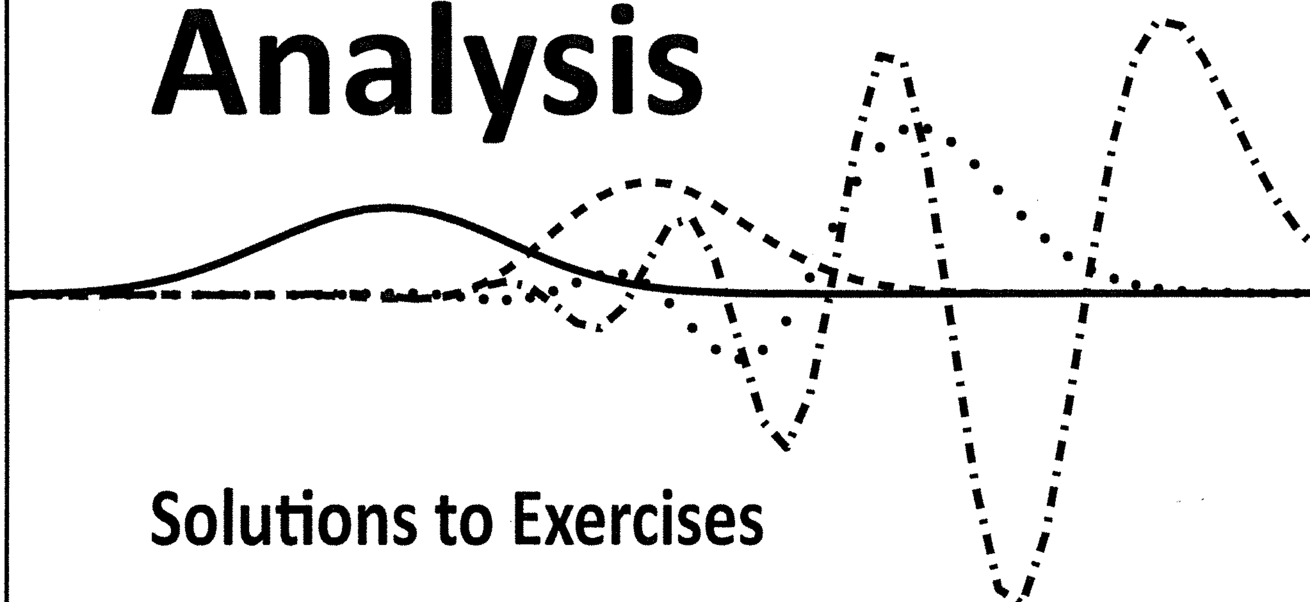
SECOND EDITION

FUNDAMENTALS OF

Engineering Numerical Analysis



Solutions to Exercises

Parviz Moin

CAMBRIDGE

Instructor's manual for

FUNDAMENTALS OF

ENGINEERING NUMERICAL ANALYSIS

PARVIZ MOIN

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Preface

This instructor's guide contains complete solutions to all the exercises in "Fundamentals of Engineering Numerical Analysis" by P. Moin. Virtually all exercises involve computations using a personal computer and the results are presented graphically. It is essential for students to have access to a plotting package (such as that available in MATLAB) to readily display and digest the numerical output. Most exercises are intended to reinforce the concepts introduced in the text, others involve numerical solution of non-trivial physical problems which are intended to remind the students of the usefulness and power of numerical methods in solving interesting problems. These latter problems are solved rather easily using the numerical packages recommended, and are great motivators for engineering and science students for studying the subject. The computer programs used to solve the computational problems are provided on an accompanying disk for the instructor's convenience.

The computer programs are written in MATLAB and Fortran. The specific files for each problem can be found in the directory 'exercises' while the common files in 'library'. Problems with more than one file have these in a directory named with the letter 'e' followed by the chapter number, underscore, and then the problem number. The names of the files to be executed (MATLAB) or compiled (Fortran) begin with the label described above. Executable files containing the word 'plot' should be ran at last. Sometimes files are generated during the execution procedure. Their names begin with the word 'data'.

