Instructor's Guide with Full Solutions
for

COMAP's
For All Practical Purposes

Tenth Edition

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Urban Services

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Chapter Summary

Management science, or operations research, is a branch of mathematics that uses mathematical methods to find optimal solutions to management problems. Some of these problems involve finding efficient routes for services such as collecting coins from parking meters, collecting garbage, and delivering mail. The mathematical structure known as a *graph* is useful in analyzing routes.

A graph consists of a finite set of *vertices* together with edges connecting (some, all, or no) pairs of vertices. A path in a graph is a connected sequence of edges that begins and ends at a vertex. A path is called a *circuit* if it begins and ends at the same vertex. In many routing applications (e.g., mail delivery or garbage collection), the best solution would be a circuit that uses each edge (e.g., sidewalk or street) of an appropriate graph exactly once. Such circuits are called *Euler circuits*, in honor of the Swiss mathematician Leonhard Euler.

In order to have an Euler circuit, a graph must satisfy two conditions: It must be *connected* (i.e., it must have a path between any pair of its vertices); and each of its vertices must have even *valences* (the number of edges meeting at that vertex). If the graph for a particular routing application does not have an Euler circuit, the best we can hope to do is find a circuit of minimum length. The problem of finding such a circuit is known as the *Chinese postman problem*. The solution process relies on "*eulerizing*" a graph, judiciously duplicating edges of the graph to produce a connected graph with even valences so that the total length of the duplicated edges (total number if all edges of the graph have the same length) is as small as possible. The Euler tour in the "new" graph can be traced on the original by interpreting the use of a duplicated edge as a reuse of the original edge it duplicates.

There are efficient procedures for finding good eulerizations and for solving the Chinese postman problem. The "*edge-walker*" method in the text is good for rectangular networks. More sophisticated procedures are needed in general.

Skill Objectives

1. Determine by observation if a graph is connected.

2. Identify vertices and edges of a given graph.

3. Construct the graph of a given street network.

4. Determine by observation the valence of each vertex of a graph.

5. Define an Euler circuit.

6. List the two conditions for the existence of an Euler circuit.

7. Determine whether a graph contains an Euler circuit.

8. If a graph contains an Euler circuit, list one such circuit by identifying the order in which the vertices are used by the circuit, or by identifying the order in which the edges are to be used.

9. If a graph does not contain an Euler circuit, add a minimum number of edges to "eulerize" the graph.

10. Find an Euler circuit in an eulerized graph and "squeeze" it onto the original graph. Be able to interpret, in terms of the original graph, the use of duplicated edges in the eulerization.

11. Identify management science problems whose solutions involve Euler circuits.

Teaching Tips

1. The concept of connectedness could be explored further by considering trees as opposed to circuits. This could begin to prepare the student for the work on trees in connectedness.

2. In preparation for assigning Exercise 12, you may want to explore in class whether the placement of the vertices representing the cities affects the graph. In particular, consider three cities whose positions are collinear.

3. Figure 1.13 demonstrates the process of adding an edge in order to eulerize a graph. Students are sometimes confused by the fact that this added edge is curved rather than straight and attempt to attach unwarranted significance to this. A helpful explanation is that it is curved only so that it won't be confused with the original segment. In addition, you may want to emphasize that the curve could be drawn on either side of the original edge and that it indicates a retracing of that edge.

4. Ask students to construct a graph of a several-block area of their neighborhood and then look for an Euler circuit for the letter carrier to use. If an Euler circuit does not exist, ask students to produce optimal eulerizations.

5. An underlying principle of this chapter is the fact that a mathematical model, in this case a graph, is an abstraction of reality. The solution suggested by the model need not imply that streets would be added to an existing street network simply for the purpose of creating an Euler circuit.

6. Emphasize the differences between Skills Check Exercises 27 and 29. The two exercises are asking two different questions for the same graph.

Research Papers

Exercise 48 indicates that the problem situation resembles the one that inspired Euler. This refers to the historical Königsberg Bridge Problem (mentioned in Spotlight 1.1). You may choose to ask students to research the history of this town (once the capital of East Prussia) and this problem. Websites can be found that contain a map of the area, including the bridges. Also, you may choose to ask students to further investigate Euler's 1736 paper on the problem, which was divided into 21 paragraphs. Students also can research the life and contributions of the French mathematician Louis Poinsot (1777–1859). Other problems posed by Euler, such as the Thirty-Six Officers Problem, may also be of interest to students.

In 2007, UPS implemented an idea to save the company money resulting from rising gas prices. The idea was to minimize the number of left-hand turns. In the first year, this initiative saved the company about 30 million miles and about 3.3 million gallons of fuel. Have students research "turn penalties." They may wish to include in their discussion the biography of J. Edgar Hoover entitled *No Left Turns*, by Joseph L. Schott. With the rising cost of gas, have students write about how such a plan saves money and reduces emissions. Students may choose to make a map of the city/town in which they live and determine routes that implement this policy and see how they compare to routes that don't. An extension to this paper can be researching the impact of roundabouts on time and cost of traveling.

Collaborative Learning

Euler Circuits

As an introduction to this chapter, duplicate the following exercise on Euler circuits and ask your students to answer the questions after discussing them in groups. (Do not introduce technical terms such as graph, edge, vertex, valence, or Euler circuit yet.)

