Problem 1.8-2: Absorption in a Window

Figure P1.8-2 illustrates a thin, disk-shaped window that is used to provide optical access to a combustion chamber. The thickness of the window is b and the outer radius of the window is R_o. The window is composed of material with conductivity k and absorption coefficient α. The combustion chamber side of the window is exposed to convection with hot gas at T_g and heat transfer coefficient h. Convection with the air outside of the chamber can be neglected. There is a radiation heat flux, $\dot{q}''_{rad}$, that is incident on the combustion chamber side of the glass. The amount of this radiation that is absorbed by the glass is, approximately, $\dot{q}''_{rad}\,\alpha\, b$. The remainder of this radiation, $\dot{q}''_{rad}\left(1-\alpha\, b\right)$, exits the opposite surface of the glass. The outer edge (at $r = R_o$) of the glass is held at temperature T_{edge}.

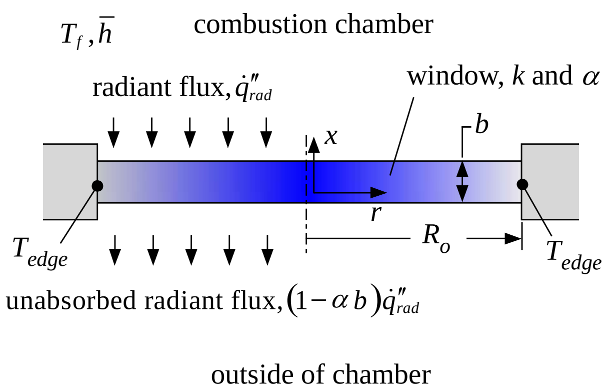

Figure P1.8-2: Disk-shaped window.

You are to develop a 1-D, steady state analytical model that can predict the temperature distribution in the glass as a function of radial position, r.

a.) How would you justify using a 1-D model of the glass? What number would you calculate in order to verify that the temperature does not vary substantially in the x direction?

The Biot number should be computed in order to justify the extended surface approximation. The Biot number is the ratio of conduction resistance in the axial direction to convection resistance from the inner surface of the window:

$$Bi = \frac{R_{cond,x}}{R_{conv}} = \frac{b}{k\,A}\frac{h\,A}{1} = \frac{b\,h}{k} \tag{1}$$

Anything within a factor of 2 of Eq. (1) would be sufficient

b.) Derive the ordinary differential equation in r that must be solved. Make sure that your differential equation includes the effect of conduction, convection with the gas within the chamber, and generation of thermal energy due to absorption.

A differentially small control volume is shown in Figure 2.

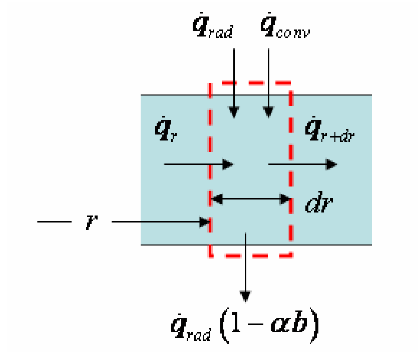

Figure 2: Differentially small control volume.

The energy balance suggested by Figure 2 is:

$$\dot{q}_r + \dot{q}_{rad} + \dot{q}_{conv} = \dot{q}_{r+dr} + \dot{q}_{rad}\left(1-\alpha\, b\right) \tag{2}$$

Expanding the $r + dr$ term and simplifying leads to:

$$\dot{q}_{rad} + \dot{q}_{conv} = \frac{d\dot{q}}{dr}dr + \dot{q}_{rad}\left(1-\alpha\, b\right) \tag{3}$$

The rate equations are:

$$\dot{q}_{rad} = \dot{q}''_{rad}\, 2\,\pi\, r\, dr \tag{4}$$

$$\dot{q} = -k\, 2\,\pi\, r\, b\frac{dT}{dr} \tag{5}$$

$$\dot{q}_{conv} = h\, 2\,\pi\, r\, dr\left(T_f - T\right) \tag{6}$$

Substituting Eqs. (4) through (6) into Eq. (3) leads to:

$$\dot{q}''_{rad}\, 2\,\pi\, r\, dr + h\, 2\,\pi\, r\, dr\left(T_f - T\right) = \frac{d}{dr}\left[-k\, 2\,\pi\, r\, b\frac{dT}{dr}\right]dr + \dot{q}''_{rad}\, 2\,\pi\, r\, dr\left(1-\alpha\, b\right) \tag{7}$$