Before executing any MATLAB program, the files in the directory 'library' must be added to the MATLAB's path. The files needed for each Fortran program are listed inside of it.

The MATLAB version used was 5.3.1.29215a, October 1999, and the Fortran compiler was MIPSpro 7 Fortran 90, on an IRIX 6.5 Silicon Graphics machine. The Numerical Recipes routines are from Press et al, *Numerical Recipes in Fortran 77*, Second Edition, Cambridge University Press, software version 2.04.

Contents

Chapter 1

INTERPOLATION

1. (a) Using `polint`, the interpolated value is 1.577.

   (b) See Fig. 1.1. Comparing to Example 1.1, the current interpolation is better around the center but much worse near the end points.

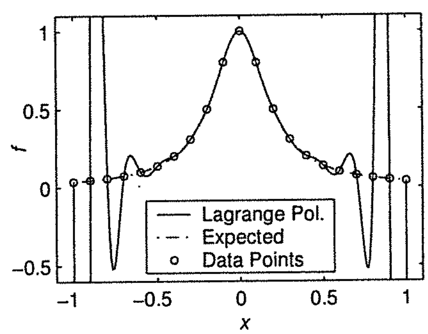

Figure 1.1: Exercise 1.

2. Differentiating $P(x) = \sum_{j=0}^{n} y_j \alpha_j \prod_{\substack{i=0 \\ i \neq j}}^{n} (x - x_i)$ gives

$$P'(x) = \sum_{j=0}^{n} y_j \alpha_j \frac{d}{dx} \prod_{\substack{i=0 \\ i \neq j}}^{n} (x - x_i) = \sum_{j=0}^{n} y_j \alpha_j \Big[\sum_{\substack{k=0 \\ k \neq j}}^{n} \prod_{\substack{i=0 \\ i \neq k,j}}^{n} (x - x_i) \Big].$$

3. When $g''(x_i) = g''(x_{i+1})$, the x^3 terms in (1.6) cancel out and $g_i(x)$ becomes a parabola:

$$\begin{aligned} g_i(x) &= \frac{g''(x_i)}{6} \Big[3x^2 - 3x(x_i + x_{i+1}) + 3x_i x_{i+1} \Big] + \\ & \quad f(x_i) \frac{x_{i+1} - x}{\Delta_i} + f(x_{i+1}) \frac{x - x_i}{\Delta_i}. \end{aligned}$$

4. (a) Continuity of the first derivative.

(b) For $x_i \le x \le x_{i+1}$:

$$g_i'(x) = g'(x_i)\frac{x - x_{i+1}}{x_i - x_{i+1}} + g'(x_{i+1})\frac{x - x_i}{x_{i+1} - x_i}.$$

Integrating and substituting $g_i(x_i) = f(x_i)$ and $g_i(x_{i+1}) = f(x_{i+1})$, we obtain

$$g'(x_i) + g'(x_{i+1}) = 2\frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}, \quad i = 0, \ldots, N-1$$

These are N equations for the $N+1$ unknowns $g'(x_0), \ldots, g'(x_N)$. One additional equation is required and it can be $g'(x_0) = g'(x_1)$, which means that the interpolant in the first interval is a straight line.

(c) For non-periodic equally-spaced data, the solution of (1.7) requires $O(2N)$ divisions and $O(3N)$ of each additions and multiplications, ignoring the effort in computing the right-hand side. Solving the system in (b) is only $O(N)$ additions.

5. Solve first for $g''(x_0), \ldots, g''(x_N)$ as explained in the text and then differentiate (1.6) to get the first derivative at the data points.

