COMPLETE SOLUTIONS MANUAL

for Stewart's

SINGLE VARIABLE CALCULUS CONCEPTS AND CONTEXTS

FOURTH EDITION

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PREFACE

This Complete Solutions Manual contains solutions to all exercises in the texts *Single Variable Calculus: Concepts and Contexts,* Fourth Edition, and Chapters 1–8 of *Calculus: Concepts and Contexts,* Fourth Edition, by James Stewart. A student version of this manual is also available; it contains solutions to the odd-numbered exercises in each chapter section, the review sections, the True-False Quizzes, and the Focus on Problem Solving sections, as well as solutions to all the exercises in the Concept Checks. No solutions to the Projects appear in the student version. It is our hope that by browsing through the solutions, professors will save time in determining appropriate assignments for their particular classes.

Some nonstandard notation is used in order to save space. If you see a symbol that you don't recognize, refer to the Table of Abbreviations and Symbols on page v.

We appreciate feedback concerning errors, solution correctness or style, and manual style. Any comments may be sent directly to us at jeff.cole@anokaramsey.edu or tim@andrew.cmu.edu, or in care of the publisher: Cengage Learning Brooks/Cole, 20 Davis Drive, Belmont, CA 94002.

We would like to thank Jim Stewart, for his guidance; Brian Betsill, Kathi Townes, and Rebekah Million, of TECH-arts, for their production services; and Richard Stratton and Jeannine Lawless, of Cengage Learning Brooks/Cole, for entrusting us with this project as well as for their patience and support.

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ABBREVIATIONS AND SYMBOLS

CD	concave downward
CU	concave upward
D	the domain of f
FDT	First Derivative Test
HA	horizontal asymptote(s)
I	interval of convergence
I/D	Increasing/Decreasing Test
IP	inflection point(s)
R	radius of convergence
VA	vertical asymptote(s)
$\overset{\text{CAS}}{=}$	indicates the use of a computer algebra system.
$\overset{\text{H}}{=}$	indicates the use of l'Hospital's Rule.
$\overset{j}{=}$	indicates the use of Formula j in the Table of Integrals in the back endpapers.
$\overset{\text{s}}{=}$	indicates the use of the substitution $\{u = \sin x, du = \cos x\, dx\}$.
$\overset{\text{c}}{=}$	indicates the use of the substitution $\{u = \cos x, du = -\sin x\, dx\}$.

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▫ APPENDIXES 753

☐ DIAGNOSTIC TESTS

Test A Algebra

1. (a) $(-3)^4 = (-3)(-3)(-3)(-3) = 81$

(b) $-3^4 = -(3)(3)(3)(3) = -81$

(c) $3^{-4} = \dfrac{1}{3^4} = \dfrac{1}{81}$

(d) $\dfrac{5^{23}}{5^{21}} = 5^{23-21} = 5^2 = 25$

(e) $\left(\frac{2}{3}\right)^{-2} = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$

(f) $16^{-3/4} = \dfrac{1}{16^{3/4}} = \dfrac{1}{\left(\sqrt[4]{16}\right)^3} = \dfrac{1}{2^3} = \dfrac{1}{8}$

2. (a) Note that $\sqrt{200} = \sqrt{100 \cdot 2} = 10\sqrt{2}$ and $\sqrt{32} = \sqrt{16 \cdot 2} = 4\sqrt{2}$. Thus $\sqrt{200} - \sqrt{32} = 10\sqrt{2} - 4\sqrt{2} = 6\sqrt{2}$.

(b) $(3a^3b^3)(4ab^2)^2 = 3a^3b^3 16a^2b^4 = 48a^5b^7$

(c) $\left(\dfrac{3x^{3/2}y^3}{x^2y^{-1/2}}\right)^{-2} = \left(\dfrac{x^2y^{-1/2}}{3x^{3/2}y^3}\right)^2 = \dfrac{(x^2y^{-1/2})^2}{(3x^{3/2}y^3)^2} = \dfrac{x^4y^{-1}}{9x^3y^6} = \dfrac{x^4}{9x^3y^6y} = \dfrac{x}{9y^7}$

3. (a) $3(x+6) + 4(2x-5) = 3x + 18 + 8x - 20 = 11x - 2$

(b) $(x+3)(4x-5) = 4x^2 - 5x + 12x - 15 = 4x^2 + 7x - 15$

(c) $\left(\sqrt{a}+\sqrt{b}\right)\left(\sqrt{a}-\sqrt{b}\right) = \left(\sqrt{a}\right)^2 - \sqrt{a}\sqrt{b} + \sqrt{a}\sqrt{b} - \left(\sqrt{b}\right)^2 = a - b$

Or: Use the formula for the difference of two squares to see that $\left(\sqrt{a}+\sqrt{b}\right)\left(\sqrt{a}-\sqrt{b}\right) = \left(\sqrt{a}\right)^2 - \left(\sqrt{b}\right)^2 = a - b$.

(d) $(2x+3)^2 = (2x+3)(2x+3) = 4x^2 + 6x + 6x + 9 = 4x^2 + 12x + 9$.

Note: A quicker way to expand this binomial is to use the formula $(a+b)^2 = a^2 + 2ab + b^2$ with $a = 2x$ and $b = 3$:
$(2x+3)^2 = (2x)^2 + 2(2x)(3) + 3^2 = 4x^2 + 12x + 9$

(e) See Reference Page 1 for the binomial formula $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$. Using it, we get
$(x+2)^3 = x^3 + 3x^2(2) + 3x(2^2) + 2^3 = x^3 + 6x^2 + 12x + 8$.

4. (a) Using the difference of two squares formula, $a^2 - b^2 = (a+b)(a-b)$, we have
$4x^2 - 25 = (2x)^2 - 5^2 = (2x+5)(2x-5)$.

(b) Factoring by trial and error, we get $2x^2 + 5x - 12 = (2x-3)(x+4)$.

(c) Using factoring by grouping and the difference of two squares formula, we have
$x^3 - 3x^2 - 4x + 12 = x^2(x-3) - 4(x-3) = (x^2-4)(x-3) = (x-2)(x+2)(x-3)$.

(d) $x^4 + 27x = x(x^3 + 27) = x(x+3)(x^2 - 3x + 9)$

This last expression was obtained using the sum of two cubes formula, $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$ with $a = x$ and $b = 3$. [See Reference Page 1 in the textbook.]

(e) The smallest exponent on x is $-\frac{1}{2}$, so we will factor out $x^{-1/2}$.
$3x^{3/2} - 9x^{1/2} + 6x^{-1/2} = 3x^{-1/2}(x^2 - 3x + 2) = 3x^{-1/2}(x-1)(x-2)$

(f) $x^3y - 4xy = xy(x^2 - 4) = xy(x-2)(x+2)$

5. (a) $\dfrac{x^2+3x+2}{x^2-x-2} = \dfrac{(x+1)(x+2)}{(x+1)(x-2)} = \dfrac{x+2}{x-2}$

(b) $\dfrac{2x^2-x-1}{x^2-9} \cdot \dfrac{x+3}{2x+1} = \dfrac{(2x+1)(x-1)}{(x-3)(x+3)} \cdot \dfrac{x+3}{2x+1} = \dfrac{x-1}{x-3}$

(c) $\dfrac{x^2}{x^2-4} - \dfrac{x+1}{x+2} = \dfrac{x^2}{(x-2)(x+2)} - \dfrac{x+1}{x+2} = \dfrac{x^2}{(x-2)(x+2)} - \dfrac{x+1}{x+2} \cdot \dfrac{x-2}{x-2} = \dfrac{x^2-(x+1)(x-2)}{(x-2)(x+2)}$

$$= \frac{x^2-(x^2-x-2)}{(x+2)(x-2)} = \frac{x+2}{(x+2)(x-2)} = \frac{1}{x-2}$$

(d) $\dfrac{\dfrac{y}{x}-\dfrac{x}{y}}{\dfrac{1}{y}-\dfrac{1}{x}} = \dfrac{\dfrac{y}{x}-\dfrac{x}{y}}{\dfrac{1}{y}-\dfrac{1}{x}} \cdot \dfrac{xy}{xy} = \dfrac{y^2-x^2}{x-y} = \dfrac{(y-x)(y+x)}{-(y-x)} = \dfrac{y+x}{-1} = -(x+y)$

6. (a) $\dfrac{\sqrt{10}}{\sqrt{5}-2} = \dfrac{\sqrt{10}}{\sqrt{5}-2} \cdot \dfrac{\sqrt{5}+2}{\sqrt{5}+2} = \dfrac{\sqrt{50}+2\sqrt{10}}{\left(\sqrt{5}\right)^2-2^2} = \dfrac{5\sqrt{2}+2\sqrt{10}}{5-4} = 5\sqrt{2}+2\sqrt{10}$

(b) $\dfrac{\sqrt{4+h}-2}{h} = \dfrac{\sqrt{4+h}-2}{h} \cdot \dfrac{\sqrt{4+h}+2}{\sqrt{4+h}+2} = \dfrac{4+h-4}{h\left(\sqrt{4+h}+2\right)} = \dfrac{h}{h\left(\sqrt{4+h}+2\right)} = \dfrac{1}{\sqrt{4+h}+2}$

7. (a) $x^2+x+1 = \left(x^2+x+\frac{1}{4}\right)+1-\frac{1}{4} = \left(x+\frac{1}{2}\right)^2+\frac{3}{4}$

(b) $2x^2-12x+11 = 2(x^2-6x)+11 = 2(x^2-6x+9-9)+11 = 2(x^2-6x+9)-18+11 = 2(x-3)^2-7$

8. (a) $x+5 = 14-\frac{1}{2}x \quad\Leftrightarrow\quad x+\frac{1}{2}x = 14-5 \quad\Leftrightarrow\quad \frac{3}{2}x = 9 \quad\Leftrightarrow\quad x = \frac{2}{3}\cdot 9 \quad\Leftrightarrow\quad x = 6$

(b) $\dfrac{2x}{x+1} = \dfrac{2x-1}{x} \quad\Rightarrow\quad 2x^2 = (2x-1)(x+1) \quad\Leftrightarrow\quad 2x^2 = 2x^2+x-1 \quad\Leftrightarrow\quad x = 1$

(c) $x^2-x-12 = 0 \quad\Leftrightarrow\quad (x+3)(x-4) = 0 \quad\Leftrightarrow\quad x+3 = 0$ or $x-4 = 0 \quad\Leftrightarrow\quad x = -3$ or $x = 4$

(d) By the quadratic formula, $2x^2+4x+1 = 0 \quad\Leftrightarrow$

$$x = \frac{-4\pm\sqrt{4^2-4(2)(1)}}{2(2)} = \frac{-4\pm\sqrt{8}}{4} = \frac{-4\pm 2\sqrt{2}}{4} = \frac{2\left(-2\pm\sqrt{2}\right)}{4} = \frac{-2\pm\sqrt{2}}{2} = -1\pm\tfrac{1}{2}\sqrt{2}.$$

(e) $x^4-3x^2+2 = 0 \quad\Leftrightarrow\quad (x^2-1)(x^2-2) = 0 \quad\Leftrightarrow\quad x^2-1 = 0$ or $x^2-2 = 0 \quad\Leftrightarrow\quad x^2 = 1$ or $x^2 = 2 \quad\Leftrightarrow$

$x = \pm 1$ or $x = \pm\sqrt{2}$

(f) $3\,|x-4| = 10 \quad\Leftrightarrow\quad |x-4| = \frac{10}{3} \quad\Leftrightarrow\quad x-4 = -\frac{10}{3}$ or $x-4 = \frac{10}{3} \quad\Leftrightarrow\quad x = \frac{2}{3}$ or $x = \frac{22}{3}$

(g) Multiplying through $2x(4-x)^{-1/2}-3\sqrt{4-x} = 0$ by $(4-x)^{1/2}$ gives $2x-3(4-x) = 0 \quad\Leftrightarrow$

$2x-12+3x = 0 \quad\Leftrightarrow\quad 5x-12 = 0 \quad\Leftrightarrow\quad 5x = 12 \quad\Leftrightarrow\quad x = \frac{12}{5}$.

9. (a) $-4 < 5-3x \le 17 \quad\Leftrightarrow\quad -9 < -3x \le 12 \quad\Leftrightarrow\quad 3 > x \ge -4$ or $-4 \le x < 3$.

In interval notation, the answer is $[-4, 3)$.

(b) $x^2 < 2x+8 \quad\Leftrightarrow\quad x^2-2x-8 < 0 \quad\Leftrightarrow\quad (x+2)(x-4) < 0$. Now, $(x+2)(x-4)$ will change sign at the critical values $x = -2$ and $x = 4$. Thus the possible intervals of solution are $(-\infty, -2)$, $(-2, 4)$, and $(4, \infty)$. By choosing a single test value from each interval, we see that $(-2, 4)$ is the only interval that satisfies the inequality.

(c) The inequality $x(x-1)(x+2) > 0$ has critical values of $-2, 0$, and 1. The corresponding possible intervals of solution are $(-\infty, -2)$, $(-2, 0)$, $(0, 1)$ and $(1, \infty)$. By choosing a single test value from each interval, we see that both intervals $(-2, 0)$ and $(1, \infty)$ satisfy the inequality. Thus, the solution is the union of these two intervals: $(-2, 0) \cup (1, \infty)$.

(d) $|x-4| < 3 \quad \Leftrightarrow \quad -3 < x - 4 < 3 \quad \Leftrightarrow \quad 1 < x < 7$. In interval notation, the answer is $(1, 7)$.

(e) $\dfrac{2x-3}{x+1} \le 1 \quad \Leftrightarrow \quad \dfrac{2x-3}{x+1} - 1 \le 0 \quad \Leftrightarrow \quad \dfrac{2x-3}{x+1} - \dfrac{x+1}{x+1} \le 0 \quad \Leftrightarrow \quad \dfrac{2x-3-x-1}{x+1} \le 0 \quad \Leftrightarrow \quad \dfrac{x-4}{x+1} \le 0.$

Now, the expression $\dfrac{x-4}{x+1}$ may change signs at the critical values $x = -1$ and $x = 4$, so the possible intervals of solution are $(-\infty, -1)$, $(-1, 4]$, and $[4, \infty)$. By choosing a single test value from each interval, we see that $(-1, 4]$ is the only interval that satisfies the inequality.

10. (a) False. In order for the statement to be true, it must hold for all real numbers, so, to show that the statement is false, pick $p = 1$ and $q = 2$ and observe that $(1+2)^2 \ne 1^2 + 2^2$. In general, $(p+q)^2 = p^2 + 2pq + q^2$.

(b) True as long as a and b are nonnegative real numbers. To see this, think in terms of the laws of exponents: $\sqrt{ab} = (ab)^{1/2} = a^{1/2}b^{1/2} = \sqrt{a}\,\sqrt{b}$.

(c) False. To see this, let $p = 1$ and $q = 2$, then $\sqrt{1^2 + 2^2} \ne 1 + 2$.

(d) False. To see this, let $T = 1$ and $C = 2$, then $\dfrac{1 + 1(2)}{2} \ne 1 + 1$.

(e) False. To see this, let $x = 2$ and $y = 3$, then $\dfrac{1}{2-3} \ne \dfrac{1}{2} - \dfrac{1}{3}$.

(f) True since $\dfrac{1/x}{a/x - b/x} \cdot \dfrac{x}{x} = \dfrac{1}{a-b}$, as long as $x \ne 0$ and $a - b \ne 0$.

Test B Analytic Geometry

1. (a) Using the point $(2, -5)$ and $m = -3$ in the point-slope equation of a line, $y - y_1 = m(x - x_1)$, we get $y - (-5) = -3(x-2) \quad \Rightarrow \quad y + 5 = -3x + 6 \quad \Rightarrow \quad y = -3x + 1$.

(b) A line parallel to the x-axis must be horizontal and thus have a slope of 0. Since the line passes through the point $(2, -5)$, the y-coordinate of every point on the line is -5, so the equation is $y = -5$.

(c) A line parallel to the y-axis is vertical with undefined slope. So the x-coordinate of every point on the line is 2 and so the equation is $x = 2$.

(d) Note that $2x - 4y = 3 \quad \Rightarrow \quad -4y = -2x + 3 \quad \Rightarrow \quad y = \frac{1}{2}x - \frac{3}{4}$. Thus the slope of the given line is $m = \frac{1}{2}$. Hence, the slope of the line we're looking for is also $\frac{1}{2}$ (since the line we're looking for is required to be parallel to the given line). So the equation of the line is $y - (-5) = \frac{1}{2}(x - 2) \quad \Rightarrow \quad y + 5 = \frac{1}{2}x - 1 \quad \Rightarrow \quad y = \frac{1}{2}x - 6$.

2. First we'll find the distance between the two given points in order to obtain the radius, r, of the circle: $r = \sqrt{[3 - (-1)]^2 + (-2-4)^2} = \sqrt{4^2 + (-6)^2} = \sqrt{52}$. Next use the standard equation of a circle, $(x-h)^2 + (y-k)^2 = r^2$, where (h, k) is the center, to get $(x+1)^2 + (y-4)^2 = 52$.

3. We must rewrite the equation in standard form in order to identify the center and radius. Note that $x^2 + y^2 - 6x + 10y + 9 = 0 \quad \Rightarrow \quad x^2 - 6x + 9 + y^2 + 10y = 0$. For the left-hand side of the latter equation, we factor the first three terms and complete the square on the last two terms as follows: $x^2 - 6x + 9 + y^2 + 10y = 0 \quad \Rightarrow$ $(x-3)^2 + y^2 + 10y + 25 = 25 \quad \Rightarrow \quad (x-3)^2 + (y+5)^2 = 25$. Thus, the center of the circle is $(3, -5)$ and the radius is 5.

4. (a) $A(-7, 4)$ and $B(5, -12) \quad \Rightarrow \quad m_{AB} = \dfrac{-12-4}{5-(-7)} = \dfrac{-16}{12} = -\dfrac{4}{3}$

(b) $y - 4 = -\frac{4}{3}[x - (-7)] \quad \Rightarrow \quad y - 4 = -\frac{4}{3}x - \frac{28}{3} \quad \Rightarrow \quad 3y - 12 = -4x - 28 \quad \Rightarrow \quad 4x + 3y + 16 = 0$. Putting $y = 0$, we get $4x + 16 = 0$, so the x-intercept is -4, and substituting 0 for x results in a y-intercept of $-\frac{16}{3}$.

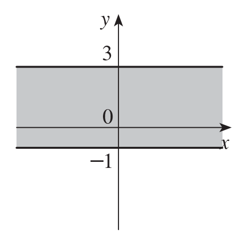
(c) The midpoint is obtained by averaging the corresponding coordinates of both points: $\left(\frac{-7+5}{2}, \frac{4+(-12)}{2}\right) = (-1, -4)$.

(d) $d = \sqrt{[5-(-7)]^2 + (-12-4)^2} = \sqrt{12^2 + (-16)^2} = \sqrt{144 + 256} = \sqrt{400} = 20$

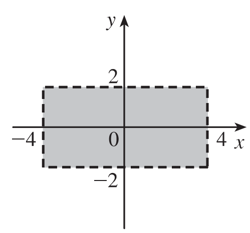
(e) The perpendicular bisector is the line that intersects the line segment $\overline{AB}$ at a right angle through its midpoint. Thus the perpendicular bisector passes through $(-1, -4)$ and has slope $\frac{3}{4}$ [the slope is obtained by taking the negative reciprocal of the answer from part (a)]. So the perpendicular bisector is given by $y + 4 = \frac{3}{4}[x - (-1)]$ or $3x - 4y = 13$.

(f) The center of the required circle is the midpoint of $\overline{AB}$, and the radius is half the length of $\overline{AB}$, which is 10. Thus, the equation is $(x+1)^2 + (y+4)^2 = 100$.

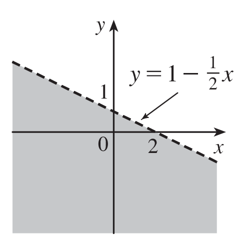
5. (a) Graph the corresponding horizontal lines (given by the equations $y = -1$ and $y = 3$) as solid lines. The inequality $y \geq -1$ describes the points (x, y) that lie on or *above* the line $y = -1$. The inequality $y \leq 3$ describes the points (x, y) that lie on or *below* the line $y = 3$. So the pair of inequalities $-1 \leq y \leq 3$ describes the points that lie on or *between* the lines $y = -1$ and $y = 3$.



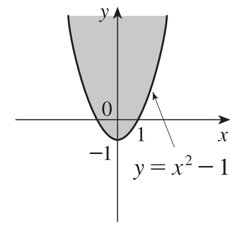
(b) Note that the given inequalities can be written as $-4 < x < 4$ and $-2 < y < 2$, respectively. So the region lies between the vertical lines $x = -4$ and $x = 4$ and between the horizontal lines $y = -2$ and $y = 2$. As shown in the graph, the region common to both graphs is a rectangle (minus its edges) centered at the origin.



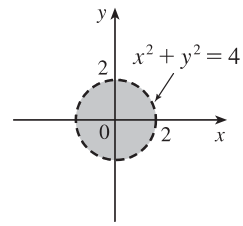
(c) We first graph $y = 1 - \frac{1}{2}x$ as a dotted line. Since $y < 1 - \frac{1}{2}x$, the points in the region lie *below* this line.



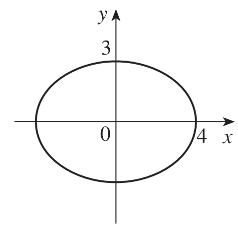
(d) We first graph the parabola $y = x^2 - 1$ using a solid curve. Since $y \geq x^2 - 1$, the points in the region lie on or *above* the parabola.



(e) We graph the circle $x^2 + y^2 = 4$ using a dotted curve. Since $\sqrt{x^2 + y^2} < 2$, the region consists of points whose distance from the origin is less than 2, that is, the points that lie *inside* the circle.



(f) The equation $9x^2 + 16y^2 = 144$ is an ellipse centered at $(0, 0)$. We put it in standard form by dividing by 144 and get $\frac{x^2}{16} + \frac{y^2}{9} = 1$. The x-intercepts are located at a distance of $\sqrt{16} = 4$ from the center while the y-intercepts are a distance of $\sqrt{9} = 3$ from the center (see the graph).



Test C Functions

1. (a) Locate -1 on the x-axis and then go down to the point on the graph with an x-coordinate of -1. The corresponding y-coordinate is the value of the function at $x = -1$, which is -2. So, $f(-1) = -2$.

(b) Using the same technique as in part (a), we get $f(2) \approx 2.8$.

(c) Locate 2 on the y-axis and then go left and right to find all points on the graph with a y-coordinate of 2. The corresponding x-coordinates are the x-values we are searching for. So $x = -3$ and $x = 1$.

(d) Using the same technique as in part (c), we get $x \approx -2.5$ and $x \approx 0.3$.

(e) The domain is all the x-values for which the graph exists, and the range is all the y-values for which the graph exists. Thus, the domain is $[-3, 3]$, and the range is $[-2, 3]$.

2. Note that $f(2 + h) = (2 + h)^3$ and $f(2) = 2^3 = 8$. So the difference quotient becomes

$$\frac{f(2+h) - f(2)}{h} = \frac{(2+h)^3 - 8}{h} = \frac{8 + 12h + 6h^2 + h^3 - 8}{h} = \frac{12h + 6h^2 + h^3}{h} = \frac{h(12 + 6h + h^2)}{h} = 12 + 6h + h^2.$$

3. (a) Set the denominator equal to 0 and solve to find restrictions on the domain: $x^2 + x - 2 = 0 \quad \Rightarrow$ $(x - 1)(x + 2) = 0 \quad \Rightarrow \quad x = 1$ or $x = -2$. Thus, the domain is all real numbers except 1 or -2 or, in interval notation, $(-\infty, -2) \cup (-2, 1) \cup (1, \infty)$.

(b) Note that the denominator is always greater than or equal to 1, and the numerator is defined for all real numbers. Thus, the domain is $(-\infty, \infty)$.

(c) Note that the function h is the sum of two root functions. So h is defined on the intersection of the domains of these two root functions. The domain of a square root function is found by setting its radicand greater than or equal to 0. Now,

$4 - x \geq 0 \quad \Rightarrow \quad x \leq 4$ and $x^2 - 1 \geq 0 \quad \Rightarrow \quad (x-1)(x+1) \geq 0 \quad \Rightarrow \quad x \leq -1$ or $x \geq 1$. Thus, the domain of h is $(-\infty, -1] \cup [1, 4]$.

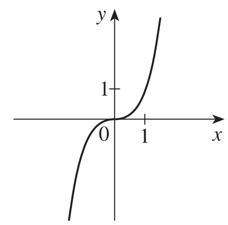
4. (a) Reflect the graph of f about the x-axis.

(b) Stretch the graph of f vertically by a factor of 2, then shift 1 unit downward.

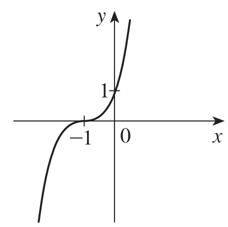
(c) Shift the graph of f right 3 units, then up 2 units.

5. (a) Make a table and then connect the points with a smooth curve:

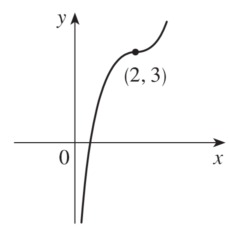
x	-2	-1	0	1	2
y	-8	-1	0	1	8



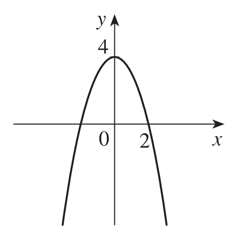
(b) Shift the graph from part (a) left 1 unit.



(c) Shift the graph from part (a) right 2 units and up 3 units.

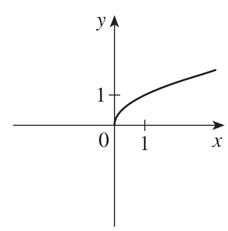


(d) First plot $y = x^2$. Next, to get the graph of $f(x) = 4 - x^2$, reflect f about the x-axis and then shift it upward 4 units.

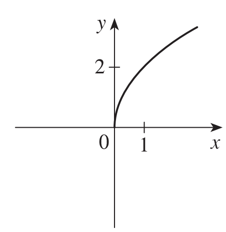


(e) Make a table and then connect the points with a smooth curve:

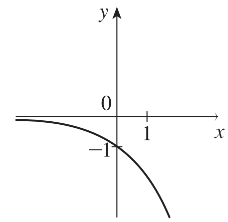
x	0	1	4	9
y	0	1	2	3



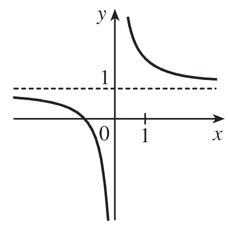
(f) Stretch the graph from part (e) vertically by a factor of two.



(g) First plot $y = 2^x$. Next, get the graph of $y = -2^x$ by reflecting the graph of $y = 2^x$ about the x-axis.

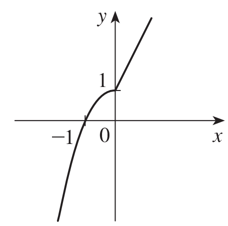


(h) Note that $y = 1 + x^{-1} = 1 + 1/x$. So first plot $y = 1/x$ and then shift it upward 1 unit.



6. (a) $f(-2) = 1 - (-2)^2 = -3$ and $f(1) = 2(1) + 1 = 3$

(b) For $x \le 0$ plot $f(x) = 1 - x^2$ and, on the same plane, for $x > 0$ plot the graph of $f(x) = 2x + 1$.



7. (a) $(f \circ g)(x) = f(g(x)) = f(2x - 3) = (2x - 3)^2 + 2(2x - 3) - 1 = 4x^2 - 12x + 9 + 4x - 6 - 1 = 4x^2 - 8x + 2$

(b) $(g \circ f)(x) = g(f(x)) = g(x^2 + 2x - 1) = 2(x^2 + 2x - 1) - 3 = 2x^2 + 4x - 2 - 3 = 2x^2 + 4x - 5$

(c) $(g \circ g \circ g)(x) = g(g(g(x))) = g(g(2x - 3)) = g(2(2x - 3) - 3) = g(4x - 9) = 2(4x - 9) - 3$

$= 8x - 18 - 3 = 8x - 21$

Test D Trigonometry

1. (a) $300^\circ = 300^\circ\left(\dfrac{\pi}{180^\circ}\right) = \dfrac{300\pi}{180} = \dfrac{5\pi}{3}$

(b) $-18^\circ = -18^\circ\left(\dfrac{\pi}{180^\circ}\right) = -\dfrac{18\pi}{180} = -\dfrac{\pi}{10}$

2. (a) $\dfrac{5\pi}{6} = \dfrac{5\pi}{6}\left(\dfrac{180^\circ}{\pi}\right) = 150^\circ$

(b) $2 = 2\left(\dfrac{180^\circ}{\pi}\right) = \dfrac{360^\circ}{\pi} \approx 114.6^\circ$

3. We will use the arc length formula, $s = r\theta$, where s is arc length, r is the radius of the circle, and θ is the measure of the central angle in radians. First, note that $30^\circ = 30^\circ\left(\dfrac{\pi}{180^\circ}\right) = \dfrac{\pi}{6}$. So $s = (12)\left(\dfrac{\pi}{6}\right) = 2\pi$ cm.

4. (a) $\tan(\pi/3) = \sqrt{3}$ [You can read the value from a right triangle with sides 1, 2, and $\sqrt{3}$.]

(b) Note that $7\pi/6$ can be thought of as an angle in the third quadrant with reference angle $\pi/6$. Thus, $\sin(7\pi/6) = -\frac{1}{2}$, since the sine function is negative in the third quadrant.

(c) Note that $5\pi/3$ can be thought of as an angle in the fourth quadrant with reference angle $\pi/3$. Thus,

$\sec(5\pi/3) = \dfrac{1}{\cos(5\pi/3)} = \dfrac{1}{1/2} = 2$, since the cosine function is positive in the fourth quadrant.

5. $\sin\theta = a/24 \quad\Rightarrow\quad a = 24\sin\theta$ and $\cos\theta = b/24 \quad\Rightarrow\quad b = 24\cos\theta$

6. $\sin x = \frac{1}{3}$ and $\sin^2 x + \cos^2 x = 1 \quad \Rightarrow \quad \cos x = \sqrt{1 - \frac{1}{9}} = \dfrac{2\sqrt{2}}{3}$. Also, $\cos y = \frac{4}{5} \quad \Rightarrow \quad \sin y = \sqrt{1 - \frac{16}{25}} = \frac{3}{5}$.

So, using the sum identity for the sine, we have

$$\sin(x + y) = \sin x \, \cos y + \cos x \, \sin y = \frac{1}{3} \cdot \frac{4}{5} + \frac{2\sqrt{2}}{3} \cdot \frac{3}{5} = \frac{4 + 6\sqrt{2}}{15} = \frac{1}{15}\left(4 + 6\sqrt{2}\right)$$

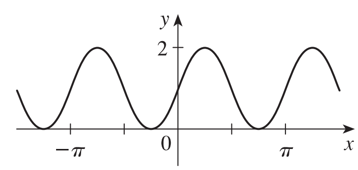
7. (a) $\tan\theta \, \sin\theta + \cos\theta = \dfrac{\sin\theta}{\cos\theta} \sin\theta + \cos\theta = \dfrac{\sin^2\theta}{\cos\theta} + \dfrac{\cos^2\theta}{\cos\theta} = \dfrac{1}{\cos\theta} = \sec\theta$

(b) $\dfrac{2\tan x}{1 + \tan^2 x} = \dfrac{2\sin x/(\cos x)}{\sec^2 x} = 2\,\dfrac{\sin x}{\cos x}\,\cos^2 x = 2\sin x \, \cos x = \sin 2x$

8. $\sin 2x = \sin x \quad \Leftrightarrow \quad 2\sin x \, \cos x = \sin x \quad \Leftrightarrow \quad 2\sin x \, \cos x - \sin x = 0 \quad \Leftrightarrow \quad \sin x\,(2\cos x - 1) = 0 \quad \Leftrightarrow$

$\sin x = 0$ or $\cos x = \frac{1}{2} \quad \Rightarrow \quad x = 0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi$.

9. We first graph $y = \sin 2x$ (by compressing the graph of $\sin x$ by a factor of 2) and then shift it upward 1 unit.



1 ☐ FUNCTIONS AND MODELS

1.1 Four Ways To Represent a Function

In exercises requiring estimations or approximations, your answers may vary slightly from the answers given here.

1. (a) The point $(1, 3)$ is on the graph of f, so $f(1) = 3$.

(b) When $x = -1$, y is about -0.2, so $f(-1) \approx -0.2$.

(c) $f(x) = 1$ is equivalent to $y = 1$. When $y = 1$, we have $x = 0$ and $x = 3$.

(d) A reasonable estimate for x when $y = 0$ is $x = -0.8$.

(e) The domain of f consists of all x-values on the graph of f. For this function, the domain is $-2 \le x \le 4$, or $[-2, 4]$. The range of f consists of all y-values on the graph of f. For this function, the range is $-1 \le y \le 3$, or $[-1, 3]$.

(f) As x increases from -2 to 1, y increases from -1 to 3. Thus, f is increasing on the interval $[-2, 1]$.

2. (a) The point $(-4, -2)$ is on the graph of f, so $f(-4) = -2$. The point $(3, 4)$ is on the graph of g, so $g(3) = 4$.

(b) We are looking for the values of x for which the y-values are equal. The y-values for f and g are equal at the points $(-2, 1)$ and $(2, 2)$, so the desired values of x are -2 and 2.

(c) $f(x) = -1$ is equivalent to $y = -1$. When $y = -1$, we have $x = -3$ and $x = 4$.

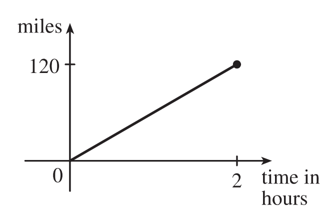
(d) As x increases from 0 to 4, y decreases from 3 to -1. Thus, f is decreasing on the interval $[0, 4]$.

(e) The domain of f consists of all x-values on the graph of f. For this function, the domain is $-4 \le x \le 4$, or $[-4, 4]$. The range of f consists of all y-values on the graph of f. For this function, the range is $-2 \le y \le 3$, or $[-2, 3]$.

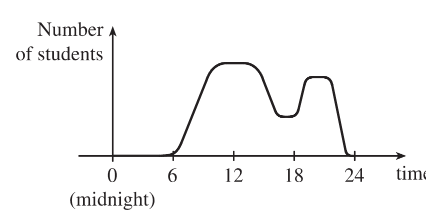
(f) The domain of g is $[-4, 3]$ and the range is $[0.5, 4]$.

3. From Figure 1 in the text, the lowest point occurs at about $(t, a) = (12, -85)$. The highest point occurs at about $(17, 115)$. Thus, the range of the vertical ground acceleration is $-85 \le a \le 115$. Written in interval notation, we get $[-85, 115]$.

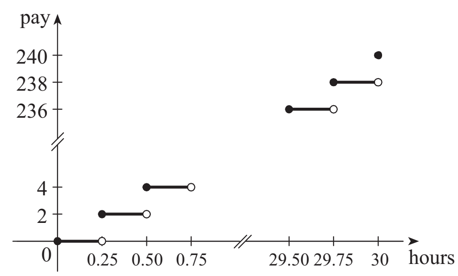
4. *Example 1:* A car is driven at 60 mi/h for 2 hours. The distance d traveled by the car is a function of the time t. The domain of the function is $\{t \mid 0 \le t \le 2\}$, where t is measured in hours. The range of the function is $\{d \mid 0 \le d \le 120\}$, where d is measured in miles.



Example 2: At a certain university, the number of students N on campus at any time on a particular day is a function of the time t after midnight. The domain of the function is $\{t \mid 0 \le t \le 24\}$, where t is measured in hours. The range of the function is $\{N \mid 0 \le N \le k\}$, where N is an integer and k is the largest number of students on campus at once.



Example 3: A certain employee is paid \$8.00 per hour and works a maximum of 30 hours per week. The number of hours worked is rounded down to the nearest quarter of an hour. This employee's gross weekly pay P is a function of the number of hours worked h. The domain of the function is $[0, 30]$ and the range of the function is $\{0, 2.00, 4.00, \ldots, 238.00, 240.00\}$.



5. No, the curve is not the graph of a function because a vertical line intersects the curve more than once. Hence, the curve fails the Vertical Line Test.

6. Yes, the curve is the graph of a function because it passes the Vertical Line Test. The domain is $[-2, 2]$ and the range is $[-1, 2]$.

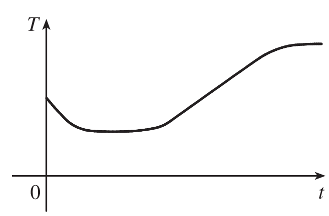
7. Yes, the curve is the graph of a function because it passes the Vertical Line Test. The domain is $[-3, 2]$ and the range is $[-3, -2) \cup [-1, 3]$.

8. No, the curve is not the graph of a function since for $x = 0$, ± 1, and ± 2, there are infinitely many points on the curve.

9. The person's weight increased to about 160 pounds at age 20 and stayed fairly steady for 10 years. The person's weight dropped to about 120 pounds for the next 5 years, then increased rapidly to about 170 pounds. The next 30 years saw a gradual increase to 190 pounds. Possible reasons for the drop in weight at 30 years of age: diet, exercise, health problems.

10. First, the tub was filled with water to a height of 15 in. Then a person got into the tub, raising the water level to 20 in. At around 12 minutes, the person stood up in the tub but then immediately sat down. Finally, at around 17 minutes, the person got out of the tub, and then drained the water.

11. The water will cool down almost to freezing as the ice melts. Then, when the ice has melted, the water will slowly warm up to room temperature.

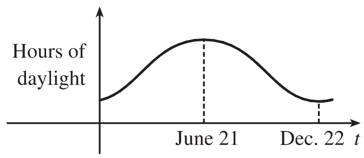


12. Runner A won the race, reaching the finish line at 100 meters in about 15 seconds, followed by runner B with a time of about 19 seconds, and then by runner C who finished in around 23 seconds. B initially led the race, followed by C, and then A. C then passed B to lead for a while. Then A passed first B, and then passed C to take the lead and finish first. Finally, B passed C to finish in second place. All three runners completed the race.

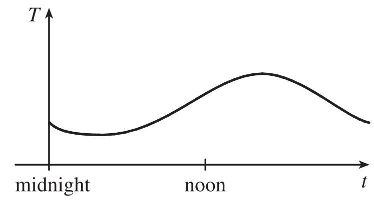
13. (a) The power consumption at 6 AM is 500 MW, which is obtained by reading the value of power P when $t = 6$ from the graph. At 6 PM we read the value of P when $t = 18$, obtaining approximately 730 MW.

(b) The minimum power consumption is determined by finding the time for the lowest point on the graph, $t = 4$, or 4 AM. The maximum power consumption corresponds to the highest point on the graph, which occurs just before $t = 12$, or right before noon. These times are reasonable, considering the power consumption schedules of most individuals and businesses.

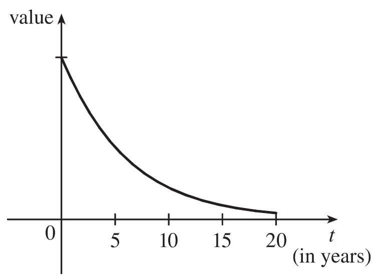
14. The summer solstice (the longest day of the year) is around June 21, and the winter solstice (the shortest day) is around December 22. (Exchange the dates for the southern hemisphere.)



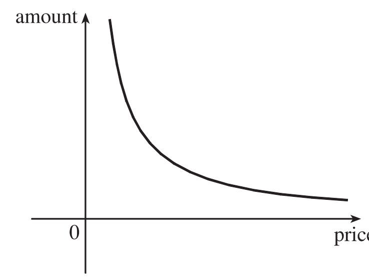
15. Of course, this graph depends strongly on the geographical location!



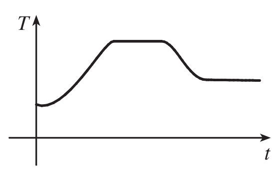
16. The value of the car decreases fairly rapidly initially, then somewhat less rapidly.



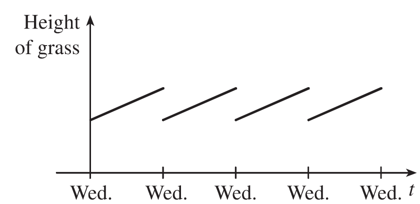
17. As the price increases, the amount sold decreases.



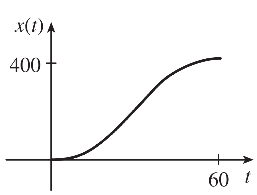
18. The temperature of the pie would increase rapidly, level off to oven temperature, decrease rapidly, and then level off to room temperature.



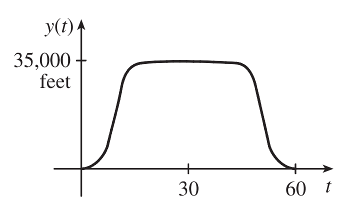
19.



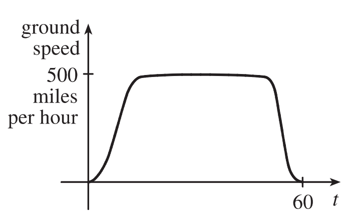
20. (a)



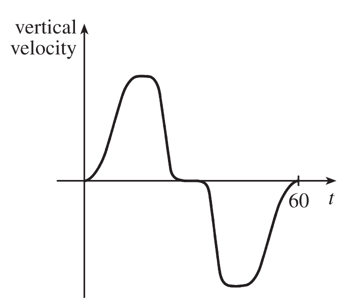
(b)



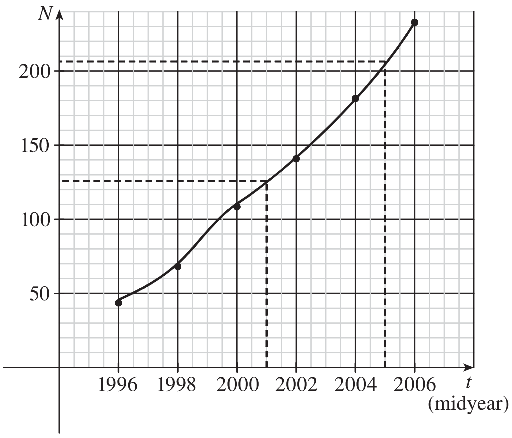
(c)



(d)

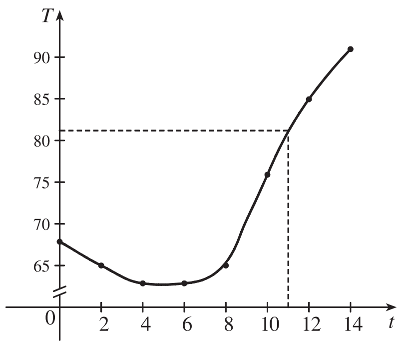


21. (a)



(b) From the graph, we estimate the number of US cell-phone subscribers to be about 126 million in 2001 and 207 million in 2005.

22. (a)



(b) From the graph in part (a), we estimate the temperature at 11:00 AM to be about 81°F.

23. $f(x) = 3x^2 - x + 2.$

$f(2) = 3(2)^2 - 2 + 2 = 12 - 2 + 2 = 12.$

$f(-2) = 3(-2)^2 - (-2) + 2 = 12 + 2 + 2 = 16.$

$f(a) = 3a^2 - a + 2.$

$f(-a) = 3(-a)^2 - (-a) + 2 = 3a^2 + a + 2.$

$f(a+1) = 3(a+1)^2 - (a+1) + 2 = 3(a^2 + 2a + 1) - a - 1 + 2 = 3a^2 + 6a + 3 - a + 1 = 3a^2 + 5a + 4.$

$2f(a) = 2 \cdot f(a) = 2(3a^2 - a + 2) = 6a^2 - 2a + 4.$

$f(2a) = 3(2a)^2 - (2a) + 2 = 3(4a^2) - 2a + 2 = 12a^2 - 2a + 2.$

$f(a^2) = 3(a^2)^2 - (a^2) + 2 = 3(a^4) - a^2 + 2 = 3a^4 - a^2 + 2.$

$$\begin{aligned}[f(a)]^2 &= \left[3a^2 - a + 2\right]^2 = \left(3a^2 - a + 2\right)\left(3a^2 - a + 2\right) \\ &= 9a^4 - 3a^3 + 6a^2 - 3a^3 + a^2 - 2a + 6a^2 - 2a + 4 = 9a^4 - 6a^3 + 13a^2 - 4a + 4.\end{aligned}$$

$f(a+h) = 3(a+h)^2 - (a+h) + 2 = 3(a^2 + 2ah + h^2) - a - h + 2 = 3a^2 + 6ah + 3h^2 - a - h + 2.$

24. A spherical balloon with radius $r+1$ has volume $V(r+1) = \frac{4}{3}\pi(r+1)^3 = \frac{4}{3}\pi(r^3 + 3r^2 + 3r + 1)$. We wish to find the amount of air needed to inflate the balloon from a radius of r to $r+1$. Hence, we need to find the difference $V(r+1) - V(r) = \frac{4}{3}\pi(r^3 + 3r^2 + 3r + 1) - \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(3r^2 + 3r + 1).$

25. $f(x) = 4 + 3x - x^2$, so $f(3+h) = 4 + 3(3+h) - (3+h)^2 = 4 + 9 + 3h - (9 + 6h + h^2) = 4 - 3h - h^2$, and $\dfrac{f(3+h) - f(3)}{h} = \dfrac{(4 - 3h - h^2) - 4}{h} = \dfrac{h(-3-h)}{h} = -3 - h.$

26. $f(x) = x^3$, so $f(a+h) = (a+h)^3 = a^3 + 3a^2h + 3ah^2 + h^3$,

and $\dfrac{f(a+h) - f(a)}{h} = \dfrac{(a^3 + 3a^2h + 3ah^2 + h^3) - a^3}{h} = \dfrac{h(3a^2 + 3ah + h^2)}{h} = 3a^2 + 3ah + h^2$.

27. $\dfrac{f(x) - f(a)}{x - a} = \dfrac{\dfrac{1}{x} - \dfrac{1}{a}}{x - a} = \dfrac{\dfrac{a - x}{xa}}{x - a} = \dfrac{a - x}{xa(x - a)} = \dfrac{-1(x - a)}{xa(x - a)} = -\dfrac{1}{ax}$

28. $\dfrac{f(x) - f(1)}{x - 1} = \dfrac{\dfrac{x+3}{x+1} - 2}{x - 1} = \dfrac{\dfrac{x + 3 - 2(x+1)}{x+1}}{x - 1} = \dfrac{x + 3 - 2x - 2}{(x+1)(x-1)}$

$= \dfrac{-x + 1}{(x+1)(x-1)} = \dfrac{-(x-1)}{(x+1)(x-1)} = -\dfrac{1}{x+1}$

29. $f(x) = (x+4)/(x^2 - 9)$ is defined for all x except when $0 = x^2 - 9 \iff 0 = (x+3)(x-3) \iff x = -3$ or 3, so the domain is $\{x \in \mathbb{R} \mid x \neq -3, 3\} = (-\infty, -3) \cup (-3, 3) \cup (3, \infty)$.

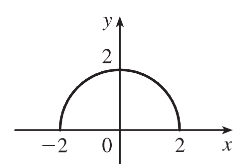
30. $f(x) = (2x^3 - 5)/(x^2 + x - 6)$ is defined for all x except when $0 = x^2 + x - 6 \iff 0 = (x+3)(x-2) \iff x = -3$ or 2, so the domain is $\{x \in \mathbb{R} \mid x \neq -3, 2\} = (-\infty, -3) \cup (-3, 2) \cup (2, \infty)$.

31. $f(t) = \sqrt[3]{2t - 1}$ is defined for all real numbres. In fact $\sqrt[3]{p(t)}$, where $p(t)$ is a polynomial, is defined for all real numbers. Thus, the domain is $\mathbb{R}$, or $(-\infty, \infty)$.

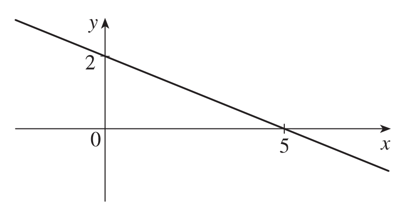
32. $g(t) = \sqrt{3 - t} - \sqrt{2 + t}$ is defined when $3 - t \geq 0 \iff t \leq 3$ and $2 + t \geq 0 \iff t \geq -2$. Thus, the domain is $-2 \leq t \leq 3$, or $[-2, 3]$.

33. $h(x) = 1 \big/ \sqrt[4]{x^2 - 5x}$ is defined when $x^2 - 5x > 0 \iff x(x - 5) > 0$. Note that $x^2 - 5x \neq 0$ since that would result in division by zero. The expression $x(x - 5)$ is positive if $x < 0$ or $x > 5$. (See Appendix A for methods for solving inequalities.) Thus, the domain is $(-\infty, 0) \cup (5, \infty)$.

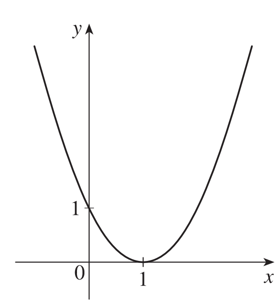
34. $h(x) = \sqrt{4 - x^2}$. Now $y = \sqrt{4 - x^2} \Rightarrow y^2 = 4 - x^2 \iff x^2 + y^2 = 4$, so the graph is the top half of a circle of radius 2 with center at the origin. The domain is $\{x \mid 4 - x^2 \geq 0\} = \{x \mid 4 \geq x^2\} = \{x \mid 2 \geq |x|\} = [-2, 2]$. From the graph, the range is $0 \leq y \leq 2$, or $[0, 2]$.



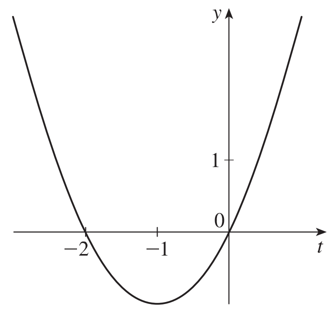
35. $f(x) = 2 - 0.4x$ is defined for all real numbers, so the domain is $\mathbb{R}$, or $(-\infty, \infty)$. The graph of f is a line with slope -0.4 and y-intercept 2.



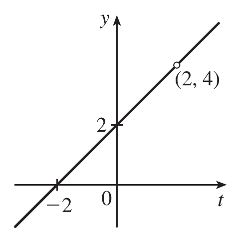
36. $F(x) = x^2 - 2x + 1 = (x - 1)^2$ is defined for all real numbers, so the domain is $\mathbb{R}$, or $(-\infty, \infty)$. The graph of F is a parabola with vertex $(1, 0)$.



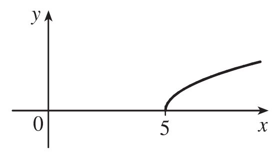
37. $f(t) = 2t + t^2$ is defined for all real numbers, so the domain is $\mathbb{R}$, or $(-\infty, \infty)$. The graph of f is a parabola opening upward since the coefficient of t^2 is positive. To find the t-intercepts, let $y = 0$ and solve for t. $0 = 2t + t^2 = t(2 + t) \quad \Rightarrow \quad t = 0$ or $t = -2$. The t-coordinate of the vertex is halfway between the t-intercepts, that is, at $t = -1$. Since $f(-1) = 2(-1) + (-1)^2 = -2 + 1 = -1$, the vertex is $(-1, -1)$.



38. $H(t) = \dfrac{4 - t^2}{2 - t} = \dfrac{(2 + t)(2 - t)}{2 - t}$, so for $t \neq 2$, $H(t) = 2 + t$. The domain is $\{t \mid t \neq 2\}$. So the graph of H is the same as the graph of the function $f(t) = t + 2$ (a line) except for the hole at $(2, 4)$.

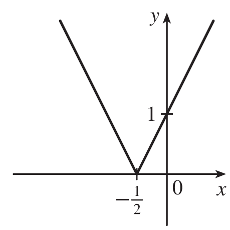


39. $g(x) = \sqrt{x - 5}$ is defined when $x - 5 \geq 0$ or $x \geq 5$, so the domain is $[5, \infty)$. Since $y = \sqrt{x - 5} \quad \Rightarrow \quad y^2 = x - 5 \quad \Rightarrow \quad x = y^2 + 5$, we see that g is the top half of a parabola.



40. $$F(x) = |2x + 1| = \begin{cases} 2x + 1 & \text{if } 2x + 1 \geq 0 \\ -(2x + 1) & \text{if } 2x + 1 < 1 \end{cases}$$
$$= \begin{cases} 2x + 1 & \text{if } x \geq -\frac{1}{2} \\ -2x - 1 & \text{if } x < -\frac{1}{2} \end{cases}$$

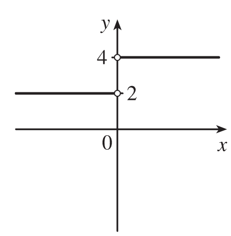
The domain is $\mathbb{R}$, or $(-\infty, \infty)$.



41. $G(x) = \dfrac{3x + |x|}{x}$. Since $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$, we have

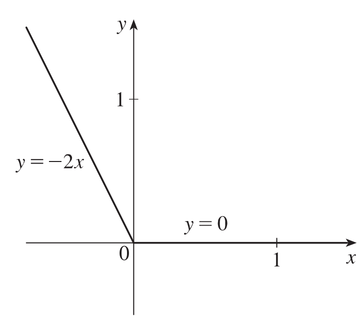
$$G(x) = \begin{cases} \dfrac{3x + x}{x} & \text{if } x > 0 \\ \dfrac{3x - x}{x} & \text{if } x < 0 \end{cases} = \begin{cases} \dfrac{4x}{x} & \text{if } x > 0 \\ \dfrac{2x}{x} & \text{if } x < 0 \end{cases} = \begin{cases} 4 & \text{if } x > 0 \\ 2 & \text{if } x < 0 \end{cases}$$

Note that G is not defined for $x = 0$. The domain is $(-\infty, 0) \cup (0, \infty)$.



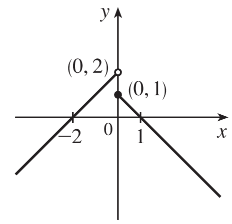
42. $g(x) = |x| - x = \begin{cases} x - x & \text{if } x \geq 0 \\ -x - x & \text{if } x < 0 \end{cases} = \begin{cases} 0 & \text{if } x \geq 0 \\ -2x & \text{if } x < 0 \end{cases}$.

The domain is $\mathbb{R}$, or $(-\infty, \infty)$.



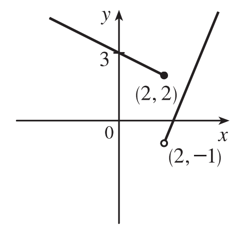
43. $f(x) = \begin{cases} x+2 & \text{if } x < 0 \\ 1-x & \text{if } x \geq 0 \end{cases}$

The domain is $\mathbb{R}$.



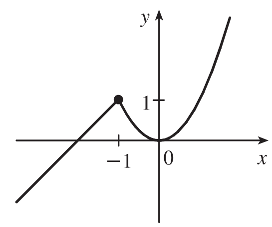
44. $f(x) = \begin{cases} 3 - \frac{1}{2}x & \text{if } x \leq 2 \\ 2x-5 & \text{if } x > 2 \end{cases}$

The domain is $\mathbb{R}$.



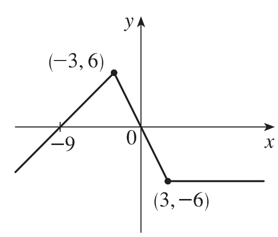
45. $f(x) = \begin{cases} x+2 & \text{if } x \leq -1 \\ x^2 & \text{if } x > -1 \end{cases}$

Note that for $x = -1$, both $x+2$ and x^2 are equal to 1. The domain is $\mathbb{R}$.



46. $f(x) = \begin{cases} x+9 & \text{if } \ x < -3 \\ -2x & \text{if } |x| \leq 3 \\ -6 & \text{if } \ x > 3 \end{cases}$

Note that for $x = -3$, both $x+9$ and $-2x$ are equal to 6; and for $x = 3$, both $-2x$ and -6 are equal to -6. The domain is $\mathbb{R}$.



47. Recall that the slope m of a line between the two points (x_1, y_1) and (x_2, y_2) is $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ and an equation of the line connecting those two points is $y - y_1 = m(x - x_1)$. The slope of the line segment joining the points $(1, -3)$ and $(5, 7)$ is $\dfrac{7-(-3)}{5-1} = \dfrac{5}{2}$, so an equation is $y - (-3) = \frac{5}{2}(x-1)$. The function is $f(x) = \frac{5}{2}x - \frac{11}{2}$, $1 \leq x \leq 5$.

48. The slope of the line segment joining the points $(-5, 10)$ and $(7, -10)$ is $\dfrac{-10-10}{7-(-5)} = -\dfrac{5}{3}$, so an equation is $y - 10 = -\frac{5}{3}[x-(-5)]$. The function is $f(x) = -\frac{5}{3}x + \frac{5}{3}$, $-5 \leq x \leq 7$.

49. We need to solve the given equation for y. $\quad x + (y-1)^2 = 0 \quad \Leftrightarrow \quad (y-1)^2 = -x \quad \Leftrightarrow \quad y - 1 = \pm\sqrt{-x} \quad \Leftrightarrow$ $y = 1 \pm \sqrt{-x}$. The expression with the positive radical represents the top half of the parabola, and the one with the negative radical represents the bottom half. Hence, we want $f(x) = 1 - \sqrt{-x}$. Note that the domain is $x \leq 0$.

50. $x^2 + (y-2)^2 = 4 \quad \Leftrightarrow \quad (y-2)^2 = 4 - x^2 \quad \Leftrightarrow \quad y - 2 = \pm\sqrt{4-x^2} \quad \Leftrightarrow \quad y = 2 \pm \sqrt{4-x^2}$. The top half is given by the function $f(x) = 2 + \sqrt{4-x^2}$, $-2 \le x \le 2$.

51. For $0 \le x \le 3$, the graph is the line with slope -1 and y-intercept 3, that is, $y = -x + 3$. For $3 < x \le 5$, the graph is the line with slope 2 passing through $(3, 0)$; that is, $y - 0 = 2(x-3)$, or $y = 2x - 6$. So the function is

$$f(x) = \begin{cases} -x+3 & \text{if } 0 \le x \le 3 \\ 2x-6 & \text{if } 3 < x \le 5 \end{cases}$$

52. For $-4 \le x \le -2$, the graph is the line with slope $-\frac{3}{2}$ passing through $(-2, 0)$; that is, $y - 0 = -\frac{3}{2}[x - (-2)]$, or $y = -\frac{3}{2}x - 3$. For $-2 < x < 2$, the graph is the top half of the circle with center $(0, 0)$ and radius 2. An equation of the circle is $x^2 + y^2 = 4$, so an equation of the top half is $y = \sqrt{4-x^2}$. For $2 \le x \le 4$, the graph is the line with slope $\frac{3}{2}$ passing through $(2, 0)$; that is, $y - 0 = \frac{3}{2}(x-2)$, or $y = \frac{3}{2}x - 3$. So the function is

$$f(x) = \begin{cases} -\frac{3}{2}x - 3 & \text{if } -4 \le x \le -2 \\ \sqrt{4-x^2} & \text{if } -2 < x < 2 \\ \frac{3}{2}x - 3 & \text{if } \ 2 \le x \le 4 \end{cases}$$

53. Let the length and width of the rectangle be L and W. Then the perimeter is $2L + 2W = 20$ and the area is $A = LW$. Solving the first equation for W in terms of L gives $W = \dfrac{20 - 2L}{2} = 10 - L$. Thus, $A(L) = L(10 - L) = 10L - L^2$. Since lengths are positive, the domain of A is $0 < L < 10$. If we further restrict L to be larger than W, then $5 < L < 10$ would be the domain.

54. Let the length and width of the rectangle be L and W. Then the area is $LW = 16$, so that $W = 16/L$. The perimeter is $P = 2L + 2W$, so $P(L) = 2L + 2(16/L) = 2L + 32/L$, and the domain of P is $L > 0$, since lengths must be positive quantities. If we further restrict L to be larger than W, then $L > 4$ would be the domain.

55. Let the length of a side of the equilateral triangle be x. Then by the Pythagorean Theorem, the height y of the triangle satisfies $y^2 + \left(\frac{1}{2}x\right)^2 = x^2$, so that $y^2 = x^2 - \frac{1}{4}x^2 = \frac{3}{4}x^2$ and $y = \frac{\sqrt{3}}{2}x$. Using the formula for the area A of a triangle, $A = \frac{1}{2}(\text{base})(\text{height})$, we obtain $A(x) = \frac{1}{2}(x)\left(\frac{\sqrt{3}}{2}x\right) = \frac{\sqrt{3}}{4}x^2$, with domain $x > 0$.

56. Let the volume of the cube be V and the length of an edge be L. Then $V = L^3$ so $L = \sqrt[3]{V}$, and the surface area is $S(V) = 6\left(\sqrt[3]{V}\right)^2 = 6V^{2/3}$, with domain $V > 0$.

57. Let each side of the base of the box have length x, and let the height of the box be h. Since the volume is 2, we know that $2 = hx^2$, so that $h = 2/x^2$, and the surface area is $S = x^2 + 4xh$. Thus, $S(x) = x^2 + 4x(2/x^2) = x^2 + (8/x)$, with domain $x > 0$.

58. The area of the window is $A = xh + \frac{1}{2}\pi\left(\frac{1}{2}x\right)^2 = xh + \dfrac{\pi x^2}{8}$, where h is the height of the rectangular portion of the window.

The perimeter is $P = 2h + x + \frac{1}{2}\pi x = 30 \quad \Leftrightarrow \quad 2h = 30 - x - \frac{1}{2}\pi x \quad \Leftrightarrow \quad h = \frac{1}{4}(60 - 2x - \pi x)$. Thus,

$$A(x) = x\,\frac{60 - 2x - \pi x}{4} + \frac{\pi x^2}{8} = 15x - \tfrac{1}{2}x^2 - \tfrac{\pi}{4}x^2 + \tfrac{\pi}{8}x^2 = 15x - \tfrac{4}{8}x^2 - \tfrac{\pi}{8}x^2 = 15x - x^2\left(\frac{\pi + 4}{8}\right).$$

Since the lengths x and h must be positive quantities, we have $x > 0$ and $h > 0$. For $h > 0$, we have $2h > 0 \quad \Leftrightarrow$

$30 - x - \frac{1}{2}\pi x > 0 \quad \Leftrightarrow \quad 60 > 2x + \pi x \quad \Leftrightarrow \quad x < \dfrac{60}{2 + \pi}$. Hence, the domain of A is $0 < x < \dfrac{60}{2 + \pi}$.

59. The height of the box is x and the length and width are $L = 20 - 2x$, $W = 12 - 2x$. Then $V = LWx$ and so

$V(x) = (20 - 2x)(12 - 2x)(x) = 4(10 - x)(6 - x)(x) = 4x(60 - 16x + x^2) = 4x^3 - 64x^2 + 240x$.

The sides L, W, and x must be positive. Thus, $L > 0 \quad \Leftrightarrow \quad 20 - 2x > 0 \quad \Leftrightarrow \quad x < 10$;

$W > 0 \quad \Leftrightarrow \quad 12 - 2x > 0 \quad \Leftrightarrow \quad x < 6$; and $x > 0$. Combining these restrictions gives us the domain $0 < x < 6$.

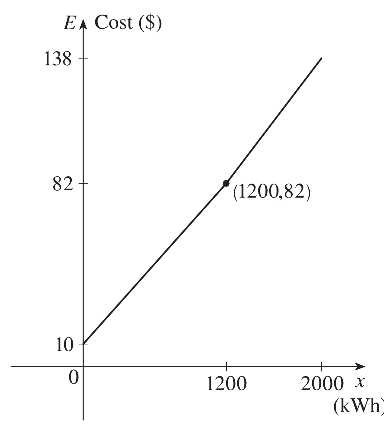
60. For the first 1200 kWh, $E(x) = 10 + 0.06x$.

For usage over 1200 kWh, the cost is

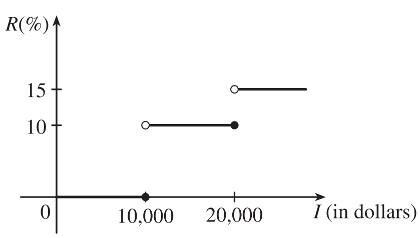
$E(x) = 10 + 0.06(1200) + 0.07(x - 1200) = 82 + 0.07(x - 1200)$.

Thus,

$$E(x) = \begin{cases} 10 + 0.06x & \text{if } 0 \le x \le 1200 \\ 82 + 0.07(x - 1200) & \text{if } x > 1200 \end{cases}$$



61. (a)

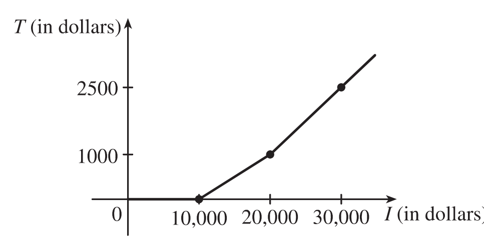


(b) On \$14,000, tax is assessed on \$4000, and $10\%(\$4000) = \$400$.

On \$26,000, tax is assessed on \$16,000, and

$10\%(\$10{,}000) + 15\%(\$6000) = \$1000 + \$900 = \$1900$.

(c) As in part (b), there is \$1000 tax assessed on \$20,000 of income, so the graph of T is a line segment from (10,000, 0) to (20,000, 1000). The tax on \$30,000 is \$2500, so the graph of T for $x > 20{,}000$ is the ray with initial point (20,000, 1000) that passes through (30,000, 2500).



62. One example is the amount paid for cable or telephone system repair in the home, usually measured to the nearest quarter hour. Another example is the amount paid by a student in tuition fees, if the fees vary according to the number of credits for which the student has registered.

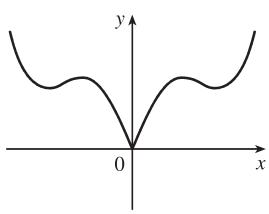
63. f is an odd function because its graph is symmetric about the origin. g is an even function because its graph is symmetric with respect to the y-axis.

64. f is not an even function since it is not symmetric with respect to the y-axis. f is not an odd function since it is not symmetric about the origin. Hence, f is *neither* even nor odd. g is an even function because its graph is symmetric with respect to the y-axis.

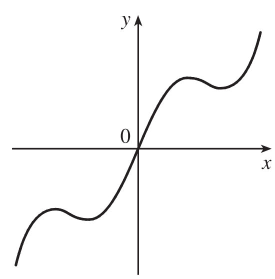
65. (a) Because an even function is symmetric with respect to the y-axis, and the point $(5, 3)$ is on the graph of this even function, the point $(-5, 3)$ must also be on its graph.

(b) Because an odd function is symmetric with respect to the origin, and the point $(5, 3)$ is on the graph of this odd function, the point $(-5, -3)$ must also be on its graph.

66. (a) If f is even, we get the rest of the graph by reflecting about the y-axis.



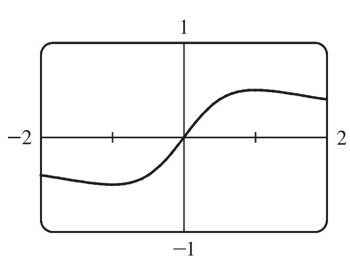
(b) If f is odd, we get the rest of the graph by rotating $180°$ about the origin.



67. $f(x) = \dfrac{x}{x^2 + 1}$.

$$f(-x) = \frac{-x}{(-x)^2 + 1} = \frac{-x}{x^2 + 1} = -\frac{x}{x^2 + 1} = -f(x).$$

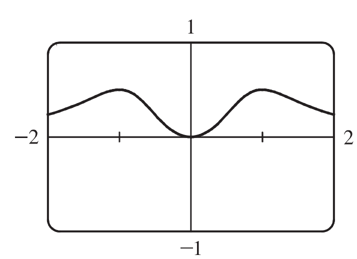
So f is an odd function.



68. $f(x) = \dfrac{x^2}{x^4 + 1}$.

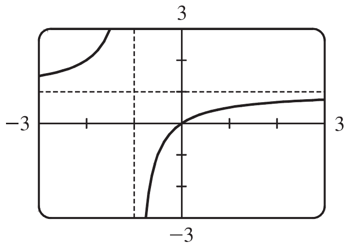
$$f(-x) = \frac{(-x)^2}{(-x)^4 + 1} = \frac{x^2}{x^4 + 1} = f(x).$$

So f is an even function.



69. $f(x) = \dfrac{x}{x+1}$, so $f(-x) = \dfrac{-x}{-x+1} = \dfrac{x}{x-1}$.

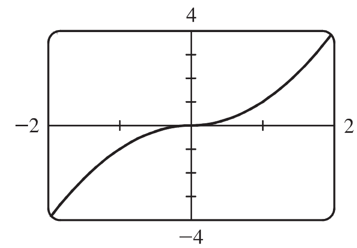
Since this is neither $f(x)$ nor $-f(x)$, the function f is neither even nor odd.



70. $f(x) = x\,|x|$.

$$f(-x) = (-x)\,|-x| = (-x)\,|x| = -(x\,|x|) = -f(x)$$

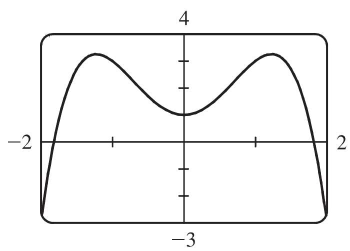
So f is an odd function.



71. $f(x) = 1 + 3x^2 - x^4$.

$f(-x) = 1+3(-x)^2-(-x)^4 = 1+3x^2-x^4 = f(x)$.

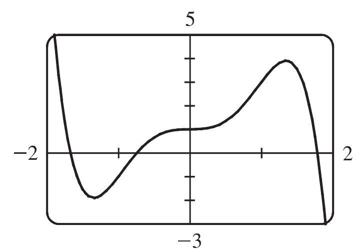
So f is an even function.



72. $f(x) = 1 + 3x^3 - x^5$, so

$$f(-x) = 1 + 3(-x)^3 - (-x)^5 = 1 + 3(-x^3) - (-x^5) = 1 - 3x^3 + x^5$$

Since this is neither $f(x)$ nor $-f(x)$, the function f is neither even nor odd.



73. (i) If f and g are both even functions, then $f(-x) = f(x)$ and $g(-x) = g(x)$. Now $(f+g)(-x) = f(-x) + g(-x) = f(x) + g(x) = (f+g)(x)$, so $f+g$ is an *even* function.

(ii) If f and g are both odd functions, then $f(-x) = -f(x)$ and $g(-x) = -g(x)$. Now $(f+g)(-x) = f(-x) + g(-x) = -f(x) + [-g(x)] = -[f(x) + g(x)] = -(f+g)(x)$, so $f+g$ is an *odd* function.

(iii) If f is an even function and g is an odd function, then $(f+g)(-x) = f(-x) + g(-x) = f(x) + [-g(x)] = f(x) - g(x)$, which is not $(f+g)(x)$ nor $-(f+g)(x)$, so $f+g$ is *neither* even nor odd. (Exception: if f is the zero function, then $f+g$ will be *odd*. If g is the zero function, then $f+g$ will be *even*.)

74. (i) If f and g are both even functions, then $f(-x) = f(x)$ and $g(-x) = g(x)$. Now $(fg)(-x) = f(-x)g(-x) = f(x)g(x) = (fg)(x)$, so fg is an *even* function.

(ii) If f and g are both odd functions, then $f(-x) = -f(x)$ and $g(-x) = -g(x)$. Now $(fg)(-x) = f(-x)g(-x) = [-f(x)][-g(x)] = f(x)g(x) = (fg)(x)$, so fg is an *even* function.

(iii) If f is an even function and g is an odd function, then $(fg)(-x) = f(-x)g(-x) = f(x)[-g(x)] = -[f(x)g(x)] = -(fg)(x)$, so fg is an *odd* function.

1.2 Mathematical Models: A Catalog of Essential Functions

1. (a) $f(x) = \log_2 x$ is a logarithmic function.

(b) $g(x) = \sqrt[4]{x}$ is a root function with $n = 4$.

(c) $h(x) = \dfrac{2x^3}{1 - x^2}$ is a rational function because it is a ratio of polynomials.

(d) $u(t) = 1 - 1.1t + 2.54t^2$ is a polynomial of degree 2 (also called a *quadratic function*).

(e) $v(t) = 5^t$ is an exponential function.

(f) $w(\theta) = \sin\theta\,\cos^2\theta$ is a trigonometric function.

2. (a) $y = \pi^x$ is an exponential function (notice that x is the *exponent*).

(b) $y = x^\pi$ is a power function (notice that x is the *base*).

(c) $y = x^2(2 - x^3) = 2x^2 - x^5$ is a polynomial of degree 5.

(d) $y = \tan t - \cos t$ is a trigonometric function.

(e) $y = s/(1 + s)$ is a rational function because it is a ratio of polynomials.

(f) $y = \sqrt{x^3 - 1}/(1 + \sqrt[3]{x})$ is an algebraic function because it involves polynomials and roots of polynomials.

3. We notice from the figure that g and h are even functions (symmetric with respect to the y-axis) and that f is an odd function (symmetric with respect to the origin). So (b) $\left[y = x^5\right]$ must be f. Since g is flatter than h near the origin, we must have (c) $\left[y = x^8\right]$ matched with g and (a) $\left[y = x^2\right]$ matched with h.

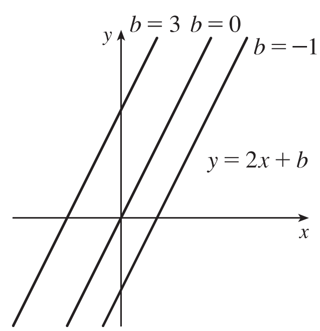
4. (a) The graph of $y = 3x$ is a line (choice G).

(b) $y = 3^x$ is an exponential function (choice f).

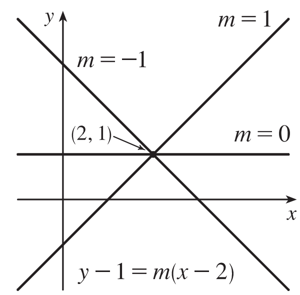
(c) $y = x^3$ is an odd polynomial function or power function (choice F).

(d) $y = \sqrt[3]{x} = x^{1/3}$ is a root function (choice g).

5. (a) An equation for the family of linear functions with slope 2 is $y = f(x) = 2x + b$, where b is the y-intercept.

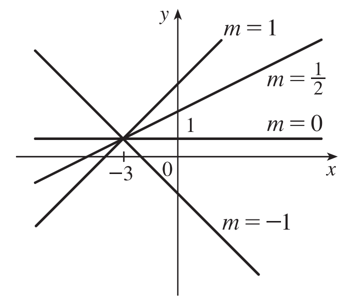


(b) $f(2) = 1$ means that the point $(2, 1)$ is on the graph of f. We can use the point-slope form of a line to obtain an equation for the family of linear functions through the point $(2, 1)$. $y - 1 = m(x - 2)$, which is equivalent to $y = mx + (1 - 2m)$ in slope-intercept form.

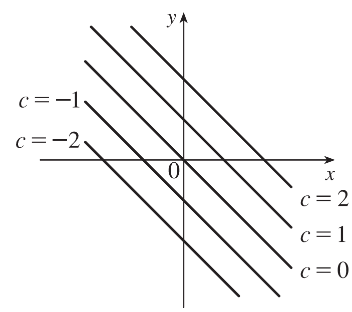


(c) To belong to both families, an equation must have slope $m = 2$, so the equation in part (b), $y = mx + (1 - 2m)$, becomes $y = 2x - 3$. It is the *only* function that belongs to both families.

6. All members of the family of linear functions $f(x) = 1 + m(x + 3)$ have graphs that are lines passing through the point $(-3, 1)$.



7. All members of the family of linear functions $f(x) = c - x$ have graphs that are lines with slope -1. The y-intercept is c.



8. The vertex of the parabola on the left is $(3, 0)$, so an equation is $y = a(x - 3)^2 + 0$. Since the point $(4, 2)$ is on the parabola, we'll substitute 4 for x and 2 for y to find a. $2 = a(4 - 3)^2 \Rightarrow a = 2$, so an equation is $f(x) = 2(x - 3)^2$.

The y-intercept of the parabola on the right is $(0, 1)$, so an equation is $y = ax^2 + bx + 1$. Since the points $(-2, 2)$ and $(1, -2.5)$ are on the parabola, we'll substitute -2 for x and 2 for y as well as 1 for x and -2.5 for y to obtain two equations with the unknowns a and b.

$(-2, 2)$: $\quad 2 = 4a - 2b + 1 \Rightarrow 4a - 2b = 1$ **(1)**

$(1, -2.5)$: $\quad -2.5 = a + b + 1 \Rightarrow a + b = -3.5$ **(2)**

$2 \cdot$ **(2)** + **(1)** gives us $6a = -6 \Rightarrow a = -1$. From **(2)**, $-1 + b = -3.5 \Rightarrow b = -2.5$, so an equation is $g(x) = -x^2 - 2.5x + 1$.

9. Since $f(-1) = f(0) = f(2) = 0$, f has zeros of -1, 0, and 2, so an equation for f is $f(x) = a[x - (-1)](x - 0)(x - 2)$, or $f(x) = ax(x + 1)(x - 2)$. Because $f(1) = 6$, we'll substitute 1 for x and 6 for $f(x)$.

