INSTRUCTOR'S SOLUTIONS MANUAL

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EXCURSIONS IN MODERN MATHEMATICS

NINTH EDITION

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Chapter 1

WALKING

1.1. Ballots and Preference Schedules

1.

Number of voters	**5**	**3**	**5**	**3**	**2**	**3**
1st choice	*A*	*A*	*C*	*D*	*D*	*B*
2nd choice	*B*	*D*	*E*	*C*	*C*	*E*
3rd choice	*C*	*B*	*D*	*B*	*B*	*A*
4th choice	*D*	*C*	*A*	*E*	*A*	*C*
5th choice	*E*	*E*	*B*	*A*	*E*	*D*

This schedule was constructed by noting, for example, that there were five ballots listing candidate *C* as the first preference, candidate *E* as the second preference, candidate *D* as the third preference, candidate *A* as the fourth preference, and candidate *B* as the last preference.

2.

Number of voters	**4**	**5**	**6**	**2**
1st choice	*A*	*B*	*C*	*A*
2nd choice	*D*	*C*	*A*	*C*
3rd choice	*B*	*D*	*D*	*D*
4th choice	*C*	*A*	*B*	*B*

3. (a) 5 + 5 + 3 + 3 + 3 + 2 = 21

(b) 11. There are 21 votes all together. A majority is more than half of the votes, or at least 11.

(c) Chavez. Argand has 3 last-place votes, Brandt has 5 last-place votes, Chavez has no last-place votes, Dietz has 3 last-place votes, and Epstein has 5 + 3 + 2 = 10 last-place votes.

4. (a) 202 + 160 + 153 + 145 + 125 + 110 + 108 + 102 + 55 = 1160

(b) 581; There are 1160 votes all together. A majority is more than half of the votes, or at least 581.

(c) Alicia. She has no last-place votes. Note that Brandy has 125 + 110 + 55 = 290 last-place votes, Cleo has 202 + 145 + 102 = 449 last-place votes, and Dionne has 160 + 153 + 108 = 421 last-place votes.

5.

Number of voters	**37**	**36**	**24**	**13**	**5**
1st choice	*B*	*A*	*B*	*E*	*C*
2nd choice	*E*	*B*	*A*	*B*	*E*
3rd choice	*A*	*D*	*D*	*C*	*A*
4th choice	*C*	*C*	*E*	*A*	*D*
5th choice	*D*	*E*	*C*	*D*	*B*

Here Brownstein was listed first by 37 voters. Those same 37 voters listed Easton as their second choice, Alvarez as their third choice, Clarkson as their fourth choice, and Dax as their last choice.

6.

Number of voters	**14**	**10**	**8**	**7**	**4**
1st choice	*B*	*B*	*A*	*D*	*E*
2nd choice	*A*	*D*	*B*	*C*	*B*
3rd choice	*E*	*A*	*E*	*B*	*A*
4th choice	*C*	*E*	*D*	*E*	*C*
5th choice	*D*	*C*	*C*	*A*	*D*

7.

Number of voters	14	10	8	7	4
A	2	3	1	5	3
B	1	1	2	3	2
C	5	5	5	2	4
D	4	2	4	1	5
E	3	4	3	4	1

Here 14 voters had the same preference ballot listing *B* as their first choice, *A* as their second choice, *E* as their third choice, *D* as their fourth choice, and *C* as their fifth and last choice.

8.

Number of voters	37	36	24	13	5
A	1	2	5	2	4
B	3	1	2	4	1
C	2	4	3	1	5
D	5	3	1	5	2
E	4	5	4	3	3

9.

Number of voters	255	480	765
1st choice	*L*	*C*	*M*
2nd choice	*M*	*M*	*L*
3rd choice	*C*	*L*	*C*

(0.17)(1500) = 255; (0.32)(500) = 480; The remaining voters (51% of 1500 or 1500-255-480=765) prefer *M* the most, *C* the least, so that *L* is their second choice.

10.

Number of voters	450	900	225	675
1st choice	*A*	*B*	*C*	*C*
2nd choice	*C*	*C*	*B*	*A*
3rd choice	*B*	*A*	*A*	*B*

100% - 20% - 40% = 40% of the voters number 225 + 675 = 900. So, if *N* represents the total number of voters, then $(0.40)N = 900$. This means there are *N* = 2250 total voters. 20% of 2250 is 450 (these voters have preference ballots *A*, *C*, *B*). 40% of 2250 is 900 (these voters have preference ballots *B*, *C*, *A*).

1.2. Plurality Method

11. **(a)** *C*. *A* has 15 first-place votes. *B* has 11 + 8 + 1 = 20 first-place votes. *C* has 27 first-place votes. *D* has 9 first-place votes. *C* has the most first-place votes with 27 and wins the election.

(b) *C*, *B*, *A*, *D*. Candidates are ranked according to the number of first-place votes they received (27, 20, 15, and 9 for *C, B, A*, and *D* respectively).

12. **(a)** *D*. *A* has 21 first-place votes. *B* has 18 first-place votes. *C* has 10 + 1 = 11 first-place votes. *D* has 29 first-place votes. *D* has the most first-place votes with 29 and wins the election.

(b) *D*, *A*, *B*, *C*.

13. **(a)** *C*. *A* has 5 first-place votes. *B* has 4 + 2 = 6 first-place votes. *C* has 6 + 2 + 2 + 2 = 12 first-place votes. *D* has no first-place votes. *C* has the most first-place votes with 12 and wins the election.

(b) *C*, *B*, *A*, *D*. Candidates are ranked according to the number of first-place votes they received (12, 6, 5, and 0 for *C, B, A*, and *D* respectively).

14. **(a)** *B*. *A* has 6 + 3 = 9 first-place votes. *B* has 6 + 5 + 3 = 14 first-place votes. *C* has no first-place votes. *D* has 4 first-place votes. *B* has the most first-place votes with 14 and wins the election.

(b) *B, A*, *D*, *C*. Candidates are ranked according to the number of first-place votes they received (14, 9, 4 and 0 for *B, A, D*, and *C* respectively).

15. (a) *D*. *A* has 11% of the first-place votes. *B* has 14% of the first-place votes. *C* has 24% of the first-place votes. *D* has 23% + 19% + 9% = 51% of the first-place votes. *E* has no first-place votes. *D* has the largest percentage of first-place votes with 51% and wins the election.

(b) *D, C*, *B*, *A*, *E*. Candidates are ranked according to the percentage of first-place votes they received (51%, 24%, 14%, 11% and 0% for *D, C, B, A*, and *E* respectively).

16. (a) *C*. *A* has 12% of the first-place votes. *B* has 15% of the first-place votes. *C* has 25% + 10% + 9% + 8% = 52% of the first-place votes. *D* has no first-place votes. *E* has 21% of the first-place votes. *C* has the largest percentage of first-place votes with 52% and wins the election.

(b) *C, E*, *B*, *A*, *D*. Candidates are ranked according to the percentage of first-place votes they received (52%, 21%, 15%, 12%, and 0% for *C, E, B, A* and *D* respectively).

17. (a) *A*. *A* has 5 + 3 = 8 first-place votes. *B* has 3 first-place votes. *C* has 5 first-place votes. *D* has 3 + 2 = 5 first-place votes. *E* has no first-place votes. *A* has the most first-place votes with 8 and wins the election.

(b) *A*, *C, D*, *B, E*. Candidates are ranked according to the number of first-place votes they received (8, 5, 5, 3, and 0 for *A, C, D, B*, and *E* respectively). Since both candidates *C* and *D* have 5 first-place votes, the tie in ranking is broken by looking at last-place votes. Since *C* has no last-place votes and *D* has 3 last-place votes, candidate *C* is ranked above candidate *D*.

18. (a) *A*. *A* has 153 + 102 + 55 = 310 first-place votes. *B* has 202 + 108 = 310 first-place votes. *C* has 160 + 110 = 270 first-place votes. *D* has 145 + 125 = 270 first-place votes. Both *A* and *B* have the most first-place votes with 310 so the tie is broken using last-place votes. *A* has no last-place votes. *B* has 125 + 110 + 55 = 290 last-place votes. So *A* wins the election.