For $x_0 \le x_i \le x_{N-1}$:

$$g'(x_i) = g_i'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} - g''(x_i)\frac{h}{3} - g''(x_{i+1})\frac{h}{6}.$$

For x_N:

$$g'(x_N) = g_{N-1}'(x_N) = \frac{f(x_N) - f(x_{N-1})}{h} + g''(x_{N-1})\frac{h}{6} + g''(x_N)\frac{h}{3}.$$

6. (a) For $\sigma = 0$, (1.3) is recovered. For $\sigma \to \infty$ we obtain

$$g_i(x) = f(x_i)\frac{x - x_{i+1}}{x_i - x_{i+1}} + f(x_{i+1})\frac{x - x_i}{x_{i+1} - x_i},$$

which is a straight line.

(b) The given differential equation for g_i is second order, linear, and non-homogeneous. Its solution is:

$$g_i(x) = C_1 e^{\sigma x} + C_2 e^{-\sigma x} - \frac{g''(x_i) - \sigma^2 f(x_i)}{\sigma^2}\frac{x - x_{i+1}}{x_i - x_{i+1}}$$
$$- \frac{g''(x_{i+1}) - \sigma^2 f(x_{i+1})}{\sigma^2}\frac{x - x_i}{x_{i+1} - x_i}.$$

Differentiating:

$$g_i'(x) = C_1\sigma e^{\sigma x} - C_2\sigma e^{-\sigma x} + \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} - \frac{1}{\sigma^2}\frac{g''(x_{i+1}) - g''(x_i)}{x_{i+1} - x_i}.$$

C_1, C_2, and the second derivatives at the data points are determined as in Section 1.2 with (1.4) and (1.5) replaced by the two equations above.

7. (b,c) `polint`, `spline`, and `splint` are used to obtain the interpolations in Fig. 1.2. The predicted tuition in 2001 is \$10,836 using Lagrange polynomial and \$34,447 using cubic spline. The Lagrange polynomial does a pretty good job interpolating the data but behaves very poorly away from it; the predicted tuition is way too low. The cubic spline behaves well for both interpolation and extrapolation.

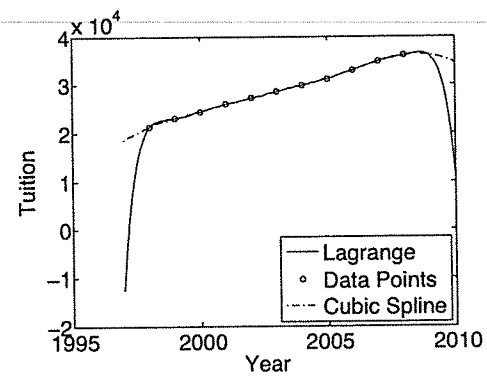

Figure 1.2: Exercise 7.

8. (a) Using `polint`, the interpolation is shown in Fig 1.3. The prediction in 2009 is −38.40 which is unrealistic.

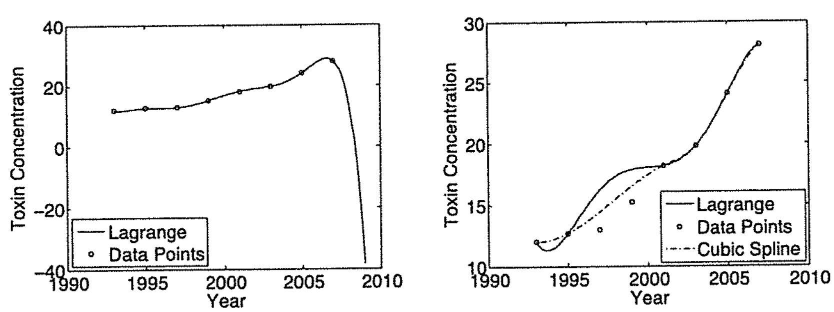

Figure 1.3: Exercise 8.

(b,c) Results are shown in Fig. 1.3. The predicted values are

	Lagrange	Spline
1997	16.23	14.44
1999	17.88	16.52

The predictions using the cubic spline are better.