$6 = a(1)(2)(-1) \Rightarrow -2a = 6 \Rightarrow a = -3$, so an equation for f is $f(x) = -3x(x + 1)(x - 2)$.

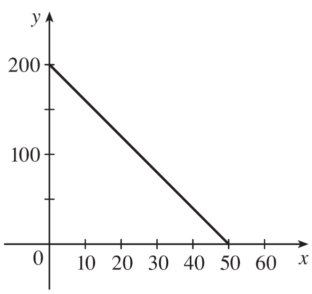
10. (a) For $T = 0.02t + 8.50$, the slope is 0.02, which means that the average surface temperature of the world is increasing at a rate of $0.02\,^\circ\text{C}$ per year. The T-intercept is 8.50, which represents the average surface temperature in $^\circ\text{C}$ in the year 1900.

(b) $t = 2100 - 1900 = 200 \quad \Rightarrow \quad T = 0.02(200) + 8.50 = 12.50\,^\circ\text{C}$

11. (a) $D = 200$, so $c = 0.0417D(a + 1) = 0.0417(200)(a + 1) = 8.34a + 8.34$. The slope is 8.34, which represents the change in mg of the dosage for a child for each change of 1 year in age.

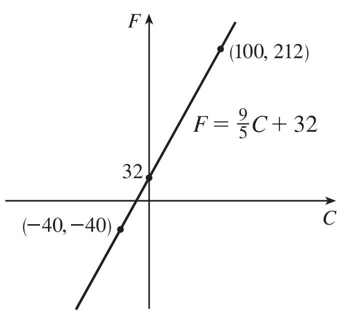
(b) For a newborn, $a = 0$, so $c = 8.34$ mg.

12. (a)



(b) The slope of -4 means that for each increase of 1 dollar for a rental space, the number of spaces rented *decreases* by 4. The y-intercept of 200 is the number of spaces that would be occupied if there were no charge for each space. The x-intercept of 50 is the smallest rental fee that results in no spaces rented.

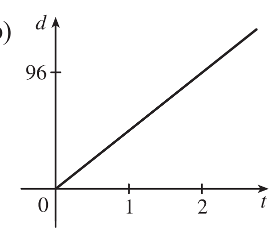
13. (a)



(b) The slope of $\frac{9}{5}$ means that F increases $\frac{9}{5}$ degrees for each increase of $1\,^\circ\text{C}$. (Equivalently, F increases by 9 when C increases by 5 and F decreases by 9 when C decreases by 5.) The F-intercept of 32 is the Fahrenheit temperature corresponding to a Celsius temperature of 0.

14. (a) Let d = distance traveled (in miles) and t = time elapsed (in hours). At $t = 0$, $d = 0$ and at $t = 50$ minutes $= 50 \cdot \frac{1}{60} = \frac{5}{6}$ h, $d = 40$. Thus we have two points: $(0, 0)$ and $\left(\frac{5}{6}, 40\right)$, so $m = \dfrac{40 - 0}{\frac{5}{6} - 0} = 48$ and so $d = 48t$.

(b)



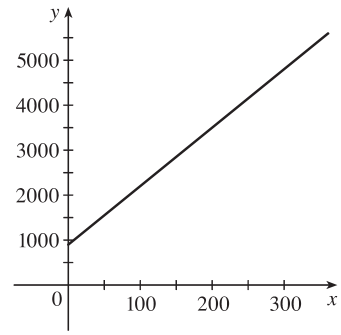
(c) The slope is 48 and represents the car's speed in mi/h.

15. (a) Using N in place of x and T in place of y, we find the slope to be $\dfrac{T_2 - T_1}{N_2 - N_1} = \dfrac{80 - 70}{173 - 113} = \dfrac{10}{60} = \dfrac{1}{6}$. So a linear equation is $T - 80 = \frac{1}{6}(N - 173) \quad \Leftrightarrow \quad T - 80 = \frac{1}{6}N - \frac{173}{6} \quad \Leftrightarrow \quad T = \frac{1}{6}N + \frac{307}{6} \quad \left[\frac{307}{6} = 51.1\overline{6}\right]$.

(b) The slope of $\frac{1}{6}$ means that the temperature in Fahrenheit degrees increases one-sixth as rapidly as the number of cricket chirps per minute. Said differently, each increase of 6 cricket chirps per minute corresponds to an increase of $1\,^\circ\text{F}$.

(c) When $N = 150$, the temperature is given approximately by $T = \frac{1}{6}(150) + \frac{307}{6} = 76.1\overline{6}\,^\circ\text{F} \approx 76\,^\circ\text{F}$.

16. (a) Let x denote the number of chairs produced in one day and y the associated cost. Using the points $(100, 2200)$ and $(300, 4800)$, we get the slope $\frac{4800-2200}{300-100} = \frac{2600}{200} = 13$. So $y - 2200 = 13(x - 100) \Leftrightarrow$ $y = 13x + 900$.



(b) The slope of the line in part (a) is 13 and it represents the cost (in dollars) of producing each additional chair.

(c) The y-intercept is 900 and it represents the fixed daily costs of operating the factory.

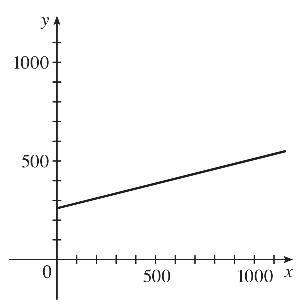
17. (a) We are given $\dfrac{\text{change in pressure}}{\text{10 feet change in depth}} = \dfrac{4.34}{10} = 0.434$. Using P for pressure and d for depth with the point $(d, P) = (0, 15)$, we have the slope-intercept form of the line, $P = 0.434d + 15$.

(b) When $P = 100$, then $100 = 0.434d + 15 \Leftrightarrow 0.434d = 85 \Leftrightarrow d = \frac{85}{0.434} \approx 195.85$ feet. Thus, the pressure is $100\ \text{lb/in}^2$ at a depth of approximately 196 feet.

18. (a) Using d in place of x and C in place of y, we find the slope to be $\dfrac{C_2 - C_1}{d_2 - d_1} = \dfrac{460 - 380}{800 - 480} = \dfrac{80}{320} = \dfrac{1}{4}$.
So a linear equation is $C - 460 = \frac{1}{4}(d - 800) \Leftrightarrow C - 460 = \frac{1}{4}d - 200 \Leftrightarrow C = \frac{1}{4}d + 260$.

(b) Letting $d = 1500$ we get $C = \frac{1}{4}(1500) + 260 = 635$.
The cost of driving 1500 miles is \$635.

(c)



The slope of the line represents the cost per mile, \$0.25.

(d) The y-intercept represents the fixed cost, \$260.

(e) A linear function gives a suitable model in this situation because you have fixed monthly costs such as insurance and car payments, as well as costs that increase as you drive, such as gasoline, oil, and tires, and the cost of these for each additional mile driven is a constant.

19. (a) The data appear to be periodic and a sine or cosine function would make the best model. A model of the form $f(x) = a\cos(bx) + c$ seems appropriate.

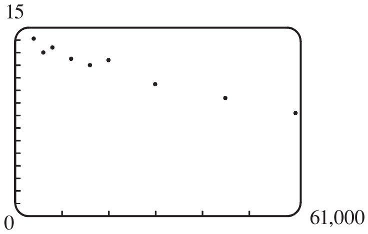
(b) The data appear to be decreasing in a linear fashion. A model of the form $f(x) = mx + b$ seems appropriate.

20. (a) The data appear to be increasing exponentially. A model of the form $f(x) = a \cdot b^x$ or $f(x) = a \cdot b^x + c$ seems appropriate.

(b) The data appear to be decreasing similarly to the values of the reciprocal function. A model of the form $f(x) = a/x$ seems appropriate.

Exercises 21 – 24: Some values are given to many decimal places. These are the results given by several computer algebra systems — rounding is left to the reader.

21. (a)

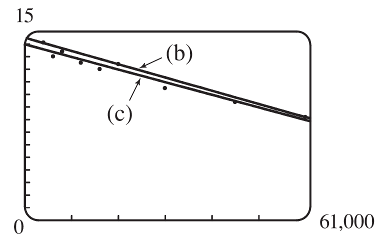


A linear model does seem appropriate.

(b) Using the points $(4000, 14.1)$ and $(60{,}000, 8.2)$, we obtain

$y - 14.1 = \dfrac{8.2 - 14.1}{60{,}000 - 4000}(x - 4000)$ or, equivalently,

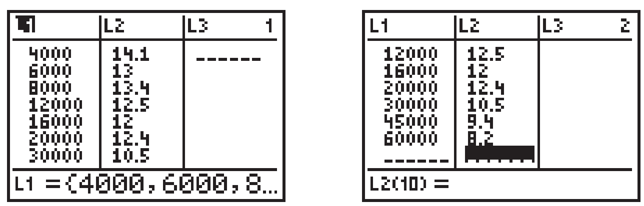
$y \approx -0.000105357x + 14.521429$.



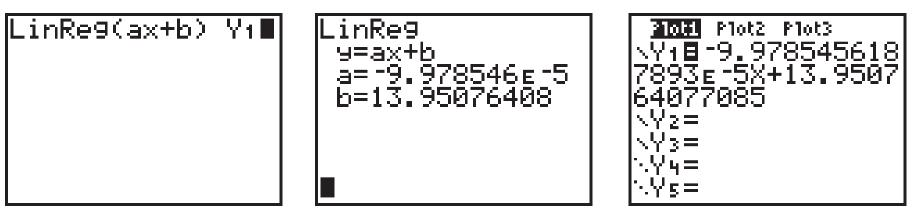
(c) Using a computing device, we obtain the least squares regression line $y = -0.0000997855x + 13.950764$.

The following commands and screens illustrate how to find the least squares regression line on a TI-84 Plus.

Enter the data into list one (L1) and list two (L2). Press [STAT] [1] to enter the editor.

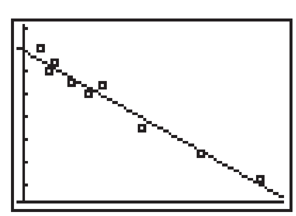


Find the regession line and store it in Y_1. Press [2nd] [QUIT] [STAT] [▶] [4] [VARS] [▶] [1] [1] [ENTER].



Note from the last figure that the regression line has been stored in Y_1 and that Plot1 has been turned on (Plot1 is highlighted). You can turn on Plot1 from the Y= menu by placing the cursor on Plot1 and pressing [ENTER] or by pressing [2nd] [STAT PLOT] [1] [ENTER].

Now press [ZOOM] [9] to produce a graph of the data and the regression line. Note that choice 9 of the ZOOM menu automatically selects a window that displays all of the data.

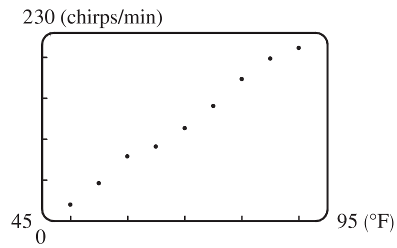


(d) When $x = 25{,}000$, $y \approx 11.456$; or about 11.5 per 100 population.

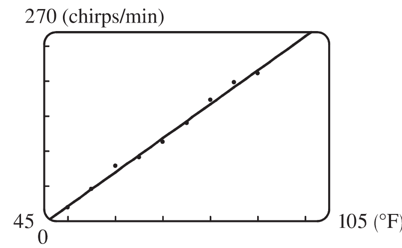
(e) When $x = 80{,}000$, $y \approx 5.968$; or about a 6% chance.

(f) When $x = 200{,}000$, y is negative, so the model does not apply.

22. (a)



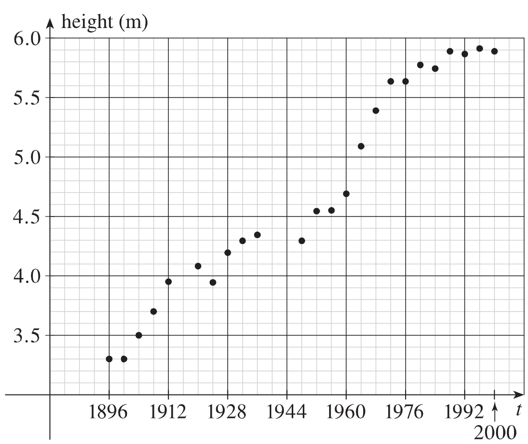
(b)



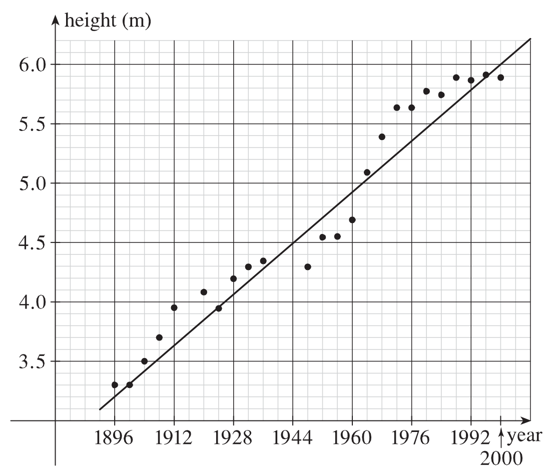
Using a computing device, we obtain the least squares regression line $y = 4.85\overline{6}x - 220.9\overline{6}$.

(c) When $x = 100°\text{F}$, $y = 264.7 \approx 265$ chirps/min.

23. (a) A linear model seems appropriate over the time interval considered.



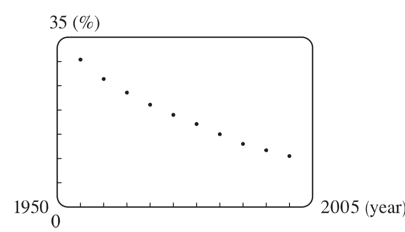
(b) Using a computing device, we obtain the least squares regression line $y \approx 0.027t - 47.758$.



(c) When $t = 2004$, $y = 6.35$, which is higher than the actual winning height of 5.95 m.

(d) No, since the times appear to be leveling off and getting further away from the model.

24. By looking at the scatter plot of the data, we rule out the power and logarithmic models.

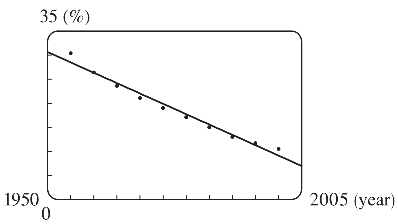


Scatter plot

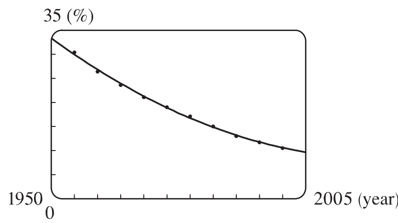
We try various models:

Linear	$y = -0.430\,545\,454\,5x + 870.183\,636\,4$
Quadratic:	$y = 0.004\,893\,939\,4x^2 - 19.786\,075\,76x + 20\,006.954\,85$
Cubic:	$y = -0.000\,073\,193\,47x^3 + 0.439\,114\,219\,1x^2 - 878.429\,871\,8x + 585\,960.983$
Quartic:	$y = 0.000\,007\,902\,097\,9x^4 - 0.062\,578\,787\,9x^3 + 185.842\,283\,8x^2 - 245\,290.9304x + 121\,409\,472.7$
Exponential:	$y = 2.618\,230\,2 \times 10^{21}(0.976\,789\,309\,4)^x$

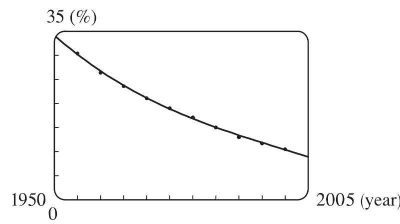
[continued]



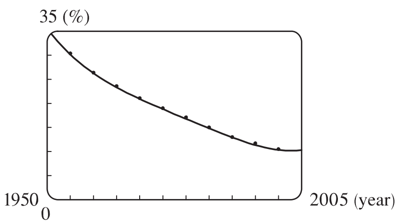
Linear model



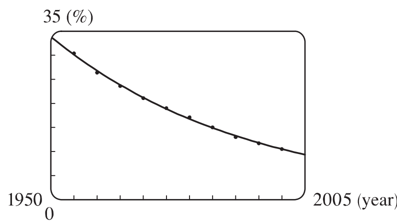
Quadratic model



Cubic model



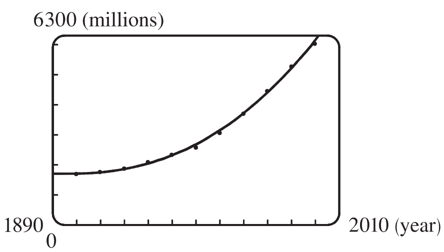
Quartic model



Exponential model

After examining the graphs of these models, we see that all the models are good and the quartic model is the best.

Using this model, we obtain estimates 13.6% and 10.2% for the rural percentages in 1988 and 2002 respectively.

25. 

Using a computing device, we obtain the cubic function $y = ax^3 + bx^2 + cx + d$ with $a = 0.0012937$, $b = -7.06142$, $c = 12{,}823$, and $d = -7{,}743{,}770$. When $x = 1925$, $y \approx 1914$ (million).

26. (a) $T = 1.000\,431\,227 d^{1.499\,528\,750}$

(b) The power model in part (a) is approximately $T = d^{1.5}$. Squaring both sides gives us $T^2 = d^3$, so the model matches Kepler's Third Law, $T^2 = kd^3$.

1.3 New Functions from Old Functions

1. (a) If the graph of f is shifted 3 units upward, its equation becomes $y = f(x) + 3$.

(b) If the graph of f is shifted 3 units downward, its equation becomes $y = f(x) - 3$.

(c) If the graph of f is shifted 3 units to the right, its equation becomes $y = f(x - 3)$.

(d) If the graph of f is shifted 3 units to the left, its equation becomes $y = f(x + 3)$.

(e) If the graph of f is reflected about the x-axis, its equation becomes $y = -f(x)$.

(f) If the graph of f is reflected about the y-axis, its equation becomes $y = f(-x)$.

(g) If the graph of f is stretched vertically by a factor of 3, its equation becomes $y = 3f(x)$.

(h) If the graph of f is shrunk vertically by a factor of 3, its equation becomes $y = \frac{1}{3}f(x)$.

2. (a) To obtain the graph of $y = f(x) + 8$ from the graph of $y = f(x)$, shift the graph 8 units upward.

(b) To obtain the graph of $y = f(x + 8)$ from the graph of $y = f(x)$, shift the graph 8 units to the left.

(c) To obtain the graph of $y = 8f(x)$ from the graph of $y = f(x)$, stretch the graph vertically by a factor of 8.

(d) To obtain the graph of $y = f(8x)$ from the graph of $y = f(x)$, shrink the graph horizontally by a factor of 8.

(e) To obtain the graph of $y = -f(x) - 1$ from the graph of $y = f(x)$, first reflect the graph about the x-axis, and then shift it 1 unit downward.

(f) To obtain the graph of $y = 8f(\frac{1}{8}x)$ from the graph of $y = f(x)$, stretch the graph horizontally and vertically by a factor of 8.

3. (a) (graph 3) The graph of f is shifted 4 units to the right and has equation $y = f(x - 4)$.

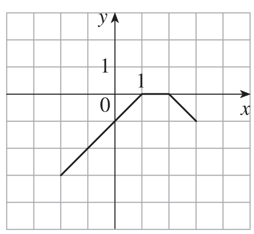
(b) (graph 1) The graph of f is shifted 3 units upward and has equation $y = f(x) + 3$.

(c) (graph 4) The graph of f is shrunk vertically by a factor of 3 and has equation $y = \frac{1}{3}f(x)$.

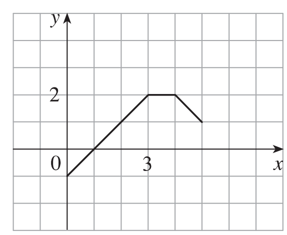
(d) (graph 5) The graph of f is shifted 4 units to the left and reflected about the x-axis. Its equation is $y = -f(x + 4)$.

(e) (graph 2) The graph of f is shifted 6 units to the left and stretched vertically by a factor of 2. Its equation is $y = 2f(x + 6)$.

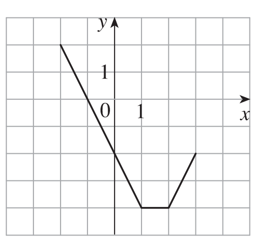
4. (a) To graph $y = f(x) - 2$, we shift the graph of f, 2 units downward. The point $(1, 2)$ on the graph of f corresponds to the point $(1, 2 - 2) = (1, 0)$.



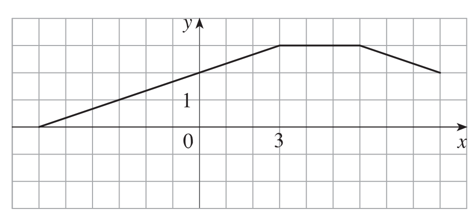
(b) To graph $y = f(x - 2)$, we shift the graph of f, 2 units to the right. The point $(1, 2)$ on the graph of f corresponds to the point $(1 + 2, 2) = (3, 2)$.



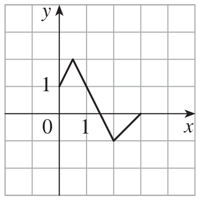
(c) To graph $y = -2f(x)$, we reflect the graph about the x-axis and stretch the graph vertically by a factor of 2. The point $(1, 2)$ on the graph of f corresponds to the point $(1, -2 \cdot 2) = (1, -4)$.



(d) To graph $y = f(\frac{1}{3}x) + 1$, we stretch the graph horizontally by a factor of 3 and shift it 1 unit upward. The point $(1, 2)$ on the graph of f corresponds to the point $(1 \cdot 3, 2 + 1) = (3, 3)$.

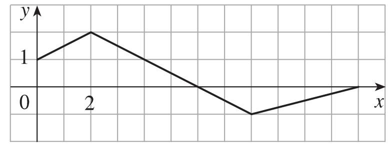


5. (a) To graph $y = f(2x)$ we shrink the graph of f horizontally by a factor of 2.



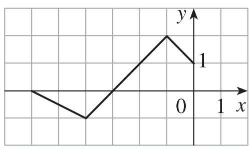
The point $(4, -1)$ on the graph of f corresponds to the point $\left(\frac{1}{2} \cdot 4, -1\right) = (2, -1)$.

(b) To graph $y = f\left(\frac{1}{2}x\right)$ we stretch the graph of f horizontally by a factor of 2.



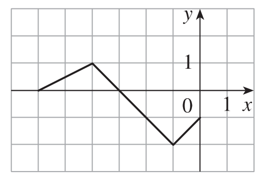
The point $(4, -1)$ on the graph of f corresponds to the point $(2 \cdot 4, -1) = (8, -1)$.

(c) To graph $y = f(-x)$ we reflect the graph of f about the y-axis.



The point $(4, -1)$ on the graph of f corresponds to the point $(-1 \cdot 4, -1) = (-4, -1)$.

(d) To graph $y = -f(-x)$ we reflect the graph of f about the y-axis, then about the x-axis.



The point $(4, -1)$ on the graph of f corresponds to the point $(-1 \cdot 4, -1 \cdot -1) = (-4, 1)$.

6. The graph of $y = f(x) = \sqrt{3x - x^2}$ has been shifted 2 units to the right and stretched vertically by a factor of 2. Thus, a function describing the graph is

$$y = 2f(x-2) = 2\sqrt{3(x-2) - (x-2)^2} = 2\sqrt{3x - 6 - (x^2 - 4x + 4)} = 2\sqrt{-x^2 + 7x - 10}$$

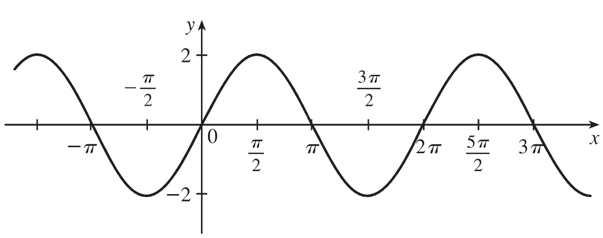
7. The graph of $y = f(x) = \sqrt{3x - x^2}$ has been shifted 4 units to the left, reflected about the x-axis, and shifted downward 1 unit. Thus, a function describing the graph is

$$y = \underbrace{-1 \cdot}_{\substack{\text{reflect} \\ \text{about } x\text{-axis}}} \quad f \underbrace{(x+4)}_{\substack{\text{shift} \\ \text{4 units left}}} \quad \underbrace{-\ 1}_{\substack{\text{shift} \\ \text{1 unit left}}}$$

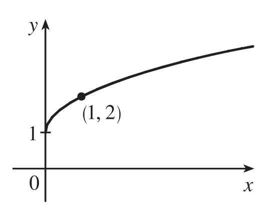
This function can be written as

$$y = -f(x+4) - 1 = -\sqrt{3(x+4) - (x+4)^2} - 1 = -\sqrt{3x + 12 - (x^2 + 8x + 16)} - 1 = -\sqrt{-x^2 - 5x - 4} - 1$$

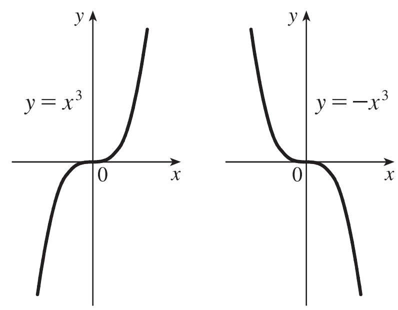
8. (a) The graph of $y = 2\sin x$ can be obtained from the graph of $y = \sin x$ by stretching it vertically by a factor of 2.



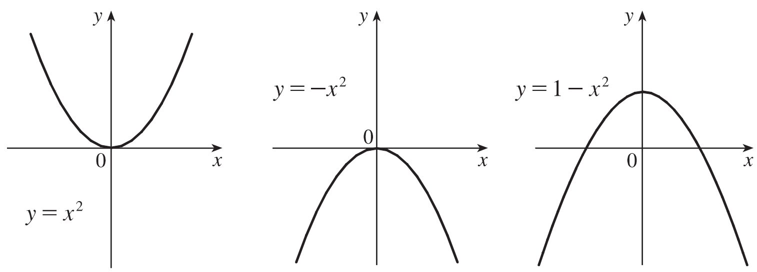
(b) The graph of $y = 1 + \sqrt{x}$ can be obtained from the graph of $y = \sqrt{x}$ by shifting it upward 1 unit.



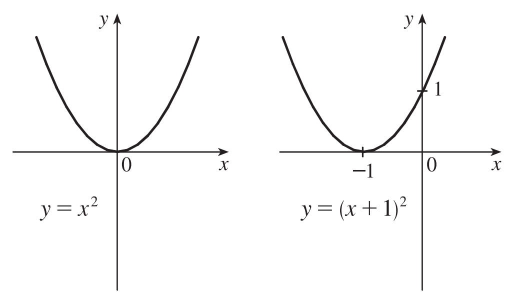
9. $y = -x^3$: Start with the graph of $y = x^3$ and reflect about the x-axis. Note: Reflecting about the y-axis gives the same result since substituting $-x$ for x gives us $y = (-x)^3 = -x^3$.



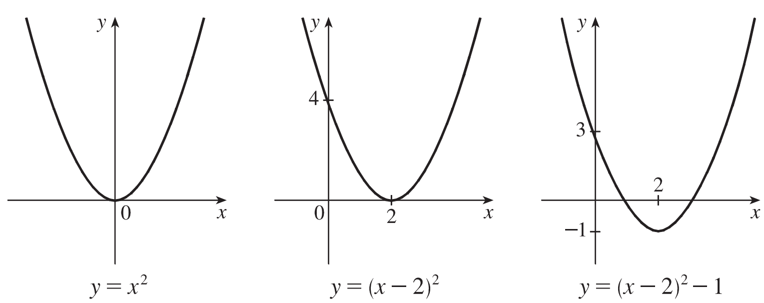
10. $y = 1 - x^2 = -x^2 + 1$: Start with the graph of $y = x^2$, reflect about the x-axis, and then shift 1 unit upward.



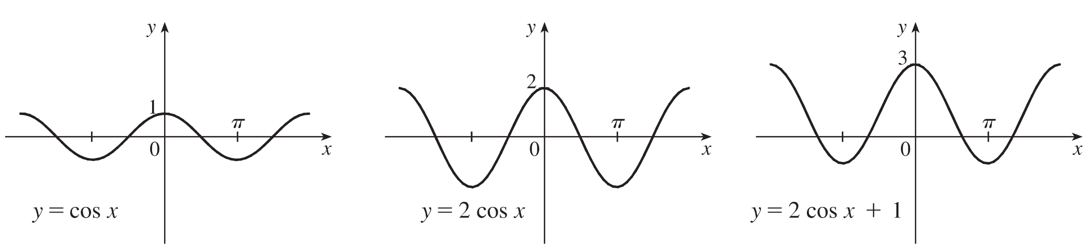
11. $y = (x + 1)^2$: Start with the graph of $y = x^2$ and shift 1 unit to the left.



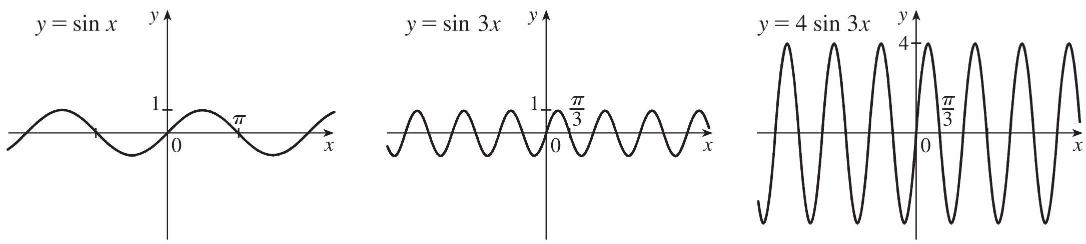
12. $y = x^2 - 4x + 3 = (x^2 - 4x + 4) - 1 = (x - 2)^2 - 1$: Start with the graph of $y = x^2$, shift 2 units to the right, and then shift 1 unit downward.



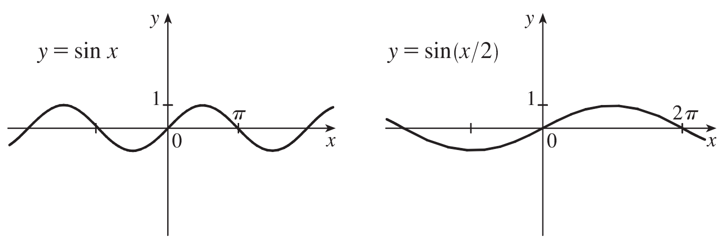
13. $y = 1 + 2\cos x$: Start with the graph of $y = \cos x$, stretch vertically by a factor of 2, and then shift 1 unit upward.



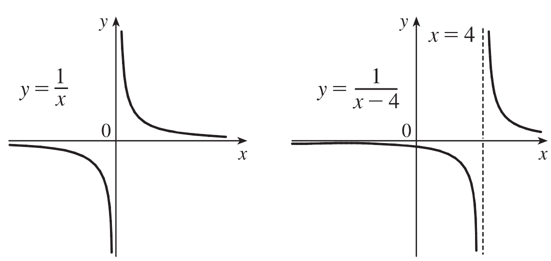
14. $y = 4\sin 3x$: Start with the graph of $y = \sin x$, compress horizontally by a factor of 3, and then stretch vertically by a factor of 4.



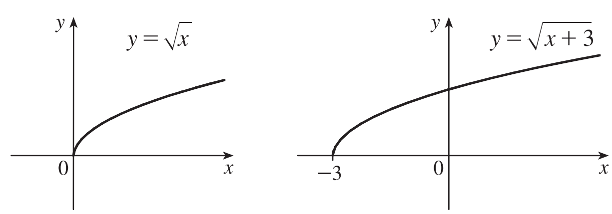
15. $y = \sin(x/2)$: Start with the graph of $y = \sin x$ and stretch horizontally by a factor of 2.



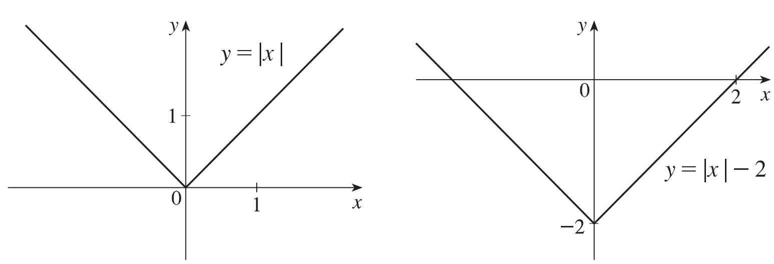
16. $y = 1/(x-4)$: Start with the graph of $y = 1/x$ and shift 4 units to the right.



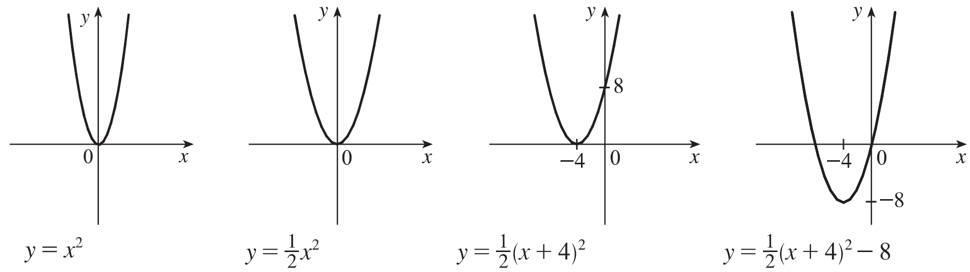
17. $y = \sqrt{x+3}$: Start with the graph of $y = \sqrt{x}$ and shift 3 units to the left.



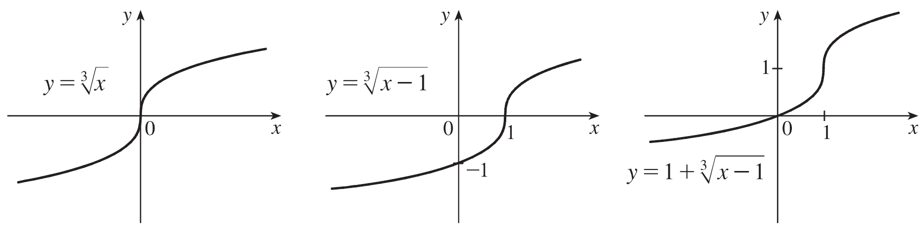
18. $y = |x| - 2$: Start with the graph of $y = |x|$ and shift 2 units downward.



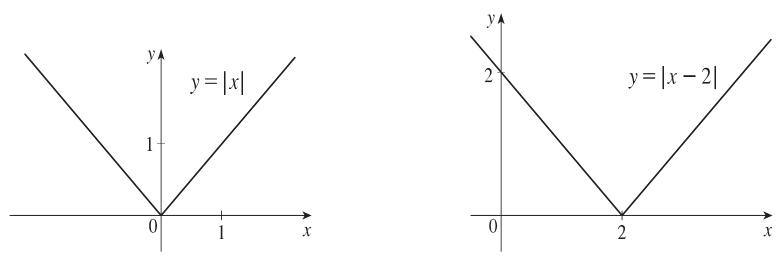
19. $y = \frac{1}{2}(x^2 + 8x) = \frac{1}{2}(x^2 + 8x + 16) - 8 = \frac{1}{2}(x + 4)^2 - 8$: Start with the graph of $y = x^2$, compress vertically by a factor of 2, shift 4 units to the left, and then shift 8 units downward.



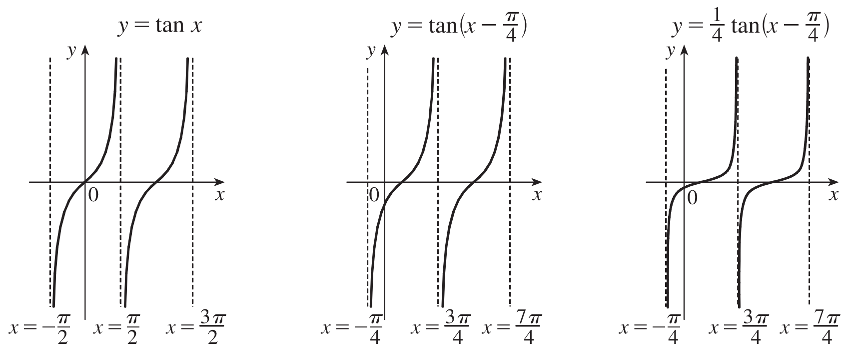
20. $y = 1 + \sqrt[3]{x - 1}$: Start with the graph of $y = \sqrt[3]{x}$, shift 1 unit to the right, and then shift 1 unit upward.



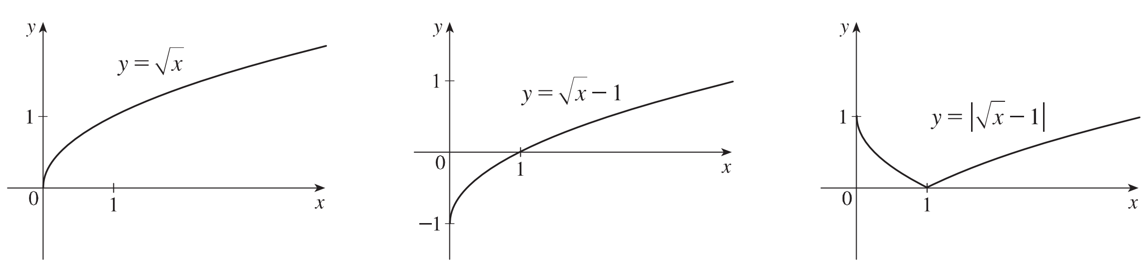
21. $y = |x - 2|$: Start with the graph of $y = |x|$ and shift 2 units to the right.



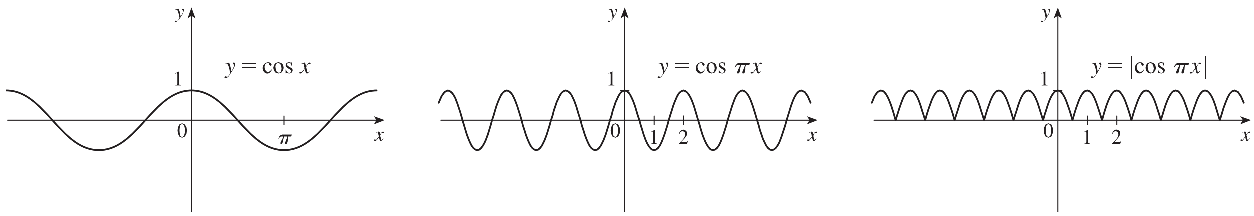
22. $y = \frac{1}{4}\tan(x - \frac{\pi}{4})$: Start with the graph of $y = \tan x$, shift $\frac{\pi}{4}$ units to the right, and then compress vertically by a factor of 4.



23. $y = |\sqrt{x} - 1|$: Start with the graph of $y = \sqrt{x}$, shift it 1 unit downward, and then reflect the portion of the graph below the x-axis about the x-axis.



24. $y = |\cos \pi x|$: Start with the graph of $y = \cos x$, shrink it horizontally by a factor of π, and reflect all the parts of the graph below the x-axis about the x-axis.

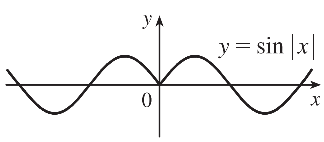


25. This is just like the solution to Example 4 except the amplitude of the curve (the 30°N curve in Figure 9 on June 21) is $14 - 12 = 2$. So the function is $L(t) = 12 + 2\sin\left[\frac{2\pi}{365}(t - 80)\right]$. March 31 is the 90th day of the year, so the model gives $L(90) \approx 12.34$ h. The daylight time (5:51 AM to 6:18 PM) is 12 hours and 27 minutes, or 12.45 h. The model value differs from the actual value by $\frac{12.45 - 12.34}{12.45} \approx 0.009$, less than 1%.

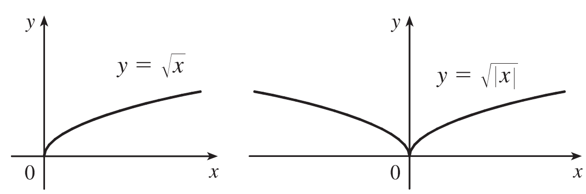
26. Using a sine function to model the brightness of Delta Cephei as a function of time, we take its period to be 5.4 days, its amplitude to be 0.35 (on the scale of magnitude), and its average magnitude to be 4.0. If we take $t = 0$ at a time of average brightness, then the magnitude (brightness) as a function of time t in days can be modeled by the formula $M(t) = 4.0 + 0.35\sin\left(\frac{2\pi}{5.4}t\right)$.

27. (a) To obtain $y = f(|x|)$, the portion of the graph of $y = f(x)$ to the right of the y-axis is reflected about the y-axis.

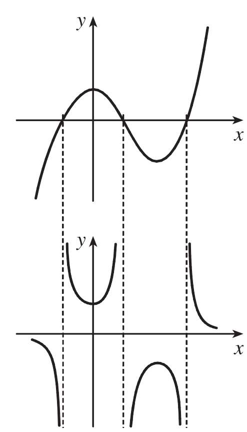
(b) $y = \sin|x|$



(c) $y = \sqrt{|x|}$



28. The most important features of the given graph are the x-intercepts and the maximum and minimum points. The graph of $y = 1/f(x)$ has vertical asymptotes at the x-values where there are x-intercepts on the graph of $y = f(x)$. The maximum of 1 on the graph of $y = f(x)$ corresponds to a minimum of $1/1 = 1$ on $y = 1/f(x)$. Similarly, the minimum on the graph of $y = f(x)$ corresponds to a maximum on the graph of $y = 1/f(x)$. As the values of y get large (positively or negatively) on the graph of $y = f(x)$, the values of y get close to zero on the graph of $y = 1/f(x)$.



29. $f(x) = x^3 + 2x^2$; $g(x) = 3x^2 - 1$. $D = \mathbb{R}$ for both f and g.

(a) $(f + g)(x) = (x^3 + 2x^2) + (3x^2 - 1) = x^3 + 5x^2 - 1$, $D = \mathbb{R}$.

(b) $(f - g)(x) = (x^3 + 2x^2) - (3x^2 - 1) = x^3 - x^2 + 1$, $D = \mathbb{R}$.

(c) $(fg)(x) = (x^3 + 2x^2)(3x^2 - 1) = 3x^5 + 6x^4 - x^3 - 2x^2$, $D = \mathbb{R}$.

(d) $\left(\dfrac{f}{g}\right)(x) = \dfrac{x^3 + 2x^2}{3x^2 - 1}$, $D = \left\{x \mid x \neq \pm\dfrac{1}{\sqrt{3}}\right\}$ since $3x^2 - 1 \neq 0$.

30. $f(x) = \sqrt{3-x},\ D = (-\infty, 3];\quad g(x) = \sqrt{x^2-1},\ D = (-\infty, -1] \cup [1, \infty)$.

(a) $(f+g)(x) = \sqrt{3-x} + \sqrt{x^2-1},\ D = (-\infty, -1] \cup [1, 3]$, which is the intersection of the domains of f and g.

(b) $(f-g)(x) = \sqrt{3-x} - \sqrt{x^2-1},\ D = (-\infty, -1] \cup [1, 3]$.

(c) $(fg)(x) = \sqrt{3-x} \cdot \sqrt{x^2-1},\ D = (-\infty, -1] \cup [1, 3]$.

(d) $\left(\dfrac{f}{g}\right)(x) = \dfrac{\sqrt{3-x}}{\sqrt{x^2-1}},\ D = (-\infty, -1) \cup (1, 3]$. We must exclude $x = \pm 1$ since these values would make $\dfrac{f}{g}$ undefined.

31. $f(x) = x^2 - 1,\ D = \mathbb{R};\quad g(x) = 2x + 1,\ D = \mathbb{R}$.

(a) $(f \circ g)(x) = f(g(x)) = f(2x+1) = (2x+1)^2 - 1 = (4x^2 + 4x + 1) - 1 = 4x^2 + 4x,\ D = \mathbb{R}$.

(b) $(g \circ f)(x) = g(f(x)) = g(x^2-1) = 2(x^2-1) + 1 = (2x^2 - 2) + 1 = 2x^2 - 1,\ D = \mathbb{R}$.

(c) $(f \circ f)(x) = f(f(x)) = f(x^2-1) = (x^2-1)^2 - 1 = (x^4 - 2x^2 + 1) - 1 = x^4 - 2x^2,\ D = \mathbb{R}$.

(d) $(g \circ g)(x) = g(g(x)) = g(2x+1) = 2(2x+1) + 1 = (4x+2) + 1 = 4x + 3,\ D = \mathbb{R}$.

32. $f(x) = x - 2;\ g(x) = x^2 + 3x + 4$. $\quad D = \mathbb{R}$ for both f and g, and hence for their composites.

(a) $(f \circ g)(x) = f(g(x)) = f(x^2 + 3x + 4) = (x^2 + 3x + 4) - 2 = x^2 + 3x + 2$.

(b) $(g \circ f)(x) = g(f(x)) = g(x-2) = (x-2)^2 + 3(x-2) + 4 = x^2 - 4x + 4 + 3x - 6 + 4 = x^2 - x + 2$.

(c) $(f \circ f)(x) = f(f(x)) = f(x-2) = (x-2) - 2 = x - 4$.

(d)
$$\begin{aligned}(g \circ g)(x) = g(g(x)) = g(x^2 + 3x + 4) &= (x^2 + 3x + 4)^2 + 3(x^2 + 3x + 4) + 4 \\ &= (x^4 + 9x^2 + 16 + 6x^3 + 8x^2 + 24x) + 3x^2 + 9x + 12 + 4 \\ &= x^4 + 6x^3 + 20x^2 + 33x + 32\end{aligned}$$

33. $f(x) = 1 - 3x;\ g(x) = \cos x$. $\quad D = \mathbb{R}$ for both f and g, and hence for their composites.

(a) $(f \circ g)(x) = f(g(x)) = f(\cos x) = 1 - 3\cos x$.

(b) $(g \circ f)(x) = g(f(x)) = g(1 - 3x) = \cos(1 - 3x)$.

(c) $(f \circ f)(x) = f(f(x)) = f(1 - 3x) = 1 - 3(1 - 3x) = 1 - 3 + 9x = 9x - 2$.

(d) $(g \circ g)(x) = g(g(x)) = g(\cos x) = \cos(\cos x)$ [Note that this is *not* $\cos x \cdot \cos x$.]

34. $f(x) = \sqrt{x},\ D = [0, \infty);\quad g(x) = \sqrt[3]{1-x},\ D = \mathbb{R}$.

(a) $(f \circ g)(x) = f(g(x)) = f\left(\sqrt[3]{1-x}\right) = \sqrt{\sqrt[3]{1-x}} = \sqrt[6]{1-x}$.

The domain of $f \circ g$ is $\{x \mid \sqrt[3]{1-x} \geq 0\} = \{x \mid 1 - x \geq 0\} = \{x \mid x \leq 1\} = (-\infty, 1]$.

(b) $(g \circ f)(x) = g(f(x)) = g(\sqrt{x}) = \sqrt[3]{1 - \sqrt{x}}$.

The domain of $g \circ f$ is $\{x \mid x$ is in the domain of f and $f(x)$ is in the domain of $g\}$. This is the domain of f, that is, $[0, \infty)$.

(c) $(f \circ f)(x) = f(f(x)) = f(\sqrt{x}) = \sqrt{\sqrt{x}} = \sqrt[4]{x}$. The domain of $f \circ f$ is $\{x \mid x \geq 0$ and $\sqrt{x} \geq 0\} = [0, \infty)$.

(d) $(g \circ g)(x) = g(g(x)) = g\left(\sqrt[3]{1-x}\right) = \sqrt[3]{1 - \sqrt[3]{1-x}}$, and the domain is $(-\infty, \infty)$.

35. $f(x) = x + \dfrac{1}{x}$, $D = \{x \mid x \neq 0\}$; $g(x) = \dfrac{x+1}{x+2}$, $D = \{x \mid x \neq -2\}$

(a) $(f \circ g)(x) = f(g(x)) = f\left(\dfrac{x+1}{x+2}\right) = \dfrac{x+1}{x+2} + \dfrac{1}{\frac{x+1}{x+2}} = \dfrac{x+1}{x+2} + \dfrac{x+2}{x+1}$

$$= \frac{(x+1)(x+1)+(x+2)(x+2)}{(x+2)(x+1)} = \frac{(x^2+2x+1)+(x^2+4x+4)}{(x+2)(x+1)} = \frac{2x^2+6x+5}{(x+2)(x+1)}$$

Since $g(x)$ is not defined for $x = -2$ and $f(g(x))$ is not defined for $x = -2$ and $x = -1$,

the domain of $(f \circ g)(x)$ is $D = \{x \mid x \neq -2, -1\}$.

(b) $(g \circ f)(x) = g(f(x)) = g\left(x + \dfrac{1}{x}\right) = \dfrac{\left(x+\frac{1}{x}\right)+1}{\left(x+\frac{1}{x}\right)+2} = \dfrac{\frac{x^2+1+x}{x}}{\frac{x^2+1+2x}{x}} = \dfrac{x^2+x+1}{x^2+2x+1} = \dfrac{x^2+x+1}{(x+1)^2}$

Since $f(x)$ is not defined for $x = 0$ and $g(f(x))$ is not defined for $x = -1$,
the domain of $(g \circ f)(x)$ is $D = \{x \mid x \neq -1, 0\}$.

(c) $(f \circ f)(x) = f(f(x)) = f\left(x + \dfrac{1}{x}\right) = \left(x + \dfrac{1}{x}\right) + \dfrac{1}{x + \frac{1}{x}} = x + \dfrac{1}{x} + \dfrac{1}{\frac{x^2+1}{x}} = x + \dfrac{1}{x} + \dfrac{x}{x^2+1}$

$$= \frac{x(x)(x^2+1) + 1\,(x^2+1) + x(x)}{x(x^2+1)} = \frac{x^4+x^2+x^2+1+x^2}{x(x^2+1)}$$

$$= \frac{x^4+3x^2+1}{x(x^2+1)}, \quad D = \{x \mid x \neq 0\}$$

(d) $(g \circ g)(x) = g(g(x)) = g\left(\dfrac{x+1}{x+2}\right) = \dfrac{\frac{x+1}{x+2}+1}{\frac{x+1}{x+2}+2} = \dfrac{\frac{x+1+1(x+2)}{x+2}}{\frac{x+1+2(x+2)}{x+2}} = \dfrac{x+1+x+2}{x+1+2x+4} = \dfrac{2x+3}{3x+5}$

Since $g(x)$ is not defined for $x = -2$ and $g(g(x))$ is not defined for $x = -\frac{5}{3}$,

the domain of $(g \circ g)(x)$ is $D = \left\{x \mid x \neq -2, -\frac{5}{3}\right\}$.

36. $f(x) = \dfrac{x}{1+x}$, $D = \{x \mid x \neq -1\}$; $g(x) = \sin 2x$, $D = \mathbb{R}$.

(a) $(f \circ g)(x) = f(g(x)) = f(\sin 2x) = \dfrac{\sin 2x}{1 + \sin 2x}$

Domain: $1 + \sin 2x \neq 0 \;\Rightarrow\; \sin 2x \neq -1 \;\Rightarrow\; 2x \neq \dfrac{3\pi}{2} + 2\pi n \;\Rightarrow\; x \neq \dfrac{3\pi}{4} + \pi n$ [n an integer].

(b) $(g \circ f)(x) = g(f(x)) = g\left(\dfrac{x}{1+x}\right) = \sin\left(\dfrac{2x}{1+x}\right)$.

Domain: $\{x \mid x \neq -1\}$

(c) $(f \circ f)(x) = f(f(x)) = f\left(\dfrac{x}{1+x}\right) = \dfrac{\frac{x}{1+x}}{1 + \frac{x}{1+x}} = \dfrac{\left(\frac{x}{1+x}\right)\cdot(1+x)}{\left(1 + \frac{x}{1+x}\right)\cdot(1+x)} = \dfrac{x}{1+x+x} = \dfrac{x}{2x+1}$

Since $f(x)$ is not defined for $x = -1$, and $f(f(x))$ is not defined for $x = -\frac{1}{2}$,

the domain of $(f \circ f)(x)$ is $D = \{x \mid x \neq -1, -\frac{1}{2}\}$.

(d) $(g \circ g)(g) = g(g(x)) = g(\sin 2x) = \sin(2 \sin 2x)$.
Domain: $\mathbb{R}$

37. $(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x-1)) = f(2(x-1)) = 2(x-1)+1 = 2x-1$

38. $(f \circ g \circ h)(x) = f(g(h(x))) = f(g(1-x)) = f((1-x)^2) = 2(1-x)^2 - 1 = 2x^2 - 4x + 1$

39. $(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x^3+2)) = f[(x^3+2)^2]$

$= f(x^6+4x^3+4) = \sqrt{(x^6+4x^3+4)-3} = \sqrt{x^6+4x^3+1}$

40. $(f \circ g \circ h)(x) = f(g(h(x))) = f(g(\sqrt[3]{x})) = f\left(\dfrac{\sqrt[3]{x}}{\sqrt[3]{x}-1}\right) = \tan\left(\dfrac{\sqrt[3]{x}}{\sqrt[3]{x}-1}\right)$

41. Let $g(x) = 2x + x^2$ and $f(x) = x^4$. Then $(f \circ g)(x) = f(g(x)) = f(2x+x^2) = (2x+x^2)^4 = F(x)$.

42. Let $g(x) = \cos x$ and $f(x) = x^2$. Then $(f \circ g)(x) = f(g(x)) = f(\cos x) = (\cos x)^2 = \cos^2 x = F(x)$.

43. Let $g(x) = \sqrt[3]{x}$ and $f(x) = \dfrac{x}{1+x}$. Then $(f \circ g)(x) = f(g(x)) = f(\sqrt[3]{x}) = \dfrac{\sqrt[3]{x}}{1+\sqrt[3]{x}} = F(x)$.

44. Let $g(x) = \dfrac{x}{1+x}$ and $f(x) = \sqrt[3]{x}$. Then $(f \circ g)(x) = f(g(x)) = f\left(\dfrac{x}{1+x}\right) = \sqrt[3]{\dfrac{x}{1+x}} = G(x)$.

45. Let $g(t) = \cos t$ and $f(t) = \sqrt{t}$. Then $(f \circ g)(t) = f(g(t)) = f(\cos t) = \sqrt{\cos t} = u(t)$.

46. Let $g(t) = \tan t$ and $f(t) = \dfrac{t}{1+t}$. Then $(f \circ g)(t) = f(g(t)) = f(\tan t) = \dfrac{\tan t}{1+\tan t} = u(t)$.

47. Let $h(x) = x^2$, $g(x) = 3^x$, and $f(x) = 1 - x$. Then

$(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x^2)) = f\left(3^{x^2}\right) = 1 - 3^{x^2} = H(x)$.

48. Let $h(x) = |x|$, $g(x) = 2 + x$, and $f(x) = \sqrt[8]{x}$. Then

$(f \circ g \circ h)(x) = f(g(h(x))) = f(g(|x|)) = f(2+|x|) = \sqrt[8]{2+|x|} = H(x)$.

49. Let $h(x) = \sqrt{x}$, $g(x) = \sec x$, and $f(x) = x^4$. Then

$(f \circ g \circ h)(x) = f(g(h(x))) = f(g(\sqrt{x})) = f(\sec\sqrt{x}) = (\sec\sqrt{x})^4 = \sec^4(\sqrt{x}) = H(x)$.

50. (a) $f(g(1)) = f(6) = 5$

(b) $g(f(1)) = g(3) = 2$

(c) $f(f(1)) = f(3) = 4$

(d) $g(g(1)) = g(6) = 3$

(e) $(g \circ f)(3) = g(f(3)) = g(4) = 1$

(f) $(f \circ g)(6) = f(g(6)) = f(3) = 4$

51. (a) $g(2) = 5$, because the point $(2, 5)$ is on the graph of g. Thus, $f(g(2)) = f(5) = 4$, because the point $(5, 4)$ is on the graph of f.

(b) $g(f(0)) = g(0) = 3$

(c) $(f \circ g)(0) = f(g(0)) = f(3) = 0$

(d) $(g \circ f)(6) = g(f(6)) = g(6)$. This value is not defined, because there is no point on the graph of g that has x-coordinate 6.

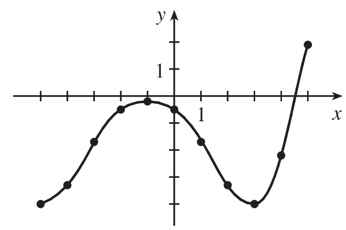
(e) $(g \circ g)(-2) = g(g(-2)) = g(1) = 4$

(f) $(f \circ f)(4) = f(f(4)) = f(2) = -2$

52. To find a particular value of $f(g(x))$, say for $x = 0$, we note from the graph that $g(0) \approx 2.8$ and $f(2.8) \approx -0.5$. Thus, $f(g(0)) \approx f(2.8) \approx -0.5$. The other values listed in the table were obtained in a similar fashion.

x	$g(x)$	$f(g(x))$
-5	-0.2	-4
-4	1.2	-3.3
-3	2.2	-1.7
-2	2.8	-0.5
-1	3	-0.2

x	$g(x)$	$f(g(x))$
0	2.8	-0.5
1	2.2	-1.7
2	1.2	-3.3
3	-0.2	-4
4	-1.9	-2.2
5	-4.1	1.9



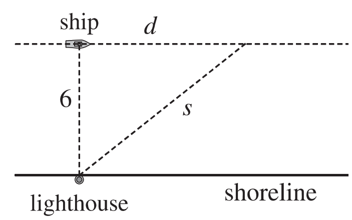
53. (a) Using the relationship *distance* $=$ *rate* $\cdot$ *time* with the radius r as the distance, we have $r(t) = 60t$.

(b) $A = \pi r^2 \;\Rightarrow\; (A \circ r)(t) = A(r(t)) = \pi(60t)^2 = 3600\pi t^2$. This formula gives us the extent of the rippled area (in cm^2) at any time t.

54. (a) The radius r of the balloon is increasing at a rate of 2 cm/s, so $r(t) = (2\text{ cm/s})(t\text{ s}) = 2t$ (in cm).

(b) Using $V = \frac{4}{3}\pi r^3$, we get $(V \circ r)(t) = V(r(t)) = V(2t) = \frac{4}{3}\pi(2t)^3 = \frac{32}{3}\pi t^3$.
The result, $V = \frac{32}{3}\pi t^3$, gives the volume of the balloon (in cm^3) as a function of time (in s).

55. (a) From the figure, we have a right triangle with legs 6 and d, and hypotenuse s.
By the Pythagorean Theorem, $d^2 + 6^2 = s^2 \;\Rightarrow\; s = f(d) = \sqrt{d^2 + 36}$.



(b) Using $d = rt$, we get $d = (30\text{ km/h})(t\text{ hours}) = 30t$ (in km). Thus, $d = g(t) = 30t$.

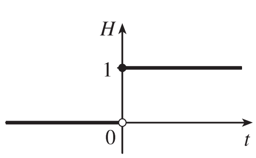
(c) $(f \circ g)(t) = f(g(t)) = f(30t) = \sqrt{(30t)^2 + 36} = \sqrt{900t^2 + 36}$. This function represents the distance between the lighthouse and the ship as a function of the time elapsed since noon.

56. (a) $d = rt \;\Rightarrow\; d(t) = 350t$

(b) There is a Pythagorean relationship involving the legs with lengths d and 1 and the hypotenuse with length s: $d^2 + 1^2 = s^2$. Thus, $s(d) = \sqrt{d^2 + 1}$.

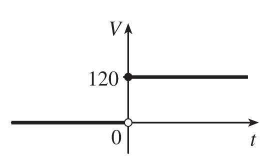
(c) $(s \circ d)(t) = s(d(t)) = s(350t) = \sqrt{(350t)^2 + 1}$

57. (a)



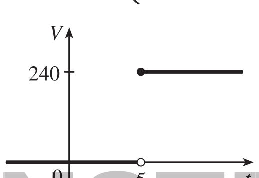
$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

(b)



$$V(t) = \begin{cases} 0 & \text{if } t < 0 \\ 120 & \text{if } t \geq 0 \end{cases} \quad \text{so } V(t) = 120H(t).$$

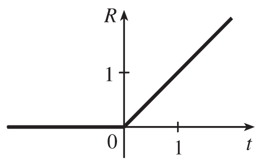
(c)



Starting with the formula in part (b), we replace 120 with 240 to reflect the different voltage. Also, because we are starting 5 units to the right of $t = 0$, we replace t with $t - 5$. Thus, the formula is $V(t) = 240H(t - 5)$.

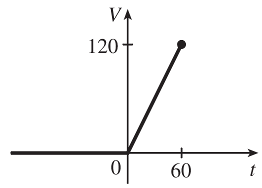
58. (a) $R(t) = tH(t)$

$$= \begin{cases} 0 & \text{if } t < 0 \\ t & \text{if } t \geq 0 \end{cases}$$



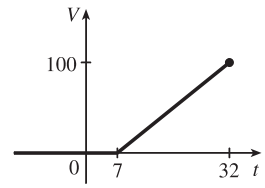
(b) $V(t) = \begin{cases} 0 & \text{if } t < 0 \\ 2t & \text{if } 0 \leq t \leq 60 \end{cases}$

so $V(t) = 2tH(t),\, t \leq 60$.



(c) $V(t) = \begin{cases} 0 & \text{if } t < 7 \\ 4\,(t-7) & \text{if } 7 \leq t \leq 32 \end{cases}$

so $V(t) = 4(t-7)H(t-7),\, t \leq 32$.



59. If $f(x) = m_1x + b_1$ and $g(x) = m_2x + b_2$, then

$(f \circ g)(x) = f(g(x)) = f(m_2x + b_2) = m_1(m_2x + b_2) + b_1 = m_1m_2x + m_1b_2 + b_1$.

So $f \circ g$ is a linear function with slope m_1m_2.

60. If $A(x) = 1.04x$, then

$(A \circ A)(x) = A(A(x)) = A(1.04x) = 1.04(1.04x) = (1.04)^2x$,

$(A \circ A \circ A)(x) = A((A \circ A)(x)) = A((1.04)^2x) = 1.04(1.04)^2x = (1.04)^3x$, and

$(A \circ A \circ A \circ A)(x) = A((A \circ A \circ A)(x)) = A((1.04)^3x) = 1.04(1.04)^3x, = (1.04)^4x$.

These compositions represent the amount of the investment after 2, 3, and 4 years.

Based on this pattern, when we compose n copies of A, we get the formula $\underbrace{(A \circ A \circ \cdots \circ A)}_{n\ A's}(x) = (1.04)^nx$.

61. (a) By examining the variable terms in g and h, we deduce that we must square g to get the terms $4x^2$ and $4x$ in h. If we let $f(x) = x^2 + c$, then $(f \circ g)(x) = f(g(x)) = f(2x+1) = (2x+1)^2 + c = 4x^2 + 4x + (1+c)$. Since $h(x) = 4x^2 + 4x + 7$, we must have $1 + c = 7$. So $c = 6$ and $f(x) = x^2 + 6$.

(b) We need a function g so that $f(g(x)) = 3(g(x)) + 5 = h(x)$. But

$h(x) = 3x^2 + 3x + 2 = 3(x^2 + x) + 2 = 3(x^2 + x - 1) + 5$, so we see that $g(x) = x^2 + x - 1$.

62. We need a function g so that $g(f(x)) = g(x+4) = h(x) = 4x - 1 = 4(x+4) - 17$. So we see that the function g must be $g(x) = 4x - 17$.

63. We need to examine $h(-x)$.

$$h(-x) = (f \circ g)(-x) = f(g(-x)) = f(g(x)) \quad \text{[because } g \text{ is even]} \quad = h(x)$$

Because $h(-x) = h(x)$, h is an even function.

64. $h(-x) = f(g(-x)) = f(-g(x))$. At this point, we can't simplify the expression, so we might try to find a counterexample to show that h is not an odd function. Let $g(x) = x$, an odd function, and $f(x) = x^2 + x$. Then $h(x) = x^2 + x$, which is neither even nor odd.

Now suppose f is an odd function. Then $f(-g(x)) = -f(g(x)) = -h(x)$. Hence, $h(-x) = -h(x)$, and so h is odd if both f and g are odd.

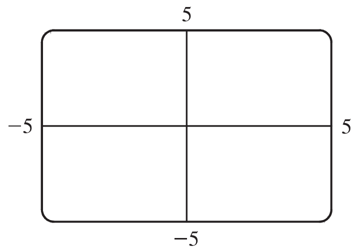
Now suppose f is an even function. Then $f(-g(x)) = f(g(x)) = h(x)$. Hence, $h(-x) = h(x)$, and so h is even if g is odd and f is even.

1.4 Graphing Calculators and Computers

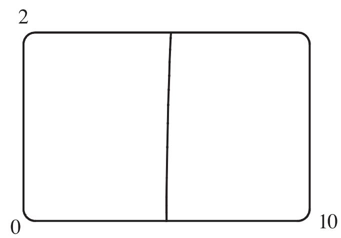
1. $f(x) = \sqrt{x^3 - 5x^2}$

(a) $[-5, 5]$ by $[-5, 5]$

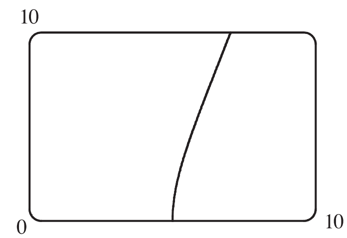
(There is no graph shown.)



(b) $[0, 10]$ by $[0, 2]$



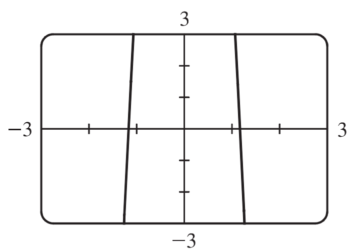
(c) $[0, 10]$ by $[0, 10]$



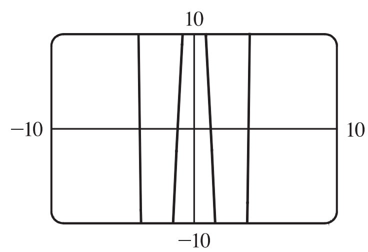
The most appropriate graph is produced in viewing rectangle (c).

2. $f(x) = x^4 - 16x^2 + 20$

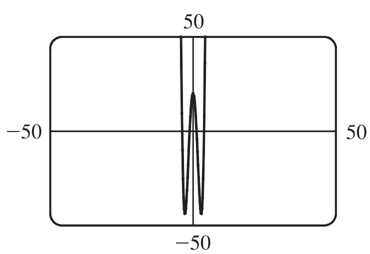
(a) $[-3, 3]$ by $[-3, 3]$



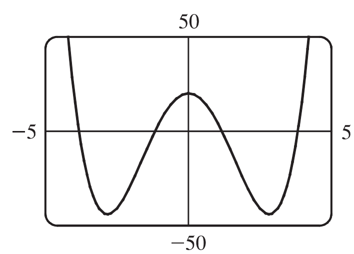
(b) $[-10, 10]$ by $[-10, 10]$



(c) $[-50, 50]$ by $[-50, 50]$

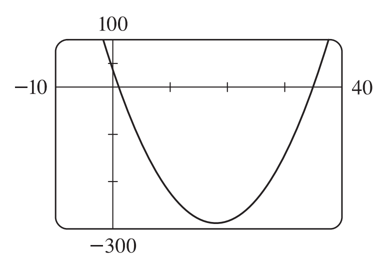


(d) $[-5, 5]$ by $[-50, 50]$

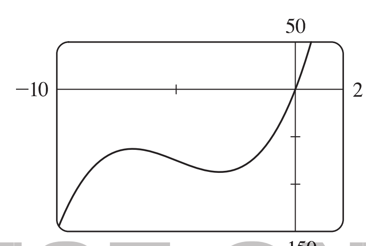


The most appropriate graph is produced in viewing rectangle (d).

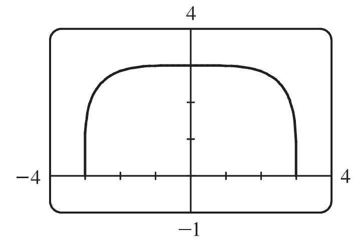
3. Since the graph of $f(x) = x^2 - 36x + 32$ is a parabola opening upward, an appropriate viewing rectangle should include the minimum point. Completing the square, we get $f(x) = (x - 18)^2 - 292$, and so the minimum point is $(18, -292)$.



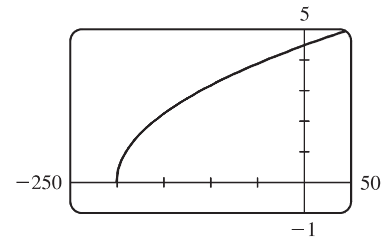
4. An appropriate viewing rectangle for $f(x) = x^3 + 15x^2 + 65x$ should include the high and low points.



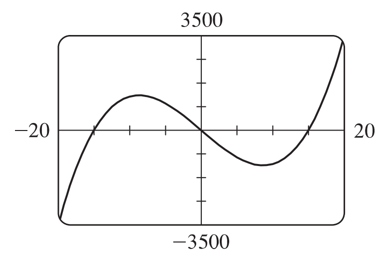
5. $f(x) = \sqrt[4]{81 - x^4}$ is defined when $81 - x^4 \geq 0 \quad \Leftrightarrow \quad x^4 \leq 81 \quad \Leftrightarrow$ $|x| \leq 3$, so the domain of f is $[-3, 3]$. Also $0 \leq \sqrt[4]{81 - x^4} \leq \sqrt[4]{81} = 3$, so the range is $[0, 3]$.



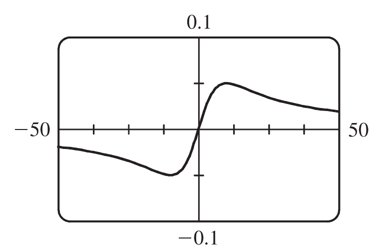
6. $f(x) = \sqrt{0.1x + 20}$ is defined when $0.1x + 20 \geq 0 \quad \Leftrightarrow \quad x \geq -200$, so the domain of f is $[-200, \infty)$.



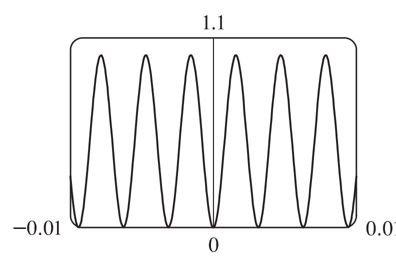
7. The graph of $f(x) = x^3 - 225x$ is symmetric with respect to the origin. Since $f(x) = x^3 - 225x = x(x^2 - 225) = x(x + 15)(x - 15)$, there are x-intercepts at 0, -15, and 15. $f(20) = 3500$.



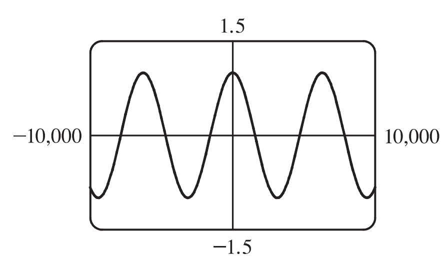
8. The graph of $f(x) = x/(x^2 + 100)$ is symmetric with respect to the origin.



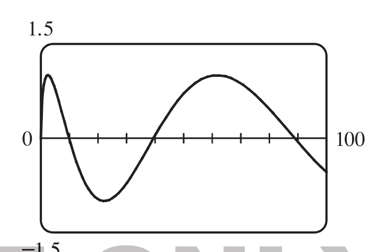
9. The period of $g(x) = \sin(1000x)$ is $\frac{2\pi}{1000} \approx 0.0063$ and its range is $[-1, 1]$. Since $f(x) = \sin^2(1000x)$ is the square of g, its range is $[0, 1]$ and a viewing rectangle of $[-0.01, 0.01]$ by $[0, 1.1]$ seems appropriate.



10. The period of $f(x) = \cos(0.001x)$ is $\frac{2\pi}{0.001} \approx 6300$ and its range is $[-1, 1]$, so a viewing rectangle of $[-10{,}000, 10{,}000]$ by $[-1.5, 1.5]$ seems appropriate.



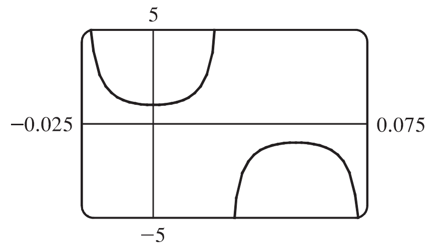
11. The domain of $y = \sqrt{x}$ is $x \geq 0$, so the domain of $f(x) = \sin\sqrt{x}$ is $[0, \infty)$ and the range is $[-1, 1]$. With a little trial-and-error experimentation, we find that an Xmax of 100 illustrates the general shape of f, so an appropriate viewing rectangle is $[0, 100]$ by $[-1.5, 1.5]$.



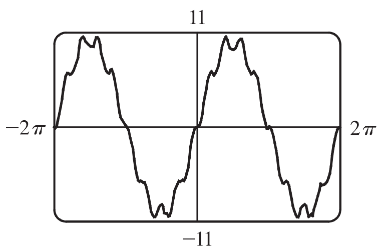
12. One period of $y = \sec x$ occurs on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

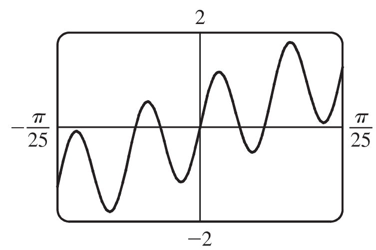
$-\frac{\pi}{2} < 20\pi x < \frac{3\pi}{2} \quad \Rightarrow \quad -\frac{1}{40} < x < \frac{3}{40}$, or equivalently,

$-0.025 < x < 0.075$.

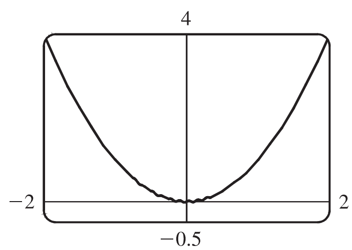


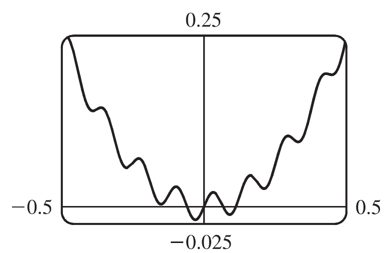
13. The first term, $10 \sin x$, has period 2π and range $[-10, 10]$. It will be the dominant term in any "large" graph of $y = 10 \sin x + \sin 100x$, as shown in the first figure. The second term, $\sin 100x$, has period $\frac{2\pi}{100} = \frac{\pi}{50}$ and range $[-1, 1]$. It causes the bumps in the first figure and will be the dominant term in any "small" graph, as shown in the view near the origin in the second figure.



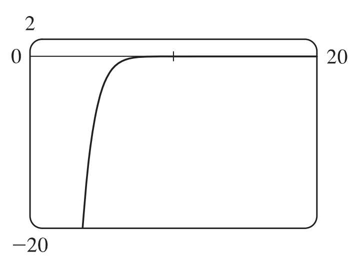


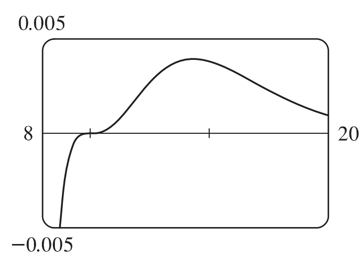
14. $y = x^2 + 0.02 \sin(50x)$





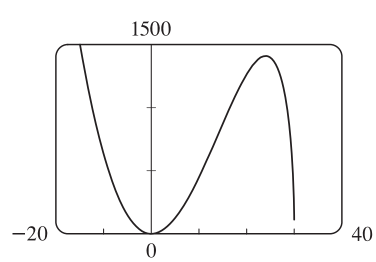
15. (a) The first figure shows the "big picture" for $f(x) = (x - 10)^3 2^{-x}$. The second figure shows a maximum near $x = 10$.





(b) You need more than one window because no single window can show what the function looks like globally and the detail of the function near $x = 10$.

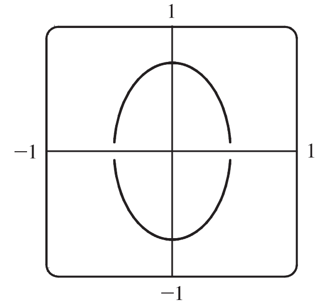
16. The function $f(x) = x^2 \sqrt{30 - x}$ has domain $(-\infty, 30]$. Its graph is very steep near $x = 30$, so part of the graph may appear to be missing.



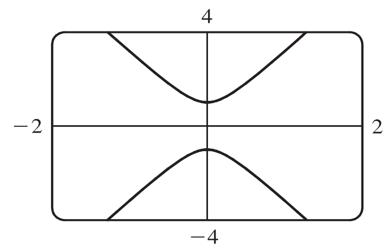
17. We must solve the given equation for y to obtain equations for the upper and lower halves of the ellipse.