(b) *A*, *B, D, C*. Candidates are ranked according to the number of first-place votes they received (310, 310, 270 and 270 for *A, B, D*, and *C* respectively). In part (a), we saw that the tie between *A* and *B* was broken in favor of *A*. Since both candidates *C* and *D* have 270 first-place votes, the tie in ranking is broken by looking at last-place votes. Since *C* has 202 + 145 + 102 = 449 last-place votes and *D* has 160 + 153 + 108 = 421 last-place votes, candidate *D* is ranked above candidate *C*.

19. (a) *A*. *A* has 5 + 3 = 8 first-place votes. *B* has 3 first-place votes. *C* has 5 first-place votes. *D* has 3 + 2 = 5 first-place votes. *E* has no first-place votes. *A* has the most first-place votes with 8 and wins the election. (Note: This is exactly the same as in Exercise 17(a).)

(b) *A*, *C, D*, *B, E*. Candidates are ranked according to the number of first-place votes they received (8, 5, 5, 3, and 0 for *A, C, D, B*, and *E* respectively). Since both candidates *C* and *D* have 5 first-place votes, the tie in ranking is broken by a head-to-head comparison between the two. But candidate *C* is ranked higher than *D* on 5 + 5 + 3 = 13 of the 21 ballots (a majority). Therefore, candidate *C* is ranked above candidate *D*.

20. (a) *B*. *A* has 153 + 102 + 55 = 310 first-place votes. *B* has 202 + 108 = 310 first-place votes. *C* has 160 + 110 = 270 first-place votes. *D* has 145 + 125 = 270 first-place votes. Both *A* and *B* have the most first-place votes with 310 so the tie is broken by head-to-head comparison. But candidate *A* is ranked higher than *B* on 153 + 125 + 110 + 102 + 55 = 545 of the 1160 ballots (less than a majority). So *B* wins the tiebreaker and the election.

(b) *B, A, D, C.* Candidates are ranked according to the number of first-place votes they received (310, 310, 270 and 270 for *B*, *A*, *D*, and *C* respectively). In part (a), we saw that the tie between *A* and *B* was broken in favor of *B*. Since both candidates *C* and *D* have 270 first-place votes, the tie in ranking is broken by head-to-head comparison. Candidate *C* is ranked higher than *D* on 160 + 153 + 110 + 108 = 531 of the 1160 ballots (less than a majority). So in the final ranking, candidate *D* is ranked above candidate *C*.

1.3. Borda Count

21. (a) *A* has $4\times15+3\times(9+8+1)+2\times11+1\times27=163$ points.
B has $4\times(11+8+1)+3\times15+2\times(27+9)+1\times0=197$ points.
C has $4\times27+3\times0+2\times8+1\times(15+11+9+1)=160$ points.
D has $4\times9+3\times(27+11)+2\times(15+1)+1\times8=190$ points.
The winner is *B*.

(b) *B*, *D, A*, *C.* Candidates are ranked according to the number of Borda points they received.

22. (a) *A* has $4\times21+3\times18+2\times(29+10)+1\times1=217$ points.
B has $4\times18+3\times(10+1)+2\times21+1\times29=176$ points.
C has $4\times(10+1)+3\times(29+21)+2\times18+1\times0=230$ points.
D has $4\times29+3\times0+2\times1+1\times(21+18+10)=167$ points.
The winner is *C*.

(b) *C, A, B, D.* Candidates are ranked according to the number of Borda points they received.

23. (a) *A* has $4\times5+3\times2+2\times(6+2)+1\times(4+2+2)=50$ points.
B has $4\times(4+2)+3\times(2+2)+2\times2+1\times(6+5)=51$ points.
C has $4\times(6+2+2+2)+3\times0+2\times(5+4+2)+1\times0=70$ points.
D has $4\times0+3\times(6+5+4+2)+2\times2+1\times(2+2)=59$ points.
The winner is *C*.

(b) *C, D, B, A.* Candidates are ranked according to the number of Borda points they received.

24. (a) *A* has $4\times(6+3)+3\times(4+3)+2\times6+1\times5=74$ points.
B has $4\times(6+5+3)+3\times3+2\times0+1\times(6+4)=75$ points.
C has $4\times0+3\times(6+6+5)+2\times(4+3+3)+1\times0=71$ points.
D has $4\times4+3\times0+2\times(6+5)+1\times(6+3+3)=50$ points.
The winner is B.

(b) *B, A, C, D.* Candidates are ranked according to the number of Borda points they received.

25. Here we can use a total of 100 voters for simplicity.
A has $5\times11+4\times(24+23+19)+3\times(14+9)+2\times0+1\times0=388$ points.
B has $5\times14+4\times0+3\times(24+11)+2\times23+1\times(19+9)=249$ points.
C has $5\times24+4\times(14+11+9)+3\times23+2\times19+1\times0=363$ points.
D has $5\times(23+19+9)+4\times0+3\times0+2\times14+1\times(24+11)=318$ points.
E has $5\times0+4\times0+3\times19+2\times(24+11+9)+1\times(23+14)=182$ points.
The ranking (according to Borda points) is *A, C, D, B, E*.

26. We use a total of 100 voters for simplicity.
A has $5\times12+4\times0+3\times9+2\times(25+21+10)+1\times(15+8)=222$ points.
B has $5\times15+4\times9+3\times(21+12)+2\times8+1\times(25+10)=261$ points.
C has $5\times(25+10+9+8)+4\times0+3\times0+2\times15+1\times(21+12)=323$ points.
D has $5\times0+4\times(21+15+12+10)+3\times(25+8)+2\times0+1\times9=340$ points.
E has $5\times21+4\times(25+8)+3\times(15+10)+2\times(12+9)+1\times0=354$ points.
The ranking (according to Borda points) is *E, D, C, B, A.*

27. Cooper had $3\times49+2\times280+1\times316=1023$ points.
Gordon had $3\times37+2\times432+1\times275=1250$ points.
Mariota had $3\times788+2\times74+1\times22=2534$ points.
The ranking (according to Borda points) is Mariota (2534), Gordon (1250), and Cooper (1023).

28. Hernandez had $7\times13+4\times17+3\times0+2\times0+1\times0=159$ points.
Kluber had $7\times17+4\times11+3\times2+2\times0+1\times0=169$ points.
Lester had $7\times0+4\times0+3\times3+2\times15+1\times7=46$ points.
Sale had $7\times0+4\times2+3\times19+2\times5+1\times3=78$ points.
Scherzer had $7\times0+4\times0+3\times4+2\times6+1\times8=32$ points.
As in Example 1.12, this uses a modified Borda count. In this case, first-place votes count 7 points rather than the usual 5. The ranking (according to modified Borda points) is Kluber (169), Hernandez (159), Sale (78), Lester (46), and Sherzer (32).

29. Each ballot has $4\times1+3\times1+2\times1+1\times1=10$ points that are awarded to candidates according to the Borda count. With 110 voters, there are a total of $110\times10=1100$ Borda points. So *D* has 1100 – 320 – 290 – 180 = 310 Borda points. The ranking is thus *A* (320), *D* (310), *B* (290), and *C* (180).

30. Each ballot has $7\times1+4\times1+3\times1+2\times1+1\times1=17$ points that are awarded to candidates according to the Borda count. With 50 voters, there are a total of $50\times17=850$ Borda points. So *E* has 850 – 152 – 133 – 191 – 175 = 199 Borda points. The ranking is thus *E* (199), *C* (191), *D* (175), *A* (152), and *B* (133).

1.4. Plurality-with-Elimination

31. (a) *A* is the winner. Round 1:

Candidate	*A*	*B*	*C*	*D*
Number of first-place votes	15	20	27	9

D is eliminated.
Round 2: The 9 first-place votes originally going to *D* now go to *A*.