9. The second order Lagrange polynomial passing through x_{i-1}, x_i, and x_{i+1} is

$$P(x) = \frac{(x-x_i)(x-x_{i+1})}{(x_{i-1}-x_i)(x_{i-1}-x_{i+1})}y_{i-1} + \frac{(x-x_{i-1})(x-x_{i+1})}{(x_i-x_{i-1})(x_i-x_{i+1})}y_i + \frac{(x-x_{i-1})(x-x_i)}{(x_{i+1}-x_{i-1})(x_{i+1}-x_i)}y_{i+1}.$$

Differentiating and evaluating at $x = x_i$, we obtain:

$$P'(x_i) = \frac{(x_i-x_{i+1})y_{i-1}}{(x_{i-1}-x_i)(x_{i-1}-x_{i+1})} + \frac{(x_i-x_{i-1})+(x_i-x_{i+1})}{(x_i-x_{i-1})(x_i-x_{i+1})}y_i + \frac{(x_i-x_{i-1})y_{i+1}}{(x_{i+1}-x_{i-1})(x_{i+1}-x_i)}$$

$$P''(x_i) = \frac{2y_{i-1}}{(x_{i-1}-x_i)(x_{i-1}-x_{i+1})} + \frac{2y_i}{(x_i-x_{i-1})(x_i-x_{i+1})} + \frac{2y_{i+1}}{(x_{i+1}-x_{i-1})(x_{i+1}-x_i)}.$$

For uniformly spaced data, these reduce to:

$$P'(x_i) = \frac{y_{i+1}-y_{i-1}}{2\Delta} \quad \text{and} \quad P''(x_i) = \frac{y_{i+1}-2y_i+y_{i-1}}{\Delta^2}.$$

10. Let $\mathbf{v}$ be the vector whose points are the values of the polynomial $L_k(x)$ at the grid points $x_0, \ldots, x_N$, i.e. $v_i = L_k(x_i) = \delta_{ik}$. The derivative of $L_k(x)$ at x_j is $\left.\frac{d}{dx}L_k(x)\right|_{x=x_j} = L_k'(x_j)$ which is also given by

$$(D\mathbf{v})_j = \sum_{l=0}^{N} d_{jl}v_l = \sum_{l=0}^{N} d_{jl}\delta_{lk} = d_{jk}.$$

Thus $d_{jk} = L_k'(x_j)$. Now, taking the logarithm of $L_k(x) = \alpha_k \prod_{\substack{i=0 \\ i\neq k}}^{N}(x-x_i)$ and differentiating gives

$$\log L_k(x) = \log\alpha_k + \sum_{\substack{i=0 \\ i\neq k}}^{N}\log(x-x_i) \quad \text{and} \quad \frac{L_k'(x)}{L_k(x)} = \sum_{\substack{i=0 \\ i\neq k}}^{N}\frac{1}{x-x_i}.$$

Evaluating the last expression at $x = x_k$ gives (3):

$$L_k'(x_k) = d_{kk} = \sum_{\substack{i=0 \\ i\neq k}}^{N}\frac{1}{x_k-x_i}.$$

The same expression cannot be evaluated at $x \neq x_k$ since the denominator will be zero. We proceed further as follows:

$$L'_k(x) = L_k(x) \sum_{\substack{i=0 \\ i\neq k}}^{N} \frac{1}{x - x_i} = \alpha_k \prod_{\substack{l=0 \\ l\neq k}}^{N} (x - x_l) \sum_{\substack{i=0 \\ i\neq k}}^{N} \frac{1}{x - x_i} = \alpha_k \sum_{\substack{i=0 \\ i\neq k}}^{N} \prod_{\substack{l=0 \\ l\neq i,k}}^{N} (x - x_l).$$

This gives

$$L'_k(x_j) = \alpha_k \sum_{\substack{i=0 \\ i\neq k}}^{N} \prod_{\substack{l=0 \\ l\neq i,k}}^{N} (x_j - x_l).$$

The product is non zero only when $i = j$. Thus:

$$L'_k(x_j) = d_{jk} = \alpha_k \prod_{\substack{l=0 \\ l\neq j,k}}^{N} (x_j - x_l) = \frac{\alpha_k}{x_j - x_k} \prod_{\substack{l=0 \\ l\neq j}}^{N} (x_j - x_l) = \frac{\alpha_k}{\alpha_j (x_j - x_k)}.$$

11. (a) Looking at the contour plot (figure 1.4) we can estimate the value of $f(1.5, 1.5)$ to be 2.7.

