Problem 1.8-5 (1-18 in text): Optimizing a Fin

Figure P1.8-5 illustrates a fin that is to be used in the evaporator of a space conditioning system for a space-craft.

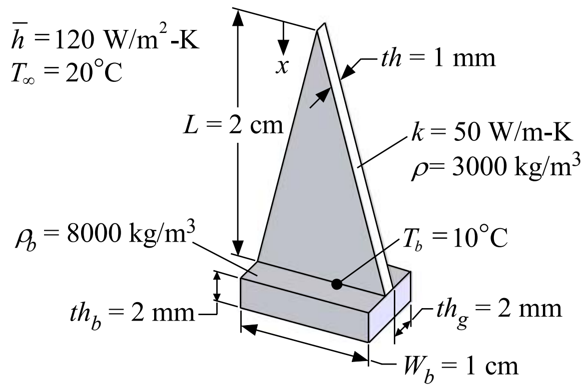


Figure P1.8-5: Fin on an evaporator.

The fin is a plate with a triangular shape. The thickness of the plate is th = 1 mm and the width of the fin at the base is W_b = 1 cm. The length of the fin is L = 2 cm. The fin material has conductivity k = 50 W/m-K. The average heat transfer coefficient between the fin surface and the air in the space-craft is $\bar{h}$ = 120 W/m^2-K. The air is at T_∞ = 20°C and the base of the fin is at T_b = 10°C. Assume that the temperature distribution in the fin is 1-D in x. Neglect convection from the edges of the fin.

a.) Obtain an analytical solution for the temperature distribution in the fin. Plot the temperature as a function of position.

The inputs are entered in EES:

```
$UnitSystem SI MASS RAD PA  K J
$TABSTOPS   0.2 0.4 0.6 0.8 3.5 in

h_bar=120 [W/m^2-K]                          "average heat transfer coefficient"
k=50 [W/m-K]                                 "conductivity"
T_infinity=converttemp(C,K,20[C])            "air material"
T_b=converttemp(C,K,10[C])                   "base temperature"
th_mm= 1 [mm]                                "fin thickness in mm"
th=th_mm*convert(mm,m)                       "fin thickness"
L_cm=2 [cm]                                  "fin length in cm"
L=L_cm*convert(cm,m)                         "fin length"
W_b=1 [cm]*convert(cm,m)                     "fin base width"
```

The differential control volume shown in Figure P1.8-5-2 is used to derive the governing differential equation:

$$\dot{q}_x = \dot{q}_{x+dx} + \dot{q}_{conv} \tag{1}$$

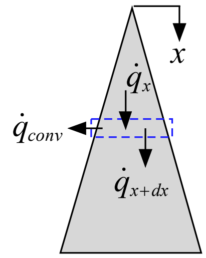


Figure P1.8-5-2: Differential control volume.

The rate of conduction and convection are:

$$\dot{q}_x = -k\, A_c \frac{dT}{dx} \tag{2}$$

$$\dot{q}_{conv} = \bar{h}\; per\left(T - T_\infty\right) dx \tag{3}$$

where A_c is the cross-sectional area for conduction and *per* is the perimeter. The width of the fin is a function of x:

$$W = W_b \frac{x}{L} \tag{4}$$

Therefore, A_c and *per* are:

$$A_c = W_b\, th \frac{x}{L} \tag{5}$$

$$per = 2\, W_b \frac{x}{L} \tag{6}$$

Substituting Eqs. (5) and (6) into Eq. (2) and (3) leads to:

$$\dot{q}_x = -k\, W_b\, th \frac{x}{L}\frac{dT}{dx} \tag{7}$$

$$\dot{q}_{conv} = \bar{h}\, 2\, W_b \frac{x}{L}\left(T - T_\infty\right) dx \tag{8}$$

Substituting Eqs. (7) and (8) into Eq. (1) leads to:

$$0 = \frac{d}{dx}\left[-k\, W_b\, th \frac{x}{L}\frac{dT}{dx}\right] dx + \bar{h}\, 2\, W_b \frac{x}{L}\left(T - T_\infty\right) dx \tag{9}$$

Simplifying:

$$\frac{d}{dx}\left(x\frac{dT}{dx}\right)-m^2\,x\,T=-m^2\,x\,T_\infty \tag{10}$$

where

$$m^2=\frac{2\,\overline{h}}{k\,th} \tag{11}$$

```
m=sqrt(2*h_bar/(k*th))                                  "solution parameter"
```

Maple is used to identify the solution to Eq. (10):

```
> restart;
> ODE:=diff(x*diff(T(x),x),x)-m^2*x*T(x)=-m^2*x*T_infinity;
```

$$ODE := \left(\frac{d}{dx}\,\mathrm{T}(x)\right)+x\left(\frac{d^2}{dx^2}\,\mathrm{T}(x)\right)-m^2\,x\,\mathrm{T}(x)=-m^2\,x\,T\_infinity$$

```
> Ts:=dsolve(ODE);
```

$$Ts := \mathrm{T}(x)=\mathrm{BesselI}(0,\,m\,x)\,\_C2+\mathrm{BesselK}(0,\,m\,x)\,\_C1+T\_infinity$$