Candidate	*A*	*B*	*C*	*D*
Number of first-place votes	24	20	27	

B is eliminated.
Round 3: There are 8 + 1 = 9 first-place votes originally going to *B* that now go to *A*. There are also 11 first-place votes going to *B* that would now go to *D*. But, since D is already eliminated, these 11 first-place votes go to *A*.

Candidate	*A*	*B*	*C*	*D*
Number of first-place votes	44		27	

Candidate *A* now has a majority of the first-place votes and is declared the winner.

(b) A complete ranking of the candidates can be found by noting in part (a) when each candidate was eliminated. Since *D* was eliminated first, it is ranked last. Since *B* was eliminated next, it is ranked next to last. The final ranking is hence *A, C, B, D*.

32. (a) *B* is the winner. Round 1:

Candidate	*A*	*B*	*C*	*D*
Number of first-place votes	21	18	11	29

C is eliminated.
Round 2: The 11 first-place votes originally going to *C* would next go to *B*.

Candidate	*A*	*B*	*C*	*D*
Number of first-place votes	21	29		29

Round 3: The 21 first-place votes going to *A* would next go to *C*. But *C* has been eliminated. So these 21 first-place votes go to *B*.

Candidate	*A*	*B*	*C*	*D*
Number of first-place votes		50		29

Candidate *B* now has a majority of the first-place votes and is declared the winner.

(b) A complete ranking of the candidates can be found by noting in part (a) when each candidate was eliminated. Since *C* was eliminated first, it is ranked last. Since *A* was eliminated next, it is ranked next to last. The final ranking is hence *B, D, A, C*.

33. (a) *C* is the winner. Round 1:

Candidate	*A*	*B*	*C*	*D*
Number of first-place votes	5	6	12	0

Candidate *C* has a majority of the first-place votes and is declared the winner.

(b) To determine a ranking, we ignore the fact that *C* wins and at the end of round 1, *D* is the first candidate eliminated.
Round 2: No first-place votes are changed.

Candidate	*A*	*B*	*C*	*D*
Number of first-place votes	5	6	12	

A is eliminated.
Round 3: There are 5 first-place votes originally going to *A* that now go to *C*.

Candidate	*A*	*B*	*C*	*D*
Number of first-place votes		6	17	

The final ranking is *C, B, A, D*.

34. (a) *B* is the winner. Round 1:

Candidate	*A*	*B*	*C*	*D*
Number of first-place votes	9	14	0	4

Candidate *B* has a majority of the first-place votes and is declared the winner.

(b) To determine a ranking, we ignore the fact that *B* wins and at the end of round 1, *C* is the first candidate eliminated.
Round 2: No first-place votes are changed.

Candidate	*A*	*B*	*C*	*D*
Number of first-place votes	9	14		4

D is eliminated.
Round 3: There are 4 first-place votes originally going to *D* that now go to *A*.

Candidate	*A*	*B*	*C*	*D*
Number of first-place votes	13	14		

The final ranking is *B, A, D, C*.

35. Round 1:

Candidate	*A*	*B*	*C*	*D*	*E*
Number of first-place votes	8	8	7	6	0

E is eliminated.
Round 2: No first-place votes change.

Candidate	*A*	*B*	*C*	*D*	*E*
Number of first-place votes	8	8	7	6	

D is eliminated.
Round 3: There are 4 first-place votes for *D* that move to candidate *C* and there are 2 first-place votes for *D* that move to candidate *B*.

Candidate	*A*	*B*	*C*	*D*	*E*
Number of first-place votes	8	10	11		

A is eliminated.
Round 4: There are 5 first-place votes for *A* that move to candidate *B* and there are 3 first-place votes for *A* that move to candidate *C* (since *D* and *E* have both been eliminated).

Candidate	*A*	*B*	*C*	*D*	*E*
Number of first-place votes		15	14		

B now has a majority of the first-place votes and is declared the winner. The final ranking is *B, C, A, D, E*.

36. Round 1:

Candidate	*A*	*B*	*C*	*D*	*E*
Number of first-place votes	8	4	11	12	5

B is eliminated.
Round 2: There are 4 first-place votes for *B* that move to candidate *E*.

Candidate	*A*	*B*	*C*	*D*	*E*
Number of first-place votes	8		11	12	9

A is eliminated.
Round 3: There are 5 first-place votes for *A* that move to candidate *E* (since *B* is eliminated), 2 first-place votes for *A* that move to candidate *D* (since *B* is eliminated), and 1 first-place vote for *A* that moves to *C*.

Candidate	*A*	*B*	*C*	*D*	*E*
Number of first-place votes			12	14	14

C is eliminated.
Round 4: There are 6 first-place votes for *C* that move to candidate *E* (since *A* is eliminated), 5 first-place votes for *C* that move to candidate *D* (since both *A* and *B* are eliminated), and there is 1 first-place vote for *C* that moves to candidate *E* (*C* earned this vote earlier when candidate *A* was eliminated).

Candidate	*A*	*B*	*C*	*D*	*E*
Number of first-place votes				19	21

E now has a majority of the first-place votes and is declared the winner. The final ranking is *E, D, C, A, B*.

37. **(a)** *D* is the winner. Round 1:

Candidate	*A*	*B*	*C*	*D*	*E*
Percentage of first-place votes	11	14	24	51	0

Candidate *D* has a majority of the first-place votes and is declared the winner.

(b) To determine a ranking, we ignore the fact that *D* wins and at the end of round 1, *E* is the first candidate eliminated.
Round 2: No first-place votes are changed.

Candidate	*A*	*B*	*C*	*D*	*E*
Percentage of first-place votes	11	14	24	51	

A is eliminated.
Round 3: The 11% of the first-place votes that went to *A* now go to *C*.

Candidate	*A*	*B*	*C*	*D*	*E*
Percentage of first-place votes		14	35	51	

B is eliminated.
Round 4: The 14% of the first-place votes that went to *B* now go to *C*.

Candidate	*A*	*B*	*C*	*D*	*E*
Percentage of first-place votes			49	51	

A complete ranking of the candidates can be found by noting when each candidate was eliminated. The final ranking is hence *D, C, B, A, E*.

38. (a) *C* is the winner. Round 1:

Candidate	*A*	*B*	*C*	*D*	*E*
Percentage of first-place votes	12	15	52	0	21

Candidate *C* has a majority of the first-place votes and is declared the winner.

(b) To determine a ranking, we ignore the fact that *C* wins and at the end of round 1, *D* is the first candidate eliminated.
Round 2: No first-place votes are changed.

Candidate	*A*	*B*	*C*	*D*	*E*
Percentage of first-place votes	12	15	52		21

A is eliminated.
Round 3: The 12% of the first-place votes that went to *A* now go to *B* (since *D* has already been eliminated).

Candidate	*A*	*B*	*C*	*D*	*E*
Percentage of first-place votes		27	52		21

E is eliminated.
Round 4: The 14% of the first-place votes that went to *E* now go to *B* (since *D* has already been eliminated).

Candidate	*A*	*B*	*C*	*D*	*E*
Percentage of first-place votes		48	52		

A complete ranking of the candidates can be found by noting when each candidate was eliminated. The final ranking is hence *C, B, E, A, D*.

39. Round 1:

Candidate	*A*	*B*	*C*	*D*	*E*
Number of first-place votes	8	8	7	6	0

Candidates *E, D,* and *C* are all eliminated.
Round 2: There are 4 first-place votes for *D* that go to *B* (since *C* has been eliminated). There are 2 first-place votes for *D* that go to *B*. There are 7 first-place votes for *C* that go to *A* (since both *D* and *E* are eliminated).

Candidate	*A*	*B*	*C*	*D*	*E*
Number of first-place votes	15	14			

A now has a majority of the first-place votes and is declared the winner.

