PROBLEM SOLUTION MANUAL FOR

Fundamentals of Nuclear Science and Engineering

Third Edition

by

J. Kenneth Shultis

and

Richard E. Faw

Dept. of Mechanical and Nuclear Engineering
Kansas State University
Manhattan, KS 66506

email: `jks@ksu.edu`

Revised June 2016

Notice

This collection contains solutions to most of the problems in our book *Fundamentals of Nuclear Science and Engineering, 3/e* (Taylor & Francis, Boca Raton, Florida, 2007. We do not warrant that all the solutions are correct or that other approaches could give equally valid results. This collection is provided to you solely as an aid in your teaching, and we ask that you do not copy this set for others without our permission. If, in your teaching, you develop better solutions than are presented here or find corrections are needed, we would appreciate receiving copies so that, over time, this collection will be improved.

A sporadically updated errata for the book can be found on the world wide web at `http://www.mne.ksu.edu/~jks/books.htm`

Chapter 1

Fundamental Concepts

PROBLEMS

1. Both the hertz and the curie have dimensions of s^{-1}. Explain the difference between these two units.

Solution:

The *hertz* is used for periodic phenomena and equals the number of "cycles per second." The *curie* is used for the random or stochastic rate at which a radioactive source decays, specifically, 1 Ci $= 3.7 \times 10^{10}$ decays/second.

2. Advantages of SI units are apparent when one is presented with units of *barrels*, *ounces*, *tons*, and many others.

(a) Compare the British and U.S. units for the gallon and barrel (liquid and dry measure) in SI units of liters (L).

(b) Compare the *long ton*, *short ton*, and *metric ton* in SI units of kg.

Solution:

Unit conversions are taken from the handbook *Conversion Factors and Tables*, 3d ed., by O.T. Zimmerman and I. Lavine, published by Industrial Research Service, Inc., 1961.

(a) In both British and U.S. units, the gallon is equivalent to 4 quarts, eight pints, etc. However, the quart and pint units differ in the two systems. The U.S. gallon measures 3.7853 L, while the British measures 4.546 L. Note that the gallon is sometimes used for dry measure, 4.405 L U.S. measure. The barrel in British units is the same for liquid and dry measure, namely, 163.65 L. The U.S. barrel (dry) is exactly 7056 in^3, 115.62 L. The U.S. barrel (liq) is 42 gallons (158.98 L) for petroleum measure, but otherwise (usually) is 31.5 gallons (119.24 L).

(b) The common U.S. unit is the short ton of 2000 lb, 907.185 kg, 20 short hundredweight (cwt). The metric ton is exactly 1000 kg, and the long ton is 20 long cwt, 22.4 short cwt, 2240 lb, or 1016 kg.

3. Compare the U.S. and British units of *ounce* (fluid), (apoth), (troy), and (avdp).

Solution:

The U.S. and British fluid ounces are, respectively, 1/32 U.S. quarts (0.02957 L) and 1/40 British quarts (0.02841 L). The oz (avdp.) is exactly 1/16 lb (avdp), i.e., 0.02834 kg. Avdp., abbreviation for *avoirdupois* refers to a system of weights with 16 oz to the pound. The apoth. *apothecary* or troy ounce is exactly 480 grains, 0.03110 kg.

4. Explain the SI errors (if any) in and give the correct equivalent units for the following units: (a) mgrams/kiloL, (b) megaohms/nm, (c) N·m/s/s, (d) gram cm/(s^{-1}/mL), and (e) Bq/milli-Curie.

Solution:

(a) Don't mix unit abbreviations and names; SI prefixes only in numerator: correct form is μ**g/L**.

(b) Don't mix names and abbreviations and don't use SI prefixes in denominator: correct form **nohm/m**.

(c) Don't use hyphen and don't use multiple solidi: correct form $\mathbf{N\,m\,s^{-2}}$.

(d) Don't mix names and abbreviations, don't use multiple solidi, and don't use parentheses: correct form g cm s mL or better **10 μg m s L**.

(e) Don't mix names with abbreviations, and SI prefix should be in numerator: correct form **kBq/Ci**.

5. Consider H_2, D_2, and H_2O, treated as ideal gases at pressures of 1 atm and temperatures of 293.2°K . What are the molecular and mass densities of each.

Solution:

According to the ideal gas law, molar densities are identical for ideal gases under the same conditions, i.e., $\rho_m = p/RT$. From Table 1.5, $R = 8.314472$ Pa m^3/K. For $p = 0.101325$ MPa= 1 atm., and $T = 293.2$°K , $\rho_m = 41.56$ mol/m^3. Multiplication by molecular weights yield, respectively, 83.78 , 167.4, and 749.0 g/m^3 for the three gases.

6. In vacuum, how far does light move in 1 ns?

Solution:

$\Delta x = c\Delta t = (3 \times 10^8 \text{ m/s}) \times (10^{-9} \text{ s}) = 3 \times 10^{-4} \text{ m} = \mathbf{30\ cm}$.

7. In a medical test for a certain molecule, the concentration in the blood is reported as 57 mcg/dL. What is the concentration in proper SI notation?

Solution:

123 mcg/dL $= 10^{-3}10^{-2}$ g/10^{-1} L $= 1.23 \times 10^{-4}$ g/L $=$ **57 μg/L**.

8. How many neutrons and protons are there in each of the following nuclides: (a) ^{11}B, (b) ^{24}Na, (c) ^{60}Co, (d) ^{207}Pb, and (e) ^{238}U?

Solution:

Nuclide	neutrons	protons
^{11}B	6	5
^{24}Na	13	11
^{60}Co	33	27
^{207}Pb	125	82
^{238}U	146	92

9. Consider the nuclide ^{71}Ge. Use the Chart of the Nuclides to find a nuclide (a) that is in the same isobar, (b) that is in the same isotone, and (c) that is an isomer.

Solution: (a) ^{71}As, (b) ^{59}Ga, and (c) ^{71m}Ge

10. Examine the Chart of the Nuclides to find any elements, with Z less that that of lead ($Z = 82$), that have no stable nuclides. Such an element can have no standard relative atomic mass.

Solution: Promethium ($Z = 61$) and Technetium ($Z = 43$)

11. What are the molecular weights of (a) H_2 gas, (b) H_2O, and (c) HDO?

Solution:

From Table A.3, $\mathcal{A}$(O) = 15.9994 g/mol; from Table B.1 $\mathcal{A}$(H) = 1.007825 g/mol and $\mathcal{A}$(D) = 2.014102 g/mol.

(a) $\mathcal{A}(H_2) = 2\ \mathcal{A}(H) = 2 \times 1.007825 =$ **2.01565 g/mol**

(b) $\mathcal{A}(H_2O) = 2\ \mathcal{A}(H) + \mathcal{A}(O) = 2 \times 1.007825 + 15.9994 =$ **18.0151 g/mol**

(c) $\mathcal{A}(HDO) = \mathcal{A}(H) + \mathcal{A}(D) + \mathcal{A}(O) = 1.007825 + 2.014102 + 15.9994$
$=$ **19.0213 g/mol**

12. What is the mass in kg of a molecule of uranyl sulfate UO_2SO_4?

Solution:

From Table A.3, $\mathcal{A}(\text{U}) = 238.0289$ g/mol, $\mathcal{A}(\text{O}) = 15.9994$ g/mol, and $\mathcal{A}(\text{S}) = 32.066$ g/mol.

