Problem 1.7-5

A flux meter is illustrated in Figure P1.7-5.

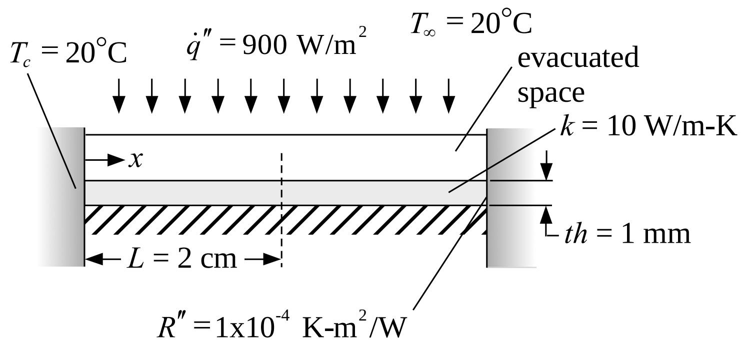

Figure P1.7-5: Flux meter.

A thin plate is clamped on either end to a casing that is maintained at T_c = 20ºC. There is a contact resistance between the plate and the casing, R'' = 1x10^{-4} K-m^2/W. The thickness of the plate is th = 1 mm and the half-width of the plate is L = 2 cm. The plate conductivity is k = 10 W/m-K. The back of the plate is insulated and the front of the plate is mounted within an evacuated enclosure in order to eliminate convection. You may model radiation loss from the plate surface using an effective "radiation" heat transfer coefficient, calculated according to: $\overline{h}_{rad} \approx 4\,\varepsilon\,\sigma\,\overline{T}^3$, where $\overline{T}$ is the average absolute temperature of the plate and the surroundings (take $\overline{T}$ = 300 K for this problem). The plate radiates to surroundings at T_∞ = 20ºC. The nominal flux on the plate is $\dot{q}''$ = 900 W/m^2. You may assume that the plate temperature distribution is 1-D in the x-direction. You may also assume that the emissivity of the plate, ε, is one and therefore all of the flux on the plate is absorbed. The flux meter operates by correlating the difference between the temperature at the center of the plate and the casing with the applied heat flux.

a.) Derive the governing differential equation that governs the temperature within the plate. Clearly show your steps.

An energy balance on a differential control volume (see Figure 2) leads to:

$$\dot{q}_x + \dot{q}''\,dx\,W = \dot{q}_{x+dx} + \overline{h}_{rad}\,dx\,W\left(T - T_\infty\right) \quad (1)$$

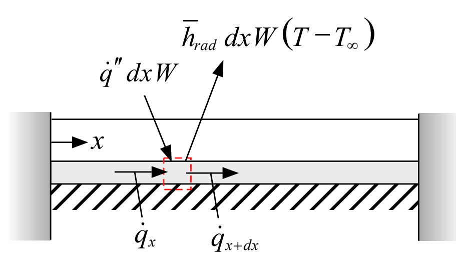

Figure 2: Differential energy balance.

Expanding Eq. (1) leads to:

$$\dot{q}_x + \dot{q}''\,dx\,W = \dot{q}_x + \frac{d\dot{q}}{dx}dx + \overline{h}_{rad}\,dx\,W\left(T - T_\infty\right) \tag{2}$$

Substituting Fourier's law into Eq. (2) leads to:

$$\dot{q}''\,dx\,W = \frac{d}{dx}\left(-k\,W\,th\frac{dT}{dx}\right)dx + \overline{h}_{rad}\,dx\,W\left(T - T_\infty\right) \tag{3}$$

or

$$\frac{d^2T}{dx^2} - \frac{\overline{h}_{rad}}{k\,th}T = -\frac{\dot{q}''}{k\,th} - \frac{\overline{h}_{rad}}{k\,th}T_\infty \tag{4}$$

b.) What are the boundary conditions for the differential equation?

An interface energy balance at $x = 0$ leads to:

$$\frac{\left(T_c - T_{x=0}\right)}{R''} = -k\left(\frac{dT}{dx}\right)_{x=0} \tag{5}$$

and an interface energy balance at $x = L$ leads to:

$$\left(\frac{dT}{dx}\right)_{x=L} = 0 \tag{6}$$

c.) Determine the solution to the differential equation from (a) subject to the boundary conditions from (b) without using Maple.