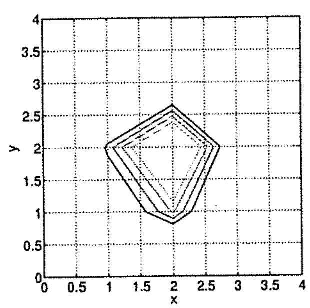

Figure 1.4: Contour plot on course data; from dark to light: $f = 2.4, 2.6, 2.8, 3.0$.

(b) Using equation (1.7) in the text, the following linear system should be solved for the second derivative.

$$\begin{pmatrix} 2/3 & 1/6 & 0 & 1/6 \\ 1/6 & 2/3 & 1/6 & 0 \\ 0 & 1/6 & 2/3 & 1/6 \\ 1/6 & 0 & 1/6 & 2/3 \end{pmatrix} \begin{pmatrix} g_{xx}(0,i) \\ g_{xx}(1,i) \\ g_{xx}(2,i) \\ g_{xx}(3,i) \end{pmatrix} = \begin{pmatrix} f(3,i) - 2f(0,i) + f(1,i) \\ f(2,i) - 2f(1,i) + f(0,i) \\ f(3,i) - 2f(2,i) + f(1,i) \\ f(0,i) - 2f(3,i) + f(2,i) \end{pmatrix}$$

For example, for $i = 0$ the solution to this system is

$g_{xx}(0,0) = 0.8466, \quad g_{xx}(1,0) = -0.0233, \quad g_{xx}(2,0) = -0.8460, \quad g_{xx}(3,0) = 0.0226,$

and from equation (1.6) in the text, $g(x, 0)$ for $1 \leq x \leq 2$ will be:

$$g(x,0)|_{1\leq x\leq 2} = \frac{g_{xx}(1,0)}{6}\left[(2-x)^3 - (2-x)\right] + \frac{g_{xx}(2,0)}{6}\left[(x-1)^3 - (x-1)\right] + g(1,0)\,(2-x) + g(2,0)\,(x-1)\,.$$

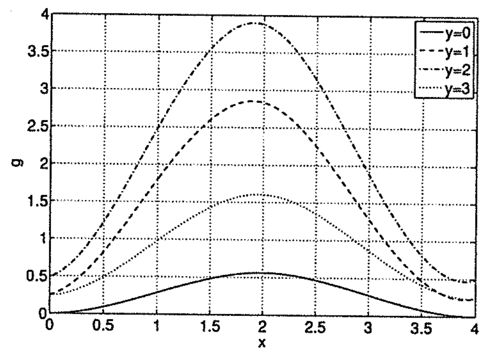

Figure 1.5: $g(x, i)$ for $i = 1, 2, 3, 4$.

The same procedure can be repeated for other intervals.

(c) From solution of part (b) we obtain:

$$g(1.5, 0) = 0.4819, \quad g(1.5, 1) = 2.6082, \quad g(1.5, 2) = 3.5588, \quad g(1.5, 3) = 1.4326.$$

The following system has to be solved for g_{yy} values.

$$\begin{pmatrix} 2/3 & 1/6 & 0 & 1/6 \\ 1/6 & 2/3 & 1/6 & 0 \\ 0 & 1/6 & 2/3 & 1/6 \\ 1/6 & 0 & 1/6 & 2/3 \end{pmatrix} \begin{pmatrix} g_{yy}(1.5,0) \\ g_{yy}(1.5,1) \\ g_{yy}(1.5,2) \\ g_{yy}(1.5,3) \end{pmatrix} = \begin{pmatrix} g(1.5,3) - 2g(1.5,0) + g(1.5,1) \\ g(1.5,0) - 2g(1.5,1) + g(1.5,2) \\ g(1.5,1) - 2g(1.5,2) + g(1.5,3) \\ g(1.5,2) - 2g(1.5,3) + g(1.5,0) \end{pmatrix}. \tag{1.1}$$