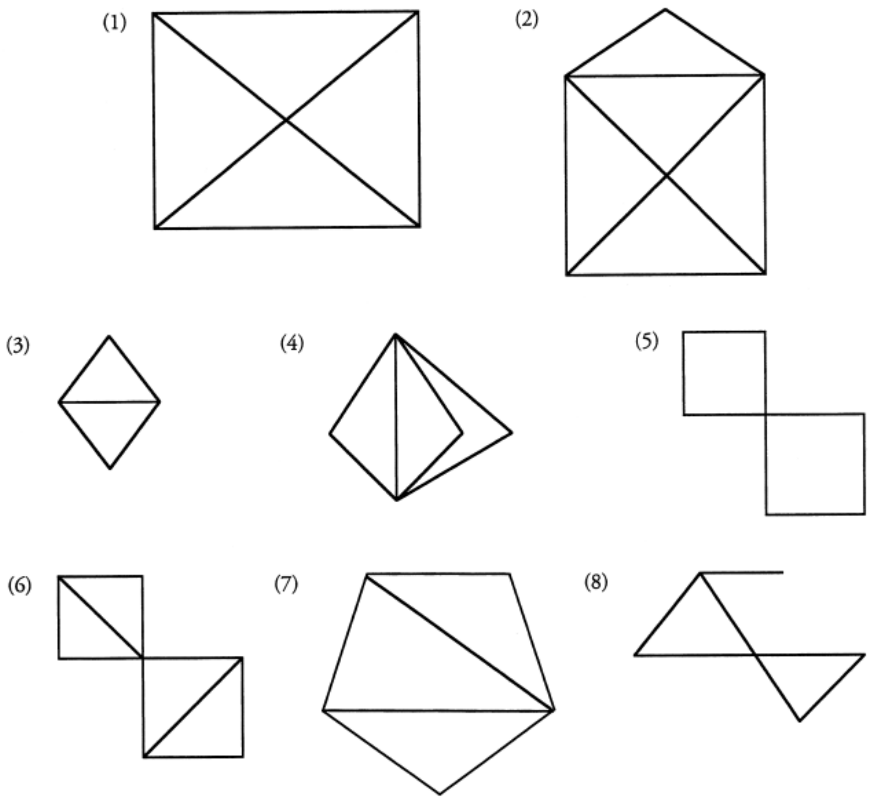
You will find that most of the students are successful in answering questions (a) and (b), but perhaps not question (c). If in fact they do have difficulty answering part (c), call their attention to the number of edges meeting at each of the vertices. (Resist using the word *valence*.) Ask them to count these numbers for each of the graphs and then try again to answer part (c) by looking for a pattern.

Eulerization

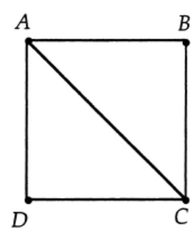
a. Which of the following diagrams can be drawn without lifting your pencil from the paper and without retracing any lines?

b. Which can be drawn as in part (a), but with the additional requirement that you end the drawing at the starting point?

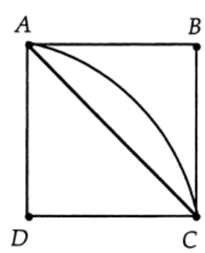
c. What do the diagrams that could be drawn in part (a) have in common? What about the diagrams that could be drawn in part (b)? In other words, try to determine simple conditions on a diagram that enable you to predict, in advance, whether or not it can be drawn according to the requirements in parts (a) or (b).



In each of the graphs below, an Euler circuit does not exist because there are vertices with odd valences. It is possible to convert such graphs to graphs having all vertices of even valence by duplicating one or more edges. For example, in the graph

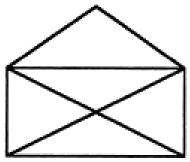


vertices *A* and *C* each have valence 3. Hence, if we duplicate edge *AC*, we obtain a modified graph in which each vertex has an even valence.

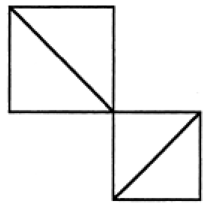


Do the same for each of the following graphs and then answer the questions that follow.

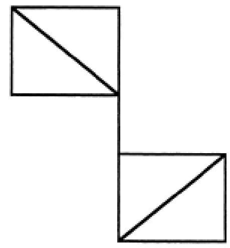
(1)



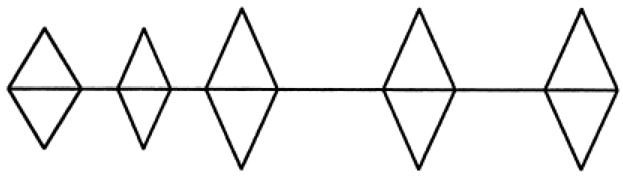
(2)



(3)



(4)



1. If a graph has 4 vertices with odd valences, what is the minimum number of duplications necessary to convert the graph to one in which all of the vertices are of even valence? Can this minimum number always be achieved?

2. Take a campus map (often found in the university catalog or directory) and pick out several key locations (library, dormitories, computer center, bookstore, etc.) and the streets or paths that connect them. Then ask the students whether an Euler circuit exists for this network. If not, how many edges have to be duplicated? By changing landmarks, you can obtain several new problems.