The molecular weight of UO_2SO_4 is thus $\mathcal{A}(\text{UO}_2\text{SO}_4) = \mathcal{A}(\text{U}) + 6\mathcal{A}(\text{O}) + \mathcal{A}(\text{S}) = 238.0289 + 6(15.994) + 32.066 = 366.091$ g/mol $= 0.336091$ kg/mol.

Since one mol contains $N_a = 6.022 \times 10^{23}$ molecules, the mass of one molecule of UO_2SO_4 $= \mathcal{A}(\text{UO}_2\text{SO}_4)/N_a = 0.366091/6.002 \times 10^{23} = \mathbf{6.079 \times 10^{-25}}$ **kg/molecule**.

13. Show by argument that the reciprocal of Avogadro's constant is the gram equivalent of 1 atomic mass unit.

Solution:

By definition one gram atomic weight of ^{12}C is 12 g/mol. Thus the mass of one atom of ^{12}C is

$$M(^{12}_{6}\text{C}) = \frac{12 \text{ g/mol}}{N_a \text{ atoms/mol}} = \frac{12}{N_a} \text{ g/atom.}$$

But by definition, one atom of ^{12}C has a mass of 12 u. Therefore,

$$1 \text{ u} = \frac{1 \text{ u}}{12 \text{ u/}(^{12}\text{C atom})} \left(\frac{12}{N_a} \text{ g/}(^{12}\text{C atom}) \right) = \frac{1}{N_a} \text{ g.}$$

14. Prior to 1961 the physical standard for atomic masses was 1/16 the mass of the $^{16}_{8}$O atom. The new standard is 1/12 the mass of the $^{12}_{6}$C atom. The change led to advantages in mass spectrometry. Determine the conversion factor needed to convert from old to new atomic mass units. How did this change affect the value of the Avogadro constant?

Solution

From Table B.1, the $^{16}_{8}$O atom has a mass of 15.9949146 amu. Thus, the pre-1961 atomic mass unit was 15.9949146/16 post-1961 units, and the conversion factor is thus 1 amu (^{16}O) = 0.99968216 amu (^{12}C).

The Avogadro constant is defined as the number of atoms in 12 g of unbound carbon-12 in its rest-energy electronic state, i.e., the number of atomic mass units per gram. Using data from Table 1.5, one finds that N_a is given by the reciprocal of the atomic mass unit, namely, $[1.6605387 \times 10^{-24}]^{-1} = 6.0221420 \times 10^{23}$ mol^{-1}. Pre-1961, the Avogadro constant was more loosely defined as the number of atoms per mol of any element, and had the best value 6.02486×10^{23}.

15. How many atoms of ^{234}U are there in 1 kg of natural uranium?

Solution:

From Table A.3, the natural abundance of ^{234}U in uranium is found to be $f(^{234}\text{U}) = 0.0055$ atom-%. A mass m of uranium contains $[m/\mathcal{A}(\text{U})]N_a$ uranium atoms. Thus, the number of ^{234}U atoms in the mass $m = 1000$ g are

$$N(^{234}\text{U}) = f(^{234}\text{U})\frac{mN_a}{\mathcal{A}(\text{U})}$$

$$= 0.000055\frac{1000 \times (6.022 \times 10^{23})}{238.0289} = \mathbf{1.392 \times 10^{20} \text{ atoms}}.$$

16. A bucket contains 1 L of water at 4 °C where water has a denisty of 1 g cm^3. (a) How many moles of H_2O are there in the bucket? (b) How many atoms of $^1_1$H and $^2_1$D are there in the bucket?

Solution:

(a) The relative atomic weight of water $\mathcal{A}(\text{H}_2\text{O}) = 2\mathcal{A}(\text{H}) + \mathcal{A}(\text{O}) = 2(1.00794) + (15.9994) = 18.01528$. Then the number of water molecules

$$\text{mols of H}_2\text{O} = \frac{\text{mass(H}_2\text{O)}}{\mathcal{A}(\text{H}_2\text{O})} = \frac{1000 \text{ g}}{18.01258 \text{ g/mol}} = \mathbf{55.5 \text{ mol}}.$$

(b) Number of molecules of H_2O = 55.5 mol $\times N_a$ mol^{-1} = $55.5 \times 6.60221 \times 10^{23} = 3.343 \times 10^{25}$ molecules. Then the number of atoms of both $^1_1$H and $^2_1$D atoms = 2 × no. of H_2O molecules = 6.6856×10^{25} atoms. From Table A.4, the isoptopic abundances are found to be $\gamma(^1_1\text{H}) = 0.999885$ and $\gamma(^2_1\text{D}) = 0.000115$. Then

$$N(^1_1\text{H}) = (0.999885)(6.6856 \times 10^{25}) = \mathbf{6.69 \times 10^{25} \text{ atoms}}$$

and

$$N(^2_1\text{D}) = (0.000115)(6.6856 \times 10^{25}) = \mathbf{7.69 \times 10^{21} \text{ atoms}}.$$

17. How many atoms of deuterium are there in 2 kg of water?

Solution:

Water is mostly H_2O, and so we first calculate the number of atoms of hydrogen $N(\text{H})$ in a mass $m = 2000$ g of H_2O is

$$N(\text{H}) = 2N(\text{H}_2\text{O}) = 2\frac{mN_a}{\mathcal{A}(\text{H}_2\text{O})} \simeq 2\frac{mN_a}{A(\text{H}_2\text{O})}$$

$$= 2\frac{2000 \times (6.022 \times 10^{23})}{18} = 1.34 \times 10^{26} \text{ atoms of H}.$$

From Table A.4, the natural isotopic abundance of deuterium (D) is 0.015 atom-% in elemental hydrogen. Thus, the number of deuterium atoms in 2 kg of water is

$$N(\mathrm{D}) = 0.00015 \times N(\mathrm{H}) = \mathbf{2.01 \times 10^{22}\ atoms}.$$

18. Estimate the number of atoms in a 3000 pound automobile. State any assumptions you make.

Solution:

The car mass $m = 3000/2.2 = 1365$ kg. Assume most the this mass is iron. If the atoms in non-iron materials (e.g., glass, plastic, rubber, etc.) were converted to iron, the car mass would increase to about $m_{equiv} = 1500$ kg. Thus the number of atoms in the car is

$$N = \frac{m_{equiv} N_a}{A(\mathrm{Fe})} = \frac{(1.5 \times 10^6)(6.022 \times 1023)}{56} = \mathbf{1.6 \times 10^{28}\ atoms}.$$

19. Calculate the relative atomic weight of oxygen.

Solution

From Table A.4, oxygen has three stable isotopes: ^{16}O, ^{17}O, and ^{18}O with percent abundances of 99.757, 0.038, and 0.205, respectively. Their atomic masses, in u, are found from Table B.1 and equal their relative atomic weights. Then from Eq. (1.2)

$$\begin{aligned}\mathcal{A}(\mathrm{O}) &= \frac{\gamma(^{16}\mathrm{O})}{100}\mathcal{A}(^{16}\mathrm{O}) + \frac{\gamma(^{17}\mathrm{O})}{100}\mathcal{A}(^{17}\mathrm{O}) + \frac{\gamma(^{18}\mathrm{O})}{100}\mathcal{A}(^{18}\mathrm{O}) \\ &= \frac{99.757}{100}15.994915 + \frac{0.038}{100}16.999132 + \frac{0.205}{100}17.999160 = \mathbf{15.999405}.\end{aligned}$$

20. Natural uranium contains the isotopes ^{234}U, ^{235}U and ^{238}U. Calculate the relative atomic weight of natural uranium.