$4x^2 + 2y^2 = 1 \quad \Leftrightarrow \quad 2y^2 = 1 - 4x^2 \quad \Leftrightarrow \quad y^2 = \dfrac{1 - 4x^2}{2} \quad \Leftrightarrow$

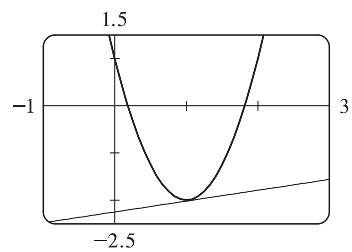
$y = \pm\sqrt{\dfrac{1 - 4x^2}{2}}$

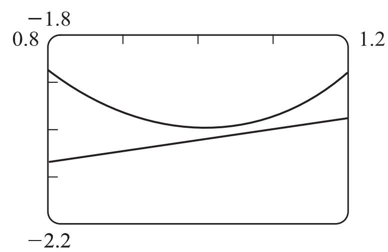


18. $y^2 - 9x^2 = 1 \quad \Leftrightarrow \quad y^2 = 1 + 9x^2 \quad \Leftrightarrow \quad y = \pm\sqrt{1 + 9x^2}$

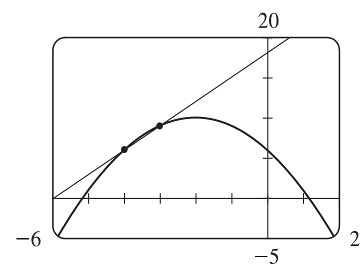


19. From the graph of $y = 3x^2 - 6x + 1$ and $y = 0.23x - 2.25$ in the viewing rectangle $[-1, 3]$ by $[-2.5, 1.5]$, it is difficult to see if the graphs intersect. If we zoom in on the fourth quadrant, we see the graphs do not intersect.

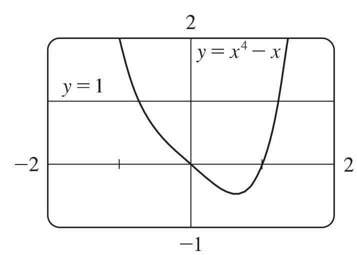




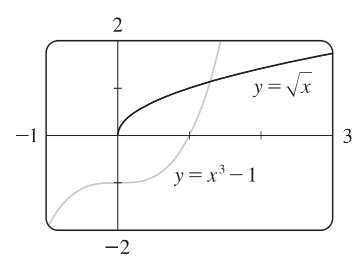
20. From the graph of $y = 6 - 4x - x^2$ and $y = 3x + 18$ in the viewing rectangle $[-6, 2]$ by $[-5, 20]$, we see that the graphs intersect twice. The points of intersection are $(-4, 6)$ and $(-3, 9)$.



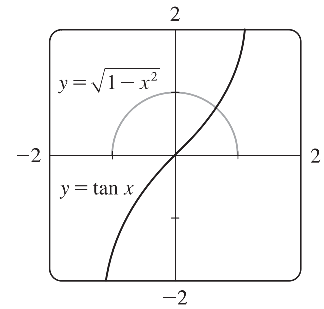
21. We see that the graphs of $f(x) = x^4 - x$ and $g(x) = 1$ intersect twice. The x-coordinates of these points (which are the solutions of the equations) are approximately -0.72 and 1.22. Alternatively, we could find these values by finding the zeros of $h(x) = x^4 - x - 1$.



22. We see that the graphs of $f(x) = \sqrt{x}$ and $g(x) = x^3 - 1$ intersect once. The x-coordinate of this point (which is the solution of the equation) is approximately 1.29. Alternatively, we could find this value by finding the zero of $h(x) = \sqrt{x} - x^3 + 1$.



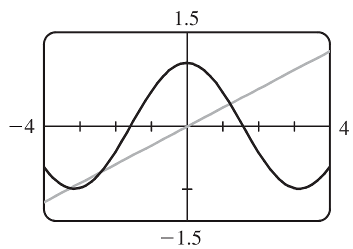
23. We see that the graphs of $f(x) = \tan x$ and $g(x) = \sqrt{1 - x^2}$ intersect once. Using an intersect feature or zooming in, we find this value to be approximately 0.65. Alternatively, we could find this value by finding the positive zero of $h(x) = \tan x - \sqrt{1 - x^2}$.



Note: After producing the graph on a TI-84 Plus, we can find the approximate value 0.65 by using the following keystrokes:

[2nd] [CALC] [5] [ENTER] [ENTER] **.6** [ENTER] . The ".6" is just a guess for 0.65.

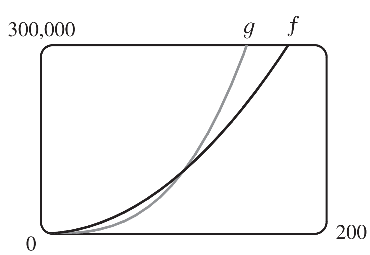
24. (a)



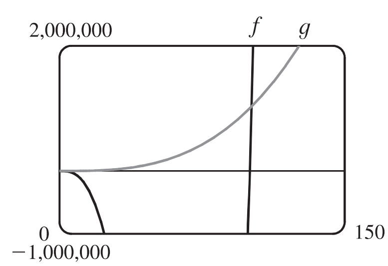
The x-coordinates of the three points of intersection are $x \approx -3.29, -2.36$ and 1.20.

(b) Using trial and error, we find that $m \approx 0.3365$. Note that m could also be negative.

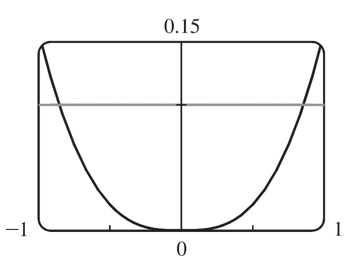
25. $g(x) = x^3/10$ is larger than $f(x) = 10x^2$ whenever $x > 100$.



26. $f(x) = x^4 - 100x^3$ is larger than $g(x) = x^3$ whenever $x > 101$.

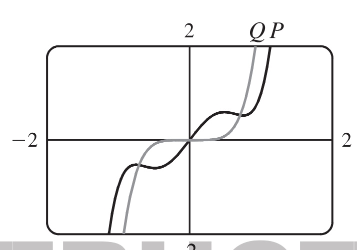


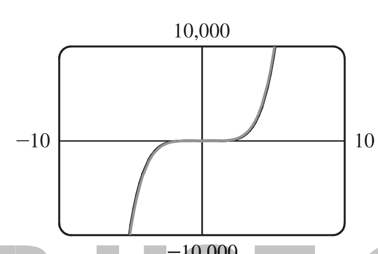
27.



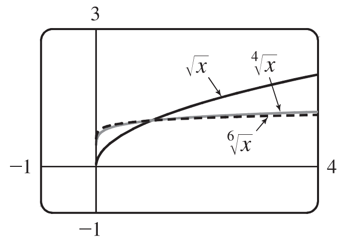
We see from the graphs of $y = |\sin x - x|$ and $y = 0.1$ that there are two solutions to the equation $|\sin x - x| = 0.1$: $x \approx -0.85$ and $x \approx 0.85$. The condition $|\sin x - x| < 0.1$ holds for any x lying between these two values, that is, $-0.85 < x < 0.85$.

28. $P(x) = 3x^5 - 5x^3 + 2x$, $Q(x) = 3x^5$. These graphs are significantly different only in the region close to the origin. The larger a viewing rectangle one chooses, the more similar the two graphs look.

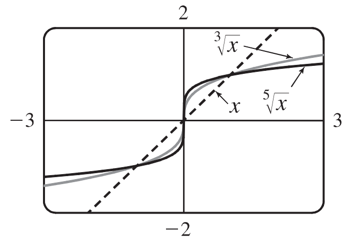




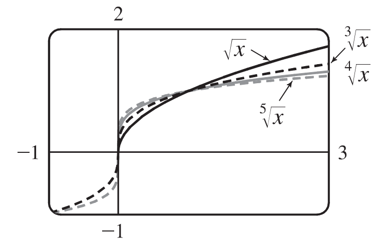
29. (a) The root functions $y = \sqrt{x}$, $y = \sqrt[4]{x}$ and $y = \sqrt[6]{x}$



(b) The root functions $y = x$, $y = \sqrt[3]{x}$ and $y = \sqrt[5]{x}$



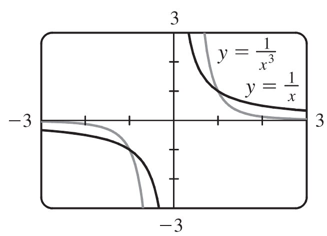
(c) The root functions $y = \sqrt{x}$, $y = \sqrt[3]{x}$, $y = \sqrt[4]{x}$ and $y = \sqrt[5]{x}$



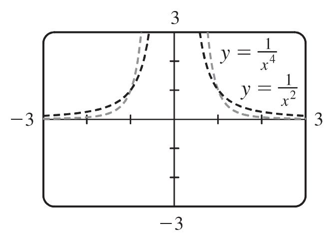
(d) • For any n, the nth root of 0 is 0 and the nth root of 1 is 1; that is, all nth root functions pass through the points $(0, 0)$ and $(1, 1)$.

- For odd n, the domain of the nth root function is $\mathbb{R}$, while for even n, it is $\{x \in \mathbb{R} \mid x \geq 0\}$.
- Graphs of even root functions look similar to that of $\sqrt{x}$, while those of odd root functions resemble that of $\sqrt[3]{x}$.
- As n increases, the graph of $\sqrt[n]{x}$ becomes steeper near 0 and flatter for $x > 1$.

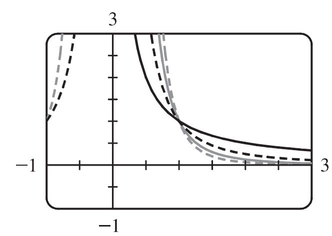
30. (a) The functions $y = 1/x$ and $y = 1/x^3$



(b) The functions $y = 1/x^2$ and $y = 1/x^4$



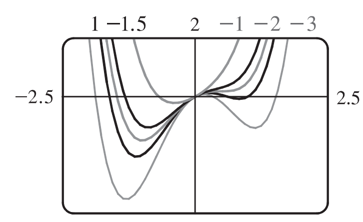
(c) The functions $y = 1/x$, $y = 1/x^2$, $y = 1/x^3$ and $y = 1/x^4$



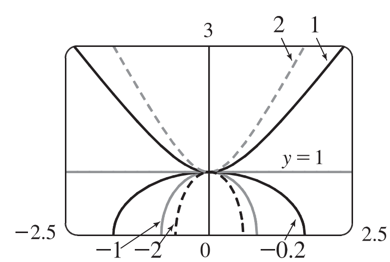
(d) • The graphs of all functions of the form $y = 1/x^n$ pass through the point $(1, 1)$.

- If n is even, the graph of the function is entirely above the x-axis. The graphs of $1/x^n$ for n even are similar to one another.
- If n is odd, the function is positive for positive x and negative for negative x. The graphs of $1/x^n$ for n odd are similar to one another.
- As n increases, the graphs of $1/x^n$ approach 0 faster as $x \to \infty$.

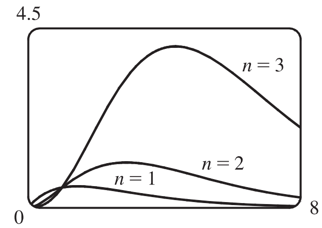
31. $f(x) = x^4 + cx^2 + x$. If $c < -1.5$, there are three humps: two minimum points and a maximum point. These humps get flatter as c increases, until at $c = -1.5$ two of the humps disappear and there is only one minimum point. This single hump then moves to the right and approaches the origin as c increases.

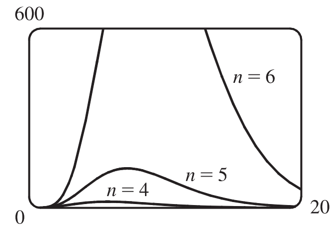


32. $f(x) = \sqrt{1 + cx^2}$. If $c < 0$, the function is only defined on $\left[-1/\sqrt{-c}\,, 1/\sqrt{-c}\,\right]$, and its graph is the top half of an ellipse. If $c = 0$, the graph is the line $y = 1$. If $c > 0$, the graph is the top half of a hyperbola. As c approaches 0, these curves become flatter and approach the line $y = 1$.

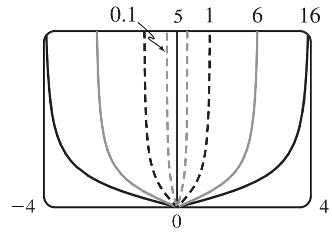


33. $y = x^n 2^{-x}$. As n increases, the maximum of the function moves further from the origin, and gets larger. Note, however, that regardless of n, the function approaches 0 as $x \to \infty$.

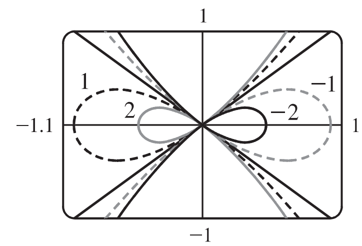




34. $y = \dfrac{|x|}{\sqrt{c - x^2}}$. The "bullet" becomes broader as c increases.

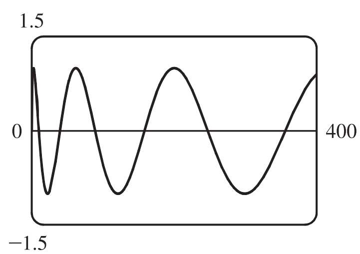


35. $y^2 = cx^3 + x^2$. If $c < 0$, the loop is to the right of the origin, and if c is positive, it is to the left. In both cases, the closer c is to 0, the larger the loop is. (In the limiting case, $c = 0$, the loop is "infinite," that is, it doesn't close.) Also, the larger $|c|$ is, the steeper the slope is on the loopless side of the origin.



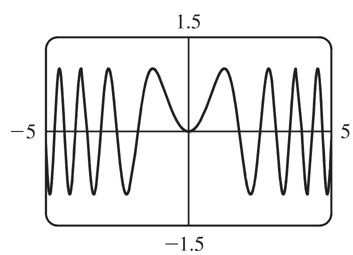
36. (a) $y = \sin(\sqrt{x})$

This function is not periodic; it oscillates less frequently as x increases.



(b) $y = \sin(x^2)$

This function oscillates more frequently as $|x|$ increases. Note also that this function is even, whereas $\sin x$ is odd.



37. The graphing window is 95 pixels wide and we want to start with $x = 0$ and end with $x = 2\pi$. Since there are 94 "gaps" between pixels, the distance between pixels is $\frac{2\pi - 0}{94}$. Thus, the x-values that the calculator actually plots are $x = 0 + \frac{2\pi}{94} \cdot n$, where $n = 0, 1, 2, \ldots, 93, 94$. For $y = \sin 2x$, the actual points plotted by the calculator are $\left(\frac{2\pi}{94} \cdot n, \sin\left(2 \cdot \frac{2\pi}{94} \cdot n\right)\right)$ for $n = 0, 1, \ldots, 94$. For $y = \sin 96x$, the points plotted are $\left(\frac{2\pi}{94} \cdot n, \sin\left(96 \cdot \frac{2\pi}{94} \cdot n\right)\right)$ for $n = 0, 1, \ldots, 94$. But

$$\begin{aligned} \sin\left(96 \cdot \tfrac{2\pi}{94} \cdot n\right) &= \sin\left(94 \cdot \tfrac{2\pi}{94} \cdot n + 2 \cdot \tfrac{2\pi}{94} \cdot n\right) = \sin\left(2\pi n + 2 \cdot \tfrac{2\pi}{94} \cdot n\right) \\ &= \sin\left(2 \cdot \tfrac{2\pi}{94} \cdot n\right) \quad \text{[by periodicity of sine]}, \quad n = 0, 1, \ldots, 94 \end{aligned}$$

So the y-values, and hence the points, plotted for $y = \sin 96x$ are identical to those plotted for $y = \sin 2x$.

Note: Try graphing $y = \sin 94x$. Can you see why all the y-values are zero?

38. As in Exercise 37, we know that the points being plotted for $y = \sin 45x$ are $\left(\frac{2\pi}{94} \cdot n, \sin\left(45 \cdot \frac{2\pi}{94} \cdot n\right)\right)$ for $n = 0, 1, \ldots, 94$. But

$$\begin{aligned}\sin\left(45 \cdot \tfrac{2\pi}{94} \cdot n\right) &= \sin\left(47 \cdot \tfrac{2\pi}{94} \cdot n - 2 \cdot \tfrac{2\pi}{94} \cdot n\right) = \sin\left(n\pi - 2 \cdot \tfrac{2\pi}{94} \cdot n\right) \\ &= \sin(n\pi)\cos\left(2 \cdot \tfrac{2\pi}{94} \cdot n\right) - \cos(n\pi)\sin\left(2 \cdot \tfrac{2\pi}{94} \cdot n\right) \quad \text{[Subtraction formula for the sine]} \\ &= 0 \cdot \cos\left(2 \cdot \tfrac{2\pi}{94} \cdot n\right) - (\pm 1)\sin\left(2 \cdot \tfrac{2\pi}{94} \cdot n\right) \\ &= \pm\sin\left(2 \cdot \tfrac{2\pi}{94} \cdot n\right), \quad n = 0, 1, \ldots, 94\end{aligned}$$

So the y-values, and hence the points, plotted for $y = \sin 45x$ lie on either $y = \sin 2x$ or $y = -\sin 2x$.

1.5 Exponential Functions

1. (a) $\dfrac{4^{-3}}{2^{-8}} = \dfrac{2^8}{4^3} = \dfrac{2^8}{(2^2)^3} = \dfrac{2^8}{2^6} = 2^{8-6} = 2^2 = 4$ (b) $\dfrac{1}{\sqrt[3]{x^4}} = \dfrac{1}{x^{4/3}} = x^{-4/3}$

2. (a) $8^{4/3} = (8^{1/3})^4 = 2^4 = 16$ (b) $x(3x^2)^3 = x \cdot 3^3(x^2)^3 = 27x \cdot x^6 = 27x^7$

3. (a) $b^8(2b)^4 = b^8 \cdot 2^4 b^4 = 16b^{12}$ (b) $\dfrac{(6y^3)^4}{2y^5} = \dfrac{6^4(y^3)^4}{2y^5} = \dfrac{1296y^{12}}{2y^5} = 648y^7$

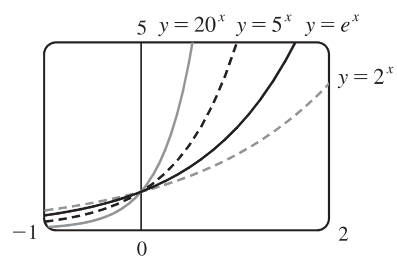
4. (a) $\dfrac{x^{2n} \cdot x^{3n-1}}{x^{n+2}} = \dfrac{x^{2n+3n-1}}{x^{n+2}} = \dfrac{x^{5n-1}}{x^{n+2}} = x^{4n-3}$

(b) $\dfrac{\sqrt{a\sqrt{b}}}{\sqrt[3]{ab}} = \dfrac{\sqrt{a}\sqrt{\sqrt{b}}}{\sqrt[3]{a}\sqrt[3]{b}} = \dfrac{a^{1/2}b^{1/4}}{a^{1/3}b^{1/3}} = a^{(1/2-1/3)}b^{(1/4-1/3)} = a^{1/6}b^{-1/12}$

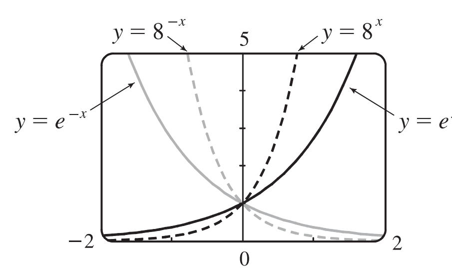
5. (a) $f(x) = a^x,\ a > 0$ (b) $\mathbb{R}$ (c) $(0, \infty)$ (d) See Figures 4(c), 4(b), and 4(a), respectively.

6. (a) The number e is the value of a such that the slope of the tangent line at $x = 0$ on the graph of $y = a^x$ is exactly 1.
(b) $e \approx 2.71828$ (c) $f(x) = e^x$

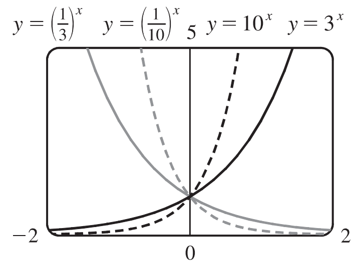
7. All of these graphs approach 0 as $x \to -\infty$, all of them pass through the point $(0, 1)$, and all of them are increasing and approach ∞ as $x \to \infty$. The larger the base, the faster the function increases for $x > 0$, and the faster it approaches 0 as $x \to -\infty$.



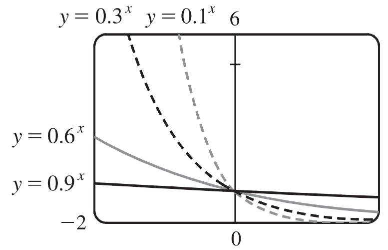
8. The graph of e^{-x} is the reflection of the graph of e^x about the y-axis, and the graph of 8^{-x} is the reflection of that of 8^x about the y-axis. The graph of 8^x increases more quickly than that of e^x for $x > 0$, and approaches 0 faster as $x \to -\infty$.



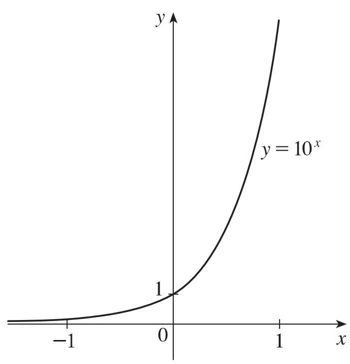
9. The functions with bases greater than 1 (3^x and 10^x) are increasing, while those with bases less than 1 $\left[\left(\frac{1}{3}\right)^x \text{ and } \left(\frac{1}{10}\right)^x\right]$ are decreasing. The graph of $\left(\frac{1}{3}\right)^x$ is the reflection of that of 3^x about the y-axis, and the graph of $\left(\frac{1}{10}\right)^x$ is the reflection of that of 10^x about the y-axis. The graph of 10^x increases more quickly than that of 3^x for $x > 0$, and approaches 0 faster as $x \to -\infty$.

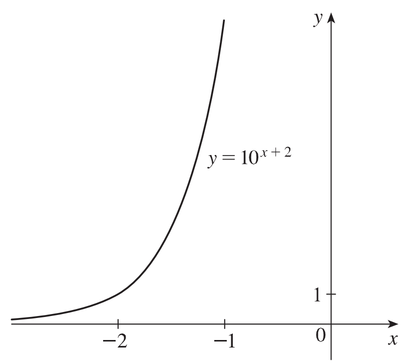


10. Each of the graphs approaches ∞ as $x \to -\infty$, and each approaches 0 as $x \to \infty$. The smaller the base, the faster the function grows as $x \to -\infty$, and the faster it approaches 0 as $x \to \infty$.

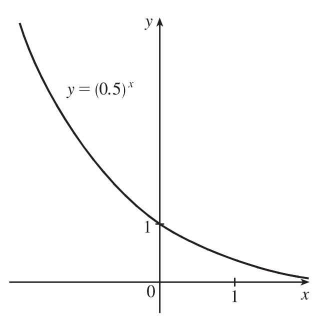


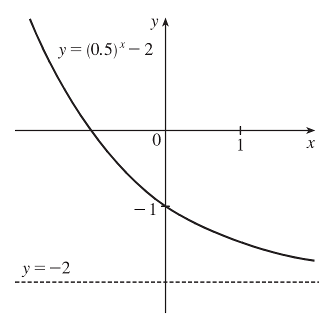
11. We start with the graph of $y = 10^x$ (Figure 3) and shift it 2 units to the left to obtain the graph of $y = 10^{x+2}$.



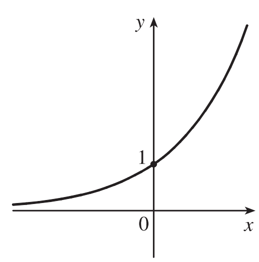


12. We start with the graph of $y = (0.5)^x$ (Figure 3) and shift it 2 units downward to obtain the graph of $y = (0.5)^x - 2$. The horizontal asymptote of the final graph is $y = -2$.

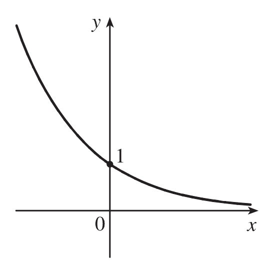




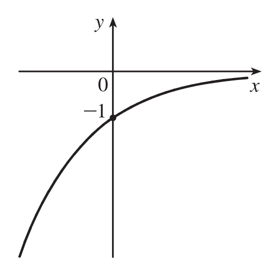
13. We start with the graph of $y = 2^x$ (Figure 3), reflect it about the y-axis, and then about the x-axis (or just rotate 180° to handle both reflections) to obtain the graph of $y = -2^{-x}$. In each graph, $y = 0$ is the horizontal asymptote.



$y = 2^x$

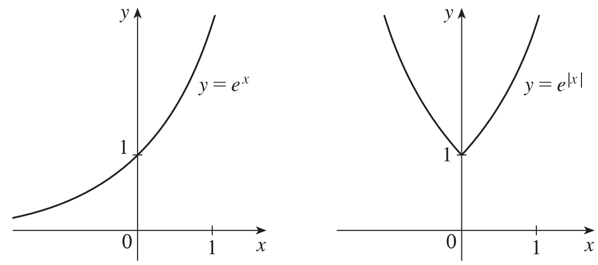


$y = 2^{-x}$

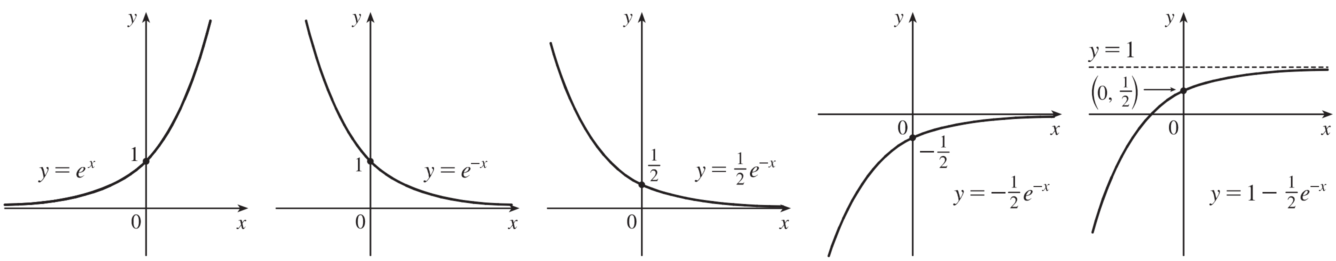


$y = -2^{-x}$

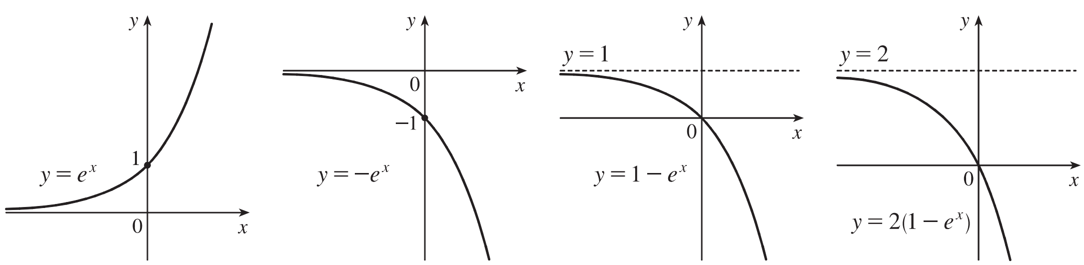
14. We start with the graph of $y = e^x$ (Figure 13) and reflect the portion of the graph in the first quadrant about the y-axis to obtain the graph of $y = e^{|x|}$.



15. We start with the graph of $y = e^x$ (Figure 13) and reflect about the y-axis to get the graph of $y = e^{-x}$. Then we compress the graph vertically by a factor of 2 to obtain the graph of $y = \frac{1}{2}e^{-x}$ and then reflect about the x-axis to get the graph of $y = -\frac{1}{2}e^{-x}$. Finally, we shift the graph upward one unit to get the graph of $y = 1 - \frac{1}{2}e^{-x}$.



16. We start with the graph of $y = e^x$ (Figure 13) and reflect about the x-axis to get the graph of $y = -e^x$. Then shift the graph upward one unit to get the graph of $y = 1 - e^x$. Finally, we stretch the graph vertically by a factor of 2 to obtain the graph of $y = 2(1 - e^x)$.



17. (a) To find the equation of the graph that results from shifting the graph of $y = e^x$ 2 units downward, we subtract 2 from the original function to get $y = e^x - 2$.

(b) To find the equation of the graph that results from shifting the graph of $y = e^x$ 2 units to the right, we replace x with $x - 2$ in the original function to get $y = e^{(x-2)}$.

(c) To find the equation of the graph that results from reflecting the graph of $y = e^x$ about the x-axis, we multiply the original function by -1 to get $y = -e^x$.

(d) To find the equation of the graph that results from reflecting the graph of $y = e^x$ about the y-axis, we replace x with $-x$ in the original function to get $y = e^{-x}$.

(e) To find the equation of the graph that results from reflecting the graph of $y = e^x$ about the x-axis and then about the y-axis, we first multiply the original function by -1 (to get $y = -e^x$) and then replace x with $-x$ in this equation to get $y = -e^{-x}$.

18. (a) This reflection consists of first reflecting the graph about the x-axis (giving the graph with equation $y = -e^x$) and then shifting this graph $2 \cdot 4 = 8$ units upward. So the equation is $y = -e^x + 8$.

(b) This reflection consists of first reflecting the graph about the y-axis (giving the graph with equation $y = e^{-x}$) and then shifting this graph $2 \cdot 2 = 4$ units to the right. So the equation is $y = e^{-(x-4)}$.

19. (a) The denominator is zero when $1 - e^{1-x^2} = 0 \quad \Leftrightarrow \quad e^{1-x^2} = 1 \quad \Leftrightarrow \quad 1 - x^2 = 0 \quad \Leftrightarrow \quad x = \pm 1$. Thus, the function $f(x) = \dfrac{1 - e^{x^2}}{1 - e^{1-x^2}}$ has domain $\{x \mid x \neq \pm 1\} = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

(b) The denominator is never equal to zero, so the function $f(x) = \dfrac{1 + x}{e^{\cos x}}$ has domain $\mathbb{R}$, or $(-\infty, \infty)$.

20. (a) The sine and exponential functions have domain $\mathbb{R}$, so $g(t) = \sin(e^{-t})$ also has domain $\mathbb{R}$.

(b) The function $g(t) = \sqrt{1 - 2^t}$ has domain $\{t \mid 1 - 2^t \geq 0\} = \{t \mid 2^t \leq 1\} = \{t \mid t \leq 0\} = (-\infty, 0]$.

21. Use $y = Ca^x$ with the points $(1, 6)$ and $(3, 24)$. $6 = Ca^1 \quad \left[C = \frac{6}{a}\right]$ and $24 = Ca^3 \quad \Rightarrow \quad 24 = \left(\dfrac{6}{a}\right)a^3 \quad \Rightarrow$ $4 = a^2 \quad \Rightarrow \quad a = 2 \quad [\text{since } a > 0]$ and $C = \frac{6}{2} = 3$. The function is $f(x) = 3 \cdot 2^x$.

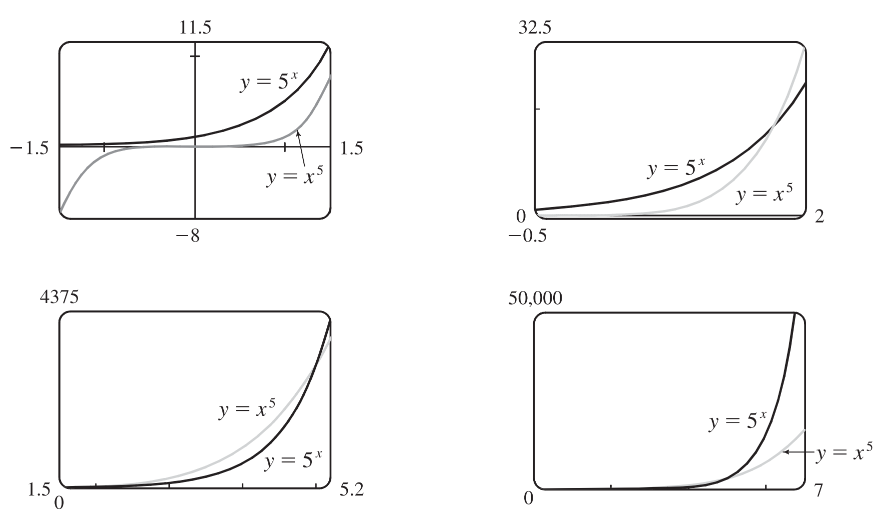
22. Use $y = Ca^x$ with the points $(-1, 3)$ and $\left(1, \frac{4}{3}\right)$. From the point $(-1, 3)$, we have $3 = Ca^{-1}$, hence $C = 3a$. Using this and the point $\left(1, \frac{4}{3}\right)$, we get $\frac{4}{3} = Ca^1 \quad \Rightarrow \quad \frac{4}{3} = (3a)a \quad \Rightarrow \quad \frac{4}{9} = a^2 \quad \Rightarrow \quad a = \frac{2}{3} \quad [\text{since } a > 0]$ and $C = 3(\frac{2}{3}) = 2$. The function is $f(x) = 2(\frac{2}{3})^x$.

23. If $f(x) = 5^x$, then $\dfrac{f(x+h) - f(x)}{h} = \dfrac{5^{x+h} - 5^x}{h} = \dfrac{5^x 5^h - 5^x}{h} = \dfrac{5^x\left(5^h - 1\right)}{h} = 5^x\left(\dfrac{5^h - 1}{h}\right)$.

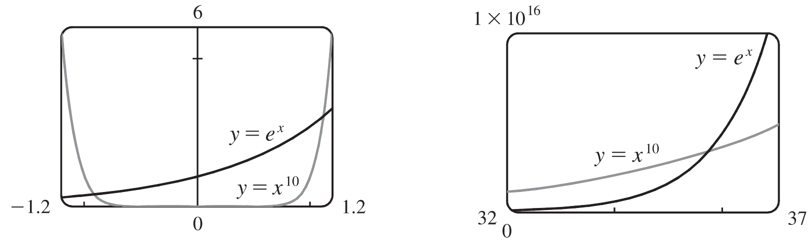
24. Suppose the month is February. Your payment on the 28th day would be $2^{28-1} = 2^{27} = 134{,}217{,}728$ cents, or \$1,342,177.28. Clearly, the second method of payment results in a larger amount for any month.

25. 2 ft $= 24$ in, $f(24) = 24^2$ in $= 576$ in $= 48$ ft. $g(24) = 2^{24}$ in $= 2^{24}/(12 \cdot 5280)$ mi ≈ 265 mi

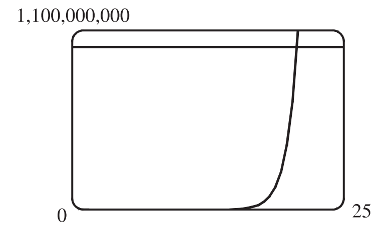
26. We see from the graphs that for x less than about 1.8, $g(x) = 5^x > f(x) = x^5$, and then near the point $(1.8, 17.1)$ the curves intersect. Then $f(x) > g(x)$ from $x \approx 1.8$ until $x = 5$. At $(5, 3125)$ there is another point of intersection, and for $x > 5$ we see that $g(x) > f(x)$. In fact, g increases much more rapidly than f beyond that point.



27. The graph of g finally surpasses that of f at $x \approx 35.8$.



28. We graph $y = e^x$ and $y = 1{,}000{,}000{,}000$ and determine where $e^x = 1 \times 10^9$. This seems to be true at $x \approx 20.723$, so $e^x > 1 \times 10^9$ for $x > 20.723$.

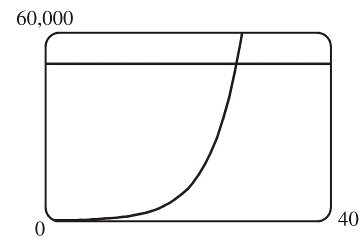


29. (a) Fifteen hours represents 5 doubling periods (one doubling period is three hours). $100 \cdot 2^5 = 3200$

(b) In t hours, there will be $t/3$ doubling periods. The initial population is 100, so the population y at time t is $y = 100 \cdot 2^{t/3}$.

(c) $t = 20 \quad \Rightarrow \quad y = 100 \cdot 2^{20/3} \approx 10{,}159$

(d) We graph $y_1 = 100 \cdot 2^{x/3}$ and $y_2 = 50{,}000$. The two curves intersect at $x \approx 26.9$, so the population reaches 50,000 in about 26.9 hours.

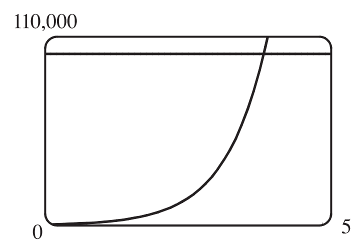


30. (a) Three hours represents 6 doubling periods (one doubling period is 30 minutes). $500 \cdot 2^6 = 32{,}000$

(b) In t hours, there will be $2t$ doubling periods. The initial population is 500, so the population y at time t is $y = 500 \cdot 2^{2t}$.

(c) $t = \frac{40}{60} = \frac{2}{3} \quad \Rightarrow \quad y = 500 \cdot 2^{2(2/3)} \approx 1260$

(d) We graph $y_1 = 500 \cdot 2^{2t}$ and $y_2 = 100{,}000$. The two curves intersect at $t \approx 3.82$, so the population reaches 100,000 in about 3.82 hours.

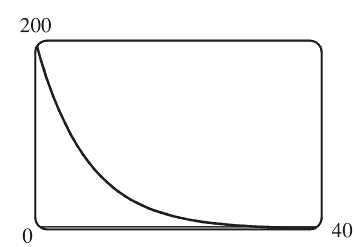


31. (a) Fifteen days represents 3 half-life periods (one half-life period is 5 days). $200\left(\frac{1}{2}\right)^3 = 25$ mg

(b) In t hours, there will be $t/5$ half-life periods. The initial amount is 200 mg, so the amount remaining after t days is $y = 200\left(\frac{1}{2}\right)^{t/5}$, or equivalently, $y = 200 \cdot 2^{-t/5}$.

(c) $t = 3$ weeks $= 21$ days $\quad \Rightarrow \quad y = 200 \cdot 2^{-21/5} \approx 10.9$ mg

(d) We graph $y_1 = 200 \cdot 2^{-t/5}$ and $y_2 = 1$. The two curves intersect at $t \approx 38.2$, so the mass will be reduced to 1 mg in about 38.2 days.

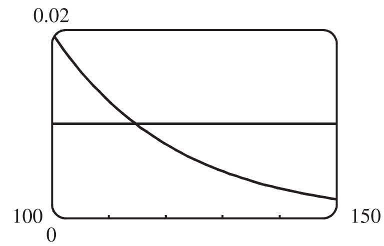


32. (a) Sixty hours represents 4 half-life periods. $2 \cdot \left(\frac{1}{2}\right)^4 = \frac{1}{8}$ g

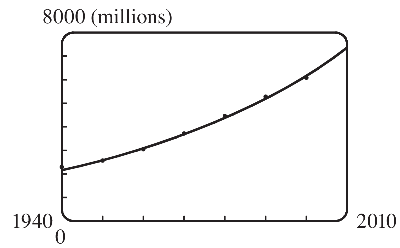
(b) In t hours, there will be $t/15$ half-life periods. The initial mass is 2 g, so the mass y at time t is $y = 2 \cdot \left(\frac{1}{2}\right)^{t/15}$.

(c) 4 days $= 4 \cdot 24 = 96$ hours. $t = 96 \quad \Rightarrow \quad y = 2 \cdot \left(\frac{1}{2}\right)^{96/15} \approx 0.024$ g

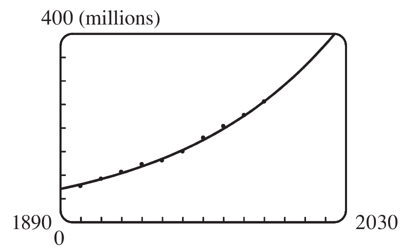
(d) $y = 0.01 \quad \Rightarrow \quad t \approx 114.7$ hours



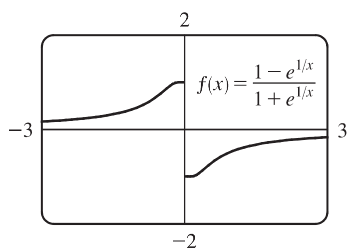
33. An exponential model is $y = ab^t$, where $a = 3.154832569 \times 10^{-12}$ and $b = 1.017764706$. This model gives $y(1993) \approx 5498$ million and $y(2010) \approx 7417$ million.



34. An exponential model is $y = ab^t$, where $a = 1.9976760197589 \times 10^{-9}$ and $b = 1.0129334321697$. This model gives $y(1925) \approx 111$ million, $y(2010) \approx 330$ million, and $y(2020) \approx 375$ million.



35.



From the graph, it appears that f is an odd function (f is undefined for $x = 0$). To prove this, we must show that $f(-x) = -f(x)$.

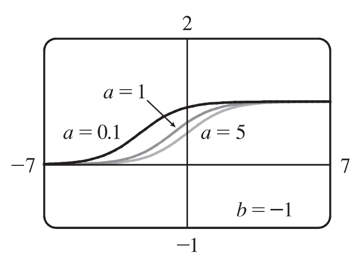
$$f(-x) = \frac{1 - e^{1/(-x)}}{1 + e^{1/(-x)}} = \frac{1 - e^{(-1/x)}}{1 + e^{(-1/x)}} = \frac{1 - \dfrac{1}{e^{1/x}}}{1 + \dfrac{1}{e^{1/x}}} \cdot \frac{e^{1/x}}{e^{1/x}} = \frac{e^{1/x} - 1}{e^{1/x} + 1}$$

$$= -\frac{1 - e^{1/x}}{1 + e^{1/x}} = -f(x)$$

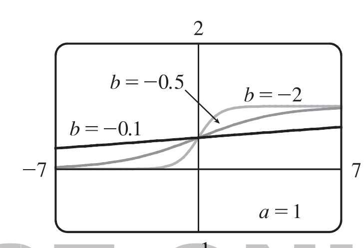
so f is an odd function.

36. We'll start with $b = -1$ and graph $f(x) = \dfrac{1}{1 + ae^{bx}}$ for $a = 0.1$, 1, and 5.

From the graph, we see that there is a horizontal asymptote $y = 0$ as $x \to -\infty$ and a horizontal asymptote $y = 1$ as $x \to \infty$. If $a = 1$, the y-intercept is $\left(0, \frac{1}{2}\right)$. As a gets smaller (close to 0), the graph of f moves left. As a gets larger, the graph of f moves right.

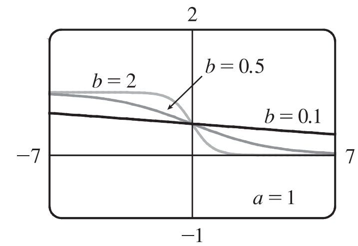


As b changes from -1 to 0, the graph of f is stretched horizontally. As b changes through large negative values, the graph of f is compressed horizontally. (This takes care of negatives values of b.)



If b is positive, the graph of f is reflected through the y-axis.

Last, if $b = 0$, the graph of f is the horizontal line $y = 1/(1+a)$.



1.6 Inverse Functions and Logarithms

1. (a) See Definition 1.

(b) It must pass the Horizontal Line Test.

2. (a) $f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y$ for any y in B. The domain of f^{-1} is B and the range of f^{-1} is A.

(b) See the steps in (5).

(c) Reflect the graph of f about the line $y = x$.

3. f is not one-to-one because $2 \neq 6$, but $f(2) = 2.0 = f(6)$.

4. f is one-to-one because it never takes on the same value twice.

5. We could draw a horizontal line that intersects the graph in more than one point. Thus, by the Horizontal Line Test, the function is not one-to-one.

6. No horizontal line intersects the graph more than once. Thus, by the Horizontal Line Test, the function is one-to-one.

7. No horizontal line intersects the graph more than once. Thus, by the Horizontal Line Test, the function is one-to-one.

8. We could draw a horizontal line that intersects the graph in more than one point. Thus, by the Horizontal Line Test, the function is not one-to-one.

9. The graph of $f(x) = x^2 - 2x$ is a parabola with axis of symmetry $x = -\dfrac{b}{2a} = -\dfrac{-2}{2(1)} = 1$. Pick any x-values equidistant from 1 to find two equal function values. For example, $f(0) = 0$ and $f(2) = 0$, so f is not one-to-one.

10. The graph of $f(x) = 10 - 3x$ is a line with slope -3. It passes the Horizontal Line Test, so f is one-to-one.

Algebraic solution: If $x_1 \neq x_2$, then $-3x_1 \neq -3x_2 \quad \Rightarrow \quad 10 - 3x_1 \neq 10 - 3x_2 \quad \Rightarrow \quad f(x_1) \neq f(x_2)$, so f is one-to-one.

11. $g(x) = 1/x. \quad x_1 \neq x_2 \quad \Rightarrow \quad 1/x_1 \neq 1/x_2 \quad \Rightarrow \quad g(x_1) \neq g(x_2)$, so g is one-to-one.

Geometric solution: The graph of g is the hyperbola shown in Figure 14 in Section 1.2. It passes the Horizontal Line Test, so g is one-to-one.

12. $g(x) = \cos x. \quad g(0) = 1 = g(2\pi)$, so g is not one-to-one.

13. A football will attain every height h up to its maximum height twice: once on the way up, and again on the way down. Thus, even if t_1 does not equal t_2, $f(t_1)$ may equal $f(t_2)$, so f is not 1-1.

14. f is not 1-1 because eventually we all stop growing and therefore, there are two times at which we have the same height.

15. Since $f(2) = 9$ and f is 1-1, we know that $f^{-1}(9) = 2$. Remember, if the point $(2, 9)$ is on the graph of f, then the point $(9, 2)$ is on the graph of f^{-1}.

16. First, we must determine x such that $f(x) = 3$. By inspection, we see that if $x = 1$, then $f(1) = 3$. Since f is 1-1 (f is an increasing function), it has an inverse, and $f^{-1}(3) = 1$. If f is a 1-1 function, then $f(f^{-1}(a)) = a$, so $f(f^{-1}(2)) = 2$.

17. First, we must determine x such that $g(x) = 4$. By inspection, we see that if $x = 0$, then $g(x) = 4$. Since g is 1-1 (g is an increasing function), it has an inverse, and $g^{-1}(4) = 0$.

18. (a) f is 1-1 because it passes the Horizontal Line Test.

(b) Domain of $f = [-3, 3] =$ Range of f^{-1}. Range of $f = [-1, 3] =$ Domain of f^{-1}.

(c) Since $f(0) = 2$, $f^{-1}(2) = 0$.

(d) Since $f(-1.7) \approx 0$, $f^{-1}(0) = -1.7$.

19. We solve $C = \frac{5}{9}(F - 32)$ for F: $\frac{9}{5}C = F - 32 \quad \Rightarrow \quad F = \frac{9}{5}C + 32$. This gives us a formula for the inverse function, that is, the Fahrenheit temperature F as a function of the Celsius temperature C. $F \geq -459.67 \quad \Rightarrow \quad \frac{9}{5}C + 32 \geq -459.67 \quad \Rightarrow$ $\frac{9}{5}C \geq -491.67 \quad \Rightarrow \quad C \geq -273.15$, the domain of the inverse function.

20. $m = \dfrac{m_0}{\sqrt{1 - v^2/c^2}} \quad \Rightarrow \quad 1 - \dfrac{v^2}{c^2} = \dfrac{m_0^2}{m^2} \quad \Rightarrow \quad \dfrac{v^2}{c^2} = 1 - \dfrac{m_0^2}{m^2} \quad \Rightarrow \quad v^2 = c^2\left(1 - \dfrac{m_0^2}{m^2}\right) \quad \Rightarrow \quad v = c\sqrt{1 - \dfrac{m_0^2}{m^2}}$.

This formula gives us the speed v of the particle in terms of its mass m, that is, $v = f^{-1}(m)$.

21. $y = f(x) = 1 + \sqrt{2 + 3x} \quad (y \geq 1) \quad \Rightarrow \quad y - 1 = \sqrt{2 + 3x} \quad \Rightarrow \quad (y - 1)^2 = 2 + 3x \quad \Rightarrow \quad (y - 1)^2 - 2 = 3x \quad \Rightarrow$ $x = \frac{1}{3}(y - 1)^2 - \frac{2}{3}$. Interchange x and y: $y = \frac{1}{3}(x - 1)^2 - \frac{2}{3}$. So $f^{-1}(x) = \frac{1}{3}(x - 1)^2 - \frac{2}{3}$. Note that the domain of f^{-1} is $x \geq 1$.

22. $y = f(x) = \dfrac{4x - 1}{2x + 3} \quad \Rightarrow \quad y(2x + 3) = 4x - 1 \quad \Rightarrow \quad 2xy + 3y = 4x - 1 \quad \Rightarrow \quad 3y + 1 = 4x - 2xy \quad \Rightarrow$

$3y + 1 = (4 - 2y)x \quad \Rightarrow \quad x = \dfrac{3y + 1}{4 - 2y}$. Interchange x and y: $y = \dfrac{3x + 1}{4 - 2x}$. So $f^{-1}(x) = \dfrac{3x + 1}{4 - 2x}$.

23. $y = f(x) = e^{2x - 1} \quad \Rightarrow \quad \ln y = 2x - 1 \quad \Rightarrow \quad 1 + \ln y = 2x \quad \Rightarrow \quad x = \frac{1}{2}(1 + \ln y)$.

Interchange x and y: $y = \frac{1}{2}(1 + \ln x)$. So $f^{-1}(x) = \frac{1}{2}(1 + \ln x)$.

24. $y = f(x) = x^2 - x \quad (x \geq \frac{1}{2}) \quad \Rightarrow \quad y = x^2 - x + \frac{1}{4} - \frac{1}{4} \quad \Rightarrow \quad y = (x - \frac{1}{2})^2 - \frac{1}{4} \quad \Rightarrow$

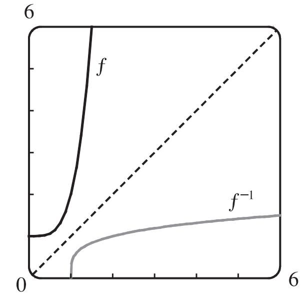
$y + \frac{1}{4} = (x - \frac{1}{2})^2 \quad \Rightarrow \quad x - \frac{1}{2} = \sqrt{y + \frac{1}{4}} \quad \Rightarrow \quad x = \frac{1}{2} + \sqrt{y + \frac{1}{4}}$. Interchange x and y: $y = \frac{1}{2} + \sqrt{x + \frac{1}{4}}$. So $f^{-1}(x) = \frac{1}{2} + \sqrt{x + \frac{1}{4}}$.

25. $y = f(x) = \ln(x + 3) \quad \Rightarrow \quad x + 3 = e^y \quad \Rightarrow \quad x = e^y - 3$. Interchange x and y: $y = e^x - 3$. So $f^{-1}(x) = e^x - 3$.

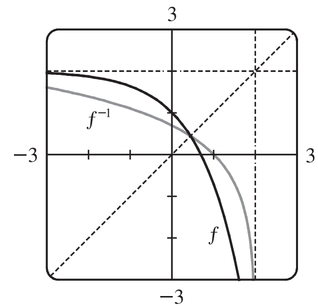
26. $y = f(x) = \dfrac{e^x}{1 + 2e^x} \quad \Rightarrow \quad y + 2ye^x = e^x \quad \Rightarrow \quad y = e^x - 2ye^x \quad \Rightarrow \quad y = e^x(1 - 2y) \quad \Rightarrow \quad e^x = \dfrac{y}{1 - 2y} \quad \Rightarrow$

$x = \ln\left(\dfrac{y}{1 - 2y}\right)$. Interchange x and y: $y = \ln\left(\dfrac{x}{1 - 2x}\right)$. So $f^{-1}(x) = \ln\left(\dfrac{x}{1 - 2x}\right)$. Note that the range of f and the domain of f^{-1} is $(0, \frac{1}{2})$.

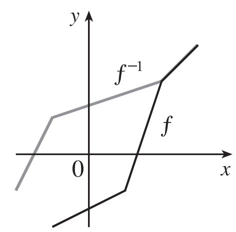
27. $y = f(x) = x^4 + 1 \;\Rightarrow\; y - 1 = x^4 \;\Rightarrow\; x = \sqrt[4]{y-1}$ [not $\pm$ since $x \geq 0$]. Interchange x and y: $y = \sqrt[4]{x-1}$. So $f^{-1}(x) = \sqrt[4]{x-1}$. The graph of $y = \sqrt[4]{x-1}$ is just the graph of $y = \sqrt[4]{x}$ shifted right one unit. From the graph, we see that f and f^{-1} are reflections about the line $y = x$.



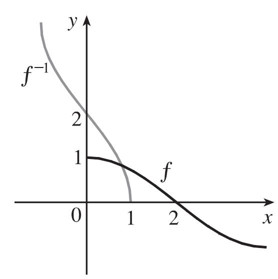
28. $y = f(x) = 2 - e^x \;\Rightarrow\; e^x = 2 - y \;\Rightarrow\; x = \ln(2-y)$. Interchange x and y: $y = \ln(2-x)$. So $f^{-1}(x) = \ln(2-x)$. From the graph, we see that f and f^{-1} are reflections about the line $y = x$.



29. Reflect the graph of f about the line $y = x$. The points $(-1,-2)$, $(1,-1)$, $(2,2)$, and $(3,3)$ on f are reflected to $(-2,-1)$, $(-1,1)$, $(2,2)$, and $(3,3)$ on f^{-1}.



30. Reflect the graph of f about the line $y = x$.

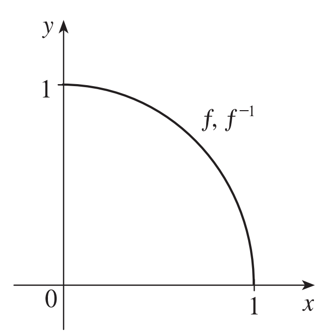


31. (a) $y = f(x) = \sqrt{1-x^2}$ $\quad (0 \leq x \leq 1$ and note that $y \geq 0) \;\Rightarrow$

$y^2 = 1 - x^2 \;\Rightarrow\; x^2 = 1 - y^2 \;\Rightarrow\; x = \sqrt{1-y^2}$. So

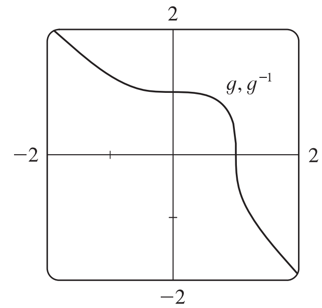
$f^{-1}(x) = \sqrt{1-x^2}$, $0 \leq x \leq 1$. We see that f^{-1} and f are the same function.

(b) The graph of f is the portion of the circle $x^2 + y^2 = 1$ with $0 \leq x \leq 1$ and $0 \leq y \leq 1$ (quarter-circle in the first quadrant). The graph of f is symmetric with respect to the line $y = x$, so its reflection about $y = x$ is itself, that is, $f^{-1} = f$.



32. (a) $y = g(x) = \sqrt[3]{1 - x^3} \quad \Rightarrow \quad y^3 = 1 - x^3 \quad \Rightarrow \quad x^3 = 1 - y^3 \quad \Rightarrow \quad x = \sqrt[3]{1 - y^3}$. So $g^{-1}(x) = \sqrt[3]{1 - x^3}$. We see that g and g^{-1} are the same function.

(b) The graph of g is symmetric with respect to the line $y = x$, so its reflection about $y = x$ is itself, that is, $g^{-1} = g$.



33. (a) It is defined as the inverse of the exponential function with base a, that is, $\log_a x = y \quad \Leftrightarrow \quad a^y = x$.

(b) $(0, \infty)$ (c) $\mathbb{R}$ (d) See Figure 11.

34. (a) The natural logarithm is the logarithm with base e, denoted $\ln x$.

(b) The common logarithm is the logarithm with base 10, denoted $\log x$.

(c) See Figure 13.

35. (a) $\log_5 125 = 3$ since $5^3 = 125$. (b) $\log_3 \dfrac{1}{27} = -3$ since $3^{-3} = \dfrac{1}{3^3} = \dfrac{1}{27}$.

36. (a) $\ln(1/e) = \ln 1 - \ln e = 0 - 1 = -1$ (b) $\log_{10} \sqrt{10} = \log_{10} 10^{1/2} = \frac{1}{2}$ by (7) .

37. (a)
$$\begin{aligned} \log_2 6 - \log_2 15 + \log_2 20 &= \log_2(\tfrac{6}{15}) + \log_2 20 && \text{[by Law 2]} \\ &= \log_2(\tfrac{6}{15} \cdot 20) && \text{[by Law 1]} \\ &= \log_2 8, \text{ and } \log_2 8 = 3 \text{ since } 2^3 = 8. \end{aligned}$$

(b)
$$\begin{aligned} \log_3 100 - \log_3 18 - \log_3 50 &= \log_3\left(\tfrac{100}{18}\right) - \log_3 50 = \log_3\left(\tfrac{100}{18 \cdot 50}\right) \\ &= \log_3(\tfrac{1}{9}), \text{ and } \log_3\left(\tfrac{1}{9}\right) = -2 \text{ since } 3^{-2} = \tfrac{1}{9}. \end{aligned}$$

38. (a) $e^{-2\ln 5} = \left(e^{\ln 5}\right)^{-2} \overset{(9)}{=} 5^{-2} = \dfrac{1}{5^2} = \dfrac{1}{25}$ (b) $\ln\left(\ln e^{e^{10}}\right) \overset{(9)}{=} \ln(e^{10}) \overset{(9)}{=} 10$

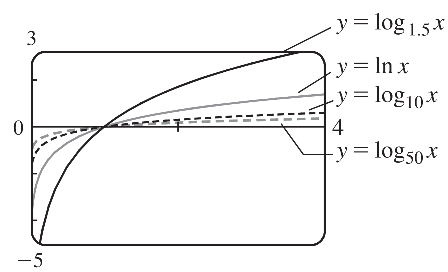
39.
$$\begin{aligned} \ln 5 + 5\ln 3 &= \ln 5 + \ln 3^5 && \text{[by Law 3]} \\ &= \ln(5 \cdot 3^5) && \text{[by Law 1]} \\ &= \ln 1215 \end{aligned}$$

40.
$$\begin{aligned} \ln(a+b) + \ln(a-b) - 2\ln c &= \ln[(a+b)(a-b)] - \ln c^2 && \text{[by Laws 1, 3]} \\ &= \ln \frac{(a+b)(a-b)}{c^2} && \text{[by Law 2]} \\ \text{or } &\ln \frac{a^2 - b^2}{c^2} \end{aligned}$$

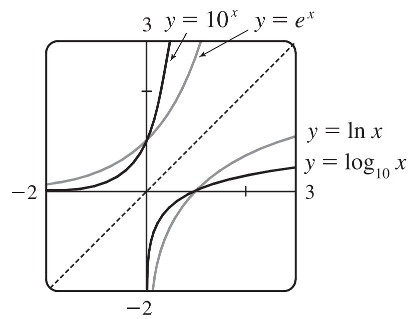
41. $\ln(1 + x^2) + \frac{1}{2}\ln x - \ln \sin x = \ln(1 + x^2) + \ln x^{1/2} - \ln \sin x = \ln[(1 + x^2)\sqrt{x}\,] - \ln \sin x = \ln \dfrac{(1 + x^2)\sqrt{x}}{\sin x}$

42. (a) $\log_{12} 10 = \dfrac{\ln 10}{\ln 12} \approx 0.926628$ (b) $\log_2 8.4 = \dfrac{\ln 8.4}{\ln 2} \approx 3.070389$

43. To graph these functions, we use $\log_{1.5} x = \dfrac{\ln x}{\ln 1.5}$ and $\log_{50} x = \dfrac{\ln x}{\ln 50}$. These graphs all approach $-\infty$ as $x \to 0^+$, and they all pass through the point $(1, 0)$. Also, they are all increasing, and all approach ∞ as $x \to \infty$. The functions with larger bases increase extremely slowly, and the ones with smaller bases do so somewhat more quickly. The functions with large bases approach the y-axis more closely as $x \to 0^+$.



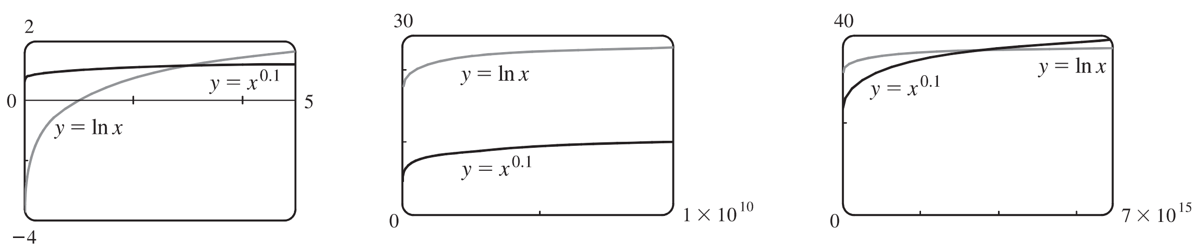
44. We see that the graph of $\ln x$ is the reflection of the graph of e^x about the line $y = x$, and that the graph of $\log_{10} x$ is the reflection of the graph of 10^x about the same line. The graph of 10^x increases more quickly than that of e^x. Also note that $\log_{10} x \to \infty$ as $x \to \infty$ more slowly than $\ln x$.



45. 3 ft $=$ 36 in, so we need x such that $\log_2 x = 36 \quad \Leftrightarrow \quad x = 2^{36} = 68{,}719{,}476{,}736$. In miles, this is

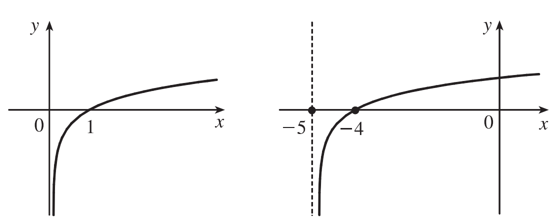
$$68{,}719{,}476{,}736 \text{ in} \cdot \frac{1 \text{ ft}}{12 \text{ in}} \cdot \frac{1 \text{ mi}}{5280 \text{ ft}} \approx 1{,}084{,}587.7 \text{ mi}.$$

46.



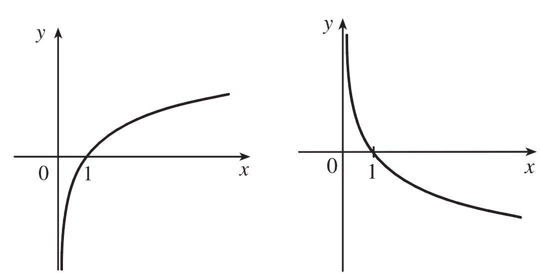
From the graphs, we see that $f(x) = x^{0.1} > g(x) = \ln x$ for approximately $0 < x < 3.06$, and then $g(x) > f(x)$ for $3.06 < x < 3.43 \times 10^{15}$ (approximately). At that point, the graph of f finally surpasses the graph of g for good.

47. (a) Shift the graph of $y = \log_{10} x$ five units to the left to obtain the graph of $y = \log_{10}(x + 5)$. Note the vertical asymptote of $x = -5$.



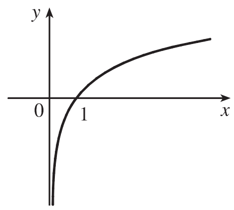
$y = \log_{10} x$ $\qquad$ $y = \log_{10}(x + 5)$

(b) Reflect the graph of $y = \ln x$ about the x-axis to obtain the graph of $y = -\ln x$.

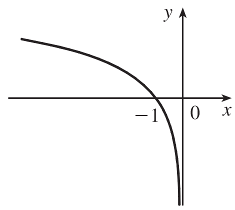


$y = \ln x$ $\qquad$ $y = -\ln x$

48. (a) Reflect the graph of $y = \ln x$ about the y-axis to obtain the graph of $y = \ln(-x)$.

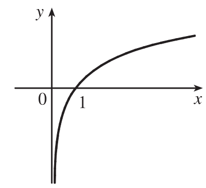


$y = \ln x$

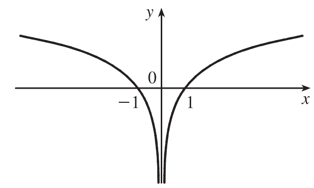


$y = \ln(-x)$

(b) Reflect the portion of the graph of $y = \ln x$ to the right of the y-axis about the y-axis. The graph of $y = \ln|x|$ is that reflection in addition to the original portion.



$y = \ln x$



$y = \ln|x|$

49. (a) $e^{7-4x} = 6 \quad \Leftrightarrow \quad 7 - 4x = \ln 6 \quad \Leftrightarrow \quad 7 - \ln 6 = 4x \quad \Leftrightarrow \quad x = \frac{1}{4}(7 - \ln 6)$

(b) $\ln(3x - 10) = 2 \quad \Leftrightarrow \quad 3x - 10 = e^2 \quad \Leftrightarrow \quad 3x = e^2 + 10 \quad \Leftrightarrow \quad x = \frac{1}{3}(e^2 + 10)$

50. (a) $\ln(x^2 - 1) = 3 \quad \Leftrightarrow \quad x^2 - 1 = e^3 \quad \Leftrightarrow \quad x^2 = 1 + e^3 \quad \Leftrightarrow \quad x = \pm\sqrt{1 + e^3}$.

(b) $e^{2x} - 3e^x + 2 = 0 \Leftrightarrow (e^x - 1)(e^x - 2) = 0 \Leftrightarrow e^x = 1$ or $e^x = 2 \Leftrightarrow x = \ln 1$ or $x = \ln 2$, so $x = 0$ or $\ln 2$.

51. (a) $2^{x-5} = 3 \quad \Leftrightarrow \quad \log_2 3 = x - 5 \quad \Leftrightarrow \quad x = 5 + \log_2 3$.

Or: $2^{x-5} = 3 \quad \Leftrightarrow \quad \ln\left(2^{x-5}\right) = \ln 3 \quad \Leftrightarrow \quad (x-5)\ln 2 = \ln 3 \quad \Leftrightarrow \quad x - 5 = \dfrac{\ln 3}{\ln 2} \quad \Leftrightarrow \quad x = 5 + \dfrac{\ln 3}{\ln 2}$

(b) $\ln x + \ln(x-1) = \ln(x(x-1)) = 1 \quad \Leftrightarrow \quad x(x-1) = e^1 \quad \Leftrightarrow \quad x^2 - x - e = 0$. The quadratic formula (with $a = 1$, $b = -1$, and $c = -e$) gives $x = \frac{1}{2}\left(1 \pm \sqrt{1 + 4e}\right)$, but we reject the negative root since the natural logarithm is not defined for $x < 0$. So $x = \frac{1}{2}\left(1 + \sqrt{1 + 4e}\right)$.

52. (a) $\ln(\ln x) = 1 \quad \Leftrightarrow \quad e^{\ln(\ln x)} = e^1 \quad \Leftrightarrow \quad \ln x = e^1 = e \quad \Leftrightarrow \quad e^{\ln x} = e^e \quad \Leftrightarrow \quad x = e^e$

(b) $e^{ax} = Ce^{bx} \quad \Leftrightarrow \quad \ln e^{ax} = \ln[C(e^{bx})] \quad \Leftrightarrow \quad ax = \ln C + bx + \ln e^{bx} \quad \Leftrightarrow \quad ax = \ln C + bx \quad \Leftrightarrow$

$ax - bx = \ln C \quad \Leftrightarrow \quad (a - b)x = \ln C \quad \Leftrightarrow \quad x = \dfrac{\ln C}{a - b}$

53. (a) $e^x < 10 \quad \Rightarrow \quad \ln e^x < \ln 10 \quad \Rightarrow \quad x < \ln 10 \quad \Rightarrow \quad x \in (-\infty, \ln 10)$

(b) $\ln x > -1 \quad \Rightarrow \quad e^{\ln x} > e^{-1} \quad \Rightarrow \quad x > e^{-1} \quad \Rightarrow \quad x \in (1/e, \infty)$

54. (a) $2 < \ln x < 9 \quad \Rightarrow \quad e^2 < e^{\ln x} < e^9 \quad \Rightarrow \quad e^2 < x < e^9 \quad \Rightarrow \quad x \in \left(e^2, e^9\right)$

(b) $e^{2-3x} > 4 \quad \Rightarrow \quad \ln e^{2-3x} > \ln 4 \quad \Rightarrow \quad 2 - 3x > \ln 4 \quad \Rightarrow \quad -3x > \ln 4 - 2 \quad \Rightarrow \quad x < -\frac{1}{3}(\ln 4 - 2) \quad \Rightarrow$

$x \in \left(-\infty, \frac{1}{3}(2 - \ln 4)\right)$

55. (a) For $f(x) = \sqrt{3 - e^{2x}}$, we must have $3 - e^{2x} \geq 0 \quad \Rightarrow \quad e^{2x} \leq 3 \quad \Rightarrow \quad 2x \leq \ln 3 \quad \Rightarrow \quad x \leq \frac{1}{2}\ln 3$.

Thus, the domain of f is $(-\infty, \frac{1}{2}\ln 3]$.

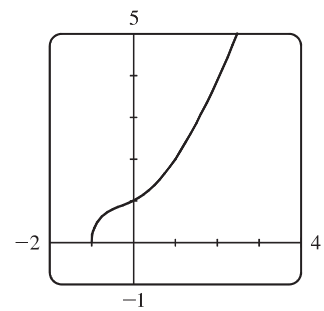
(b) $y = f(x) = \sqrt{3 - e^{2x}}$ [note that $y \geq 0$] $\quad \Rightarrow \quad y^2 = 3 - e^{2x} \quad \Rightarrow \quad e^{2x} = 3 - y^2 \quad \Rightarrow \quad 2x = \ln(3 - y^2) \quad \Rightarrow$

$x = \frac{1}{2}\ln(3 - y^2)$. Interchange x and y: $y = \frac{1}{2}\ln(3 - x^2)$. So $f^{-1}(x) = \frac{1}{2}\ln(3 - x^2)$. For the domain of f^{-1}, we must have $3 - x^2 > 0 \Rightarrow x^2 < 3 \Rightarrow |x| < \sqrt{3} \Rightarrow -\sqrt{3} < x < \sqrt{3} \Rightarrow 0 \le x < \sqrt{3}$ since $x \ge 0$. Note that the domain of f^{-1}, $[0, \sqrt{3})$, equals the range of f.

56. (a) For $f(x) = \ln(2 + \ln x)$, we must have $2 + \ln x > 0 \Rightarrow \ln x > -2 \Rightarrow x > e^{-2}$. Thus, the domain of f is (e^{-2}, ∞).

(b) $y = f(x) = \ln(2 + \ln x) \Rightarrow e^y = 2 + \ln x \Rightarrow \ln x = e^y - 2 \Rightarrow x = e^{e^y - 2}$. Interchange x and y: $y = e^{e^x - 2}$. So $f^{-1}(x) = e^{e^x - 2}$. The domain of f^{-1}, as well as the range of f, is $\mathbb{R}$.

57. We see that the graph of $y = f(x) = \sqrt{x^3 + x^2 + x + 1}$ is increasing, so f is 1-1. Enter $x = \sqrt{y^3 + y^2 + y + 1}$ and use your CAS to solve the equation for y. Using Derive, we get two (irrelevant) solutions involving imaginary expressions, as well as one which can be simplified to the following:



$y = f^{-1}(x) = -\frac{\sqrt[3]{4}}{6}\left(\sqrt[3]{D - 27x^2 + 20} - \sqrt[3]{D + 27x^2 - 20} + \sqrt[3]{2}\right)$

where $D = 3\sqrt{3}\sqrt{27x^4 - 40x^2 + 16}$.

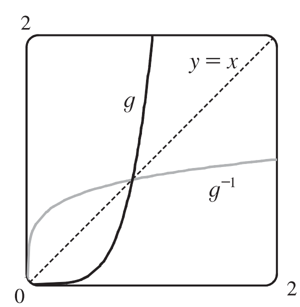
Maple and Mathematica each give two complex expressions and one real expression, and the real expression is equivalent to that given by Derive. For example, Maple's expression simplifies to $\frac{1}{6}\frac{M^{2/3} - 8 - 2M^{1/3}}{2M^{1/3}}$, where $M = 108x^2 + 12\sqrt{48 - 120x^2 + 81x^4} - 80$.

58. (a) If we use Derive, then solving $x = y^6 + y^4$ for y gives us six solutions of the form $y = \pm\frac{\sqrt{3}}{3}\sqrt{B - 1}$, where $B \in \left\{-2\sin\frac{A}{3}, 2\sin\left(\frac{A}{3} + \frac{\pi}{3}\right), -2\cos\left(\frac{A}{3} + \frac{\pi}{6}\right)\right\}$ and $A = \sin^{-1}\left(\frac{27x - 2}{2}\right)$. The inverse for $y = x^6 + x^4$ $(x \ge 0)$ is $y = \frac{\sqrt{3}}{3}\sqrt{B - 1}$ with $B = 2\sin\left(\frac{A}{3} + \frac{\pi}{3}\right)$, but because the domain of A is $\left[0, \frac{4}{27}\right]$, this expression is only valid for $x \in \left[0, \frac{4}{27}\right]$.

Happily, Maple gives us the rest of the solution! We solve $x = y^6 + y^4$ for y to get the two real solutions $\pm\frac{\sqrt{6}}{6}\frac{\sqrt{C^{1/3}\left(C^{2/3} - 2C^{1/3} + 4\right)}}{C^{1/3}}$, where $C = 108x + 12\sqrt{3}\sqrt{x(27x - 4)}$, and the inverse for $y = x^6 + x^4$ $(x \ge 0)$ is the positive solution, whose domain is $\left[\frac{4}{27}, \infty\right)$.

Mathematica also gives two real solutions, equivalent to those of Maple. The positive one is $\frac{\sqrt{6}}{6}\left(\sqrt[3]{4}D^{1/3} + 2\sqrt[3]{2}D^{-1/3} - 2\right)$, where $D = -2 + 27x + 3\sqrt{3}\sqrt{x}\sqrt{27x - 4}$. Although this expression also has domain $\left[\frac{4}{27}, \infty\right)$, Mathematica is mysteriously able to plot the solution for all $x \ge 0$.

(b)



59. (a) $n = 100 \cdot 2^{t/3} \Rightarrow \dfrac{n}{100} = 2^{t/3} \Rightarrow \log_2\left(\dfrac{n}{100}\right) = \dfrac{t}{3} \Rightarrow t = 3\log_2\left(\dfrac{n}{100}\right)$. Using formula (10), we can write this as $t = f^{-1}(n) = 3 \cdot \dfrac{\ln(n/100)}{\ln 2}$. This function tells us how long it will take to obtain n bacteria (given the number n).

(b) $n = 50{,}000 \Rightarrow t = f^{-1}(50{,}000) = 3 \cdot \dfrac{\ln\left(\frac{50{,}000}{100}\right)}{\ln 2} = 3\left(\dfrac{\ln 500}{\ln 2}\right) \approx 26.9$ hours

60. (a) $Q = Q_0(1 - e^{-t/a}) \Rightarrow \dfrac{Q}{Q_0} = 1 - e^{-t/a} \Rightarrow e^{-t/a} = 1 - \dfrac{Q}{Q_0} \Rightarrow -\dfrac{t}{a} = \ln\left(1 - \dfrac{Q}{Q_0}\right) \Rightarrow$ $t = -a\ln(1 - Q/Q_0)$. This gives us the time t necessary to obtain a given charge Q.

(b) $Q = 0.9Q_0$ and $a = 2 \Rightarrow t = -2\ln(1 - 0.9(Q_0/Q_0)) = -2\ln 0.1 \approx 4.6$ seconds.

61. (a) To find the equation of the graph that results from shifting the graph of $y = \ln x$ 3 units upward, we add 3 to the original function to get $y = \ln x + 3$.

(b) To find the equation of the graph that results from shifting the graph of $y = \ln x$ 3 units to the left, we replace x with $x + 3$ in the original function to get $y = \ln(x + 3)$.

(c) To find the equation of the graph that results from reflecting the graph of $y = \ln x$ about the x-axis, we multiply the original equation by -1 to get $y = -\ln x$.

(d) To find the equation of the graph that results from reflecting the graph of $y = \ln x$ about the y-axis, we replace x with $-x$ in the original equation to get $y = \ln(-x)$.

(e) To find the equation of the graph that results from reflecting the graph of $y = \ln x$ about the line $y = x$, we interchange x and y in the original equation to get $x = \ln y \Leftrightarrow y = e^x$.

(f) To find the equation of the graph that results from reflecting the graph of $y = \ln x$ about the x-axis and then about the line $y = x$, we first multiply the original equation by -1 [to get $y = -\ln x$] and then interchange x and y in this equation to get $x = -\ln y \Leftrightarrow \ln y = -x \Leftrightarrow y = e^{-x}$.

(g) To find the equation of the graph that results from reflecting the graph of $y = \ln x$ about the y-axis and then about the line $y = x$, we first replace x with $-x$ in the original equation [to get $y = \ln(-x)$] and then interchange x and y to get $x = \ln(-y) \Leftrightarrow -y = e^x \Leftrightarrow y = -e^x$.

(h) To find the equation of the graph that results from shifting the graph of $y = \ln x$ 3 units to the left and then reflecting it about the line $y = x$, we first replace x with $x + 3$ in the original equation [to get $y = \ln(x + 3)$] and then interchange x and y in this equation to get $x = \ln(y + 3) \Leftrightarrow y + 3 = e^x \Leftrightarrow y = e^x - 3$.