The solution to Eq. (4) is split into a homogeneous and particular solution:

$$T = T_h + T_p \tag{7}$$

Substituting Eq. (7) into Eq. (4) leads to:

$$\underbrace{\frac{d^2T_h}{dx^2} - \frac{\overline{h}_{rad}}{k\,th}T_h}_{=0 \text{ for homogeneous equation}} + \underbrace{\frac{d^2T_p}{dx^2} - \frac{\overline{h}_{rad}}{k\,th}T_p = -\frac{\dot{q}''}{k\,th} - \frac{\overline{h}_{rad}}{k\,th}T_\infty}_{\text{whatever is left must be the particular differential equation}} \tag{8}$$

The solution to the homogeneous differential equation:

$$\frac{d^2T_h}{dx^2} - \frac{\overline{h}_{rad}}{k\,th}T_h = 0 \tag{9}$$

is an exponential:

$$T_h = C \exp(m\,x) \tag{10}$$

Substituting Eq. (10) into Eq. (9) leads to:

$$C\,m^2 \exp(m\,x) - \frac{\overline{h}_{rad}}{k\,th} C \exp(m\,x) = 0 \tag{11}$$

which is solved by:

$$m^2 = \frac{\overline{h}_{rad}}{k\,th} \tag{12}$$

Therefore, there are two solutions corresponding to the two roots of Eq. (12):

$$T_h = C_1 \exp(m\,x) + C_2 \exp(-m\,x) \tag{13}$$

where C_1 and C_2 are undetermined constants and m is:

$$m = \sqrt{\frac{\overline{h}_{rad}}{k\,th}} \tag{14}$$

The solution to the particular differential equation:

$$\frac{d^2 T_p}{dx^2} - \frac{\overline{h}_{rad}}{k\,th} T_p = -\frac{\dot{q}''}{k\,th} - \frac{\overline{h}_{rad}}{k\,th} T_\infty \tag{15}$$

is, by inspection:

$$T_p = \frac{\dot{q}''}{\overline{h}_{rad}} + T_\infty \tag{16}$$

Substituting Eqs. (13) and (16) into Eq. (7) leads to:

$$T = C_1 \exp(m\,x) + C_2 \exp(-m\,x) + \frac{\dot{q}''}{\overline{h}_{rad}} + T_\infty \tag{17}$$

Substituting Eq. (17) into Eq. (5) leads to:

$$\frac{\left(T_c - C_1 - C_2 - \frac{\dot{q}''}{\bar{h}_{rad}} - T_\infty\right)}{R''} = -k\left(C_1\, m - C_2\, m\right) \tag{18}$$

Substituting Eq. (17) into Eq. (6) leads to:

$$C_1\, m \exp\left(m\, L\right) - C_2\, m \exp\left(-m\, L\right) = 0 \tag{19}$$

d.) Plot the temperature in the plate as a function of position.

The inputs are entered in EES:

```
"P1.7-5"
$UnitSystem SI MASS  DEG PA C J
$Tabstops 0.2 0.4 0.6 0.8 3.5

"Inputs"
q_flux=900 [W/m^2]                                          "solar flux"
th=1 [mm]*convert(mm,m)                                     "thickness of plate"
L=2 [cm]*convert(cm,m)                                      "half-length of plate"
k=10 [W/m-K]                                                "conductivity of plate"
Rc=1e-4 [K-m^2/W]                                           "contact resistance at clamped edges"
T_c=converttemp(C,K,20[C])                                  "casing temperature"
T_infinity=converttemp(C,K,20[C])                           "surrounding temperature"
e=1 [-]                                                     "emissivity"
T_bar=300 [K]                                               "average temperature"
h_rad=4*sigma#*e*T_bar^3                                    "radiation heat transfer coefficient"
```

Equations (18) and (19) are entered in EES in order to determine the constants C_1 and C_2:

```
"boundary conditions"
m=sqrt(h_rad/(k*th))                                        "fin constant"
(T_c-C_1-C_2-q_flux/h_rad-T_infinity)/Rc=-k*(C_1*m-C_2*m)   "at x=0"
C_1*m*exp(m*L)-C_2*m*exp(-m*L)=0                            "at x=L"
```

The solution is programmed in EES:

```
"solution"
{x_bar=0 [-]}                                               "dimensionless position"
x=x_bar*L                                                   "position"
T=C_1*exp(m*x)+C_2*exp(-m*x)+q_flux/h_rad+T_infinity        "temperature solution"
T_Celsius=converttemp(K,C,T)                                "in C"
```

Figure 3 illustrates the temperature as a function of position in the plate.