Solution

From Table A.4, the three isotopes ^{234}U, ^{235}U, and ^{238}U have isotopic abundances of 0.0055%, 0.720%, and 99.2745%, respectively. Their atomic masses, in u, are found from Table B.1 and equal their relative atomic weights. Then from Eq. (1.2)

$$\begin{aligned}\mathcal{A}(\mathrm{O}) &= \frac{\gamma(^{234}\mathrm{U})}{100}\mathcal{A}(^{234}\mathrm{U}) + \frac{\gamma(^{235}\mathrm{U})}{100}\mathcal{A}(^{235}\mathrm{U}) + \frac{\gamma(^{238}\mathrm{U})}{100}\mathcal{A}(^{238}\mathrm{U}) \\ &= \frac{0.0055}{100}234.040945 + \frac{0.720}{100}235.043923 + \frac{99.2745}{100}238.050783 \\ &= \mathbf{238.02891}.\end{aligned}$$

21. Does a sample of carbon extracted from coal have the same relative atomic weight as a sample of carbon extracted from a plant? Explain.

Solution

The carbon extracted from coal has only two isotopes, namely ^{12}C and ^{13}C with with abundances of 98.93% and 1.07%, respectively. The relative atomic weight is thus is slightly larger than 12 that would result if there were no ^{13}C, namely 12.0107. Carbon extracted from plant material, however, also contains the radioactive isotope ^{14}C produced in the atmosphere by cosmic rays. Thus, the relative atomic weight is *conceptually* greater than that of carbon from coal in which all the ^{14}C has radioactively decayed away.

However, as discussed in Section 5.8.1, the amount of ^{14}C in plant material is extremely small (1.23×10^{-12} atoms per atom of stable carbon). Thus, ^{14}C would increase the atomic weight only in the 12th significant figure!

22. Dry air at normal temperature and pressure has a mass density of 0.0012 g/cm^3 with a mass fraction of oxygen of 0.23. What is the atom density (atom/cm^3) of ^{18}O?

Solution:

From Eq. (1.5), the atom density of oxygen is

$$N(\mathrm{O}) = \frac{w_o \rho N_a}{\mathcal{A}(\mathrm{O})} = \frac{0.23 \times 0.0012 \times (6.022 \times 10^{23})}{15.9994} = 1.04 \times 10^{19} \text{ atoms/cm}^3.$$

From Table A.4 isotopic abundance of ^{18}O in elemental oxygen is $f_{18} = 0.2$ atom-% of all oxygen atoms. Thus, the atom density of ^{18}O is

$$N(^{18}\mathrm{O}) = f_{18} N(\mathrm{O}) = 0.002 \times 1.04 \times 10^{19} = \mathbf{2.08 \times 10^{16} \text{ atoms/cm}^3}.$$

23. A reactor is fueled with 4 kg uranium enriched to 20 atom-percent in ^{235}U. The remainder of the fuel is ^{238}U. The fuel has a mass density of 19.2 g/cm^3. (a) What is the mass of ^{235}U in the reactor? (b) What are the atom densities of ^{235}U and ^{238}U in the fuel?

Solution:

(a) Let m_5 and m_8 be the mass in kg of an atom of ^{235}U and ^{238}U, and let n_5 and n_8 be the total number of atoms of ^{235}U and ^{238}U in the uranium mass $M_U = 4$ kg. For 20% enrichment, $n_8 = 4n_5$, so that

$$M_U = n_5 m_5 + n_8 m_8 = n_5 m_5 + 4 n_5 m_8 = n_5 m_5 \left(1 + 4\frac{m_8}{m_5}\right).$$

Here $n_5 m_5 = M_5$ is the mass of ^{235}U in the uranium mass M_U. From this result we obtain using $m_5/m_8 \simeq 235/238$

$$M_5 = M_U \left[1 + 4\frac{m_8}{m_5}\right]^{-1} = 4 \text{ kg} \left[1 + 4\left(\frac{238}{235}\right)\right]^{-1} = \mathbf{0.7919 \text{ kg}}.$$

The mass of ^{238}U $M_8 = M_U - M_5 = 3.208$ kg.

(b) The volume V of the uranium is $V = M_U/\rho_U = (4000 \text{ g})/(19.2 \text{ g/cm}^3) = 208.3 \text{ cm}^3$. Hence the atom densities are

$$N_5 = \frac{M_5 N_a}{A_5 V} = \frac{(791.9 \text{ g})(6.022 \times 10^{23} \text{ atoms/mol})}{(235 \text{ g/mol})(208.3 \text{ cm}^3)} = \mathbf{9.740 \times 10^{21} \ cm^{-3}}$$

$$N_8 = \frac{M_8 N_a}{A_8 V} = \frac{(3208 \text{ g})(6.022 \times 10^{23} \text{ atoms/mol})}{(238 \text{ g/mol})(208.3 \text{ cm}^3)} = \mathbf{3.896 \times 10^{22} \ cm^{-3}}$$

24. A sample of uranium is enriched to 3.2 atom-percent in ^{235}U with the remainder being ^{238}U. What is the enrichment of ^{235}U in weight-percent?

Solution:

Let the subscripts 5, 8 and U refer to ^{235}U, ^{238}U, and uranium, respectively. For the given atom-% enrichment, The number of atoms in a sample of the uranium are

$$N_5 = 0.0320 N_U \qquad \text{and} \qquad N_8 = 0.9680 N_U.$$

The mass M_5 and M_8 of ^{235}U and ^{238}U in the sample is

$$M_5 = 0.0320 N_U m_5 \qquad \text{and} \qquad M_8 = 0.9680 N_U m_8,$$

where m_5 and m_8 is the mass of an atom of ^{235}U and ^{238}U, respectively.

The enrichment in weight-% is thus

$$\begin{aligned} e(\text{wt-\%}) &= 100 \times \frac{M_5}{M_5 + M_8} = 100 \times \frac{0.0320 m_5}{0.0320 m_5 + 0.9680 m_8} \\ &= \frac{100 \times 0.0320}{0.0320 + 0.9680(m_8/m_5)} \simeq \frac{100 \times 0.0320}{0.0320 + 0.9680(238/235)} \\ &= \mathbf{3.16 \ wt\text{-}\%}. \end{aligned}$$

25. A crystal of NaCl has a density of 2.17 g/cm^3. What is the atom density of sodium in the crystal?

Solution:

Atomic weights for Na and Cl are obtained from Table A.3, so that $\mathcal{A}(\text{NaCl}) = \mathcal{A}(\text{Na}) + \mathcal{A}(\text{Cl}) = 22.990 + 35.453 = 58.443$ g/mol. Thus the atom density of Na is

$$N(\text{Na}) = N(\text{NaCl}) = \frac{\rho_{\text{NaCl}} N_a}{\mathcal{A}(\text{NaCl})} = \frac{2.17 \times 6.022 \times 10^{23}}{58.443} = \mathbf{2.24 \times 10^{22} \ cm^{-3}}.$$

26. A concrete with a density of 2.35 g/cm^3 has a hydrogen content of 0.0085 weight fraction. What is the atom density of hydrogen in the concrete?