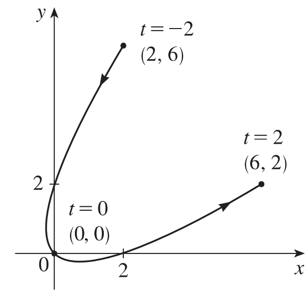
62. (a) If the point (x, y) is on the graph of $y = f(x)$, then the point $(x - c, y)$ is that point shifted c units to the left. Since f is 1-1, the point (y, x) is on the graph of $y = f^{-1}(x)$ and the point corresponding to $(x - c, y)$ on the graph of f is $(y, x - c)$ on the graph of f^{-1}. Thus, the curve's reflection is shifted *down* the same number of units as the curve itself is shifted to the left. So an expression for the inverse function is $g^{-1}(x) = f^{-1}(x) - c$.

(b) If we compress (or stretch) a curve horizontally, the curve's reflection in the line $y = x$ is compressed (or stretched) *vertically* by the same factor. Using this geometric principle, we see that the inverse of $h(x) = f(cx)$ can be expressed as $h^{-1}(x) = (1/c)\, f^{-1}(x)$.

1.7 Parametric Curves

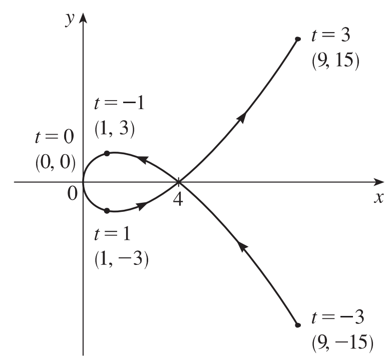
1. $x = t^2 + t, \quad y = t^2 - t, \quad -2 \le t \le 2$

t	-2	-1	0	1	2
x	2	0	0	2	6
y	6	2	0	0	2



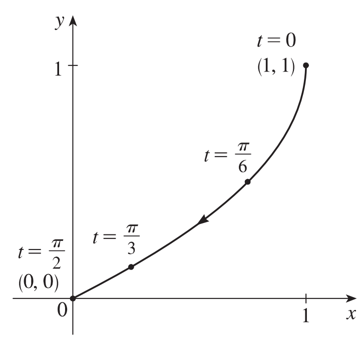
2. $x = t^2, \quad y = t^3 - 4t, \quad -3 \le t \le 3$

t	± 3	± 2	± 1	0
x	9	4	1	0
y	± 15	0	∓ 3	0



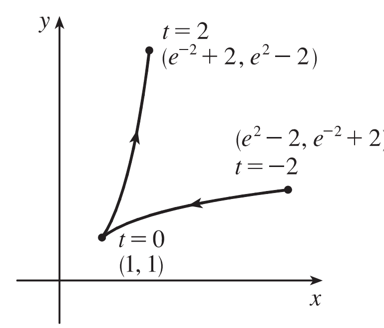
3. $x = \cos^2 t, \quad y = 1 - \sin t, \quad 0 \le t \le \pi/2$

t	0	$\pi/6$	$\pi/3$	$\pi/2$
x	1	$3/4$	$1/4$	0
y	1	$1/2$	$1 - \frac{\sqrt{3}}{2} \approx 0.13$	0



4. $x = e^{-t} + t, \quad y = e^t - t, \quad -2 \le t \le 2$

t	-2	-1	0	1	2
x	$e^2 - 2$ 5.39	$e - 1$ 1.72	1	$e^{-1} + 1$ 1.37	$e^{-2} + 2$ 2.14
y	$e^{-2} + 2$ 2.14	$e^{-1} + 1$ 1.37	1	$e - 1$ 1.72	$e^2 - 2$ 5.39



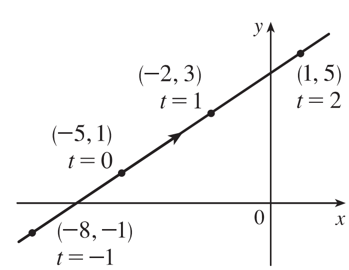
5. $x = 3t - 5, \quad y = 2t + 1$

(a)

t	-2	-1	0	1	2	3	4
x	-11	-8	-5	-2	1	4	7
y	-3	-1	1	3	5	7	9

(b) $x = 3t - 5 \quad \Rightarrow \quad 3t = x + 5 \quad \Rightarrow \quad t = \frac{1}{3}(x + 5) \quad \Rightarrow$

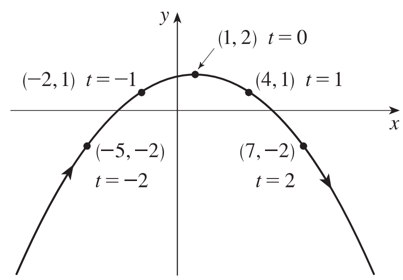
$y = 2 \cdot \frac{1}{3}(x + 5) + 1$, so $y = \frac{2}{3}x + \frac{13}{3}$.



6. $x = 1 + 3t, \quad y = 2 - t^2$

(a)

t	-3	-2	-1	0	1	2	3
x	-8	-5	-2	1	4	7	10
y	-7	-2	1	2	1	-2	-7



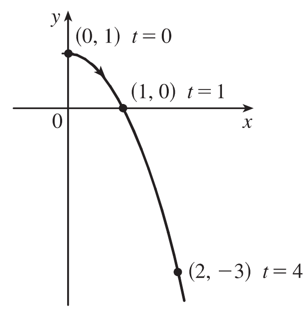
(b) $x = 1 + 3t \quad \Rightarrow \quad t = \frac{1}{3}(x-1) \quad \Rightarrow \quad y = 2 - \left[\frac{1}{3}(x-1)\right]^2$,

so $y = -\frac{1}{9}(x-1)^2 + 2$.

7. $x = \sqrt{t},\ y = 1 - t$

(a)

t	0	1	2	3	4
x	0	1	1.414	1.732	2
y	1	0	-1	-2	-3



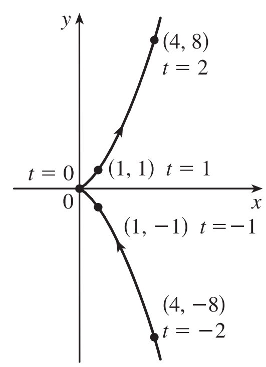
(b) $x = \sqrt{t} \quad \Rightarrow \quad t = x^2 \quad \Rightarrow \quad y = 1 - t = 1 - x^2$. Since $t \geq 0$, $x \geq 0$.

So the curve is the right half of the parabola $y = 1 - x^2$.

8. $x = t^2, y = t^3$

(a)

t	-2	-1	0	1	2
x	4	1	0	1	4
y	-8	-1	0	1	8

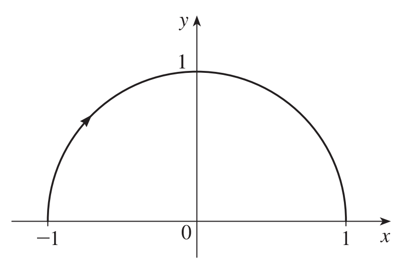


(b) $y = t^3 \quad \Rightarrow \quad t = \sqrt[3]{y} \quad \Rightarrow \quad x = t^2 = \left(\sqrt[3]{y}\right)^2 = y^{2/3}$. $\quad t \in \mathbb{R}, y \in \mathbb{R}, x \geq 0$.

9. (a) $x = \sin \frac{1}{2}\theta, y = \cos \frac{1}{2}\theta, -\pi \leq \theta \leq \pi$.

$x^2 + y^2 = \sin^2 \frac{1}{2}\theta + \cos^2 \frac{1}{2}\theta = 1$. For $-\pi \leq \theta \leq 0$, we have $-1 \leq x \leq 0$ and $0 \leq y \leq 1$. For $0 < \theta \leq \pi$, we have $0 < x \leq 1$ and $1 > y \geq 0$. The graph is a semicircle.

(b)

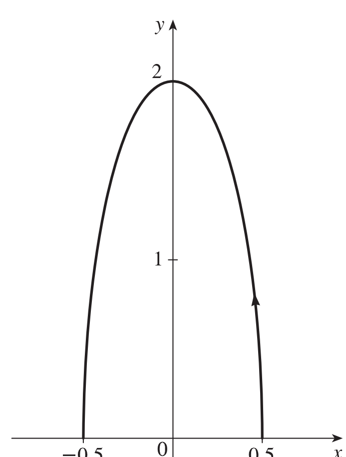


10. (a) $x = \frac{1}{2}\cos\theta, y = 2\sin\theta, 0 \leq \theta \leq \pi$.

$(2x)^2 + \left(\frac{1}{2}y\right)^2 = \cos^2\theta + \sin^2\theta = 1 \quad \Rightarrow \quad 4x^2 + \frac{1}{4}y^2 = 1 \quad \Rightarrow$

$\dfrac{x^2}{(1/2)^2} + \dfrac{y^2}{2^2} = 1$, which is an equation of an ellipse with x-intercepts $\pm\frac{1}{2}$ and y-intercepts ± 2. For $0 \leq \theta \leq \pi/2$, we have $\frac{1}{2} \geq x \geq 0$ and $0 \leq y \leq 2$. For $\pi/2 < \theta \leq \pi$, we have $0 > x \geq -\frac{1}{2}$ and $2 > y \geq 0$. So the graph is the top half of the ellipse.

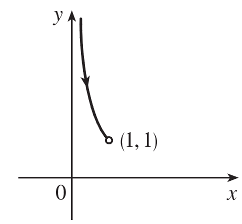
(b)



11. (a) $x = \sin t$, $y = \csc t$, $0 < t < \frac{\pi}{2}$. $y = \csc t = \dfrac{1}{\sin t} = \dfrac{1}{x}$.

For $0 < t < \frac{\pi}{2}$, we have $0 < x < 1$ and $y > 1$. Thus, the curve is the portion of the hyperbola $y = 1/x$ with $y > 1$.

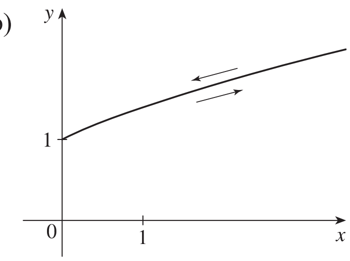
(b)



12. (a) $x = \tan^2\theta$, $y = \sec\theta$, $-\pi/2 < \theta < \pi/2$.

$1 + \tan^2\theta = \sec^2\theta \quad\Rightarrow\quad 1 + x = y^2 \quad\Rightarrow\quad x = y^2 - 1$. For $-\pi/2 < \theta \le 0$, we have $x \ge 0$ and $y \ge 1$. For $0 < \theta < \pi/2$, we have $0 < x$ and $1 < y$. Thus, the curve is the portion of the parabola $x = y^2 - 1$ in the first quadrant. As θ increases from $-\pi/2$ to 0, the point (x, y) approaches $(0, 1)$ along the parabola. As θ increases from 0 to $\pi/2$, the point (x, y) retreats from $(0, 1)$ along the parabola.

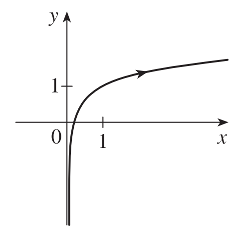
(b)



13. (a) $x = e^{2t} \quad\Rightarrow\quad 2t = \ln x \quad\Rightarrow\quad t = \frac{1}{2}\ln x$.

$y = t + 1 = \frac{1}{2}\ln x + 1$.

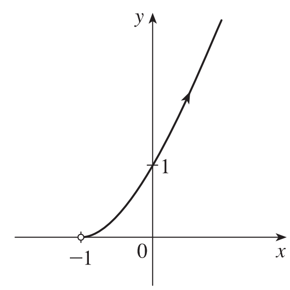
(b)



14. (a) $x = e^t - 1$, $y = e^{2t}$.

$y = (e^t)^2 = (x + 1)^2$ and since $x > -1$, we have the right side of the parabola $y = (x + 1)^2$.

(b)

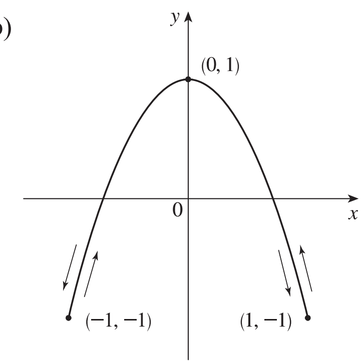


15. (a) $x = \sin\theta,\ y = \cos 2\theta$.

$$y = \cos^2\theta - \sin^2\theta = 1 - \sin^2\theta - \sin^2\theta$$
$$= 1 - 2\sin^2\theta = 1 - 2x^2.$$

Since $-1 \le \sin\theta \le 1$ and $-1 \le \cos 2\theta \le 1$, $-1 \le x \le 1$, and $-1 \le y \le 1$. The point (x, y) moves back and forth infinitely often along the parabola $y = 1 - 2x^2$ from $(1, -1)$ to $(-1, -1)$.

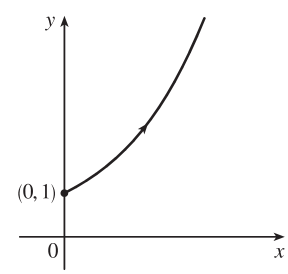
(b)



16. (a) $x = \ln t$, $y = \sqrt{t}$, $t \ge 1$.

$x = \ln t \quad\Rightarrow\quad t = e^x \quad\Rightarrow\quad y = \sqrt{t} = e^{x/2}$, $x \ge 0$.

(b)



17. $x = 3 + 2\cos t$, $y = 1 + 2\sin t$, $\pi/2 \le t \le 3\pi/2$. By Example 4 with $r = 2$, $h = 3$, and $k = 1$, the motion of the particle takes place on a circle centered at $(3, 1)$ with a radius of 2. As t goes from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$, the particle starts at the point $(3, 3)$ and moves counterclockwise to $(3, -1)$ [one-half of a circle].

18. $x = 2\sin t$, $y = 4 + \cos t \;\Rightarrow\; \sin t = \dfrac{x}{2}$, $\cos t = y - 4$. $\sin^2 t + \cos^2 t = 1 \;\Rightarrow\; \left(\dfrac{x}{2}\right)^2 + (y-4)^2 = 1$. The motion of the particle takes place on an ellipse centered at $(0, 4)$. As t goes from 0 to $\frac{3\pi}{2}$, the particle starts at the point $(0, 5)$ and moves clockwise to $(-2, 4)$ [three-quarters of an ellipse].

19. $x = 5\sin t$, $y = 2\cos t \;\Rightarrow\; \sin t = \dfrac{x}{5}$, $\cos t = \dfrac{y}{2}$. $\sin^2 t + \cos^2 t = 1 \;\Rightarrow\; \left(\dfrac{x}{5}\right)^2 + \left(\dfrac{y}{2}\right)^2 = 1$. The motion of the particle takes place on an ellipse centered at $(0, 0)$. As t goes from $-\pi$ to 5π, the particle starts at the point $(0, -2)$ and moves clockwise around the ellipse 3 times.

20. $y = \cos^2 t = 1 - \sin^2 t = 1 - x^2$. The motion of the particle takes place on the parabola $y = 1 - x^2$. As t goes from -2π to $-\pi$, the particle starts at the point $(0, 1)$, moves to $(1, 0)$, and goes back to $(0, 1)$. As t goes from $-\pi$ to 0, the particle moves to $(-1, 0)$ and goes back to $(0, 1)$. The particle repeats this motion as t goes from 0 to 2π.

21. We must have $1 \le x \le 4$ and $2 \le y \le 3$. So the graph of the curve must be contained in the rectangle $[1, 4]$ by $[2, 3]$.

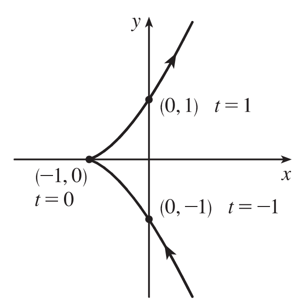
22. (a) From the first graph, we have $1 \le x \le 2$. From the second graph, we have $-1 \le y \le 1$. The only choice that satisfies either of those conditions is III.

(b) From the first graph, the values of x cycle through the values from -2 to 2 four times. From the second graph, the values of y cycle through the values from -2 to 2 six times. Choice I satisfies these conditions.

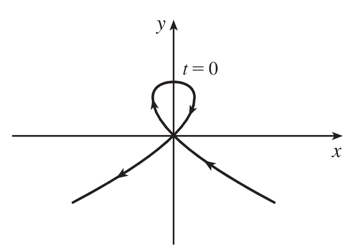
(c) From the first graph, the values of x cycle through the values from -2 to 2 three times. From the second graph, we have $0 \le y \le 2$. Choice IV satisfies these conditions.

(d) From the first graph, the values of x cycle through the values from -2 to 2 two times. From the second graph, the values of y do the same thing. Choice II satisfies these conditions.

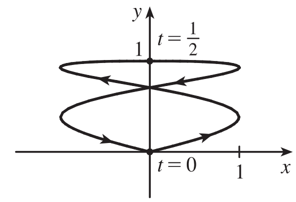
23. When $t = -1$, $(x, y) = (0, -1)$. As t increases to 0, x decreases to -1 and y increases to 0. As t increases from 0 to 1, x increases to 0 and y increases to 1. As t increases beyond 1, both x and y increase. For $t < -1$, x is positive and decreasing and y is negative and increasing. We could achieve greater accuracy by estimating x- and y-values for selected values of t from the given graphs and plotting the corresponding points.



24. For $t < -1$, x is positive and decreasing, while y is negative and increasing (these points are in Quadrant IV). When $t = -1$, $(x, y) = (0, 0)$ and, as t increases from -1 to 0, x becomes negative and y increases from 0 to 1. At $t = 0$, $(x, y) = (0, 1)$ and, as t increases from 0 to 1, y decreases from 1 to 0 and x is positive. At $t = 1$, $(x, y) = (0, 0)$ again, so the loop is completed. For $t > 1$, x and y both become large negative. This enables us to draw a rough sketch. We could achieve greater accuracy by estimating x- and y-values for selected values of t from the given graphs and plotting the corresponding points.



25. When $t = 0$ we see that $x = 0$ and $y = 0$, so the curve starts at the origin. As t increases from 0 to $\frac{1}{2}$, the graphs show that y increases from 0 to 1 while x increases from 0 to 1, decreases to 0 and to -1, then increases back to 0, so we arrive at the point $(0, 1)$. Similarly, as t increases from $\frac{1}{2}$ to 1, y decreases from 1 to 0 while x repeats its pattern, and we arrive back at the origin. We could achieve greater accuracy by estimating x- and y-values for selected values of t from the given graphs and plotting the corresponding points.



26. (a) $x = t^4 - t + 1 = (t^4 + 1) - t > 0$ [think of the graphs of $y = t^4 + 1$ and $y = t$] and $y = t^2 \geq 0$, so these equations are matched with graph V.

(b) $y = \sqrt{t} \geq 0$. $x = t^2 - 2t = t(t - 2)$ is negative for $0 < t < 2$, so these equations are matched with graph I.

(c) $x = \sin 2t$ has period $2\pi/2 = \pi$. Note that

$y(t + 2\pi) = \sin[t + 2\pi + \sin 2(t + 2\pi)] = \sin(t + 2\pi + \sin 2t) = \sin(t + \sin 2t) = y(t)$, so y has period 2π.

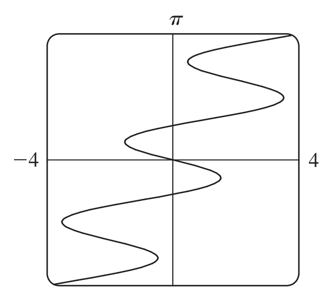
These equations match graph II since x cycles through the values -1 to 1 twice as y cycles through those values once.

(d) $x = \cos 5t$ has period $2\pi/5$ and $y = \sin 2t$ has period π, so x will take on the values -1 to 1, and then 1 to -1, before y takes on the values -1 to 1. Note that when $t = 0$, $(x, y) = (1, 0)$. These equations are matched with graph VI.

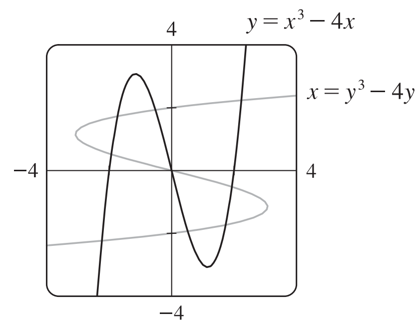
(e) $x = t + \sin 4t$, $y = t^2 + \cos 3t$. As t becomes large, t and t^2 become the dominant terms in the expressions for x and y, so the graph will look like the graph of $y = x^2$, but with oscillations. These equations are matched with graph IV.

(f) $x = \dfrac{\sin 2t}{4 + t^2}$, $y = \dfrac{\cos 2t}{4 + t^2}$. As $t \to \infty$, x and y both approach 0. These equations are matched with graph III.

27. Use $y = t$ and $x = t - 2\sin \pi t$ with a t-interval of $[-\pi, \pi]$.



28. Use $x_1 = t$, $y_1 = t^3 - 4t$ and $x_2 = t^3 - 4t$, $y_2 = t$ with a t-interval of $[-3, 3]$. There are 9 points of intersection; $(0, 0)$ is fairly obvious. The point in quadrant I is approximately $(2.2, 2.2)$, and by symmetry, the point in quadrant III is approximately $(-2.2, -2.2)$. The other six points are approximately $(\mp 1.9, \pm 0.5)$, $(\mp 1.7, \pm 1.7)$, and $(\mp 0.5, \pm 1.9)$.



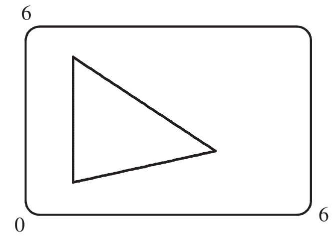
29. (a) $x = x_1 + (x_2 - x_1)t$, $y = y_1 + (y_2 - y_1)t$, $0 \le t \le 1$. Clearly the curve passes through $P_1(x_1, y_1)$ when $t = 0$ and through $P_2(x_2, y_2)$ when $t = 1$. For $0 < t < 1$, x is strictly between x_1 and x_2 and y is strictly between y_1 and y_2. For every value of t, x and y satisfy the relation $y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)$, which is the equation of the line through $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$.

Finally, any point (x, y) on that line satisfies $\dfrac{y - y_1}{y_2 - y_1} = \dfrac{x - x_1}{x_2 - x_1}$; if we call that common value t, then the given parametric equations yield the point (x, y); and any (x, y) on the line between $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ yields a value of t in $[0, 1]$. So the given parametric equations exactly specify the line segment from $P_1(x_1, y_1)$ to $P_2(x_2, y_2)$.

(b) $x = -2 + [3 - (-2)]t = -2 + 5t$ and $y = 7 + (-1 - 7)t = 7 - 8t$ for $0 \le t \le 1$.

30. For the side of the triangle from A to B, use $(x_1, y_1) = (1, 1)$ and $(x_2, y_2) = (4, 2)$. Hence, the equations are

$$x = x_1 + (x_2 - x_1)\,t = 1 + (4 - 1)\,t = 1 + 3t,$$
$$y = y_1 + (y_2 - y_1)\,t = 1 + (2 - 1)\,t = 1 + t.$$



Graphing $x = 1 + 3t$ and $y = 1 + t$ with $0 \le t \le 1$ gives us the side of the triangle from A to B. Similarly, for the side BC we use $x = 4 - 3t$ and $y = 2 + 3t$, and for the side AC we use $x = 1$ and $y = 1 + 4t$.

31. The circle $x^2 + (y - 1)^2 = 4$ has center $(0, 1)$ and radius 2, so by Example 4 it can be represented by $x = 2\cos t$, $y = 1 + 2\sin t$, $0 \le t \le 2\pi$. This representation gives us the circle with a counterclockwise orientation starting at $(2, 1)$.

(a) To get a clockwise orientation, we could change the equations to $x = 2\cos t$, $y = 1 - 2\sin t$, $0 \le t \le 2\pi$.

(b) To get three times around in the counterclockwise direction, we use the original equations $x = 2\cos t$, $y = 1 + 2\sin t$ with the domain expanded to $0 \le t \le 6\pi$.

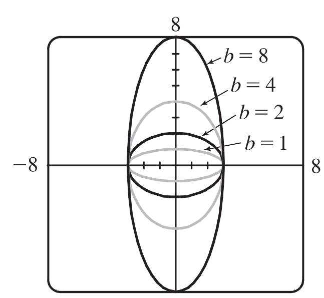
(c) To start at $(0, 3)$ using the original equations, we must have $x_1 = 0$; that is, $2\cos t = 0$. Hence, $t = \frac{\pi}{2}$. So we use $x = 2\cos t$, $y = 1 + 2\sin t$, $\frac{\pi}{2} \le t \le \frac{3\pi}{2}$.

Alternatively, if we want t to start at 0, we could change the equations of the curve. For example, we could use $x = -2\sin t$, $y = 1 + 2\cos t$, $0 \le t \le \pi$.

32. (a) Let $x^2/a^2 = \sin^2 t$ and $y^2/b^2 = \cos^2 t$ to obtain $x = a\sin t$ and $y = b\cos t$ with $0 \le t \le 2\pi$ as possible parametric equations for the ellipse $x^2/a^2 + y^2/b^2 = 1$.

(b) The equations are $x = 3\sin t$ and $y = b\cos t$ for $b \in \{1, 2, 4, 8\}$.

(c) As b increases, the ellipse stretches vertically.



33. *Big circle:* It's centered at $(2, 2)$ with a radius of 2, so by Example 4, parametric equations are

$$x = 2 + 2\cos t, \qquad y = 2 + 2\sin t, \qquad 0 \le t \le 2\pi$$

Small circles: They are centered at $(1, 3)$ and $(3, 3)$ with a radius of 0.1. By Example 4, parametric equations are

(left) $\quad x = 1 + 0.1\cos t, \qquad y = 3 + 0.1\sin t, \qquad 0 \le t \le 2\pi$

and *(right)* $\quad x = 3 + 0.1\cos t, \qquad y = 3 + 0.1\sin t, \qquad 0 \le t \le 2\pi$

Semicircle: It's the lower half of a circle centered at $(2, 2)$ with radius 1. By Example 4, parametric equations are

$$x = 2 + 1\cos t, \qquad y = 2 + 1\sin t, \qquad \pi \le t \le 2\pi$$

To get all four graphs on the same screen with a typical graphing calculator, we need to change the last t-interval to $[0, 2\pi]$ in order to match the others. We can do this by changing t to $0.5t$. This change gives us the upper half. There are several ways to get the lower half—one is to change the "$+$" to a "$-$" in the y-assignment, giving us

$$x = 2 + 1\cos(0.5t), \qquad y = 2 - 1\sin(0.5t), \qquad 0 \le t \le 2\pi$$

34. If you are using a calculator or computer that can overlay graphs (using multiple t-intervals), the following is appropriate.

Left side: $x = 1$ and y goes from 1.5 to 4, so use

$$x = 1, \qquad y = t, \qquad 1.5 \le t \le 4$$

Right side: $x = 10$ and y goes from 1.5 to 4, so use

$$x = 10, \qquad y = t, \qquad 1.5 \le t \le 4$$

Bottom: x goes from 1 to 10 and $y = 1.5$, so use

$$x = t, \qquad y = 1.5, \qquad 1 \le t \le 10$$

Handle: It starts at $(10, 4)$ and ends at $(13, 7)$, so use

$$x = 10 + t, \qquad y = 4 + t, \qquad 0 \le t \le 3$$

Left wheel: It's centered at $(3, 1)$, has a radius of 1, and appears to go about 30° above the horizontal, so use

$$x = 3 + 1\cos t, \qquad y = 1 + 1\sin t, \qquad \tfrac{5\pi}{6} \le t \le \tfrac{13\pi}{6}$$

Right wheel: Similar to the left wheel with center $(8, 1)$, so use

$$x = 8 + 1\cos t, \qquad y = 1 + 1\sin t, \qquad \tfrac{5\pi}{6} \le t \le \tfrac{13\pi}{6}$$

If you are using a calculator or computer that cannot overlay graphs (using one t-interval), the following is appropriate. We'll start by picking the t-interval $[0, 2.5]$ since it easily matches the t-values for the two sides. We now need to find parametric equations for all graphs with $0 \le t \le 2.5$.

Left side: $x = 1$ and y goes from 1.5 to 4, so use

$$x = 1, \qquad y = 1.5 + t, \qquad 0 \le t \le 2.5$$

Right side: $x = 10$ and y goes from 1.5 to 4, so use

$$x = 10, \qquad y = 1.5 + t, \qquad 0 \le t \le 2.5$$

Bottom: x goes from 1 to 10 and $y = 1.5$, so use

$$x = 1 + 3.6t, \qquad y = 1.5, \qquad 0 \le t \le 2.5$$

To get the x-assignment, think of creating a linear function such that when $t = 0$, $x = 1$ and when $t = 2.5$, $x = 10$. We can use the point-slope form of a line with $(t_1, x_1) = (0, 1)$ and $(t_2, x_2) = (2.5, 10)$.

$x - 1 = \dfrac{10 - 1}{2.5 - 0}(t - 0) \quad \Rightarrow \quad x = 1 + 3.6t.$

Handle: It starts at $(10, 4)$ and ends at $(13, 7)$, so use

$$x = 10 + 1.2t, \qquad y = 4 + 1.2t, \qquad 0 \le t \le 2.5$$

$(t_1, x_1) = (0, 10)$ and $(t_2, x_2) = (2.5, 13)$ gives us $x - 10 = \dfrac{13 - 10}{2.5 - 0}(t - 0) \quad \Rightarrow \quad x = 10 + 1.2t.$

$(t_1, y_1) = (0, 4)$ and $(t_2, y_2) = (2.5, 7)$ gives us $y - 4 = \dfrac{7 - 4}{2.5 - 0}(t - 0) \quad \Rightarrow \quad y = 4 + 1.2t.$

Left wheel: It's centered at $(3, 1)$, has a radius of 1, and appears to go about $30°$ above the horizontal, so use

$$x = 3 + 1\cos\left(\tfrac{8\pi}{15}t + \tfrac{5\pi}{6}\right), \qquad y = 1 + 1\sin\left(\tfrac{8\pi}{15}t + \tfrac{5\pi}{6}\right), \qquad 0 \le t \le 2.5$$

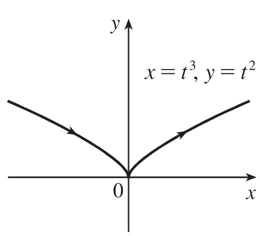
$(t_1, \theta_1) = \left(0, \frac{5\pi}{6}\right)$ and $(t_2, \theta_2) = \left(\frac{5}{2}, \frac{13\pi}{6}\right)$ gives us $\theta - \frac{5\pi}{6} = \dfrac{\frac{13\pi}{6} - \frac{5\pi}{6}}{\frac{5}{2} - 0}(t - 0) \quad \Rightarrow \quad \theta = \frac{5\pi}{6} + \frac{8\pi}{15}t.$

Right wheel: Similar to the left wheel with center $(8, 1)$, so use

$$x = 8 + 1\cos\left(\tfrac{8\pi}{15}t + \tfrac{5\pi}{6}\right), \qquad y = 1 + 1\sin\left(\tfrac{8\pi}{15}t + \tfrac{5\pi}{6}\right), \qquad 0 \le t \le 2.5$$

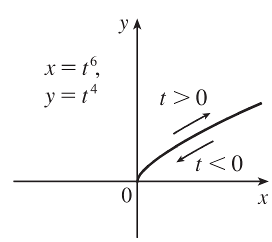
35. (a) $x = t^3 \quad \Rightarrow \quad t = x^{1/3}$, so $y = t^2 = x^{2/3}$.

We get the entire curve $y = x^{2/3}$ traversed in a left to right direction.



(b) $x = t^6 \quad \Rightarrow \quad t = x^{1/6}$, so $y = t^4 = x^{4/6} = x^{2/3}$.

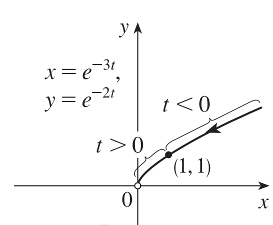
Since $x = t^6 \ge 0$, we only get the right half of the curve $y = x^{2/3}$.



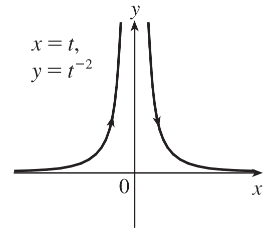
(c) $x = e^{-3t} = (e^{-t})^3 \quad$ [so $e^{-t} = x^{1/3}$],

$y = e^{-2t} = (e^{-t})^2 = (x^{1/3})^2 = x^{2/3}$.

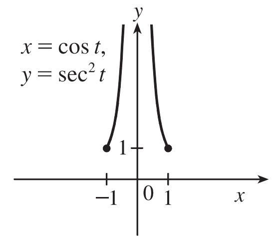
If $t < 0$, then x and y are both larger than 1. If $t > 0$, then x and y are between 0 and 1. Since $x > 0$ and $y > 0$, the curve never quite reaches the origin.



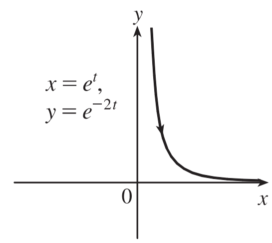
36. (a) $x = t$, so $y = t^{-2} = x^{-2}$. We get the entire curve $y = 1/x^2$ traversed in a left-to-right direction.



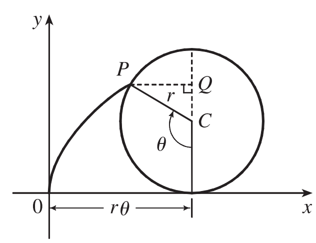
(b) $x = \cos t$, $y = \sec^2 t = \dfrac{1}{\cos^2 t} = \dfrac{1}{x^2}$. Since $\sec t \geq 1$, we only get the parts of the curve $y = 1/x^2$ with $y \geq 1$. We get the first quadrant portion of the curve when $x > 0$, that is, $\cos t > 0$, and we get the second quadrant portion of the curve when $x < 0$, that is, $\cos t < 0$.



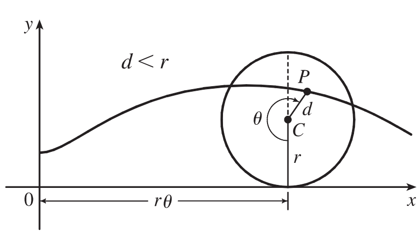
(c) $x = e^t$, $y = e^{-2t} = (e^t)^{-2} = x^{-2}$. Since e^t and e^{-2t} are both positive, we only get the first quadrant portion of the curve $y = 1/x^2$.

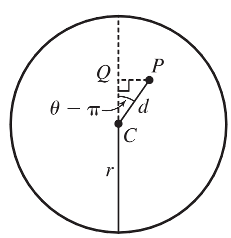


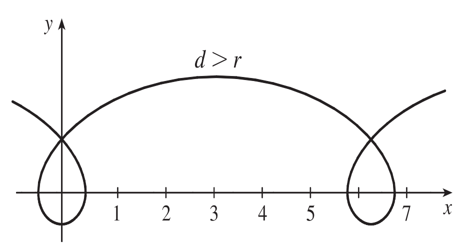
37. The case $\frac{\pi}{2} < \theta < \pi$ is illustrated. C has coordinates $(r\theta, r)$ as in Example 7, and Q has coordinates $(r\theta, r + r\cos(\pi - \theta)) = (r\theta, r(1 - \cos\theta))$ [since $\cos(\pi - \alpha) = \cos\pi\cos\alpha + \sin\pi\sin\alpha = -\cos\alpha$], so P has coordinates $(r\theta - r\sin(\pi - \theta), r(1 - \cos\theta)) = (r(\theta - \sin\theta), r(1 - \cos\theta))$ [since $\sin(\pi - \alpha) = \sin\pi\cos\alpha - \cos\pi\sin\alpha = \sin\alpha$]. Again we have the parametric equations $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$.



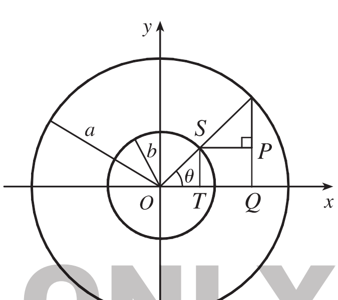
38. The first two diagrams depict the case $\pi < \theta < \frac{3\pi}{2}$, $d < r$. As in Example 7, C has coordinates $(r\theta, r)$. Now Q (in the second diagram) has coordinates $(r\theta, r + d\cos(\theta - \pi)) = (r\theta, r - d\cos\theta)$, so a typical point P of the trochoid has coordinates $(r\theta + d\sin(\theta - \pi), r - d\cos\theta)$. That is, P has coordinates (x, y), where $x = r\theta - d\sin\theta$ and $y = r - d\cos\theta$. When $d = r$, these equations agree with those of the cycloid.



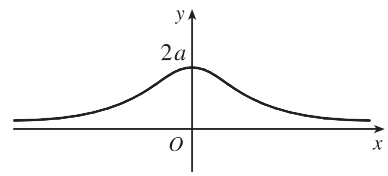




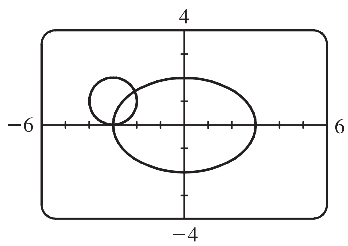
39. It is apparent that $x = |OQ|$ and $y = |QP| = |ST|$. From the diagram, $x = |OQ| = a\cos\theta$ and $y = |ST| = b\sin\theta$. Thus, the parametric equations are $x = a\cos\theta$ and $y = b\sin\theta$. To eliminate θ we rearrange: $\sin\theta = y/b \Rightarrow \sin^2\theta = (y/b)^2$ and $\cos\theta = x/a \Rightarrow \cos^2\theta = (x/a)^2$. Adding the two equations: $\sin^2\theta + \cos^2\theta = 1 = x^2/a^2 + y^2/b^2$. Thus, we have an ellipse.



40. $C = (2a\cot\theta, 2a)$, so the x-coordinate of P is $x = 2a\cot\theta$. Let $B = (0, 2a)$. Then $\angle OAB$ is a right angle and $\angle OBA = \theta$, so $|OA| = 2a\sin\theta$ and $A = ((2a\sin\theta)\cos\theta, (2a\sin\theta)\sin\theta)$. Thus, the y-coordinate of P is $y = 2a\sin^2\theta$.



41. (a)



There are 2 points of intersection: $(-3, 0)$ and approximately $(-2.1, 1.4)$.

(b) A collision point occurs when $x_1 = x_2$ and $y_1 = y_2$ for the same t. So solve the equations:

$$3\sin t = -3 + \cos t \quad \textbf{(1)}$$
$$2\cos t = 1 + \sin t \quad \textbf{(2)}$$

From **(2)**, $\sin t = 2\cos t - 1$. Substituting into **(1)**, we get $3(2\cos t - 1) = -3 + \cos t \Rightarrow 5\cos t = 0 \;(\star) \Rightarrow \cos t = 0 \Rightarrow t = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. We check that $t = \frac{3\pi}{2}$ satisfies **(1)** and **(2)** but $t = \frac{\pi}{2}$ does not. So the only collision point occurs when $t = \frac{3\pi}{2}$, and this gives the point $(-3, 0)$. [We could check our work by graphing x_1 and x_2 together as functions of t and, on another plot, y_1 and y_2 as functions of t. If we do so, we see that the only value of t for which *both* pairs of graphs intersect is $t = \frac{3\pi}{2}$.]

(c) The circle is centered at $(3, 1)$ instead of $(-3, 1)$. There are still 2 intersection points: $(3, 0)$ and $(2.1, 1.4)$, but there are no collision points, since $(\star)$ in part (b) becomes $5\cos t = 6 \Rightarrow \cos t = \frac{6}{5} > 1$.

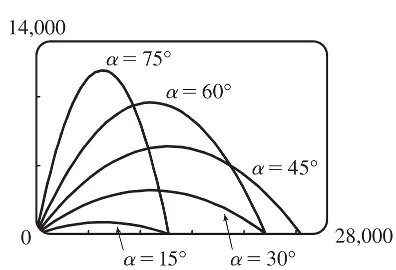
42. (a) If $\alpha = 30^\circ$ and $v_0 = 500$ m/s, then the equations become $x = (500\cos 30^\circ)t = 250\sqrt{3}t$ and $y = (500\sin 30^\circ)t - \frac{1}{2}(9.8)t^2 = 250t - 4.9t^2$. $y = 0$ when $t = 0$ (when the gun is fired) and again when $t = \frac{250}{4.9} \approx 51$ s. Then $x = \left(250\sqrt{3}\right)\left(\frac{250}{4.9}\right) \approx 22{,}092$ m, so the bullet hits the ground about 22 km from the gun.

The formula for y is quadratic in t. To find the maximum y-value, we will complete the square:

$$y = -4.9\left(t^2 - \frac{250}{4.9}t\right) = -4.9\left[t^2 - \frac{250}{4.9}t + \left(\frac{125}{4.9}\right)^2\right] + \frac{125^2}{4.9} = -4.9\left(t - \frac{125}{4.9}\right)^2 + \frac{125^2}{4.9} \le \frac{125^2}{4.9}$$

with equality when $t = \frac{125}{4.9}$ s, so the maximum height attained is $\frac{125^2}{4.9} \approx 3189$ m.

(b)



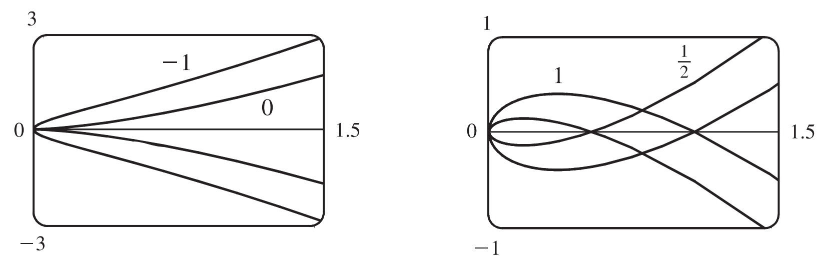
As α $(0^\circ < \alpha < 90^\circ)$ increases up to 45°, the projectile attains a greater height and a greater range. As α increases past 45°, the projectile attains a greater height, but its range decreases.

(c) $x = (v_0 \cos\alpha)t \quad\Rightarrow\quad t = \dfrac{x}{v_0 \cos\alpha}$.

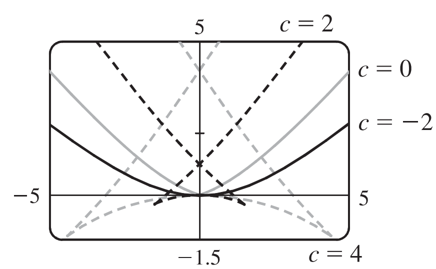
$$y = (v_0 \sin\alpha)t - \tfrac{1}{2}gt^2 \quad\Rightarrow\quad y = (v_0 \sin\alpha)\,\frac{x}{v_0\cos\alpha} - \frac{g}{2}\left(\frac{x}{v_0\cos\alpha}\right)^2 = (\tan\alpha)x - \left(\frac{g}{2v_0^2\cos^2\alpha}\right)x^2,$$

which is the equation of a parabola (quadratic in x).

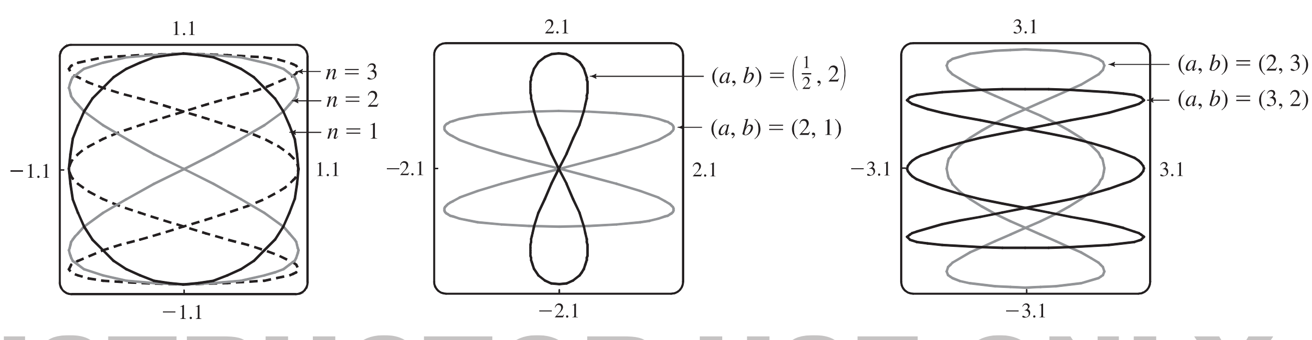
43. $x = t^2, y = t^3 - ct$. We use a graphing device to produce the graphs for various values of c with $-\pi \le t \le \pi$. Note that all the members of the family are symmetric about the x-axis. For $c < 0$, the graph does not cross itself, but for $c = 0$ it has a cusp at $(0, 0)$ and for $c > 0$ the graph crosses itself at $x = c$, so the loop grows larger as c increases.



44. $x = 2ct - 4t^3, y = -ct^2 + 3t^4$. We use a graphing device to produce the graphs for various values of c with $-\pi \le t \le \pi$. Note that all the members of the family are symmetric about the y-axis. When $c < 0$, the graph resembles that of a polynomial of even degree, but when $c = 0$ there is a corner at the origin, and when $c > 0$, the graph crosses itself at the origin, and has two cusps below the x-axis. The size of the "swallowtail" increases as c increases.

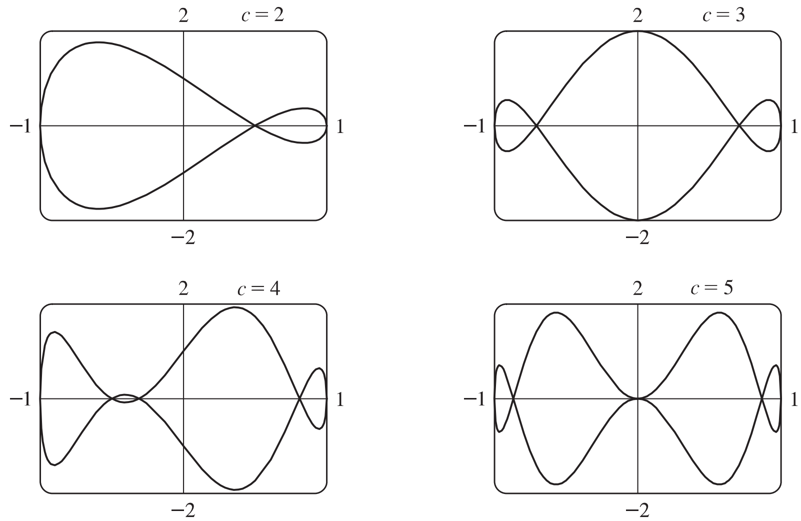


45. Note that all the Lissajous figures are symmetric about the x-axis. The parameters a and b simply stretch the graph in the x- and y-directions respectively. For $a = b = n = 1$ the graph is simply a circle with radius 1. For $n = 2$ the graph crosses itself at the origin and there are loops above and below the x-axis. In general, the figures have $n - 1$ points of intersection, all of which are on the y-axis, and a total of n closed loops.

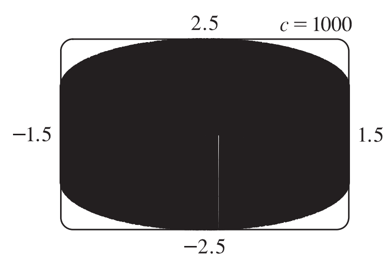


$a = b = 1$ $\qquad$ $n = 2$ $\qquad$ $n = 3$

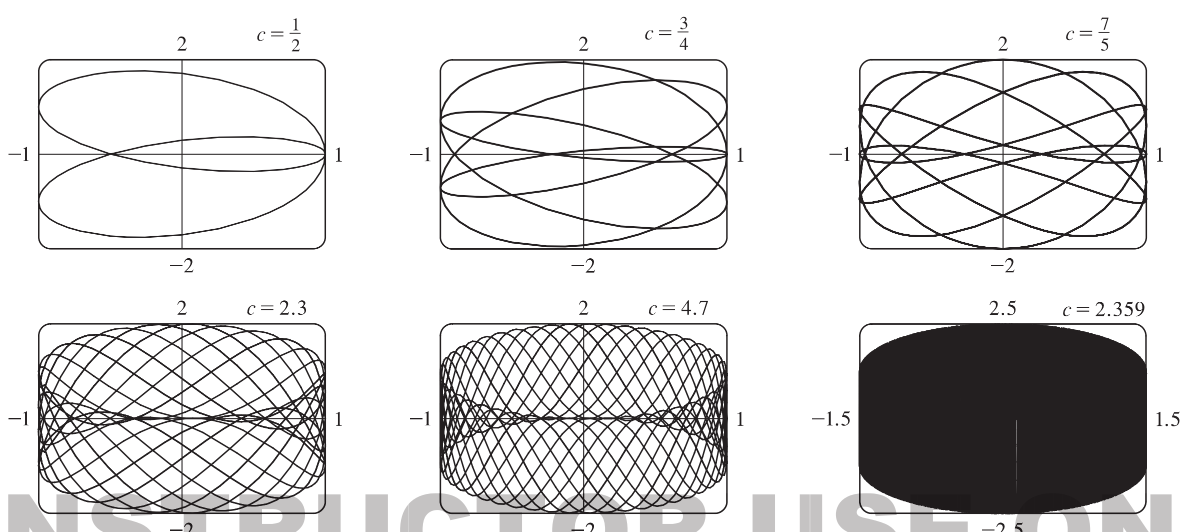
46. $x = \cos t$, $y = \sin t - \sin ct$. If $c = 1$, then $y = 0$, and the curve is simply the line segment from $(-1, 0)$ to $(1, 0)$. The graphs are shown for $c = 2, 3, 4$ and 5.



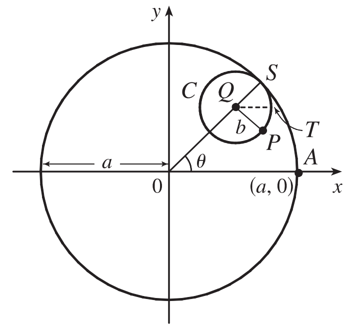
It is easy to see that all the curves lie in the rectangle $[-1, 1]$ by $[-2, 2]$. When c is an integer, $x(t + 2\pi) = x(t)$ and $y(t + 2\pi) = y(t)$, so the curve is closed. When c is a positive integer greater than 1, the curve intersects the x-axis $c + 1$ times and has c loops (one of which degenerates to a tangency at the origin when c is an odd integer of the form $4k + 1$). As c increases, the curve's loops become thinner, but stay in the region bounded by the semicircles $y = \pm\left(1 + \sqrt{1 - x^2}\,\right)$ and the line segments from $(-1, -1)$ to $(-1, 1)$ and from $(1, -1)$ to $(1, 1)$. This is true because $|y| = |\sin t - \sin ct| \leq |\sin t| + |\sin ct| \leq \sqrt{1 - x^2} + 1$. This curve appears to fill the entire region when c is very large, as shown in the figure for $c = 1000$.



When c is a fraction, we get a variety of shapes with multiple loops, but always within the same region. For some fractional values, such as $c = 2.359$, the curve again appears to fill the region.



LABORATORY PROJECT Running Circles Around Circles

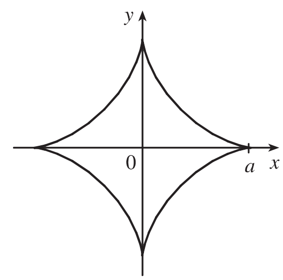


1. The center Q of the smaller circle has coordinates $((a-b)\cos\theta, (a-b)\sin\theta)$. Arc PS on circle C has length $a\theta$ since it is equal in length to arc AS (the smaller circle rolls without slipping against the larger.)

Thus, $\angle PQS = \frac{a}{b}\theta$ and $\angle PQT = \frac{a}{b}\theta - \theta$, so P has coordinates

$$x = (a-b)\cos\theta + b\cos(\angle PQT) = (a-b)\cos\theta + b\cos\left(\frac{a-b}{b}\theta\right)$$

and $$y = (a-b)\sin\theta - b\sin(\angle PQT) = (a-b)\sin\theta - b\sin\left(\frac{a-b}{b}\theta\right).$$

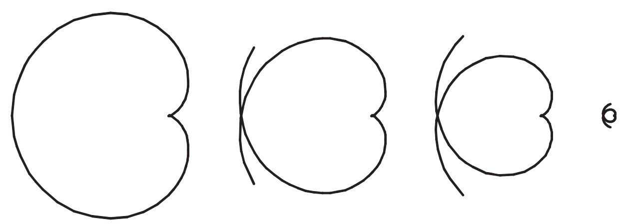


2. With $b = 1$ and a a positive integer greater than 2, we obtain a hypocycloid of a cusps. Shown in the figure is the graph for $a = 4$. Let $a = 4$ and $b = 1$. Using the sum identities to expand $\cos 3\theta$ and $\sin 3\theta$, we obtain

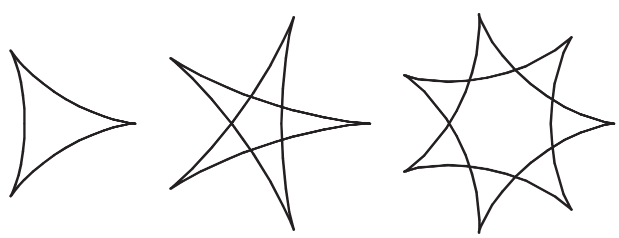
$$x = 3\cos\theta + \cos 3\theta = 3\cos\theta + \left(4\cos^3\theta - 3\cos\theta\right) = 4\cos^3\theta$$

and $$y = 3\sin\theta - \sin 3\theta = 3\sin\theta - \left(3\sin\theta - 4\sin^3\theta\right) = 4\sin^3\theta.$$

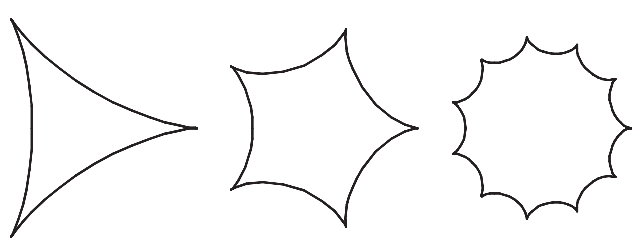
3. The graphs at the right are obtained with $b = 1$ and $a = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, and $\frac{1}{10}$ with $-2\pi \le \theta \le 2\pi$. We conclude that as the denominator d increases, the graph gets smaller, but maintains the basic shape shown.



Letting $d = 2$ and $n = 3$, 5, and 7 with $-2\pi \le \theta \le 2\pi$ gives us the following:



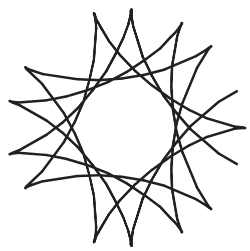
So if d is held constant and n varies, we get a graph with n cusps (assuming n/d is in lowest form). When $n = d + 1$, we obtain a hypocycloid of n cusps. As n increases, we must expand the range of θ in order to get a closed curve. The following graphs have $a = \frac{3}{2}, \frac{5}{4}$, and $\frac{11}{10}$.



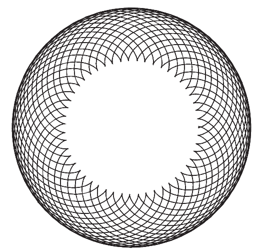
4. If $b = 1$, the equations for the hypocycloid

are $\qquad x = (a-1)\cos\theta + \cos((a-1)\theta) \qquad y = (a-1)\sin\theta - \sin((a-1)\theta)$

which is a hypocycloid of a cusps (from Problem 2). In general, if $a > 1$, we get a figure with cusps on the "outside ring" and if $a < 1$, the cusps are on the "inside ring". In any case, as the values of θ get larger, we get a figure that looks more and more like a washer. If we were to graph the hypocycloid for all values of θ, every point on the washer would eventually be arbitrarily close to a point on the curve.



$a = \sqrt{2}, \quad -10\pi \le \theta \le 10\pi$

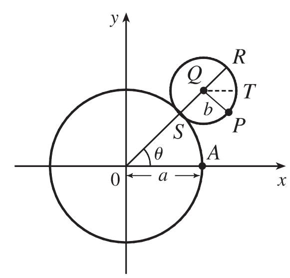


$a = e - 2, \quad 0 \le \theta \le 446$

5. The center Q of the smaller circle has coordinates $((a+b)\cos\theta, (a+b)\sin\theta)$.

Arc PS has length $a\theta$ (as in Problem 1), so that $\angle PQS = \dfrac{a\theta}{b}$, $\angle PQR = \pi - \dfrac{a\theta}{b}$,

and $\angle PQT = \pi - \dfrac{a\theta}{b} - \theta = \pi - \left(\dfrac{a+b}{b}\right)\theta$ since $\angle RQT = \theta$.



Thus, the coordinates of P are

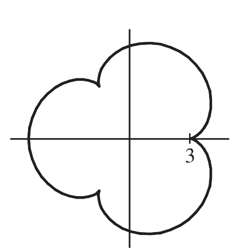
$$x = (a+b)\cos\theta + b\cos\left(\pi - \frac{a+b}{b}\theta\right) = (a+b)\cos\theta - b\cos\left(\frac{a+b}{b}\theta\right)$$

and $$y = (a+b)\sin\theta - b\sin\left(\pi - \frac{a+b}{b}\theta\right) = (a+b)\sin\theta - b\sin\left(\frac{a+b}{b}\theta\right).$$

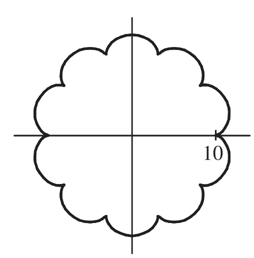
6. Let $b = 1$ and the equations become

$$x = (a+1)\cos\theta - \cos((a+1)\theta) \qquad y = (a+1)\sin\theta - \sin((a+1)\theta)$$

If $a = 1$, we have a cardioid. If a is a positive integer greater than 1, we get the graph of an "a-leafed clover", with cusps that are a units from the origin. (Some of the pairs of figures are not to scale.)

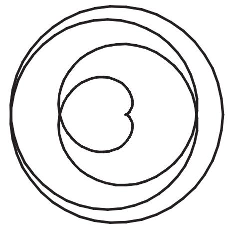


$a = 3, -2\pi \le \theta \le 2\pi$

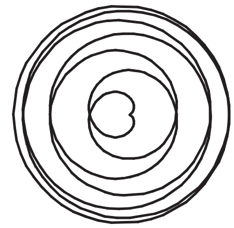


$a = 10, -2\pi \le \theta \le 2\pi$

If $a = n/d$ with $n = 1$, we obtain a figure that does not increase in size and requires $-d\pi \le \theta \le d\pi$ to be a closed curve traced exactly once.

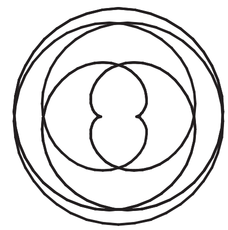


$a = \frac{1}{4}, -4\pi \le \theta \le 4\pi$

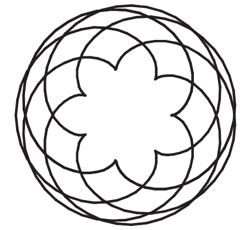


$a = \frac{1}{7}, -7\pi \le \theta \le 7\pi$

Next, we keep d constant and let n vary. As n increases, so does the size of the figure. There is an n-pointed star in the middle.

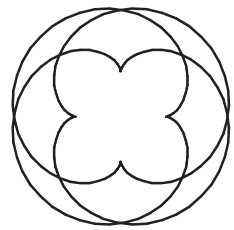


$a = \frac{2}{5}, -5\pi \le \theta \le 5\pi$

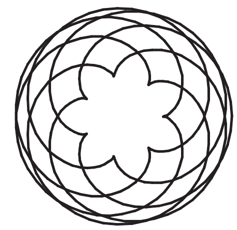


$a = \frac{7}{5}, -5\pi \le \theta \le 5\pi$

Now if $n = d + 1$ we obtain figures similar to the previous ones, but the size of the figure does not increase.

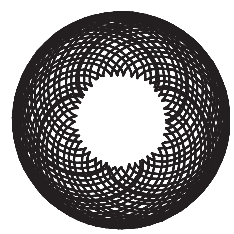


$a = \frac{4}{3}, -3\pi \le \theta \le 3\pi$

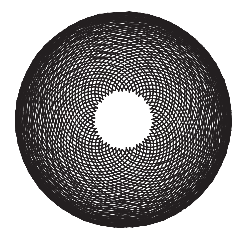


$a = \frac{7}{6}, -6\pi \le \theta \le 6\pi$

If a is irrational, we get washers that increase in size as a increases.



$a = \sqrt{2}, 0 \le \theta \le 200$



$a = e - 2, 0 \le \theta \le 446$

1 Review

CONCEPT CHECK

1. (a) A **function** f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B. The set A is called the **domain** of the function. The **range** of f is the set of all possible values of $f(x)$ as x varies throughout the domain.

(b) If f is a function with domain A, then its **graph** is the set of ordered pairs $\{(x, f(x)) \mid x \in A\}$.

(c) Use the Vertical Line Test on page 17.

2. The four ways to represent a function are: verbally, numerically, visually, and algebraically. An example of each is given below.

Verbally: An assignment of students to chairs in a classroom (a description in words)

Numerically: A tax table that assigns an amount of tax to an income (a table of values)

Visually: A graphical history of the Dow Jones average (a graph)

Algebraically: A relationship between distance, rate, and time: $d = rt$ (an explicit formula)

3. (a) An **even function** f satisfies $f(-x) = f(x)$ for every number x in its domain. It is symmetric with respect to the y-axis.

(b) An **odd function** g satisfies $g(-x) = -g(x)$ for every number x in its domain. It is symmetric with respect to the origin.

4. A function f is called **increasing** on an interval I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in I.

5. A **mathematical model** is a mathematical description (often by means of a function or an equation) of a real-world phenomenon.

6. (a) Linear function: $f(x) = 2x + 1$, $f(x) = ax + b$

(b) Power function: $f(x) = x^2$, $f(x) = x^a$

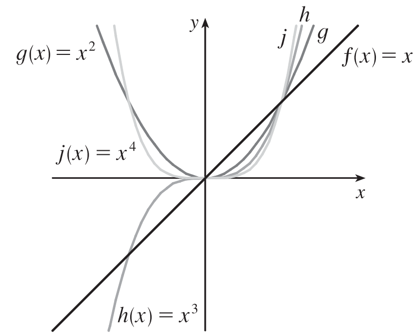
(c) Exponential function: $f(x) = 2^x$, $f(x) = a^x$

(d) Quadratic function: $f(x) = x^2 + x + 1$, $f(x) = ax^2 + bx + c$

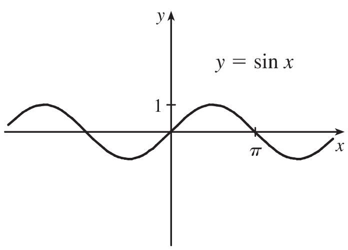
(e) Polynomial of degree 5: $f(x) = x^5 + 2$

(f) Rational function: $f(x) = \dfrac{x}{x+2}$, $f(x) = \dfrac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials

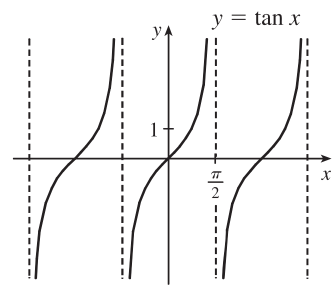
7.



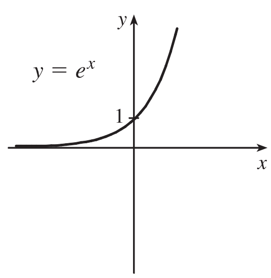
8. (a)



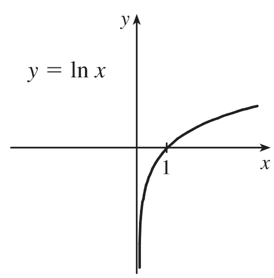
(b)



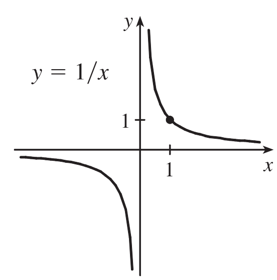
(c)



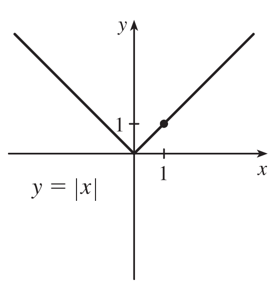
(d)



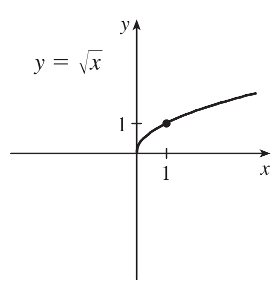
(e)



(f)



(g)



9. (a) The domain of $f + g$ is the intersection of the domain of f and the domain of g; that is, $A \cap B$.

(b) The domain of fg is also $A \cap B$.

(c) The domain of f/g must exclude values of x that make g equal to 0; that is, $\{x \in A \cap B \mid g(x) \neq 0\}$.

10. Given two functions f and g, the **composite** function $f \circ g$ is defined by $(f \circ g)(x) = f(g(x))$. The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f.

11. (a) If the graph of f is shifted 2 units upward, its equation becomes $y = f(x) + 2$.

(b) If the graph of f is shifted 2 units downward, its equation becomes $y = f(x) - 2$.

(c) If the graph of f is shifted 2 units to the right, its equation becomes $y = f(x - 2)$.

(d) If the graph of f is shifted 2 units to the left, its equation becomes $y = f(x + 2)$.

(e) If the graph of f is reflected about the x-axis, its equation becomes $y = -f(x)$.

(f) If the graph of f is reflected about the y-axis, its equation becomes $y = f(-x)$.

(g) If the graph of f is stretched vertically by a factor of 2, its equation becomes $y = 2f(x)$.

(h) If the graph of f is shrunk vertically by a factor of 2, its equation becomes $y = \frac{1}{2}f(x)$.

(i) If the graph of f is stretched horizontally by a factor of 2, its equation becomes $y = f\left(\frac{1}{2}x\right)$.

(j) If the graph of f is shrunk horizontally by a factor of 2, its equation becomes $y = f(2x)$.

12. (a) A function f is called a *one-to-one function* if it never takes on the same value twice; that is, if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$. (Or, f is 1-1 if each output corresponds to only one input.)

Use the Horizontal Line Test: A function is one-to-one if and only if no horizontal line intersects its graph more than once.

(b) If f is a one-to-one function with domain A and range B, then its *inverse function* f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y$$

for any y in B. The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.

13. (a) A parametric curve is a set of points of the form $(x, y) = (f(t), g(t))$, where f and g are continuous functions of a variable t.

(b) Sketching a parametric curve, like sketching the graph of a function, is difficult to do in general. We can plot points on the curve by finding $f(t)$ and $g(t)$ for various values of t, either by hand or with a calculator or computer. Sometimes, when f and g are given by formulas, we can eliminate t from the equations $x = f(t)$ and $y = g(t)$ to get a Cartesian equation relating x and y. It may be easier to graph that equation than to work with the original formulas for x and y in terms of t.

(c) See the margin note on page 72.

TRUE-FALSE QUIZ

1. False. Let $f(x) = x^2$, $s = -1$, and $t = 1$. Then $f(s + t) = (-1 + 1)^2 = 0^2 = 0$, but $f(s) + f(t) = (-1)^2 + 1^2 = 2 \neq 0 = f(s + t)$.

2. False. Let $f(x) = x^2$. Then $f(-2) = 4 = f(2)$, but $-2 \neq 2$.

3. False. Let $f(x) = x^2$. Then $f(3x) = (3x)^2 = 9x^2$ and $3f(x) = 3x^2$. So $f(3x) \neq 3f(x)$.

4. True. If $x_1 < x_2$ and f is a decreasing function, then the y-values get smaller as we move from left to right. Thus, $f(x_1) > f(x_2)$.

5. True. See the Vertical Line Test.

6. False. Let $f(x) = x^2$ and $g(x) = 2x$. Then $(f \circ g)(x) = f(g(x)) = f(2x) = (2x)^2 = 4x^2$ and $(g \circ f)(x) = g(f(x)) = g(x^2) = 2x^2$. So $f \circ g \neq g \circ f$.

7. False. Let $f(x) = x^3$. Then f is one-to-one and $f^{-1}(x) = \sqrt[3]{x}$. But $1/f(x) = 1/x^3$, which is not equal to $f^{-1}(x)$.

8. True. We can divide by e^x since $e^x \neq 0$ for every x.

9. True. The function $\ln x$ is an increasing function on $(0, \infty)$.

10. False. Let $x = e$. Then $(\ln x)^6 = (\ln e)^6 = 1^6 = 1$, but $6 \ln x = 6 \ln e = 6 \cdot 1 = 6 \neq 1 = (\ln x)^6$.

11. False. Let $x = e^2$ and $a = e$. Then $\dfrac{\ln x}{\ln a} = \dfrac{\ln e^2}{\ln e} = \dfrac{2 \ln e}{\ln e} = 2$ and $\ln \dfrac{x}{a} = \ln \dfrac{e^2}{e} = \ln e = 1$, so in general the statement is false. What *is* true, however, is that $\ln \dfrac{x}{a} = \ln x - \ln a$.

12. False. The first pair of equations gives the portion of the parabola $y = x^2$ with $x \geq 0$, whereas the second pair of equations traces out the whole parabola $y = x^2$.

EXERCISES

1. (a) When $x = 2$, $y \approx 2.7$. Thus, $f(2) \approx 2.7$.

(b) $f(x) = 3 \;\Rightarrow\; x \approx 2.3, 5.6$

(c) The domain of f is $-6 \leq x \leq 6$, or $[-6, 6]$.

(d) The range of f is $-4 \leq y \leq 4$, or $[-4, 4]$.

(e) f is increasing on $[-4, 4]$, that is, on $-4 \leq x \leq 4$.

(f) f is not one-to-one since it fails the Horizontal Line Test.

(g) f is odd since its graph is symmetric about the origin.

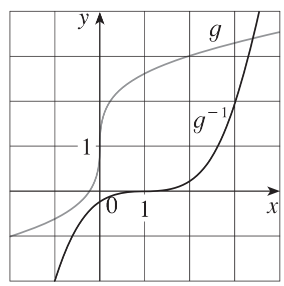
2. (a) When $x = 2$, $y = 3$. Thus, $g(2) = 3$.

(b) g is one-to-one because it passes the Horizontal Line Test.

(c) When $y = 2$, $x \approx 0.2$. So $g^{-1}(2) \approx 0.2$.

(d) The range of g is $[-1, 3.5]$, which is the same as the domain of g^{-1}.

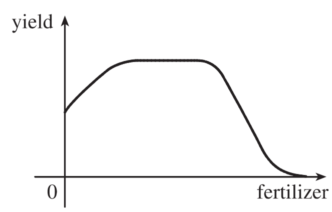
(e) We reflect the graph of g through the line $y = x$ to obtain the graph of g^{-1}.



3. $f(x) = x^2 - 2x + 3$, so $f(a + h) = (a + h)^2 - 2(a + h) + 3 = a^2 + 2ah + h^2 - 2a - 2h + 3$, and

$$\frac{f(a+h) - f(a)}{h} = \frac{(a^2 + 2ah + h^2 - 2a - 2h + 3) - (a^2 - 2a + 3)}{h} = \frac{h(2a + h - 2)}{h} = 2a + h - 2.$$

4. There will be some yield with no fertilizer, increasing yields with increasing fertilizer use, a leveling-off of yields at some point, and disaster with too much fertilizer use.



5. $f(x) = 2/(3x - 1)$. Domain: $3x - 1 \neq 0 \;\Rightarrow\; 3x \neq 1 \;\Rightarrow\; x \neq \frac{1}{3}$. $D = \left(-\infty, \frac{1}{3}\right) \cup \left(\frac{1}{3}, \infty\right)$

Range: all reals except 0 ($y = 0$ is the horizontal asymptote for f.) $R = (-\infty, 0) \cup (0, \infty)$

6. $g(x) = \sqrt{16 - x^4}$. Domain: $16 - x^4 \geq 0 \;\Rightarrow\; x^4 \leq 16 \;\Rightarrow\; |x| \leq \sqrt[4]{16} \;\Rightarrow\; |x| \leq 2$. $D = [-2, 2]$

Range: $y \geq 0$ and $y \leq \sqrt{16} \;\Rightarrow\; 0 \leq y \leq 4$. $R = [0, 4]$

7. $h(x) = \ln(x+6)$. Domain: $x + 6 > 0 \Rightarrow x > -6$. $D = (-6, \infty)$

Range: $x + 6 > 0$, so $\ln(x+6)$ takes on all real numbers and, hence, the range is $\mathbb{R}$.

$R = (-\infty, \infty)$

8. $y = F(t) = 3 + \cos 2t$. Domain: $\mathbb{R}$. $D = (-\infty, \infty)$

Range: $-1 \le \cos 2t \le 1 \Rightarrow 2 \le 3 + \cos 2t \le 4 \Rightarrow 2 \le y \le 4$. $R = [2, 4]$

9. (a) To obtain the graph of $y = f(x) + 8$, we shift the graph of $y = f(x)$ up 8 units.

(b) To obtain the graph of $y = f(x+8)$, we shift the graph of $y = f(x)$ left 8 units.

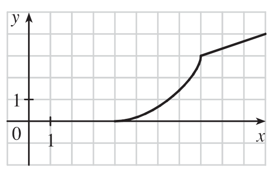
(c) To obtain the graph of $y = 1 + 2f(x)$, we stretch the graph of $y = f(x)$ vertically by a factor of 2, and then shift the resulting graph 1 unit upward.

(d) To obtain the graph of $y = f(x-2) - 2$, we shift the graph of $y = f(x)$ right 2 units (for the "-2" inside the parentheses), and then shift the resulting graph 2 units downward.

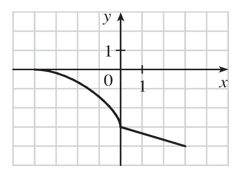
(e) To obtain the graph of $y = -f(x)$, we reflect the graph of $y = f(x)$ about the x-axis.

(f) To obtain the graph of $y = f^{-1}(x)$, we reflect the graph of $y = f(x)$ about the line $y = x$ (assuming f is one–to-one).

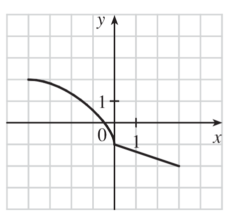
10. (a) To obtain the graph of $y = f(x-8)$, we shift the graph of $y = f(x)$ right 8 units.



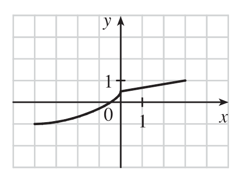
(b) To obtain the graph of $y = -f(x)$, we reflect the graph of $y = f(x)$ about the x-axis.



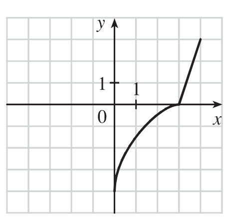
(c) To obtain the graph of $y = 2 - f(x)$, we reflect the graph of $y = f(x)$ about the x-axis, and then shift the resulting graph 2 units upward.



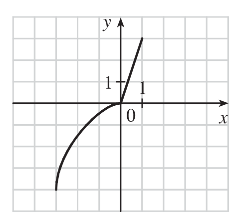
(d) To obtain the graph of $y = \frac{1}{2}f(x) - 1$, we shrink the graph of $y = f(x)$ by a factor of 2, and then shift the resulting graph 1 unit downward.



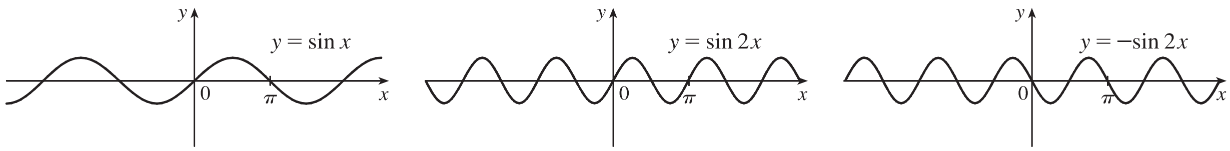
(e) To obtain the graph of $y = f^{-1}(x)$, we reflect the graph of $y = f(x)$ about the line $y = x$.



(f) To obtain the graph of $y = f^{-1}(x+3)$, we reflect the graph of $y = f(x)$ about the line $y = x$ [see part (e)], and then shift the resulting graph left 3 units.

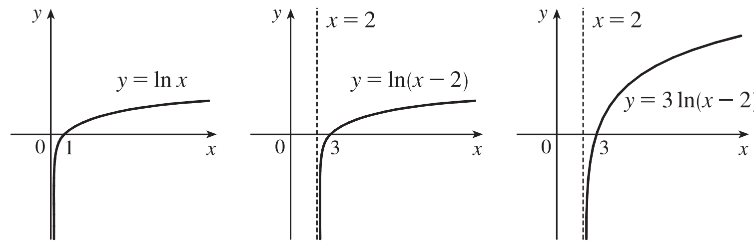


11. $y = -\sin 2x$: Start with the graph of $y = \sin x$, compress horizontally by a factor of 2, and reflect about the x-axis.



