Problem 1.3-12

Figure P1.3-12 illustrates a plane wall. The temperature distribution in the wall is 1-D and the problem is steady state.

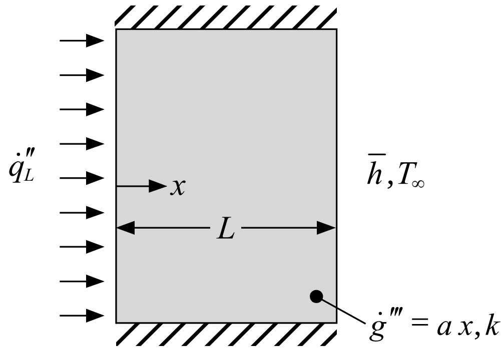

Figure P1.3-12: Plane wall.

There is generation of thermal energy in the wall. The generation per unit volume is not uniform but rather depends on position according to:

$$\dot{g}''' = a\,x \tag{1}$$

where a is a constant and x is position. The left side of the wall experiences a specified heat flux, $\dot{q}''_L$. The right side of the wall experiences convection with heat transfer coefficient $\overline{h}$ to fluid at temperature T_∞. The thickness of the wall is L and the conductivity of the wall material, k, is constant.

a.) Derive the ordinary differential equation that governs this problem. Clearly show your steps.

A differential control volume is shown in Figure 2 and leads to:

$$\dot{q}_x + \dot{g} = \dot{q}_{x+dx} \tag{2}$$

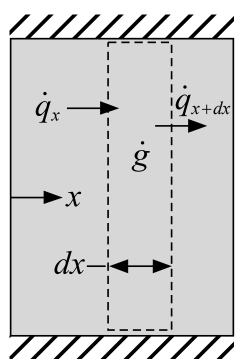

Figure 2: Differential control volume with energy terms.

After expanding the $x + dx$ term:

$$\dot{q}_x + \dot{g} = \dot{q}_x + \frac{d\dot{q}}{dx}dx \tag{3}$$

The rate of thermal energy generation within the control volume is:

$$\dot{g} = \dot{g}''' A_c \, dx \tag{4}$$

where A_c is the cross-sectional area of the wall. The conduction term is expressed using Fourier's law:

$$\dot{q} = -k \, A_c \frac{dT}{dx} \tag{5}$$

Substituting Eqs. (5) and (4) into Eq. (3) results in

$$\dot{g}''' A_c \, dx = \frac{d}{dx}\left(-k \, A_c \frac{dT}{dx}\right)dx \tag{6}$$

which can be simplified:

$$\frac{d}{dx}\left(\frac{dT}{dx}\right) = -\frac{\dot{g}'''}{k} \tag{7}$$

Substituting the position dependent generation into Eq. (7) leads to:

$$\boxed{\frac{d}{dx}\left(\frac{dT}{dx}\right) = -\frac{a\,x}{k}} \tag{8}$$

b.) Solve the differential equation that you obtained in (a). Your solution should include two undetermined constants.

Equation (7) is separated and integrated:

$$\int d\left(\frac{dT}{dx}\right) = \int -\frac{a\,x}{k}dx \tag{9}$$

which leads to:

$$\frac{dT}{dx} = -\frac{a}{2\,k}x^2 + C_1 \tag{10}$$

where C_1 is a constant of integration. Equation (10) is integrated again:

$$\int dT = \int \left(-\frac{a}{2\,k} x^2 + C_1 \right) dx \tag{11}$$

which leads to:

$$T = -\frac{a}{6\,k} x^3 + C_1\,x + C_2 \tag{12}$$

c.) Specify the boundary conditions for the differential equation that you derived in (a).

An interface energy balance at $x = 0$ leads to:

$$\dot{q}''_L = -k \left.\frac{dT}{dx}\right|_{x=0} \tag{13}$$

An interface energy balance at $x = L$ leads to:

$$-k \left.\frac{dT}{dx}\right|_{x=L} = \bar{h}\left(T_{x=L} - T_\infty\right) \tag{14}$$

d.) Use the results of (b) and (c) to obtain two equations that can be solved for the two undetermined constants.

Substituting Eq. (10) into Eq. (13) leads to:

$$\boxed{\dot{q}''_L = -k\,C_1} \tag{15}$$

Substituting Eqs. (10) and (12) into Eq. (14) leads to:

$$\boxed{-k\left(-\frac{a}{2\,k} L^2 + C_1 \right) = \bar{h}\left(-\frac{a}{6\,k} L^3 + C_1\,L + C_2 - T_\infty \right)} \tag{16}$$

Equations (15) and (16) can be solved for C_1 and C_2.