Dividing through by ($-k$ 2 π b dr) leads to:

$$\boxed{\frac{d}{dr}\left[r\frac{dT}{dr}\right] - \frac{h}{k\, b}r\, T = -r\left(\frac{\dot{q}''_{rad}\,\alpha}{k} + \frac{h}{k\, b}T_f\right)} \tag{8}$$

c.) What are the boundary conditions for the ordinary differential equation that you derived in part (b)?

At $r = 0$ the temperature must be finite. At the edge, the temperature is specified:

$$T_{r=R_o} = T_{edge} \tag{9}$$

d.) Solve the ordinary differential equation in order to obtain an expression for the temperature as a function of radius.

The solution is split into its homogeneous (u) and particular (v) parts:

$$T = u + v \tag{10}$$

The particular solution is a constant:

$$-\frac{h}{k\,b} r\,v = -r\left(\frac{\dot{q}''_{rad}\,\alpha}{k} + \frac{h}{k\,b} T_f\right) \tag{11}$$

or

$$v = T_f + \frac{\dot{q}''_{rad}\,\alpha\,b}{h} \tag{12}$$

The homogeneous form of the differential equation is:

$$\frac{d}{dr}\left[r\frac{du}{dr}\right] - m^2\,r\,u = 0 \tag{13}$$

where

$$m = \sqrt{\frac{h}{k\,b}} \tag{14}$$

Equation (13) is a form of Bessel's equation:

$$\frac{d}{dx}\left(x^p\,\frac{d\theta}{dx}\right) \pm c^2\,x^s\,\theta = 0 \tag{15}$$

where

$$\theta = u \tag{16}$$

$$x = r \tag{17}$$

$$p = 1 \tag{18}$$

$$c = m \tag{19}$$

$$s = 1 \tag{20}$$

The solution can be obtained by using the chart found in the notes:

$$n = \frac{1-p}{s-p+2} = \frac{1-1}{1-1+2} = 0 \tag{21}$$

$$a = \frac{2}{1-1+2} = 1 \tag{22}$$

$$\frac{n}{a} = \frac{1-1}{2} = 0 \tag{23}$$

so that

$$u = C_1 \,\mathrm{BesselI}\left(0, m\,r\right) + C_2 \,\mathrm{BesselK}\left(0, m\,r\right) \tag{1-24}$$

and the solution is:

$$T = C_1 \,\mathrm{BesselI}\left(0, m\,r\right) + C_2 \,\mathrm{BesselK}\left(0, m\,r\right) + T_f + \frac{\dot{q}''_{rad}\,\alpha\, b}{h} \tag{1-25}$$

Because BesselK(0,0) becomes infinite, $C_2 = 0$ and:

$$T = C_1 \,\mathrm{BesselI}\left(0, m\,r\right) + T_f + \frac{\dot{q}''_{rad}\,\alpha\, b}{h} \tag{1-26}$$

The boundary condition associated with Eq. (9) leads to:

$$T_{edge} = C_1 \,\mathrm{BesselI}\left(0, m\,R_o\right) + T_f + \frac{\dot{q}''_{rad}\,\alpha\, b}{h} \tag{1-27}$$

so that:

$$C_1 = \frac{T_{edge} - T_f - \dfrac{\dot{q}''_{rad}\,\alpha\, b}{h}}{\mathrm{BesselI}\left(0, m\,R_o\right)} \tag{1-28}$$

and

$$\boxed{T = \left(T_{edge} - T_f - \frac{\dot{q}''_{rad}\,\alpha\, b}{h}\right)\frac{\mathrm{BesselI}\left(0, m\,r\right)}{\mathrm{BesselI}\left(0, m\,R_o\right)} + T_f + \frac{\dot{q}''_{rad}\,\alpha\, b}{h}} \tag{1-29}$$