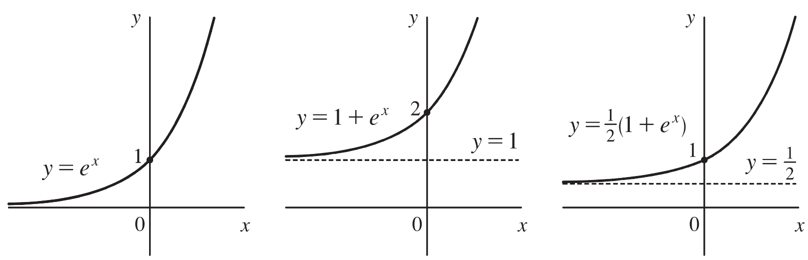
12. $y = 3\ln(x-2)$:

Start with the graph of $y = \ln x$, shift 2 units to the right, and stretch vertically by a factor of 3.



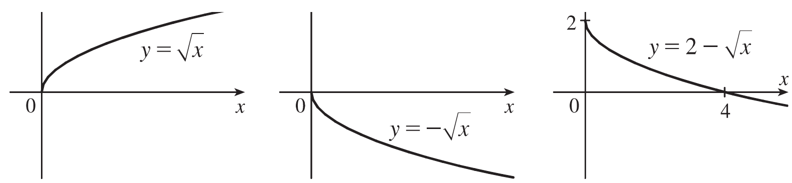
13. $y = \frac{1}{2}(1 + e^x)$:

Start with the graph of $y = e^x$, shift 1 unit upward, and compress vertically by a factor of 2.



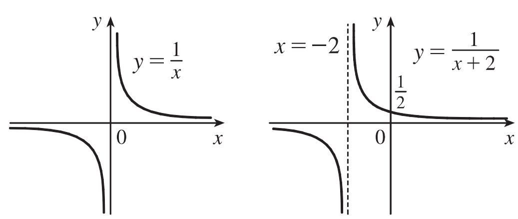
14. $y = 2 - \sqrt{x}$:

Start with the graph of $y = \sqrt{x}$, reflect about the x-axis, and shift 2 units upward.



15. $f(x) = \dfrac{1}{x+2}$:

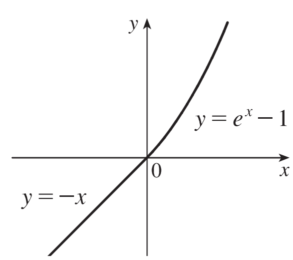
Start with the graph of $f(x) = 1/x$ and shift 2 units to the left.



16. $f(x) = \begin{cases} -x & \text{if } x < 0 \\ e^x - 1 & \text{if } x \geq 0 \end{cases}$

On $(-\infty, 0)$, graph $y = -x$ (the line with slope -1 and y-intercept 0) with open endpoint $(0, 0)$.

On $[0, \infty)$, graph $y = e^x - 1$ (the graph of $y = e^x$ shifted 1 unit downward) with closed endpoint $(0, 0)$.



17. (a) The terms of f are a mixture of odd and even powers of x, so f is neither even nor odd.

(b) The terms of f are all odd powers of x, so f is odd.

(c) $f(-x) = e^{-(-x)^2} = e^{-x^2} = f(x)$, so f is even.

(d) $f(-x) = 1 + \sin(-x) = 1 - \sin x$. Now $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$, so f is neither even nor odd.

18. For the line segment from $(-2, 2)$ to $(-1, 0)$, the slope is $\dfrac{0-2}{-1+2} = -2$, and an equation is $y - 0 = -2(x+1)$ or, equivalently, $y = -2x - 2$. The circle has equation $x^2 + y^2 = 1$; the top half has equation $y = \sqrt{1-x^2}$ (we have solved for positive y). Thus, $f(x) = \begin{cases} -2x-2 & \text{if } -2 \le x \le -1 \\ \sqrt{1-x^2} & \text{if } -1 < x \le 1 \end{cases}$

19. $f(x) = \ln x, \quad D = (0, \infty); \quad g(x) = x^2 - 9, \quad D = \mathbb{R}$.

(a) $(f \circ g)(x) = f(g(x)) = f(x^2 - 9) = \ln(x^2 - 9)$.

Domain: $x^2 - 9 > 0 \;\Rightarrow\; x^2 > 9 \;\Rightarrow\; |x| > 3 \;\Rightarrow\; x \in (-\infty, -3) \cup (3, \infty)$

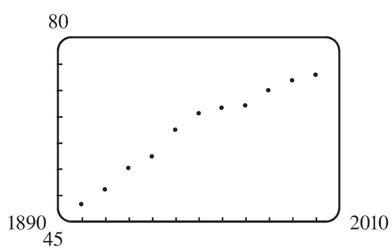
(b) $(g \circ f)(x) = g(f(x)) = g(\ln x) = (\ln x)^2 - 9$. Domain: $x > 0$, or $(0, \infty)$

(c) $(f \circ f)(x) = f(f(x)) = f(\ln x) = \ln(\ln x)$. Domain: $\ln x > 0 \;\Rightarrow\; x > e^0 = 1$, or $(1, \infty)$

(d) $(g \circ g)(x) = g(g(x)) = g(x^2 - 9) = (x^2 - 9)^2 - 9$. Domain: $x \in \mathbb{R}$, or $(-\infty, \infty)$

20. Let $h(x) = x + \sqrt{x}$, $g(x) = \sqrt{x}$, and $f(x) = 1/x$. Then $(f \circ g \circ h)(x) = \dfrac{1}{\sqrt{x + \sqrt{x}}} = F(x)$.

21.

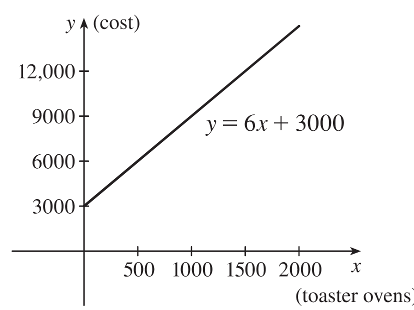


Many models appear to be plausible. Your choice depends on whether you think medical advances will keep increasing life expectancy, or if there is bound to be a natural leveling-off of life expectancy. A linear model, $y = 0.2493x - 423.4818$, gives us an estimate of 77.6 years for the year 2010.

22. (a) Let x denote the number of toaster ovens produced in one week and y the associated cost. Using the points $(1000, 9000)$ and $(1500, 12{,}000)$, we get an equation of a line:

$$y - 9000 = \frac{12{,}000 - 9000}{1500 - 1000}(x - 1000) \quad\Rightarrow$$

$$y = 6(x - 1000) + 9000 \quad\Rightarrow\quad y = 6x + 3000.$$



(b) The slope of 6 means that each additional toaster oven produced adds \$6 to the weekly production cost.

(c) The y-intercept of 3000 represents the overhead cost—the cost incurred without producing anything.

23. We need to know the value of x such that $f(x) = 2x + \ln x = 2$. Since $x = 1$ gives us $y = 2$, $f^{-1}(2) = 1$.

24. $y = \dfrac{x+1}{2x+1}$. Interchanging x and y gives us $x = \dfrac{y+1}{2y+1} \;\Rightarrow\; 2xy + x = y + 1 \;\Rightarrow\; 2xy - y = 1 - x \;\Rightarrow$

$y(2x - 1) = 1 - x \;\Rightarrow\; y = \dfrac{1-x}{2x-1} = f^{-1}(x)$.

25. (a) $e^{2\ln 3} = (e^{\ln 3})^2 = 3^2 = 9$

(b) $\log_{10} 25 + \log_{10} 4 = \log_{10}(25 \cdot 4) = \log_{10} 100 = \log_{10} 10^2 = 2$

26. (a) $e^x = 5 \;\Rightarrow\; x = \ln 5$

(b) $\ln x = 2 \;\Rightarrow\; x = e^2$

(c) $e^{e^x} = 2 \;\Rightarrow\; e^x = \ln 2 \;\Rightarrow\; x = \ln(\ln 2)$

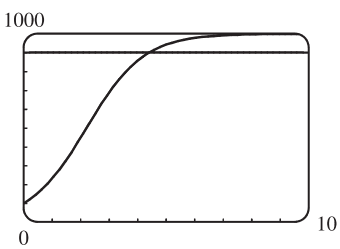
27. (a) After 4 days, $\frac{1}{2}$ gram remains; after 8 days, $\frac{1}{4}$ g; after 12 days, $\frac{1}{8}$ g; after 16 days, $\frac{1}{16}$ g.

(b) $m(4) = \dfrac{1}{2}$, $m(8) = \dfrac{1}{2^2}$, $m(12) = \dfrac{1}{2^3}$, $m(16) = \dfrac{1}{2^4}$. From the pattern, we see that $m(t) = \dfrac{1}{2^{t/4}}$, or $2^{-t/4}$.

(c) $m = 2^{-t/4} \;\Rightarrow\; \log_2 m = -t/4 \;\Rightarrow\; t = -4\log_2 m$; this is the time elapsed when there are m grams of ^{100}Pd.

(d) $m = 0.01 \;\Rightarrow\; t = -4\log_2 0.01 = -4\left(\dfrac{\ln 0.01}{\ln 2}\right) \approx 26.6$ days

28. (a)



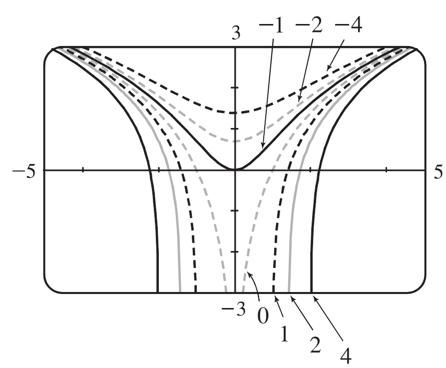
The population would reach 900 in about 4.4 years.

(b) $P = \dfrac{100{,}000}{100 + 900e^{-t}} \;\Rightarrow\; 100P + 900Pe^{-t} = 100{,}000 \Rightarrow\; 900Pe^{-t} = 100{,}000 - 100P \;\Rightarrow$

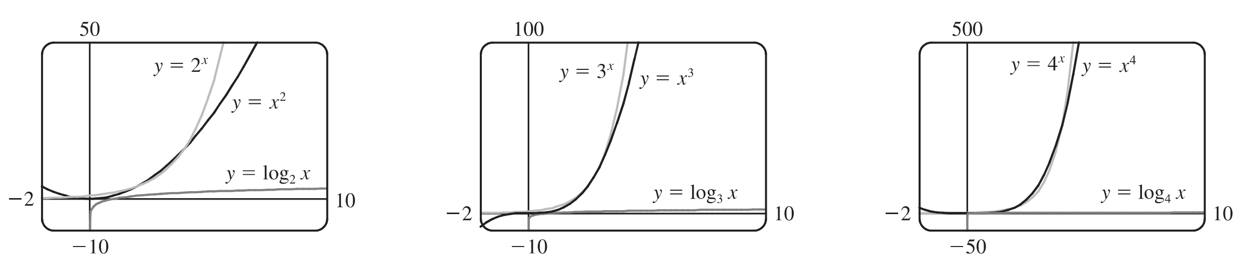
$e^{-t} = \dfrac{100{,}000 - 100P}{900P} \;\Rightarrow\; -t = \ln\left(\dfrac{1000 - P}{9P}\right) \;\Rightarrow\; t = -\ln\left(\dfrac{1000 - P}{9P}\right)$, or $\ln\left(\dfrac{9P}{1000 - P}\right)$; this is the time required for the population to reach a given number P.

(c) $P = 900 \;\Rightarrow\; t = \ln\left(\dfrac{9 \cdot 900}{1000 - 900}\right) = \ln 81 \approx 4.4$ years, as in part (a).

29. $f(x) = \ln(x^2 - c)$. If $c < 0$, the domain of f is $\mathbb{R}$. If $c = 0$, the domain of f is $(-\infty, 0) \cup (0, \infty)$. If $c > 0$, the domain of f is $(-\infty, -\sqrt{c}) \cup (\sqrt{c}, \infty)$. As c increases, the dip at $x = 0$ becomes deeper. For $c \geq 0$, the graph has asymptotes at $x = \pm\sqrt{c}$.

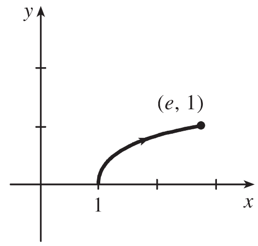


30.



For large values of x, $y = a^x$ has the largest y-values and $y = \log_a x$ has the smallest y-values. This makes sense because they are inverses of each other.

31. (a)

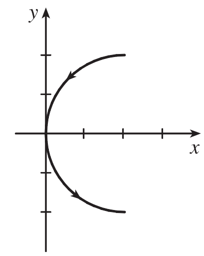


$0 \le t \le 1 \quad \Rightarrow \quad 0 \le y \le 1$ and $1 \le x \le e$.

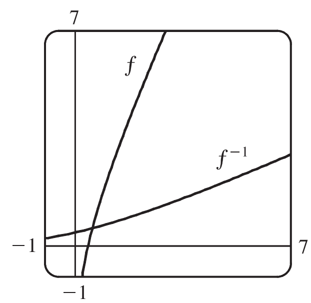
(b) $x = e^t \quad \Rightarrow \quad t = \ln x$; $y = \sqrt{t}$ so $y = \sqrt{\ln x}$.

32. (a) $(x-2)^2 + y^2 = 4 \quad \Rightarrow \quad \dfrac{(x-2)^2}{4} + \dfrac{y^2}{4} = 1$. Let $\dfrac{(x-2)^2}{4} = \sin^2 t$ and $\dfrac{y^2}{4} = \cos^2 t$ (since $\sin^2 t + \cos^2 t = 1$). Solving for x and y gives $x = 2 \pm 2\sin t$ and $y = \pm 2\cos t$. We want to move from $(2, 2)$ to $(2, -2)$ and pass through $(0, 0)$. When $t = 0$, we want $y = 2$, so choose $y = 2\cos t$. When $t = \frac{\pi}{2}$, we want $x = 0$, so choose $x = 2 - 2\sin t$. Thus, parametric equations are $x = 2 - 2\sin t$, $y = 2\cos t$, $0 \le t \le \pi$. Another possibility is $x = 2 + 2\cos t$, $y = 2\sin t$, $\frac{\pi}{2} \le t \le \frac{3\pi}{2}$.

(b)



33. We sketch $x = t$, $y = 2t + \ln t$ (the function) and $x = 2t + \ln t$, $y = t$ (its inverse) for $t > 0$.



34. (a) Let θ be the angle of inclination of segment OP. Then $|OB| = \dfrac{2a}{\cos\theta}$.

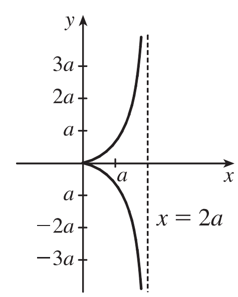
Let $C = (2a, 0)$. Then by use of right triangle OAC we see that $|OA| = 2a\cos\theta$.

Now

$$\begin{aligned} |OP| &= |AB| = |OB| - |OA| \\ &= 2a\left(\frac{1}{\cos\theta} - \cos\theta\right) = 2a\,\frac{1-\cos^2\theta}{\cos\theta} = 2a\,\frac{\sin^2\theta}{\cos\theta} = 2a\sin\theta\,\tan\theta \end{aligned}$$

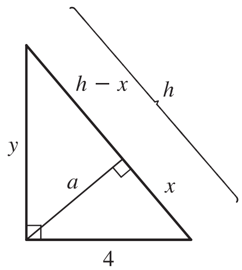
So P has coordinates $x = 2a\sin\theta\,\tan\theta \cdot \cos\theta = 2a\sin^2\theta$ and $y = 2a\sin\theta\,\tan\theta \cdot \sin\theta = 2a\sin^2\theta\,\tan\theta$.

(b)



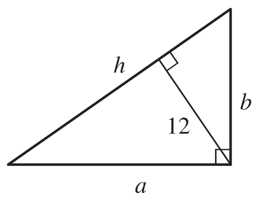
PRINCIPLES OF PROBLEM SOLVING

1.



By using the area formula for a triangle, $\frac{1}{2}$ (base) (height), in two ways, we see that $\frac{1}{2}(4)(y) = \frac{1}{2}(h)(a)$, so $a = \dfrac{4y}{h}$. Since $4^2 + y^2 = h^2$, $y = \sqrt{h^2 - 16}$, and $a = \dfrac{4\sqrt{h^2-16}}{h}$.

2.



Refer to Example 1, where we obtained $h = \dfrac{P^2 - 100}{2P}$. The 100 came from 4 times the area of the triangle. In this case, the area of the triangle is $\frac{1}{2}(h)(12) = 6h$. Thus, $h = \dfrac{P^2 - 4(6h)}{2P} \quad \Rightarrow \quad 2Ph = P^2 - 24h \quad \Rightarrow$

$2Ph + 24h = P^2 \quad \Rightarrow \quad h(2P + 24) = P^2 \quad \Rightarrow \quad h = \dfrac{P^2}{2P+24}$.

3. $|2x-1| = \begin{cases} 2x-1 & \text{if } x \geq \frac{1}{2} \\ 1-2x & \text{if } x < \frac{1}{2} \end{cases}$ and $|x+5| = \begin{cases} x+5 & \text{if } x \geq -5 \\ -x-5 & \text{if } x < -5 \end{cases}$

Therefore, we consider the three cases $x < -5$, $-5 \leq x < \frac{1}{2}$, and $x \geq \frac{1}{2}$.

If $x < -5$, we must have $1 - 2x - (-x-5) = 3 \quad \Leftrightarrow \quad x = 3$, which is false, since we are considering $x < -5$.

If $-5 \leq x < \frac{1}{2}$, we must have $1 - 2x - (x+5) = 3 \quad \Leftrightarrow \quad x = -\frac{7}{3}$.

If $x \geq \frac{1}{2}$, we must have $2x - 1 - (x+5) = 3 \quad \Leftrightarrow \quad x = 9$.

So the two solutions of the equation are $x = -\frac{7}{3}$ and $x = 9$.

4. $|x-1| = \begin{cases} x-1 & \text{if } x \geq 1 \\ 1-x & \text{if } x < 1 \end{cases}$ and $|x-3| = \begin{cases} x-3 & \text{if } x \geq 3 \\ 3-x & \text{if } x < 3 \end{cases}$

Therefore, we consider the three cases $x < 1$, $1 \leq x < 3$, and $x \geq 3$.

If $x < 1$, we must have $1 - x - (3 - x) \geq 5 \quad \Leftrightarrow \quad 0 \geq 7$, which is false.

If $1 \leq x < 3$, we must have $x - 1 - (3 - x) \geq 5 \quad \Leftrightarrow \quad x \geq \frac{9}{2}$, which is false because $x < 3$.

If $x \geq 3$, we must have $x - 1 - (x - 3) \geq 5 \quad \Leftrightarrow \quad 2 \geq 5$, which is false.

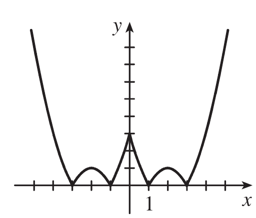
All three cases lead to falsehoods, so the inequality has no solution.

5. $f(x) = \left|x^2 - 4|x| + 3\right|$. If $x \geq 0$, then $f(x) = \left|x^2 - 4x + 3\right| = |(x-1)(x-3)|$.

Case (i): If $0 < x \leq 1$, then $f(x) = x^2 - 4x + 3$.

Case (ii): If $1 < x \leq 3$, then $f(x) = -(x^2 - 4x + 3) = -x^2 + 4x - 3$.

Case (iii): If $x > 3$, then $f(x) = x^2 - 4x + 3$.

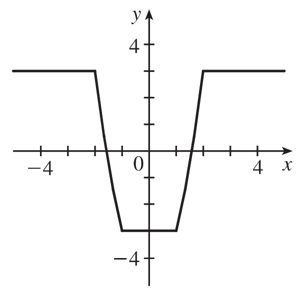


This enables us to sketch the graph for $x \geq 0$. Then we use the fact that f is an even function to reflect this part of the graph about the y-axis to obtain the entire graph. Or, we could consider also the cases $x < -3$, $-3 \leq x < -1$, and $-1 \leq x < 0$.

6. $g(x) = |x^2 - 1| - |x^2 - 4|$.

$$|x^2 - 1| = \begin{cases} x^2 - 1 & \text{if } |x| \geq 1 \\ 1 - x^2 & \text{if } |x| < 1 \end{cases} \text{ and } |x^2 - 4| = \begin{cases} x^2 - 4 & \text{if } |x| \geq 2 \\ 4 - x^2 & \text{if } |x| < 2 \end{cases}$$

So for $0 \leq |x| < 1, g(x) = 1 - x^2 - (4 - x^2) = -3$, for $1 \leq |x| < 2, g(x) = x^2 - 1 - (4 - x^2) = 2x^2 - 5$, and for $|x| \geq 2, g(x) = x^2 - 1 - (x^2 - 4) = 3$.



7. Remember that $|a| = a$ if $a \geq 0$ and that $|a| = -a$ if $a < 0$. Thus,

$$x + |x| = \begin{cases} 2x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \quad \text{and} \quad y + |y| = \begin{cases} 2y & \text{if } y \geq 0 \\ 0 & \text{if } y < 0 \end{cases}$$

We will consider the equation $x + |x| = y + |y|$ in four cases.

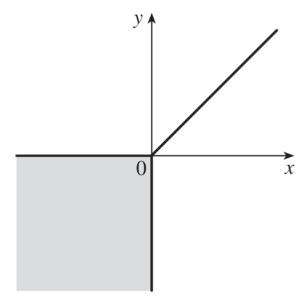
(1) $x \geq 0, y \geq 0$	(2) $x \geq 0, y < 0$	(3) $x < 0, y \geq 0$	(4) $x < 0, y < 0$
$2x = 2y$	$2x = 0$	$0 = 2y$	$0 = 0$
$x = y$	$x = 0$	$0 = y$	

Case 1 gives us the line $y = x$ with nonnegative x and y.

Case 2 gives us the portion of the y-axis with y negative.

Case 3 gives us the portion of the x-axis with x negative.

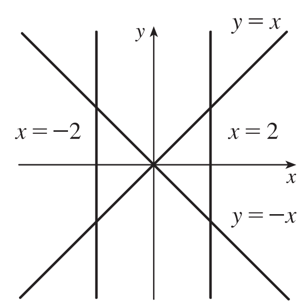
Case 4 gives us the entire third quadrant.



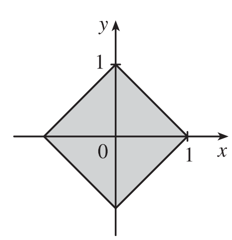
8. $x^4 - 4x^2 - x^2y^2 + 4y^2 = 0 \quad \Leftrightarrow \quad x^2(x^2 - 4) - y^2(x^2 - 4) = 0 \quad \Leftrightarrow$

$(x^2 - y^2)(x^2 - 4) = 0 \quad \Leftrightarrow \quad (x + y)(x - y)(x + 2)(x - 2) = 0$.

So the graph of the equation consists of the graphs of the four lines $y = -x$, $y = x$, $x = -2$, and $x = 2$.



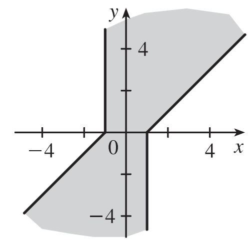
9. $|x| + |y| \leq 1$. The boundary of the region has equation $|x| + |y| = 1$. In quadrants I, II, III, and IV, this becomes the lines $x + y = 1$, $-x + y = 1$, $-x - y = 1$, and $x - y = 1$ respectively.



10. $|x - y| + |x| - |y| \leq 2$

Case (i):	$x > y > 0$	$\Leftrightarrow$	$x - y + x - y \leq 2$	$\Leftrightarrow$	$x - y \leq 1$	$\Leftrightarrow$	$y \geq x - 1$
Case (ii):	$y > x > 0$	$\Leftrightarrow$	$y - x + x - y \leq 2$	$\Leftrightarrow$	$0 \leq 2$ (true)		
Case (iii):	$x > 0$ and $y < 0$	$\Leftrightarrow$	$x - y + x + y \leq 2$	$\Leftrightarrow$	$2x \leq 2$	$\Leftrightarrow$	$x \leq 1$
Case (iv):	$x < 0$ and $y > 0$	$\Leftrightarrow$	$y - x - x - y \leq 2$	$\Leftrightarrow$	$-2x \leq 2$	$\Leftrightarrow$	$x \geq -1$
Case (v):	$y < x < 0$	$\Leftrightarrow$	$x - y - x + y \leq 2$	$\Leftrightarrow$	$0 \leq 2$ (true)		
Case (vi):	$x < y < 0$	$\Leftrightarrow$	$y - x - x + y \leq 2$	$\Leftrightarrow$	$y - x \leq 1$	$\Leftrightarrow$	$y \leq x + 1$

Note: Instead of considering cases (iv), (v), and (vi), we could have noted that the region is unchanged if x and y are replaced by $-x$ and $-y$, so the region is symmetric about the origin. Therefore, we need only draw cases (i), (ii), and (iii), and rotate through $180°$ about the origin.



11. $(\log_2 3)(\log_3 4)(\log_4 5)\cdots(\log_{31} 32) = \left(\dfrac{\ln 3}{\ln 2}\right)\left(\dfrac{\ln 4}{\ln 3}\right)\left(\dfrac{\ln 5}{\ln 4}\right)\cdots\left(\dfrac{\ln 32}{\ln 31}\right) = \dfrac{\ln 32}{\ln 2} = \dfrac{\ln 2^5}{\ln 2} = \dfrac{5\ln 2}{\ln 2} = 5$

12. (a) $f(-x) = \ln\left(-x + \sqrt{(-x)^2+1}\right) = \ln\left(-x + \sqrt{x^2+1}\cdot\dfrac{-x-\sqrt{x^2+1}}{-x-\sqrt{x^2+1}}\right)$

$$= \ln\left(\frac{x^2-(x^2+1)}{-x-\sqrt{x^2+1}}\right) = \ln\left(\frac{-1}{-x-\sqrt{x^2+1}}\right) = \ln\left(\frac{1}{x+\sqrt{x^2+1}}\right)$$

$$= \ln 1 - \ln\left(x+\sqrt{x^2+1}\right) = -\ln\left(x+\sqrt{x^2-1}\right) = -f(x)$$

(b) $y = \ln\left(x+\sqrt{x^2+1}\right)$. Interchanging x and y, we get $x = \ln\left(y+\sqrt{y^2+1}\right) \Rightarrow e^x = y + \sqrt{y^2+1} \Rightarrow$ $e^x - y = \sqrt{y^2+1} \Rightarrow e^{2x} - 2ye^x + y^2 = y^2 + 1 \Rightarrow e^{2x} - 1 = 2ye^x \Rightarrow y = \dfrac{e^{2x}-1}{2e^x} = f^{-1}(-x)$

13. $\ln\left(x^2-2x-2\right) \le 0 \Rightarrow x^2-2x-2 \le e^0 = 1 \Rightarrow x^2-2x-3 \le 0 \Rightarrow (x-3)(x+1) \le 0 \Rightarrow x \in [-1, 3]$.

Since the argument must be positive, $x^2-2x-2 > 0 \Rightarrow \left[x-\left(1-\sqrt{3}\right)\right]\left[x-\left(1+\sqrt{3}\right)\right] > 0 \Rightarrow$

$x \in \left(-\infty, 1-\sqrt{3}\right) \cup \left(1+\sqrt{3}, \infty\right)$. The intersection of these intervals is $\left[-1, 1-\sqrt{3}\right) \cup \left(1+\sqrt{3}, 3\right]$.

14. Assume that $\log_2 5$ is rational. Then $\log_2 5 = m/n$ for natural numbers m and n. Changing to exponential form gives us $2^{m/n} = 5$ and then raising both sides to the nth power gives $2^m = 5^n$. But 2^m is even and 5^n is odd. We have arrived at a contradiction, so we conclude that our hypothesis, that $\log_2 5$ is rational, is false. Thus, $\log_2 5$ is irrational.

15. Let d be the distance traveled on each half of the trip. Let t_1 and t_2 be the times taken for the first and second halves of the trip. For the first half of the trip we have $t_1 = d/30$ and for the second half we have $t_2 = d/60$. Thus, the average speed for the entire trip is $\dfrac{\text{total distance}}{\text{total time}} = \dfrac{2d}{t_1+t_2} = \dfrac{2d}{\dfrac{d}{30}+\dfrac{d}{60}}\cdot\dfrac{60}{60} = \dfrac{120d}{2d+d} = \dfrac{120d}{3d} = 40$. The average speed for the entire trip is 40 mi/h.

16. Let $f = \sin$, $g = x$, and $h = x$. Then the left-hand side of the equation is $f\circ(g+h) = \sin(x+x) = \sin 2x = 2\sin x\cos x$; and the right-hand side is $f\circ g + f\circ h = \sin x + \sin x = 2\sin x$. The two sides are not equal, so the given statement is false.

17. Let S_n be the statement that $7^n - 1$ is divisible by 6.

- S_1 is true because $7^1 - 1 = 6$ is divisible by 6.
- Assume S_k is true, that is, $7^k - 1$ is divisible by 6. In other words, $7^k - 1 = 6m$ for some positive integer m. Then $7^{k+1} - 1 = 7^k\cdot 7 - 1 = (6m+1)\cdot 7 - 1 = 42m + 6 = 6(7m+1)$, which is divisible by 6, so S_{k+1} is true.
- Therefore, by mathematical induction, $7^n - 1$ is divisible by 6 for every positive integer n.

18. Let S_n be the statement that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

- S_1 is true because $[2(1) - 1] = 1 = 1^2$.
- Assume S_k is true, that is, $1 + 3 + 5 + \cdots + (2k - 1) = k^2$. Then

  $$1 + 3 + 5 + \cdots + (2k - 1) + [2(k + 1) - 1] = 1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) = k^2 + (2k + 1) = (k + 1)^2$$

  which shows that S_{k+1} is true.
- Therefore, by mathematical induction, $1 + 3 + 5 + \cdots + (2n - 1) = n^2$ for every positive integer n.

19. $f_0(x) = x^2$ and $f_{n+1}(x) = f_0(f_n(x))$ for $n = 0, 1, 2, \ldots$.

$f_1(x) = f_0(f_0(x)) = f_0\left(x^2\right) = \left(x^2\right)^2 = x^4$, $f_2(x) = f_0(f_1(x)) = f_0(x^4) = (x^4)^2 = x^8$,

$f_3(x) = f_0(f_2(x)) = f_0(x^8) = (x^8)^2 = x^{16}, \ldots$. Thus, a general formula is $f_n(x) = x^{2^{n+1}}$.

20. (a) $f_0(x) = 1/(2 - x)$ and $f_{n+1} = f_0 \circ f_n$ for $n = 0, 1, 2, \ldots$.

$$f_1(x) = f_0\left(\frac{1}{2 - x}\right) = \frac{1}{2 - \dfrac{1}{2 - x}} = \frac{2 - x}{2(2 - x) - 1} = \frac{2 - x}{3 - 2x},$$

$$f_2(x) = f_0\left(\frac{2 - x}{3 - 2x}\right) = \frac{1}{2 - \dfrac{2 - x}{3 - 2x}} = \frac{3 - 2x}{2(3 - 2x) - (2 - x)} = \frac{3 - 2x}{4 - 3x},$$

$$f_3(x) = f_0\left(\frac{3 - 2x}{4 - 3x}\right) = \frac{1}{2 - \dfrac{3 - 2x}{4 - 3x}} = \frac{4 - 3x}{2(4 - 3x) - (3 - 2x)} = \frac{4 - 3x}{5 - 4x}, \ldots$$

Thus, we conjecture that the general formula is $f_n(x) = \dfrac{n + 1 - nx}{n + 2 - (n + 1)x}$.

To prove this, we use the Principle of Mathematical Induction. We have already verified that f_n is true for $n = 1$.

Assume that the formula is true for $n = k$; that is, $f_k(x) = \dfrac{k + 1 - kx}{k + 2 - (k + 1)x}$. Then

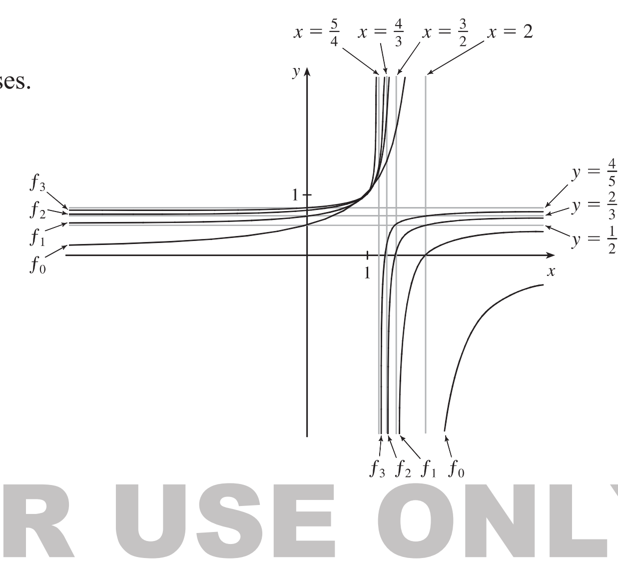
$$f_{k+1}(x) = (f_0 \circ f_k)(x) = f_0(f_k(x)) = f_0\left(\frac{k + 1 - kx}{k + 2 - (k + 1)x}\right) = \frac{1}{2 - \dfrac{k + 1 - kx}{k + 2 - (k + 1)x}}$$

$$= \frac{k + 2 - (k + 1)x}{2\,[k + 2 - (k + 1)x] - (k + 1 - kx)} = \frac{k + 2 - (k + 1)x}{k + 3 - (k + 2)x}$$

This shows that the formula for f_n is true for $n = k + 1$. Therefore, by mathematical induction, the formula is true for all positive integers n.

(b) From the graph, we can make several observations:

- The values at each fixed $x = a$ keep increasing as n increases.
- The vertical asymptote gets closer to $x = 1$ as n increases.
- The horizontal asymptote gets closer to $y = 1$ as n increases.
- The x-intercept for f_{n+1} is the value of the vertical asymptote for f_n.
- The y-intercept for f_n is the value of the horizontal asymptote for f_{n+1}.



2 ☐ LIMITS AND DERIVATIVES

2.1 The Tangent and Velocity Problems

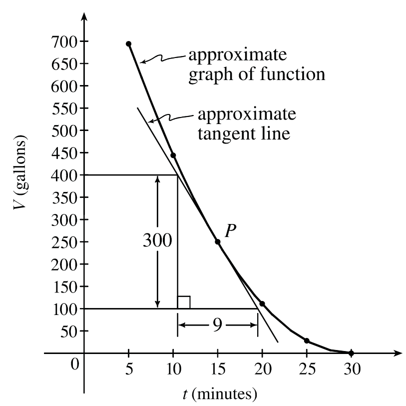
1. (a) Using $P(15, 250)$, we construct the following table:

t	Q	slope $= m_{PQ}$
5	$(5, 694)$	$\frac{694-250}{5-15} = -\frac{444}{10} = -44.4$
10	$(10, 444)$	$\frac{444-250}{10-15} = -\frac{194}{5} = -38.8$
20	$(20, 111)$	$\frac{111-250}{20-15} = -\frac{139}{5} = -27.8$
25	$(25, 28)$	$\frac{28-250}{25-15} = -\frac{222}{10} = -22.2$
30	$(30, 0)$	$\frac{0-250}{30-15} = -\frac{250}{15} = -16.\overline{6}$

(b) Using the values of t that correspond to the points closest to P ($t = 10$ and $t = 20$), we have

$$\frac{-38.8 + (-27.8)}{2} = -33.3$$

(c) From the graph, we can estimate the slope of the tangent line at P to be $\frac{-300}{9} = -33.\overline{3}$.



2. (a) Slope $= \frac{2948 - 2530}{42 - 36} = \frac{418}{6} \approx 69.67$

(b) Slope $= \frac{2948 - 2661}{42 - 38} = \frac{287}{4} = 71.75$

(c) Slope $= \frac{2948 - 2806}{42 - 40} = \frac{142}{2} = 71$

(d) Slope $= \frac{3080 - 2948}{44 - 42} = \frac{132}{2} = 66$

From the data, we see that the patient's heart rate is decreasing from 71 to 66 heartbeats/minute after 42 minutes. After being stable for a while, the patient's heart rate is dropping.

3. (a) $y = \dfrac{x}{1+x}$, $P(1, \frac{1}{2})$

	x	Q	m_{PQ}
(i)	0.5	$(0.5, 0.333333)$	0.333333
(ii)	0.9	$(0.9, 0.473684)$	0.263158
(iii)	0.99	$(0.99, 0.497487)$	0.251256
(iv)	0.999	$(0.999, 0.499750)$	0.250125
(v)	1.5	$(1.5, 0.6)$	0.2
(vi)	1.1	$(1.1, 0.523810)$	0.238095
(vii)	1.01	$(1.01, 0.502488)$	0.248756
(viii)	1.001	$(1.001, 0.500250)$	0.249875

(b) The slope appears to be $\frac{1}{4}$.

(c) $y - \frac{1}{2} = \frac{1}{4}(x - 1)$ or $y = \frac{1}{4}x + \frac{1}{4}$.

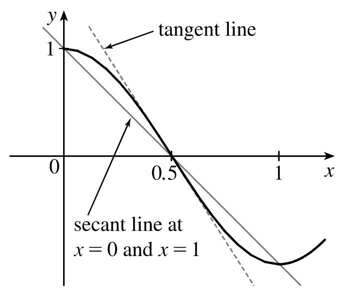
4. (a) $y = \cos \pi x$, $P(0.5, 0)$

	x	Q	m_{PQ}
(i)	0	$(0, 1)$	-2
(ii)	0.4	$(0.4, 0.309017)$	-3.090170
(iii)	0.49	$(0.49, 0.031411)$	-3.141076
(iv)	0.499	$(0.499, 0.003142)$	-3.141587
(v)	1	$(1, -1)$	-2
(vi)	0.6	$(0.6, -0.309017)$	-3.090170
(vii)	0.51	$(0.51, -0.031411)$	-3.141076
(viii)	0.501	$(0.501, -0.003142)$	-3.141587

(b) The slope appears to be $-\pi$.

(c) $y - 0 = -\pi(x - 0.5)$ or $y = -\pi x + \frac{1}{2}\pi$.

(d)



5. (a) $y = y(t) = 40t - 16t^2$. At $t = 2$, $y = 40(2) - 16(2)^2 = 16$. The average velocity between times 2 and $2 + h$ is

$$v_{\text{ave}} = \frac{y(2+h) - y(2)}{(2+h) - 2} = \frac{\left[40(2+h) - 16(2+h)^2\right] - 16}{h} = \frac{-24h - 16h^2}{h} = -24 - 16h, \text{ if } h \neq 0.$$

(i) $[2, 2.5]$: $h = 0.5$, $v_{\text{ave}} = -32$ ft/s

(ii) $[2, 2.1]$: $h = 0.1$, $v_{\text{ave}} = -25.6$ ft/s

(iii) $[2, 2.05]$: $h = 0.05$, $v_{\text{ave}} = -24.8$ ft/s

(iv) $[2, 2.01]$: $h = 0.01$, $v_{\text{ave}} = -24.16$ ft/s

(b) The instantaneous velocity when $t = 2$ (h approaches 0) is -24 ft/s.

6. (a) $y = y(t) = 10t - 1.86t^2$. At $t = 1$, $y = 10(1) - 1.86(1)^2 = 8.14$. The average velocity between times 1 and $1 + h$ is

$$v_{\text{ave}} = \frac{y(1+h) - y(1)}{(1+h) - 1} = \frac{\left[10(1+h) - 1.86(1+h)^2\right] - 8.14}{h} = \frac{6.28h - 1.86h^2}{h} = 6.28 - 1.86h, \text{ if } h \neq 0.$$

(i) $[1, 2]$: $h = 1$, $v_{\text{ave}} = 4.42$ m/s

(ii) $[1, 1.5]$: $h = 0.5$, $v_{\text{ave}} = 5.35$ m/s

(iii) $[1, 1.1]$: $h = 0.1$, $v_{\text{ave}} = 6.094$ m/s

(iv) $[1, 1.01]$: $h = 0.01$, $v_{\text{ave}} = 6.2614$ m/s

(v) $[1, 1.001]$: $h = 0.001$, $v_{\text{ave}} = 6.27814$ m/s

(b) The instantaneous velocity when $t = 1$ (h approaches 0) is 6.28 m/s.

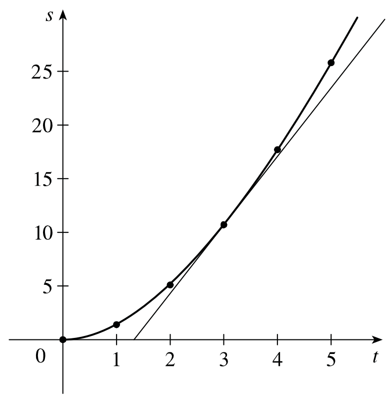
7. (a) (i) On the interval $[1, 3]$, $v_{\text{ave}} = \dfrac{s(3) - s(1)}{3 - 1} = \dfrac{10.7 - 1.4}{2} = \dfrac{9.3}{2} = 4.65$ m/s.

(ii) On the interval $[2, 3]$, $v_{\text{ave}} = \dfrac{s(3) - s(2)}{3 - 2} = \dfrac{10.7 - 5.1}{1} = 5.6$ m/s.

(iii) On the interval $[3, 5]$, $v_{\text{ave}} = \dfrac{s(5) - s(3)}{5 - 3} = \dfrac{25.8 - 10.7}{2} = \dfrac{15.1}{2} = 7.55$ m/s.

(iv) On the interval $[3, 4]$, $v_{\text{ave}} = \dfrac{s(4) - s(3)}{4 - 3} = \dfrac{17.7 - 10.7}{1} = 7$ m/s.

(b)



Using the points $(2, 4)$ and $(5, 23)$ from the approximate tangent line, the instantaneous velocity at $t = 3$ is about $\dfrac{23 - 4}{5 - 2} \approx 6.3 \text{ m/s}$.

8. (a) (i) $s = s(t) = 2\sin \pi t + 3\cos \pi t$. On the interval $[1, 2]$, $v_{\text{ave}} = \dfrac{s(2) - s(1)}{2 - 1} = \dfrac{3 - (-3)}{1} = 6 \text{ cm/s}$.

(ii) On the interval $[1, 1.1]$, $v_{\text{ave}} = \dfrac{s(1.1) - s(1)}{1.1 - 1} \approx \dfrac{-3.471 - (-3)}{0.1} = -4.71 \text{ cm/s}$.

(iii) On the interval $[1, 1.01]$, $v_{\text{ave}} = \dfrac{s(1.01) - s(1)}{1.01 - 1} \approx \dfrac{-3.0613 - (-3)}{0.01} = -6.13 \text{ cm/s}$.

(iv) On the interval $[1, 1.001]$, $v_{\text{ave}} = \dfrac{s(1.001) - s(1)}{1.001 - 1} \approx \dfrac{-3.00627 - (-3)}{1.001 - 1} = -6.27 \text{ cm/s}$.

(b) The instantaneous velocity of the particle when $t = 1$ appears to be about -6.3 cm/s.

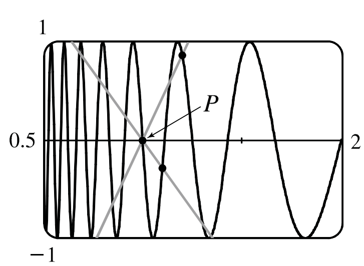
9. (a) For the curve $y = \sin(10\pi/x)$ and the point $P(1, 0)$:

x	Q	m_{PQ}
2	$(2, 0)$	0
1.5	$(1.5, 0.8660)$	1.7321
1.4	$(1.4, -0.4339)$	-1.0847
1.3	$(1.3, -0.8230)$	-2.7433
1.2	$(1.2, 0.8660)$	4.3301
1.1	$(1.1, -0.2817)$	-2.8173

x	Q	m_{PQ}
0.5	$(0.5, 0)$	0
0.6	$(0.6, 0.8660)$	-2.1651
0.7	$(0.7, 0.7818)$	-2.6061
0.8	$(0.8, 1)$	-5
0.9	$(0.9, -0.3420)$	3.4202

As x approaches 1, the slopes do not appear to be approaching any particular value.

(b)



We see that problems with estimation are caused by the frequent oscillations of the graph. The tangent is so steep at P that we need to take x-values much closer to 1 in order to get accurate estimates of its slope.

(c) If we choose $x = 1.001$, then the point Q is $(1.001, -0.0314)$ and $m_{PQ} \approx -31.3794$. If $x = 0.999$, then Q is $(0.999, 0.0314)$ and $m_{PQ} = -31.4422$. The average of these slopes is -31.4108. So we estimate that the slope of the tangent line at P is about -31.4.

2.2 The Limit of a Function

1. As x approaches 2, $f(x)$ approaches 5. [Or, the values of $f(x)$ can be made as close to 5 as we like by taking x sufficiently close to 2 (but $x \neq 2$).] Yes, the graph could have a hole at $(2, 5)$ and be defined such that $f(2) = 3$.

2. As x approaches 1 from the left, $f(x)$ approaches 3; and as x approaches 1 from the right, $f(x)$ approaches 7. No, the limit does not exist because the left- and right-hand limits are different.

3. (a) $f(x)$ approaches 2 as x approaches 1 from the left, so $\lim_{x \to 1^-} f(x) = 2$.

(b) $f(x)$ approaches 3 as x approaches 1 from the right, so $\lim_{x \to 1^+} f(x) = 3$.

(c) $\lim_{x \to 1} f(x)$ does not exist because the limits in part (a) and part (b) are not equal.

(d) $f(x)$ approaches 4 as x approaches 5 from the left and from the right, so $\lim_{x \to 5} f(x) = 4$.

(e) $f(5)$ is not defined, so it doesn't exist.

4. (a) $\lim_{x \to 0} f(x) = 3$ (b) $\lim_{x \to 3^-} f(x) = 4$ (c) $\lim_{x \to 3^+} f(x) = 2$

(d) $\lim_{x \to 3} f(x)$ does not exist because the limits in part (b) and part (c) are not equal.

(e) $f(3) = 3$

5. (a) $\lim_{t \to 0^-} g(t) = -1$ (b) $\lim_{t \to 0^+} g(t) = -2$

(c) $\lim_{t \to 0} g(t)$ does not exist because the limits in part (a) and part (b) are not equal.

(d) $\lim_{t \to 2^-} g(t) = 2$ (e) $\lim_{t \to 2^+} g(t) = 0$

(f) $\lim_{t \to 2} g(t)$ does not exist because the limits in part (d) and part (e) are not equal.

(g) $g(2) = 1$ (h) $\lim_{t \to 4} g(t) = 3$

6. (a) $h(x)$ approaches 4 as x approaches -3 from the left, so $\lim_{x \to -3^-} h(x) = 4$.

(b) $h(x)$ approaches 4 as x approaches -3 from the right, so $\lim_{x \to -3^+} h(x) = 4$.

(c) $\lim_{x \to -3} h(x) = 4$ because the limits in part (a) and part (b) are equal.

(d) $h(-3)$ is not defined, so it doesn't exist.

(e) $h(x)$ approaches 1 as x approaches 0 from the left, so $\lim_{x \to 0^-} h(x) = 1$.

(f) $h(x)$ approaches -1 as x approaches 0 from the right, so $\lim_{x \to 0^+} h(x) = -1$.

(g) $\lim_{x \to 0} h(x)$ does not exist because the limits in part (e) and part (f) are not equal.

(h) $h(0) = 1$ since the point $(0, 1)$ is on the graph of h.

(i) Since $\lim_{x\to 2^-} h(x) = 2$ and $\lim_{x\to 2^+} h(x) = 2$, we have $\lim_{x\to 2} h(x) = 2$.

(j) $h(2)$ is not defined, so it doesn't exist.

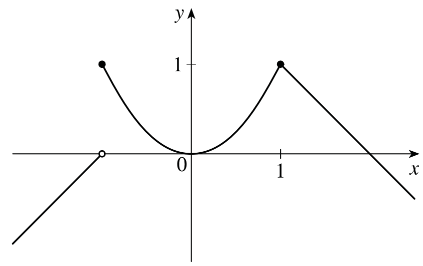
(k) $h(x)$ approaches 3 as x approaches 5 from the right, so $\lim_{x\to 5^+} h(x) = 3$.

(l) $h(x)$ does not approach any one number as x approaches 5 from the left, so $\lim_{x\to 5^-} h(x)$ does not exist.

7. From the graph of

$$f(x) = \begin{cases} 1+x & \text{if } x < -1 \\ x^2 & \text{if } -1 \le x < 1, \\ 2-x & \text{if } x \ge 1 \end{cases}$$

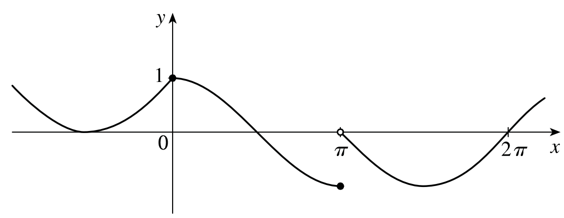
we see that $\lim_{x\to a} f(x)$ exists for all a except $a = -1$. Notice that the right and left limits are different at $a = -1$.



8. From the graph of

$$f(x) = \begin{cases} 1+\sin x & \text{if } x < 0 \\ \cos x & \text{if } 0 \le x \le \pi, \\ \sin x & \text{if } x > \pi \end{cases}$$

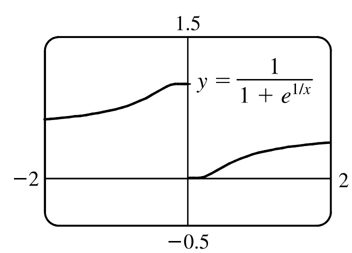
we see that $\lim_{x\to a} f(x)$ exists for all a except $a = \pi$. Notice that the right and left limits are different at $a = \pi$.



9. (a) $\lim_{x\to 0^-} f(x) = 1$

(b) $\lim_{x\to 0^+} f(x) = 0$

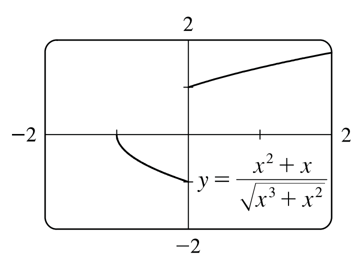
(c) $\lim_{x\to 0} f(x)$ does not exist because the limits in part (a) and part (b) are not equal.



10. (a) $\lim_{x\to 0^-} f(x) = -1$

(b) $\lim_{x\to 0^+} f(x) = 1$

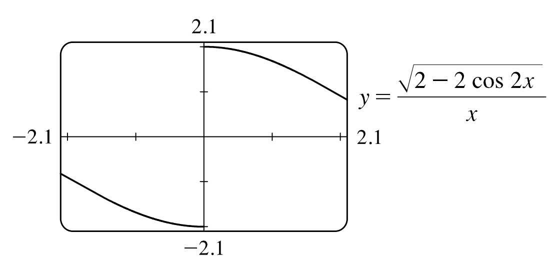
(c) $\lim_{x\to 0} f(x)$ does not exist because the limits in part (a) and part (b) are not equal.



11. (a) $\lim_{x\to 0^-} f(x) = -2$

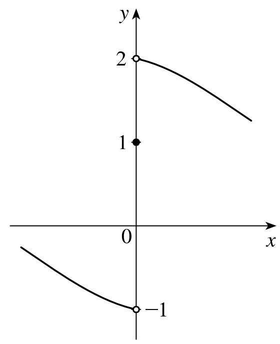
(b) $\lim_{x\to 0^+} f(x) = 2$

(c) $\lim_{x\to 0} f(x)$ does not exist because the limits in part (a) and part (b) are not equal.

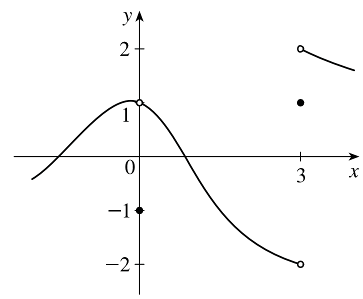


12. $\lim_{t\to 12^-} f(t) = 150$ mg and $\lim_{t\to 12^+} f(t) = 300$ mg. These limits show that there is an abrupt change in the amount of drug in the patient's bloodstream at $t = 12$ h. The left-hand limit represents the amount of the drug just before the fourth injection. The right-hand limit represents the amount of the drug just after the fourth injection.

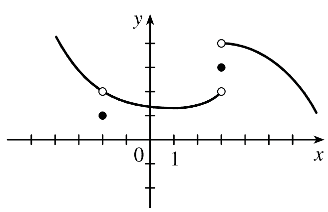
13. $\lim_{x\to 0^-} f(x) = -1, \quad \lim_{x\to 0^+} f(x) = 2, \quad f(0) = 1$



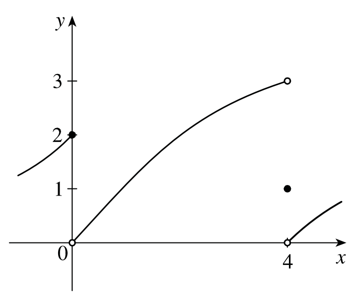
14. $\lim_{x\to 0} f(x) = 1$, $\lim_{x\to 3^-} f(x) = -2$, $\lim_{x\to 3^+} f(x) = 2$, $f(0) = -1$, $f(3) = 1$



15. $\lim_{x\to 3^+} f(x) = 4, \quad \lim_{x\to 3^-} f(x) = 2$, $\lim_{x\to -2} f(x) = 2$, $f(3) = 3, \quad f(-2) = 1$



16. $\lim_{x\to 0^-} f(x) = 2$, $\lim_{x\to 0^+} f(x) = 0$, $\lim_{x\to 4^-} f(x) = 3$, $\lim_{x\to 4^+} f(x) = 0$, $f(0) = 2$, $f(4) = 1$



17. For $f(x) = \dfrac{x^2 - 2x}{x^2 - x - 2}$:

x	$f(x)$
2.5	0.714286
2.1	0.677419
2.05	0.672131
2.01	0.667774
2.005	0.667221
2.001	0.666778

x	$f(x)$
1.9	0.655172
1.95	0.661017
1.99	0.665552
1.995	0.666110
1.999	0.666556

It appears that $\lim_{x\to 2} \dfrac{x^2 - 2x}{x^2 - x - 2} = 0.\bar{6} = \frac{2}{3}$.

18. For $f(x) = \dfrac{x^2 - 2x}{x^2 - x - 2}$:

x	$f(x)$
0	0
−0.5	−1
−0.9	−9
−0.95	−19
−0.99	−99
−0.999	−999

x	$f(x)$
−2	2
−1.5	3
−1.1	11
−1.01	101
−1.001	1001

It appears that $\lim_{x\to -1} \dfrac{x^2 - 2x}{x^2 - x - 2}$ *does not exist* since $f(x) \to \infty$ as $x \to -1^-$ and $f(x) \to -\infty$ as $x \to -1^+$.

19. For $f(t) = \dfrac{e^{5t} - 1}{t}$:

t	$f(t)$
0.5	22.364988
0.1	6.487213
0.01	5.127110
0.001	5.012521
0.0001	5.001250

t	$f(t)$
−0.5	1.835830
−0.1	3.934693
−0.01	4.877058
−0.001	4.987521
−0.0001	4.998750

It appears that $\lim\limits_{t \to 0} \dfrac{e^{5t} - 1}{t} = 5$.

20. For $f(h) = \dfrac{(2+h)^5 - 32}{h}$:

h	$f(h)$
0.5	131.312500
0.1	88.410100
0.01	80.804010
0.001	80.080040
0.0001	80.008000

h	$f(h)$
−0.5	48.812500
−0.1	72.390100
−0.01	79.203990
−0.001	79.920040
−0.0001	79.992000

It appears that $\lim\limits_{h \to 0} \dfrac{(2+h)^5 - 32}{h} = 80$.

21. For $f(x) = \dfrac{\sqrt{x+4} - 2}{x}$:

x	$f(x)$
1	0.236068
0.5	0.242641
0.1	0.248457
0.05	0.249224
0.01	0.249844

x	$f(x)$
−1	0.267949
−0.5	0.258343
−0.1	0.251582
−0.05	0.250786
−0.01	0.250156

It appears that $\lim\limits_{x \to 0} \dfrac{\sqrt{x+4} - 2}{x} = 0.25 = \frac{1}{4}$.

22. For $f(x) = \dfrac{\tan 3x}{\tan 5x}$:

x	$f(x)$
±0.2	0.439279
±0.1	0.566236
±0.05	0.591893
±0.01	0.599680
±0.001	0.599997

It appears that $\lim\limits_{x \to 0} \dfrac{\tan 3x}{\tan 5x} = 0.6 = \frac{3}{5}$.

23. For $f(x) = \dfrac{x^6 - 1}{x^{10} - 1}$:

x	$f(x)$
0.5	0.985337
0.9	0.719397
0.95	0.660186
0.99	0.612018
0.999	0.601200

x	$f(x)$
1.5	0.183369
1.1	0.484119
1.05	0.540783
1.01	0.588022
1.001	0.598800

It appears that $\lim\limits_{x \to 1} \dfrac{x^6 - 1}{x^{10} - 1} = 0.6 = \frac{3}{5}$.

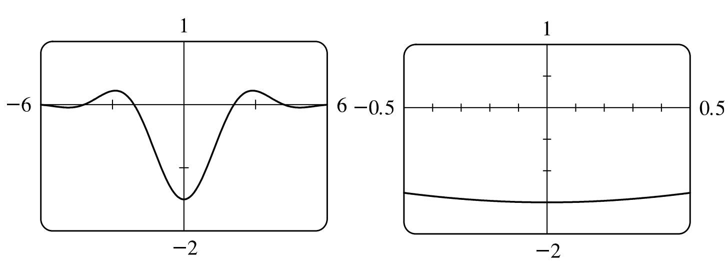
24. For $f(x) = \dfrac{9^x - 5^x}{x}$:

x	$f(x)$
0.5	1.527864
0.1	0.711120
0.05	0.646496
0.01	0.599082
0.001	0.588906

x	$f(x)$
−0.5	0.227761
−0.1	0.485984
−0.05	0.534447
−0.01	0.576706
−0.001	0.586669

It appears that $\lim\limits_{x \to 0} \dfrac{9^x - 5^x}{x} = 0.59$. Later we will be able to show that the exact value is $\ln(9/5)$.

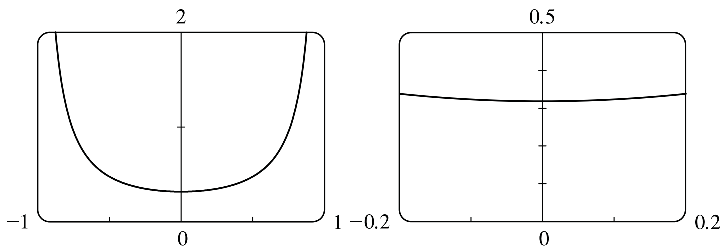
25. (a) From the graphs, it seems that $\lim\limits_{x \to 0} \dfrac{\cos 2x - \cos x}{x^2} = -1.5$.



(b)

x	$f(x)$
±0.1	−1.493759
±0.01	−1.499938
±0.001	−1.499999
±0.0001	−1.500000

26. (a) From the graphs, it seems that $\lim_{x \to 0} \dfrac{\sin x}{\sin \pi x} = 0.32$.



(b)

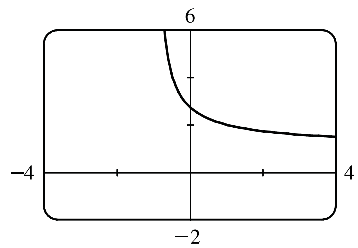
x	$f(x)$
± 0.1	0.323068
± 0.01	0.318357
± 0.001	0.318310
± 0.0001	0.318310

Later we will be able to show that the exact value is $\dfrac{1}{\pi}$.

27. (a) Let $h(x) = (1 + x)^{1/x}$.

x	$h(x)$
-0.001	2.71964
-0.0001	2.71842
-0.00001	2.71830
-0.000001	2.71828
0.000001	2.71828
0.00001	2.71827
0.0001	2.71815
0.001	2.71692

(b)



It appears that $\lim_{x \to 0} (1 + x)^{1/x} \approx 2.71828$, which is approximately e.

In Section 3.7 we will see that the value of the limit is exactly e.

28. For the curve $y = 2^x$ and the points $P(0, 1)$ and $Q(x, 2^x)$:

x	Q	m_{PQ}
0.1	$(0.1, 1.0717735)$	0.71773
0.01	$(0.01, 1.0069556)$	0.69556
0.001	$(0.001, 1.0006934)$	0.69339
0.0001	$(0.0001, 1.0000693)$	0.69317

The slope appears to be about 0.693.

29. For $f(x) = x^2 - (2^x/1000)$:

(a)

x	$f(x)$
1	0.998000
0.8	0.638259
0.6	0.358484
0.4	0.158680
0.2	0.038851
0.1	0.008928
0.05	0.001465

It appears that $\lim_{x \to 0} f(x) = 0$.

(b)

x	$f(x)$
0.04	0.000572
0.02	-0.000614
0.01	-0.000907
0.005	-0.000978
0.003	-0.000993
0.001	-0.001000

It appears that $\lim_{x \to 0} f(x) = -0.001$.

30. For $h(x) = \dfrac{\tan x - x}{x^3}$:

(a)

x	$h(x)$
1.0	0.55740773
0.5	0.37041992
0.1	0.33467209
0.05	0.33366700
0.01	0.33334667
0.005	0.33333667

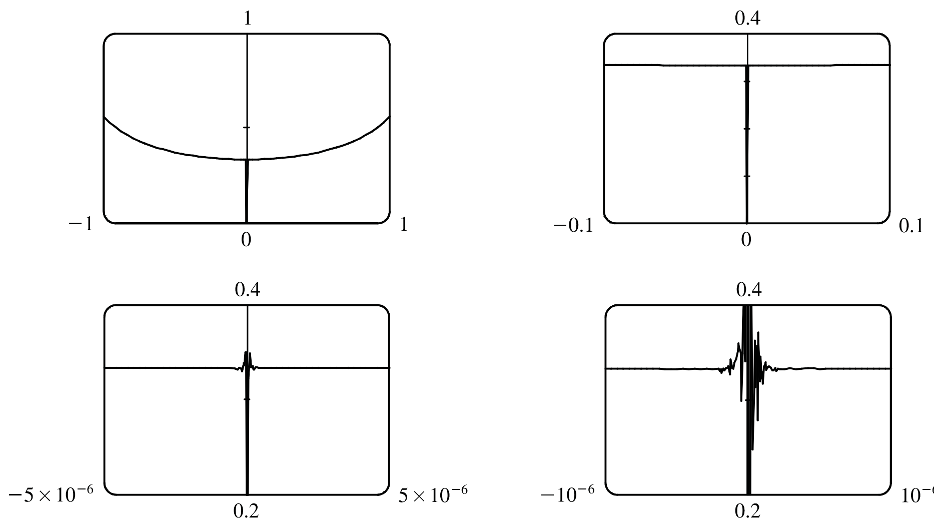
(b) It seems that $\lim\limits_{x \to 0} h(x) = \frac{1}{3}$.

(c)

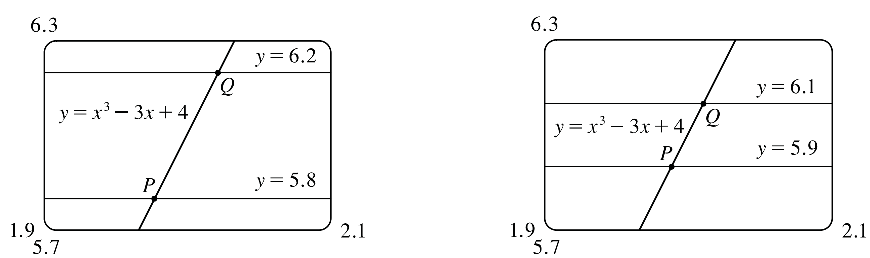
x	$h(x)$
0.001	0.33333350
0.0005	0.33333344
0.0001	0.33333000
0.00005	0.33333600
0.00001	0.33300000
0.000001	0.00000000

Here the values will vary from one calculator to another. Every calculator will eventually give *false values*.

(d) As in part (c), when we take a small enough viewing rectangle we get incorrect output.



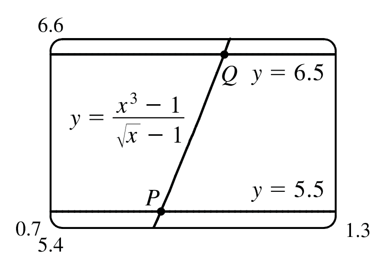
31. We need to have $5.8 < x^3 - 3x + 4 < 6.2$. From the graph we obtain the approximate points of intersection $P(1.9774, 5.8)$ and $Q(2.0219, 6.2)$. So if x is within 0.021 of 2, then y will be within 0.2 of 6. If we must have $x^3 - 3x + 4$ within 0.1 of 6, we get $P(1.9888, 5.9)$ and $Q(2.0110, 6.1)$. We would then need x to be within 0.011 of 2.



32. (a) Let $y = \dfrac{x^3 - 1}{\sqrt{x} - 1}$.

From the table and the graph, we guess that the limit of y as x approaches 1 is 6.

x	y
0.99	5.92531
0.999	5.99250
0.9999	5.99925
1.01	6.07531
1.001	6.00750
1.0001	6.00075



(b) We need to have $5.5 < \dfrac{x^3 - 1}{\sqrt{x} - 1} < 6.5$. From the graph we obtain the approximate points of intersection $P(0.9314, 5.5)$ and $Q(1.0649, 6.5)$. Now $1 - 0.9314 = 0.0686$ and $1.0649 - 1 = 0.0649$, so by requiring that x be within 0.0649 of 1, we ensure that y is within 0.5 of 6.

2.3 Calculating Limits Using the Limit Laws

1. (a) $\lim\limits_{x \to 2} [f(x) + 5g(x)] = \lim\limits_{x \to 2} f(x) + \lim\limits_{x \to 2} [5g(x)]$ [Limit Law 1]

$= \lim\limits_{x \to 2} f(x) + 5 \lim\limits_{x \to 2} g(x)$ [Limit Law 3]

$= 4 + 5(-2) = -6$

(b) $\lim\limits_{x \to 2} [g(x)]^3 = \left[\lim\limits_{x \to 2} g(x)\right]^3$ [Limit Law 6]

$= (-2)^3 = -8$

(c) $\lim\limits_{x \to 2} \sqrt{f(x)} = \sqrt{\lim\limits_{x \to 2} f(x)}$ [Limit Law 11]

$= \sqrt{4} = 2$

(d) $\lim\limits_{x \to 2} \dfrac{3f(x)}{g(x)} = \dfrac{\lim\limits_{x \to 2} [3f(x)]}{\lim\limits_{x \to 2} g(x)}$ [Limit Law 5]

$= \dfrac{3 \lim\limits_{x \to 2} f(x)}{\lim\limits_{x \to 2} g(x)}$ [Limit Law 3]

$= \dfrac{3(4)}{-2} = -6$

(e) Because the limit of the denominator is 0, we can't use Limit Law 5. The given limit, $\lim\limits_{x \to 2} \dfrac{g(x)}{h(x)}$, does not exist because the denominator approaches 0 while the numerator approaches a nonzero number.

(f) $\lim\limits_{x \to 2} \dfrac{g(x)\, h(x)}{f(x)} = \dfrac{\lim\limits_{x \to 2} [g(x)\, h(x)]}{\lim\limits_{x \to 2} f(x)}$ [Limit Law 5]

$= \dfrac{\lim\limits_{x \to 2} g(x) \cdot \lim\limits_{x \to 2} h(x)}{\lim\limits_{x \to 2} f(x)}$ [Limit Law 4]

$= \dfrac{-2 \cdot 0}{4} = 0$

2. (a) $\lim\limits_{x \to 2} [f(x) + g(x)] = \lim\limits_{x \to 2} f(x) + \lim\limits_{x \to 2} g(x) = 2 + 0 = 2$

(b) $\lim\limits_{x \to 1} g(x)$ does not exist since its left- and right-hand limits are not equal, so the given limit does not exist.

(c) $\lim\limits_{x \to 0} [f(x) g(x)] = \lim\limits_{x \to 0} f(x) \cdot \lim\limits_{x \to 0} g(x) = 0 \cdot 1.3 = 0$

(d) Since $\lim\limits_{x \to -1} g(x) = 0$ and g is in the denominator, but $\lim\limits_{x \to -1} f(x) = -1 \neq 0$, the given limit does not exist.

(e) $\lim\limits_{x\to 2} x^3 f(x) = \left[\lim\limits_{x\to 2} x^3\right]\left[\lim\limits_{x\to 2} f(x)\right] = 2^3 \cdot 2 = 16$

(f) $\lim\limits_{x\to 1} \sqrt{3 + f(x)} = \sqrt{3 + \lim\limits_{x\to 1} f(x)} = \sqrt{3+1} = 2$

3.
$$\begin{aligned}\lim_{x\to -2}(3x^4 + 2x^2 - x + 1) &= \lim_{x\to -2} 3x^4 + \lim_{x\to -2} 2x^2 - \lim_{x\to -2} x + \lim_{x\to -2} 1 && \text{[Limit Laws 1 and 2]}\\ &= 3\lim_{x\to -2} x^4 + 2\lim_{x\to -2} x^2 - \lim_{x\to -2} x + \lim_{x\to -2} 1 && \text{[3]}\\ &= 3(-2)^4 + 2(-2)^2 - (-2) + (1) && \text{[9, 8, and 7]}\\ &= 48 + 8 + 2 + 1 = 59\end{aligned}$$

4.
$$\begin{aligned}\lim_{t\to -1}(t^2+1)^3(t+3)^5 &= \lim_{t\to -1}(t^2+1)^3 \cdot \lim_{t\to -1}(t+3)^5 && \text{[Limit Law 4]}\\ &= \left[\lim_{t\to -1}(t^2+1)\right]^3 \cdot \left[\lim_{t\to -1}(t+3)\right]^5 && \text{[6]}\\ &= \left[\lim_{t\to -1} t^2 + \lim_{t\to -1} 1\right]^3 \cdot \left[\lim_{t\to -1} t + \lim_{t\to -1} 3\right]^5 && \text{[1]}\\ &= \left[(-1)^2 + 1\right]^3 \cdot [-1+3]^5 = 8 \cdot 32 = 256 && \text{[9, 7, and 8]}\end{aligned}$$

5.
$$\begin{aligned}\lim_{x\to 8}(1+\sqrt[3]{x})(2-6x^2+x^3) &= \lim_{x\to 8}(1+\sqrt[3]{x}) \cdot \lim_{x\to 8}(2-6x^2+x^3) && \text{[Limit Law 4]}\\ &= \left(\lim_{x\to 8} 1 + \lim_{x\to 8}\sqrt[3]{x}\right) \cdot \left(\lim_{x\to 8} 2 - 6\lim_{x\to 8} x^2 + \lim_{x\to 8} x^3\right) && \text{[1, 2, and 3]}\\ &= \left(1+\sqrt[3]{8}\right) \cdot \left(2 - 6 \cdot 8^2 + 8^3\right) && \text{[7, 10, 9]}\\ &= (3)(130) = 390\end{aligned}$$

6.
$$\begin{aligned}\lim_{u\to -2}\sqrt{u^4+3u+6} &= \sqrt{\lim_{u\to -2}(u^4+3u+6)} && \text{[11]}\\ &= \sqrt{\lim_{u\to -2} u^4 + 3\lim_{u\to -2} u + \lim_{u\to -2} 6} && \text{[1, 2, and 3]}\\ &= \sqrt{(-2)^4 + 3(-2) + 6} && \text{[9, 8, and 7]}\\ &= \sqrt{16-6+6} = \sqrt{16} = 4\end{aligned}$$

7.
$$\begin{aligned}\lim_{x\to 2}\sqrt{\frac{2x^2+1}{3x-2}} &= \sqrt{\lim_{x\to 2}\frac{2x^2+1}{3x-2}} && \text{[Limit Law 11]}\\ &= \sqrt{\frac{\lim_{x\to 2}(2x^2+1)}{\lim_{x\to 2}(3x-2)}} && \text{[5]}\\ &= \sqrt{\frac{2\lim_{x\to 2} x^2 + \lim_{x\to 2} 1}{3\lim_{x\to 2} x - \lim_{x\to 2} 2}} && \text{[1, 2, and 3]}\\ &= \sqrt{\frac{2(2)^2+1}{3(2)-2}} = \sqrt{\frac{9}{4}} = \frac{3}{2} && \text{[9, 8, and 7]}\end{aligned}$$

8. (a) The left-hand side of the equation is not defined for $x = 2$, but the right-hand side is.

(b) Since the equation holds for all $x \neq 2$, it follows that both sides of the equation approach the same limit as $x \to 2$, just as in Example 3. Remember that in finding $\lim\limits_{x \to a} f(x)$, we never consider $x = a$.

9. $\lim\limits_{x \to 5} \dfrac{x^2 - 6x + 5}{x - 5} = \lim\limits_{x \to 5} \dfrac{(x-5)(x-1)}{x-5} = \lim\limits_{x \to 5} (x - 1) = 5 - 1 = 4$

10. $\lim\limits_{x \to 4} \dfrac{x^2 - 4x}{x^2 - 3x - 4} = \lim\limits_{x \to 4} \dfrac{x(x-4)}{(x-4)(x+1)} = \lim\limits_{x \to 4} \dfrac{x}{x+1} = \dfrac{4}{4+1} = \dfrac{4}{5}$

11. $\lim\limits_{x \to 5} \dfrac{x^2 - 5x + 6}{x - 5}$ does not exist since $x - 5 \to 0$, but $x^2 - 5x + 6 \to 6$ as $x \to 5$.

12. $\lim\limits_{x \to -1} \dfrac{2x^2 + 3x + 1}{x^2 - 2x - 3} = \lim\limits_{x \to -1} \dfrac{(2x+1)(x+1)}{(x-3)(x+1)} = \lim\limits_{x \to -1} \dfrac{2x+1}{x-3} = \dfrac{2(-1)+1}{-1-3} = \dfrac{-1}{-4} = \dfrac{1}{4}$

13. $\lim\limits_{t \to -3} \dfrac{t^2 - 9}{2t^2 + 7t + 3} = \lim\limits_{t \to -3} \dfrac{(t+3)(t-3)}{(2t+1)(t+3)} = \lim\limits_{t \to -3} \dfrac{t-3}{2t+1} = \dfrac{-3-3}{2(-3)+1} = \dfrac{-6}{-5} = \dfrac{6}{5}$

14. $\lim\limits_{x \to -1} \dfrac{x^2 - 4x}{x^2 - 3x - 4}$ does not exist since $x^2 - 3x - 4 \to 0$ but $x^2 - 4x \to 5$ as $x \to -1$.