Solutions

Skills Check:

1. c
2. 7; 8
3. b
4. 3
5. a
6. 16
7. a
8. *B*; *E*
9. c
10. 7; 7
11. b
12. 1; 2; 4; 5; 10
13. c
14. 6
15. a
16. 9; 22
17. c
18. 8
19. c
20. a
21. 4
22. 12
23. a
24. 4
25. a
26. graph; digraph; graph
27. b
28. 9; 15
29. a
30. 13

Collaborative Learning:

Euler Circuits:

a. Diagrams (2), (3), (4), (5), (7), and (8) can be drawn without lifting the pencil from the paper.

b. In (4) and (5) you can return to the starting point.

c. In part (a), there are at most 2 vertices with odd valences, while in part (b) there are no vertices with odd valences.

Eulerization:

(1) Duplicate 1 edge.

(2) Duplicate 2 edges.

(3) Duplicate 3 edges.

(4) Duplicate 9 edges.

If there are 4 vertices with odd valences, then there will be a *minimum* of two duplications. However, this minimum can't always be achieved, as we see from number (3).

Exercises:

1. **(a)** There are 6 vertices, each marked with a capital letter.

(b) There are 9 edges.

2. **(a)** Three ways to get from C to A by traversing 4 edges are:

CDBEA and CDFEA and CDEBA

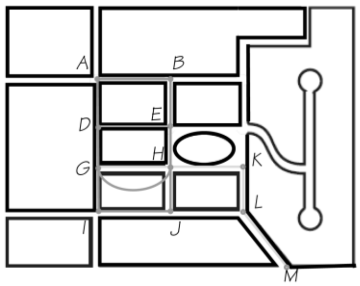
(b) Three ways to get from C to A by traversing three edges are:

CDBA and CBEA and CDEA

(c) Three ways to start and end at B traversing four edges are:

BDFEB and BCDEB and BAEDB

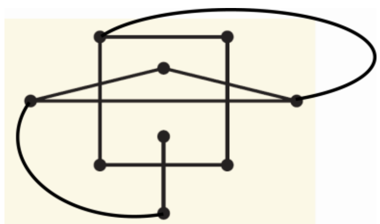
3. (a)



(b) *ADGIJHKLMLKHG* (using other edges) *HEDEBA* is such a path. Four of the edges are deadheaded.

4. (a) There are 9 vertices and 8 edges.

(b) You would need to add two edges. One possibility is as follows:



(c) There are seven vertices of valence 2 and two vertices of valence 1.

5. (a) This diagram fails to be a graph because a line segment joins a single vertex to itself. The definition being used does not allow this.

(b) The edge *EC* crosses edges *AD* and *BD* at points that are not vertices; edge *AC* crosses *BD* at a point that is not a vertex.

(c) This graph has 5 vertices and 6 edges.

6. (a) 7 stores

(b) 10 roads

(c) *CBF* is a path.

(d) *EDFB* is a path.

7. (a) 8 vertices

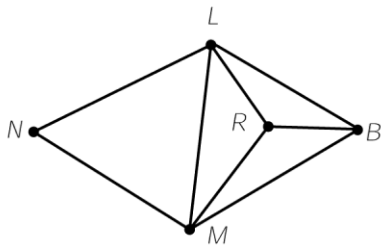
(b) 13 edges

(c) *A*: 4; *B*: 2; *C*: 3; *D*: 3; *E*: 4; *F*: 4; *G*: 3; *H*: 3

(d) *A*, *D*, and *F*

(e) *E*, *G*, and *H*

8. **(a)**



(b) Assuming that no cities are revisited and that one goes directly from an American city to a European one the routes are:

NL, *NLR*, *NLB*, *NLBR*, *NLRB*, *ML*, *MR*, *MB*, *MLR*, *MLB*, *MRL*, *MRB*, *MBL*, *MBR*, *MLRB*, *MLBR*, *MRLB*, *MRBL*, *MBLR*, *MBRL*

9. *E* has valence 0; *A* has valence 1; *H*, *D*, and *G* have valence 2; *B* and *F* have valence 3; *C* has valence 5. *E* is "isolated." *E* might have valence 0 because it is on an island with no road access.

10. **(a)** Yes.

(b) No. There is no way to get from *A* to *C*, *E*, or *H*.

(c) No. There is no way to get from *A* to *F*, *H*, or *E*.

11. **(a)** There are 4 vertices and 4 edges.

(b) There are 7 vertices and 6 edges.

(c) There are 10 vertices and 14 edges.

12. **(a)** There are five paths to choose from if you assume that no cities are revisited and that one goes directly from an American city to a European city. These are *MLB*, *MLRB*, *MRB*, *MRLB*, and *MB*. If you assume that you can travel between Miami and New York prior to going to Europe then you would consider the paths *MNLB* and *MNLRB*. If revisiting a city is allowed, then there are infinitely many paths.

(b) There are five paths if domestic travel is not considered and seven paths to choose from if domestic travel is allowed.

(c) In such a route, time and probably cost would be greater if you repeat a city prior to getting to the destination.