Solution:

From Eq. (1.5), the atom density of hydrogen is

$$N(\text{H}) = \frac{w_H \rho M_a}{\mathcal{A}(\text{H})} = \frac{(0.0085)(2.35\text{ g/cm}^3)(6.022 \times 10^{23}\text{ atoms/mol})}{1\text{ g/mol}}$$

$$= \mathbf{1.20 \times 10^{22}\ atoms/cm^3}.$$

27. How much larger in diameter is a uranium nucleus compared to an iron nucleus?

Solution:

From Eq. (1.7) the nuclear diameter is $D = 2R_o A^{1/3}$ so that

$$\frac{D_U}{D_{Fe}} = \left(\frac{A_U}{A_{Fe}}\right)^{1/3} \simeq \left(\frac{238}{56}\right)^{1/3} = 1.62.$$

Thus, $\mathbf{D_U \simeq 1.62\, D_{Fe}}$.

28. By inspecting the chart of the nuclides, determine which element has the most stable isotopes?

Solution:

The element **tin (Sn)** has 10 stable isotopes.

29. Find an internet site where the isotopic abundances of mercury may be found.

Solution: http://www.nndc.bnl.gov

30. The earth has a radius of about 6.35×10^6 m and a mass of 5.98×10^{24} kg. What would be the radius if the earth had the same mass density as matter in a nucleus?

Solution:

From the text, the density of matter in a nucleus is $\rho_n \simeq 2.4 \times 10^{14}$ g/cm^3. The mass of the earth $M = \rho \times V$ where the volume $V = (4/3)\pi R^3$. Combining these results and solving for the radius gives

$$R = \left(\frac{3M}{4\pi\rho}\right)^{1/3} = \left(\frac{3(5.98 \times 10^{27}\text{ g})}{4\pi(2.4 \times 10^{14}\text{ g/cm}^3)}\right)^{1/3} = 1.81 \times 10^4\text{ cm} = \mathbf{181\ m}.$$

Chapter 2

Modern Physics Concepts

PROBLEMS

1. An accelerator increases the kinetic energy of electrons uniformly to 10 GeV over a 3000 m path. That means that at 30 m, 300 m, and 3000 m, the kinetic energy is 10^8, 10^9, and 10^{10} eV, respectively. At each of these distances, compute the velocity, relative to light (v/c), and the mass in atomic mass units.

Solution:

From Eq. (2.10) in the text $T = mc^2 - m_o c^2$ we obtain

$$m = T/c^2 + m_o. \tag{P2.1}$$

From Eq. (2.5) in the text $m = m_o/\sqrt{1 - v^2/c^2}$, which can be solved for v/c to give

$$\frac{v}{c} = \sqrt{1 - \frac{m_o^2}{m^2}} \simeq 1 - \frac{1}{2}\frac{m_o^2}{m^2}, \quad \text{if} \quad \frac{m_o}{m} << 1. \tag{P2.2}$$

(a) For an electron ($m_o = m_e$) with $T = 10^8$ eV $= 100$ MeV, Eq. (P2.1) gives

$$m = \frac{100 \text{ MeV}}{931.5 \text{ MeV/u}} + m_e = 0.1074 \text{ u} + 0.0005486 \text{ u} = \mathbf{0.1079 \text{ u}}.$$

Then $m_e^2/m^2 = (0.0005486/0.1079)^2 = 2.59 \times 10^{-5}$. Finally, from Eq. (P2.2) above, we obtain

$$\frac{v}{c} \simeq 1 - \frac{1}{2}\frac{m_o^2}{m^2} = 1 - 1.29 \times 10^{-5} = \mathbf{0.999987}.$$

(b) For an electron with $T = 10^9$ eV $= 1000$ MeV, we similarly obtain $m = \mathbf{1.0741}$ u and $v/c = \mathbf{0.99999987}$.

(c) For an electron with $T = 10^{10}$ eV $= 10^4$ MeV, we similarly obtain $m = \mathbf{10.736}$ u and $v/c = \mathbf{0.9999999987}$.

Alternative solution: Use Eq. (P2.4) developed in Problem 2-3, namely

$$\frac{v}{c} = \left\{1 - \left[\frac{m_e c^2}{T + m_e c^2}\right]^2\right\}^{1/2}.$$

2. Consider a fast moving particle whose relativistic mass m is 100ϵ percent greater than its rest mass m_o, i.e., $m = m_o(1+\epsilon)$. (a) Show that the particle's speed v, relative to that of light, is

$$\frac{v}{c} = \sqrt{1 - \frac{1}{(1+\epsilon)^2}}.$$

(b) For $v/c << 1$, show that this exact result reduces to $v/c \simeq \sqrt{2\epsilon}$.

Solution:

(a) We are given

$$\frac{m - m_o}{m_o} = \frac{m_o((1+\epsilon)-1)}{m_o} = \epsilon.$$

But we also have

$$\frac{m - m_o}{m_o} = \frac{1}{m_o}\left[\frac{m_o}{\sqrt{1 - v^2/c^2}} - m_o\right].$$

Equating these two results yields

$$\epsilon = \frac{1}{\sqrt{1-v^2/c^2}} - 1.$$

Solving this result for v/c gives

$$\frac{v}{c} = \sqrt{1 - \frac{1}{(1+\epsilon)^2}}. \qquad \text{(P2.3)}$$

(b) For $\epsilon << 1$ we have $(1+\epsilon)^{-2} \simeq 1 - 2\epsilon + \cdots$. Substitution of the approximation into Eq. (P2.3) above gives

$$\frac{v}{c} \simeq \sqrt{1-(1-2\epsilon)} = \sqrt{2\epsilon}.$$

3. In fission reactors one deals with neutrons having kinetic energies as high as 10 MeV. How much error is incurred in computing the speed of 10-MeV neutrons by using the classical expression rather than the relativistic expression for kinetic energy?

Solution:

A neutron with rest mass $m_n = 1.6749288 \times 10^{-27}$ kg has a kinetic energy $T = (10^7 \text{ eV})(1.602177 \times 10^{-19} \text{ J/eV}) = 1.602177 \times 10^{-12}$ J. For the neutron $m_n c^2 = 939.56536$ MeV.

Classically:

$$v_c = \sqrt{2T/m_n} = \left[\frac{2 \times 1.602177 \times 10^{-12}}{1.6749288 \times 10^{-27}}\right]^{1/2} = 4.373993 \times 10^7 \text{ m/s}.$$

Relativistically: From the text we have

$$T = mc^2 - m_oc^2 = \frac{m_oc^2}{\sqrt{1 - v^2/c^2}} - m_oc^2.$$

Solving this equation for v yields the relativistic speed v_r

$$v_r = c\left\{1 - \left[\frac{m_oc^2}{T + m_oc^2}\right]^2\right\}^{1/2}. \qquad \text{(P2.4)}$$

Substitution then gives

$$v_r = c\left\{1 - \left[\frac{939.56536}{10 + 939.56536}\right]^2\right\}^{1/2} = 0.1447459c = 4.339373 \times 10^7 \text{ m/s}.$$

Thus the percent error in the classical speed is $= 100(v_c - v_r)/v_r = \mathbf{0.798\%}$.