40. Round 1:

Candidate	*A*	*B*	*C*	*D*	*E*
Number of first-place votes	8	4	11	12	5

Candidates *B, E,* and *A* are all eliminated.
Round 2: There are 4 first-place votes for *B* that go to *C* (since *E* has been eliminated). There are 5 first-place votes for *E* that go to *D* (since *A* has been eliminated). There are 5 first-place votes for *A* that go to *C* (since both *B* and *E* are eliminated), there are 2 first-place votes for *A* that go to *D* (since *B* is eliminated), and 1 first-place vote for *A* that goes to *C*.

Candidate	*A*	*B*	*C*	*D*	*E*
Number of first-place votes			21	19	

C now has a majority of the first-place votes and is declared the winner.

1.5. Pairwise Comparisons

41. **(a)** Candidate *D* is the winner.
A versus *B*: 15 + 9 = 24 votes to 27 + 11 + 8 + 1 = 47 votes (*B* wins). *B* gets 1 point.
A versus *C*: 15 + 11 + 9 + 8 + 1 = 44 votes to 27 votes (*A* wins). *A* gets 1 point.
A versus *D*: 15 + 8 + 1 = 24 votes to 27 + 11 + 9 = 47 votes (*D* wins). *D* gets 1 point.
B versus *C*: 15 + 11 + 9 + 8 + 1 = 44 votes to 27 votes (*B* wins). *B* gets 1 point.
B versus *D*: 15 + 11 + 8 + 1 = 35 votes to 27 + 9 = 36 votes (*D* wins). *D* gets 1 point.
C versus *D*: 27 + 8 = 35 votes to 15 + 11 + 9 + 1 = 36 votes. (*D* wins). *D* gets 1 point.
The final tally is 1 point for *A*, 2 points for *B*, 0 points for *C*, and 3 points for *D*.

(b) A complete ranking for the candidates is found by tallying points. In this case, the final ranking is *D* (3 points), *B* (2 points), *A* (1 point), and *C* (0 points).

42. **(a)** Candidate *C* is the winner.
A versus *B*: 29 + 21 = 50 votes to 18 + 10 + 1 = 29 votes (*A* wins). *A* gets 1 point.
A versus *C*: 21 + 18 = 39 votes to 29 + 10 + 1 = 40 votes (*C* wins). *C* gets 1 point.
A versus *D*: 21 + 18 + 10 = 49 votes to 29 + 1 = 30 votes (*A* wins). *A* gets 1 point.
B versus *C*: 18 votes to 29 + 21 + 10 + 1 = 61 votes (*C* wins). *C* gets 1 point.
B versus *D*: 21 + 18 + 10 + 1 = 50 votes to 29 votes (*B* wins). *B* gets 1 point.
C versus *D*: 21 + 18 + 10 + 1 = 50 votes to 29 votes. (*C* wins). *C* gets 1 point.
The final tally is 2 points for *A*, 1 point for *B*, 3 points for *C*, and 0 points for *D*.

(b) A complete ranking for the candidates is found by tallying points. In this case, the final ranking is *C* (3 points), *A* (2 points), *B* (1 point), and *D* (0 points).

43. **(a)** Candidate *C* is the winner.
A versus *B*: 6 + 5 = 11 votes to 4 + 2 + 2 + 2 + 2 = 12 votes (*B* wins). *B* gets 1 point.
A versus *C*: 5 + 2 = 7 votes to 6 + 4 + 2 + 2 + 2 = 16 votes (*C* wins). *C* gets 1 point.
A versus *D*: 5 + 2 + 2 = 9 votes to 6 + 4 + 2 + 2 = 14 votes (*D* wins). *D* gets 1 point.
B versus *C*: 4 + 2 = 6 votes to 6 + 5 + 2 + 2 + 2 = 17 votes (*C* wins). *C* gets 1 point.
B versus *D*: 4 + 2 + 2 + 2 = 10 votes to 6 + 5 + 2 = 13 votes (*D* wins). *D* gets 1 point.
C versus *D*: 6 + 2 + 2 + 2 + 2 = 10 votes to 5 + 4 = 9 votes. (*C* wins). *C* gets 1 point.
The final tally is 0 points for *A*, 1 point for *B*, 3 points for *C*, and 2 points for *D*.

(b) A complete ranking for the candidates is found by tallying points. In this case, the final ranking is *C* (3 points), *D* (2 points), *B* (1 point), and *A* (0 points).

44. **(a)** Candidate *B* is the winner.
A versus *B*: 6 + 4 + 3 = 13 votes to 6 + 5 + 3 = 14 votes (*B* wins). *B* gets 1 point.
A versus *C*: 6 + 4 + 3 + 3 = 16 votes to 6 + 5 = 11 votes (*A* wins). *A* gets 1 point.
A versus *D*: 6 + 6 + 3 + 3 = 18 votes to 5 + 4 = 9 votes (*A* wins). *A* gets 1 point.
B versus *C*: 6 + 5 + 3 = 14 votes to 6 + 4 + 3 = 13 votes (*B* wins). *B* gets 1 point.
B versus *D*: 6 + 5 + 3 = 14 votes to 6 + 4 + 3 = 13 votes (*B* wins). *B* gets 1 point.
C versus *D*: 6 + 6 + 5 + 3 + 3 = 23 votes to 4 votes. (*C* wins). *C* gets 1 point.
The final tally is 2 points for *A*, 3 points for *B*, 1 point for *C*, and 0 points for *D*.

(b) The final ranking is *B* (3 points), *A* (2 points), *C* (1 point), and *D* (0 points).

45. Candidate *D* is the winner.
A versus *B*: 24% + 23% + 19% + 11% + 9% = 86% of the votes to 14% of the votes (*A* wins).
A versus *C*: 23% + 19% + 11% = 53% of the votes to 24% + 14% + 9% = 47% of the votes (*A* wins).
A versus *D*: 24% + 14% + 11% = 49% of the votes to 23% + 19% + 9% = 51% of the votes (*D* wins).
A versus *E*: 24% + 23% + 19% + 14% + 11% + 9% = 100% of the votes to 0% of the votes (*A* wins).
B versus *C*: 14% of the votes to 24% + 23% + 19% + 11% + 9% = 86% of the votes (*C* wins).
B versus *D*: 24% + 14% + 11% = 49% of the votes to 23% + 19% + 9% = 51% of the votes (*D* wins).
B versus *E*: 24% + 23% + 14% + 11% = 72% of the votes to 19% + 9% = 28% of the votes (*B* wins).
C versus *D*: 24% + 14% + 11% = 49% of the votes to 23% + 19% + 9% = 51% of the votes (*D* wins).
C versus *E*: 24% + 23% + 14% + 11% + 9% = 81% of the votes to 19% of the votes (*C* wins).
D versus *E*: 23% + 19% + 14% + 9% = 65% of the votes to 24% + 11% = 35% of the votes (*D* wins).
The final tally is 3 points for *A*, 1 point for *B*, 2 points for *C*, 4 points for *D*, and 0 points for *E*.

46. Candidate *C* is the winner.
A versus *B*: 25% + 12% + 10% = 47% of the votes to 21% + 15% + 9% + 8% = 53% of the votes (*B* wins).
A versus *C*: 21% + 12% = 33% of the votes to 25% + 15% + 10% + 9% + 8% = 67% of the votes (*C* wins).
A versus *D*: 12% + 9% = 21% of the votes to 25% + 21% + 15% + 10% + 8% = 79% of the votes (*D* wins).
A versus *E*: 12% + 9% = 21% of the votes to 25% + 21% + 15% + 10% + 8% = 79% of the votes (*E* wins).
B versus *C*: 21% + 15% + 12% = 48% of the votes to 25% + 10% + 9% + 8% = 52% of the votes (*C* wins).
B versus *D*: 15% + 9% = 24% of the votes to 25% + 21% + 12% + 10% + 8% = 76% of the votes (*D* wins).
B versus *E*: 15% + 12% + 9% = 36% of the votes to 25% + 21% + 10% + 8% = 64% of the votes (*E* wins).
C versus *D*: 25% + 10% + 9% + 8% = 52% of the votes to 21% + 15% + 12% = 48% of the votes (*C* wins).
C versus *E*: 25% + 10% + 9% + 8% = 52% of the votes to 21% + 15% + 12% = 48% of the votes (*C* wins).
D versus *E*: 15% + 12% + 10% = 37% of the votes to 25% + 21% + 9% + 8% = 63% of the votes (*E* wins).
The final tally is 0 points for *A*, 1 point for *B*, 4 points for *C*, 2 points for *D*, and 3 points for *E*.