13. **(a)** Answers will vary.
Possible answers include *CGDBC.*

(b) **(i)** *BD*; *BFD*

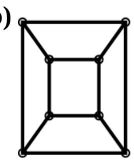
(ii) *CBF*; *CGDF*; *CGDBF*

(iii) *GDBCG*

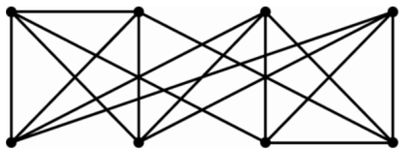
14. In any of these graphs, two edges can be removed and the graph will become disconnected. One of the disconnected pieces will be a single vertex.

15. **(a)**

(b)



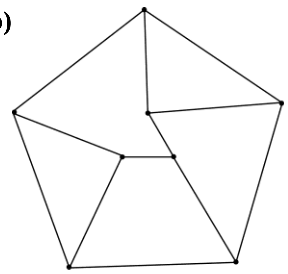
(c)



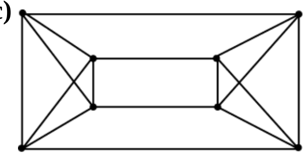
Note: Drawing and renderings can vary. The text answer for part (c) offers another drawing.

Some other graphs for parts (b) and (c) could be as follows:

(b)



(c)



(d) Yes. The sum of the valences of a graph with 8 vertices, each of valence 2, is 16. Thus, all such graphs have 8 edges.

16. **(a)** 2 + 3 + 3 + 0 = 8

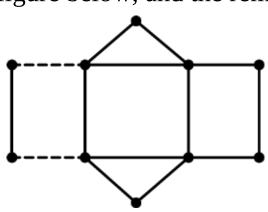
(b) 2 + 2 + 2 + 2 + 2 + 1 + 1 = 12

(c) 28

(d) The number we obtain is twice the number of edges in the graph.

(e) The fact that the sum of the valences of the vertices of a graph is always twice the number of edges in the graph follows from noticing that each vertex of an edge contributes a total of two to the sum because the edge has two endpoints.

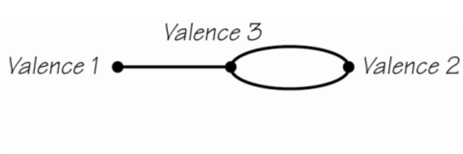
17. Remove the two edges dotted in the figure below, and the remaining graph will be disconnected.



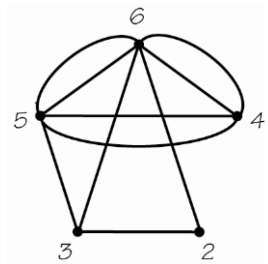
Note: Other pairs of edges will also disconnect the graph.

18. **(a)** Yes, it is possible.

(b)



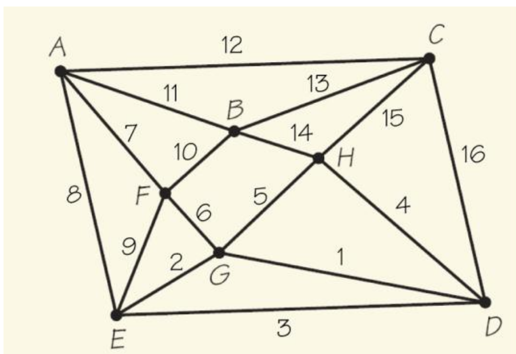
19. **(a)** Yes, it is possible.



(b) No. To have an Euler circuit, a graph cannot have odd-valent vertices.

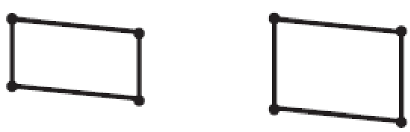
20. **(a)** There are 8 vertices and 16 edges.

(b)

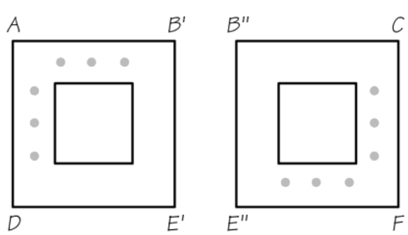


DGEDHGFAEFBACBHCD

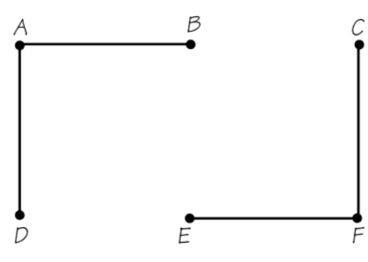
21. Drawings can vary. Possible renderings include the following:



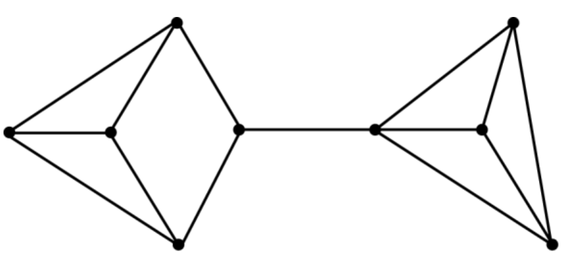
22. **(a)** Yes, a disconnected graph can arise.
One possible example is shown below:



This gives rise to the disconnected graph:

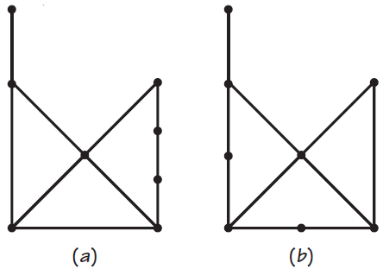


(b)



(c) The edge might represent a bridge or tunnel. Recently, when a bridge collapsed because it was hit by a barge, there was a major disruption to the communities near the bridge on opposite sides of the river.