4. What speed (m s^{-1}) and kinetic energy (MeV) would a neutron have if its relativistic mass were 10% greater than its rest mass?

Solution:

We are given $(m - m_o)/m_o \equiv \epsilon = 0.1$. From Problem 2-2

$$\frac{v}{c} = \sqrt{1 - \frac{1}{(1+\epsilon)^2}} = \sqrt{1 - \frac{1}{1.1^2}} = 0.4167.$$

Thus the neutron's speed is $v = 0.4167c = \mathbf{1.25 \times 10^8}$ **m/s**.

The kinetic energy can be calculated from

$$T = mc^2 - m_oc^2 = m_oc^2\left[\frac{1}{\sqrt{1 - v^2/c^2}} - 1\right].$$

For $m_oc^2 = 939.6$ MeV and $v/c = 0.4167$ we obtain

$$T = 939.6\left[\frac{1}{\sqrt{1 - 0.4167^2}} - 1\right] = \mathbf{94.0 \text{ MeV}}.$$

5. Show that for a relativistic particle the kinetic energy is given in terms of the particl's momentum by

$$T = \sqrt{p^2c^2 + m_o^2c^4} - m_cc^2.$$

Solution:

Squaring Eq. (2.17) and rearranging the terms one obtains

$$T^2 + 2Tm_oc^2 - p^2c^2 = 0$$

The solution of this quadratic equation gives

$$T = \frac{1}{2}\left\{-2m_oc^2 \pm \sqrt{4m_o^2c^4 + 4p^2c^2}\right\}$$

Only the + sign gives a physically meaningful result. Rearrangement gives the desired realtion.

6. For a relativistic particle show that Eq. (2.21) is valid.

Solution:

From the definition of η one has

$$\eta^2+1 = \frac{P^2}{(m_oc)^2}+1 = \frac{p^2c^2}{(m_oc^2)^2}+1 = \frac{(mc^2)^2-(m_oc^2)^2}{(m_oc^2)^2}+1 = (W^2-1)+1 = W^2.$$

7. Prove the relationships given in (a) Eq. (2.19), (b) Eq. (2.20), and (c) Eq. (2.21).

Solution:

(a) From the definition of η and W one immediately has

$$\beta = \frac{v}{c} = \frac{p}{mc} = \frac{\eta}{W}.$$

(b) Because $W^2 = 1+\eta^2$, then

$$\beta^2 = \left(\frac{v}{c}\right)^2 = \frac{\eta^2}{W^2} = \frac{\eta^2}{1+\eta^2}.$$

(c) Because $\beta = \eta/W$ and $W^2 = 1+\eta^2$, one has

$$\frac{\beta^2}{1-\beta^2} = \frac{\eta^2/W^2}{1-\eta^2/W^2} = \frac{\eta^2/(1+\eta^2)}{1-\eta^2/(1+\eta^2)} = \frac{\eta^2}{(1+\eta^2)-\eta^2} = \eta^2.$$

From this result we see

$$\frac{\beta^2}{1-\beta^2} = \frac{p^2}{m_o^2c^2} = \frac{c^2p^2}{(m_oc^2)^2},$$

but we know $p^2c^2 = T^2 + 2Tm_oc^2$, so

$$\frac{\beta^2}{1-\beta^2} = \frac{T^2+2Tm_oc^2}{(m_oc^2)^2} = \left(\frac{T}{m_oc^2}\right)^2 + \frac{2T}{m_oc^2} = \left(\frac{T}{m_oc^2}\right)^2\left(1+\frac{2m_oc^2}{T}\right).$$

8. In the Relativistic Heavy Ion Collider, nuclei of gold are accelerated to speeds of 99.95% the speed of light. These nuclei are almost spherical when at rest; however, as they move past the experimenters they appear considerably flattened in the direction of motion because of relativistic effects. Calculate the apparent diameter of such a gold nucleus in its direction of motion relative to that perpendicular to the motion.

Solution: The relativistically contracted diameter D to the uncontracted diameter D_o when $v/c = 0.9995$ is

$$D/D_o = \sqrt{1 - v^2/c^2} = \sqrt{1 - 0.9995^2} = \sqrt{1 - (1 - 0.0005)^2}$$
$$\simeq \sqrt{1 - (1 - 2 \times 0.0005)} = \sqrt{0.001} = 0.031.$$

Hence the gold nucleus appears to flatten to **3.1%** of its at-rest width.

9. Muons are subatomic particles that have the negative charge of an electron but are 206.77 times more massive. They are produced high in the atmosphere by cosmic rays colliding with nuclei of oxygen or nitrogen, and muons are the dominant cosmic-ray contribution to background radiation at the earth's surface. A muon, however, rapidly decays into an energetic electron, existing, from its point of view, for only 2.20 μs, on the average. Cosmic-ray generated muons typically have speeds of about $0.998c$ and thus should travel only a few hundred meters in air before decaying. Yet muons travel through several kilometers of air to reach the earth's surface. Using the results of special relativity explain how this is possible. HINT: consider the atmospheric travel distance as it appears to a muon, and the muon lifetime as it appears to an observer on the earth's surface.

Solution:

Muon's Point of View: A muon, with a lifetime $t_o = 2.20 \times 10^{-6}$ s and traveling with a speed $v = 0.998c$, travels on the average a distance $d = vt_o = 0.998(3.00 \times 10^8 \text{ m/s})(2.29 \times 10^{-6} \text{ s}) = 660$ m.

If the muon is created at an altitude L_o, from the muon's point of view the distance to the surface (approaching with speed $v = 0.998c$) is relativistically narrowed or contracted to a distance

$$L = L_o\sqrt{1 - v^2/c^2} = L_o\sqrt{1 - 0.998^2} = 0.063L_o.$$

For example, if $L_o = 10$ km, $L = 630$ m, so that, on the average, almost half of the muons will reach the surface.

Surface Observer's Point of View: An observer on the earth's surface observes the muon approaching at a speed $v = 0.998c$ and the muon's lifetime appears to expand (the muon's internal clock appears to slow) as

$$t = \frac{t_o}{\sqrt{1 - v^2/c^2}} = \frac{t_o}{\sqrt{1 - 0.998^2}} = 15.9t_o = 3.49 \times 10^{-5} \text{ s}.$$

In such a lifetime, the muon can travel $d = 0.998c \times t = 10,500$ m so that it can reach the surface from an altitude of 10 km before decaying.

