end
```

The radial position of each node is stored in the vector r.

```
Deltar=R_bale/(N-1);    %distance between adjacent nodes (m)
for i=1:N
    r(i,1)=Deltar*(i-1);    %radial location of each node (m)
end
```

The problem is nonlinear because the generation rate depends on temperature; therefore, the method of successive substitution is used. An initial guess for the temperature distribution is stored in the vector T_g:

```
%initial guess for temperature distribution
for i=1:N
    T_g(i,1)=T_infinity;
end
```

The guess values for temperature are used to setup the matrix A and vector b which contain the matrix formulation of the equations. The energy balance for node 1 is placed in row 1 of A.

$$\frac{k\,2\,\pi\left(r_1+\frac{\Delta r}{2}\right)L}{\Delta r}\left(T_2-T_1\right)+\dot{g}'''\left(T_1^*\right)\pi\,L\left(r_1+\frac{\Delta r}{2}\right)^2=0 \tag{16}$$

where T_1^* is the guess value of the temperature or

$$T_1\underbrace{\left[-\frac{k\,2\,\pi\left(r_1+\frac{\Delta r}{2}\right)L}{\Delta r}\right]}_{A(1,1)}+T_2\underbrace{\left[\frac{k\,2\,\pi\left(r_1+\frac{\Delta r}{2}\right)L}{\Delta r}\right]}_{A(1,2)}=\underbrace{-\dot{g}'''\left(T_1^*\right)\pi\,L\left(r_1+\frac{\Delta r}{2}\right)^2}_{b(1)} \tag{17}$$

The energy balances for the internal nodes are:

$$\frac{k\,2\,\pi\left(r_i-\frac{\Delta r}{2}\right)L}{\Delta r}\left(T_{i-1}-T_i\right)+\frac{k\,2\,\pi\left(r_i+\frac{\Delta r}{2}\right)L}{\Delta r}\left(T_{i+1}-T_i\right)+\dot{g}'''\left(T_i^*\right)\pi\,L\left[\left(r_i+\frac{\Delta r}{2}\right)^2-\left(r_i-\frac{\Delta r}{2}\right)^2\right]=0 \tag{18}$$

$$\text{for } i=2...(N-1)$$

or

$$T_i \underbrace{\left[-\frac{k\,2\pi\left(r_i - \frac{\Delta r}{2}\right)L}{\Delta r} - \frac{k\,2\pi\left(r_i + \frac{\Delta r}{2}\right)L}{\Delta r} \right]}_{A(i,i)} + T_{i-1} \underbrace{\left[\frac{k\,2\pi\left(r_i - \frac{\Delta r}{2}\right)L}{\Delta r} \right]}_{A(i,i-1)} +$$

$$T_{i+1} \underbrace{\left[\frac{k\,2\pi\left(r_i + \frac{\Delta r}{2}\right)L}{\Delta r} \right]}_{A(i,i+1)} = \underbrace{-\dot{g}'''\left(T_i^*\right)\pi\,L\left[\left(r_i + \frac{\Delta r}{2}\right)^2 - \left(r_i - \frac{\Delta r}{2}\right)^2\right]}_{b(i)} \tag{19}$$

$$\text{for } i = 2\ ..\ \left(N-1\right)$$

The energy balance for node *N* is:

$$\frac{k\,2\pi\left(r_N - \frac{\Delta r}{2}\right)L}{\Delta r}\left(T_{N-1} - T_N\right) + \dot{g}'''\left(T_N^*\right)\pi\,L\left[r_N^2 - \left(r_N - \frac{\Delta r}{2}\right)^2\right] = \frac{\left(T_N - T_\infty\right)}{R_p + R_{conv}} \tag{20}$$

or

$$T_N \underbrace{\left[-\frac{k\,2\pi\left(r_N - \frac{\Delta r}{2}\right)L}{\Delta r} - \frac{1}{R_p + R_{conv}} \right]}_{A(N,N)} + T_{N-1} \underbrace{\left[\frac{k\,2\pi\left(r_N - \frac{\Delta r}{2}\right)L}{\Delta r} \right]}_{A(N,N-1)} =$$

$$\underbrace{-\dot{g}'''\left(T_N^*\right)\pi\,L\left[r_N^2 - \left(r_N - \frac{\Delta r}{2}\right)^2\right] - \frac{T_\infty}{R_p + R_{conv}}}_{b(N)} \tag{21}$$

The matrices *A* and *b* are initialized and the resistances due to convection and conduction through the plastic are computed:

```
A=spalloc(N,N,3*N);
b=zeros(N,1);
R_p=t_p/(k_p*2*pi*R_bale*L);    %resistance through plastic
R_conv=1/(h*2*pi*R_bale*L);     %resistance due to convection
```

The matrices *A* and *b* are filled in according to Eqs. (17), (19), and (21):

```
    %Node 1
    A(1,1)=-k*2*pi*(r(1)+Deltar/2)*L/Deltar;
    A(1,2)=k*2*pi*(r(1)+Deltar/2)*L/Deltar;
    b(1)=-gen(T_g(i))*pi*L*(r(1)+Deltar/2)^2;
```

```
    %Nodes 2 to (N-1)
    for i=2:(N-1)
        A(i,i)=-k*2*pi*(r(i)-Deltar/2)*L/Deltar-
                  k*2*pi*(r(i)+Deltar/2)*L/Deltar;
        A(i,i-1)=k*2*pi*(r(i)-Deltar/2)*L/Deltar;
        A(i,i+1)=k*2*pi*(r(i)+Deltar/2)*L/Deltar;
        b(i)=-gen(T_g(i))*pi*L*((r(i)+Deltar/2)^2-(r(i)-Deltar/2)^2);
    end