23. Drawings can vary. Possible renderings include the following:

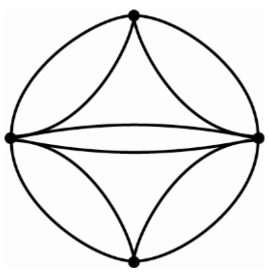


24. The street direction will matter for a problem involving how long it will take to get between two street intersections and for routing a street sweeper that follows traffic rules. The street direction may not matter for an inspector checking manholes located in the middle of streets or a service that involves walking along either side of the street, such as inspecting sidewalks.

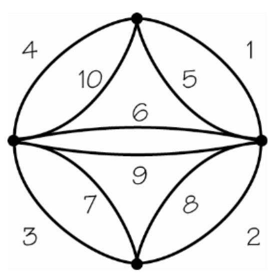
25. **(a)** The supervisor is not satisfied because all of the edges are not traveled upon by the postal worker.

(b) The worker is unhappy because the end of the worker's route wasn't the same point as where the worker began. The original job description is unrealistic because there is no Euler circuit in the graph.

26.

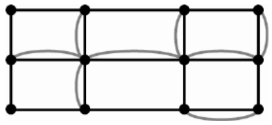


27. There is such an efficient route. The appropriate graph model has an additional edge joining the same pair of vertices for each of the edges shown in the graph of Exercise 25. Adding those edges gives you the graph for Exercise 26. Since this graph is connected and even-valent, it has an Euler circuit, any one of which will provide a route for the snowplow. Routes without 180-degree turns are better choices. One possible route is indicated by the numbered routes.

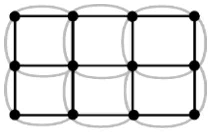


28. **(a)** Pothole inspection or inspecting the centerline for possible repainting because it had faded.

(b) Street sweeping, snow removal, and curb inspections in urban areas.

29. 

30. **(a)** The graph is a rectangular network with two rows and three columns No extra edges need to be added.

(b) 

31. **(a)** The largest number of such paths is 3. One set of such paths is *AGC*, *AHFC*, and *ABEIJC*.

(b) This task is simplified by noticing there are many symmetries in this graph. You may notice that starting with *A*, there are only three directions to go. Any number of paths greater than three would involve repeating an edge starting from *A*.

(c) In a communication system such a graph offers redundant ways to get messages between pairs of points even when the failure of some of the communication links (edges removed) occurs.

32. Both are circuits; however, only graph (b) is an Euler circuit.

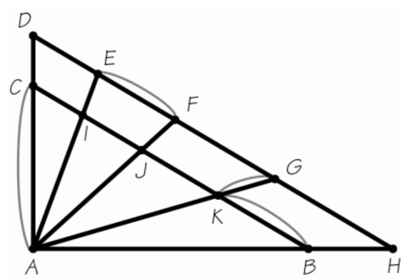
33. **(a)** No. It is not possible. There are odd-valent vertices.

(b) The minimum number is four.

(c) Answers will vary.

One solution would involve duplicating edges *CD*, *DE*, *FG*, and *AB*.

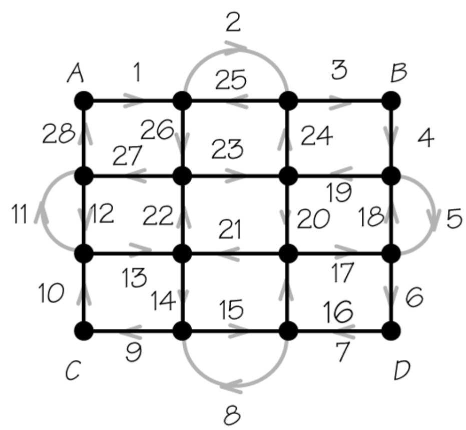
Another possible Euler circuit starting and ending at *C* is *CDEFGHBKGKBAKJFEIJAICAC.* The duplicated edges are shown below:



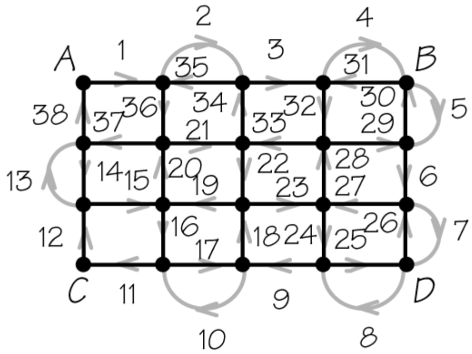
34. Do not choose edge 3, but edges 9 or 10 could be chosen.