10. A 1-MeV gamma ray loses 200 keV in a Compton scatter. Calculate the scattering angle.

Solution:

From Eq. (2.26) in the text we find

$$1 - \cos\theta_s = m_e c^2 \left[\frac{1}{E'} - \frac{1}{E}\right]$$

or

$$\cos\theta_s = 1 - m_e c^2 \left[\frac{1}{E'} - \frac{1}{E}\right].$$

Here $m_e c^2 = 0.511$ MeV, $E' = 0.8$ MeV, and $E = 1$ MeV so that

$$\cos\theta_s = 1 - 0.511 \left[\frac{1}{0.8} - \frac{1}{1}\right] = 0.87225.$$

Thus the scattering angle $\theta_s = \cos^{-1}(0.87225) = 29.3^{\mathbf{o}}$

11. At what energy (in MeV) can a photon lose at most one-half of its energy in Compton scattering?

Solution:

Eq. (2.26) in the text gives the basic Compton scattering relation:

$$\frac{1}{E'} - \frac{1}{E} = \frac{1}{m_e c^2}(1 - \cos\theta_s).$$

By inspection, the maximum energy loss (the smallest E') occurs when $\theta_s = \pi$. Here we are told $E' = E/2$

$$\frac{2}{E} - \frac{1}{E} = \frac{1}{E} = \frac{2}{m_e c^2} = \frac{2}{0.511 \text{ MeV}}.$$

From this result, we find $E =$ **0.255 MeV**. *Above* this incident photon energy, the minimum scattered photon energy is less than one-half of the initial energy.

12. Derive for the Compton scattering process the recoil electron energy T as a function of the incident photon energy E and the electron angle of scattering ϕ_e. Show that ϕ_e is never greater than $\pi/2$ radians.

Solution:

Application of the law of cosines to the triangle in text Fig. 2.5 leads to

$${p_{\lambda'}}^2 = {p_\lambda}^2 + {p_e}^2 - 2 p_\lambda p_e \cos\phi_e.$$

Substitute E/c for p_λ, $(E - T)/c$ for $p_{\lambda'}$, and $(1/c)\sqrt{T^2 + 2T m_e c^2}$ for p_e. Then solve for T, with the result

$$T = \frac{2 m_e c^2 E^2 \cos^2\phi_e}{(E + m_e c^2)^2 - E^2 \cos^2\phi_e}.$$

Examination of the triangle in Fig. 2.5 reveals that, since $p_{\lambda'} \le p_\lambda$, $0 \le \phi_e \le \pi/2$, confirming the commonsense observation that the target electron, initially at rest, can recoil only in the forward hemisphere.

13. A 1 MeV photon is Compton scattered at an angle of 55 degrees. Calculate (a) the energy of the scattered photon, (b) the change in wavelength, and (c) the recoil energy of the electron.

Solution:

(a) From Eq. (2.26)

$$\frac{1}{E'} = \frac{1}{E} + \frac{1-\cos\theta_s}{m_e c^2} = \frac{1}{1\ \text{MeV}} + \frac{1-\cos 55}{0.511\ \text{MeV}} = 1.835\ \text{MeV}^{-1}.$$

Thus the scattered photon energy is $E' = 1/1.835 = \mathbf{0.545\ MeV}$.

(b) From Eq. (2.25) we have

$$\begin{aligned}\Delta\lambda = \lambda' - \lambda &= \frac{h}{m_e c}(1-\cos\theta_s) = \frac{hc}{m_e c^2}(1-\cos\theta_s)\\ &= \frac{(4.135\times10^{-21}\ \text{MeV s})(3.00\times10^8\ \text{m/s})}{0.511\ \text{MeV}}(1-\cos 55)\\ &= \mathbf{1.04\times10^{-12}\ m}.\end{aligned}$$

(c) The kinetic energy of the recoil electron is $E_r = E - E' = 1 - 0.545 =$ $\mathbf{0.455\ MeV}$.

14. When light with wavelengths > 475 nm $= \lambda_{\text{max}}$ impinges on of a certain metalic surface photoelectrons are observed to be emitted. What is the work functiion of this metal in eV?

Solution:

The frequency of light corresponding the the maximum wavelgth is $\nu_{\text{min}} = c/\lambda_{\text{max}} = (2.998\times10^8\text{m s}^{-1}/(475\times10^{-9}\ \text{m}) = 6.31\times10^{14}\ \text{s}^{-1}$. From Example 2.3, the work function is $A = h\nu_{\text{min}} = (4.136\times10^{-15}\ \text{eV s})(6.31\times10^{14}\ \text{s}^{-1}) =$ $\mathbf{2.61\ eV}$.

15. Consider the experimental arrangement shown in Fig. 2.3. The surface of a sodium sample was illuminated by monochromatic light of various wavelengths, and the retarding potentials required to stop the collection of the photoelectrons were observed. The results are shown below.

wavelemgth (nm)	253.6	283.0	303.9	330.2	366.3	435.8
retarding potential (V)	2.60	2.11	1.81	1.47	1.10	0.57

Present these data graphically to verify the photoelectric equation $eV_o = h\nu - A$. From the graph estimate the value of Planck's constant h and the work function A for sodium.

Solution:

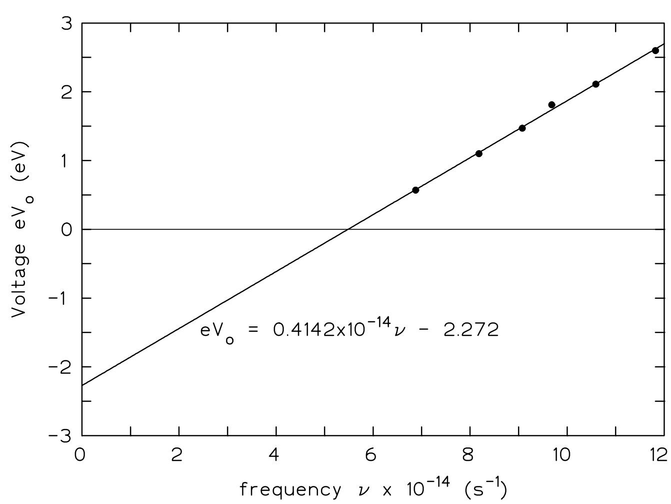
The frequency of the light is related to the wavelength by

$$[\nu = \frac{c}{\lambda} = \frac{2.997 \times 10^{17}}{\lambda \text{ (nm)}} \text{ s}^{-1}.$$

Then plot the following data:

eV_o (eV)	2.60	2.11	1.81	1.47	1.10	0.57
$\nu \times 10^{-14}$	11.82	10.59	9.682	9.076	8.182	6.877

Fit a straight line to the plotted data as shown below.



From the least-squares fit it is found that $h = \mathbf{4.142 \times 10^{-15}}$ **eV s** and that the work function for sodium is $A = \mathbf{2.271}$ **eV**.

16. Consider the electron scattering experiment of Davisson and Germer described in Section 2.2.4. For the nickel crystal they used the interatomic spacing was $d = 2.15$ Å $= 2.15 \times 10^{-10}$ m. (a) For an incident electrons with an arbitrary energy of T eV, show that the constructive interference peaks occur at angles

$$\theta = \sin^{-1}\left(\frac{n\lambda}{d}\right) = \sin^{-1}\left(\frac{5.705n}{\sqrt{T \text{ eV}}}\right), \qquad n = 1, 2, 3, \ldots.$$

(b) What are the angles of the peaks when $T = 54$ eV (as used by Davisson and Germer) and when $T = 300$ eV?

