Problem 1.4-7

Figure P1.4-7 illustrates a plane wall. The temperature distribution in the wall is 1-D and the problem is steady state.

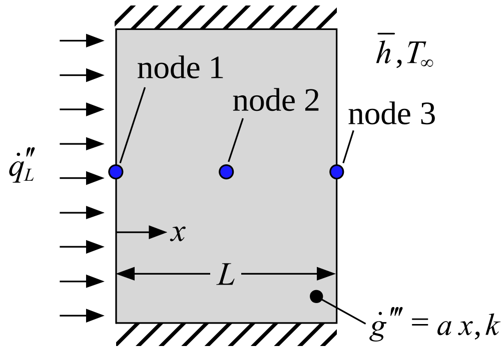

Figure P1.4-7: Three-node model of a plane wall.

There is generation of thermal energy in the wall. The generation per unit volume is not uniform but rather depends on position according to:

$$\dot{g}''' = a\,x \tag{1}$$

where a is a constant and x is position. The left side of the wall experiences a specified heat flux, $\dot{q}''_L$. The right side of the wall experiences convection with heat transfer coefficient $\bar{h}$ to fluid at temperature T_∞. The thickness of the wall is L and the conductivity of the wall material, k, is constant. You are going to develop a numerical model with 3 nodes, as shown in Figure P1.4-7. The nodes are distributed uniformly throughout the domain. Derive the three equations that must be solved in order to implement the numerical model. Do not solve these equations.

An energy balance on node 1 leads to:

$$\dot{q}''_L\,A_c + \dot{g}'''_{x=0}\,A_c\,\frac{\Delta x}{2} + \frac{k\,A_c}{\Delta x}\left(T_2 - T_1\right) = 0 \tag{2}$$

Substituting Eq. (1) into Eq. (2) and dividing through by A_c leads to:

$$\boxed{\dot{q}''_L + \frac{k}{\Delta x}\left(T_2 - T_1\right) = 0} \tag{3}$$

An energy balance on node 2 leads to:

$$\frac{k\,A_c}{\Delta x}\left(T_1 - T_2\right) + \dot{g}'''_{x=L/2}\,A_c\,\Delta x + \frac{k\,A_c}{\Delta x}\left(T_3 - T_2\right) = 0 \tag{4}$$

Substituting Eq. (1) into Eq. (4) and dividing through by A_c leads to:

$$\boxed{\frac{k}{\Delta x}\left(T_1 - T_2\right) + a\frac{L}{2}\Delta x + \frac{k}{\Delta x}\left(T_3 - T_2\right) = 0} \tag{5}$$

An energy balance on node 3 leads to:

$$\frac{k\,A_c}{\Delta x}\left(T_2 - T_3\right) + \dot{g}'''_{x=L}\,A_c\,\frac{\Delta x}{2} + \bar{h}\,A_c\left(T_\infty - T_3\right) = 0 \tag{6}$$

Substituting Eq. (1) into Eq. (6) and dividing through by A_c leads to:

$$\boxed{\frac{k}{\Delta x}\left(T_2 - T_3\right) + a\,L\,\frac{\Delta x}{2} + \bar{h}\left(T_\infty - T_3\right) = 0} \tag{7}$$

Equations (3), (5), and (7) can be solved to provide T_1, T_2, and T_3.