    %Node N
    A(N,N)=-k*2*pi*(r(N)-Deltar/2)*L/Deltar-1/(R_p+R_conv);
    A(N,N-1)=k*2*pi*(r(N)-Deltar/2)*L/Deltar;
    b(N)=-gen(T_g(N))*pi*L*(r(N)^2-(r(N)-Deltar/2)^2)-
            T_infinity/(R_p+R_conv);
```

The temperature distribution is obtained according to:

```
    T=A\b;
```

The successive substitution process occurs within a while loop that is terminated when some convergence error, err, goes below a tolerance, tol. The tolerance is set and the error is initialized to a value that will ensure that the loop executes at least once. Once the solution is obtained, it is compared with the guess value to determine an error. The guess values are reset and, if the error is not sufficiently small then the process is repeated. The code is shown below; the new lines are shown in bold:

```
function[r,T]=P1p5_1(N)

L = 1;                  %per unit length of bale (m)
R_bale= 1.524;          %bale radius (m)
t_p=0.00114;            %plastic thickness (m)
k_p=0.15;               %plastic conductivity (W/m-K)
T_infinity=293.2;       %ambient temperature (K)
h=10;                   %heat transfer coefficient (W/m^2-K)
k=0.04;                 %hay conductivity (W/m-K)

Deltar=R_bale/(N-1);    %distance between adjacent nodes (m)
for i=1:N
    r(i,1)=Deltar*(i-1);    %radial location of each node (m)
end

%initial guess for temperature distribution
for i=1:N
    T_g(i,1)=T_infinity;
end

A=spalloc(N,N,3*N);
b=zeros(N,1);
R_p=t_p/(k_p*2*pi*R_bale*L);    %resistance through plastic
R_conv=1/(h*2*pi*R_bale*L);     %resistance due to convection

tol=0.1;                %tolerance for convergence (K)
```

```
err=2*tol;          %error initialization
while(err>tol)
    %Node 1
    A(1,1)=-k*2*pi*(r(1)+Deltar/2)*L/Deltar;
    A(1,2)=k*2*pi*(r(1)+Deltar/2)*L/Deltar;
    b(1)=-gen(T_g(i))*pi*L*(r(1)+Deltar/2)^2;

    %Nodes 2 to (N-1)
    for i=2:(N-1)
        A(i,i)=-k*2*pi*(r(i)-Deltar/2)*L/Deltar-
k*2*pi*(r(i)+Deltar/2)*L/Deltar;
        A(i,i-1)=k*2*pi*(r(i)-Deltar/2)*L/Deltar;
        A(i,i+1)=k*2*pi*(r(i)+Deltar/2)*L/Deltar;
        b(i)=-gen(T_g(i))*pi*L*((r(i)+Deltar/2)^2-(r(i)-Deltar/2)^2);
    end

    %Node N
    A(N,N)=-k*2*pi*(r(N)-Deltar/2)*L/Deltar-1/(R_p+R_conv);
    A(N,N-1)=k*2*pi*(r(N)-Deltar/2)*L/Deltar;
    b(N)=-gen(T_g(N))*pi*L*(r(N)^2-(r(N)-Deltar/2)^2)-
T_infinity/(R_p+R_conv);

    T=A\b;  %obtain temperature distribution
    err=sum(abs(T-T_g))/N   %calculate error
    T_g=T;
end

end

function[gv]=gen(T)

%coefficients of function
a=-1;               %(W/m^3)
b=0.01;             %(W/m^3-K)
gv=a+b*T;
end
```

The temperature as a function of radius is shown in Figure 5.

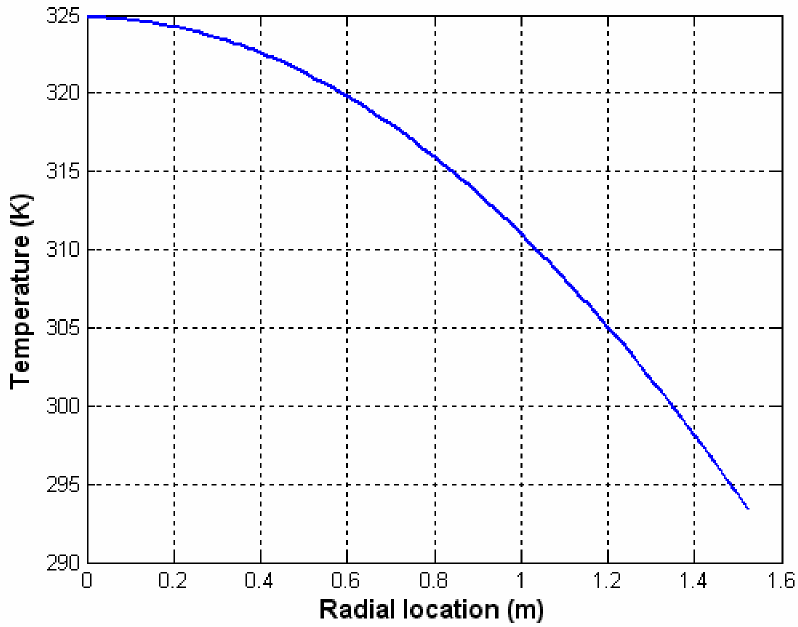

Figure 5: Predicted temperature as a function of radial position