15. $\lim\limits_{h \to 0} \dfrac{(4+h)^2 - 16}{h} = \lim\limits_{h \to 0} \dfrac{(16 + 8h + h^2) - 16}{h} = \lim\limits_{h \to 0} \dfrac{8h + h^2}{h} = \lim\limits_{h \to 0} \dfrac{h(8+h)}{h} = \lim\limits_{h \to 0} (8 + h) = 8 + 0 = 8$

16.
$$\begin{aligned}\lim_{h \to 0} \frac{(2+h)^3 - 8}{h} &= \lim_{h \to 0} \frac{\left(8 + 12h + 6h^2 + h^3\right) - 8}{h} = \lim_{h \to 0} \frac{12h + 6h^2 + h^3}{h} \\ &= \lim_{h \to 0} \left(12 + 6h + h^2\right) = 12 + 0 + 0 = 12\end{aligned}$$

17. By the formula for the sum of cubes, we have

$$\lim_{x \to -2} \frac{x+2}{x^3+8} = \lim_{x \to -2} \frac{x+2}{(x+2)(x^2 - 2x + 4)} = \lim_{x \to -2} \frac{1}{x^2 - 2x + 4} = \frac{1}{4 + 4 + 4} = \frac{1}{12}.$$

18.
$$\begin{aligned}\lim_{h \to 0} \frac{\sqrt{1+h} - 1}{h} &= \lim_{h \to 0} \frac{\sqrt{1+h} - 1}{h} \cdot \frac{\sqrt{1+h} + 1}{\sqrt{1+h} + 1} = \lim_{h \to 0} \frac{(1+h) - 1}{h\left(\sqrt{1+h} + 1\right)} = \lim_{h \to 0} \frac{h}{h\left(\sqrt{1+h} + 1\right)} \\ &= \lim_{h \to 0} \frac{1}{\sqrt{1+h} + 1} = \frac{1}{\sqrt{1} + 1} = \frac{1}{2}\end{aligned}$$

19. $\lim\limits_{x \to -4} \dfrac{\frac{1}{4} + \frac{1}{x}}{4 + x} = \lim\limits_{x \to -4} \dfrac{\frac{x+4}{4x}}{4 + x} = \lim\limits_{x \to -4} \dfrac{x+4}{4x(4+x)} = \lim\limits_{x \to -4} \dfrac{1}{4x} = \dfrac{1}{4(-4)} = -\dfrac{1}{16}$

20. $\lim\limits_{x \to -1} \dfrac{x^2 + 2x + 1}{x^4 - 1} = \lim\limits_{x \to -1} \dfrac{(x+1)^2}{(x^2+1)(x^2-1)} = \lim\limits_{x \to -1} \dfrac{(x+1)^2}{(x^2+1)(x+1)(x-1)} = \lim\limits_{x \to -1} \dfrac{x+1}{(x^2+1)(x-1)} = \dfrac{0}{2(-2)} = 0$

21. $\displaystyle\lim_{x\to 16}\frac{4-\sqrt{x}}{16x-x^2}=\lim_{x\to 16}\frac{(4-\sqrt{x})(4+\sqrt{x})}{(16x-x^2)(4+\sqrt{x})}=\lim_{x\to 16}\frac{16-x}{x(16-x)(4+\sqrt{x})}$

$$=\lim_{x\to 16}\frac{1}{x(4+\sqrt{x})}=\frac{1}{16\left(4+\sqrt{16}\right)}=\frac{1}{16(8)}=\frac{1}{128}$$

22. $\displaystyle\lim_{t\to 0}\left(\frac{1}{t}-\frac{1}{t^2+t}\right)=\lim_{t\to 0}\frac{(t^2+t)-t}{t(t^2+t)}=\lim_{t\to 0}\frac{t^2}{t\cdot t(t+1)}=\lim_{t\to 0}\frac{1}{t+1}=\frac{1}{0+1}=1$

23. $\displaystyle\lim_{t\to 0}\left(\frac{1}{t\sqrt{1+t}}-\frac{1}{t}\right)=\lim_{t\to 0}\frac{1-\sqrt{1+t}}{t\sqrt{1+t}}=\lim_{t\to 0}\frac{\left(1-\sqrt{1+t}\right)\left(1+\sqrt{1+t}\right)}{t\sqrt{t+1}\left(1+\sqrt{1+t}\right)}=\lim_{t\to 0}\frac{-t}{t\sqrt{1+t}\left(1+\sqrt{1+t}\right)}$

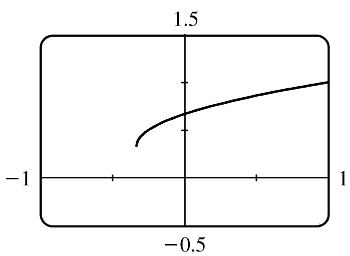
$$=\lim_{t\to 0}\frac{-1}{\sqrt{1+t}\left(1+\sqrt{1+t}\right)}=\frac{-1}{\sqrt{1+0}\left(1+\sqrt{1+0}\right)}=-\frac{1}{2}$$

24. $\displaystyle\lim_{x\to -4}\frac{\sqrt{x^2+9}-5}{x+4}=\lim_{x\to -4}\frac{\left(\sqrt{x^2+9}-5\right)\left(\sqrt{x^2+9}+5\right)}{(x+4)\left(\sqrt{x^2+9}+5\right)}=\lim_{x\to -4}\frac{(x^2+9)-25}{(x+4)\left(\sqrt{x^2+9}+5\right)}$

$$=\lim_{x\to -4}\frac{x^2-16}{(x+4)\left(\sqrt{x^2+9}+5\right)}=\lim_{x\to -4}\frac{(x+4)(x-4)}{(x+4)\left(\sqrt{x^2+9}+5\right)}$$

$$=\lim_{x\to -4}\frac{x-4}{\sqrt{x^2+9}+5}=\frac{-4-4}{\sqrt{16+9}+5}=\frac{-8}{5+5}=-\frac{4}{5}$$

25. (a)



$\displaystyle\lim_{x\to 0}\frac{x}{\sqrt{1+3x}-1}\approx\frac{2}{3}$

(b)

x	$f(x)$
−0.001	0.6661663
−0.0001	0.6666167
−0.00001	0.6666617
−0.000001	0.6666662
0.000001	0.6666672
0.00001	0.6666717
0.0001	0.6667167
0.001	0.6671663

The limit appears to be $\frac{2}{3}$.

(c) $\displaystyle\lim_{x\to 0}\left(\frac{x}{\sqrt{1+3x}-1}\cdot\frac{\sqrt{1+3x}+1}{\sqrt{1+3x}+1}\right)=\lim_{x\to 0}\frac{x\left(\sqrt{1+3x}+1\right)}{(1+3x)-1}=\lim_{x\to 0}\frac{x\left(\sqrt{1+3x}+1\right)}{3x}$

$\displaystyle=\frac{1}{3}\lim_{x\to 0}\left(\sqrt{1+3x}+1\right)$ [Limit Law 3]

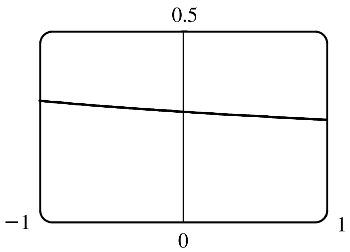
$\displaystyle=\frac{1}{3}\left[\sqrt{\lim_{x\to 0}(1+3x)}+\lim_{x\to 0}1\right]$ [1 and 11]

$\displaystyle=\frac{1}{3}\left(\sqrt{\lim_{x\to 0}1+3\lim_{x\to 0}x}+1\right)$ [1, 3, and 7]

$\displaystyle=\frac{1}{3}\left(\sqrt{1+3\cdot 0}+1\right)$ [7 and 8]

$\displaystyle=\frac{1}{3}(1+1)=\frac{2}{3}$

26. (a)



$$\lim_{x\to 0}\frac{\sqrt{3+x}-\sqrt{3}}{x}\approx 0.29$$

(b)

x	$f(x)$
-0.001	0.2886992
-0.0001	0.2886775
-0.00001	0.2886754
-0.000001	0.2886752
0.000001	0.2886751
0.00001	0.2886749
0.0001	0.2886727
0.001	0.2886511

The limit appears to be approximately 0.2887.

(c) $\displaystyle\lim_{x\to 0}\left(\frac{\sqrt{3+x}-\sqrt{3}}{x}\cdot\frac{\sqrt{3+x}+\sqrt{3}}{\sqrt{3+x}+\sqrt{3}}\right)=\lim_{x\to 0}\frac{(3+x)-3}{x\left(\sqrt{3+x}+\sqrt{3}\right)}=\lim_{x\to 0}\frac{1}{\sqrt{3+x}+\sqrt{3}}$

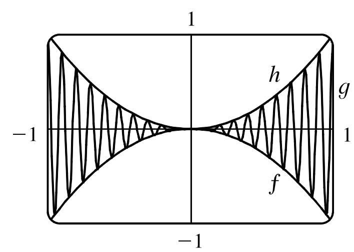
$\displaystyle=\frac{\lim_{x\to 0}1}{\lim_{x\to 0}\sqrt{3+x}+\lim_{x\to 0}\sqrt{3}}$ [Limit Laws 5 and 1]

$\displaystyle=\frac{1}{\sqrt{\lim_{x\to 0}(3+x)}+\sqrt{3}}$ [7 and 11]

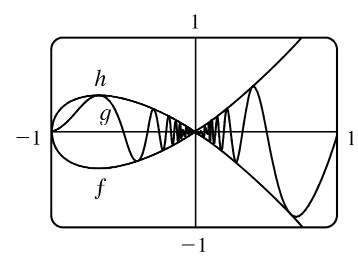
$\displaystyle=\frac{1}{\sqrt{3+0}+\sqrt{3}}$ [1, 7, and 8]

$\displaystyle=\frac{1}{2\sqrt{3}}$

27. Let $f(x)=-x^2$, $g(x)=x^2\cos 20\pi x$ and $h(x)=x^2$. Then $-1\le\cos 20\pi x\le 1\ \Rightarrow\ -x^2\le x^2\cos 20\pi x\le x^2\ \Rightarrow\ f(x)\le g(x)\le h(x)$. So since $\lim_{x\to 0}f(x)=\lim_{x\to 0}h(x)=0$, by the Squeeze Theorem we have $\lim_{x\to 0}g(x)=0$.



28. Let $f(x)=-\sqrt{x^3+x^2}$, $g(x)=\sqrt{x^3+x^2}\,\sin(\pi/x)$, and $h(x)=\sqrt{x^3+x^2}$. Then $-1\le\sin(\pi/x)\le 1\ \Rightarrow\ -\sqrt{x^3+x^2}\le\sqrt{x^3+x^2}\sin(\pi/x)\le\sqrt{x^3+x^2}\ \Rightarrow$ $f(x)\le g(x)\le h(x)$. So since $\lim_{x\to 0}f(x)=\lim_{x\to 0}h(x)=0$, by the Squeeze Theorem we have $\lim_{x\to 0}g(x)=0$.



29. We have $\lim_{x\to 4}(4x-9)=4(4)-9=7$ and $\lim_{x\to 4}\left(x^2-4x+7\right)=4^2-4(4)+7=7$. Since $4x-9\le f(x)\le x^2-4x+7$ for $x\ge 0$, $\lim_{x\to 4}f(x)=7$ by the Squeeze Theorem.

30. We have $\lim_{x\to 1}(2x)=2(1)=2$ and $\lim_{x\to 1}(x^4-x^2+2)=1^4-1^2+2=2$. Since $2x\le g(x)\le x^4-x^2+2$ for all x, $\lim_{x\to 1}g(x)=2$ by the Squeeze Theorem.

31. $-1\le\cos(2/x)\le 1\ \Rightarrow\ -x^4\le x^4\cos(2/x)\le x^4$. Since $\lim_{x\to 0}\left(-x^4\right)=0$ and $\lim_{x\to 0}x^4=0$, we have $\lim_{x\to 0}\left[x^4\cos(2/x)\right]=0$ by the Squeeze Theorem.

32. $-1 \le \sin(\pi/x) \le 1 \;\Rightarrow\; e^{-1} \le e^{\sin(\pi/x)} \le e^1 \;\Rightarrow\; \sqrt{x}/e \le \sqrt{x}\, e^{\sin(\pi/x)} \le \sqrt{x}\, e$. Since $\lim\limits_{x\to 0^+} (\sqrt{x}/e) = 0$ and $\lim\limits_{x\to 0^+} (\sqrt{x}\, e) = 0$, we have $\lim\limits_{x\to 0^+} \left[\sqrt{x}\, e^{\sin(\pi/x)}\right] = 0$ by the Squeeze Theorem.

33. $|x-3| = \begin{cases} x-3 & \text{if } x-3 \ge 0 \\ -(x-3) & \text{if } x-3 < 0 \end{cases} = \begin{cases} x-3 & \text{if } x \ge 3 \\ 3-x & \text{if } x < 3 \end{cases}$

Thus, $\lim\limits_{x\to 3^+} (2x + |x-3|) = \lim\limits_{x\to 3^+} (2x + x - 3) = \lim\limits_{x\to 3^+} (3x-3) = 3(3) - 3 = 6$ and $\lim\limits_{x\to 3^-} (2x + |x-3|) = \lim\limits_{x\to 3^-} (2x + 3 - x) = \lim\limits_{x\to 3^-} (x+3) = 3+3 = 6$. Since the left and right limits are equal, $\lim\limits_{x\to 3} (2x + |x-3|) = 6$.

34. $|x+6| = \begin{cases} x+6 & \text{if } x+6 \ge 0 \\ -(x+6) & \text{if } x+6 < 0 \end{cases} = \begin{cases} x+6 & \text{if } x \ge -6 \\ -(x+6) & \text{if } x < -6 \end{cases}$

We'll look at the one-sided limits.

$\lim\limits_{x\to -6^+} \dfrac{2x+12}{|x+6|} = \lim\limits_{x\to -6^+} \dfrac{2(x+6)}{x+6} = 2$ and $\lim\limits_{x\to -6^-} \dfrac{2x+12}{|x+6|} = \lim\limits_{x\to -6^-} \dfrac{2(x+6)}{-(x+6)} = -2$

The left and right limits are different, so $\lim\limits_{x\to -6} \dfrac{2x+12}{|x+6|}$ does not exist.

35. Since $|x| = -x$ for $x < 0$, we have $\lim\limits_{x\to 0^-} \left(\dfrac{1}{x} - \dfrac{1}{|x|}\right) = \lim\limits_{x\to 0^-} \left(\dfrac{1}{x} - \dfrac{1}{-x}\right) = \lim\limits_{x\to 0^-} \dfrac{2}{x}$, which does not exist since the denominator approaches 0 and the numerator does not.

36. Since $|x| = -x$ for $x < 0$, we have $\lim\limits_{x\to -2} \dfrac{2-|x|}{2+x} = \lim\limits_{x\to -2} \dfrac{2-(-x)}{2+x} = \lim\limits_{x\to -2} \dfrac{2+x}{2+x} = \lim\limits_{x\to -2} 1 = 1$.

37. (a) (i) $\lim\limits_{x\to 1^-} g(x) = \lim\limits_{x\to 1^-} x = 1$

(ii) $\lim\limits_{x\to 1^+} g(x) = \lim\limits_{x\to 1^+} (2 - x^2) = 2 - 1^2 = 1$. Since $\lim\limits_{x\to 1^-} g(x) = 1$ and $\lim\limits_{x\to 1^+} g(x) = 1$, we have $\lim\limits_{x\to 1} g(x) = 1$.

Note that the fact $g(1) = 3$ does not affect the value of the limit.

(iii) When $x = 1$, $g(x) = 3$, so $g(1) = 3$.

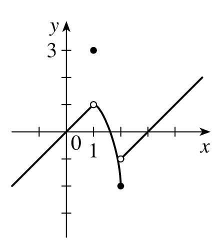
(iv) $\lim\limits_{x\to 2^-} g(x) = \lim\limits_{x\to 2^-} (2 - x^2) = 2 - 2^2 = 2 - 4 = -2$

(v) $\lim\limits_{x\to 2^+} g(x) = \lim\limits_{x\to 2^+} (x - 3) = 2 - 3 = -1$

(vi) $\lim\limits_{x\to 2} g(x)$ does not exist since $\lim\limits_{x\to 2^-} g(x) \ne \lim\limits_{x\to 2^+} g(x)$.

(b)

$$g(x) = \begin{cases} x & \text{if } x < 1 \\ 3 & \text{if } x = 1 \\ 2 - x^2 & \text{if } 1 < x \le 2 \\ x - 3 & \text{if } x > 2 \end{cases}$$

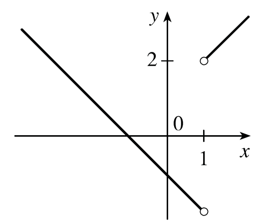


38. (a) (i) $\lim\limits_{x\to1^+} F(x) = \lim\limits_{x\to1^+} \dfrac{x^2-1}{|x-1|} = \lim\limits_{x\to1^+} \dfrac{x^2-1}{x-1} = \lim\limits_{x\to1^+} (x+1) = 2$

(ii) $\lim\limits_{x\to1^-} F(x) = \lim\limits_{x\to1^-} \dfrac{x^2-1}{|x-1|} = \lim\limits_{x\to1^-} \dfrac{x^2-1}{-(x-1)} = \lim\limits_{x\to1^-} -(x+1) = -2$

(b) No, $\lim\limits_{x\to1} F(x)$ does not exist since $\lim\limits_{x\to1^+} F(x) \neq \lim\limits_{x\to1^-} F(x)$.

(c)



39. (a) (i) $[\![x]\!] = -2$ for $-2 \le x < -1$, so $\lim\limits_{x\to-2^+} [\![x]\!] = \lim\limits_{x\to-2^+} (-2) = -2$

(ii) $[\![x]\!] = -3$ for $-3 \le x < -2$, so $\lim\limits_{x\to-2^-} [\![x]\!] = \lim\limits_{x\to-2^-} (-3) = -3$.

The right and left limits are different, so $\lim\limits_{x\to-2} [\![x]\!]$ does not exist.

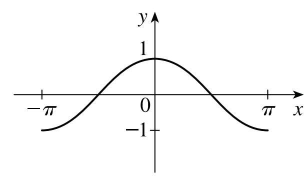
(iii) $[\![x]\!] = -3$ for $-3 \le x < -2$, so $\lim\limits_{x\to-2.4} [\![x]\!] = \lim\limits_{x\to-2.4} (-3) = -3$.

(b) (i) $[\![x]\!] = n-1$ for $n-1 \le x < n$, so $\lim\limits_{x\to n^-} [\![x]\!] = \lim\limits_{x\to n^-} (n-1) = n-1$.

(ii) $[\![x]\!] = n$ for $n \le x < n+1$, so $\lim\limits_{x\to n^+} [\![x]\!] = \lim\limits_{x\to n^+} n = n$.

(c) $\lim\limits_{x\to a} [\![x]\!]$ exists $\Leftrightarrow$ a is not an integer.

40. (a) See the graph of $y = \cos x$.

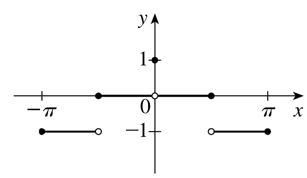


Since $-1 \le \cos x < 0$ on $[-\pi, -\pi/2)$, we have $y = f(x) = [\![\cos x]\!] = -1$ on $[-\pi, -\pi/2)$.

Since $0 \le \cos x < 1$ on $[-\pi/2, 0) \cup (0, \pi/2]$, we have $f(x) = 0$ on $[-\pi/2, 0) \cup (0, \pi/2]$.

Since $-1 \le \cos x < 0$ on $(\pi/2, \pi]$, we have $f(x) = -1$ on $(\pi/2, \pi]$.

Note that $f(0) = 1$.



(b) (i) $\lim\limits_{x\to0^-} f(x) = 0$ and $\lim\limits_{x\to0^+} f(x) = 0$, so $\lim\limits_{x\to0} f(x) = 0$.

(ii) As $x \to (\pi/2)^-$, $f(x) \to 0$, so $\lim\limits_{x\to(\pi/2)^-} f(x) = 0$.

(iii) As $x \to (\pi/2)^+$, $f(x) \to -1$, so $\lim\limits_{x\to(\pi/2)^+} f(x) = -1$.

(iv) Since the answers in parts (ii) and (iii) are not equal, $\lim\limits_{x\to\pi/2} f(x)$ does not exist.

(c) $\lim\limits_{x\to a} f(x)$ exists for all a in the open interval $(-\pi, \pi)$ except $a = -\pi/2$ and $a = \pi/2$.

41. The graph of $f(x) = [\![x]\!] + [\![-x]\!]$ is the same as the graph of $g(x) = -1$ with holes at each integer, since $f(a) = 0$ for any integer a. Thus, $\lim\limits_{x\to2^-} f(x) = -1$ and $\lim\limits_{x\to2^+} f(x) = -1$, so $\lim\limits_{x\to2} f(x) = -1$. However, $f(2) = [\![2]\!] + [\![-2]\!] = 2 + (-2) = 0$, so $\lim\limits_{x\to2} f(x) \neq f(2)$.

42. $\lim\limits_{v\to c^-} \left(L_0\sqrt{1 - \dfrac{v^2}{c^2}} \right) = L_0\sqrt{1-1} = 0$. As the velocity approaches the speed of light, the length approaches 0.

A left-hand limit is necessary since L is not defined for $v > c$.

43. Since $p(x)$ is a polynomial, $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$. Thus, by the Limit Laws,

$$\lim_{x\to a} p(x) = \lim_{x\to a} \left(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n\right) = a_0 + a_1 \lim_{x\to a} x + a_2 \lim_{x\to a} x^2 + \cdots + a_n \lim_{x\to a} x^n$$

$$= a_0 + a_1a + a_2a^2 + \cdots + a_na^n = p(a)$$

Thus, for any polynomial p, $\lim\limits_{x\to a} p(x) = p(a)$.

44. Let $r(x) = \dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are any polynomials, and suppose that $q(a) \neq 0$. Thus,

$$\lim_{x\to a} r(x) = \lim_{x\to a} \frac{p(x)}{q(x)} = \frac{\lim\limits_{x\to a} p(x)}{\lim\limits_{x\to a} q(x)} \quad \text{[Limit Law 5]} \quad = \frac{p(a)}{q(a)} \quad \text{[Exercise 43]} \quad = r(a).$$

45. $\lim\limits_{x\to 1} [f(x) - 8] = \lim\limits_{x\to 1} \left[\dfrac{f(x) - 8}{x - 1} \cdot (x - 1)\right] = \lim\limits_{x\to 1} \dfrac{f(x) - 8}{x - 1} \cdot \lim\limits_{x\to 1}(x - 1) = 10 \cdot 0 = 0.$

Thus, $\lim\limits_{x\to 1} f(x) = \lim\limits_{x\to 1} \{[f(x) - 8] + 8\} = \lim\limits_{x\to 1}[f(x) - 8] + \lim\limits_{x\to 1} 8 = 0 + 8 = 8.$

Note: The value of $\lim\limits_{x\to 1} \dfrac{f(x) - 8}{x - 1}$ does not affect the answer since it's multiplied by 0. What's important is that $\lim\limits_{x\to 1} \dfrac{f(x) - 8}{x - 1}$ exists.

46. (a) $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} \left[\dfrac{f(x)}{x^2} \cdot x^2\right] = \lim\limits_{x\to 0} \dfrac{f(x)}{x^2} \cdot \lim\limits_{x\to 0} x^2 = 5 \cdot 0 = 0$

(b) $\lim\limits_{x\to 0} \dfrac{f(x)}{x} = \lim\limits_{x\to 0} \left[\dfrac{f(x)}{x^2} \cdot x\right] = \lim\limits_{x\to 0} \dfrac{f(x)}{x^2} \cdot \lim\limits_{x\to 0} x = 5 \cdot 0 = 0$

47. Let $f(x) = [\![x]\!]$ and $g(x) = -[\![x]\!]$. Then $\lim\limits_{x\to 3} f(x)$ and $\lim\limits_{x\to 3} g(x)$ do not exist [Example 9]

but $\lim\limits_{x\to 3} [f(x) + g(x)] = \lim\limits_{x\to 3} ([\![x]\!] - [\![x]\!]) = \lim\limits_{x\to 3} 0 = 0.$

48. Let $f(x) = H(x)$ and $g(x) = 1 - H(x)$, where H is the Heaviside function defined in Example 6 in Section 2.2.

Thus, either f or g is 0 for any value of x. Then $\lim\limits_{x\to 0} f(x)$ and $\lim\limits_{x\to 0} g(x)$ do not exist, but $\lim\limits_{x\to 0} [f(x)g(x)] = \lim\limits_{x\to 0} 0 = 0.$

49. Since the denominator approaches 0 as $x \to -2$, the limit will exist only if the numerator also approaches 0 as $x \to -2$. In order for this to happen, we need $\lim\limits_{x\to -2} \left(3x^2 + ax + a + 3\right) = 0 \quad \Leftrightarrow$

$3(-2)^2 + a(-2) + a + 3 = 0 \quad \Leftrightarrow \quad 12 - 2a + a + 3 = 0 \quad \Leftrightarrow \quad a = 15$. With $a = 15$, the limit becomes

$$\lim_{x\to -2} \frac{3x^2 + 15x + 18}{x^2 + x - 2} = \lim_{x\to -2} \frac{3(x + 2)(x + 3)}{(x - 1)(x + 2)} = \lim_{x\to -2} \frac{3(x + 3)}{x - 1} = \frac{3(-2 + 3)}{-2 - 1} = \frac{3}{-3} = -1.$$

50. *Solution 1:* First, we find the coordinates of P and Q as functions of r. Then we can find the equation of the line determined by these two points, and thus find the x-intercept (the point R), and take the limit as $r \to 0$. The coordinates of P are $(0, r)$. The point Q is the point of intersection of the two circles $x^2 + y^2 = r^2$ and $(x-1)^2 + y^2 = 1$. Eliminating y from these equations, we get $r^2 - x^2 = 1 - (x-1)^2 \quad \Leftrightarrow \quad r^2 = 1 + 2x - 1 \quad \Leftrightarrow \quad x = \frac{1}{2}r^2$. Substituting back into the equation of the shrinking circle to find the y-coordinate, we get $\left(\frac{1}{2}r^2\right)^2 + y^2 = r^2 \quad \Leftrightarrow \quad y^2 = r^2\left(1 - \frac{1}{4}r^2\right) \quad \Leftrightarrow \quad y = r\sqrt{1 - \frac{1}{4}r^2}$ (the positive y-value). So the coordinates of Q are $\left(\frac{1}{2}r^2, r\sqrt{1 - \frac{1}{4}r^2}\right)$. The equation of the line joining P and Q is thus

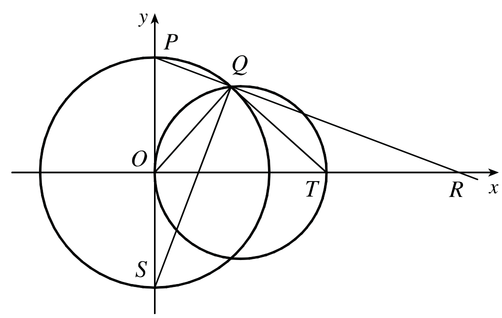
$y - r = \dfrac{r\sqrt{1 - \frac{1}{4}r^2} - r}{\frac{1}{2}r^2 - 0}(x - 0)$. We set $y = 0$ in order to find the x-intercept, and get

$$x = -r\frac{\frac{1}{2}r^2}{r\left(\sqrt{1 - \frac{1}{4}r^2} - 1\right)} = \frac{-\frac{1}{2}r^2\left(\sqrt{1 - \frac{1}{4}r^2} + 1\right)}{1 - \frac{1}{4}r^2 - 1} = 2\left(\sqrt{1 - \frac{1}{4}r^2} + 1\right)$$

Now we take the limit as $r \to 0^+$: $\lim\limits_{r \to 0^+} x = \lim\limits_{r \to 0^+} 2\left(\sqrt{1 - \frac{1}{4}r^2} + 1\right) = \lim\limits_{r \to 0^+} 2\left(\sqrt{1} + 1\right) = 4$.

So the limiting position of R is the point $(4, 0)$.

Solution 2: We add a few lines to the diagram, as shown. Note that $\angle PQS = 90°$ (subtended by diameter PS). So $\angle SQR = 90° = \angle OQT$ (subtended by diameter OT). It follows that $\angle OQS = \angle TQR$. Also $\angle PSQ = 90° - \angle SPQ = \angle ORP$. Since $\triangle QOS$ is isosceles, so is $\triangle QTR$, implying that $QT = TR$. As the circle C_2 shrinks, the point Q plainly approaches the origin, so the point R must approach a point twice as far from the origin as T, that is, the point $(4, 0)$, as above.

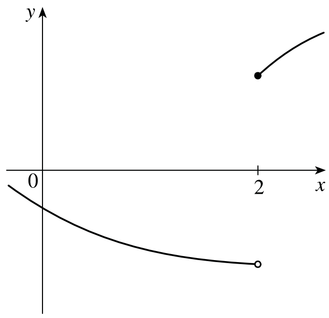


2.4 Continuity

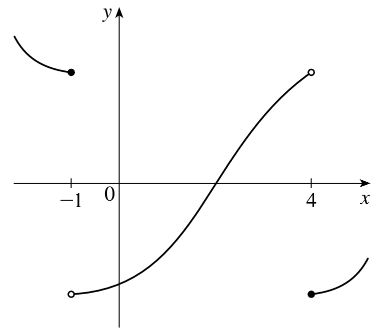
1. From Definition 1, $\lim\limits_{x \to 4} f(x) = f(4)$.
2. The graph of f has no hole, jump, or vertical asymptote.
3. (a) f is discontinuous at -4 since $f(-4)$ is not defined and at -2, 2, and 4 since the limit does not exist (the left and right limits are not the same).

   (b) f is continuous from the left at -2 since $\lim\limits_{x \to -2^-} f(x) = f(-2)$. f is continuous from the right at 2 and 4 since $\lim\limits_{x \to 2^+} f(x) = f(2)$ and $\lim\limits_{x \to 4^+} f(x) = f(4)$. It is continuous from neither side at -4 since $f(-4)$ is undefined.
4. g is continuous on $[-4, -2)$, $(-2, 2)$, $[2, 4)$, $(4, 6)$, and $(6, 8)$.

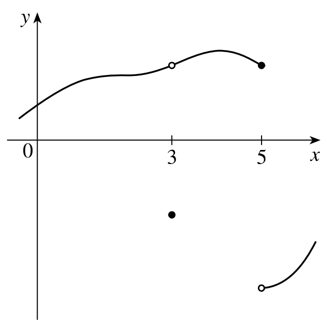
5. The graph of $y = f(x)$ must have a discontinuity at $x = 2$ and must show that $\lim_{x \to 2^+} f(x) = f(2)$.



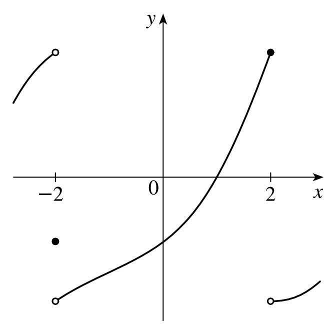
6. The graph of $y = f(x)$ must have discontinuities at $x = -1$ and $x = 4$. It must show that $\lim_{x \to -1^-} f(x) = f(-1)$ and $\lim_{x \to 4^+} f(x) = f(4)$.



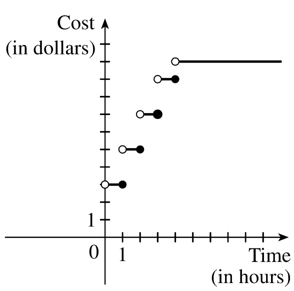
7. The graph of $y = f(x)$ must have a removable discontinuity (a hole) at $x = 3$ and a jump discontinuity at $x = 5$.



8. The graph of $y = f(x)$ must have a discontinuity at $x = -2$ with $\lim_{x \to -2^-} f(x) \neq f(-2)$ and $\lim_{x \to -2^+} f(x) \neq f(-2)$. It must also show that $\lim_{x \to 2^-} f(x) = f(2)$ and $\lim_{x \to 2^+} f(x) \neq f(2)$.



9. (a)



(b) There are discontinuities at times $t = 1$, 2, 3, and 4. A person parking in the lot would want to keep in mind that the charge will jump at the beginning of each hour.

10. (a) Continuous; at the location in question, the temperature changes smoothly as time passes, without any instantaneous jumps from one temperature to another.

(b) Continuous; the temperature at a specific time changes smoothly as the distance due west from New York City increases, without any instantaneous jumps.

(c) Discontinuous; as the distance due west from New York City increases, the altitude above sea level may jump from one height to another without going through all of the intermediate values — at a cliff, for example.

(d) Discontinuous; as the distance traveled increases, the cost of the ride jumps in small increments.

(e) Discontinuous; when the lights are switched on (or off), the current suddenly changes between 0 and some nonzero value, without passing through all of the intermediate values. This is debatable, though, depending on your definition of current.

11. Since f and g are continuous functions,

$$\begin{aligned} \lim_{x\to 3}[2f(x)-g(x)] &= 2\lim_{x\to 3}f(x)-\lim_{x\to 3}g(x) && \text{[by Limit Laws 2 and 3]} \\ &= 2f(3)-g(3) && \text{[by continuity of } f \text{ and } g \text{ at } x=3] \\ &= 2\cdot 5-g(3)=10-g(3) \end{aligned}$$

Since it is given that $\lim_{x\to 3}[2f(x)-g(x)]=4$, we have $10-g(3)=4$, so $g(3)=6$.

12. $\displaystyle \lim_{t\to 1}h(t)=\lim_{t\to 1}\frac{2t-3t^2}{1+t^3}=\frac{\lim_{t\to 1}(2t-3t^2)}{\lim_{t\to 1}(1+t^3)}=\frac{2\lim_{t\to 1}t-3\lim_{t\to 1}t^2}{\lim_{t\to 1}1+\lim_{t\to 1}t^3}=\frac{2(1)-3(1)^2}{1+(1)^3}=\frac{-1}{2}=h(1).$

By the definition of continuity, h is continuous at $a=1$.

13. $\displaystyle \lim_{x\to -1}f(x)=\lim_{x\to -1}\left(x+2x^3\right)^4=\left(\lim_{x\to -1}x+2\lim_{x\to -1}x^3\right)^4=\left[-1+2(-1)^3\right]^4=(-3)^4=81=f(-1).$

By the definition of continuity, f is continuous at $a=-1$.

14. For $a<3$, we have

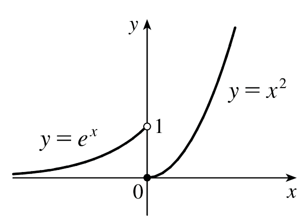
$$\begin{aligned} \lim_{x\to a}g(x) &= \lim_{x\to a}2\sqrt{3-x} \\ &= 2\lim_{x\to a}\sqrt{3-x} && \text{[Limit Law 3]} \\ &= 2\sqrt{\lim_{x\to a}(3-x)} && [11] \\ &= 2\sqrt{\lim_{x\to a}3-\lim_{x\to a}x} && [2] \\ &= 2\sqrt{3-a} && [7 \text{ and } 8] \\ &= g(a) \end{aligned}$$

So g is continuous at $x=a$ for every a in $(-\infty,3)$. Also, $\lim_{x\to 3^-}g(x)=0=g(3)$, so g is continuous from the left at 3.

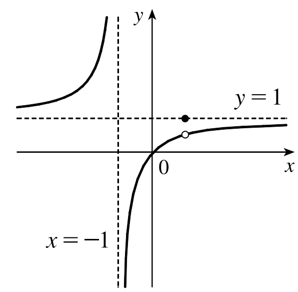
Thus, g is continuous on $(-\infty,3]$.

15. $f(x)=\begin{cases} e^x & \text{if } x<0 \\ x^2 & \text{if } x\geq 0 \end{cases}$

The left-hand limit of f at $a=0$ is $\lim_{x\to 0^-}f(x)=\lim_{x\to 0^-}e^x=1$. The right-hand limit of f at $a=0$ is $\lim_{x\to 0^+}f(x)=\lim_{x\to 0^+}x^2=0$. Since these limits are not equal, $\lim_{x\to 0}f(x)$ does not exist and f is discontinuous at 0.



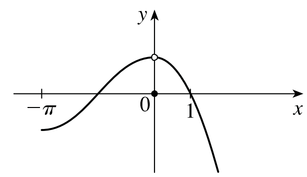
16. $f(x) = \begin{cases} \dfrac{x^2 - x}{x^2 - 1} & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}$



$\lim\limits_{x \to 1} f(x) = \lim\limits_{x \to 1} \dfrac{x^2 - x}{x^2 - 1} = \lim\limits_{x \to 1} \dfrac{x(x-1)}{(x+1)(x-1)} = \lim\limits_{x \to 1} \dfrac{x}{x+1} = \dfrac{1}{2}$,

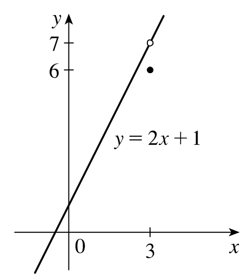
but $f(1) = 1$, so f is discontinous at 1.

17. $f(x) = \begin{cases} \cos x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 - x^2 & \text{if } x > 0 \end{cases}$



$\lim\limits_{x \to 0} f(x) = 1$, but $f(0) = 0 \neq 1$, so f is discontinuous at 0.

18. $f(x) = \begin{cases} \dfrac{2x^2 - 5x - 3}{x - 3} & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases}$



$\lim\limits_{x \to 3} f(x) = \lim\limits_{x \to 3} \dfrac{2x^2 - 5x - 3}{x - 3} = \lim\limits_{x \to 3} \dfrac{(2x+1)(x-3)}{x - 3} = \lim\limits_{x \to 3} (2x + 1) = 7$,

but $f(3) = 6$, so f is discontinuous at 3.

19. By Theorem 5, the polynomials x^2 and $2x - 1$ are continuous on $(-\infty, \infty)$. By Theorem 7, the root function $\sqrt{x}$ is continuous on $[0, \infty)$. By Theorem 9, the composite function $\sqrt{2x - 1}$ is continuous on its domain, $\left[\frac{1}{2}, \infty\right)$. By part 1 of Theorem 4, the sum $R(x) = x^2 + \sqrt{2x - 1}$ is continuous on $\left[\frac{1}{2}, \infty\right)$.

20. By Theorem 7, the root function $\sqrt[3]{x}$ and the polynomial function $1 + x^3$ are continuous on $\mathbb{R}$. By part 4 of Theorem 4, the product $G(x) = \sqrt[3]{x}\,(1 + x^3)$ is continuous on its domain, $\mathbb{R}$.

21. By Theorem 7, the exponential function e^{-5t} and the trigonometric function $\cos 2\pi t$ are continuous on $(-\infty, \infty)$. By part 4 of Theorem 4, $L(t) = e^{-5t} \cos 2\pi t$ is continuous on $(-\infty, \infty)$.

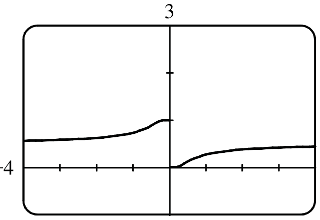
22. By Theorem 7, the trigonometric function $\sin x$ and the polynomial function $x + 1$ are continuous on $\mathbb{R}$. By part 5 of Theorem 4, $h(x) = \dfrac{\sin x}{x + 1}$ is continuous on its domain, $\{x \mid x \neq -1\}$.

23. By Theorem 5, the polynomial $t^4 - 1$ is continuous on $(-\infty, \infty)$. By Theorem 7, $\ln x$ is continuous on its domain, $(0, \infty)$. By Theorem 9, $\ln(t^4 - 1)$ is continuous on its domain, which is

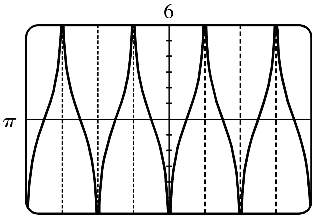
$$\{t \mid t^4 - 1 > 0\} = \{t \mid t^4 > 1\} = \{t \mid |t| > 1\} = (-\infty, -1) \cup (1, \infty)$$

24. The sine and cosine functions are continuous everywhere by Theorem 7, so $F(x) = \sin(\cos(\sin x))$, which is the composite of sine, cosine, and (once again) sine, is continuous everywhere by Theorem 9.

25. The function $y = \dfrac{1}{1 + e^{1/x}}$ is discontinuous at $x = 0$ because the left- and right-hand limits at $x = 0$ are different.



26. The function $y = \tan^2 x$ is discontinuous at $x = \frac{\pi}{2} + \pi k$, where k is any integer. The function $y = \ln(\tan^2 x)$ is also discontinuous where $\tan^2 x$ is 0, that is, at $x = \pi k$. So $y = \ln(\tan^2 x)$ is discontinuous at $x = \frac{\pi}{2} n$, n any integer.



27. Because we are dealing with root functions, $5 + \sqrt{x}$ is continuous on $[0, \infty)$, $\sqrt{x+5}$ is continuous on $[-5, \infty)$, so the quotient $f(x) = \dfrac{5 + \sqrt{x}}{\sqrt{5 + x}}$ is continuous on $[0, \infty)$. Since f is continuous at $x = 4$, $\lim\limits_{x \to 4} f(x) = f(4) = \frac{7}{3}$.

28. Because x is continuous on $\mathbb{R}$, $\sin x$ is continuous on $\mathbb{R}$, and $x + \sin x$ is continuous on $\mathbb{R}$, the composite function $f(x) = \sin(x + \sin x)$ is continuous on $\mathbb{R}$, so $\lim\limits_{x \to \pi} f(x) = f(\pi) = \sin(\pi + \sin \pi) = \sin \pi = 0$.

29. Because $x^2 - x$ is continuous on $\mathbb{R}$, the composite function $f(x) = e^{x^2 - x}$ is continuous on $\mathbb{R}$, so $\lim\limits_{x \to 1} f(x) = f(1) = e^{1-1} = e^0 = 1$.

30. $x^3 - 3x + 1 = 0$ for three values of x, but 2 is not one of them. Thus, $f(x) = (x^3 - 3x + 1)^{-3}$ is continuous at $x = 2$ and $\lim\limits_{x \to 2} f(x) = f(2) = (8 - 6 + 1)^{-3} = 3^{-3} = \frac{1}{27}$.

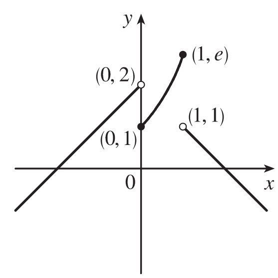
31. $f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ \sqrt{x} & \text{if } x \geq 1 \end{cases}$

By Theorem 5, since $f(x)$ equals the polynomial x^2 on $(-\infty, 1)$, f is continuous on $(-\infty, 1)$. By Theorem 7, since $f(x)$ equals the root function $\sqrt{x}$ on $(1, \infty)$, f is continuous on $(1, \infty)$. At $x = 1$, $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} x^2 = 1$ and $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} \sqrt{x} = 1$. Thus, $\lim\limits_{x \to 1} f(x)$ exists and equals 1. Also, $f(1) = \sqrt{1} = 1$. Thus, f is continuous at $x = 1$. We conclude that f is continuous on $(-\infty, \infty)$.

32. $f(x) = \begin{cases} \sin x & \text{if } x < \pi/4 \\ \cos x & \text{if } x \geq \pi/4 \end{cases}$

By Theorem 7, the trigonometric functions are continuous. Since $f(x) = \sin x$ on $(-\infty, \pi/4)$ and $f(x) = \cos x$ on $(\pi/4, \infty)$, f is continuous on $(-\infty, \pi/4) \cup (\pi/4, \infty)$. $\lim\limits_{x\to(\pi/4)^-} f(x) = \lim\limits_{x\to(\pi/4)^-} \sin x = \sin\frac{\pi}{4} = 1/\sqrt{2}$ since the sine function is continuous at $\pi/4$. Similarly, $\lim\limits_{x\to(\pi/4)^+} f(x) = \lim\limits_{x\to(\pi/4)^+} \cos x = 1/\sqrt{2}$ by continuity of the cosine function at $\pi/4$. Thus, $\lim\limits_{x\to(\pi/4)} f(x)$ exists and equals $1/\sqrt{2}$, which agrees with the value $f(\pi/4)$. Therefore, f is continuous at $\pi/4$, so f is continuous on $(-\infty, \infty)$.

33. $f(x) = \begin{cases} x+2 & \text{if } x < 0 \\ e^x & \text{if } 0 \leq x \leq 1 \\ 2-x & \text{if } x > 1 \end{cases}$



f is continuous on $(-\infty, 0)$ and $(1, \infty)$ since on each of these intervals it is a polynomial; it is continuous on $(0, 1)$ since it is an exponential.

Now $\lim\limits_{x\to0^-} f(x) = \lim\limits_{x\to0^-} (x+2) = 2$ and $\lim\limits_{x\to0^+} f(x) = \lim\limits_{x\to0^+} e^x = 1$, so f is discontinuous at 0. Since $f(0) = 1$, f is continuous from the right at 0. Also $\lim\limits_{x\to1^-} f(x) = \lim\limits_{x\to1^-} e^x = e$ and $\lim\limits_{x\to1^+} f(x) = \lim\limits_{x\to1^+} (2-x) = 1$, so f is discontinuous at 1. Since $f(1) = e$, f is continuous from the left at 1.

34. By Theorem 5, each piece of F is continuous on its domain. We need to check for continuity at $r = R$.

$\lim\limits_{r\to R^-} F(r) = \lim\limits_{r\to R^-} \frac{GMr}{R^3} = \frac{GM}{R^2}$ and $\lim\limits_{r\to R^+} F(r) = \lim\limits_{r\to R^+} \frac{GM}{r^2} = \frac{GM}{R^2}$, so $\lim\limits_{r\to R} F(r) = \frac{GM}{R^2}$. Since $F(R) = \frac{GM}{R^2}$, F is continuous at R. Therefore, F is a continuous function of r.

35. $f(x) = \begin{cases} cx^2 + 2x & \text{if } x < 2 \\ x^3 - cx & \text{if } x \geq 2 \end{cases}$

f is continuous on $(-\infty, 2)$ and $(2, \infty)$. Now $\lim\limits_{x\to2^-} f(x) = \lim\limits_{x\to2^-} \left(cx^2 + 2x\right) = 4c + 4$ and $\lim\limits_{x\to2^+} f(x) = \lim\limits_{x\to2^+} \left(x^3 - cx\right) = 8 - 2c$. So f is continuous $\Leftrightarrow$ $4c + 4 = 8 - 2c$ $\Leftrightarrow$ $6c = 4$ $\Leftrightarrow$ $c = \frac{2}{3}$. Thus, for f to be continuous on $(-\infty, \infty)$, $c = \frac{2}{3}$.

36. $f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & \text{if } x < 2 \\ ax^2 - bx + 3 & \text{if } 2 \leq x < 3 \\ 2x - a + b & \text{if } x \geq 3 \end{cases}$

At $x = 2$: $\lim\limits_{x\to2^-} f(x) = \lim\limits_{x\to2^-} \frac{x^2-4}{x-2} = \lim\limits_{x\to2^-} \frac{(x+2)(x-2)}{x-2} = \lim\limits_{x\to2^-} (x+2) = 2 + 2 = 4$

$\lim\limits_{x\to2^+} f(x) = \lim\limits_{x\to2^+} (ax^2 - bx + 3) = 4a - 2b + 3$

We must have $4a - b + 3 = 4$, or $\mathbf{4a - 2b = 1}$ **(1)**.

At $x = 3$: $\lim\limits_{x \to 3^-} f(x) = \lim\limits_{x \to 3^-} (ax^2 - bx + 3) = 9a - 3b + 3$

$\lim\limits_{x \to 3^+} f(x) = \lim\limits_{x \to 3^+} (2x - a + b) = 6 - a + b$

We must have $9a - 3b + 3 = 6 - a + b$, or $\mathbf{10a - 4b = 3}$ **(2)**.

Now solve the system of equations by adding -2 times equation **(1)** to equation **(2)**.

$$\begin{array}{rcr} -8a + 4b & = & -2 \\ 10a - 4b & = & 3 \\ \hline 2a & = & 1 \end{array}$$

So $a = \frac{1}{2}$. Substituting $\frac{1}{2}$ for a in **(1)** gives us $-2b = -1$, so $b = \frac{1}{2}$ as well. Thus, for f to be continuous on $(-\infty, \infty)$, $a = b = \frac{1}{2}$.

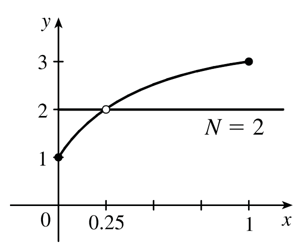
37. (a) $f(x) = \dfrac{x^4 - 1}{x - 1} = \dfrac{(x^2 + 1)(x^2 - 1)}{x - 1} = \dfrac{(x^2 + 1)(x + 1)(x - 1)}{x - 1} = (x^2 + 1)(x + 1)$ [or $x^3 + x^2 + x + 1$]

for $x \neq 1$. The discontinuity is removable and $g(x) = x^3 + x^2 + x + 1$ agrees with f for $x \neq 1$ and is continuous on $\mathbb{R}$.

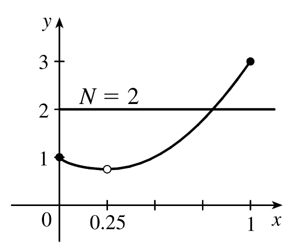
(b) $f(x) = \dfrac{x^3 - x^2 - 2x}{x - 2} = \dfrac{x(x^2 - x - 2)}{x - 2} = \dfrac{x(x - 2)(x + 1)}{x - 2} = x(x + 1)$ [or $x^2 + x$] for $x \neq 2$. The discontinuity is removable and $g(x) = x^2 + x$ agrees with f for $x \neq 2$ and is continuous on $\mathbb{R}$.

(c) $\lim\limits_{x \to \pi^-} f(x) = \lim\limits_{x \to \pi^-} [\![\sin x]\!] = \lim\limits_{x \to \pi^-} 0 = 0$ and $\lim\limits_{x \to \pi^+} f(x) = \lim\limits_{x \to \pi^+} [\![\sin x]\!] = \lim\limits_{x \to \pi^+} (-1) = -1$, so $\lim\limits_{x \to \pi} f(x)$ does not exist. The discontinuity at $x = \pi$ is a jump discontinuity.

38.



f does not satisfy the conclusion of the Intermediate Value Theorem.



f does satisfy the conclusion of the Intermediate Value Theorem.

39. $f(x) = x^2 + 10 \sin x$ is continuous on the interval $[31, 32]$, $f(31) \approx 957$, and $f(32) \approx 1030$. Since $957 < 1000 < 1030$, there is a number c in $(31, 32)$ such that $f(c) = 1000$ by the Intermediate Value Theorem. *Note:* There is also a number c in $(-32, -31)$ such that $f(c) = 1000$.

40. Suppose that $f(3) < 6$. By the Intermediate Value Theorem applied to the continuous function f on the closed interval $[2, 3]$, the fact that $f(2) = 8 > 6$ and $f(3) < 6$ implies that there is a number c in $(2, 3)$ such that $f(c) = 6$. This contradicts the fact that the only solutions of the equation $f(x) = 6$ are $x = 1$ and $x = 4$. Hence, our supposition that $f(3) < 6$ was incorrect. It follows that $f(3) \geq 6$. But $f(3) \neq 6$ because the only solutions of $f(x) = 6$ are $x = 1$ and $x = 4$. Therefore, $f(3) > 6$.

41. $f(x) = x^4 + x - 3$ is continuous on the interval $[1, 2]$, $f(1) = -1$, and $f(2) = 15$. Since $-1 < 0 < 15$, there is a number c in $(1, 2)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $x^4 + x - 3 = 0$ in the interval $(1, 2)$.

42. $f(x) = \sqrt[3]{x} + x - 1$ is continuous on the interval $[0, 1]$, $f(0) = -1$, and $f(1) = 1$. Since $-1 < 0 < 1$, there is a number c in $(0, 1)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\sqrt[3]{x} + x - 1 = 0$, or $\sqrt[3]{x} = 1 - x$, in the interval $(0, 1)$.

43. The equation $e^x = 3 - 2x$ is equivalent to the equation $e^x + 2x - 3 = 0$. $f(x) = e^x + 2x - 3$ is continuous on the interval $[0, 1]$, $f(0) = -2$, and $f(1) = e - 1 \approx 1.72$. Since $-2 < 0 < e - 1$, there is a number c in $(0, 1)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $e^x + 2x - 3 = 0$, or $e^x = 3 - 2x$, in the interval $(0, 1)$.

44. The equation $\sin x = x^2 - x$ is equivalent to the equation $\sin x - x^2 + x = 0$. $f(x) = \sin x - x^2 + x$ is continuous on the interval $[1, 2]$, $f(1) = \sin 1 \approx 0.84$, and $f(2) = \sin 2 - 2 \approx -1.09$. Since $\sin 1 > 0 > \sin 2 - 2$, there is a number c in $(1, 2)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\sin x - x^2 + x = 0$, or $\sin x = x^2 - x$, in the interval $(1, 2)$.

45. (a) $f(x) = \cos x - x^3$ is continuous on the interval $[0, 1]$, $f(0) = 1 > 0$, and $f(1) = \cos 1 - 1 \approx -0.46 < 0$. Since $1 > 0 > -0.46$, there is a number c in $(0, 1)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\cos x - x^3 = 0$, or $\cos x = x^3$, in the interval $(0, 1)$.

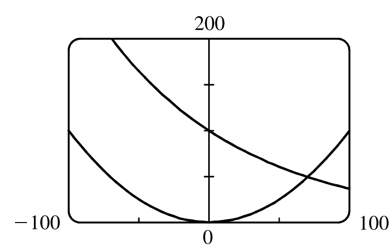
(b) $f(0.86) \approx 0.016 > 0$ and $f(0.87) \approx -0.014 < 0$, so there is a root between 0.86 and 0.87, that is, in the interval $(0.86, 0.87)$.

46. (a) $f(x) = \ln x - 3 + 2x$ is continuous on the interval $[1, 2]$, $f(1) = -1 < 0$, and $f(2) = \ln 2 + 1 \approx 1.7 > 0$. Since $-1 < 0 < 1.7$, there is a number c in $(1, 2)$ such that $f(c) = 0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\ln x - 3 + 2x = 0$, or $\ln x = 3 - 2x$, in the interval $(1, 2)$.

(b) $f(1.34) \approx -0.03 < 0$ and $f(1.35) \approx 0.0001 > 0$, so there is a root between 1.34 and 1.35, that is, in the interval $(1.34, 1.35)$.

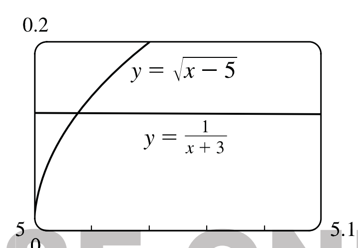
47. (a) Let $f(x) = 100e^{-x/100} - 0.01x^2$. Then $f(0) = 100 > 0$ and $f(100) = 100e^{-1} - 100 \approx -63.2 < 0$. So by the Intermediate Value Theorem, there is a number c in $(0, 100)$ such that $f(c) = 0$. This implies that $100e^{-c/100} = 0.01c^2$.

(b) Using the intersect feature of the graphing device, we find that the root of the equation is $x = 70.347$, correct to three decimal places.



48. (a) Let $f(x) = \sqrt{x - 5} - \dfrac{1}{x + 3}$. Then $f(5) = -\frac{1}{8} < 0$ and $f(6) = \frac{8}{9} > 0$, and f is continuous on $[5, \infty)$. So by the Intermediate Value Theorem, there is a number c in $(5, 6)$ such that $f(c) = 0$. This implies that $\dfrac{1}{c + 3} = \sqrt{c - 5}$.

(b) Using the intersect feature of the graphing device, we find that the root of the equation is $x = 5.016$, correct to three decimal places.



49. $\lim_{h\to 0}\sin(a+h) = \lim_{h\to 0}(\sin a\cos h + \cos a\sin h) = \lim_{h\to 0}(\sin a\cos h) + \lim_{h\to 0}(\cos a\sin h)$

$$= \left(\lim_{h\to 0}\sin a\right)\left(\lim_{h\to 0}\cos h\right) + \left(\lim_{h\to 0}\cos a\right)\left(\lim_{h\to 0}\sin h\right) = (\sin a)(1) + (\cos a)(0) = \sin a$$

50. As in the previous exercise, we must show that $\lim_{h\to 0}\cos(a+h) = \cos a$ to prove that the cosine function is continuous.

$$\lim_{h\to 0}\cos(a+h) = \lim_{h\to 0}(\cos a\cos h - \sin a\sin h) = \lim_{h\to 0}(\cos a\cos h) - \lim_{h\to 0}(\sin a\sin h)$$

$$= \left(\lim_{h\to 0}\cos a\right)\left(\lim_{h\to 0}\cos h\right) - \left(\lim_{h\to 0}\sin a\right)\left(\lim_{h\to 0}\sin h\right) = (\cos a)(1) - (\sin a)(0) = \cos a$$

51. If there is such a number, it satisfies the equation $x^3 + 1 = x \quad\Leftrightarrow\quad x^3 - x + 1 = 0$. Let the left-hand side of this equation be called $f(x)$. Now $f(-2) = -5 < 0$, and $f(-1) = 1 > 0$. Note also that $f(x)$ is a polynomial, and thus continuous. So by the Intermediate Value Theorem, there is a number c between -2 and -1 such that $f(c) = 0$, so that $c = c^3 + 1$.

52. $\dfrac{a}{x^3 + 2x^2 - 1} + \dfrac{b}{x^3 + x - 2} = 0 \quad\Rightarrow\quad a(x^3 + x - 2) + b(x^3 + 2x^2 - 1) = 0$. Let $p(x)$ denote the left side of the last equation. Since p is continuous on $[-1, 1]$, $p(-1) = -4a < 0$, and $p(1) = 2b > 0$, there exists a c in $(-1, 1)$ such that $p(c) = 0$ by the Intermediate Value Theorem. Note that the only root of either denominator that is in $(-1, 1)$ is $(-1 + \sqrt{5})/2 = r$, but $p(r) = (3\sqrt{5} - 9)a/2 \neq 0$. Thus, c is not a root of either denominator, so $p(c) = 0 \quad\Rightarrow$ $x = c$ is a root of the given equation.

53. $f(x) = x^4\sin(1/x)$ is continuous on $(-\infty, 0) \cup (0, \infty)$ since it is the product of a polynomial and a composite of a trigonometric function and a rational function. Now since $-1 \le \sin(1/x) \le 1$, we have $-x^4 \le x^4\sin(1/x) \le x^4$. Because $\lim_{x\to 0}(-x^4) = 0$ and $\lim_{x\to 0}x^4 = 0$, the Squeeze Theorem gives us $\lim_{x\to 0}(x^4\sin(1/x)) = 0$, which equals $f(0)$. Thus, f is continuous at 0 and, hence, on $(-\infty, \infty)$.

54. (a) $\lim_{x\to 0^+}F(x) = 0$ and $\lim_{x\to 0^-}F(x) = 0$, so $\lim_{x\to 0}F(x) = 0$, which is $F(0)$, and hence F is continuous at $x = a$ if $a = 0$. For $a > 0$, $\lim_{x\to a}F(x) = \lim_{x\to a}x = a = F(a)$. For $a < 0$, $\lim_{x\to a}F(x) = \lim_{x\to a}(-x) = -a = F(a)$. Thus, F is continuous at $x = a$; that is, continuous everywhere.

(b) Assume that f is continuous on the interval I. Then for $a \in I$, $\lim_{x\to a}|f(x)| = \left|\lim_{x\to a}f(x)\right| = |f(a)|$ by Theorem 8. (If a is an endpoint of I, use the appropriate one-sided limit.) So $|f|$ is continuous on I.

(c) No, the converse is false. For example, the function $f(x) = \begin{cases} 1 & \text{if } x \ge 0 \\ -1 & \text{if } x < 0 \end{cases}$ is not continuous at $x = 0$, but $|f(x)| = 1$ is continuous on $\mathbb{R}$.

55. Define $u(t)$ to be the monk's distance from the monastery, as a function of time, on the first day, and define $d(t)$ to be his distance from the monastery, as a function of time, on the second day. Let D be the distance from the monastery to the top of the mountain. From the given information we know that $u(0) = 0$, $u(12) = D$, $d(0) = D$ and $d(12) = 0$. Now consider the function $u - d$, which is clearly continuous. We calculate that $(u - d)(0) = -D$ and $(u - d)(12) = D$. So by the Intermediate Value Theorem, there must be some time t_0 between 0 and 12 such that $(u - d)(t_0) = 0 \quad\Leftrightarrow\quad u(t_0) = d(t_0)$. So at time t_0 after 7:00 AM, the monk will be at the same place on both days.

2.5 Limits Involving Infinity

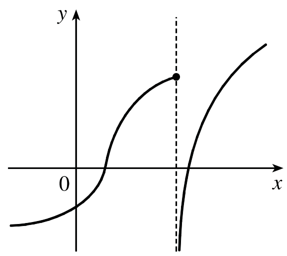
1. (a) As x approaches 2 (from the right or the left), the values of $f(x)$ become large.

(b) As x approaches 1 from the right, the values of $f(x)$ become large negative.

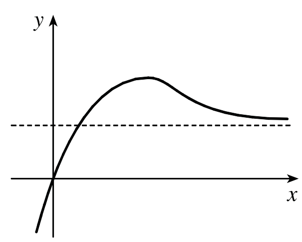
(c) As x becomes large, the values of $f(x)$ approach 5.

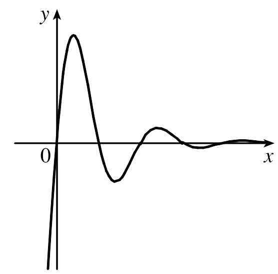
(d) As x becomes large negative, the values of $f(x)$ approach 3.

2. (a) The graph of a function can intersect a vertical asymptote in the sense that it can meet but not cross it.

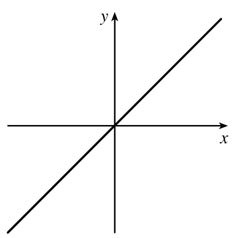


The graph of a function can intersect a horizontal asymptote. It can even intersect its horizontal asymptote an infinite number of times.

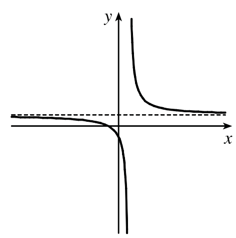




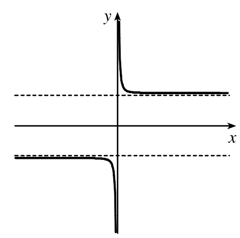
(b) The graph of a function can have 0, 1, or 2 horizontal asymptotes. Representative examples are shown.



No horizontal asymptote



One horizontal asymptote



Two horizontal asymptotes

3. (a) $\lim_{x\to 2} f(x) = \infty$ (b) $\lim_{x\to -1^-} f(x) = \infty$ (c) $\lim_{x\to -1^+} f(x) = -\infty$ (d) $\lim_{x\to\infty} f(x) = 1$

(e) $\lim_{x\to -\infty} f(x) = 2$ (f) Vertical: $x = -1$, $x = 2$; Horizontal: $y = 1$, $y = 2$

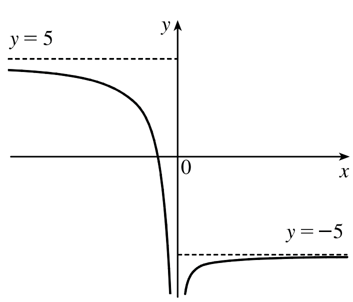
4. (a) $\lim_{x\to\infty} g(x) = 2$ (b) $\lim_{x\to -\infty} g(x) = -2$ (c) $\lim_{x\to 3} g(x) = \infty$

(d) $\lim_{x\to 0} g(x) = -\infty$ (e) $\lim_{x\to -2^+} g(x) = -\infty$ (f) Vertical: $x = -2$, $x = 0$, $x = 3$; Horizontal: $y = -2$, $y = 2$

5. $\lim_{x\to 0} f(x) = -\infty$,

$\lim_{x\to -\infty} f(x) = 5$,

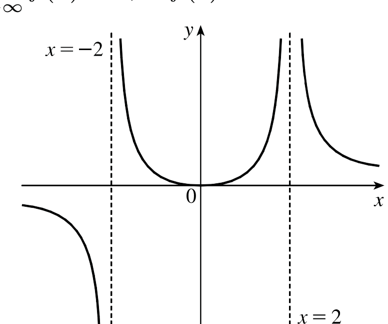
$\lim_{x\to\infty} f(x) = -5$



6. $\lim_{x\to 2} f(x) = \infty$, $\lim_{x\to -2^+} f(x) = \infty$,

$\lim_{x\to -2^-} f(x) = -\infty$, $\lim_{x\to -\infty} f(x) = 0$,

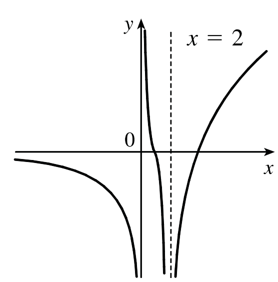
$\lim_{x\to\infty} f(x) = 0$, $f(0) = 0$



7. $\lim_{x\to 2} f(x) = -\infty$, $\lim_{x\to\infty} f(x) = \infty$,

$\lim_{x\to -\infty} f(x) = 0$, $\lim_{x\to 0^+} f(x) = \infty$,

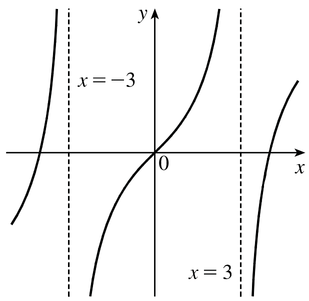
$\lim_{x\to 0^-} f(x) = -\infty$



8. $\lim_{x\to 3^+} f(x) = -\infty$,

$\lim_{x\to 3^-} f(x) = \infty$,

$\lim_{x\to \infty} f(x) = \infty$, f is odd

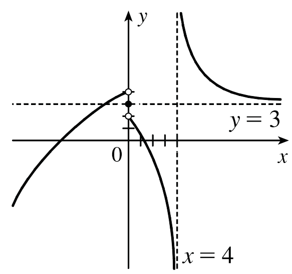


9. $f(0) = 3$, $\lim_{x\to 0^-} f(x) = 4$,

$\lim_{x\to 0^+} f(x) = 2$,

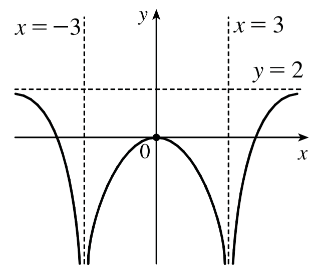
$\lim_{x\to -\infty} f(x) = -\infty$, $\lim_{x\to 4^-} f(x) = -\infty$,

$\lim_{x\to 4^+} f(x) = \infty$, $\lim_{x\to \infty} f(x) = 3$



10. $\lim_{x\to 3} f(x) = -\infty$, $\lim_{x\to \infty} f(x) = 2$,

$f(0) = 0$, f is even



11. If $f(x) = x^2/2^x$, then a calculator gives $f(0) = 0$, $f(1) = 0.5$, $f(2) = 1$, $f(3) = 1.125$, $f(4) = 1$, $f(5) = 0.78125$, $f(6) = 0.5625$, $f(7) = 0.3828125$, $f(8) = 0.25$, $f(9) = 0.158203125$, $f(10) = 0.09765625$, $f(20) \approx 0.00038147$, $f(50) \approx 2.2204 \times 10^{-12}$, $f(100) \approx 7.8886 \times 10^{-27}$.

It appears that $\lim_{x\to\infty} \left(x^2/2^x\right) = 0$.

12. (a) $f(x) = \dfrac{1}{x^3 - 1}$.

From these calculations, it seems that

$\lim_{x\to 1^-} f(x) = -\infty$ and $\lim_{x\to 1^+} f(x) = \infty$.

x	$f(x)$
0.5	−1.14
0.9	−3.69
0.99	−33.7
0.999	−333.7
0.9999	−3333.7
0.99999	−33,333.7

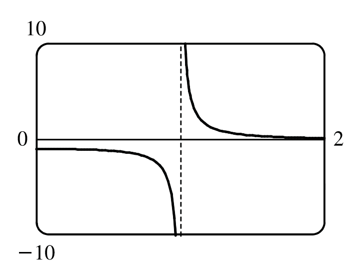
x	$f(x)$
1.5	0.42
1.1	3.02
1.01	33.0
1.001	333.0
1.0001	3333.0
1.00001	33,333.3

(b) If x is slightly smaller than 1, then $x^3 - 1$ will be a negative number close to 0, and the reciprocal of $x^3 - 1$, that is, $f(x)$, will be a negative number with large absolute value. So $\lim_{x\to 1^-} f(x) = -\infty$.

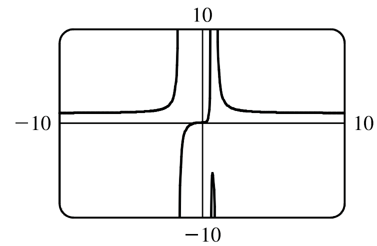
If x is slightly larger than 1, then $x^3 - 1$ will be a small positive number, and its reciprocal, $f(x)$, will be a large positive number. So $\lim_{x\to 1^+} f(x) = \infty$.

(c) It appears from the graph of f that

$\lim_{x\to 1^-} f(x) = -\infty$ and $\lim_{x\to 1^+} f(x) = \infty$.



13. Vertical: $x \approx -1.62$, $x \approx 0.62$, $x = 1$;
Horizontal: $y = 1$



14. (a) From a graph of $f(x) = (1 - 2/x)^x$ in a window of $[0, 10{,}000]$ by $[0, 0.2]$, we estimate that $\lim_{x \to \infty} f(x) = 0.14$ (to two decimal places.)

(b)

x	$f(x)$
10,000	0.135308
100,000	0.135333
1,000,000	0.135335

From the table, we estimate that $\lim_{x \to \infty} f(x) = 0.1353$ (to four decimal places.)

15. $\lim_{x \to 1} \dfrac{2 - x}{(x - 1)^2} = \infty$ since the numerator is positive and the denominator approaches 0 through positive values as $x \to 1$.

16. $\lim_{x \to -3^-} \dfrac{x + 2}{x + 3} = \infty$ since the numerator is negative and the denominator approaches 0 from the negative side as $x \to -3^-$.

17. Let $t = 3/(2 - x)$. As $x \to 2^+$, $t \to -\infty$. So $\lim_{x \to 2^+} e^{3/(2-x)} = \lim_{t \to -\infty} e^t = 0$ by (7).

18. $\lim_{x \to \pi^-} \cot x = \lim_{x \to \pi^-} \dfrac{\cos x}{\sin x} = -\infty$ since the numerator is negative and the denominator approaches 0 through positive values as $x \to \pi^-$.

19. Let $t = x^2 - 9$. Then as $x \to 3^+$, $t \to 0^+$, and $\lim_{x \to 3^+} \ln(x^2 - 9) = \lim_{t \to 0^+} \ln t = -\infty$ by (3).

20. $\lim_{x \to 2^-} \dfrac{x^2 - 2x}{x^2 - 4x + 4} = \lim_{x \to 2^-} \dfrac{x(x-2)}{(x-2)^2} = \lim_{x \to 2^-} \dfrac{x}{x - 2} = -\infty$ since the numerator is positive and the denominator approaches 0 through negative values as $x \to 2^-$.

21. $\lim_{x \to 2\pi^-} x \csc x = \lim_{x \to 2\pi^-} \dfrac{x}{\sin x} = -\infty$ since the numerator is positive and the denominator approaches 0 through negative values as $x \to 2\pi^-$.

22. $\lim_{x \to \infty} \dfrac{3x + 5}{x - 4} = \lim_{x \to \infty} \dfrac{(3x + 5)/x}{(x - 4)/x} = \lim_{x \to \infty} \dfrac{3 + 5/x}{1 - 4/x} = \dfrac{\lim_{x \to \infty} 3 + 5 \lim_{x \to \infty} \frac{1}{x}}{\lim_{x \to \infty} 1 - 4 \lim_{x \to \infty} \frac{1}{x}} = \dfrac{3 + 5(0)}{1 - 4(0)} = 3$

23. Divide both the numerator and denominator by x^3 (the highest power of x that occurs in the denominator).

$$\lim_{x \to \infty} \frac{x^3 + 5x}{2x^3 - x^2 + 4} = \lim_{x \to \infty} \frac{\dfrac{x^3 + 5x}{x^3}}{\dfrac{2x^3 - x^2 + 4}{x^3}} = \lim_{x \to \infty} \frac{1 + \dfrac{5}{x^2}}{2 - \dfrac{1}{x} + \dfrac{4}{x^3}} = \frac{\lim_{x \to \infty} \left(1 + \dfrac{5}{x^2}\right)}{\lim_{x \to \infty} \left(2 - \dfrac{1}{x} + \dfrac{4}{x^3}\right)}$$

$$= \frac{\lim_{x \to \infty} 1 + 5 \lim_{x \to \infty} \dfrac{1}{x^2}}{\lim_{x \to \infty} 2 - \lim_{x \to \infty} \dfrac{1}{x} + 4 \lim_{x \to \infty} \dfrac{1}{x^3}} = \frac{1 + 5(0)}{2 - 0 + 4(0)} = \frac{1}{2}$$

24. $\lim\limits_{t\to-\infty}\dfrac{t^2+2}{t^3+t^2-1}=\lim\limits_{t\to-\infty}\dfrac{(t^2+2)/t^3}{(t^3+t^2-1)/t^3}=\lim\limits_{t\to-\infty}\dfrac{1/t+2/t^3}{1+1/t-1/t^3}=\dfrac{0+0}{1+0-0}=0$

25. First, multiply the factors in the denominator. Then divide both the numerator and denominator by u^4.

$$\lim_{u\to\infty}\frac{4u^4+5}{(u^2-2)(2u^2-1)}=\lim_{u\to\infty}\frac{4u^4+5}{2u^4-5u^2+2}=\lim_{u\to\infty}\frac{\dfrac{4u^4+5}{u^4}}{\dfrac{2u^4-5u^2+2}{u^4}}=\lim_{u\to\infty}\frac{4+\dfrac{5}{u^4}}{2-\dfrac{5}{u^2}+\dfrac{2}{u^4}}$$

$$=\frac{\lim\limits_{u\to\infty}\left(4+\dfrac{5}{u^4}\right)}{\lim\limits_{u\to\infty}\left(2-\dfrac{5}{u^2}+\dfrac{2}{u^4}\right)}=\frac{\lim\limits_{u\to\infty}4+5\lim\limits_{u\to\infty}\dfrac{1}{u^4}}{\lim\limits_{u\to\infty}2-5\lim\limits_{u\to\infty}\dfrac{1}{u^2}+2\lim\limits_{u\to\infty}\dfrac{1}{u^4}}=\frac{4+5(0)}{2-5(0)+2(0)}=\frac{4}{2}=2$$

26. $\lim\limits_{x\to\infty}\dfrac{x+2}{\sqrt{9x^2+1}}=\lim\limits_{x\to\infty}\dfrac{(x+2)/x}{\sqrt{9x^2+1}/\sqrt{x^2}}=\lim\limits_{x\to\infty}\dfrac{1+2/x}{\sqrt{9+1/x^2}}=\dfrac{1+0}{\sqrt{9+0}}=\dfrac{1}{3}$

27. $\lim\limits_{x\to\infty}\left(\sqrt{9x^2+x}-3x\right)=\lim\limits_{x\to\infty}\dfrac{\left(\sqrt{9x^2+x}-3x\right)\left(\sqrt{9x^2+x}+3x\right)}{\sqrt{9x^2+x}+3x}=\lim\limits_{x\to\infty}\dfrac{\left(\sqrt{9x^2+x}\right)^2-(3x)^2}{\sqrt{9x^2+x}+3x}$

$$=\lim_{x\to\infty}\frac{(9x^2+x)-9x^2}{\sqrt{9x^2+x}+3x}=\lim_{x\to\infty}\frac{x}{\sqrt{9x^2+x}+3x}\cdot\frac{1/x}{1/x}$$

$$=\lim_{x\to\infty}\frac{x/x}{\sqrt{9x^2/x^2+x/x^2}+3x/x}=\lim_{x\to\infty}\frac{1}{\sqrt{9+1/x}+3}=\frac{1}{\sqrt{9}+3}=\frac{1}{3+3}=\frac{1}{6}$$

28. $\lim\limits_{x\to\infty}\left(\sqrt{x^2+ax}-\sqrt{x^2+bx}\right)=\lim\limits_{x\to\infty}\dfrac{\left(\sqrt{x^2+ax}-\sqrt{x^2+bx}\right)\left(\sqrt{x^2+ax}+\sqrt{x^2+bx}\right)}{\sqrt{x^2+ax}+\sqrt{x^2+bx}}$

$$=\lim_{x\to\infty}\frac{(x^2+ax)-(x^2+bx)}{\sqrt{x^2+ax}+\sqrt{x^2+bx}}=\lim_{x\to\infty}\frac{[(a-b)x]/x}{\left(\sqrt{x^2+ax}+\sqrt{x^2+bx}\right)/\sqrt{x^2}}$$

$$=\lim_{x\to\infty}\frac{a-b}{\sqrt{1+a/x}+\sqrt{1+b/x}}=\frac{a-b}{\sqrt{1+0}+\sqrt{1+0}}=\frac{a-b}{2}$$

29. Let $t=-x^2$. As $x\to\infty$, $t\to-\infty$. So $\lim\limits_{x\to\infty}e^{-x^2}=\lim\limits_{t\to-\infty}e^t=0$ by (7).

30. For $x>0$, $\sqrt{x^2+1}>\sqrt{x^2}=x$. So as $x\to\infty$, we have $\sqrt{x^2+1}\to\infty$, that is, $\lim\limits_{x\to\infty}\sqrt{x^2+1}=\infty$.

31. $\lim\limits_{x\to\infty}\cos x$ does not exist because as x increases $\cos x$ does not approach any one value, but oscillates between 1 and -1.

32. Since $0\le\sin^2 x\le 1$, we have $0\le\dfrac{\sin^2 x}{x^2}\le\dfrac{1}{x^2}$. Now $\lim\limits_{x\to\infty}0=0$ and $\lim\limits_{x\to\infty}\dfrac{1}{x^2}=0$, so by the Squeeze Theorem, $\lim\limits_{x\to\infty}\dfrac{\sin^2 x}{x^2}=0$.

33. Since $-1\le\cos x\le 1$ and $e^{-2x}>0$, we have $-e^{-2x}\le e^{-2x}\cos x\le e^{-2x}$. We know that $\lim\limits_{x\to\infty}(-e^{-2x})=0$ and $\lim\limits_{x\to\infty}\left(e^{-2x}\right)=0$, so by the Squeeze Theorem, $\lim\limits_{x\to\infty}(e^{-2x}\cos x)=0$.

34. Divide numerator and denominator by e^{3x}: $\lim\limits_{x\to\infty}\dfrac{e^{3x}-e^{-3x}}{e^{3x}+e^{-3x}}=\lim\limits_{x\to\infty}\dfrac{1-e^{-6x}}{1+e^{-6x}}=\dfrac{1-0}{1+0}=1$

35. $\lim\limits_{x\to-\infty}(x^4+x^5)=\lim\limits_{x\to-\infty}x^5(\frac{1}{x}+1)$ [factor out the largest power of x] $=-\infty$ because $x^5\to-\infty$ and $1/x+1\to1$ as $x\to-\infty$.

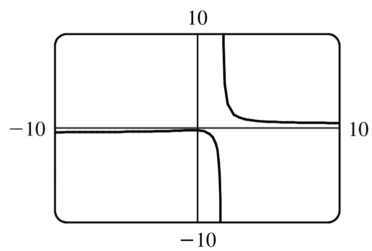
Or: $\lim\limits_{x\to-\infty}\left(x^4+x^5\right)=\lim\limits_{x\to-\infty}x^4\left(1+x\right)=-\infty$.