35. Do not choose edge 2, but edges 1 or 10 could be chosen.

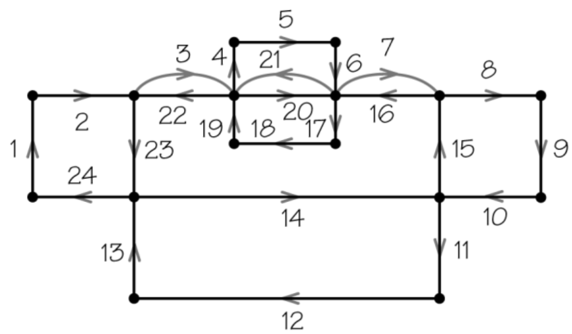
36. The following diagram shows one of many solutions for Figure 1.17a:



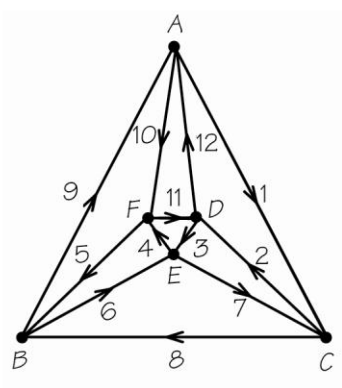
The following diagram shows one of many solutions for Figure 1.17b:



37.



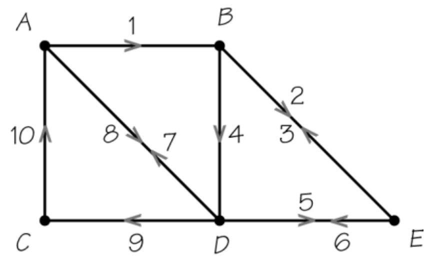
38. If one was outlining garden plots with a sprinkler hose, this tour would allow having the hoses as flat as possible because one hose would not have to cross another hose.



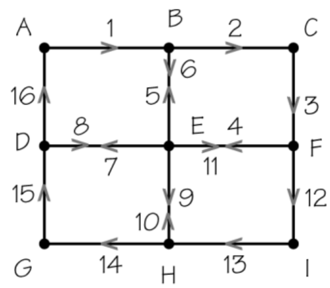
39. **(a)** *A*, *C*, *E*, and *H* are odd-valent.

(b) Two edges, *AH* and *CE*, need to be dropped to produce a graph with an Euler circuit. Persons who parked along these stretches of sidewalk without putting coins in the meters would not need to fear that they would get tickets.

40. The curved edges on the first graph become double-traversals on the straight edges of the second graph.

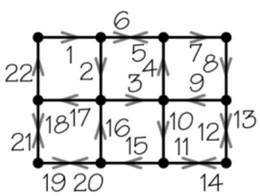


ABEBDEDADCA

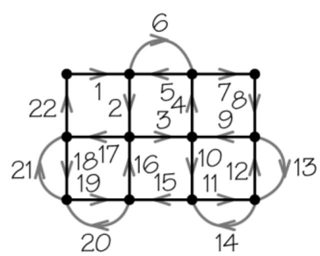


ABCFEBEDEHEFI

41. (a)



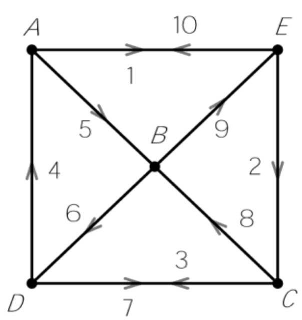
(b)



No. Five is the minimum number of edges that must be reused. Fewer than five reused edges cannot be achieved.

42. A minimum of three edges must be added: one edge along the horizontal segment in the first parallelogram and a segment along two opposite edges of the second parallelogram.

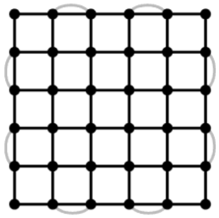
43.



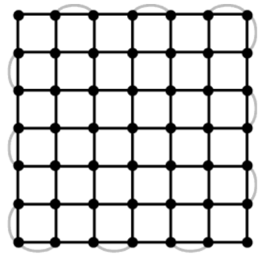
AECDABDCBEA

44. There are many different circuits which will involve three reuses of edges. These are the edges which join up the six 3-valent vertices in pairs.

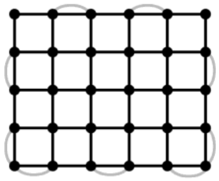
45. **(a)**



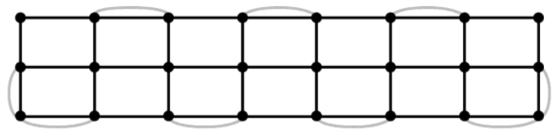
(c)



(b)

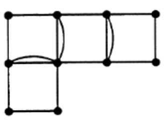


(d)

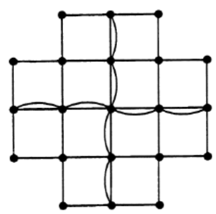


Nine is the best one can do.

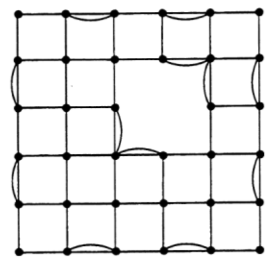
46. **(a)**



(b)



(c)



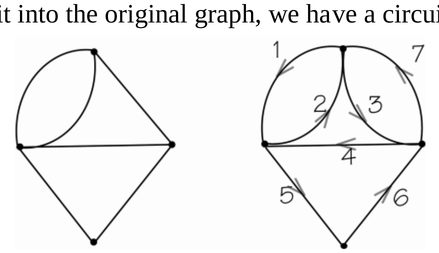
47. **(a)** There are four 3-valent vertices. By properly removing two edges adjacent to these four vertices (edge between left two 3-valent vertices and edge between right two 3-valent vertices), one can make the graph even-valent.