After solving this system we obtain

$$g_{yy}(1.5, 1) = -1.7637, \quad g_{yy}(1.5, 2) = -4.6150.$$

Therefore, $g(1.5, y)$ for $1 \leq y \leq 2$ will be:

$$g(1.5, y)|_{1\leq y \leq 2} = \frac{-1.7637}{6}\left[(2-y)^3 - (2-y)\right] + \frac{-4.6150}{6}\left[(y-1)^3 - (y-1)\right] + 2.6082\,(2-y) + 3.5588\,(y-1).$$

Substituting $y = 1.5$ results in $g(1.5, 1.5) = 3.4821$.

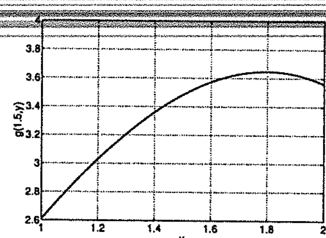

Figure 1.6: $g(1.5, y)$ for $1 \leq y \leq 2$

(d) After solving corresponding systems which are similar to (1.1) we obtain

$g_{yy}(1,1) = -1.2279, \quad g_{yy}(1,2) = -3.2994, \quad g_{yy}(2,1) = -1.8428, \quad g_{yy}(2,2) = -4.9700.$

Therefore, the polynomial expressions for $1 \le y \le 2$ will be

$$\begin{aligned} g(1,y)|_{1\le y\le 2} &= \frac{-1.2279}{6}\left[(2-y)^3-(2-y)\right] + \frac{-3.2994}{6}\left[(y-1)^3-(y-1)\right] + \\ &\quad 1.7995\,(2-y) + 2.4900\,(y-1)\,, \end{aligned} \tag{1.2}$$

$$\begin{aligned} g(2,y)|_{1\le y\le 2} &= \frac{-1.8428}{6}\left[(2-y)^3-(2-y)\right] + \frac{-4.9700}{6}\left[(y-1)^3-(y-1)\right] + \\ &\quad 2.8357\,(2-y) + 3.8781\,(y-1)\,. \end{aligned} \tag{1.3}$$

(e) In part (b) the g_{xx} values at the grid points are computed. We can use spline to interpolate these values in the y direction. We first solve the following system.

$$\begin{pmatrix} 2/3 & 1/6 & 0 & 1/6 \\ 1/6 & 2/3 & 1/6 & 0 \\ 0 & 1/6 & 2/3 & 1/6 \\ 1/6 & 0 & 1/6 & 2/3 \end{pmatrix} \begin{pmatrix} g_{xxyy}(1,0) \\ g_{xxyy}(1,1) \\ g_{xxyy}(1,2) \\ g_{xxyy}(1,3) \end{pmatrix} = \begin{pmatrix} g_{xx}(1,3) - 2g_{xx}(1,0) + g_{xx}(1,1) \\ g_{xx}(1,2) - 2g_{xx}(1,1) + g_{xx}(1,0) \\ g_{xx}(1,3) - 2g_{xx}(1,2) + g_{xx}(1,1) \\ g_{xx}(1,0) - 2g_{xx}(1,3) + g_{xx}(1,2) \end{pmatrix}$$

A similar system should be solved for $g_{xx}(2,y)$. The resulting numerical values are

$g_{xxyy}(1,1) = 0.9025, \quad g_{xxyy}(1,2) = 1.3381, \quad g_{xxyy}(2,1) = 2.7470, \quad g_{xxyy}(2,2) = 6.3511.$