47. *A* versus *B*: 7 + 5 + 3 = 15 votes to 8 + 4 + 2 = 14 votes (*A* wins). *A* gets 1 point.
A versus *C*: 8 + 5 + 3 = 16 votes to 7 + 4 + 2 = 13 votes (*A* wins). *A* gets 1 point.
A versus *D*: 8 + 5 + 3 = 16 votes to 7 + 4 + 2 = 13 votes (*A* wins). *A* gets 1 point.
A versus *E*: 5 + 3 + 2 = 10 votes to 8 + 7 + 4 = 19 votes (*E* wins). *E* gets 1 point.
B versus *C*: 8 + 5 + 2 = 15 votes to 7 + 4 + 3 = 14 votes (*B* wins). *B* gets 1 point.
B versus *D*: 8 + 5 = 13 votes to 7 + 4 + 3 + 2 = 16 votes (*D* wins). *D* gets 1 point.
B versus *E*: 8 + 5 + 4 + 2 = 19 votes to 7 + 3 = 10 votes (*B* wins). *B* gets 1 point.
C versus *D*: 8 + 7 + 5 = 20 votes to 4 + 3 + 2 = 9 votes (*C* wins). *C* gets 1 point.
C versus *E*: 7 + 5 + 4 + 2 = 18 votes to 8 + 3 = 11 votes (*C* wins). *C* gets 1 point.
D versus *E*: 5 + 4 + 3 + 2 = 14 votes to 8 + 7 = 15 votes (*E* wins). *E* gets 1 point.
The final tally is 3 points for *A*, 2 points for *B*, 2 points for *C*, 1 point for *D*, and 2 points for *E*.
Now *B, C*, and *E* each have 2 points. In head-to-head comparisons, *B* beats *C* and *B* beats *E* so that *B* is ranked higher than *C* and *E*. Also, *C* beats *E* so *C* is ranked higher than *E* as well. The final ranking is thus *A, B, C, E, D*.

48. *A* versus *B*: 6 + 5 + 5 + 5 + 5 + 2 + 1 = 29 votes to 11 votes (*A* wins). *A* gets 1 point.
A versus *C*: 7 + 5 + 5 + 2 + 1 = 20 votes to 20 votes (tie). *A* and *C* each get ½ point.
A versus *D*: 6 + 5 + 5 + 5 + 2 + 1 = 24 votes to 16 votes (*A* wins). *A* gets 1 point.
A versus *E*: 7 + 6 + 5 + 5 + 5 + 2 + 1 = 31 votes to 9 votes (*A* wins). *A* gets 1 point.
B versus *C*: 7 + 5 + 5 + 4 + 2 = 23 votes to 17 votes (*B* wins). *B* gets 1 point.
B versus *D*: 6 + 5 + 5 + 4 + 2 + 1 = 23 votes to 17 votes (*B* wins). *B* gets 1 point.
B versus *E*: 7 + 5 + 5 + 4 + 2 = 23 votes to 17 votes (*B* wins). *B* gets 1 point.
C versus *D*: 6 + 5 + 5 + 4 + 1 = 21 votes to 19 votes (*C* wins). *C* gets 1 point.
C versus *E*: 7 + 6 + 5 + 5 + 1 = 24 votes to 16 votes (*C* wins). *C* gets 1 point.
D versus *E*: 7 + 5 + 5 + 2 = 19 votes to 21 votes (*E* wins). *E* gets 1 point.
The final tally is 3.5 points for *A*, 3 points for *B*, 2.5 points for *C*, 0 points for *D*, and 1 point for *E*.
The final ranking is thus *A, B, C, E, D*.

49. **(a)** With five candidates, there are a total of 4 + 3 + 2 + 1 = 10 pairwise comparisons. Each candidate is part of 4 of these (one against each other candidate). So, to find the number of points each candidate earns, we simply subtract the losses from 4. The 10 points are distributed as follows: *E* wins $1\frac{1}{2}$ points, *D* wins $2\frac{1}{2}$ points, *C* gets 3 points, *B* gets 2 points, and *A* gets the remaining $10-1\frac{1}{2}-2\frac{1}{2}-3-2=1$ point. So *A* loses 3 pairwise comparisons.

(b) Candidate *C*, with 3 points, is the winner. [The complete ranking is *C, D, B, E, A*.]

50. **(a)** Since there are a total of $(6\times 5)/2=15$ pairwise comparisons, *F* must have won 15 – 1 – 2 – 2 - $3\frac{1}{2}$ - $2\frac{1}{2}$ = 4 of them (*A* earned 1 point, *B* and *C* each earned 2 points, *D* earned $3\frac{1}{2}$, and *E* earned $2\frac{1}{2}$ points). This means *F lost* 1 pairwise comparison.

(b) Candidate *F*, with 4 points, is the winner.

1.6. Fairness Criteria

51. First, we determine the winner using the Borda count.
A has $4\times 6+3\times 0+2\times 0+1\times(2+3)=29$ points.
B has $4\times 2+3\times 6+2\times 3+1\times 0=32$ points.
C has $4\times 3+3\times 2+2\times 6+1\times 0=30$ points.
D has $4\times 0+3\times 3+2\times 2+1\times 6=19$ points.
So candidate *B* is the winner using Borda count. However, candidate *A* has 6 of the 11 votes (a majority) and beats all three other candidates (*B*, *C*, and *D*) in head-to-head comparisons. That is, candidate *A* is a Condorcet candidate. Since this candidate did not win using the Borda count, this is a violation of the Condorcet criterion.

52. First, we determine the winner is candidate *B* using the plurality-with-elimination method (see Exercise 32(a)). In Exercise 42, however, we saw that candidate *C* was a Condorcet candidate (beating each other candidate in head-to-head comparisons and earning 3 points in the process). Since candidate *C* did not win using plurality-with-elimination, this is a violation of the Condorcet criterion.

53. The winner of this election is candidate R using the plurality method. Now F is clearly a nonwinning candidate. Removing F as a candidate leaves the following preference table.

Number of voters	**49**	**48**	**3**
1st choice	R	H	H
2nd choice	H	S	S
3rd choice	O	O	O
4th choice	S	R	R

In a recount, candidate H would be the winner using the plurality method. This is a violation of the IIA criterion.

54. First, we determine the winner using the Borda count.
A has $4\times14+3\times0+2\times0+1\times(10+8+4+1)=79$ points.
B has $4\times4+3\times(14+10)+2\times(8+1)+1\times0=106$ points.
C has $4\times(10+1)+3\times8+2\times(14+4)+1\times0=104$ points.
D has $4\times8+3\times(4+1)+2\times10+1\times14=81$ points.
So candidate B (Boris) is the winner using Borda count. Now D (Dave) is clearly a nonwinning (irrelevant) candidate. Removing D as a candidate leaves the following preference table.

Number of voters	**14**	**10**	**8**	**4**	**1**
1st choice	A	C	C	B	C
2nd choice	B	B	B	C	B
3rd choice	C	A	A	A	A

We now recount using the Borda count.
A has $3\times14+2\times0+1\times(10+8+4+1)=65$ points.
B has $3\times4+2\times(14+10+8+1)+1\times0=78$ points.
C has $3\times(10+8+1)+2\times4+1\times14=79$ points.
In a recount, candidate C (Carmen!) would be the winner using Borda count, a violation of the IIA criterion.

55. First, we determine the winner using plurality-with-elimination. Round 1:

Candidate	A	B	C	D	E
Number of first-place votes	8	3	5	5	0

Candidate E is eliminated. Round 2: No votes are shifted.