(b) Yes, because the resulting graph is connected and even-valent.

(c) It is possible to remove two edges and have the resulting graph be even-valent.

(d) No, because the resulting graph is not connected, even though it is even-valent.

48. Represent each riverbank by a vertex and each island by a vertex. Represent each bridge by an edge. This produces the graph on the left. After eulerization we produce the graph on the right. An Euler circuit is shown on this graph. After squeezing this circuit into the original graph, we have a circuit with one repeated edge.



49. **(a)** One possible answer includes *ADEADCAFCFBA.*

(b) There are nine streets being covered, with two streets being deadheaded. The total amount of time would be $9\times9+2\times7=81+14=95$ minutes.

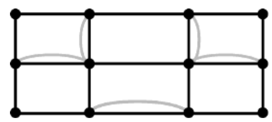
50. **(a)** **(i)** The number of edges added will always be the number of odd-valent vertices divided by 2.

(ii) Since we must duplicate existing edges to find the best eulerization, the number of edges added will always be at least the number of odd-valent vertices divided by 2, but often is a larger number.

(b) Yes

51. The following graph satisfies the condition. You would eulerize the graph by duplicating each edge exactly once. The two end vertices are odd-valent.

52. The minimum length (34,000 feet) is obtained for any Euler circuit in the graph with edges duplicated as shown below. For minimizing total length it is better to repeat many shorter edges rather than a few long ones.



53. There are many circuits that achieve a length of 44,000 feet. The number of edges reused is eight because a shorter length tour can be found by repeating more shorter edges than fewer longer edges.

54. **(a)** The cheapest route has cost 49 and repeats edges *BC*, *CD*, and *DF*.

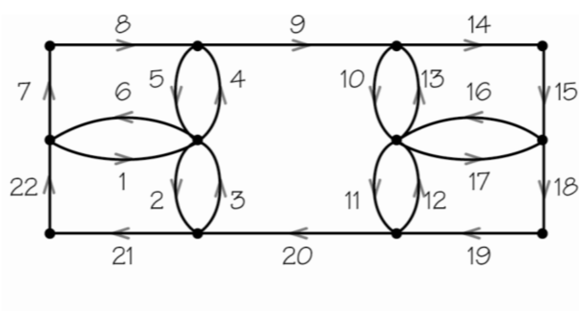
(b) Three edges

(c) When there are different weights on the edges of a graph, the discussion about good eulerizations must be modified to take the size of the weights into account. It turns out there is an efficient, though complex, algorithm for finding minimum cost solutions to such problems.

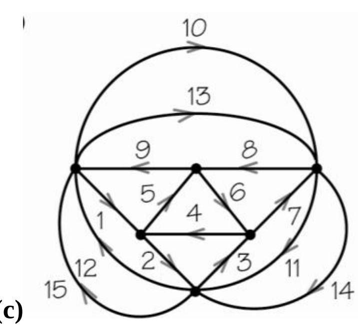
(d) The weight might represent time. Two blocks of the same physical length can take different times to traverse due to construction or other factors.

(e) The weight might represent traversal time, traffic volume, number of potholes, number of stop signs, etc.

55. Both graphs (a) and (c) have Euler circuits. The valences of all of the vertices in (b) are odd, which makes it impossible to have an Euler circuit there.

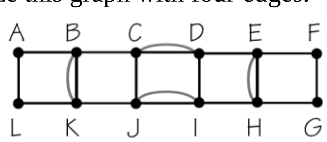


(a)



(c)

56. There are five different ways to eulerize this graph with four edges. One of them is shown below:



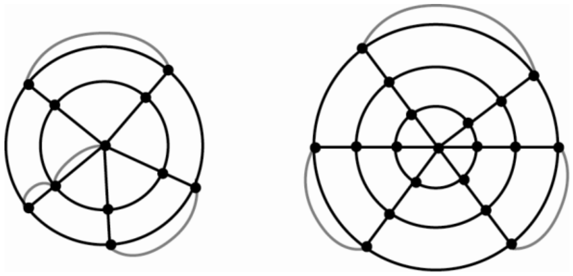
An Euler circuit in the original graph that repeats four edges is: *ABCDEHGFEHIJCDIJKBKLA.*

There are a total of 20 edges.

57. For each edge *e* of graph *G*, add an additional edge joining the vertices which are endpoints of *e* to obtain graph *H*. If *G* is connected, then *H* does have an Euler circuit because whatever the valances of *G*, the graph *H* has valences that are doubles of these and remains connected. Hence *H* is even-valent and connected.

58. A good eulerization duplicates the five "spokes" that go from the inner pentagon to the outer one. There are many Euler circuits in the eulerized graph.

59. (a)



(b) The best eulerization for the four-circle, four-ray case adds two edges.

(c) Answers will vary.

Hint: Consider the cases where *r* is even and odd separately.