The polynomial expressions for $1 \le y \le 2$ will be

$$\begin{aligned} g_{xx}(1,y)|_{1\le y\le 2} &= \frac{0.9025}{6}\left[(2-y)^3-(2-y)\right] + \frac{1.3381}{6}\left[(y-1)^3-(y-1)\right] + \\ &\quad -0.7701\,(2-y) + -0.9153\,(y-1)\,, \end{aligned} \tag{1.4}$$

$$\begin{aligned} g_{xx}(2,y)|_{1\le y\le 2} &= \frac{2.7470}{6}\left[(2-y)^3-(2-y)\right] + \frac{6.3511}{6}\left[(y-1)^3-(y-1)\right] + \\ &\quad -3.8787\,(2-y) + -5.0800\,(y-1)\,. \end{aligned} \tag{1.5}$$

(f) We can now use the information of (d) and (e) to do a cubic spline in the x direction. For $1 \le x \le 2$ and $1 \le y_0 \le 2$ we have

$$\begin{aligned} g(x,y_0)|_{1\le x\le 2} &= \frac{g_{xx}(1,y_0)}{6}[(2-x)^3-(2-x)] + \frac{g_{xx}(2,y_0)}{6}[(x-1)^3-(x-1)] + \\ &\quad g(1,y_0)(2-x) + g(2,y_0)(x-1), \end{aligned} \tag{1.6}$$

where $g(1,y_0)$, $g(2,y_0)$, $g_{xx}(1,y_0)$, and $g_{xx}(2,y_0)$ should be substituted from equations (1.2), (1.3), (1.4), and (1.5) respectively. The resulting polynomial will be of the form

$$\begin{aligned} P(x,y) &= a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{30}x^3 + a_{21}x^2y + \\ &\quad a_{12}xy^2 + a_{03}y^3 + a_{31}x^3y + a_{22}x^2y^2 + a_{13}xy^3 + a_{32}x^3y^2 + a_{23}x^2y^3 + a_{33}x^3y^3. \end{aligned}$$

Let's look at the terms in (1.6) that contribute to a_{33} (the terms that contain x^3y^3). Both

$$\frac{g_{xx}(1,y_0)}{6}(2-x)^3 \text{ and } \frac{g_{xx}(2,y_0)}{6}(x-1)^3$$

will contribute. Substituting for $g_{xx}(1,y_0)$ and $g_{xx}(2,y_0)$ from equations (1.4) and (1.5) and keeping only the terms with x^3y^3 results in

$$a_{33} = \frac{1}{36}(0.9025 - 1.3381 - 2.7470 + 6.3511) = 0.0880.$$

(g) From Equation (1.6) in part (f) we have

$$g(1.5,1.5) = \frac{g_{xx}(1,1.5)}{6}[(0.5)^3 - 0.5] + \frac{g_{xx}(2,1.5)}{6}[(0.5)^3 - (0.5)] + \\ +g(1,1.5)(0.5) + g(2,1.5)(0.5). \qquad (1.7)$$

From equations (1.2), (1.3), (1.4), and (1.5) we obtain

$$g_{xx}(1,1.5) = -0.9827, \quad g_{xx}(2,1.5) = -5.048, \quad g(1,1.5) = 2.4277, \quad g(2,1.5) = 3.7827$$

Substituting these values into (1.7) results in $g(1.5,1.5) = 3.4821$ which is the same as the result of part (c).

By interpolating the data to a fine mesh using splines, one can obtain a much smoother contour plot compared to the one shown in part (a).

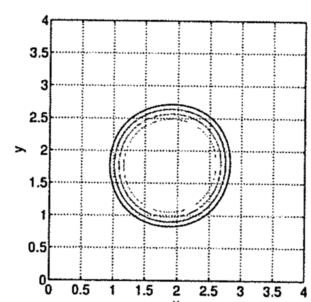

Figure 1.7: Contour plot after spline interpolation; from dark to light $f = 2.4, 2.6, 2.8, 3.0$.