Candidate	A	B	C	D	E
Number of first-place votes	8	3	5	5	

Candidate B is eliminated. Round 3: The 3 first-place votes for B now go to A (since E has been eliminated).

Candidate	A	B	C	D	E
Number of first-place votes	11		5	5	

Candidate A now has a majority (11 of the 21 votes) and is declared the winner. Now C is clearly a nonwinning (irrelevant) candidate. Removing C as a candidate leaves the following preference table.

Number of voters	**5**	**5**	**3**	**3**	**3**	**2**
1st choice	A	E	A	D	B	D
2nd choice	B	D	D	B	E	B
3rd choice	D	B	B	E	A	A
4th choice	E	A	E	A	D	E

We now recount using the plurality-with-elimination.

Round 1:

Candidate	*A*	*B*	*D*	*E*
Number of first-place votes	8	3	5	5

Candidate *B* is eliminated.
Round 2: The 3 first-place votes that went to *B* now shift to *E*.

Candidate	*A*	*B*	*D*	*E*
Number of first-place votes	8		5	8

Candidate *D* is eliminated.
Round 3: *D* had 5 first-place votes. Of these, 3 go to *E* and 2 go to *A* (since *B* was eliminated).

Candidate	*A*	*B*	*D*	*E*
Number of first-place votes	10			11

Candidate *E* now has a majority (11 of the 21 votes) and is declared the winner. Remember that candidate *A* was the winner before candidate *C* was removed. This is a violation of the IIA criterion.

56. If *X* has a majority of the first-place votes, then *X* will win every pairwise comparison (it is ranked above all other candidates on more than half the ballots) and is, therefore, the winner under the method of pairwise comparisons.

57. If *X* is the Condorcet candidate, then by definition *X* wins every pairwise comparison and is, therefore, the winner under the method of pairwise comparisons.

58. When a voter moves a candidate up in his or her ballot the number of first place votes for that candidate either increases or stays the same. It follows that if *X* had a plurality of the first place votes and a voter changes his or her ballot to rank *X* higher, then *X* still has a plurality.

59. When a voter moves a candidate up in his or her ballot, that candidate's Borda points increase. It follows that if *X* had the most Borda points and a voter changes his or her ballot to rank *X* higher, then *X* still has the most Borda points.

60. When a voter moves a candidate up in his or her ballot it can't hurt the candidate in a pairwise comparison—the candidate wins the same pairwise comparisons as before and possibly a few more. It follows that if *X* won the most pairwise comparisons and a voter changes his or her ballot to rank *X* higher, then *X* still wins the most pairwise comparisons.

JOGGING

61. Suppose the two candidates are *A* and *B* and that *A* gets a first-place votes and *B* gets b first-place votes and suppose that $a > b$. Then *A* has a majority of the votes and the preference schedule is

Number of voters	a	b
1st choice	*A*	*B*
2nd choice	*B*	*A*

It is clear that candidate *A* wins the election under the plurality method, the plurality-with-elimination method, and the method of pairwise comparisons. Under the Borda count method, *A* gets $2a + b$ points while *B* gets $2b + a$ points. Since $a > b$, $2a + b > 2b + a$ and so again *A* wins the election.

62. In this variation of the Borda count each candidate gets 1 less point per ballot. It follows that if N is the number of voters, each candidate gets N fewer points than he or she would under the standard Borda count method. Since each candidate's total points gets decreased by the same number N, the ranking of the candidates remains the same.

63. The number of points under this variation is complementary to the number of points under the standard Borda count method: a first place is worth 1 point instead of N, a second place is worth 2 points instead of $N - 1, \ldots$, a last place is worth N points instead of 1. It follows that having the fewest points here is equivalent to having the most points under the standard Borda count method, having the second fewest is equivalent to having the second most, and so on.

As another way to see this, suppose candidates C_1 and C_2 receive p_1 and p_2 points respectively using the Borda count as originally described in the chapter and r_1 and r_2 points under the variation described in this exercise. Then, for an election with k voters, we have $p_1 + r_1 = p_2 + r_2 = k(N+1)$. So if $p_1 < p_2$, we have $-p_1 > -p_2$ and so $k(N + 1) - p_1 > k(N + 1) - p_2$ which implies $r_1 > r_2$. Consequently the relative ranking of the candidates is not changed.

64. Use the Reverse Borda count method described in Exercise 63. In this variation the ranking of a candidate on a ballot equals the number of Borda points the candidate gets from that ballot. It follows that the average ranking of a candidate equals the total Borda points for that candidate divided by N (the number of voters). The candidate with the lowest average ranking is the candidate with the least total points and thus the winner of the election.

65. **(a)** Each of the voters gave Ohio State 25 points, so we compute 1625/25 to find 65 voters in the poll.

(b) Let x = the number of Florida's second-place votes. Then $1529 = 24 \cdot x + 23 \cdot (65 - x)$ and so $x = 34$. This leaves 65 – 34 = 31 third-place votes for Florida.

(c) Let y = the number of Michigan's second-place votes. Then $1526 = 24 \cdot y + 23 \cdot (65 - y)$ and so $y = 31$. This leaves 65 – 31 = 34 third-place votes for Michigan.

66. **(a)** Under the Borda count, each voter gives X more points than Y. Hence, the Borda count for X will be greater than the Borda count for Y and so X will rank above Y.

(b) Since every voter prefers candidate X to candidate Y, a pairwise comparison between X and Y results in a win for X. If Y wins a pairwise comparison against Z, then X must also win against Z (by transitivity – since X is above Y on each ballot, when Y is above Z it must be that X is also above Z). Therefore X will have at least one more point than Y (from the head-to-head between X and Y). Thus, X will rank above Y under pairwise comparisons.

67. By looking at Dwayne Wade's vote totals, it is clear that 1 point is awarded for each third-place vote (0 points is too few and 2 points is too many for each third-place vote). Let x = points awarded for each second-place vote. Then, $3x + 108 = 117$ gives $x = 3$ points for each second-place vote. Next, let y = points awarded for each first-place vote. Looking at LeBron James' votes, we see that $78y + 39 \times 3 + 1 \times 1 = 508$. Solving this equation gives $y = 5$ points for each first-place vote.

68. **(a)** Recall that using plurality-with-elimination in the Math Club election resulted in B being eliminated in the first round, C being eliminated second, A being eliminated third, and D winning (see Example 1.13). If top-two IRV is used, candidates B and D are *both* eliminated in round one. The 4 first-place votes for B would transfer to candidate C. The 8 first-place votes for D would also transfer to candidate C. The new preference table is shown below.

Number of voters	**14**	**23**
1st choice	*A*	*C*
2nd choice	*C*	*A*

We then have C winning the election -- clearly different than the plurality-with-elimination outcome.

(b) Many examples are possible. (Example 1.21 shows that plurality-with-elimination violates the monotonicity criterion and in that example, since there are only 3 candidates from the start, top-two IRV and plurality-with-elimination are identical methods and give the same result.)

(c) Consider an election with 3 candidates (*A*, *B*, and *C*) and preference schedule as follows.

Number of voters	10	8	6
1st choice	*A*	*C*	*B*
2nd choice	*B*	*B*	*C*
3rd choice	*C*	*A*	*A*

In this election *B* is a Condorcet candidate (since *B* beats *A* by a 'score' of14 to 10 and *B* beats *C* by a score of 16 to 8) and yet *B* is eliminated first using top-two IRV making *C* the winner.