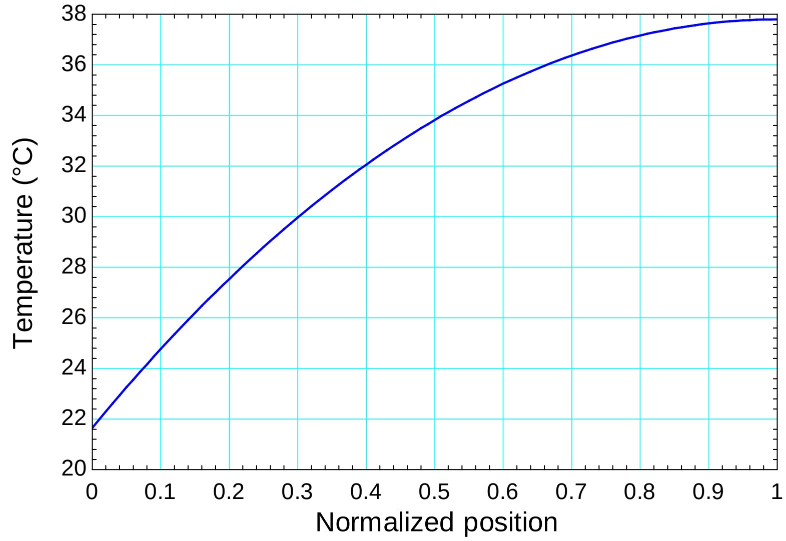

Figure 3: Temperature as a function of position in the plate.

e.) Use Maple to solve the differential equation and obtain symbolic expressions for the boundary conditions. Implement the Maple expressions into EES and show that your answer is identical to the one determined in (c).

The differential equation, Eq. (4), is entered in Maple:

```
> restart;
> ODE:=diff(diff(T(x),x),x)-h_rad*T(x)/(k*th)=-q_flux/(k*th)-h_rad*T_infinity/(k*th);
```

$$ODE := \left(\frac{d^2}{dx^2}\mathrm{T}(x)\right) - \frac{h\_rad\ \mathrm{T}(x)}{k\ th} = -\frac{q\_flux}{k\ th} - \frac{h\_rad\ T\_infinity}{k\ th}$$

and solved:

```
> Ts:=dsolve(ODE);
```

$$Ts := \mathrm{T}(x) = \mathbf{e}^{\left(\frac{\sqrt{h\_rad}\ x}{\sqrt{k}\ \sqrt{th}}\right)}\_C2 + \mathbf{e}^{\left(-\frac{\sqrt{h\_rad}\ x}{\sqrt{k}\ \sqrt{th}}\right)}\_C1 + \frac{q\_flux + h\_rad\ T\_infinity}{h\_rad}$$

Symbolic expressions for the boundary conditions are obtained:

```
> BC1:=(T_c-rhs(eval(Ts,x=0)))/Rc=-k*rhs(eval(diff(Ts,x),x=L));
```

$$BC1 := \frac{T\_c - \_C2 - \_C1 - \frac{q\_flux + h\_rad\ T\_infinity}{h\_rad}}{Rc} = -k\left(\frac{\sqrt{h\_rad}\ \mathbf{e}^{\left(\frac{\sqrt{h\_rad}\ L}{\sqrt{k}\ \sqrt{th}}\right)}\_C2}{\sqrt{k}\ \sqrt{th}} - \frac{\sqrt{h\_rad}\ \mathbf{e}^{\left(-\frac{\sqrt{h\_rad}\ L}{\sqrt{k}\ \sqrt{th}}\right)}\_C1}{\sqrt{k}\ \sqrt{th}}\right)$$

```
> BC2:=rhs(eval(diff(Ts,x),x=L))=0;
```

$$BC2 := \frac{\sqrt{h\_rad}\,\mathbf{e}^{\left(\frac{\sqrt{h\_rad}\,L}{\sqrt{k}\,\sqrt{th}}\right)}\,\_C2}{\sqrt{k}\,\sqrt{th}} - \frac{\sqrt{h\_rad}\,\mathbf{e}^{\left(-\frac{\sqrt{h\_rad}\,L}{\sqrt{k}\,\sqrt{th}}\right)}\,\_C1}{\sqrt{k}\,\sqrt{th}} = 0$$

The original equations in EES are commented out and the expressions from Maple are copied to EES:

```
{"boundary conditions"
m=sqrt(h_rad/(k*th))                                          "fin constant"
(T_c-C_1-C_2-q_flux/h_rad-T_infinity)/Rc=-k*(C_1*m-C_2*m)     "at x=0"
C_1*m*exp(m*L)-C_2*m*exp(-m*L)=0                              "at x=L"

"solution"
{x_bar=0 [-]}                                                 "dimensionless position"
x=x_bar*L                                                     "position"
T=C_1*exp(m*x)+C_2*exp(-m*x)+q_flux/h_rad+T_infinity          "temperature solution"
T_Celsius=converttemp(K,C,T)                                  "in C"}

"Maple solution"
(T_c-C_2-C_1-1/h_rad*(q_flux+h_rad*T_infinity))/Rc = -k*(1/k^(1/2)/th^(1/2)*h_rad^(1/2)*&
exp(1/k^(1/2)/th^(1/2)*h_rad^(1/2)*L)*C_2-1/k^(1/2)/th^(1/2)*h_rad^(1/2)*exp(-1/k^(1/2)/&
th^(1/2)*h_rad^(1/2)*L)*C_1)                                  "boundary condition at x=0"

1/k^(1/2)/th^(1/2)*h_rad^(1/2)*exp(1/k^(1/2)/th^(1/2)*h_rad^(1/2)*L)*C_2-1/k^(1/2)/th^(1/2)*&
h_rad^(1/2)*exp(-1/k^(1/2)/th^(1/2)*h_rad^(1/2)*L)*C_1 = 0   "boundary condition at x=L"
```