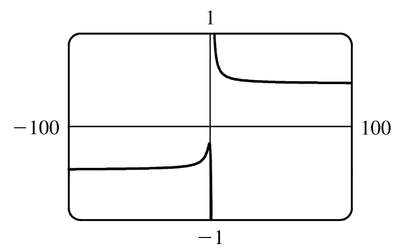
36. If we let $t = \tan x$, then as $x \to (\pi/2)^+$, $t \to -\infty$. Thus, $\displaystyle\lim_{x\to(\pi/2)^+} e^{\tan x} = \lim_{t\to-\infty} e^t = 0$.

37. $$\lim_{x\to\infty} \frac{x + x^3 + x^5}{1 - x^2 + x^4} = \lim_{x\to\infty} \frac{(x + x^3 + x^5)/x^4}{(1 - x^2 + x^4)/x^4} \quad \text{[divide by the highest power of } x \text{ in the denominator]}$$
$$= \lim_{x\to\infty} \frac{1/x^3 + 1/x + x}{1/x^4 - 1/x^2 + 1} = \infty$$

because $(1/x^3 + 1/x + x) \to \infty$ and $(1/x^4 - 1/x^2 + 1) \to 1$ as $x \to \infty$.

38. (a)





From the graph, it appears at first that there is only one horizontal asymptote, at $y \approx 0$, and a vertical asymptote at $x \approx 1.7$. However, if we graph the function with a wider viewing rectangle, we see that in fact there seem to be two horizontal asymptotes: one at $y \approx 0.5$ and one at $y \approx -0.5$. So we estimate that

$$\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5} \approx 0.5 \qquad \text{and} \qquad \lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{3x-5} \approx -0.5$$

(b) $f(1000) \approx 0.4722$ and $f(10{,}000) \approx 0.4715$, so we estimate that $\displaystyle\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5} \approx 0.47$.

$f(-1000) \approx -0.4706$ and $f(-10{,}000) \approx -0.4713$, so we estimate that $\displaystyle\lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{3x-5} \approx -0.47$.

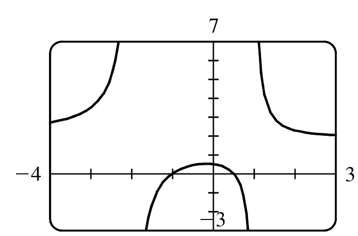
(c) $\displaystyle\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to\infty} \frac{\sqrt{2+1/x^2}}{3-5/x}$ [since $\sqrt{x^2} = x$ for $x > 0$] $\displaystyle = \frac{\sqrt{2}}{3} \approx 0.471404$.

For $x < 0$, we have $\sqrt{x^2} = |x| = -x$, so when we divide the numerator by x, with $x < 0$, we get $\displaystyle\frac{1}{x}\sqrt{2x^2+1} = -\frac{1}{\sqrt{x^2}}\sqrt{2x^2+1} = -\sqrt{2+1/x^2}$. Therefore,

$$\lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to-\infty} \frac{-\sqrt{2+1/x^2}}{3-5/x} = -\frac{\sqrt{2}}{3} \approx -0.471404.$$

39. $$\lim_{x\to\infty} \frac{2x^2+x-1}{x^2+x-2} = \lim_{x\to\infty} \frac{\dfrac{2x^2+x-1}{x^2}}{\dfrac{x^2+x-2}{x^2}} = \lim_{x\to\infty} \frac{2 + \dfrac{1}{x} - \dfrac{1}{x^2}}{1 + \dfrac{1}{x} - \dfrac{2}{x^2}} = \frac{\displaystyle\lim_{x\to\infty}\left(2 + \frac{1}{x} - \frac{1}{x^2}\right)}{\displaystyle\lim_{x\to\infty}\left(1 + \frac{1}{x} - \frac{2}{x^2}\right)}$$
$$= \frac{\displaystyle\lim_{x\to\infty} 2 + \lim_{x\to\infty}\frac{1}{x} - \lim_{x\to\infty}\frac{1}{x^2}}{\displaystyle\lim_{x\to\infty} 1 + \lim_{x\to\infty}\frac{1}{x} - 2\lim_{x\to\infty}\frac{1}{x^2}} = \frac{2+0-0}{1+0-2(0)} = 2, \quad \text{so } y = 2 \text{ is a horizontal asymptote.}$$

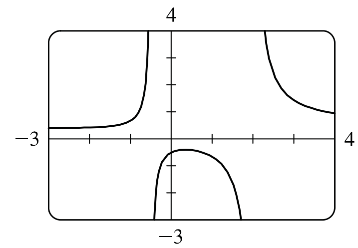
$y = f(x) = \dfrac{2x^2+x-1}{x^2+x-2} = \dfrac{(2x-1)(x+1)}{(x+2)(x-1)}$, so $\displaystyle\lim_{x\to-2^-} f(x) = \infty$, $\displaystyle\lim_{x\to-2^+} f(x) = -\infty$, $\displaystyle\lim_{x\to1^-} f(x) = -\infty$, and $\displaystyle\lim_{x\to1^+} f(x) = \infty$. Thus, $x = -2$ and $x = 1$ are vertical asymptotes. The graph confirms our work.



40. $\displaystyle \lim_{x\to\infty} \frac{x^2+1}{2x^2-3x-2} = \lim_{x\to\infty} \frac{\dfrac{x^2+1}{x^2}}{\dfrac{2x^2-3x-2}{x^2}} = \lim_{x\to\infty} \frac{1+\dfrac{1}{x^2}}{2-\dfrac{3}{x}-\dfrac{2}{x^2}} = \frac{\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x^2}\right)}{\lim\limits_{x\to\infty}\left(2-\dfrac{3}{x}-\dfrac{2}{x^2}\right)}$

$$= \frac{\lim\limits_{x\to\infty} 1 + \lim\limits_{x\to\infty} \dfrac{1}{x^2}}{\lim\limits_{x\to\infty} 2 - \lim\limits_{x\to\infty} \dfrac{3}{x} - \lim\limits_{x\to\infty} \dfrac{2}{x^2}} = \frac{1+0}{2-0-0} = \frac{1}{2},$$ so $y = \frac{1}{2}$ is a horizontal asymptote.

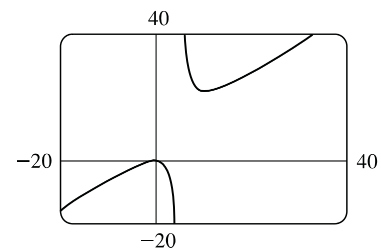
$y = f(x) = \dfrac{x^2+1}{2x^2-3x-2} = \dfrac{x^2+1}{(2x+1)(x-2)}$, so $\lim\limits_{x\to(-1/2)^-} f(x) = \infty$ because as $x \to (-1/2)^-$ the numerator is positive while the denominator approaches 0 through positive values. Similarly, $\lim\limits_{x\to(-1/2)^+} f(x) = -\infty$, $\lim\limits_{x\to 2^-} f(x) = -\infty$, and $\lim\limits_{x\to 2^+} f(x) = \infty$. Thus, $x = -\frac{1}{2}$ and $x = 2$ are vertical asymptotes. The graph confirms our work.



41. $y = f(x) = \dfrac{x^3-x}{x^2-6x+5} = \dfrac{x(x^2-1)}{(x-1)(x-5)} = \dfrac{x(x+1)(x-1)}{(x-1)(x-5)} = \dfrac{x(x+1)}{x-5} = g(x)$ for $x \neq 1$.

The graph of g is the same as the graph of f with the exception of a hole in the graph of f at $x = 1$. By long division, $g(x) = \dfrac{x^2+x}{x-5} = x + 6 + \dfrac{30}{x-5}$.

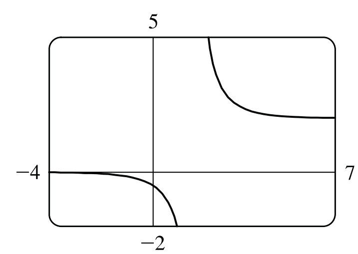
As $x \to \pm\infty$, $g(x) \to \pm\infty$, so there is no horizontal asymptote. The denominator of g is zero when $x = 5$. $\lim\limits_{x\to 5^-} g(x) = -\infty$ and $\lim\limits_{x\to 5^+} g(x) = \infty$, so $x = 5$ is a vertical asymptote. The graph confirms our work.



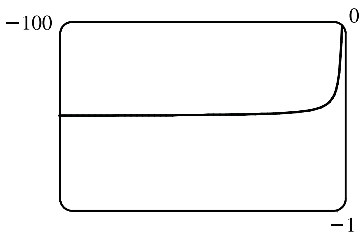
42. $\displaystyle \lim_{x\to\infty} \frac{2e^x}{e^x-5} = \lim_{x\to\infty} \frac{2e^x}{e^x-5} \cdot \frac{1/e^x}{1/e^x} = \lim_{x\to\infty} \frac{2}{1-(5/e^x)} = \frac{2}{1-0} = 2$, so $y = 2$ is a horizontal asymptote.

$\displaystyle \lim_{x\to-\infty} \frac{2e^x}{e^x-5} = \frac{2(0)}{0-5} = 0$, so $y = 0$ is a horizontal asymptote. The denominator is zero (and the numerator isn't) when $e^x - 5 = 0 \quad\Rightarrow\quad e^x = 5 \quad\Rightarrow\quad x = \ln 5$.

$\displaystyle \lim_{x\to(\ln 5)^+} \frac{2e^x}{e^x-5} = \infty$ since the numerator approaches 10 and the denominator approaches 0 through positive values as $x \to (\ln 5)^+$. Similarly, $\displaystyle \lim_{x\to(\ln 5)^-} \frac{2e^x}{e^x-5} = -\infty$. Thus, $x = \ln 5$ is a vertical asymptote. The graph confirms our work.



43. (a)



From the graph of $f(x) = \sqrt{x^2 + x + 1} + x$, we estimate the value of $\lim_{x\to-\infty} f(x)$ to be -0.5.

(b)

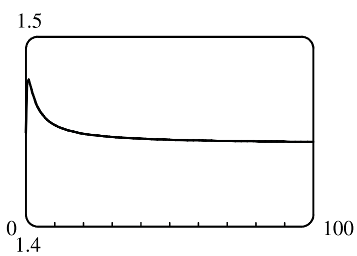
x	$f(x)$
$-10{,}000$	-0.4999625
$-100{,}000$	-0.4999962
$-1{,}000{,}000$	-0.4999996

From the table, we estimate the limit to be -0.5.

(c) $$\lim_{x\to-\infty}\left(\sqrt{x^2+x+1}+x\right) = \lim_{x\to-\infty}\left(\sqrt{x^2+x+1}+x\right)\left[\frac{\sqrt{x^2+x+1}-x}{\sqrt{x^2+x+1}-x}\right] = \lim_{x\to-\infty}\frac{(x^2+x+1)-x^2}{\sqrt{x^2+x+1}-x}$$
$$= \lim_{x\to-\infty}\frac{(x+1)(1/x)}{\left(\sqrt{x^2+x+1}-x\right)(1/x)} = \lim_{x\to-\infty}\frac{1+(1/x)}{-\sqrt{1+(1/x)+(1/x^2)}-1}$$
$$= \frac{1+0}{-\sqrt{1+0+0}-1} = -\frac{1}{2}$$

Note that for $x < 0$, we have $\sqrt{x^2} = |x| = -x$, so when we divide the radical by x, with $x < 0$, we get $\frac{1}{x}\sqrt{x^2+x+1} = -\frac{1}{\sqrt{x^2}}\sqrt{x^2+x+1} = -\sqrt{1+(1/x)+(1/x^2)}$.

44. (a)



From the graph of $f(x) = \sqrt{3x^2+8x+6} - \sqrt{3x^2+3x+1}$, we estimate (to one decimal place) the value of $\lim_{x\to\infty} f(x)$ to be 1.4.

(b)

x	$f(x)$
10,000	1.44339
100,000	1.44338
1,000,000	1.44338

From the table, we estimate (to four decimal places) the limit to be 1.4434.

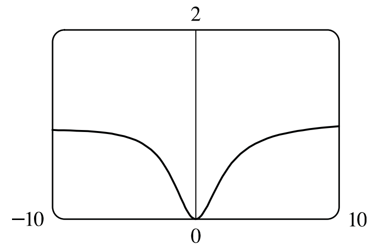
(c) $$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\frac{\left(\sqrt{3x^2+8x+6}-\sqrt{3x^2+3x+1}\right)\left(\sqrt{3x^2+8x+6}+\sqrt{3x^2+3x+1}\right)}{\sqrt{3x^2+8x+6}+\sqrt{3x^2+3x+1}}$$
$$= \lim_{x\to\infty}\frac{(3x^2+8x+6)-(3x^2+3x+1)}{\sqrt{3x^2+8x+6}+\sqrt{3x^2+3x+1}} = \lim_{x\to\infty}\frac{(5x+5)(1/x)}{\left(\sqrt{3x^2+8x+6}+\sqrt{3x^2+3x+1}\right)(1/x)}$$
$$= \lim_{x\to\infty}\frac{5+5/x}{\sqrt{3+8/x+6/x^2}+\sqrt{3+3/x+1/x^2}} = \frac{5}{\sqrt{3}+\sqrt{3}} = \frac{5}{2\sqrt{3}} = \frac{5\sqrt{3}}{6} \approx 1.443376$$

45. From the graph, it appears $y = 1$ is a horizontal asymptote.

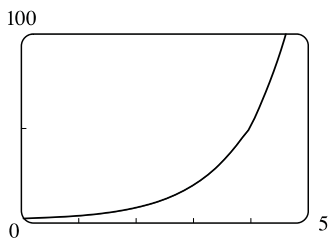
$$\lim_{x\to\infty} \frac{3x^3 + 500x^2}{x^3 + 500x^2 + 100x + 2000} = \lim_{x\to\infty} \frac{\dfrac{3x^3 + 500x^2}{x^3}}{\dfrac{x^3 + 500x^2 + 100x + 2000}{x^3}} = \lim_{x\to\infty} \frac{3 + (500/x)}{1 + (500/x) + (100/x^2) + (2000/x^3)}$$

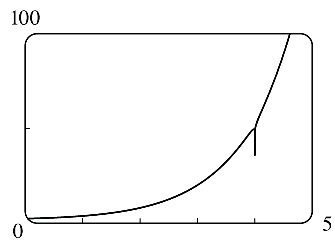
$$= \frac{3 + 0}{1 + 0 + 0 + 0} = 3, \quad \text{so } y = 3 \text{ is a horizontal asymptote.}$$

The discrepancy can be explained by the choice of the viewing window. Try $[-100{,}000,\ 100{,}000]$ by $[-1, 4]$ to get a graph that lends credibility to our calculation that $y = 3$ is a horizontal asymptote.



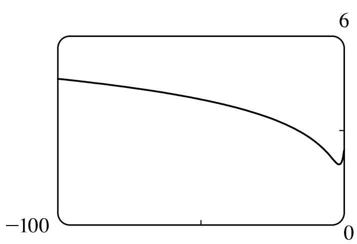
46. (a)

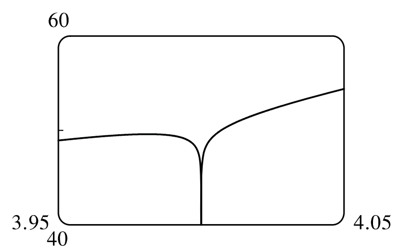




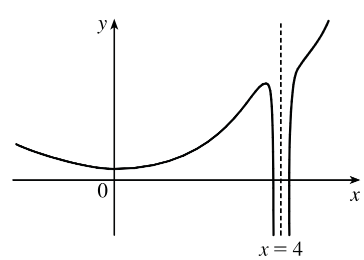
No, because the calculator-produced graph of $f(x) = e^x + \ln|x - 4|$ looks like an exponential function, but the graph of f has an infinite discontinuity at $x = 4$. A second graph, obtained by increasing the `numpoints` option in Maple, begins to reveal the discontinuity at $x = 4$.

(b) There isn't a single graph that shows all the features of f. Several graphs are needed since f looks like $\ln|x - 4|$ for large negative values of x and like e^x for $x > 5$, but yet has the infinite discontiuity at $x = 4$.





A hand-drawn graph, though distorted, might be better at revealing the main features of this function.



47. Let's look for a rational function.

(1) $\lim\limits_{x\to\pm\infty} f(x) = 0 \quad\Rightarrow\quad$ degree of numerator $<$ degree of denominator

(2) $\lim\limits_{x\to 0} f(x) = -\infty \quad\Rightarrow\quad$ there is a factor of x^2 in the denominator (not just x, since that would produce a sign change at $x = 0$), and the function is negative near $x = 0$.

(3) $\lim\limits_{x\to 3^-} f(x) = \infty$ and $\lim\limits_{x\to 3^+} f(x) = -\infty \quad\Rightarrow\quad$ vertical asymptote at $x = 3$; there is a factor of $(x - 3)$ in the denominator.

(4) $f(2) = 0 \quad \Rightarrow \quad$ 2 is an x-intercept; there is at least one factor of $(x-2)$ in the numerator.

Combining all of this information and putting in a negative sign to give us the desired left- and right-hand limits gives us $f(x) = \dfrac{2-x}{x^2(x-3)}$ as one possibility.

48. Since the function has vertical asymptotes $x = 1$ and $x = 3$, the denominator of the rational function we are looking for must have factors $(x-1)$ and $(x-3)$. Because the horizontal asymptote is $y = 1$, the degree of the numerator must equal the degree of the denominator, and the ratio of the leading coefficients must be 1. One possibility is $f(x) = \dfrac{x^2}{(x-1)(x-3)}$.

49. (a) We must first find the function f. Since f has a vertical asymptote $x = 4$ and x-intercept $x = 1$, $x - 4$ is a factor of the denominator and $x - 1$ is a factor of the numerator. There is a removable discontinuity at $x = -1$, so $x - (-1) = x + 1$ is a factor of both the numerator and denominator. Thus, f now looks like this: $f(x) = \dfrac{a(x-1)(x+1)}{(x-4)(x+1)}$, where a is still to be determined. Then $\lim\limits_{x\to-1} f(x) = \lim\limits_{x\to-1} \dfrac{a(x-1)(x+1)}{(x-4)(x+1)} = \lim\limits_{x\to-1} \dfrac{a(x-1)}{x-4} = \dfrac{a(-1-1)}{(-1-4)} = \dfrac{2}{5}a$, so $\dfrac{2}{5}a = 2$, and $a = 5$. Thus $f(x) = \dfrac{5(x-1)(x+1)}{(x-4)(x+1)}$ is a ratio of quadratic functions satisfying all the given conditions and $f(0) = \dfrac{5(-1)(1)}{(-4)(1)} = \dfrac{5}{4}$.

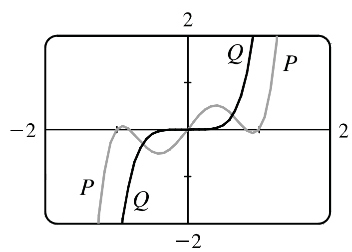
(b) $\lim\limits_{x\to\infty} f(x) = 5 \lim\limits_{x\to\infty} \dfrac{x^2-1}{x^2-3x-4} = 5 \lim\limits_{x\to\infty} \dfrac{(x^2/x^2) - (1/x^2)}{(x^2/x^2) - (3x/x^2) - (4/x^2)} = 5\dfrac{1-0}{1-0-0} = 5(1) = 5$

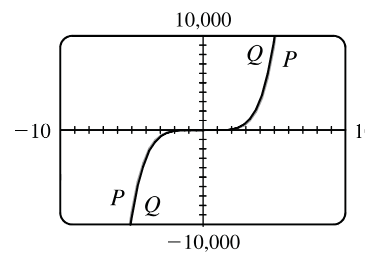
50. (a) In both viewing rectangles,

$\lim\limits_{x\to\infty} P(x) = \lim\limits_{x\to\infty} Q(x) = \infty$ and

$\lim\limits_{x\to-\infty} P(x) = \lim\limits_{x\to-\infty} Q(x) = -\infty$.

In the larger viewing rectangle, P and Q become less distinguishable.





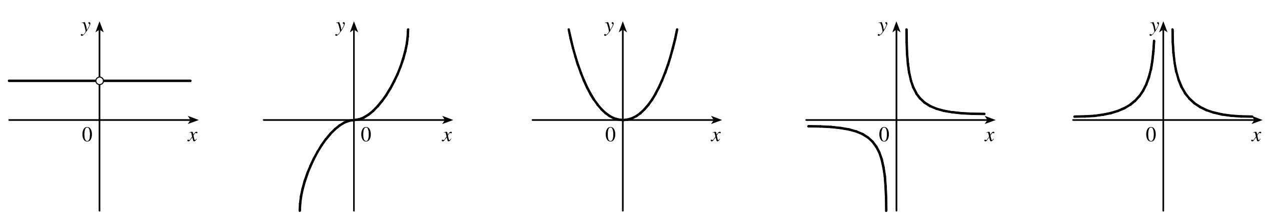
(b) $\lim\limits_{x\to\infty} \dfrac{P(x)}{Q(x)} = \lim\limits_{x\to\infty} \dfrac{3x^5 - 5x^3 + 2x}{3x^5} = \lim\limits_{x\to\infty} \left(1 - \dfrac{5}{3} \cdot \dfrac{1}{x^2} + \dfrac{2}{3} \cdot \dfrac{1}{x^4}\right) = 1 - \frac{5}{3}(0) + \frac{2}{3}(0) = 1 \quad \Rightarrow$ P and Q have the same end behavior.

51. (a) Divide the numerator and the denominator by the highest power of x in $Q(x)$.

(a) If deg $P <$ deg Q, then the numerator $\to 0$ but the denominator doesn't. So $\lim\limits_{x\to\infty} [P(x)/Q(x)] = 0$.

(b) If deg $P >$ deg Q, then the numerator $\to \pm\infty$ but the denominator doesn't, so $\lim\limits_{x\to\infty} [P(x)/Q(x)] = \pm\infty$ (depending on the ratio of the leading coefficients of P and Q).

52.



(i) $n = 0$ (ii) $n > 0$ (n odd) (iii) $n > 0$ (n even) (iv) $n < 0$ (n odd) (v) $n < 0$ (n even)

From these sketches we see that

(a) $\lim\limits_{x\to 0^+} x^n = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n > 0 \\ \infty & \text{if } n < 0 \end{cases}$

(b) $\lim\limits_{x\to 0^-} x^n = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n > 0 \\ -\infty & \text{if } n < 0,\ n \text{ odd} \\ \infty & \text{if } n < 0,\ n \text{ even} \end{cases}$

(c) $\lim\limits_{x\to \infty} x^n = \begin{cases} 1 & \text{if } n = 0 \\ \infty & \text{if } n > 0 \\ 0 & \text{if } n < 0 \end{cases}$

(d) $\lim\limits_{x\to -\infty} x^n = \begin{cases} 1 & \text{if } n = 0 \\ -\infty & \text{if } n > 0,\ n \text{ odd} \\ \infty & \text{if } n > 0,\ n \text{ even} \\ 0 & \text{if } n < 0 \end{cases}$

53. $\lim\limits_{x\to\infty} \dfrac{5\sqrt{x}}{\sqrt{x-1}} \cdot \dfrac{1/\sqrt{x}}{1/\sqrt{x}} = \lim\limits_{x\to\infty} \dfrac{5}{\sqrt{1-(1/x)}} = \dfrac{5}{\sqrt{1-0}} = 5$ and

$\lim\limits_{x\to\infty} \dfrac{10e^x - 21}{2e^x} \cdot \dfrac{1/e^x}{1/e^x} = \lim\limits_{x\to\infty} \dfrac{10-(21/e^x)}{2} = \dfrac{10-0}{2} = 5$. Since $\dfrac{10e^x - 21}{2e^x} < f(x) < \dfrac{5\sqrt{x}}{\sqrt{x-1}}$,

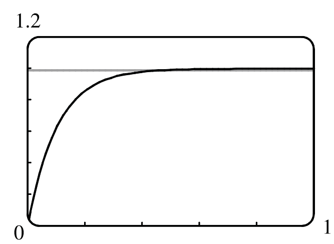
we have $\lim\limits_{x\to\infty} f(x) = 5$ by the Squeeze Theorem.

54. $\lim\limits_{v\to c^-} m = \lim\limits_{v\to c^-} \dfrac{m_0}{\sqrt{1-v^2/c^2}}$. As $v \to c^-$, $\sqrt{1-v^2/c^2} \to 0^+$, and $m \to \infty$.

55. (a) After t minutes, $25t$ liters of brine with 30 g of salt per liter has been pumped into the tank, so it contains $(5000 + 25t)$ liters of water and $25t \cdot 30 = 750t$ grams of salt. Therefore, the salt concentration at time t will be $C(t) = \dfrac{750t}{5000 + 25t} = \dfrac{30t}{200 + t}\ \dfrac{\text{g}}{\text{L}}$.

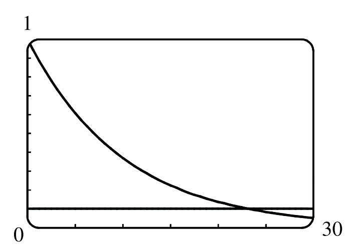
(b) $\lim\limits_{t\to\infty} C(t) = \lim\limits_{t\to\infty} \dfrac{30t}{200+t} = \lim\limits_{t\to\infty} \dfrac{30t/t}{200/t + t/t} = \dfrac{30}{0+1} = 30$. So the salt concentration approaches that of the brine being pumped into the tank.

56. (a) $\lim\limits_{t\to\infty} v(t) = \lim\limits_{t\to\infty} v^*\left(1 - e^{-gt/v^*}\right) = v^*(1-0) = v^*$



(b) We graph $v(t) = 1 - e^{-9.8t}$ and $v(t) = 0.99v^*$, or in this case, $v(t) = 0.99$. Using an intersect feature or zooming in on the point of intersection, we find that $t \approx 0.47$ s.

57. (a) If $t = -x/10$, then $x = -10t$ and as $x \to \infty$, $t \to -\infty$. Thus, $\lim\limits_{x\to\infty} e^{-x/10} = \lim\limits_{t\to-\infty} e^t = 0$ by Equation 7.



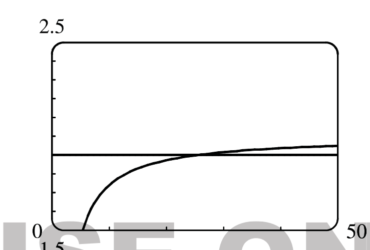
(b) $y = e^{-x/10}$ and $y = 0.1$ intersect at $x_1 \approx 23.03$.

If $x > x_1$, then $e^{-x/10} < 0.1$.

(c) $e^{-x/10} < 0.1 \quad\Rightarrow\quad -x/10 < \ln 0.1 \quad\Rightarrow$

$x > -10 \ln \frac{1}{10} = -10 \ln 10^{-1} = 10 \ln 10 \approx 23.03$

58. (a) $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{4x^2 - 5x}{2x^2 + 1} = \lim\limits_{x\to\infty} \dfrac{4 - 5/x}{2 + 1/x^2} = \dfrac{4}{2} = 2$



(b) $f(x) = 1.9 \quad\Rightarrow\quad x \approx 25.3744$, so $f(x) > 1.9$ when $x > N = 25.4$.

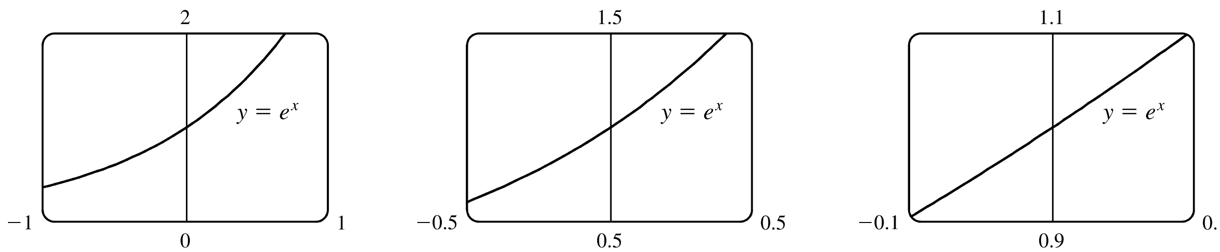
$f(x) = 1.99 \quad\Rightarrow\quad x \approx 250.3974$, so $f(x) > 1.99$ when $x > N = 250.4$.

2.6 Derivatives and Rates of Change

1. (a) This is just the slope of the line through two points: $m_{PQ} = \dfrac{\Delta y}{\Delta x} = \dfrac{f(x) - f(3)}{x - 3}$.

(b) This is the limit of the slope of the secant line PQ as Q approaches P: $m = \lim\limits_{x \to 3} \dfrac{f(x) - f(3)}{x - 3}$.

2. The curve looks more like a line as the viewing rectangle gets smaller.



3. (a) (i) Using Definition 1 with $f(x) = 4x - x^2$ and $P(1, 3)$,

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to 1} \frac{(4x - x^2) - 3}{x - 1} = \lim_{x \to 1} \frac{-(x^2 - 4x + 3)}{x - 1} = \lim_{x \to 1} \frac{-(x - 1)(x - 3)}{x - 1}$$

$$= \lim_{x \to 1} (3 - x) = 3 - 1 = 2$$

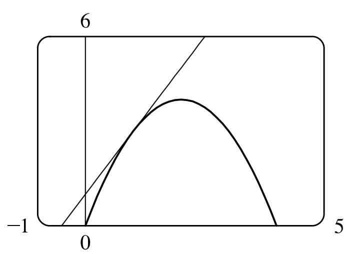
(ii) Using Equation 2 with $f(x) = 4x - x^2$ and $P(1, 3)$,

$$m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \lim_{h \to 0} \frac{f(1 + h) - f(1)}{h} = \lim_{h \to 0} \frac{\left[4(1 + h) - (1 + h)^2\right] - 3}{h}$$

$$= \lim_{h \to 0} \frac{4 + 4h - 1 - 2h - h^2 - 3}{h} = \lim_{h \to 0} \frac{-h^2 + 2h}{h} = \lim_{h \to 0} \frac{h(-h + 2)}{h} = \lim_{h \to 0} (-h + 2) = 2$$

(b) An equation of the tangent line is $y - f(a) = f'(a)(x - a) \Rightarrow y - f(1) = f'(1)(x - 1) \Rightarrow y - 3 = 2(x - 1)$, or $y = 2x + 1$.

(c)



The graph of $y = 2x + 1$ is tangent to the graph of $y = 4x - x^2$ at the point $(1, 3)$. Now zoom in toward the point $(1, 3)$ until the parabola and the tangent line are indistiguishable.

4. (a) (i) Using Definition 1 with $f(x) = x - x^3$ and $P(1, 0)$,

$$m = \lim_{x \to 1} \frac{f(x) - 0}{x - 1} = \lim_{x \to 1} \frac{x - x^3}{x - 1} = \lim_{x \to 1} \frac{x(1 - x^2)}{x - 1} = \lim_{x \to 1} \frac{x(1 + x)(1 - x)}{x - 1}$$

$$= \lim_{x \to 1} [-x(1 + x)] = -1(2) = -2$$

(ii) Using Equation 2 with $f(x) = x - x^3$ and $P(1, 0)$,

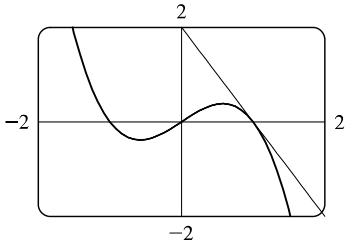
$$m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \lim_{h \to 0} \frac{f(1 + h) - f(1)}{h} = \lim_{h \to 0} \frac{\left[(1 + h) - (1 + h)^3\right] - 0}{h}$$

$$= \lim_{h \to 0} \frac{1 + h - (1 + 3h + 3h^2 + h^3)}{h} = \lim_{h \to 0} \frac{-h^3 - 3h^2 - 2h}{h} = \lim_{h \to 0} \frac{h(-h^2 - 3h - 2)}{h}$$

$$= \lim_{h \to 0} (-h^2 - 3h - 2) = -2$$

(b) An equation of the tangent line is $y - f(a) = f'(a)(x - a) \;\Rightarrow\; y - f(1) = f'(1)(x - 1) \;\Rightarrow\; y - 0 = -2(x - 1)$, or $y = -2x + 2$.

(c)



The graph of $y = -2x + 2$ is tangent to the graph of $y = x - x^3$ at the point $(1, 0)$. Now zoom in toward the point $(1, 0)$ until the cubic and the tangent line are indistinguishable.

5. Using (1) with $f(x) = 4x - 3x^2$ and $P(2, -4)$ [we could also use (2)],

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to 2} \frac{(4x - 3x^2) - (-4)}{x - 2} = \lim_{x \to 2} \frac{-3x^2 + 4x + 4}{x - 2}$$

$$= \lim_{x \to 2} \frac{(-3x - 2)(x - 2)}{x - 2} = \lim_{x \to 2} (-3x - 2) = -3(2) - 2 = -8$$

Tangent line: $y - (-4) = -8(x - 2) \;\Leftrightarrow\; y + 4 = -8x + 16 \;\Leftrightarrow\; y = -8x + 12$.

6. Using (2) with $f(x) = x^3 - 3x + 1$ and $P(2, 3)$,

$$m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \lim_{h \to 0} \frac{f(2 + h) - f(2)}{h} = \lim_{h \to 0} \frac{(2 + h)^3 - 3(2 + h) + 1 - 3}{h}$$

$$= \lim_{h \to 0} \frac{8 + 12h + 6h^2 + h^3 - 6 - 3h - 2}{h} = \lim_{h \to 0} \frac{9h + 6h^2 + h^3}{h} = \lim_{h \to 0} \frac{h(9 + 6h + h^2)}{h}$$

$$= \lim_{h \to 0} (9 + 6h + h^2) = 9$$

Tangent line: $y - 3 = 9(x - 2) \;\Leftrightarrow\; y - 3 = 9x - 18 \;\Leftrightarrow\; y = 9x - 15$

7. Using (1), $m = \lim\limits_{x \to 1} \dfrac{\sqrt{x} - \sqrt{1}}{x - 1} = \lim\limits_{x \to 1} \dfrac{(\sqrt{x} - 1)(\sqrt{x} + 1)}{(x - 1)(\sqrt{x} + 1)} = \lim\limits_{x \to 1} \dfrac{x - 1}{(x - 1)(\sqrt{x} + 1)} = \lim\limits_{x \to 1} \dfrac{1}{\sqrt{x} + 1} = \dfrac{1}{2}$.

Tangent line: $y - 1 = \frac{1}{2}(x - 1) \;\Leftrightarrow\; y = \frac{1}{2}x + \frac{1}{2}$

8. Using (1) with $f(x) = \dfrac{2x + 1}{x + 2}$ and $P(1, 1)$,

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to 1} \frac{\dfrac{2x + 1}{x + 2} - 1}{x - 1} = \lim_{x \to 1} \frac{\dfrac{2x + 1 - (x + 2)}{x + 2}}{x - 1} = \lim_{x \to 1} \frac{x - 1}{(x - 1)(x + 2)}$$

$$= \lim_{x \to 1} \frac{1}{x + 2} = \frac{1}{1 + 2} = \frac{1}{3}$$

Tangent line: $y - 1 = \frac{1}{3}(x - 1) \;\Leftrightarrow\; y - 1 = \frac{1}{3}x - \frac{1}{3} \;\Leftrightarrow\; y = \frac{1}{3}x + \frac{2}{3}$

9. (a) Using (2) with $y = f(x) = 3 + 4x^2 - 2x^3$,

$$m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \lim_{h \to 0} \frac{3 + 4(a + h)^2 - 2(a + h)^3 - (3 + 4a^2 - 2a^3)}{h}$$

$$= \lim_{h \to 0} \frac{3 + 4(a^2 + 2ah + h^2) - 2(a^3 + 3a^2h + 3ah^2 + h^3) - 3 - 4a^2 + 2a^3}{h}$$

$$= \lim_{h \to 0} \frac{3 + 4a^2 + 8ah + 4h^2 - 2a^3 - 6a^2h - 6ah^2 - 2h^3 - 3 - 4a^2 + 2a^3}{h}$$

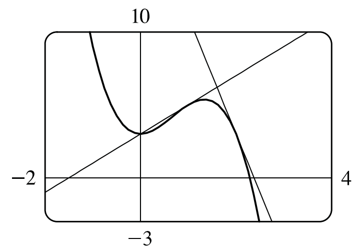
$$= \lim_{h \to 0} \frac{8ah + 4h^2 - 6a^2h - 6ah^2 - 2h^3}{h} = \lim_{h \to 0} \frac{h(8a + 4h - 6a^2 - 6ah - 2h^2)}{h}$$

$$= \lim_{h \to 0} (8a + 4h - 6a^2 - 6ah - 2h^2) = 8a - 6a^2$$

(b) At $(1, 5)$: $m = 8(1) - 6(1)^2 = 2$, so an equation of the tangent line is $y - 5 = 2(x - 1) \Leftrightarrow y = 2x + 3$.

At $(2, 3)$: $m = 8(2) - 6(2)^2 = -8$, so an equation of the tangent line is $y - 3 = -8(x - 2) \Leftrightarrow y = -8x + 19$.

(c)



10. (a) Using (1),

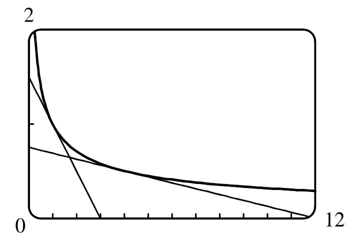
$$m = \lim_{x \to a} \frac{\dfrac{1}{\sqrt{x}} - \dfrac{1}{\sqrt{a}}}{x - a} = \lim_{x \to a} \frac{\dfrac{\sqrt{a} - \sqrt{x}}{\sqrt{ax}}}{x - a} = \lim_{x \to a} \frac{(\sqrt{a} - \sqrt{x})(\sqrt{a} + \sqrt{x})}{\sqrt{ax}\,(x - a)(\sqrt{a} + \sqrt{x})} = \lim_{x \to a} \frac{a - x}{\sqrt{ax}\,(x - a)(\sqrt{a} + \sqrt{x})}$$

$$= \lim_{x \to a} \frac{-1}{\sqrt{ax}\,(\sqrt{a} + \sqrt{x})} = \frac{-1}{\sqrt{a^2}\,(2\sqrt{a})} = -\frac{1}{2a^{3/2}} \text{ or } -\frac{1}{2}a^{-3/2}$$

(b) At $(1, 1)$: $m = -\frac{1}{2}$, so an equation of the tangent line is $y - 1 = -\frac{1}{2}(x - 1) \Leftrightarrow y = -\frac{1}{2}x + \frac{3}{2}$.

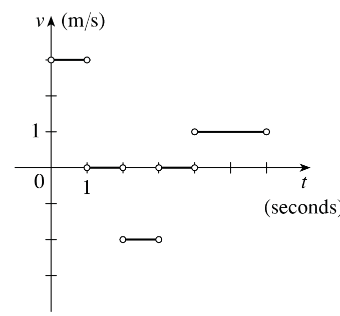
At $\left(4, \frac{1}{2}\right)$: $m = -\frac{1}{16}$, so an equation of the tangent line is $y - \frac{1}{2} = -\frac{1}{16}(x - 4) \Leftrightarrow y = -\frac{1}{16}x + \frac{3}{4}$.

(c)



11. (a) The particle is moving to the right when s is increasing; that is, on the intervals $(0, 1)$ and $(4, 6)$. The particle is moving to the left when s is decreasing; that is, on the interval $(2, 3)$. The particle is standing still when s is constant; that is, on the intervals $(1, 2)$ and $(3, 4)$.

(b) The velocity of the particle is equal to the slope of the tangent line of the graph. Note that there is no slope at the corner points on the graph. On the interval $(0, 1)$, the slope is $\dfrac{3 - 0}{1 - 0} = 3$. On the interval $(2, 3)$, the slope is $\dfrac{1 - 3}{3 - 2} = -2$. On the interval $(4, 6)$, the slope is $\dfrac{3 - 1}{6 - 4} = 1$.



12. (a) **Runner A** runs the entire 100-meter race at the same velocity since the slope of the position function is constant.
Runner B starts the race at a slower velocity than runner A, but finishes the race at a faster velocity.

(b) The distance between the runners is the greatest at the time when the largest vertical line segment fits between the two graphs—this appears to be somewhere between 9 and 10 seconds.

(c) The runners had the same velocity when the slopes of their respective position functions are equal—this also appears to be at about 9.5 s. Note that the answers for parts (b) and (c) must be the same for these graphs because as soon as the velocity for runner B overtakes the velocity for runner A, the distance between the runners starts to decrease.

13. Let $s(t) = 40t - 16t^2$.

$$v(2) = \lim_{t \to 2} \frac{s(t) - s(2)}{t - 2} = \lim_{t \to 2} \frac{(40t - 16t^2) - 16}{t - 2} = \lim_{t \to 2} \frac{-16t^2 + 40t - 16}{t - 2} = \lim_{t \to 2} \frac{-8(2t^2 - 5t + 2)}{t - 2}$$

$$= \lim_{t \to 2} \frac{-8(t - 2)(2t - 1)}{t - 2} = -8 \lim_{t \to 2}(2t - 1) = -8(3) = -24$$

Thus, the instantaneous velocity when $t = 2$ is -24 ft/s.

14. (a) Let $H(t) = 10t - 1.86t^2$.

$$v(1) = \lim_{h\to 0} \frac{H(1+h) - H(1)}{h} = \lim_{h\to 0} \frac{\left[10(1+h) - 1.86(1+h)^2\right] - (10 - 1.86)}{h}$$
$$= \lim_{h\to 0} \frac{10 + 10h - 1.86(1 + 2h + h^2) - 10 + 1.86}{h}$$
$$= \lim_{h\to 0} \frac{10 + 10h - 1.86 - 3.72h - 1.86h^2 - 10 + 1.86}{h}$$
$$= \lim_{h\to 0} \frac{6.28h - 1.86h^2}{h} = \lim_{h\to 0}(6.28 - 1.86h) = 6.28$$

The velocity of the rock after one second is 6.28 m/s.

(b) $$v(a) = \lim_{h\to 0} \frac{H(a+h) - H(a)}{h} = \lim_{h\to 0} \frac{\left[10(a+h) - 1.86(a+h)^2\right] - (10a - 1.86a^2)}{h}$$
$$= \lim_{h\to 0} \frac{10a + 10h - 1.86(a^2 + 2ah + h^2) - 10a + 1.86a^2}{h}$$
$$= \lim_{h\to 0} \frac{10a + 10h - 1.86a^2 - 3.72ah - 1.86h^2 - 10a + 1.86a^2}{h} = \lim_{h\to 0} \frac{10h - 3.72ah - 1.86h^2}{h}$$
$$= \lim_{h\to 0} \frac{h(10 - 3.72a - 1.86h)}{h} = \lim_{h\to 0}(10 - 3.72a - 1.86h) = 10 - 3.72a$$

The velocity of the rock when $t = a$ is $(10 - 3.72a)$ m/s.

(c) The rock will hit the surface when $H = 0 \iff 10t - 1.86t^2 = 0 \iff t(10 - 1.86t) = 0 \iff t = 0$ or $1.86t = 10$.

The rock hits the surface when $t = 10/1.86 \approx 5.4$ s.

(d) The velocity of the rock when it hits the surface is $v\left(\frac{10}{1.86}\right) = 10 - 3.72\left(\frac{10}{1.86}\right) = 10 - 20 = -10$ m/s.

15. $$v(a) = \lim_{h\to 0} \frac{s(a+h) - s(a)}{h} = \lim_{h\to 0} \frac{\dfrac{1}{(a+h)^2} - \dfrac{1}{a^2}}{h} = \lim_{h\to 0} \frac{\dfrac{a^2 - (a+h)^2}{a^2(a+h)^2}}{h} = \lim_{h\to 0} \frac{a^2 - (a^2 + 2ah + h^2)}{ha^2(a+h)^2}$$
$$= \lim_{h\to 0} \frac{-(2ah + h^2)}{ha^2(a+h)^2} = \lim_{h\to 0} \frac{-h(2a + h)}{ha^2(a+h)^2} = \lim_{h\to 0} \frac{-(2a + h)}{a^2(a+h)^2} = \frac{-2a}{a^2 \cdot a^2} = \frac{-2}{a^3} \text{ m/s}$$

So $v\,(1) = \dfrac{-2}{1^3} = -2$ m/s, $v(2) = \dfrac{-2}{2^3} = -\dfrac{1}{4}$ m/s, and $v(3) = \dfrac{-2}{3^3} = -\dfrac{2}{27}$ m/s.

16. (a) The average velocity between times t and $t + h$ is

$$\frac{s(t+h) - s(t)}{(t+h) - t} = \frac{(t+h)^2 - 8(t+h) + 18 - \left(t^2 - 8t + 18\right)}{h} = \frac{t^2 + 2th + h^2 - 8t - 8h + 18 - t^2 + 8t - 18}{h}$$
$$= \frac{2th + h^2 - 8h}{h} = (2t + h - 8) \text{ m/s}.$$

(i) $[3, 4]$: $t = 3$, $h = 4 - 3 = 1$, so the average velocity is $2(3) + 1 - 8 = -1$ m/s.

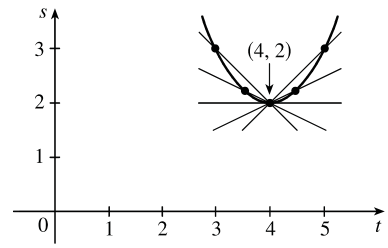
(ii) $[3.5, 4]$: $t = 3.5$, $h = 0.5$, so the average velocity is $2(3.5) + 0.5 - 8 = -0.5$ m/s.

(iii) $[4, 5]$: $t = 4$, $h = 1$, so the average velocity is $2(4) + 1 - 8 = 1$ m/s.

(iv) $[4, 4.5]$: $t = 4$, $h = 0.5$, so the average velocity is $2(4) + 0.5 - 8 = 0.5$ m/s.

(b) $v(t) = \lim\limits_{h \to 0} \dfrac{s(t+h) - s(t)}{h} = \lim\limits_{h \to 0} (2t + h - 8) = 2t - 8$, so $v(4) = 0$.

(c)



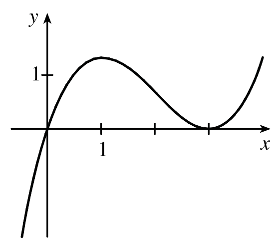
17. $g'(0)$ is the only negative value. The slope at $x = 4$ is smaller than the slope at $x = 2$ and both are smaller than the slope at $x = -2$. Thus, $g'(0) < 0 < g'(4) < g'(2) < g'(-2)$.

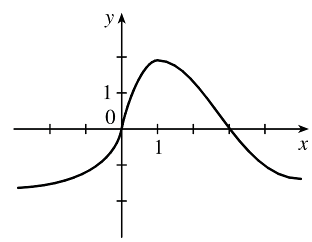
18. Since $g(5) = -3$, the point $(5, -3)$ is on the graph of g. Since $g'(5) = 4$, the slope of the tangent line at $x = 5$ is 4. Using the point-slope form of a line gives us $y - (-3) = 4(x - 5)$, or $y = 4x - 23$.

19. For the tangent line $y = 4x - 5$: when $x = 2$, $y = 4(2) - 5 = 3$ and its slope is 4 (the coefficient of x). At the point of tangency, these values are shared with the curve $y = f(x)$; that is, $f(2) = 3$ and $f'(2) = 4$.

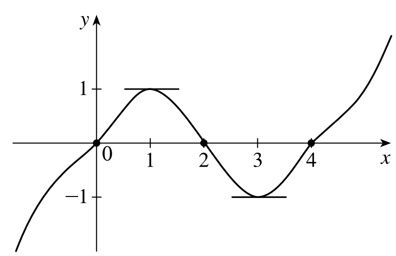
20. Since $(4, 3)$ is on $y = f(x)$, $f(4) = 3$. The slope of the tangent line between $(0, 2)$ and $(4, 3)$ is $\frac{1}{4}$, so $f'(4) = \frac{1}{4}$.

21. We begin by drawing a curve through the origin with a slope of 3 to satisfy $f(0) = 0$ and $f'(0) = 3$. Since $f'(1) = 0$, we will round off our figure so that there is a horizontal tangent directly over $x = 1$. Last, we make sure that the curve has a slope of -1 as we pass over $x = 2$. Two of the many possibilities are shown.





22. We begin by drawing a curve through the origin with a slope of 1 to satisfy $g(0) = 0$ and $g'(0) = 1$. We round off our figure at $x = 1$ to satisfy $g'(1) = 0$, and then pass through $(2, 0)$ with slope -1 to satisfy $g(2) = 0$ and $g'(2) = -1$. We round the figure at $x = 3$ to satisfy $g'(3) = 0$, and then pass through $(4, 0)$ with slope 1 to satisfy $g(4) = 0$ and $g'(4) = 1$. Finally we extend the curve on both ends to satisfy $\lim\limits_{x \to \infty} g(x) = \infty$ and $\lim\limits_{x \to -\infty} g(x) = -\infty$.



23. Using (4) with $f(x) = 3x^2 - x^3$ and $a = 1$,

$$\begin{aligned} f'(1) &= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{[3(1+h)^2 - (1+h)^3] - 2}{h} \\ &= \lim_{h \to 0} \frac{(3 + 6h + 3h^2) - (1 + 3h + 3h^2 + h^3) - 2}{h} = \lim_{h \to 0} \frac{3h - h^3}{h} = \lim_{h \to 0} \frac{h(3 - h^2)}{h} \\ &= \lim_{h \to 0} (3 - h^2) = 3 - 0 = 3 \end{aligned}$$

Tangent line: $y - 2 = 3(x - 1) \quad \Leftrightarrow \quad y - 2 = 3x - 3 \quad \Leftrightarrow \quad y = 3x - 1$

24. Using (5) with $g(x) = x^4 - 2$ and $a = 1$,

$$g'(1) = \lim_{x\to 1} \frac{g(x) - g(1)}{x - 1} = \lim_{x\to 1} \frac{(x^4 - 2) - (-1)}{x - 1} = \lim_{x\to 1} \frac{x^4 - 1}{x - 1} = \lim_{x\to 1} \frac{(x^2 + 1)(x^2 - 1)}{x - 1}$$
$$= \lim_{x\to 1} \frac{(x^2 + 1)(x + 1)(x - 1)}{x - 1} = \lim_{x\to 1} [(x^2 + 1)(x + 1)] = 2(2) = 4$$

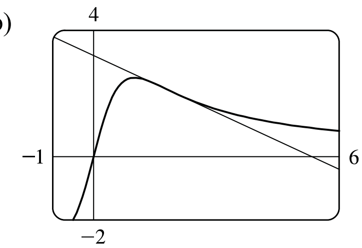
Tangent line: $y - (-1) = 4(x - 1) \quad \Leftrightarrow \quad y + 1 = 4x - 4 \quad \Leftrightarrow \quad y = 4x - 5$

25. (a) Using (4) with $F(x) = 5x/(1 + x^2)$ and the point $(2, 2)$, we have

$$F'(2) = \lim_{h\to 0} \frac{F(2 + h) - F(2)}{h} = \lim_{h\to 0} \frac{\dfrac{5(2 + h)}{1 + (2 + h)^2} - 2}{h}$$
$$= \lim_{h\to 0} \frac{\dfrac{5h + 10}{h^2 + 4h + 5} - 2}{h} = \lim_{h\to 0} \frac{\dfrac{5h + 10 - 2(h^2 + 4h + 5)}{h^2 + 4h + 5}}{h}$$
$$= \lim_{h\to 0} \frac{-2h^2 - 3h}{h(h^2 + 4h + 5)} = \lim_{h\to 0} \frac{h(-2h - 3)}{h(h^2 + 4h + 5)} = \lim_{h\to 0} \frac{-2h - 3}{h^2 + 4h + 5} = \frac{-3}{5}$$

So an equation of the tangent line at $(2, 2)$ is $y - 2 = -\frac{3}{5}(x - 2)$ or $y = -\frac{3}{5}x + \frac{16}{5}$.

(b)

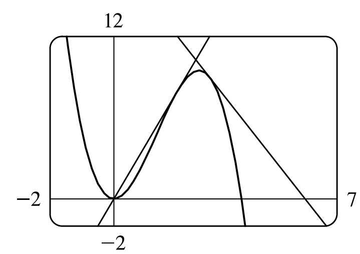


26. (a) Using (4) with $G(x) = 4x^2 - x^3$, we have

$$G'(a) = \lim_{h\to 0} \frac{G(a + h) - G(a)}{h} = \lim_{h\to 0} \frac{[4(a + h)^2 - (a + h)^3] - (4a^2 - a^3)}{h}$$
$$= \lim_{h\to 0} \frac{4a^2 + 8ah + 4h^2 - (a^3 + 3a^2h + 3ah^2 + h^3) - 4a^2 + a^3}{h} = \lim_{h\to 0} \frac{8ah + 4h^2 - 3a^2h - 3ah^2 - h^3}{h}$$
$$= \lim_{h\to 0} \frac{h(8a + 4h - 3a^2 - 3ah - h^2)}{h} = \lim_{h\to 0} (8a + 4h - 3a^2 - 3ah - h^2) = 8a - 3a^2$$

At the point $(2, 8)$, $G'(2) = 16 - 12 = 4$, and an equation of the tangent line is $y - 8 = 4(x - 2)$, or $y = 4x$. At the point $(3, 9)$, $G'(3) = 24 - 27 = -3$, and an equation of the tangent line is $y - 9 = -3(x - 3)$, or $y = -3x + 18$.

(b)



27. Use (4) with $f(x) = 3x^2 - 4x + 1$.

$$f'(a) = \lim_{h\to 0} \frac{f(a + h) - f(a)}{h} = \lim_{h\to 0} \frac{[3(a + h)^2 - 4(a + h) + 1] - (3a^2 - 4a + 1)]}{h}$$
$$= \lim_{h\to 0} \frac{3a^2 + 6ah + 3h^2 - 4a - 4h + 1 - 3a^2 + 4a - 1}{h} = \lim_{h\to 0} \frac{6ah + 3h^2 - 4h}{h}$$
$$= \lim_{h\to 0} \frac{h(6a + 3h - 4)}{h} = \lim_{h\to 0} (6a + 3h - 4) = 6a - 4$$

28. Use (4) with $f(t) = 2t^3 + t$.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{[2(a+h)^3 + (a+h)] - (2a^3+a)}{h}$$
$$= \lim_{h\to 0} \frac{2a^3 + 6a^2h + 6ah^2 + 2h^3 + a + h - 2a^3 - a}{h} = \lim_{h\to 0} \frac{6a^2h + 6ah^2 + 2h^3 + h}{h}$$
$$= \lim_{h\to 0} \frac{h(6a^2 + 6ah + 2h^2 + 1)}{h} = \lim_{h\to 0}(6a^2 + 6ah + 2h^2 + 1) = 6a^2 + 1$$

29. Use (4) with $f(t) = (2t+1)/(t+3)$.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{\dfrac{2(a+h)+1}{(a+h)+3} - \dfrac{2a+1}{a+3}}{h} = \lim_{h\to 0} \frac{(2a+2h+1)(a+3) - (2a+1)(a+h+3)}{h(a+h+3)(a+3)}$$
$$= \lim_{h\to 0} \frac{(2a^2 + 6a + 2ah + 6h + a + 3) - (2a^2 + 2ah + 6a + a + h + 3)}{h(a+h+3)(a+3)}$$
$$= \lim_{h\to 0} \frac{5h}{h(a+h+3)(a+3)} = \lim_{h\to 0} \frac{5}{(a+h+3)(a+3)} = \frac{5}{(a+3)^2}$$

30. Use (4) with $f(x) = x^{-2} = 1/x^2$.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{\dfrac{1}{(a+h)^2} - \dfrac{1}{a^2}}{h} = \lim_{h\to 0} \frac{\dfrac{a^2 - (a+h)^2}{a^2(a+h)^2}}{h} = \lim_{h\to 0} \frac{a^2 - (a^2 + 2ah + h^2)}{ha^2(a+h)^2}$$
$$= \lim_{h\to 0} \frac{-2ah - h^2}{ha^2(a+h)^2} = \lim_{h\to 0} \frac{h(-2a-h)}{ha^2(a+h)^2} = \lim_{h\to 0} \frac{-2a-h}{a^2(a+h)^2} = \frac{-2a}{a^2(a^2)} = \frac{-2}{a^3}$$

31. Use (4) with $f(x) = \sqrt{1-2x}$.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{\sqrt{1-2(a+h)} - \sqrt{1-2a}}{h}$$
$$= \lim_{h\to 0} \frac{\sqrt{1-2(a+h)} - \sqrt{1-2a}}{h} \cdot \frac{\sqrt{1-2(a+h)} + \sqrt{1-2a}}{\sqrt{1-2(a+h)} + \sqrt{1-2a}} = \lim_{h\to 0} \frac{\left(\sqrt{1-2(a+h)}\right)^2 - \left(\sqrt{1-2a}\right)^2}{h\left(\sqrt{1-2(a+h)} + \sqrt{1-2a}\right)}$$
$$= \lim_{h\to 0} \frac{(1-2a-2h) - (1-2a)}{h\left(\sqrt{1-2(a+h)} + \sqrt{1-2a}\right)} = \lim_{h\to 0} \frac{-2h}{h\left(\sqrt{1-2(a+h)} + \sqrt{1-2a}\right)}$$
$$= \lim_{h\to 0} \frac{-2}{\sqrt{1-2(a+h)} + \sqrt{1-2a}} = \frac{-2}{\sqrt{1-2a} + \sqrt{1-2a}} = \frac{-2}{2\sqrt{1-2a}} = \frac{-1}{\sqrt{1-2a}}$$

32. Use (4) with $f(x) = \dfrac{4}{\sqrt{1-x}}$.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{\dfrac{4}{\sqrt{1-(a+h)}} - \dfrac{4}{\sqrt{1-a}}}{h}$$
$$= 4\lim_{h\to 0} \frac{\dfrac{\sqrt{1-a} - \sqrt{1-a-h}}{\sqrt{1-a-h}\,\sqrt{1-a}}}{h} = 4\lim_{h\to 0} \frac{\sqrt{1-a} - \sqrt{1-a-h}}{h\sqrt{1-a-h}\,\sqrt{1-a}}$$

[continued]

$$= 4\lim_{h\to 0}\frac{\sqrt{1-a}-\sqrt{1-a-h}}{h\sqrt{1-a-h}\,\sqrt{1-a}}\cdot\frac{\sqrt{1-a}+\sqrt{1-a-h}}{\sqrt{1-a}+\sqrt{1-a-h}} = 4\lim_{h\to 0}\frac{(\sqrt{1-a})^2-(\sqrt{1-a-h})^2}{h\sqrt{1-a-h}\,\sqrt{1-a}(\sqrt{1-a}+\sqrt{1-a-h})}$$

$$= 4\lim_{h\to 0}\frac{(1-a)-(1-a-h)}{h\sqrt{1-a-h}\,\sqrt{1-a}(\sqrt{1-a}+\sqrt{1-a-h})} = 4\lim_{h\to 0}\frac{-h}{h\sqrt{1-a-h}\,\sqrt{1-a}(\sqrt{1-a}+\sqrt{1-a-h})}$$

$$= 4\lim_{h\to 0}\frac{-1}{\sqrt{1-a-h}\,\sqrt{1-a}(\sqrt{1-a}+\sqrt{1-a-h})} = 4\cdot\frac{-1}{\sqrt{1-a}\,\sqrt{1-a}(\sqrt{1-a}+\sqrt{1-a})}$$

$$= \frac{-4}{(1-a)(2\sqrt{1-a})} = \frac{-2}{(1-a)^1(1-a)^{1/2}} = \frac{-2}{(1-a)^{3/2}}$$

Note that the answers to Exercises 33 – 38 are not unique.

33. By (4), $\lim\limits_{h\to 0}\dfrac{(1+h)^{10}-1}{h} = f'(1)$, where $f(x)=x^{10}$ and $a=1$.

Or: By (4), $\lim\limits_{h\to 0}\dfrac{(1+h)^{10}-1}{h} = f'(0)$, where $f(x)=(1+x)^{10}$ and $a=0$.

34. By (4), $\lim\limits_{h\to 0}\dfrac{\sqrt[4]{16+h}-2}{h} = f'(16)$, where $f(x)=\sqrt[4]{x}$ and $a=16$.

Or: By (4), $\lim\limits_{h\to 0}\dfrac{\sqrt[4]{16+h}-2}{h} = f'(0)$, where $f(x)=\sqrt[4]{16+x}$ and $a=0$.

35. By Equation 5, $\lim\limits_{x\to 5}\dfrac{2^x-32}{x-5} = f'(5)$, where $f(x)=2^x$ and $a=5$.

36. By Equation 5, $\lim\limits_{x\to \pi/4}\dfrac{\tan x-1}{x-\pi/4} = f'(\pi/4)$, where $f(x)=\tan x$ and $a=\pi/4$.

37. By (4), $\lim\limits_{h\to 0}\dfrac{\cos(\pi+h)+1}{h} = f'(\pi)$, where $f(x)=\cos x$ and $a=\pi$.

Or: By (4), $\lim\limits_{h\to 0}\dfrac{\cos(\pi+h)+1}{h} = f'(0)$, where $f(x)=\cos(\pi+x)$ and $a=0$.

38. By Equation 5, $\lim\limits_{t\to 1}\dfrac{t^4+t-2}{t-1} = f'(1)$, where $f(t)=t^4+t$ and $a=1$.

39. $v(5)=f'(5)=\lim\limits_{h\to 0}\dfrac{f(5+h)-f(5)}{h} = \lim\limits_{h\to 0}\dfrac{[100+50(5+h)-4.9(5+h)^2]-[100+50(5)-4.9(5)^2]}{h}$

$$= \lim_{h\to 0}\frac{(100+250+50h-4.9h^2-49h-122.5)-(100+250-122.5)}{h} = \lim_{h\to 0}\frac{-4.9h^2+h}{h}$$

$$= \lim_{h\to 0}\frac{h(-4.9h+1)}{h} = \lim_{h\to 0}(-4.9h+1) = 1 \text{ m/s}$$

The speed when $t=5$ is $|1| = 1$ m/s.

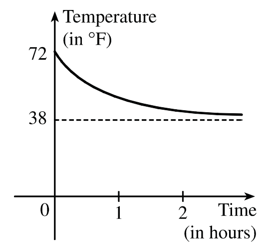
40. $v(5)=f'(5)=\lim\limits_{h\to 0}\dfrac{f(5+h)-f(5)}{h} = \lim\limits_{h\to 0}\dfrac{[(5+h)^{-1}-(5+h)]-(5^{-1}-5)}{h}$

$$= \lim_{h\to 0}\frac{\frac{1}{5+h}-5-h-\frac{1}{5}+5}{h} = \lim_{h\to 0}\frac{\frac{1}{5+h}-h-\frac{1}{5}}{h} = \lim_{h\to 0}\frac{\frac{5-5h(5+h)-(5+h)}{5(5+h)}}{h}$$

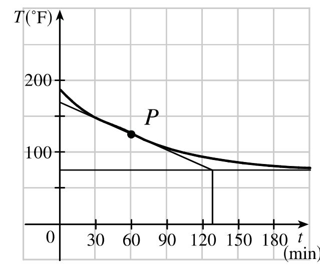
$$= \lim_{h\to 0}\frac{5-25h-5h^2-5-h}{5h(5+h)} = \lim_{h\to 0}\frac{-5h^2-26h}{5h(5+h)} = \lim_{h\to 0}\frac{h(-5h-26)}{5h(5+h)} = \lim_{h\to 0}\frac{-5h-26}{5(5+h)} = \frac{-26}{25} \text{ m/s}$$

The speed when $t=5$ is $\left|-\frac{26}{25}\right| = \frac{26}{25} = 1.04$ m/s.

41. The sketch shows the graph for a room temperature of 72° and a refrigerator temperature of 38°. The initial rate of change is greater in magnitude than the rate of change after an hour.



42. The slope of the tangent (that is, the rate of change of temperature with respect to time) at $t = 1$ h seems to be about $\dfrac{75 - 168}{132 - 0} \approx -0.7\,^\circ\text{F/min}$.



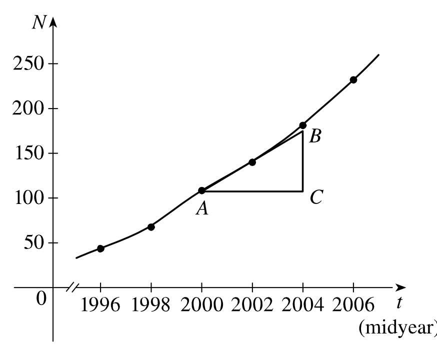
43. (a) (i) $[2002, 2006]$: $\dfrac{N(2006) - N(2002)}{2006 - 2002} = \dfrac{233 - 141}{4} = \dfrac{92}{4} = 23$ millions of cell phone subscribers per year

(ii) $[2002, 2004]$: $\dfrac{N(2004) - N(2002)}{2004 - 2002} = \dfrac{182 - 141}{2} = \dfrac{41}{2} = 20.5$ millions of cell phone subscribers per year

(iii) $[2000, 2002]$: $\dfrac{N(2002) - N(2000)}{2002 - 2000} = \dfrac{141 - 109}{2} = \dfrac{32}{2} = 16$ millions of cell phone subscribers per year

(b) Using the values from (ii) and (iii), we have $\dfrac{20.5 + 16}{2} = 18.25$ millions of cell phone subscribers per year.

(c) Estimating A as $(2000, 107)$ and B as $(2004, 175)$, the slope at 2002 is $\dfrac{175 - 107}{2004 - 2000} = \dfrac{68}{4} = 17$ millions of cell phone subscribers per year.



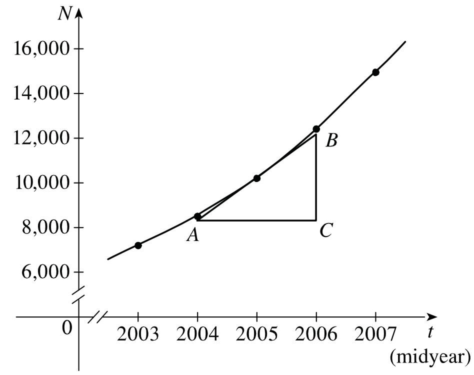
44. (a) (i) $[2005, 2007]$: $\dfrac{N(2007) - N(2005)}{2007 - 2005} = \dfrac{15{,}011 - 10{,}241}{2} = \dfrac{4770}{2} = 2385$ locations per year

(ii) $[2005, 2006]$: $\dfrac{N(2006) - N(2005)}{2006 - 2005} = \dfrac{12{,}440 - 10{,}241}{1} = 2199$ locations per year

(iii) $[2004, 2005]$: $\dfrac{N(2005) - N(2004)}{2005 - 2004} = \dfrac{10{,}241 - 8569}{1} = 1672$ locations per year

(b) Using the values from (ii) and (iii), we have $\dfrac{2199 + 1672}{2} = 1935.5$ locations per year.

(c) Estimating A as $(2004, 8300)$ and B as $(2006, 12{,}200)$, the slope at 2005 is $\dfrac{12{,}200 - 8300}{2006 - 2004} = \dfrac{3900}{2} = 1950$ locations per year.



45. (a) (i) $\dfrac{\Delta C}{\Delta x} = \dfrac{C(105) - C(100)}{105 - 100} = \dfrac{6601.25 - 6500}{5} = \$20.25/\text{unit}.$

(ii) $\dfrac{\Delta C}{\Delta x} = \dfrac{C(101) - C(100)}{101 - 100} = \dfrac{6520.05 - 6500}{1} = \$20.05/\text{unit}.$

(b) $\dfrac{C(100+h) - C(100)}{h} = \dfrac{\left[5000 + 10(100+h) + 0.05(100+h)^2\right] - 6500}{h} = \dfrac{20h + 0.05h^2}{h}$

$= 20 + 0.05h,\ h \neq 0$

So the instantaneous rate of change is $\lim\limits_{h \to 0} \dfrac{C(100+h) - C(100)}{h} = \lim\limits_{h \to 0}(20 + 0.05h) = \$20/\text{unit}.$

46. $\Delta V = V(t+h) - V(t) = 100{,}000\left(1 - \dfrac{t+h}{60}\right)^2 - 100{,}000\left(1 - \dfrac{t}{60}\right)^2$

$= 100{,}000\left[\left(1 - \dfrac{t+h}{30} + \dfrac{(t+h)^2}{3600}\right) - \left(1 - \dfrac{t}{30} + \dfrac{t^2}{3600}\right)\right] = 100{,}000\left(-\dfrac{h}{30} + \dfrac{2th}{3600} + \dfrac{h^2}{3600}\right)$

$= \dfrac{100{,}000}{3600} h\,(-120 + 2t + h) = \dfrac{250}{9} h\,(-120 + 2t + h)$

Dividing ΔV by h and then letting $h \to 0$, we see that the instantaneous rate of change is $\frac{500}{9}\,(t - 60)$ gal/min.

t	Flow rate (gal/min)	Water remaining $V(t)$ (gal)
0	$-3333.\overline{3}$	$100,000$
10	$-2777.\overline{7}$	$69,444.\overline{4}$
20	$-2222.\overline{2}$	$44,444.\overline{4}$
30	$-1666.\overline{6}$	$25,000$
40	$-1111.\overline{1}$	$11,111.\overline{1}$
50	$-555.\overline{5}$	$2,777.\overline{7}$
60	0	0

The magnitude of the flow rate is greatest at the beginning and gradually decreases to 0.

47. (a) $f'(x)$ is the rate of change of the production cost with respect to the number of ounces of gold produced. Its units are dollars per ounce.

(b) After 800 ounces of gold have been produced, the rate at which the production cost is increasing is \$17/ounce. So the cost of producing the 800th (or 801st) ounce is about \$17.

(c) In the short term, the values of $f'(x)$ will decrease because more efficient use is made of start-up costs as x increases. But eventually $f'(x)$ might increase due to large-scale operations.

48. (a) $f'(5)$ is the rate of growth of the bacteria population when $t = 5$ hours. Its units are bacteria per hour.

(b) With unlimited space and nutrients, f' should increase as t increases; so $f'(5) < f'(10)$. If the supply of nutrients is limited, the growth rate slows down at some point in time, and the opposite may be true.

49. $T'(10)$ is the rate at which the temperature is changing at 10:00 AM. To estimate the value of $T'(10)$, we will average the difference quotients obtained using the times $t = 8$ and $t = 12$. Let $A = \dfrac{T(8) - T(10)}{8 - 10} = \dfrac{65 - 76}{-2} = 5.5$ and $B = \dfrac{T(12) - T(10)}{12 - 10} = \dfrac{85 - 76}{2} = 4.5$. Then $T'(10) = \lim\limits_{t \to 10} \dfrac{T(t) - T(10)}{t - 10} \approx \dfrac{A + B}{2} = \dfrac{5.5 + 4.5}{2} = 5^\circ\text{F/h}$.

50. (a) $f'(8)$ is the rate of change of the quantity of coffee sold with respect to the price per pound when the price is \$8 per pound. The units for $f'(8)$ are pounds/(dollars/pound).

(b) $f'(8)$ is negative since the quantity of coffee sold will decrease as the price charged for it increases. People are generally less willing to buy a product when its price increases.

51. (a) $S'(T)$ is the rate at which the oxygen solubility changes with respect to the water temperature. Its units are $(\text{mg/L})/^\circ\text{C}$.

(b) For $T = 16^\circ\text{C}$, it appears that the tangent line to the curve goes through the points $(0, 14)$ and $(32, 6)$. So $S'(16) \approx \dfrac{6 - 14}{32 - 0} = -\dfrac{8}{32} = -0.25\ (\text{mg/L})/^\circ\text{C}$. This means that as the temperature increases past 16°C, the oxygen solubility is decreasing at a rate of $0.25\ (\text{mg/L})/^\circ\text{C}$.

52. (a) $S'(T)$ is the rate of change of the maximum sustainable speed of Coho salmon with respect to the temperature. Its units are $(\text{cm/s})/^\circ\text{C}$.

(b) For $T = 15^\circ\text{C}$, it appears the tangent line to the curve goes through the points $(10, 25)$ and $(20, 32)$. So $S'(15) \approx \dfrac{32 - 25}{20 - 10} = 0.7\ (\text{cm/s})/^\circ\text{C}$. This tells us that at $T = 15^\circ\text{C}$, the maximum sustainable speed of Coho salmon is changing at a rate of 0.7 $(\text{cm/s})/^\circ\text{C}$. In a similar fashion for $T = 25^\circ\text{C}$, we can use the points $(20, 35)$ and $(25, 25)$ to obtain $S'(25) \approx \dfrac{25 - 35}{25 - 20} = -2\ (\text{cm/s})/^\circ\text{C}$. As it gets warmer than 20°C, the maximum sustainable speed decreases rapidly.

53. Since $f(x) = x\sin(1/x)$ when $x \neq 0$ and $f(0) = 0$, we have
$f'(0) = \lim\limits_{h \to 0} \dfrac{f(0 + h) - f(0)}{h} = \lim\limits_{h \to 0} \dfrac{h\sin(1/h) - 0}{h} = \lim\limits_{h \to 0} \sin(1/h)$. This limit does not exist since $\sin(1/h)$ takes the values -1 and 1 on any interval containing 0. (Compare with Example 4 in Section 2.2.)

54. Since $f(x) = x^2\sin(1/x)$ when $x \neq 0$ and $f(0) = 0$, we have

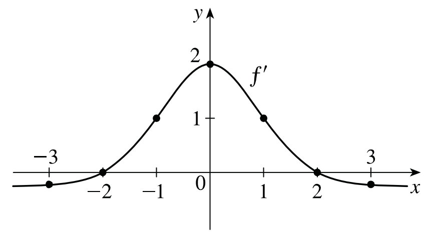
$f'(0) = \lim\limits_{h \to 0} \dfrac{f(0 + h) - f(0)}{h} = \lim\limits_{h \to 0} \dfrac{h^2\sin(1/h) - 0}{h} = \lim\limits_{h \to 0} h\sin(1/h)$. Since $-1 \leq \sin\dfrac{1}{h} \leq 1$, we have

$-|h| \leq |h|\sin\dfrac{1}{h} \leq |h| \quad \Rightarrow \quad -|h| \leq h\sin\dfrac{1}{h} \leq |h|$. Because $\lim\limits_{h \to 0} (-|h|) = 0$ and $\lim\limits_{h \to 0} |h| = 0$, we know that

$\lim\limits_{h \to 0} \left(h\sin\dfrac{1}{h}\right) = 0$ by the Squeeze Theorem. Thus, $f'(0) = 0$.

2.7 The Derivative as a Function

1. It appears that f is an odd function, so f' will be an even function—that is, $f'(-a) = f'(a)$.



(a) $f'(-3) \approx -0.2$

(b) $f'(-2) \approx 0$

(c) $f'(-1) \approx 1$

(d) $f'(0) \approx 2$

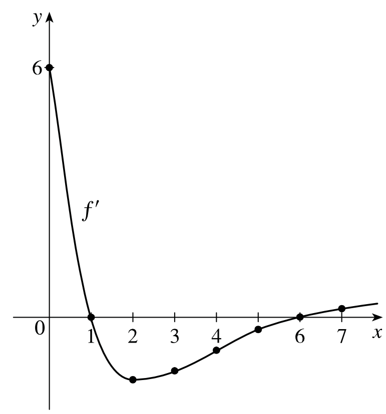
(e) $f'(1) \approx 1$

(f) $f'(2) \approx 0$

(g) $f'(3) \approx -0.2$

2. Your answers may vary depending on your estimates.

(a) *Note:* By estimating the slopes of tangent lines on the graph of f, it appears that $f'(0) \approx 6$.



(b) $f'(1) \approx 0$

(c) $f'(2) \approx -1.5$

(d) $f'(3) \approx -1.3$

(e) $f'(4) \approx -0.8$

(f) $f'(5) \approx -0.3$

(g) $f'(6) \approx 0$

(h) $f'(7) \approx 0.2$

3. (a)$'$ = II, since from left to right, the slopes of the tangents to graph (a) start out negative, become 0, then positive, then 0, then negative again. The actual function values in graph II follow the same pattern.

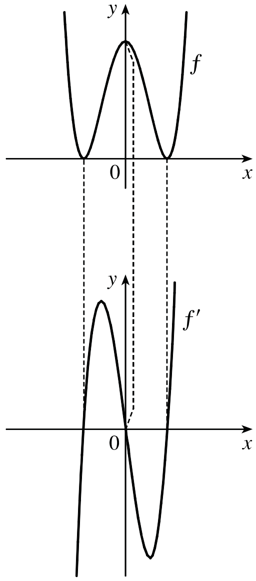
(b)$'$ = IV, since from left to right, the slopes of the tangents to graph (b) start out at a fixed positive quantity, then suddenly become negative, then positive again. The discontinuities in graph IV indicate sudden changes in the slopes of the tangents.

(c)$'$ = I, since the slopes of the tangents to graph (c) are negative for $x < 0$ and positive for $x > 0$, as are the function values of graph I.

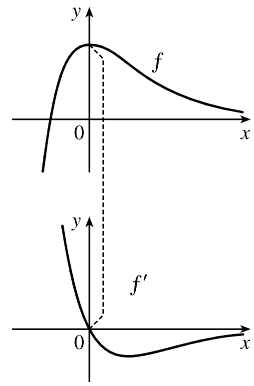
(d)$'$ = III, since from left to right, the slopes of the tangents to graph (d) are positive, then 0, then negative, then 0, then positive, then 0, then negative again, and the function values in graph III follow the same pattern.