69. (a) Round 1:

Number of voters	14	10	8	4	1
1st choice	*A*	*C*	*D*	*B*	*C*
2nd choice	*B*	*B*	*C*	*D*	*D*
3rd choice	*C*	*D*	*B*	*C*	*B*
4th choice	*D*	*A*	*A*	*A*	*A*

Candidate *A* with 23 last-place votes is eliminated. Round 2:

Number of voters	14	10	8	4	1
1st choice	*B*	*C*	*D*	*B*	*C*
2nd choice	*C*	*B*	*C*	*D*	*D*
3rd choice	*D*	*D*	*B*	*C*	*B*

Candidate *B* has 8 + 1 = 9 last-place votes, *C* has 4 last-place votes, and *D* has 14 + 10 = 24 last-place votes. So candidate *D* is eliminated. Round 3:

Number of voters	14	10	8	4	1
1st choice	*B*	*C*	*C*	*B*	*C*
2nd choice	*C*	*B*	*B*	*C*	*B*

B has more last-place votes (meaning *C* has more first-place votes) and *C* is declared the winner.

(b) Consider the election given by the preference schedule below. Here *A* is a Condorcet candidate but, having the most last-place votes, is eliminated in the first round.

Number of voters	10	6	6	3	3
1st choice	*B*	*A*	*A*	*D*	*C*
2nd choice	*C*	*B*	*C*	*A*	*A*
3rd choice	*D*	*D*	*B*	*C*	*B*
4^{th} choice	*A*	*C*	*D*	*B*	*D*

(c) Consider the election given by the preference schedule below. Here *B* wins under the Coombs method (*C* is eliminated first and *B* is preferred to *A* by 17 voters). However, if 8 voters move *B* from their 3^{rd} choice to their 2^{nd} choice, then *C* wins (since *A* would be eliminated first and *C* is preferred head-to-head over *B*).

Number of voters	10	8	7	4
1st choice	*B*	*C*	*C*	*A*
2nd choice	*A*	*A*	*B*	*B*
3rd choice	*C*	*B*	*A*	*C*

70. (a) In round 1, no candidate has a majority of the 37 votes.

Number of voters	**14**	**10**	**8**	**4**	**1**
1st choice	*A*	*C*	*D*	*B*	*C*

In round 2, counting 1st and 2nd place votes, *A* totals 14 votes, *B* totals 14 + 10 + 4 = 28 votes, *C* totals 10 + 8 + 1 = 19 votes, and *D* total 8 + 4 + 1 = 13 votes.

Number of voters	**14**	**10**	**8**	**4**	**1**
1st choice	*A*	*C*	*D*	*B*	*C*
2nd choice	*B*	*B*	*C*	*D*	*D*

Both *B* and *C* have a majority of the 37 votes. However, candidate *B* has more and is declared the winner.

(b) *C* is a Condorcet candidate in the Math Club election. However, as shown in part (a), *B* wins the election using the Bucklin method.

(c) Say a ballot has candidate *X* ranked in the *k*th position. If and when the election gets to round *k* that vote contributes to *X*'s total under the Bucklin method. It follows that if *X* wins in round *k* and a voter moves *X* up in his or her ballot then X would still win (in round *k* or possibly in an earlier round).

RUNNING

71. One reasonable approach would be to use variables x_k, $1 \le k \le 10$, to represent the number of points handed out for *k*th place. With so many variables, there is some flexibility in determining the values of these variables. For starters, choose $x_{10} = 1$, $x_9 = 2$, $x_8 = 3$, $x_7 = 4$, $x_6 = 5$, and $x_5 = 6$. In that case, voting for Keuchel forces $3x_4 + 8\times 6 + 5\times 5 + 1\times 4 + 3\times 3 = 107$. That is, $x_4 = 7$. Then, voting for Machado forces $4x_3 + 11\times 7 + 5\times 6 + 1\times 5 + 1\times 4 + 1\times 3 + 3\times 2 + 1\times 1 = 158$. So $x_3 = 8$. We check this these choices work for Cain. But $20x_3 + 8\times 7 + 1\times 5 + 1\times 4 = 225$ also gives $x_3 = 8$. For Trout and Donaldson, we have two more equations that must hold:

$7x_1 + 22x_2 + 1\times 8 = 304$

$23x_1 + 7x_2 = 385$

One technique would be to solve these two linear equations in two unknowns by solving one of the equations for one variable and substituting that expression into the other equation. Another method would be to "guess" that $x_2 = 9$ and note that $x_1 = 14$ solves both equations. The point values on each ballot are thus determined to be 14 points for 1st, 9 points for 2nd, 8 points for 3rd, 7 points for 4th, 6 points for 5th, 5 points for 6th, 4 points for 7th, 3 points for 8th, 2 points for 9th, and 1 point for 10th.

72. (a) In this election, *C* is the winner under the plurality method yet a majority of the voters (4 out of 7) prefer both *A* and *B* over *C*.

Number of voters	**2**	**2**	**3**
1st choice	*A*	*B*	*C*
2nd choice	*B*	*A*	*A*
3rd choice	*C*	*C*	*B*

(b) We may use the same example as in part (a). Both *A* and *B* are eliminated in the first round and so *C* is the winner under the plurality-with-elimination method.

(c) Suppose there are k voters and N candidates. Under the Borda count method, the total number of points for all the candidates is $kN(N+1)/2$ (each voter contributes $1+2+\cdots+N = N(N+1)/2$ points to the total), so the average number of points per candidate is $k(N+1)/2$. Now suppose that X is a candidate that loses to every other candidate in a one-to-one comparison. The claim is that under the Borda count method, X will receive less than the average $k(N+1)$ points and therefore cannot be the winner. (One way to see this is as follows: Suppose that the N candidates are presented to the voters one at a time, with each voter keeping an updated partial ranking of the candidates as they are being presented. Suppose, moreover, that X is the first candidate to be presented. When this happens, X starts with kN points (X is the only candidate in every voter's ballot and therefore has all the first-place votes.) As soon as the next candidate (say Y) is presented, X's points will drop by more than $k/2$ (X loses to Y in a one-to-one comparison means that more than $k/2$ of the voters have ranked Y above X.) By the same argument each time one of the subsequent candidates is presented X's points drop by more than $k/2$. By the time we are done, X must have less than $kN - \frac{k}{2}(N-1) = \frac{k}{2}(N+1)$ points.)

73. (a) The Math Club election serves as an example. Candidate A has a majority of last-place votes (23 of the 37) and yet wins the election when the plurality method is used.

Number of voters	**14**	**10**	**8**	**4**	**1**
1st choice	*A*	*C*	*D*	*B*	*C*
2nd choice	*B*	*B*	*C*	*D*	*D*
3rd choice	*C*	*D*	*B*	*C*	*B*
4th choice	*D*	*A*	*A*	*A*	*A*

(b) In this election, C is the winner under the plurality method (in the case of the tie in round 1, eliminate all candidates with the fewest number of votes). However, a majority of the voters (4 out of 7) prefer both A and B over C.

Number of voters	**2**	**2**	**3**
1st choice	*A*	*B*	*C*
2nd choice	*B*	*A*	*A*
3rd choice	*C*	*C*	*B*

(c) Under the method of pairwise comparisons, if a majority of voters have candidate X ranked last on their ballot, then candidate X will never win a head-to-head comparison (since any other candidate Y is preferred to X by a majority of voters). Thus, X will end up with no points under the method and cannot win the election.

(d) Suppose there are N candidates, x is the number of voters placing X last, and y is the remaining number of voters. Since a majority of the voters place X last, $x > y$. The *maximum* number of points that X can receive using a Borda count is $x + Ny$. (One point for each of the x last place votes and, assuming all other voters place X in first-place, N points for each of the other y voters.) The total number of points given out by each voter is $1 + 2 + 3 + ... + N = N(N+1)/2$ and so the total number of points given out by all $x + y$ voters is $N(N+1)(x+y)/2$. Since some candidate must receive at least $1/N$ of the total points, some candidate must receive at least $(N+1)(x+y)/2$ points. Since $x > y$, we have $(N+1)(x+y)/2 = (Nx + x + Ny + y)/2 > (Ny + x + Ny + y)/2 > x + Ny$, and consequently, X cannot be the winner of the election using the Borda count method.

74. In this election, D is preferred over A by a majority of the voters (12 to 10) and yet all four complete rankings rank A above D.