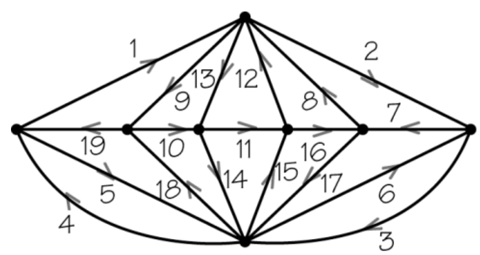
60. Pick any vertex and try to start an Euler circuit for the graph there. At some point the circuit traverses this special edge, crossing from the starting part of the graph to the other part. This special edge is the only connection between the parts, so we cannot return to the starting part and thus cannot have an Euler circuit. Since there is no Euler circuit, somewhere there must be a vertex with an odd valence.

61. (a) Yes, a graph with six vertices where each vertex is joined to every other vertex will have valence 5 for each vertex.

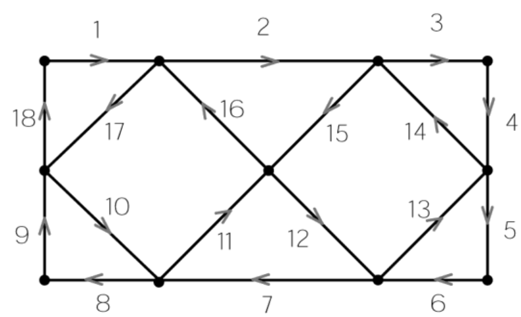
(b) The graph will have 15 edges. This is because the number of edges will be half of the total sum of the valences. Since there are six valence-5 vertices, the sum of the valences will be 30.

62. All graphs have Euler circuits because all graphs are connected and all vertices have even valences. Possible Euler circuits are shown below.

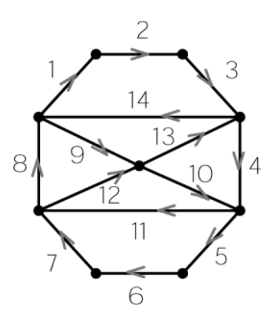
(a)



(b)



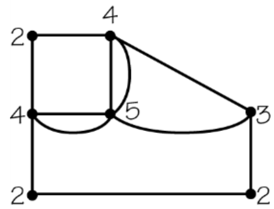
(c)



63. When you attach a new edge to an existing graph, it gets attached at two ends. At each of its ends, it makes the valence of the existing vertex go up by one. Thus the increase in the sum of the valences is two. Therefore, if the graph had an even sum of the valences before, it still does, and if its valence sum was odd before, it still is.

64. Dots without edges all have valence zero, which is an even number, so the number of odd-valent vertices is zero. As edges are added, the number of odd-valent vertices will always increase by either 0 or 2. Thus, any graph has an even number of odd-valent vertices.

65.

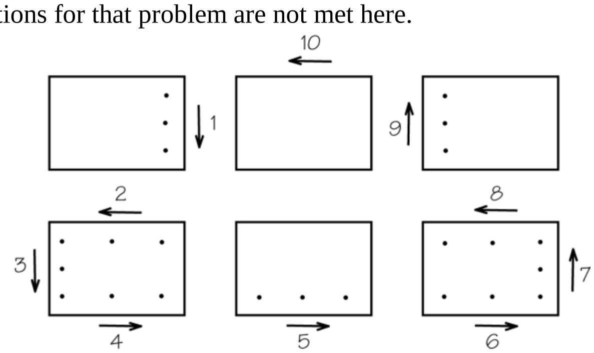


The graph is connected.

66. When $r = 1$, a formula for the number of repeated edges is $(s-1)$. If r and s are odd, where $r = 2a+1$ and $s = 2b+1$ (where a and b are positive integers that are at least 1) then a formula for the number of repeated edges is $2(a+b)$. Similar formulas hold for the cases where both r and s are even, or one of them is even and the other odd. The exact form of the formula depends on the way one expresses these situations.

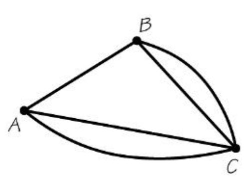
67. In chemistry when we say, for example, that hydrogen has valence 1, we mean that it forms one chemical bond with other elements. This usage has similarities with the graph theory concept of valence.

68. The thinking used to solve the Chinese postman problem does not apply directly for the situation described because the modeling assumptions for that problem are not met here.



69. A tour that begins and ends at vertex *A* and that respects the traffic directions would be *ABDEFBEBFEDBACDCBCA*. The cutting machine has to make sharp turns at some intersections.

70. Answers will vary. One such graph is:



71. **(a)** There are 10 vertices and 18 edges.

(b) The valences of vertices *A*, *F* and *D* are 4, 4, and 4 respectively.

(c) Answers will vary. One possible solution is: *EBADAGBCEGHDFHIFCIE*.

72. (a) The graph *G* has 4 odd valent vertices: *A*, *F*, *G* and *H*.

(b) *G* does not have an Euler circuit because it has odd valent vertices.

(c) The minimum number of edges that must be repeated is 2.

(d) The answer in (c) relates to part (a) by the statement given on page 20 in the text; specifically that in a best eulerization of a graph, the minimum number of edges that must be duplicated is at least half of the number of odd-valent vertices. This is a case where more than half need to be added because 4 edges will be necessary.

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