Solution:

(a) From Eq. (2.30) for non-relativistic electrons $\lambda = h/\sqrt{2m_eT}$. Recall the rest mass of the electron is $m_e/c^2 = 5.11 \times 10^6$ eV. Substitution of of these values gives

$$\begin{aligned}\theta &= \sin^{-1}\left(\frac{nhc}{d\sqrt{2m_eT}}\right)\\ &= \sin^{-1}\left(\frac{n(4.136 \times 10^{-15} \text{ eV s})(2.998 \times 10^8 \text{ m s}^{-1})}{(2.15 \times 10^{-10} \text{ m})\sqrt{(2 \times 0.555 \times 10^6 \text{ eV})(T \text{ eV})}}\right)\\ &= \sin^{-1}\left(\frac{5.705n}{\sqrt{T \text{ eV}}}\right). \qquad \text{(P2.5)}\end{aligned}$$

(b) For $T = 54$ eV the only angle is $\theta = 50.9°$ $(n = 1)$. For $T = 300$ eV the angles are $\theta = 19.2°$ $(n = 1)$, $41.2°$ $(n = 2)$, and$81.2°$ $(n = 3)$.

17. Show that the de Broglie wavelength of a particle with kinetic energy T can be written as

$$\lambda = \frac{h}{\sqrt{m_o}}\frac{1}{\sqrt{T}}\left[1 + \frac{m}{m_o}\right]^{-1/2}$$

where m_o is the particles's rest mass and m is its relativistic mass.

Solution: From Eq. (2.17)

$$p = \frac{1}{c}\sqrt{T^2 + 2Tm_oc^2} = \frac{\sqrt{T}}{c}\sqrt{T + 2m_oc^2}.$$

But $T = mc^2 - m_oc^2$ so the above result can be written as

$$p = \frac{\sqrt{T}}{c}\sqrt{mc^2 + m_oc^2} = \sqrt{T}\sqrt{m_o}\sqrt{1 + (m/m_o)}.$$

Finally, use of the de Broglie relation $\lambda = h/p$ in the above result gives

$$\lambda = \frac{h}{\sqrt{m_o}}\frac{1}{\sqrt{T}}\left[1 + \frac{m}{m_o}\right]^{-1/2}.$$

18. Apply the result of the previous problem to an electron. (a) Show that when the electron's kinetic energy is expressed in units of eV, its de Broglie wavelength can be written as

$$\lambda = \frac{17.35 \times 10^{-8}}{\sqrt{T}} \left[1 + \frac{m}{m_o}\right]^{-1/2} \text{ cm.}$$

(b) For non-relativistic electrons, i.e., $m \simeq m_o$, show that this result reduces to

$$\lambda = \frac{12.27 \times 10^{-8}}{\sqrt{T}} \text{ cm.}$$

(c) For very relativistic electrons, i.e., $m >> m_o$, show that the de Broglie wavelength is given by

$$\lambda = \frac{17.35 \times 10^{-8}}{\sqrt{T}} \sqrt{\frac{m_o}{m}} \text{ cm.}$$

Solution:

(a) Rewrite the result of Problem 2-10 as

$$\lambda = \frac{hc}{\sqrt{m_o c^2}} \frac{1}{\sqrt{T}} \left[1 + \frac{m}{m_o}\right]^{-1/2}.$$

Substitute for the constants and use $m_o = m_e = 0.511$ MeV/c^2 to obtain

$$\lambda = \frac{(4.1357 \times 10^{-15} \text{ eV s})(2.998 \times 10^{10} \text{ cm/s})}{\sqrt{0.5110 \times 10^6 \text{ eV}}} \frac{(1 + m/m_o)^{-1/2}}{\sqrt{T \text{ (eV)}}}$$

$$= \frac{17.35 \times 10^{-8}}{\sqrt{T \text{ (eV)}}} \left[1 + \frac{m}{m_o}\right]^{-1/2} \text{ cm.} \qquad \text{(P2.6)}$$

(b) For non-relativistic electrons $m \simeq m_o$, so that $1/\sqrt{1 + (m/m_o)} \simeq 1/\sqrt{2}$, and the above result becomes

$$\lambda = \frac{12.27 \times 10^{-8}}{\sqrt{T \text{ (eV)}}} \text{ cm.}$$

(c) For very relativistic particles, $m >> m_o$ so that $1/\sqrt{1 + (m/m_o)} \simeq \sqrt{m_o/m}$. Eq. (2.4) above then becomes

$$\lambda = \frac{17.35 \times \sqrt{m_o/m}}{\sqrt{T \text{ (eV)}}} \times 10^{-8} \text{ cm.}$$

19. What are the wavelengths of electrons with kinetic energies of (a) 10 eV, (b) 1000 eV, and (c) 10^7 eV?

Solution: From Eq. (2.17) $p = (1/c)\sqrt{T^2 + 2Tm_oc^2}$ and using the de Broglie relation $\lambda = h/p$ we obtain the de Broglie wavelength as

$$\lambda = \frac{hc}{\sqrt{T^2 + 2Tm_oc^2}}. \tag{P2.7}$$

Now apply this equation to the three electron energies.

(a) Substitute $m_oc^2 = m_ec^2 = 0.5110$ MeV and $T = 10$ eV into Eq. (P2.6) to obtain

$$\lambda = \frac{(4.135 \times 10^{-15} \text{ eV s})(2.998 \times 10^8 \text{ m/s})}{\sqrt{10^2 + 2(10)(0.5110 \times 10^6)} \text{ eV}} = \mathbf{3.88 \times 10^{-10} \text{ m}}.$$

(b) similarly, for $T = 10^3$ eV we find

$$\lambda = \frac{(4.135 \times 10^{-15} \text{ eV s})(2.998 \times 10^8 \text{ m/s})}{\sqrt{10^6 + 2(10^3)(0.5110 \times 10^6)} \text{ eV}} = \mathbf{3.87 \times 10^{-11} \text{ m}}.$$

(c) similarly, for $T = 10^7$ eV we find

$$\lambda = \frac{(4.135 \times 10^{-15} \text{ eV s})(2.998 \times 10^8 \text{ m/s})}{\sqrt{10^{14} + 2(10^7)(0.5110 \times 10^6)} \text{ eV}} = \mathbf{1.18 \times 10^{-13} \text{ m}}.$$

20. Low energy neutrons are often referred to by their de Broglie wavelength as measured in angstoms (Å) with 1 Å= 1×10^{-10} m. (a) Derive a formula that gives the kinetic energy of such a neutron in terms of its de Broglie wavelength. (b) What is the energy of a neutron (in eV) of a 6-Å neutron.

Solution:

(a) Equation (2.30) for a non-relativistic particle reduces to

$$\lambda = h/\sqrt{2m_oT},$$

which, upon solving to T gives

$$T = \frac{h^2}{2\lambda^2 m_o}.$$

(b) Here $\lambda = 6 \times 10^{-10}$ m and $m_o/c^2 = 931.49 \times 10^6$ eV, so

$$T = \frac{(4.135 \times 10^{-15} \text{ eV s})^2}{(2)(6 \times 10^{-10}\text{m}})^2(931.49 \times 10^6 \text{ eV})/(2.998 \times 10^8 \text{ m s}^{-1})^2$$
$$= \mathbf{0.00229 \text{ eV}}.$$

21. What is the de Broglie wavelength of a water molecule moving at a speed of 2400 m/s? What is the wavelength of a 3-g bullet moving at 400 m/s?