Therefore:

$$T=C_2\,\mathrm{BesselI}\left(0,m\,x\right)+C_1\,\mathrm{BesselK}\left(0,m\,x\right)+T_\infty \tag{12}$$

The fin temperature at the tip must be bounded:

$$T_{x=0}=C_2\,\underbrace{\mathrm{BesselI}\left(0,m\,0\right)}_{1}+C_1\,\underbrace{\mathrm{BesselK}\left(0,m\,0\right)}_{\infty}+T_\infty<\infty \tag{13}$$

The limit of the 0th order modified Bessel functions as $x \rightarrow 0$ are evaluated using Maple:

```
> limit(BesselI(0,m*x),x=0);
```

$$1$$

```
> limit(BesselK(0,m*x),x=0);
```

$$\infty$$

Therefore, C_1 must be zero:

$$T=C_2\,\mathrm{BesselI}\left(0,m\,x\right)+T_\infty \tag{14}$$

The base temperature is specified; therefore:

$$T_b = C_2 \text{BesselI}\left(0, m\,L\right) + T_\infty \tag{15}$$

so:

$$C_2 = \frac{\left(T_b - T_\infty\right)}{\text{BesselI}\left(0, m\,L\right)} \tag{16}$$

Substituting Eq. (16) into Eq. (14) leads to:

$$T = \left(T_b - T_\infty\right)\frac{\text{BesselI}\left(0, m\,x\right)}{\text{BesselI}\left(0, m\,L\right)} + T_\infty \tag{17}$$

```
x_bar=0.5 [-]                                                "dimensionless position"
x=x_bar*L                                                    "position"
T=(T_b-T_infinity)*BesselI(0,m*x)/BesselI(0,m*L)+T_infinity  "temperature"
T_C=converttemp(K,C,T)                                       "in C"
```

Figure P1.8-5-3 illustrates the temperature as a function of position normalized by the fin length.

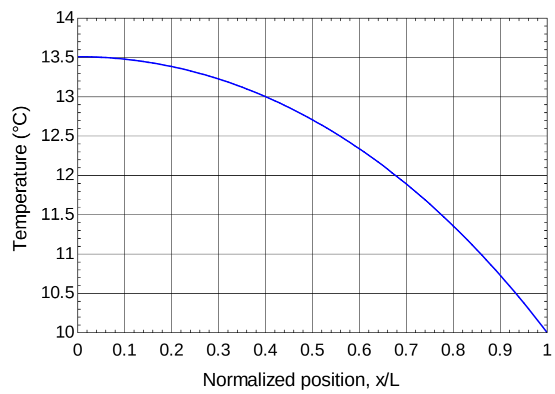


Figure P1.8-5-3: Fin temperature as a function of dimensionless position.

b.) Calculate the rate of heat transfer to the fin.

The rate of heat transfer to the fin is computed according to:

$$\dot{q}_{fin} = k\,W_b\,th \left.\frac{dT}{dx}\right|_{x=L} \tag{18}$$

Equation (18) is evaluated using Maple:

```
> restart;
> T:=(T_b-T_infinity)*BesselI(0,m*x)/BesselI(0,m*L)+T_infinity;
```

$$T := \frac{(T\_b - T\_infinity)\ \text{BesselI}(0,\ m\ x)}{\text{BesselI}(0,\ m\ L)} + T\_infinity$$

```
> q_dot_fin=k*W_b*th*eval(diff(T,x),x=L);
```

$$q\_dot\_fin = \frac{k\ W\_b\ th\ (T\_b - T\_infinity)\ \text{BesselI}(1, m\ L)\ m}{\text{BesselI}(0, m\ L)}$$

Therefore:

$$\dot{q}_{fin} = k\, W_b\, th\, m \left(T_b - T_\infty\right) \frac{\text{BesselI}\left(1, m\, L\right)}{\text{BesselI}\left(0, m\, L\right)} \tag{19}$$

```
q_dot_fin=k*W_b*th*(T_b-T_infinity)*m*BesselI(1,m*L)/BesselI(0,m*L)  "fin heat transfer rate"
```

which leads to $\dot{q}_{fin}$ = -0.196 W (the heat transfer is negative because the base temperature is less than the ambient temperature).

c.) Determine the fin efficiency.

The fin efficiency is defined according to:

$$\eta_{fin} = \frac{\dot{q}_{fin}}{\bar{h}\, A_s \left(T_b - T_\infty\right)} \tag{20}$$

where A_s is the total surface area of the fin exposed to the fluid:

$$A_s = W_b\, L \tag{21}$$

Substituting Eqs. (19) and (21) into Eq. (20) leads to:

$$\eta_{fin} = \frac{k\, W_b\, th\, m \left(T_b - T_\infty\right)}{\bar{h}\, W_b\, L \left(T_b - T_\infty\right)} \frac{\text{BesselI}\left(1, m\, L\right)}{\text{BesselI}\left(0, m\, L\right)} \tag{22}$$

Substituting Eq. (11) into Eq. (22) and simplifying leads to:

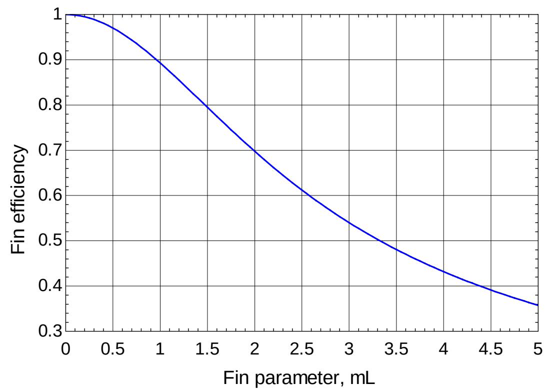
$$\eta_{fin} = \frac{k\, th}{\bar{h}\, L} \sqrt{\frac{2\,\bar{h}}{k\, th}} \frac{\text{BesselI}\left(1, m\, L\right)}{\text{BesselI}\left(0, m\, L\right)} = \frac{2}{L} \underbrace{\sqrt{\frac{th\, k}{2\,\bar{h}}}}_{1/m} \frac{\text{BesselI}\left(1, m\, L\right)}{\text{BesselI}\left(0, m\, L\right)} \tag{23}$$

or

$$\eta_{fin} = \frac{2\, \text{BesselI}\left(1, mL\right)}{mL\, \text{BesselI}\left(0, mL\right)} \tag{24}$$

```
eta_fin=2*BesselI(1,m*L)/(m*L*BesselI(0,m*L))   "fin efficiency"
```

which leads to η_{fin} = 0.8178. Figure P1.8-5-4 illustrates the fin efficiency as a function of the fin parameter mL.



Problem P1.8-5-4: Fin efficiency as a function of the fin parameter, *mL*.

The fin has density ρ = 3000 kg/m^3. The fin is installed on a base material with thickness th_b = 2 mm and density ρ_b = 8000 kg/m^3. The half-width between the gap between adjacent fins is th_g = 2 mm. Therefore, the volume of the base material associated with each fin is $th_b\, W_b\, (th + 2\, th_g)$.

d.) Determine the ratio of the absolute value of the rate of heat transfer to the fin to the total mass of material (fin and base material associated with the fin).

The additional inputs are entered in EES:

```
rho=3000 [kg/m^3]                         "density of fin material"
th_b=2 [mm]*convert(mm,m)                 "thickness of base material"
th_g=2 [mm]*convert(mm,m)                 "half-width of gap between adjacent fins"
rho_b=8000 [kg/m^3]                       "base material density"
```

The fin mass is given by:

$$M_{fin} = \frac{W_b\, L}{2} th\, \rho \tag{25}$$

The mass of the associated base material is:

$$M_b = W_b \left(th + 2\, th_g \right) th_b\, \rho_b \tag{26}$$

The ratio of rate of the fin heat transfer to mass is:

$$\frac{\dot{q}_{fin}}{M} = \frac{\dot{q}_{fin}}{\left(M_{fin} + M_b \right)} \tag{27}$$

```
M_fin=W_b*L*th*rho/2                        "fin mass"
M_b=W_b*(th+2*th_g)*th_b*rho_b              "mass of base material"
q\M=abs(q_dot_fin)/(M_fin+M_b)              "ratio of heat transfer to mass"
```

which leads to $\dot{q}_{fin}/M = 178.4$ W/kg.

e.) Prepare a contour plot that shows the ratio of the heat transfer to the fin to the total mass of material as a function of the length of the fin (L) and the fin thickness (th).

A parametric table is generated that contains the variables L_cm, th_mm and q\M and has 400 rows. The value of the variable L_cm is varied from 1 cm to 10 cm every 20 rows and the value of th_mm is varied from 0.2 mm to 2 mm in increments of 20 rows. The table is run and used to generate the contour plot shown in Figure P1.8-5-5.

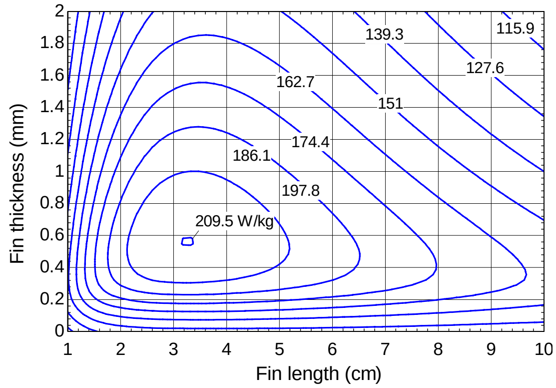


Figure P1.8-5-5: Contours of heat transfer per mass in the parameter space of fin length and thickness.

f.) What is the optimal value of L and th that maximizes the absolute value of the fin heat transfer rate to the mass of material?

According to Figure P1.8-5-5, the optimal design is approximately $L = 3.3$ cm and $th = 0.58$ mm. A more precise optimization can be carried out using EES' internal optimization feature. Maximizing $\dot{q}_{fin}/M$ by varying L and th leads to $\dot{q}_{fin}/M = 209.6$ W/kg at $L = 3.25$ cm and $th = 0.56$ mm.