The solution from Maple is copied into EES:

```
T = exp(1/k^(1/2)/th^(1/2)*h_rad^(1/2)*x)*C_2+exp(-1/k^(1/2)/th^(1/2)*h_rad^(1/2)*x)*C_1&
    +1/h_rad*(q_flux+h_rad*T_infinity)                        "solution"
x=x_bar*L                                                     "position"
T_Celsius=converttemp(K,C,T)                                  "in C"
```

The solution is overlaid onto the plot from (d) in Figure 4.

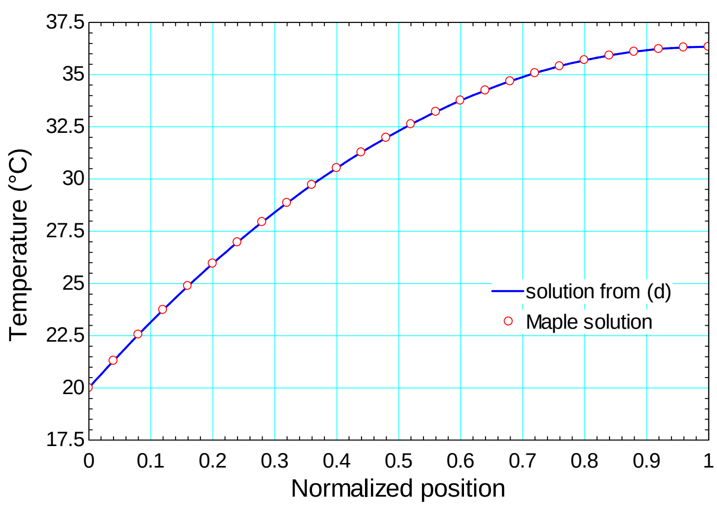

Figure 4: Temperature as a function of position in the plate using Maple solution.

f.) Prepare a calibration curve for the flux meter - plot the heat flux as a function of the difference between the temperature at the center of the plate and the casing.

Figure 5 illustrates the difference between the temperature at the center of the plate as a function of the applied heat flux.

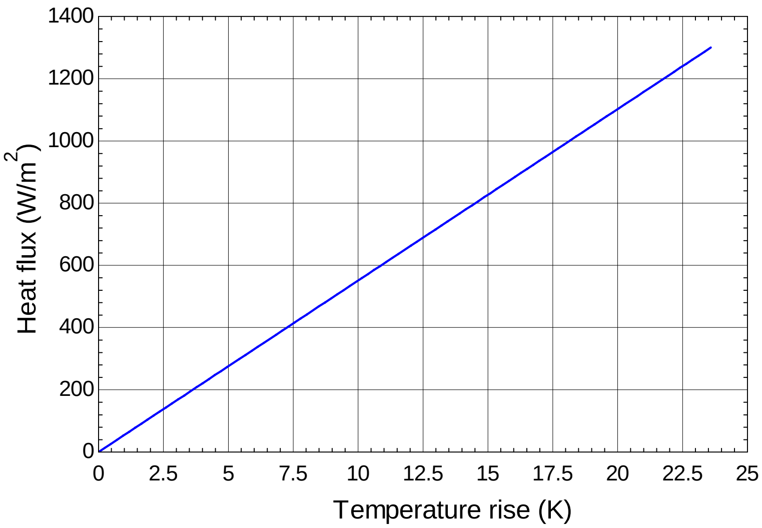

Figure 5: Calibration curve for the flux meter.

g.) If the uncertainty in the measurement of the temperature difference is $\delta\Delta T$ = 0.5 K then what is the uncertainty in the measurement of the heat flux?

The uncertainty in the measurement of the heat flux is approximately:

$$\delta \dot{q}'' = \frac{\partial \dot{q}''}{\delta\Delta T}\delta\Delta T \tag{20}$$

where $\frac{\partial \dot{q}''}{\delta\Delta T}$ = 55 W/m^2-K according to Figure 5. Therefore, the uncertainty in the heat flux is approximately 22 W/m^2.