Number of voters	**8**	**5**	**3**	**2**	**4**
1st choice	*A*	*C*	*C*	*C*	*B*
2nd choice	*B*	*D*	*B*	*B*	*D*
3rd choice	*C*	*A*	*D*	*A*	*A*
4th choice	*D*	*B*	*A*	*D*	*C*

Method	**Winner**	**2nd place**	**3rd place**	**Last place**
Plurality	*C*	*A*	*B*	*D*
Plurality with elimination	*A*	*C*	*B*	*D*
Borda count	*C,B* (tied)		*A*	*D*
Pairwise comparison	*A,B* (tied)		*C,D* (tied)	

75. Suppose there are k voters and N candidates.

Case 1: k is odd, say $k = 2t + 1$. Suppose the candidate with a majority of the first-place votes is X. The fewest possible Borda points X can have is $F(t + 1) + t$ [when there are $(t + 1)$ votes that place X first and the remaining votes place X last]. The most Borda points that any other candidate can have is $(N - 1)(t + 1) + Ft$ [when there are $(t + 1)$ voters that place the candidate second and the remaining voters place that candidate first]. Thus, the majority criterion will be satisfied when $F(t+ 1) + t > (N - 1)(t + 1) + Ft$, which after simplification implies $F > N(t + 1) - (2t + 1)$, or $F > N\left(\frac{k+1}{2}\right) - k$.

Case 2: k is even, say $k = 2t$. An argument similar to the one given in case 1 gives the inequality $F(t + 1) + (t - 1) > (N - 1)(t + 1) + F(t - 1)$ which after simplification implies $F > [N(t + 1) - 2t]/2$, or $F > \left[N\left(\frac{k}{2}+1\right) - k\right]/2$.

APPLET BYTES

76. (a) Plurality method: A has 89 first-place votes, B has 91 first-place votes, and C has 73 first-place votes. So, using the plurality method, B wins.

Borda count method: A has $3\times 49+3\times 40+2\times 40+1\times 51+2\times 43+1\times 30=514$ points.
B has $2\times 49+1\times 40+3\times 40+3\times 51+1\times 43+2\times 30=514$ points.
C has $1\times 49+2\times 40+1\times 40+2\times 51+3\times 43+3\times 30=490$ points.
So, using the Borda count method, there is a tie between candidates A and B (each with 514 points).

Plurality-with-elimination method: In round 1, B has 91 votes, A has 89 votes, and C has 73 votes. So candidate C is eliminated. In an election between A and B, A has $49 + 40 + 43 = 132$ votes and B has $40 + 51 + 30 = 121$ votes so B is eliminated and A is declared the winner.

Pairwise comparisons method:
A vs. B – A wins 132-121 and gets 1 point.
A vs. C – A wins 129-124 and gets 1 point.
B vs. C – B wins 140-113 and gets 1 point.

The final tally is *A* – 2 points, *B* – 1 point, *C* – 0 points. So, using the pairwise comparisons method, candidate *A* wins.

(b) One possible manipulation is to change two votes from *B*>*C*>*A* to *A*>*B*>*C*. This makes *A* the winner under all four voting methods studied.

Number of voters	**51**	**40**	**40**	**49**	**43**	**30**
1st choice	*A*	*A*	*B*	*B*	*C*	*C*
2nd choice	*B*	*C*	*A*	*C*	*A*	*B*
3rd choice	*C*	*B*	*C*	*A*	*B*	*A*

77. Many preference schedules are possible. Giving candidate *C* more strength relative to *A* and *B* is one approach to take. One such schedule moves 9 votes away from *A*>*B*>*C* (4 go to *A*>*C*>*B* and 5 go to *C*>*B*>*A*) and 5 votes away from *B*>*C*>*A* to *C*>*B*>*A*. In this new schedule, candidate *B* wins using the plurality method, candidate *C* wins using the Borda count method, candidate *A* wins using the plurality-with-elimination method, and candidate *C* wins using pairwise comparisons.

Number of voters	**40**	**44**	**40**	**46**	**43**	**40**
1st choice	*A*	*A*	*B*	*B*	*C*	*C*
2nd choice	*B*	*C*	*A*	*C*	*A*	*B*
3rd choice	*C*	*B*	*C*	*A*	*B*	*A*

78. (a) Plurality method: *A* has 93 first-place votes, *E* has 81 first-place votes, *B* has 44 first-place votes, *D* has 42 first-place votes, and *C* has 40 first-place votes. So, using the plurality method, *A* wins.

Borda count method: *A* has $5\times 93+1\times 44+4\times 10+2\times 30+2\times 42+1\times 81=774$ points.
B has $4\times 93+5\times 44+2\times 10+1\times 30+1\times 42+2\times 81=846$ points.
C has $3\times 93+2\times 44+5\times 10+5\times 30+4\times 42+3\times 81=978$ points.
D has $2\times 93+4\times 44+1\times 10+3\times 30+5\times 42+4\times 81=996$ points.
E has $1\times 93+3\times 44+3\times 10+4\times 30+3\times 42+5\times 81=906$ points.
So, using the Borda count method, candidate *D* wins with 996 points.

Plurality-with-elimination method: Candidate *E* wins (*C* is eliminated first, followed by *D*, *B*, and then *A*).

Pairwise comparisons:
A vs. *B* – *A* wins 175-125 and gets 1 point.
A vs. *C* – *C* wins 93-207 and gets 1 point.
A vs. *D* – *D* wins 103-197 and gets 1 point.
A vs. *E* – *E* wins 103-197 and gets 1 point.
B vs. *C* – *C* wins 137-163 and gets 1 point.
B vs. *D* – *D* wins 147-153 and gets 1 point.
B vs. *E* – *E* wins 137-163 and gets 1 point.
C vs. *D* – *D* wins 133-167 and gets 1 point.
C vs. *E* – *C* wins 175-125 and gets 1 point.
D vs. *E* – *D* wins 179-121 and gets 1 point.
So, using the pairwise comparisons method, candidate *D* wins (4 points for *D*, 3 points for *C*, 2 points for *E*, 1 point for *A*, 0 points for *B*).

Since candidate *D* is a Condorcet candidate, we conclude that plurality method and the plurality-with-elimination method both violate the Condorcet criterion in this election.

(b) We aim to make *D* a winning candidate when using the plurality method. One way to do this is to count all first place votes for candidate *A*, *D*, and *E* and divide that number by 3. We then make sure candidate *D* has at least that many first-place votes and candidates *A* and *E* have less than that many. Since 93 + 42 + 81 = 216, we will take 21 votes away from the 93 voters ranking *A*>*B*>*C*>*D*>*E* and take 10 votes away from the voters ranking *E*>*D*>*C*>*B*>*A* giving these votes to the 42 voters ranking *D*>*C*>*E*>*A*>*B*. This makes *D* the winner under all four voting methods studied in this chapter.

Number of voters	**72**	**44**	**10**	**30**	**73**	**71**
1st choice	*A*	*B*	*C*	*C*	*D*	*E*
2nd choice	*B*	*D*	*A*	*E*	*C*	*D*
3rd choice	*C*	*E*	*E*	*D*	*E*	*C*
4th choice	*D*	*C*	*B*	*A*	*A*	*B*
5th choice	*E*	*A*	*D*	*B*	*B*	*A*

79. *D* is preferred over *A* by a majority of the voters (11 to 10) but all four complete rankings rank *A* above *D*.

Number of voters	**8**	**4**	**3**	**2**	**4**
1st choice	*A*	*C*	*C*	*C*	*B*
2nd choice	*B*	*D*	*B*	*B*	*D*
3rd choice	*C*	*A*	*D*	*A*	*A*
4th choice	*D*	*B*	*A*	*D*	*C*

Method	**Winner**	**2nd place**	**3rd place**	**Last place**
Plurality	*C*	*A*	*B*	*D*
Borda count	*B*	*C*	*A*	*D*
Plurality with elimination	*A*	*C*	*B*	*D*
Pairwise comparison	*A,B* (tied)		*C,D* (tied)	