Solution:

(a) A water molecule (H_2O) has a rest mass of about $m = (18 \text{ u})(1.661 \times 10^{-27} \text{ kg/u}) = 2.989 \times 10^{-26}$ kg.

Its momentum when traveling at 2400 m/s is $p = mv = (2.989 \times 10^{-26} \text{ kg}) \times (2400 \text{ m/s}) = 7.18 \times 10^{-23} \text{ kg m s}^{-1} = 7.18 \times 10^{-23} \text{ J s m}^{-1}$.

Thus the de Broglie wavelength of the water molecule is

$$\lambda = \frac{h}{p} = \frac{6.626 \times 10^{-34} \text{ J s}}{7.18 \times 10^{-23} \text{ J s m}^{-1}} = \mathbf{9.23 \times 10^{-12} \text{ m}}.$$

(b) A 3-g bullet moving at 400 m/s has a momentum $p = mv = (0.003 \text{ kg}) \times (400 \text{ m/s}) = 1.2 \text{ kg m s}^{-1} = 1.2 \text{ J s m}^{-1}$. Its de Broglie wavelength is thus

$$\lambda = \frac{h}{p} = \frac{6.626 \times 10^{-34} \text{ J s}}{1.2 \text{ J s m}^{-1}} = \mathbf{5.53 \times 10^{-34} \text{ m}}.$$

22. If a neutron is confined somewhere inside a nucleus of characteristic dimension $\Delta x \simeq 10^{-14}$ m, what is the uncertainty in its momentum Δp? For a neutron with momentum equal to Δp, what is its total energy and its kinetic energy in MeV? Verify that classical expressions for momentum and kinetic energy may be used.

Solution:

From the uncertainty principle, $\Delta p \Delta x \gtrsim h/(2\pi)$ so that for $\Delta x \simeq 10^{-14}$ m

$$\Delta p = \frac{h}{2\pi \Delta x} = \frac{6.626 \times 10^{-34} \text{ J s}}{2\pi \times 10^{-14} \text{ m}} = 1.05 \times 10^{-20} \text{ J s m}^{-1}.$$

A non-relativistic (classical) particle has kinetic energy $T = (1/2)mv^2 = p^2/(2m)$. For a neutron with $p \simeq \Delta p = 1.05 \times 10^{-20}$ J s m^{-1}

$$T = \frac{(\Delta p)^2}{2m_n} = \frac{(1.05 \times 10^{-20} \text{ J s m}^{-1})^2}{2(1.6749 \times 10^{-27} \text{ kg})} = 3.32 \times 10^{-14} \text{ J}$$

$$= \frac{3.32 \times 10^{-14} \text{ J}}{1.602 \times 10^{-13} \text{ J/MeV}} = \mathbf{0.208 \text{ MeV}}.$$

This energy is well below the energy at which a neutron becomes relativistic, and hence justifies the use of classical mechanics.

The neutron's total energy is thus $E = T + m_n c^2 = 0.207 \text{ MeV} + 939 \text{ MeV} \simeq m_n c^2$.

23. Repeat the previous problem for an electron trapped in the nucleus. HINT: relativistic expressions for momentum and kinetic energy must be used.

Solution:

From the uncertainty principle, $\Delta p \Delta x \gtrsim h/(2\pi)$ so that for $\Delta x \simeq 10^{-14}$ m

$$\Delta p = \frac{h}{2\pi \Delta x} = \frac{6.626 \times 10^{-34} \text{ J s}}{2\pi \times 10^{-14} \text{ m}} = 1.05 \times 10^{-20} \text{ J s m}^{-1}.$$

For an electron with $p \simeq \Delta p = 1.05 \times 10^{-20}$ J s m^{-1}

$$\begin{aligned} p^2c^2 &= (1.05 \times 10^{-20} \text{ J s m}^{-1})^2 (3.00 \times 10^8 \text{ m/s})^2 \\ &= (3.15 \times 10^{-12} \text{ J})^2 = (19.7 \text{ MeV})^2. \end{aligned}$$

From the equation above Eq. (2.16) in the text, we see that $p^2c^2 = (mc^2)^2 - (m_oc^2)^2 = E^2 - (m_oc^2)^2$. We use this relation to find the electron's total energy E as

$$E = \sqrt{p^2c^2 + (m_ec^2)^2} = \sqrt{19.7^2 + 0.511^2} \text{ MeV} \simeq 20 \text{ MeV}.$$

Since the electron's total energy E is related to the kinetic energy T by $E = T + m_ec^2 = T + 0.511$ MeV, in this problem the total energy is essentially the electron's kinetic energy, i.e., $E \simeq T$.

24. The wavefunction for the electron in a hydrogen atom in its ground state (the 1s state for which $n = 0,\ \ell = 0,$ and $m = 0$ is spherically symmetric as shown in Fig. 2.14. For this state the wavefuntion is real and is given by

$$\psi_0(r) = \frac{1}{\sqrt{\pi a_0^3}} \exp[-r/a_0],$$

where $a_o = h^2\epsilon_o/(4\pi^2 m_e e^2) \simeq 5.29 \times 10^{-11}$ m. This quantity is the radius of the first Bohr orbit for hydrogen (see next chapter). Because of the spherical symmetry of ψ_o, dV in Eq. (2.40) is $dV = 4\pi r^2\, dr$ and the integral in Eq. (2.40) can be written as

$$\int_0^\infty \psi_0(r)\psi_0^*(r) 4\pi dr = \frac{4}{a_0^3} \int_0^\infty r^2 e^{-\alpha r} dr,$$

where $\alpha \equiv 2/a_0$. (a) Verify that the required normalization required by Eq. (2.40) is satisfied, i.e., the electron is somewhere in the space around the proton. (b) What is the probability the electron is found a radial distance $r < a_0$ from the proton?

Solution:

(a) Integration by parts twice gives

$$\frac{4}{a_0^3} \int_0^\infty r^2 e^{-\alpha r} dr = \frac{4}{a_0^3} \frac{2}{\alpha^3} = \frac{4}{a_0^3} \frac{a_0^3}{4} = 1.$$

(b) Replace upper limit in the above itegral by a_0. Then integration by parts twice gives

$$
\begin{aligned}
\text{Prob}\{\text{electron is inside } r \le a_0\} &= \frac{4}{a_0^3}\int_0^{a_0} r^2 e^{-\alpha r}\,dr \\
&= 1 - \frac{4}{a_0^3}e^{-\alpha a_0}\left\{\frac{a_0^2}{\alpha} + \frac{2a_0}{\alpha^2} + \frac{2}{\alpha^3}\right\} \\
&= 1 - \frac{4}{a_0^3}e^{-2}\left\{\frac{a_0^3}{2} + \frac{2a_0^3}{4} + \frac{2a_0^3}{8}\right\} \\
&= 1 - 5e^{-2} = \mathbf{0.323}.
\end{aligned}
$$

Thus the electron has a 32.3% of being at a radial distance less that a_0.