Hints for Exercises 4–11: First plot x-intercepts on the graph of f' for any horizontal tangents on the graph of f. Look for any corners on the graph of f—there will be a discontinuity on the graph of f'. On any interval where f has a tangent with positive (or negative) slope, the graph of f' will be positive (or negative). If the graph of the function is linear, the graph of f' will be a horizontal line.

4.



5.



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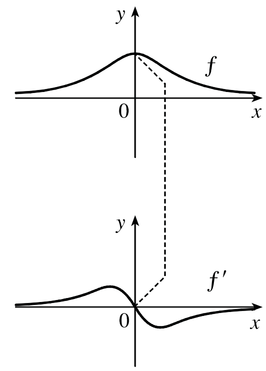

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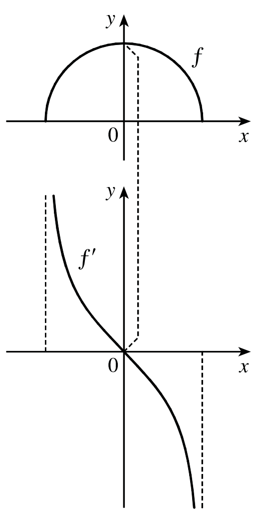
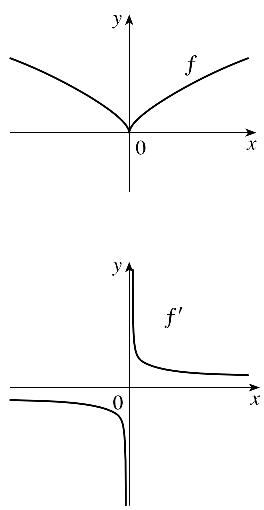
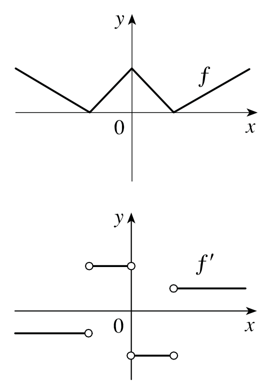


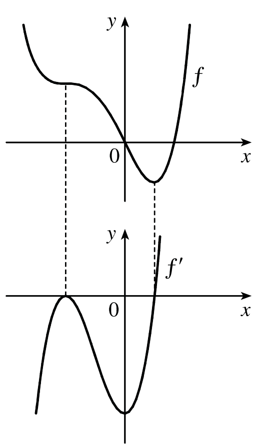
8.



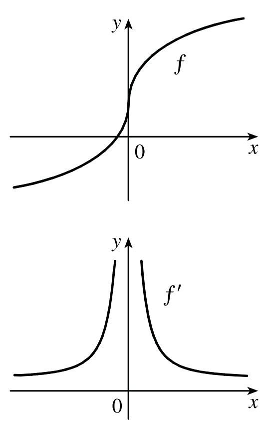
9.



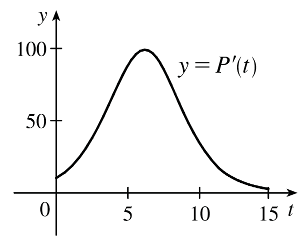
10.



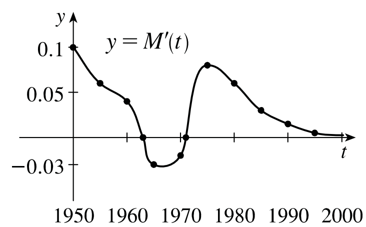
11.



12. The slopes of the tangent lines on the graph of $y = P(t)$ are always positive, so the y-values of $y = P'(t)$ are always positive. These values start out relatively small and keep increasing, reaching a maximum at about $t = 6$. Then the y-values of $y = P'(t)$ decrease and get close to zero. The graph of P' tells us that the yeast culture grows most rapidly after 6 hours and then the growth rate declines.

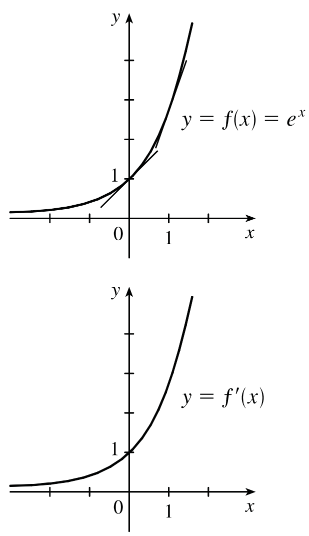


13. It appears that there are horizontal tangents on the graph of M for $t = 1963$ and $t = 1971$. Thus, there are zeros for those values of t on the graph of M'. The derivative is negative for the years 1963 to 1971.



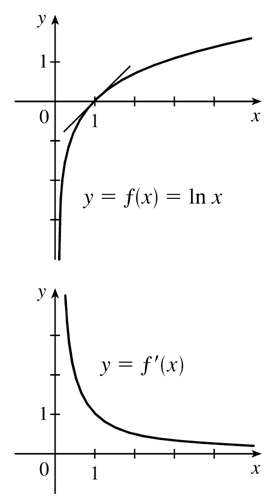
14. See Figure 1 in Section 3.3.

15.



The slope at 0 appears to be 1 and the slope at 1 appears to be 2.7. As x decreases, the slope gets closer to 0. Since the graphs are so similar, we might guess that $f'(x) = e^x$.

16.

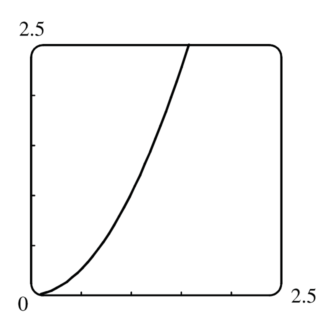


As x increases toward 1, $f'(x)$ decreases from very large numbers to 1. As x becomes large, $f'(x)$ gets closer to 0. As a guess, $f'(x) = 1/x^2$ or $f'(x) = 1/x$ makes sense.

17. (a) By zooming in, we estimate that $f'(0) = 0$, $f'\left(\frac{1}{2}\right) = 1$, $f'(1) = 2$, and $f'(2) = 4$.

(b) By symmetry, $f'(-x) = -f'(x)$. So $f'\left(-\frac{1}{2}\right) = -1$, $f'(-1) = -2$, and $f'(-2) = -4$.

(c) It appears that $f'(x)$ is twice the value of x, so we guess that $f'(x) = 2x$.

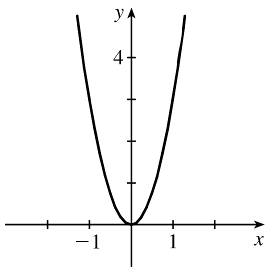


(d) $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$
$$= \lim_{h\to 0} \frac{(x^2 + 2hx + h^2) - x^2}{h} = \lim_{h\to 0} \frac{2hx + h^2}{h} = \lim_{h\to 0} \frac{h(2x+h)}{h} = \lim_{h\to 0} (2x + h) = 2x$$

18. (a) By zooming in, we estimate that $f'(0) = 0$, $f'\left(\frac{1}{2}\right) \approx 0.75$, $f'(1) \approx 3$, $f'(2) \approx 12$, and $f'(3) \approx 27$.

(b) By symmetry, $f'(-x) = f'(x)$. So $f'\left(-\frac{1}{2}\right) \approx 0.75$, $f'(-1) \approx 3$, $f'(-2) \approx 12$, and $f'(-3) \approx 27$.

(c)



(d) Since $f'(0) = 0$, it appears that f' may have the form $f'(x) = ax^2$. Using $f'(1) = 3$, we have $a = 3$, so $f'(x) = 3x^2$.

(e) $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h\to 0} \dfrac{(x+h)^3 - x^3}{h} = \lim\limits_{h\to 0} \dfrac{(x^3 + 3x^2h + 3xh^2 + h^3) - x^3}{h}$

$= \lim\limits_{h\to 0} \dfrac{3x^2h + 3xh^2 + h^3}{h} = \lim\limits_{h\to 0} \dfrac{h(3x^2 + 3xh + h^2)}{h} = \lim\limits_{h\to 0} (3x^2 + 3xh + h^2) = 3x^2$

19. $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h\to 0} \dfrac{\left[\frac{1}{2}(x+h) - \frac{1}{3}\right] - \left(\frac{1}{2}x - \frac{1}{3}\right)}{h} = \lim\limits_{h\to 0} \dfrac{\frac{1}{2}x + \frac{1}{2}h - \frac{1}{3} - \frac{1}{2}x + \frac{1}{3}}{h}$

$= \lim\limits_{h\to 0} \dfrac{\frac{1}{2}h}{h} = \lim\limits_{h\to 0} \frac{1}{2} = \frac{1}{2}$

Domain of f = domain of $f' = \mathbb{R}$.

20. $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h\to 0} \dfrac{[m(x+h) + b] - (mx + b)}{h} = \lim\limits_{h\to 0} \dfrac{mx + mh + b - mx - b}{h}$

$= \lim\limits_{h\to 0} \dfrac{mh}{h} = \lim\limits_{h\to 0} m = m$

Domain of f = domain of $f' = \mathbb{R}$.

21. $f'(t) = \lim\limits_{h\to 0} \dfrac{f(t+h) - f(t)}{h} = \lim\limits_{h\to 0} \dfrac{\left[5(t+h) - 9(t+h)^2\right] - (5t - 9t^2)}{h}$

$= \lim\limits_{h\to 0} \dfrac{5t + 5h - 9(t^2 + 2th + h^2) - 5t + 9t^2}{h} = \lim\limits_{h\to 0} \dfrac{5t + 5h - 9t^2 - 18th - 9h^2 - 5t + 9t^2}{h}$

$= \lim\limits_{h\to 0} \dfrac{5h - 18th - 9h^2}{h} = \lim\limits_{h\to 0} \dfrac{h(5 - 18t - 9h)}{h} = \lim\limits_{h\to 0} (5 - 18t - 9h) = 5 - 18t$

Domain of f = domain of $f' = \mathbb{R}$.

22. $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h\to 0} \dfrac{\left[1.5(x+h)^2 - (x+h) + 3.7\right] - \left(1.5x^2 - x + 3.7\right)}{h}$

$= \lim\limits_{h\to 0} \dfrac{1.5x^2 + 3xh + 1.5h^2 - x - h + 3.7 - 1.5x^2 + x - 3.7}{h} = \lim\limits_{h\to 0} \dfrac{3xh + 1.5h^2 - h}{h}$

$= \lim\limits_{h\to 0} (3x + 1.5h - 1) = 3x - 1$

Domain of f = domain of $f' = \mathbb{R}$.

23. $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{[(x+h)^2 - 2(x+h)^3] - (x^2 - 2x^3)}{h}$

$= \lim\limits_{h\to 0} \dfrac{x^2 + 2xh + h^2 - 2x^3 - 6x^2h - 6xh^2 - 2h^3 - x^2 + 2x^3}{h}$

$= \lim\limits_{h\to 0} \dfrac{2xh + h^2 - 6x^2h - 6xh^2 - 2h^3}{h} = \lim\limits_{h\to 0} \dfrac{h(2x + h - 6x^2 - 6xh - 2h^2)}{h}$

$= \lim\limits_{h\to 0} (2x + h - 6x^2 - 6xh - 2h^2) = 2x - 6x^2$

Domain of f = domain of $f' = \mathbb{R}$.

24. $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{\left(x+h+\sqrt{x+h}\right) - \left(x+\sqrt{x}\right)}{h}$

$= \lim\limits_{h\to 0} \left(\dfrac{h}{h} + \dfrac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \dfrac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right) = \lim\limits_{h\to 0} \left[1 + \dfrac{(x+h)-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}\right]$

$= \lim\limits_{h\to 0} \left(1 + \dfrac{1}{\sqrt{x+h}+\sqrt{x}}\right) = 1 + \dfrac{1}{\sqrt{x}+\sqrt{x}} = 1 + \dfrac{1}{2\sqrt{x}}$

Domain of $f = [0, \infty)$, domain of $f' = (0, \infty)$.

25. $g'(x) = \lim\limits_{h\to 0} \dfrac{g(x+h)-g(x)}{h} = \lim\limits_{h\to 0} \dfrac{\sqrt{1+2(x+h)} - \sqrt{1+2x}}{h} \left[\dfrac{\sqrt{1+2(x+h)} + \sqrt{1+2x}}{\sqrt{1+2(x+h)} + \sqrt{1+2x}}\right]$

$= \lim\limits_{h\to 0} \dfrac{(1+2x+2h)-(1+2x)}{h\left[\sqrt{1+2(x+h)} + \sqrt{1+2x}\right]} = \lim\limits_{h\to 0} \dfrac{2}{\sqrt{1+2x+2h}+\sqrt{1+2x}} = \dfrac{2}{2\sqrt{1+2x}} = \dfrac{1}{\sqrt{1+2x}}$

Domain of $g = \left[-\frac{1}{2}, \infty\right)$, domain of $g' = \left(-\frac{1}{2}, \infty\right)$.

26. $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{\dfrac{(x+h)^2-1}{2(x+h)-3} - \dfrac{x^2-1}{2x-3}}{h}$

$= \lim\limits_{h\to 0} \dfrac{\dfrac{[(x+h)^2-1](2x-3) - [2(x+h)-3](x^2-1)}{[2(x+h)-3](2x-3)}}{h}$

$= \lim\limits_{h\to 0} \dfrac{(x^2+2xh+h^2-1)(2x-3) - (2x+2h-3)(x^2-1)}{h[2(x+h)-3](2x-3)}$

$= \lim\limits_{h\to 0} \dfrac{(2x^3+4x^2h+2xh^2-2x-3x^2-6xh-3h^2+3) - (2x^3+2x^2h-3x^2-2x-2h+3)}{h(2x+2h-3)(2x-3)}$

$= \lim\limits_{h\to 0} \dfrac{4x^2h+2xh^2-6xh-3h^2-2x^2h+2h}{h(2x+2h-3)(2x-3)} = \lim\limits_{h\to 0} \dfrac{h(2x^2+2xh-6x-3h+2)}{h(2x+2h-3)(2x-3)}$

$= \lim\limits_{h\to 0} \dfrac{2x^2+2xh-6x-3h+2}{(2x+2h-3)(2x-3)} = \dfrac{2x^2-6x+2}{(2x-3)^2}$

Domain of f = domain of $f' = \left(-\infty, \frac{3}{2}\right) \cup \left(\frac{3}{2}, \infty\right)$.

27. $G'(t) = \lim_{h\to 0} \dfrac{G(t+h)-G(t)}{h} = \lim_{h\to 0} \dfrac{\dfrac{4(t+h)}{(t+h)+1} - \dfrac{4t}{t+1}}{h} = \lim_{h\to 0} \dfrac{\dfrac{4(t+h)(t+1)-4t(t+h+1)}{(t+h+1)(t+1)}}{h}$

$= \lim_{h\to 0} \dfrac{\left(4t^2+4ht+4t+4h\right)-\left(4t^2+4ht+4t\right)}{h(t+h+1)(t+1)} = \lim_{h\to 0} \dfrac{4h}{h(t+h+1)(t+1)}$

$= \lim_{h\to 0} \dfrac{4}{(t+h+1)(t+1)} = \dfrac{4}{(t+1)^2}$

Domain of G = domain of $G' = (-\infty,-1)\cup(-1,\infty)$.

28. $g'(t) = \lim_{h\to 0} \dfrac{g(t+h)-g(t)}{h} = \lim_{h\to 0} \dfrac{\dfrac{1}{\sqrt{t+h}} - \dfrac{1}{\sqrt{t}}}{h} = \lim_{h\to 0} \dfrac{\dfrac{\sqrt{t}-\sqrt{t+h}}{\sqrt{t+h}\,\sqrt{t}}}{h} = \lim_{h\to 0} \left(\dfrac{\sqrt{t}-\sqrt{t+h}}{h\,\sqrt{t+h}\,\sqrt{t}} \cdot \dfrac{\sqrt{t}+\sqrt{t+h}}{\sqrt{t}+\sqrt{t+h}}\right)$

$= \lim_{h\to 0} \dfrac{t-(t+h)}{h\,\sqrt{t+h}\,\sqrt{t}\left(\sqrt{t}+\sqrt{t+h}\right)} = \lim_{h\to 0} \dfrac{-h}{h\,\sqrt{t+h}\,\sqrt{t}\left(\sqrt{t}+\sqrt{t+h}\right)} = \lim_{h\to 0} \dfrac{-1}{\sqrt{t+h}\,\sqrt{t}\left(\sqrt{t}+\sqrt{t+h}\right)}$

$= \dfrac{-1}{\sqrt{t}\,\sqrt{t}\left(\sqrt{t}+\sqrt{t}\right)} = \dfrac{-1}{t\left(2\sqrt{t}\right)} = -\dfrac{1}{2t^{3/2}}$

Domain of g = domain of $g' = (0,\infty)$.

29. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{(x+h)^4-x^4}{h} = \lim_{h\to 0} \dfrac{\left(x^4+4x^3h+6x^2h^2+4xh^3+h^4\right)-x^4}{h}$

$= \lim_{h\to 0} \dfrac{4x^3h+6x^2h^2+4xh^3+h^4}{h} = \lim_{h\to 0} \left(4x^3+6x^2h+4xh^2+h^3\right) = 4x^3$

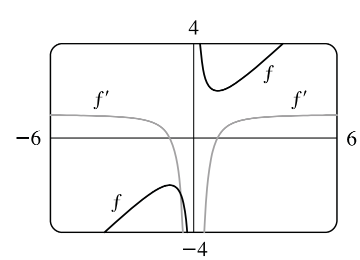
Domain of f = domain of $f' = \mathbb{R}$.

30. (a) $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{[(x+h)+1/(x+h)]-(x+1/x)}{h} = \lim_{h\to 0} \dfrac{\dfrac{(x+h)^2+1}{x+h} - \dfrac{x^2+1}{x}}{h}$

$= \lim_{h\to 0} \dfrac{x[(x+h)^2+1]-(x+h)(x^2+1)}{h(x+h)x} = \lim_{h\to 0} \dfrac{(x^3+2hx^2+xh^2+x)-(x^3+x+hx^2+h)}{h(x+h)x}$

$= \lim_{h\to 0} \dfrac{hx^2+xh^2-h}{h(x+h)x} = \lim_{h\to 0} \dfrac{h(x^2+xh-1)}{h(x+h)x} = \lim_{h\to 0} \dfrac{x^2+xh-1}{(x+h)x} = \dfrac{x^2-1}{x^2}$, or $1-\dfrac{1}{x^2}$

(b) Notice that $f'(x)=0$ when f has a horizontal tangent, $f'(x)$ is positive when the tangents have positive slope, and $f'(x)$ is negative when the tangents have negative slope. Both functions are discontinuous at $x=0$.



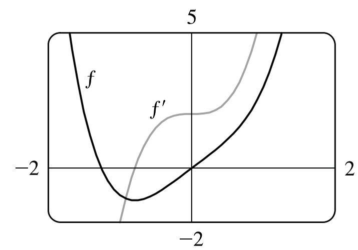
31. (a) $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{[(x+h)^4 + 2(x+h)] - (x^4+2x)}{h}$

$$= \lim_{h\to 0} \frac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 + 2x + 2h - x^4 - 2x}{h}$$

$$= \lim_{h\to 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4 + 2h}{h} = \lim_{h\to 0} \frac{h(4x^3 + 6x^2h + 4xh^2 + h^3 + 2)}{h}$$

$$= \lim_{h\to 0} (4x^3 + 6x^2h + 4xh^2 + h^3 + 2) = 4x^3 + 2$$

(b) Notice that $f'(x) = 0$ when f has a horizontal tangent, $f'(x)$ is positive when the tangents have positive slope, and $f'(x)$ is negative when the tangents have negative slope.



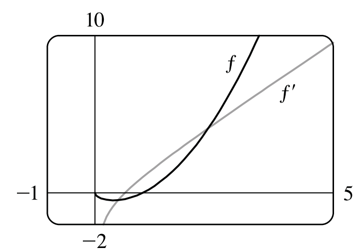
32. (a) $f'(t) = \lim\limits_{h\to 0} \dfrac{f(t+h)-f(t)}{h} = \lim\limits_{h\to 0} \dfrac{\left[(t+h)^2 - \sqrt{t+h}\,\right] - \left(t^2 - \sqrt{t}\,\right)}{h}$

$$= \lim_{h\to 0} \frac{t^2 + 2ht + h^2 - \sqrt{t+h} - t^2 + \sqrt{t}}{h} = \lim_{h\to 0} \left(\frac{2ht + h^2}{h} + \frac{\sqrt{t} - \sqrt{t+h}}{h}\right)$$

$$= \lim_{h\to 0} \left(\frac{h(2t+h)}{h} + \frac{\sqrt{t} - \sqrt{t+h}}{h} \cdot \frac{\sqrt{t} + \sqrt{t+h}}{\sqrt{t} + \sqrt{t+h}}\right) = \lim_{h\to 0} \left(2t + h + \frac{t - (t+h)}{h\left(\sqrt{t} + \sqrt{t+h}\,\right)}\right)$$

$$= \lim_{h\to 0} \left(2t + h + \frac{-h}{h\left(\sqrt{t} + \sqrt{t+h}\,\right)}\right) = \lim_{h\to 0} \left(2t + h + \frac{-1}{\sqrt{t} + \sqrt{t+h}}\right) = 2t - \frac{1}{2\sqrt{t}}$$

(b) Notice that $f'(t) = 0$ when f has a horizontal tangent, $f'(t)$ is positive when the tangents have positive slope, and $f'(t)$ is negative when the tangents have negative slope.



33. (a) $U'(t)$ is the rate at which the unemployment rate is changing with respect to time. Its units are percent per year.

(b) To find $U'(t)$, we use $\lim\limits_{h\to 0} \dfrac{U(t+h) - U(t)}{h} \approx \dfrac{U(t+h) - U(t)}{h}$ for small values of h.

For 1998: $U'(1998) \approx \dfrac{U(1999) - U(1998)}{1999 - 1998} = \dfrac{4.2 - 4.5}{1} = -0.30$

For 1999: We estimate $U'(1999)$ by using $h = -1$ and $h = 1$, and then average the two results to obtain a final estimate.

$h = -1 \quad\Rightarrow\quad U'(1999) \approx \dfrac{U(1998) - U(1999)}{1998 - 1999} = \dfrac{4.5 - 4.2}{-1} = -0.30;$

$h = 1 \quad\Rightarrow\quad U'(1999) \approx \dfrac{U(2000) - U(1999)}{2000 - 1999} = \dfrac{4.0 - 4.2}{1} = -0.20.$

So we estimate that $U'(1999) \approx \frac{1}{2}[(-0.30) + (-0.20)] = -0.25$.

t	1998	1999	2000	2001	2002	2003	2004	2005	2006	2007
$U'(t)$	−0.30	−0.25	0.25	0.90	0.65	−0.15	−0.45	−0.45	−0.25	0.00

34. (a) $P'(t)$ is the rate at which the percentage of Americans under the age of 18 is changing with respect to time. Its units are percent per year (%/yr).

(b) To find $P'(t)$, we use $\lim_{h\to 0} \frac{P(t+h)-P(t)}{h} \approx \frac{P(t+h)-P(t)}{h}$ for small values of h.

For 1950: $P'(1950) \approx \frac{P(1960)-P(1950)}{1960-1950} = \frac{35.7-31.1}{10} = 0.46$

For 1960: We estimate $P'(1960)$ by using $h=-10$ and $h=10$, and then average the two results to obtain a final estimate.

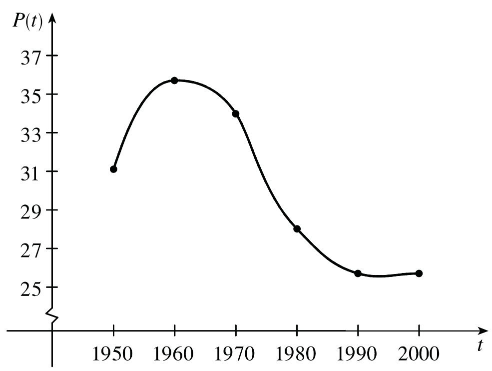
$h=-10 \quad\Rightarrow\quad P'(1960) \approx \frac{P(1950)-P(1960)}{1950-1960} = \frac{31.1-35.7}{-10} = 0.46$

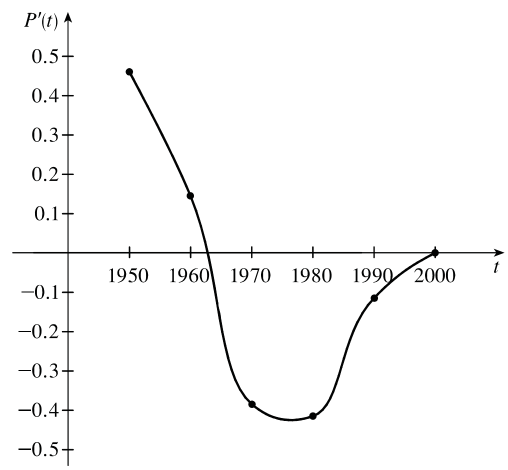
$h=10 \quad\Rightarrow\quad P'(1960) \approx \frac{P(1970)-P(1960)}{1970-1960} = \frac{34.0-35.7}{10} = -0.17$

So we estimate that $P'(1960) \approx \frac{1}{2}[0.46+(-0.17)] = 0.145$.

t	1950	1960	1970	1980	1990	2000
$P'(t)$	0.460	0.145	−0.385	−0.415	−0.115	0.000

(c)





(d) We could get more accurate values for $P'(t)$ by obtaining data for the mid-decade years 1955, 1965, 1975, 1985, and 1995.

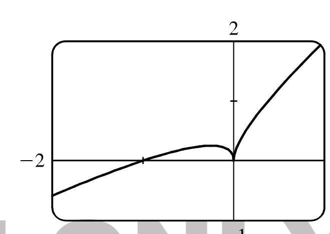
35. f is not differentiable at $x=-4$, because the graph has a corner there, and at $x=0$, because there is a discontinuity there.

36. f is not differentiable at $x=0$, because there is a discontinuity there, and at $x=3$, because the graph has a vertical tangent there.

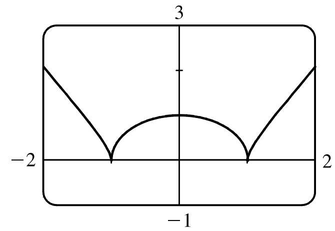
37. f is not differentiable at $x=-1$, because the graph has a vertical tangent there, and at $x=4$, because the graph has a corner there.

38. f is not differentiable at $x=-1$, because there is a discontinuity there, and at $x=2$, because the graph has a corner there.

39. As we zoom in toward $(-1,0)$, the curve appears more and more like a straight line, so $f(x)=x+\sqrt{|x|}$ is differentiable at $x=-1$. But no matter how much we zoom in toward the origin, the curve doesn't straighten out—we can't eliminate the sharp point (a cusp). So f is not differentiable at $x=0$.



40. As we zoom in toward $(0, 1)$, the curve appears more and more like a straight line, so f is differentiable at $x = 0$. But no matter how much we zoom in toward $(1, 0)$ or $(-1, 0)$, the curve doesn't straighten out—we can't eliminate the sharp point (a cusp). So f is not differentiable at $x = \pm 1$.



41. $a = f$, $b = f'$, $c = f''$. We can see this because where a has a horizontal tangent, $b = 0$, and where b has a horizontal tangent, $c = 0$. We can immediately see that c can be neither f nor f', since at the points where c has a horizontal tangent, neither a nor b is equal to 0.

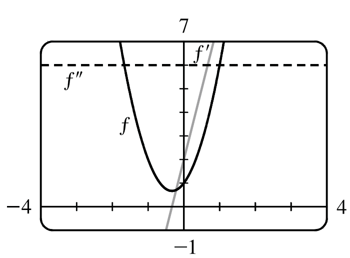
42. Where d has horizontal tangents, only c is 0, so $d' = c$. c has negative tangents for $x < 0$ and b is the only graph that is negative for $x < 0$, so $c' = b$. b has positive tangents on $\mathbb{R}$ (except at $x = 0$), and the only graph that is positive on the same domain is a, so $b' = a$. We conclude that $d = f$, $c = f'$, $b = f''$, and $a = f'''$.

43. We can immediately see that a is the graph of the acceleration function, since at the points where a has a horizontal tangent, neither c nor b is equal to 0. Next, we note that $a = 0$ at the point where b has a horizontal tangent, so b must be the graph of the velocity function, and hence, $b' = a$. We conclude that c is the graph of the position function.

44. a must be the jerk since none of the graphs are 0 at its high and low points. a is 0 where b has a maximum, so $b' = a$. b is 0 where c has a maximum, so $c' = b$. We conclude that d is the position function, c is the velocity, b is the acceleration, and a is the jerk.

45.
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[3(x+h)^2 + 2(x+h) + 1] - (3x^2 + 2x + 1)}{h}$$
$$= \lim_{h \to 0} \frac{(3x^2 + 6xh + 3h^2 + 2x + 2h + 1) - (3x^2 + 2x + 1)}{h} = \lim_{h \to 0} \frac{6xh + 3h^2 + 2h}{h}$$
$$= \lim_{h \to 0} \frac{h(6x + 3h + 2)}{h} = \lim_{h \to 0} (6x + 3h + 2) = 6x + 2$$

$$f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h \to 0} \frac{[6(x+h) + 2] - (6x + 2)}{h} = \lim_{h \to 0} \frac{(6x + 6h + 2) - (6x + 2)}{h}$$
$$= \lim_{h \to 0} \frac{6h}{h} = \lim_{h \to 0} 6 = 6$$

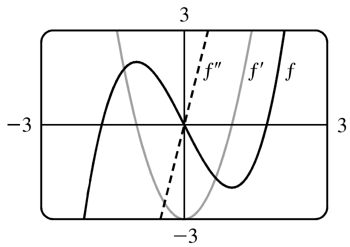


We see from the graph that our answers are reasonable because the graph of f' is that of a linear function and the graph of f'' is that of a constant function.

46.
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[(x+h)^3 - 3(x+h)] - (x^3 - 3x)}{h}$$
$$= \lim_{h \to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3 - 3x - 3h) - (x^3 - 3x)}{h} = \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3 - 3h}{h}$$
$$= \lim_{h \to 0} \frac{h(3x^2 + 3xh + h^2 - 3)}{h} = \lim_{h \to 0} (3x^2 + 3xh + h^2 - 3) = 3x^2 - 3$$

$$f''(x) = \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h\to 0} \frac{[3(x+h)^2 - 3] - (3x^2 - 3)}{h} = \lim_{h\to 0} \frac{(3x^2 + 6xh + 3h^2 - 3) - (3x^2 - 3)}{h}$$

$$= \lim_{h\to 0} \frac{6xh + 3h^2}{h} = \lim_{h\to 0} \frac{h(6x + 3h)}{h} = \lim_{h\to 0}(6x + 3h) = 6x$$



We see from the graph that our answers are reasonable because the graph of f' is that of a even function (f is an odd function) and the graph of f'' is that of an odd function. Furthermore, $f' = 0$ when f has a horizontal tangent and $f'' = 0$ when f' has a horizontal tangent.

47. $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\left[2(x+h)^2 - (x+h)^3\right] - (2x^2 - x^3)}{h}$

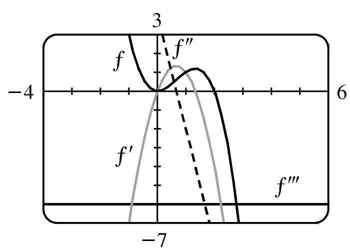
$$= \lim_{h\to 0} \frac{h(4x + 2h - 3x^2 - 3xh - h^2)}{h} = \lim_{h\to 0} (4x + 2h - 3x^2 - 3xh - h^2) = 4x - 3x^2$$

$$f''(x) = \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h\to 0} \frac{\left[4(x+h) - 3(x+h)^2\right] - (4x - 3x^2)}{h} = \lim_{h\to 0} \frac{h(4 - 6x - 3h)}{h}$$

$$= \lim_{h\to 0}(4 - 6x - 3h) = 4 - 6x$$

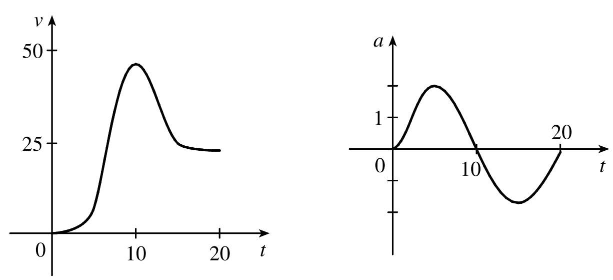
$$f'''(x) = \lim_{h\to 0} \frac{f''(x+h) - f''(x)}{h} = \lim_{h\to 0} \frac{[4 - 6(x+h)] - (4 - 6x)}{h} = \lim_{h\to 0} \frac{-6h}{h} = \lim_{h\to 0} (-6) = -6$$

$$f^{(4)}(x) = \lim_{h\to 0} \frac{f'''(x+h) - f'''(x)}{h} = \lim_{h\to 0} \frac{-6 - (-6)}{h} = \lim_{h\to 0} \frac{0}{h} = \lim_{h\to 0} (0) = 0$$



The graphs are consistent with the geometric interpretations of the derivatives because f' has zeros where f has a local minimum and a local maximum, f'' has a zero where f' has a local maximum, and f''' is a constant function equal to the slope of f''.

48. (a) Since we estimate the velocity to be a maximum at $t = 10$, the acceleration is 0 at $t = 10$.



(b) Drawing a tangent line at $t = 10$ on the graph of a, a appears to decrease by $10\ \text{ft/s}^2$ over a period of 20 s. So at $t = 10$ s, the jerk is approximately $-10/20 = -0.5\ (\text{ft/s}^2)/\text{s}$ or ft/s^3.

49. (a) Note that we have factored $x - a$ as the difference of two cubes in the third step.

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{x^{1/3} - a^{1/3}}{x - a} = \lim_{x \to a} \frac{x^{1/3} - a^{1/3}}{(x^{1/3} - a^{1/3})(x^{2/3} + x^{1/3}a^{1/3} + a^{2/3})}$$

$$= \lim_{x \to a} \frac{1}{x^{2/3} + x^{1/3}a^{1/3} + a^{2/3}} = \frac{1}{3a^{2/3}} \text{ or } \tfrac{1}{3}a^{-2/3}$$

(b) $f'(0) = \lim\limits_{h \to 0} \dfrac{f(0 + h) - f(0)}{h} = \lim\limits_{h \to 0} \dfrac{\sqrt[3]{h} - 0}{h} = \lim\limits_{h \to 0} \dfrac{1}{h^{2/3}}$. This function increases without bound, so the limit does not exist, and therefore $f'(0)$ does not exist.

(c) $\lim\limits_{x \to 0} |f'(x)| = \lim\limits_{x \to 0} \dfrac{1}{3x^{2/3}} = \infty$ and f is continuous at $x = 0$ (root function), so f has a vertical tangent at $x = 0$.

50. (a) $g'(0) = \lim\limits_{x \to 0} \dfrac{g(x) - g(0)}{x - 0} = \lim\limits_{x \to 0} \dfrac{x^{2/3} - 0}{x} = \lim\limits_{x \to 0} \dfrac{1}{x^{1/3}}$, which does not exist.

(b) $$g'(a) = \lim_{x \to a} \frac{g(x) - g(a)}{x - a} = \lim_{x \to a} \frac{x^{2/3} - a^{2/3}}{x - a} = \lim_{x \to a} \frac{(x^{1/3} - a^{1/3})(x^{1/3} + a^{1/3})}{(x^{1/3} - a^{1/3})(x^{2/3} + x^{1/3}a^{1/3} + a^{2/3})}$$

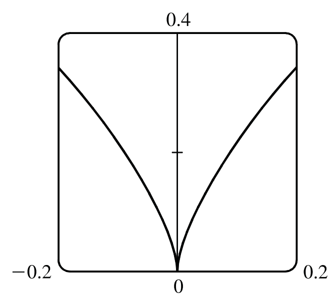
$$= \lim_{x \to a} \frac{x^{1/3} + a^{1/3}}{x^{2/3} + x^{1/3}a^{1/3} + a^{2/3}} = \frac{2a^{1/3}}{3a^{2/3}} = \frac{2}{3a^{1/3}} \text{ or } \tfrac{2}{3}a^{-1/3}$$

(c) $g(x) = x^{2/3}$ is continuous at $x = 0$ and

$\lim\limits_{x \to 0} |g'(x)| = \lim\limits_{x \to 0} \dfrac{2}{3\,|x|^{1/3}} = \infty$. This shows that

g has a vertical tangent line at $x = 0$.

(d)



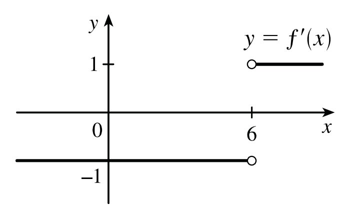
51. $f(x) = |x - 6| = \begin{cases} x - 6 & \text{if } x - 6 \geq 6 \\ -(x - 6) & \text{if } x - 6 < 0 \end{cases} = \begin{cases} x - 6 & \text{if } x \geq 6 \\ 6 - x & \text{if } x < 6 \end{cases}$

So the right-hand limit is $\lim\limits_{x \to 6^+} \dfrac{f(x) - f(6)}{x - 6} = \lim\limits_{x \to 6^+} \dfrac{|x - 6| - 0}{x - 6} = \lim\limits_{x \to 6^+} \dfrac{x - 6}{x - 6} = \lim\limits_{x \to 6^+} 1 = 1$, and the left-hand limit is $\lim\limits_{x \to 6^-} \dfrac{f(x) - f(6)}{x - 6} = \lim\limits_{x \to 6^-} \dfrac{|x - 6| - 0}{x - 6} = \lim\limits_{x \to 6^-} \dfrac{6 - x}{x - 6} = \lim\limits_{x \to 6^-} (-1) = -1$. Since these limits are not equal,

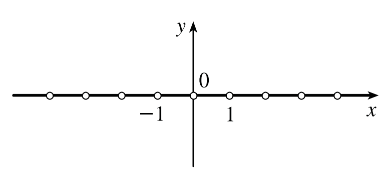
$f'(6) = \lim\limits_{x \to 6} \dfrac{f(x) - f(6)}{x - 6}$ does not exist and f is not differentiable at 6.

However, a formula for f' is $f'(x) = \begin{cases} 1 & \text{if } x > 6 \\ -1 & \text{if } x < 6 \end{cases}$

Another way of writing the formula is $f'(x) = \dfrac{x - 6}{|x - 6|}$.



52. $f(x) = [\![x]\!]$ is not continuous at any integer n, so f is not differentiable at n by the contrapositive of Theorem 4. If a is not an integer, then f is constant on an open interval containing a, so $f'(a) = 0$. Thus, $f'(x) = 0$, x not an integer.



53. (a) If f is even, then

$$\begin{aligned} f'(-x) &= \lim_{h\to 0} \frac{f(-x+h)-f(-x)}{h} = \lim_{h\to 0} \frac{f[-(x-h)]-f(-x)}{h} \\ &= \lim_{h\to 0} \frac{f(x-h)-f(x)}{h} = -\lim_{h\to 0} \frac{f(x-h)-f(x)}{-h} \qquad [\text{let } \Delta x = -h] \\ &= -\lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = -f'(x) \end{aligned}$$

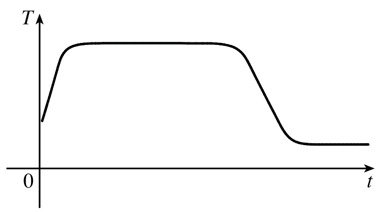
Therefore, f' is odd.

(b) If f is odd, then

$$\begin{aligned} f'(-x) &= \lim_{h\to 0} \frac{f(-x+h)-f(-x)}{h} = \lim_{h\to 0} \frac{f[-(x-h)]-f(-x)}{h} \\ &= \lim_{h\to 0} \frac{-f(x-h)+f(x)}{h} = \lim_{h\to 0} \frac{f(x-h)-f(x)}{-h} \qquad [\text{let } \Delta x = -h] \\ &= \lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = f'(x) \end{aligned}$$

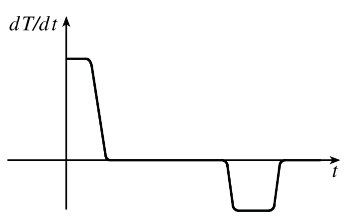
Therefore, f' is even.

54. (a)

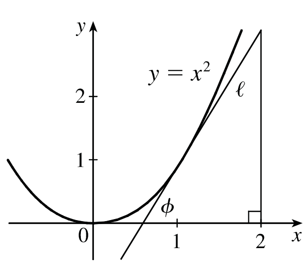


(b) The initial temperature of the water is close to room temperature because of the water that was in the pipes. When the water from the hot water tank starts coming out, dT/dt is large and positive as T increases to the temperature of the water in the tank. In the next phase, $dT/dt = 0$ as the water comes out at a constant, high temperature. After some time, dT/dt becomes small and negative as the contents of the hot water tank are exhausted. Finally, when the hot water has run out, dT/dt is once again 0 as the water maintains its (cold) temperature.

(c)



55.



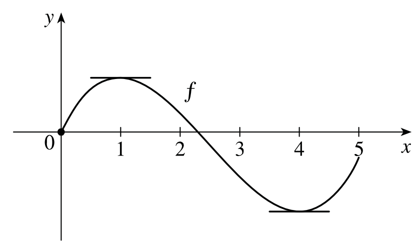
In the right triangle in the diagram, let Δy be the side opposite angle ϕ and Δx the side adjacent angle ϕ. Then the slope of the tangent line ℓ is $m = \Delta y/\Delta x = \tan\phi$. Note that $0 < \phi < \frac{\pi}{2}$. We know (see Exercise 17) that the derivative of $f(x) = x^2$ is $f'(x) = 2x$. So the slope of the tangent to the curve at the point $(1, 1)$ is 2. Thus, ϕ is the angle between 0 and $\frac{\pi}{2}$ whose tangent is 2; that is, $\phi = \tan^{-1} 2 \approx 63^\circ$.

2.8 What Does f' Say about f?

1. (a) Since $f'(x) < 0$ on $(1, 4)$, f is decreasing on this interval. Since $f'(x) > 0$ on $(0, 1)$ and $(4, 5)$, f is increasing on these intervals.

(b) At $x = 1$, $f'(x) = 0$ and f' changes from positive to negative there, f changes from increasing to decreasing and has a local maximum at $x = 1$. At $x = 4$, $f'(x) = 0$ and f' changes from negative to positive there, f changes from decreasing to increasing and has local minimum at $x = 4$.

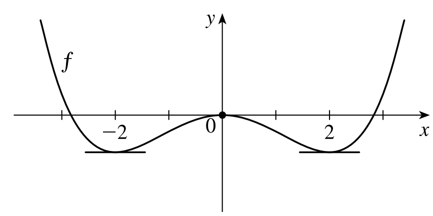
(c) Since $f(0) = 0$, start at the origin. Draw an increasing function on $(0, 1)$ with a local maximum at $x = 1$. Now draw a decreasing function on $(1, 4)$ and the steepest slope should occur at $x = 2.5$ since that's where the smallest value of f' occurs. Last, draw an increasing function on $(4, 5)$ making sure you have a local minimum at $x = 4$.



2. (a) $f'(x) > 0$ and f is increasing on $(-2, 0)$ and $(2, 3)$. $f'(x) < 0$ and f is decreasing on $(-3, -2)$ and $(0, 2)$.

(b) At $x = 0$, $f'(x) = 0$ and f' changes from positive to negative, so f has a local maximum at $x = 0$. At $x = -2$ and $x = 2$, $f'(x) = 0$ and f' changes from negative to positive, so f has local minima at $x = -2$ and $x = 2$.

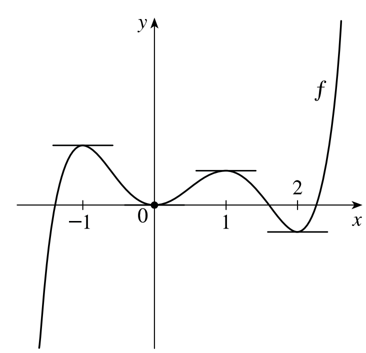
(c)



3. (a) $f'(x) > 0$ and f is increasing on $(-2, -1)$, $(0, 1)$, and $(2, 3)$. $f'(x) < 0$ and f is decreasing on $(-1, 0)$ and $(1, 2)$.

(b) At $x = -1$ and $x = 1$, $f'(x) = 0$ and f' changes from positive to negative, so f has local maxima at $x = -1$ and $x = 1$. At $x = 0$ and $x = 2$, $f'(x) = 0$ and f' changes from negative to positive, so f has local minima at $x = 0$ and $x = 2$.

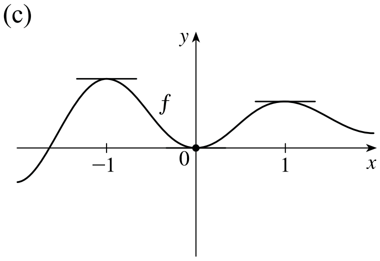
(c)



4. (a) $f'(x) > 0$ and f is increasing on $(-2, -1)$ and $(0, 1)$. $f'(x) < 0$ and f is decreasing on $(-1, 0)$ and $(1, 2)$.

(b) At $x = -1$ and $x = 1$, $f'(x) = 0$ and f' changes from positive to negative, so f has local maxima at $x = -1$ and $x = 1$. At $x = 0$, $f'(x) = 0$ and f' changes from negative to positive, so f has a local minimum at $x = 0$. (The points at $x = -2$ and $x = 2$ are not part of the graph.)

(c)

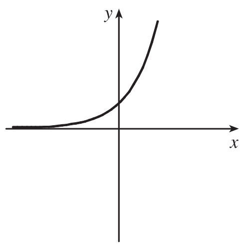


5. The derivative f' is increasing when the slopes of the tangent lines of f are becoming larger as x increases. This seems to be the case on the interval $(2, 5)$. The derivative is decreasing when the slopes of the tangent lines of f are becoming smaller as x increases, and this seems to be the case on $(-\infty, 2)$ and $(5, \infty)$. So f' is increasing on $(2, 5)$ and decreasing on $(-\infty, 2)$ and $(5, \infty)$.

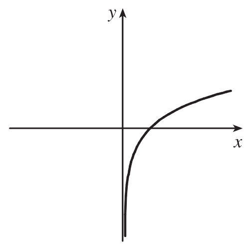
6. Call the curve with the smallest positive x-intercept g and the other curve h. Notice that where g is positive in the first quadrant, h is increasing. Thus, $h = f$ and $g = f'$. Now $f'(-1)$ is positive since f' is above the x-axis there and $f''(1)$ appears to be zero since f has an inflection point at $x = 1$. Therefore, $f''(-1)$ is greater than $f'(1)$.

7. Call the curve with the positive y-intercept g and the other curve h. Notice that g has a maximum (horizontal tangent) at $x = 0$, but $h \neq 0$, so h cannot be the derivative of g. Also notice that where g is positive, h is increasing. Thus, $h = f$ and $g = f'$. Now $f'(-1)$ is negative since f' is below the x-axis there and $f''(1)$ is positive since f is concave upward at $x = 1$. Therefore, $f''(1)$ is greater than $f'(-1)$.

8. (a)



(b)



(c) In part (a), the graph of $y = e^x$ is a curve whose slope is always positive and increasing. In part (b), the graph of $y = \ln x$ is a curve whose slope is always positive and decreasing.

9. If $D(t)$ is the size of the deficit as a function of time, then at the time of the speech $D'(t) > 0$, but $D''(t) < 0$ because $D''(t) = (D')'(t)$ is the rate of change of $D'(t)$.

10. (a) The rate of increase of the population is initially very small, then gets larger until it reaches a maximum at about $t = 8$ hours, and decreases toward 0 as the population begins to level off.

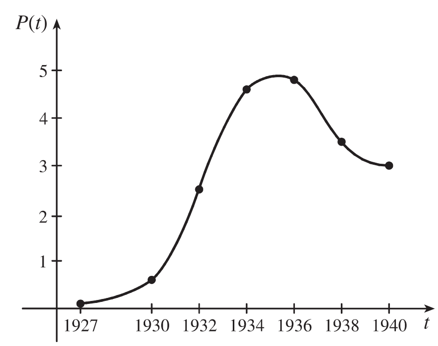
(b) The rate of increase has its maximum value at $t = 8$ hours.

(c) The population function is concave upward on $(0, 8)$ and concave downward on $(8, 18)$.

(d) At $t = 8$, the population is about 350, so the inflection point is about $(8, 350)$.

11. (a) The rate of increase of the population is initially very small, then increases rapidly until about 1932 when it starts decreasing. The rate becomes negative by 1936, peaks in magnitude in 1937, and approaches 0 in 1940.

(b) Inflection points (IP) appear to be at $(1932, 2.5)$ and $(1937, 4.3)$. The rate of change of population density starts to decrease in 1932 and starts to increase in 1937. The rates of population increase and decrease have their maximum values at those points.

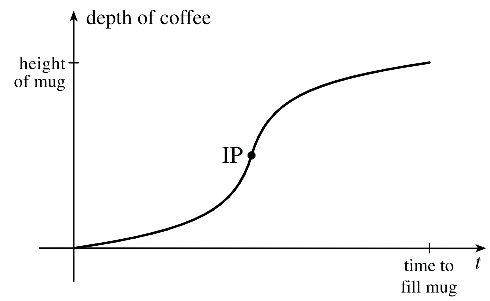


12. (a) If the position function is increasing, then the particle is moving toward the right. This occurs on t-intervals $(0, 2)$ and $(4, 6)$. If the function is decreasing, then the particle is moving toward the left—that is, on $(2, 4)$.

(b) The acceleration is the second derivative and is positive where the curve is concave upward. This occurs on $(3, 6)$. The acceleration is negative where the curve is concave downward—that is, on $(0, 3)$.

13. Most students learn more in the third hour of studying than in the eighth hour, so $K(3) - K(2)$ is larger than $K(8) - K(7)$. In other words, as you begin studying for a test, the rate of knowledge gain is large and then starts to taper off, so $K'(t)$ decreases and the graph of K is concave downward.

14. At first the depth increases slowly because the base of the mug is wide. But as the mug narrows, the coffee rises more quickly. Thus, the depth d increases at an increasing rate and its graph is concave upward. The rate of increase of d has a maximum where the mug is narrowest; that is, when the mug is half full. It is there that the inflection point (IP) occurs. Then the rate of increase of d starts to decrease as the mug widens and the graph becomes concave down.



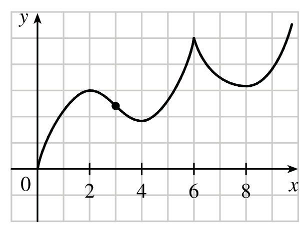
15. (a) f is increasing where f' is positive, that is, on $(0, 2)$, $(4, 6)$, and $(8, \infty)$; and decreasing where f' is negative, that is, on $(2, 4)$ and $(6, 8)$.

(b) f has local maxima where f' changes from positive to negative, at $x = 2$ and at $x = 6$, and local minima where f' changes from negative to positive, at $x = 4$ and at $x = 8$.

(c) f is concave upward (CU) where f' is increasing, that is, on $(3, 6)$ and $(6, \infty)$, and concave downward (CD) where f' is decreasing, that is, on $(0, 3)$.

(d) There is a point of inflection where f changes from being CD to being CU, that is, at $x = 3$.

(e)



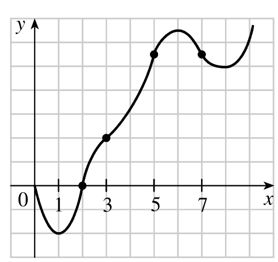
16. (a) f is increasing where f' is positive, on $(1, 6)$ and $(8, \infty)$, and decreasing where f' is negative, on $(0, 1)$ and $(6, 8)$.

(b) f has a local maximum where f' changes from positive to negative, at $x = 6$, and local minima where f' changes from negative to positive, at $x = 1$ and at $x = 8$.

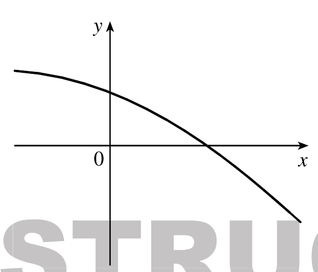
(c) f is concave upward where f' is increasing, that is, on $(0, 2)$, $(3, 5)$, and $(7, \infty)$, and concave downward where f' is decreasing, that is, on $(2, 3)$ and $(5, 7)$.

(d) There are points of inflection where f changes its direction of concavity, at $x = 2$, $x = 3$, $x = 5$ and $x = 7$.

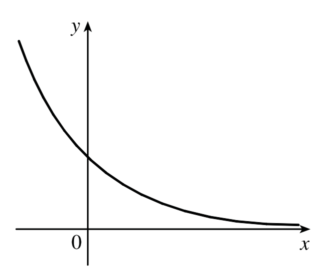
(e)



17. The function must be always decreasing (since the first derivative is always negative) and concave downward (since the second derivative is always negative).



18. The function must be always decreasing and concave upward.



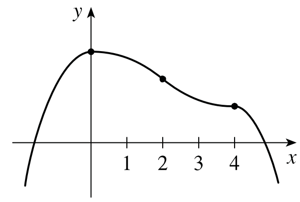
19. $f'(0) = f'(4) = 0 \quad \Rightarrow \quad$ horizontal tangents at $x = 0, 4$.

$f'(x) > 0$ if $x < 0 \quad \Rightarrow \quad f$ is increasing on $(-\infty, 0)$.

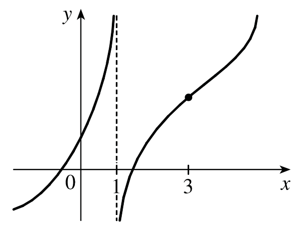
$f'(x) < 0$ if $0 < x < 4$ or if $x > 4 \quad \Rightarrow \quad f$ is decreasing on $(0, 4)$ and $(4, \infty)$.

$f''(x) > 0$ if $2 < x < 4 \quad \Rightarrow \quad f$ is concave upward on $(2, 4)$.

$f''(x) < 0$ if $x < 2$ or $x > 4 \quad \Rightarrow \quad f$ is concave downward on $(-\infty, 2)$ and $(4, \infty)$. There are inflection points when $x = 2$ and 4.



20. $f'(x) > 0$ for all $x \neq 1$ with vertical asymptote $x = 1$, so f is increasing on $(-\infty, 1)$ and $(1, \infty)$. $f''(x) > 0$ if $x < 1$ or $x > 3$, and $f''(x) < 0$ if $1 < x < 3$, so f is concave upward on $(-\infty, 1)$ and $(3, \infty)$, and concave downward on $(1, 3)$. There is an inflection point when $x = 3$.



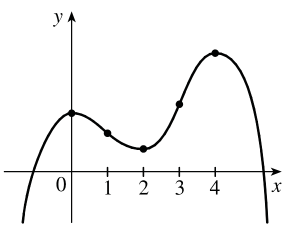
21. $f'(0) = f'(2) = f'(4) = 0 \quad \Rightarrow \quad$ horizontal tangents at $x = 0, 2, 4$.

$f'(x) > 0$ if $x < 0$ or $2 < x < 4 \quad \Rightarrow \quad f$ is increasing on $(-\infty, 0)$ and $(2, 4)$.

$f'(x) < 0$ if $0 < x < 2$ or $x > 4 \quad \Rightarrow \quad f$ is decreasing on $(0, 2)$ and $(4, \infty)$.

$f''(x) > 0$ if $1 < x < 3 \quad \Rightarrow \quad f$ is concave upward on $(1, 3)$.

$f''(x) < 0$ if $x < 1$ or $x > 3 \quad \Rightarrow \quad f$ is concave downward on $(-\infty, 1)$ and $(3, \infty)$. There are inflection points when $x = 1$ and 3.

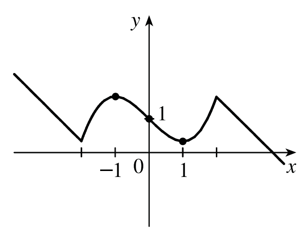


22. $f'(1) = f'(-1) = 0 \quad \Rightarrow \quad$ horizontal tangents at $x = \pm 1$.

$f'(x) < 0$ if $|x| < 1 \quad \Rightarrow \quad f$ is decreasing on $(-1, 1)$.

$f'(x) > 0$ if $1 < |x| < 2 \quad \Rightarrow \quad f$ is increasing on $(-2, -1)$ and $(1, 2)$.

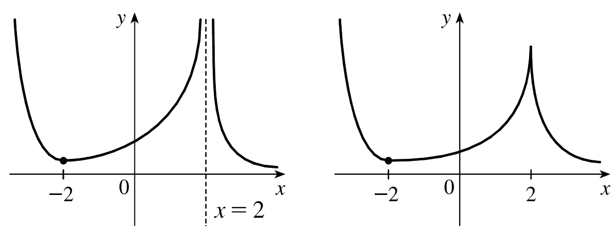
$f'(x) = -1$ if $|x| > 2 \quad \Rightarrow \quad$ the graph of f has constant slope -1 on $(-\infty, -2)$ and $(2, \infty)$.



$f''(x) < 0$ if $-2 < x < 0 \quad \Rightarrow \quad f$ is concave downward on $(-2, 0)$. The point $(0, 1)$ is an inflection point.

23. $f'(x) > 0$ if $|x| < 2 \quad \Rightarrow \quad f$ is increasing on $(-2, 2)$.

$f'(x) < 0$ if $|x| > 2 \quad \Rightarrow \quad f$ is decreasing on $(-\infty, -2)$ and $(2, \infty)$. $\quad f'(-2) = 0 \quad \Rightarrow \quad$ horizontal tangent at $x = -2$. $\lim\limits_{x \to 2} |f'(x)| = \infty \quad \Rightarrow \quad$ there is a vertical

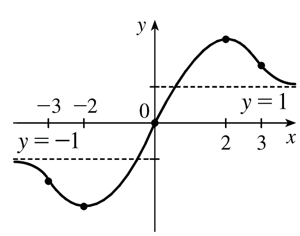


asymptote or vertical tangent (cusp) at $x = 2$. $f''(x) > 0$ if $x \neq 2 \quad \Rightarrow \quad f$ is concave upward on $(-\infty, 2)$ and $(2, \infty)$.

24. $f'(x) > 0$ if $|x| < 2 \quad \Rightarrow \quad f$ is increasing on $(-2, 2)$. $f'(x) < 0$ if $|x| > 2 \quad \Rightarrow$ f is decreasing on $(-\infty, -2)$ and $(2, \infty)$. $\quad f'(2) = 0$, so f has a horizontal tangent (and local maximum) at $x = 2$. $\lim\limits_{x \to \infty} f(x) = 1 \quad \Rightarrow \quad y = 1$ is a horizontal asymptote.

$f(-x) = -f(x) \quad \Rightarrow \quad f$ is an odd function (its graph is symmetric about the origin).

Finally, $f''(x) < 0$ if $0 < x < 3$ and $f''(x) > 0$ if $x > 3$, so f is CD on $(0, 3)$ and CU on $(3, \infty)$.



25. (a) Since e^{-x^2} is positive for all x, $f'(x) = xe^{-x^2}$ is positive where $x > 0$ and negative where $x < 0$. Thus, f is increasing on $(0, \infty)$ and decreasing on $(-\infty, 0)$.

(b) Since f changes from decreasing to increasing at $x = 0$, f has a minimum value there.

26. Since $f'(x) = e^{-x^2} > 0$ on $\mathbb{R}$, f is increasing on $\mathbb{R}$.

27. (a) To find the intervals on which f is increasing, we need to find the intervals on which $f'(x) = 3x^2 - 1$ is positive. $3x^2 - 1 > 0 \quad \Leftrightarrow \quad 3x^2 > 1 \quad \Leftrightarrow \quad x^2 > \frac{1}{3} \quad \Leftrightarrow \quad |x| > \sqrt{\frac{1}{3}}$, so $x \in \left(-\infty, -\sqrt{\frac{1}{3}}\right) \cup \left(\sqrt{\frac{1}{3}}, \infty\right)$. Thus, f is increasing on $\left(-\infty, -\sqrt{\frac{1}{3}}\right)$ and on $\left(\sqrt{\frac{1}{3}}, \infty\right)$. In a similar fashion, f is decreasing on $\left(-\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}\right)$.

(b) To find the intervals on which f is concave upward, we need to find the intervals on which $f''(x) = 6x$ is positive. $6x > 0 \quad \Leftrightarrow \quad x > 0$. So f is concave upward on $(0, \infty)$ and f is concave downward on $(-\infty, 0)$.

(c) There is an inflection point at $(0, 0)$ since f changes its direction of concavity at $x = 0$.

28. (a)
$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[(x+h)^4 - 2(x+h)^2] - (x^4 - 2x^2)}{h} \\ &= \lim_{h \to 0} \frac{(x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - 2x^2 - 4xh - 2h^2) - (x^4 - 2x^2)}{h} \\ &= \lim_{h \to 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4 - 4xh - 2h^2}{h} = \lim_{h \to 0} (4x^3 + 6x^2h + 4xh^2 + h^3 - 4x - 2h) = 4x^3 - 4x \end{aligned}$$

$$\begin{aligned} f''(x) &= \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h \to 0} \frac{[4(x+h)^3 - 4(x+h)] - (4x^3 - 4x)}{h} \\ &= \lim_{h \to 0} \frac{(4x^3 + 12x^2h + 12xh^2 + 4h^3 - 4x - 4h) - (4x^3 - 4x)}{h} = \lim_{h \to 0} \frac{12x^2h + 12xh^2 + 4h^3 - 4h}{h} \\ &= \lim_{h \to 0} (12x^2 + 12xh + 4h^2 - 4) = 12x^2 - 4 \end{aligned}$$

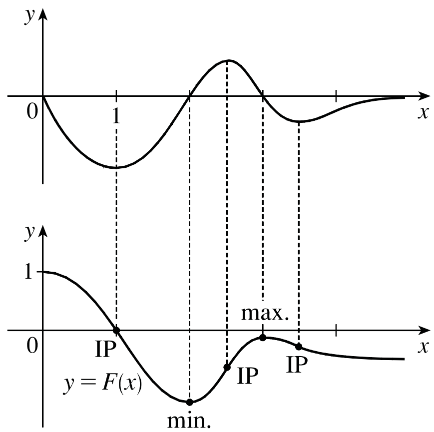
(b) $f'(x) > 0 \quad \Leftrightarrow \quad 4x^3 - 4x > 0 \quad \Leftrightarrow \quad 4x(x^2 - 1) > 0 \quad \Leftrightarrow \quad 4x(x+1)(x-1) > 0$, so f is increasing on $(-1, 0)$ and $(1, \infty)$ and f is decreasing on $(-\infty, -1)$ and $(0, 1)$.

(c) $f''(x) > 0 \quad \Leftrightarrow \quad 12x^2 - 4 > 0 \quad \Leftrightarrow \quad 12x^2 > 4 \quad \Leftrightarrow \quad x^2 > \frac{1}{3} \quad \Leftrightarrow \quad |x| > \sqrt{\frac{1}{3}}$, so f is CU on $\left(-\infty, -\sqrt{\frac{1}{3}}\right)$ and $\left(\sqrt{\frac{1}{3}}, \infty\right)$ and f is CD on $\left(-\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}\right)$.

29. b is the antiderivative of f. For small x, f is negative, so the graph of its antiderivative must be decreasing. But both a and c are increasing for small x, so only b can be f's antiderivative. Also, f is positive where b is increasing, which supports our conclusion.

30. We know right away that c cannot be f's antiderivative, since the slope of c is not zero at the x-value where $f = 0$. Now f is positive when a is increasing and negative when a is decreasing, so a is the antiderivative of f.

31.



The graph of F must start at $(0, 1)$. Where the given graph, $y = f(x)$, has a local minimum or maximum, the graph of F will have an inflection point.

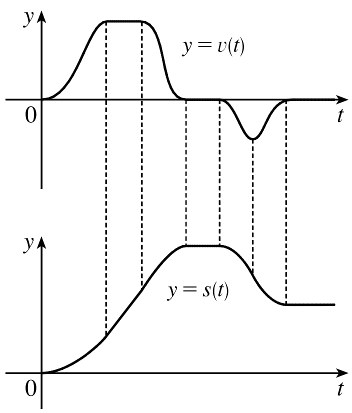
Where f is negative (positive), F is decreasing (increasing).

Where f changes from negative to positive, F will have a minimum.

Where f changes from positive to negative, F will have a maximum.

Where f is decreasing (increasing), F is concave downward (upward).

32.



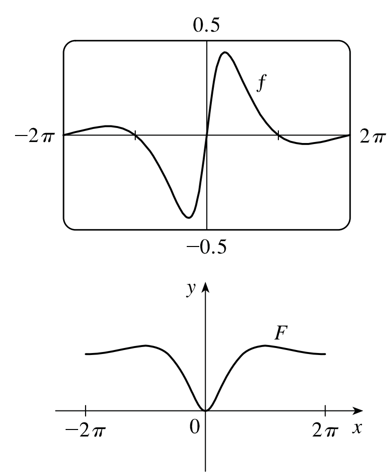
Where v is positive (negative), s is increasing (decreasing).

Where v is increasing (decreasing), s is concave upward (downward).

Where v is horizontal (a steady velocity), s is linear.

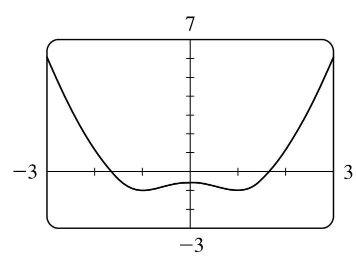
33. $f(x) = \dfrac{\sin x}{1 + x^2}$, $-2\pi \le x \le 2\pi$

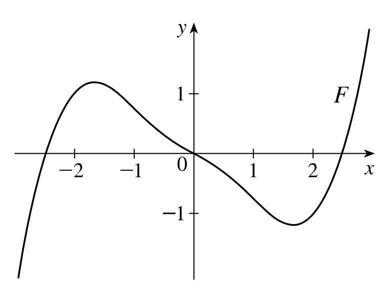
Note that the graph of f is one of an odd function, so the graph of F will be one of an even function.



34. $f(x) = \sqrt{x^4 - 2x^2 + 2} - 2$, $-3 \le x \le 3$

Note that the graph of f is one of an even function, so the graph of F will be one of an odd function.





2 Review

CONCEPT CHECK

1. (a) $\lim\limits_{x \to a} f(x) = L$: See Definition 2.2.1 and Figures 1 and 2 in Section 2.2.

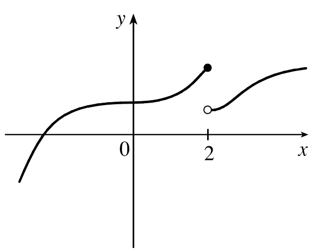
(b) $\lim\limits_{x \to a^+} f(x) = L$: See the paragraph after Definition 2.2.2 and Figure 9(b) in Section 2.2.

(c) $\lim\limits_{x \to a^-} f(x) = L$: See Definition 2.2.2 and Figure 9(a) in Section 2.2.

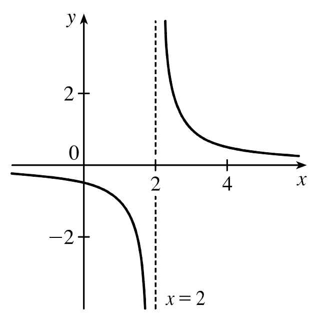
(d) $\lim\limits_{x \to a} f(x) = \infty$: See Definition 2.5.1 and Figure 2 in Section 2.5.

(e) $\lim\limits_{x \to \infty} f(x) = L$: See Definition 2.5.4 and Figure 9 in Section 2.5.

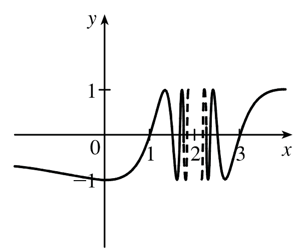
2. In general, the limit of a function fails to exist when the function does not approach a fixed number. For each of the following functions, the limit fails to exist at $x = 2$.



The left- and right-hand limits are not equal.



There is an infinite discontinuity.



There are an infinite number of oscillations.

3. (a) – (g) See the statements of Limit Laws 1 – 6 and 11 in Section 2.3.

4. See Theorem 3 in Section 2.3.

5. (a) See Definition 2.5.2 and Figures 2 – 4 in Section 2.5.

(b) See Definition 2.5.5 and Figures 9 and 10 in Section 2.5.

6. (a) $y = x^4$: No asymptote

(b) $y = \sin x$: No asymptote

(c) $y = \tan x$: Vertical asymptotes $x = \frac{\pi}{2} + \pi n$, n an integer

(d) $y = e^x$: Horizontal asymptote $y = 0$ $\left(\lim\limits_{x \to -\infty} e^x = 0\right)$

(e) $y = \ln x$: Vertical asymptote $x = 0$ $\left(\lim\limits_{x \to 0^+} \ln x = -\infty\right)$

(f) $y = 1/x$: Vertical asymptote $x = 0$, horizontal asymptote $y = 0$

(g) $y = \sqrt{x}$: No asymptote

7. (a) A function f is continuous at a number a if $f(x)$ approaches $f(a)$ as x approaches a; that is, $\lim\limits_{x \to a} f(x) = f(a)$.

(b) A function f is continuous on the interval $(-\infty, \infty)$ if f is continuous at every real number a. The graph of such a function has no breaks and every vertical line crosses it.

8. See Theorem 2.4.10.

9. See Definition 2.6.1.

10. See the paragraph containing Formula 3 in Section 2.6.

11. (a) The average rate of change of y with respect to x over the interval $[x_1, x_2]$ is $\dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$.

(b) The instantaneous rate of change of y with respect to x at $x = x_1$ is $\lim\limits_{x_2 \to x_1} \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$.

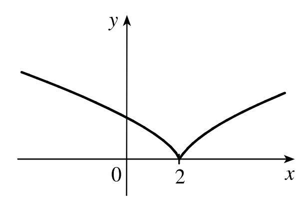
12. See Definition 2.7.2. The pages following the definition discuss interpretations of $f'(a)$ as the slope of a tangent line to the graph of f at $x = a$ and as an instantaneous rate of change of $f(x)$ with respect to x when $x = a$.

13. See the paragraphs before and after Example 7 in Section 2.7.

14. (a) A function f is differentiable at a number a if its derivative f' exists at $x = a$; that is, if $f'(a)$ exists.

(b) See Theorem 2.7.4. This theorem also tells us that if f is *not* continuous at a, then f is *not* differentiable at a.

(c)



15. See the discussion and Figure 8 on page 152.

16. (a) See the first box in Section 2.8.

(b) See the second box in Section 2.8.

17. (a) An antiderivative of a function f is a function F such that $F' = f$.

(b) The antiderivative of a velocity function is a position function (the derivative of a position function is a velocity function). The antiderivative of an acceleration function is a velocity function (the derivative of a velocity function is an acceleration function).

TRUE-FALSE QUIZ

1. False. Limit Law 2 applies only if the individual limits exist (these don't).

2. False. Limit Law 5 cannot be applied if the limit of the denominator is 0 (it is).

3. True. Limit Law 5 applies.

4. True. The limit doesn't exist since $f(x)/g(x)$ doesn't approach any real number as x approaches 5. (The denominator approaches 0 and the numerator doesn't.)

5. False. Consider $\lim\limits_{x \to 5} \dfrac{x(x-5)}{x-5}$ or $\lim\limits_{x \to 5} \dfrac{\sin(x-5)}{x-5}$. The first limit exists and is equal to 5. By Example 3 in Section 2.2, we know that the latter limit exists (and it is equal to 1).

6. False. Consider $\lim\limits_{x \to 6} [f(x)g(x)] = \lim\limits_{x \to 6} \left[(x-6)\dfrac{1}{x-6}\right]$. It exists (its value is 1) but $f(6) = 0$ and $g(6)$ does not exist, so $f(6)g(6) \neq 1$.

7. True. A polynomial is continuous everywhere, so $\lim\limits_{x \to b} p(x)$ exists and is equal to $p(b)$.

8. False. Consider $\lim_{x \to 0} [f(x) - g(x)] = \lim_{x \to 0} \left(\frac{1}{x^2} - \frac{1}{x^4} \right)$. This limit is $-\infty$ (not 0), but each of the individual functions approaches ∞.

9. True. See Figure 11 in Section 2.5.

10. False. Consider $f(x) = \sin x$ for $x \geq 0$. $\lim_{x \to \infty} f(x) \neq \pm\infty$ and f has no horizontal asymptote.

11. False. Consider $f(x) = \begin{cases} 1/(x-1) & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$

12. False. The function f must be *continuous* in order to use the Intermediate Value Theorem. For example, let $f(x) = \begin{cases} 1 & \text{if } 0 \leq x < 3 \\ -1 & \text{if } x = 3 \end{cases}$ There is no number $c \in [0, 3]$ with $f(c) = 0$.

13. True. Use Theorem 2.4.8 with $a = 2$, $b = 5$, and $g(x) = 4x^2 - 11$. Note that $f(4) = 3$ is not needed.

14. True. Use the Intermediate Value Theorem with $a = -1$, $b = 1$, and $N = \pi$, since $3 < \pi < 4$.

15. False. See the note after Theorem 4 in Section 2.7.

16. True. $f'(r)$ exists $\Rightarrow$ f is differentiable at r $\Rightarrow$ f is continuous at r $\Rightarrow$ $\lim_{x \to r} f(x) = f(r)$.

17. False. $\frac{d^2y}{dx^2}$ is the second derivative while $\left(\frac{dy}{dx}\right)^2$ is the first derivative squared. For example, if $y = x$, then $\frac{d^2y}{dx^2} = 0$, but $\left(\frac{dy}{dx}\right)^2 = 1$.

18. False. For example, let $f(x) = \begin{cases} x^2 + 1 & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases}$

Then $f(x) > 1$ for all x, but $\lim_{x \to 0} f(x) = \lim_{x \to 0} (x^2 + 1) = 1$.

EXERCISES

1. (a) (i) $\lim_{x \to 2^+} f(x) = 3$ (ii) $\lim_{x \to -3^+} f(x) = 0$

(iii) $\lim_{x \to -3} f(x)$ does not exist since the left and right limits are not equal. (The left limit is -2.)

(iv) $\lim_{x \to 4} f(x) = 2$

(v) $\lim_{x \to 0} f(x) = \infty$ (vi) $\lim_{x \to 2^-} f(x) = -\infty$

(vii) $\lim_{x \to \infty} f(x) = 4$ (viii) $\lim_{x \to -\infty} f(x) = -1$

(b) The equations of the horizontal asymptotes are $y = -1$ and $y = 4$.

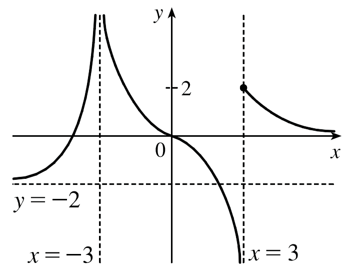
(c) The equations of the vertical asymptotes are $x = 0$ and $x = 2$.

(d) f is discontinuous at $x = -3$, 0, 2, and 4. The discontinuities are jump, infinite, infinite, and removable, respectively.

2. $\lim_{x\to-\infty} f(x) = -2, \quad \lim_{x\to\infty} f(x) = 0, \quad \lim_{x\to-3} f(x) = \infty,$

$\lim_{x\to 3^-} f(x) = -\infty, \quad \lim_{x\to 3^+} f(x) = 2,$

f is continuous from the right at 3



3. Since the exponential function is continuous, $\lim_{x\to 1} e^{x^3-x} = e^{1-1} = e^0 = 1$.

4. Since rational functions are continuous, $\lim_{x\to 3} \dfrac{x^2-9}{x^2+2x-3} = \dfrac{3^2-9}{3^2+2(3)-3} = \dfrac{0}{12} = 0$.

5. $\lim_{x\to-3} \dfrac{x^2-9}{x^2+2x-3} = \lim_{x\to-3} \dfrac{(x+3)(x-3)}{(x+3)(x-1)} = \lim_{x\to-3} \dfrac{x-3}{x-1} = \dfrac{-3-3}{-3-1} = \dfrac{-6}{-4} = \dfrac{3}{2}$

6. $\lim_{x\to 1^+} \dfrac{x^2-9}{x^2+2x-3} = -\infty$ since $x^2+2x-3 \to 0$ as $x \to 1^+$ and $\dfrac{x^2-9}{x^2+2x-3} < 0$ for $1 < x < 3$.

7. $\lim_{h\to 0} \dfrac{(h-1)^3+1}{h} = \lim_{h\to 0} \dfrac{\left(h^3-3h^2+3h-1\right)+1}{h} = \lim_{h\to 0} \dfrac{h^3-3h^2+3h}{h} = \lim_{h\to 0} \left(h^2-3h+3\right) = 3$

Another solution: Factor the numerator as a sum of two cubes and then simplify.

$$\begin{aligned}\lim_{h\to 0} \frac{(h-1)^3+1}{h} &= \lim_{h\to 0} \frac{(h-1)^3+1^3}{h} = \lim_{h\to 0} \frac{[(h-1)+1]\left[(h-1)^2-1(h-1)+1^2\right]}{h} \\ &= \lim_{h\to 0} \left[(h-1)^2-h+2\right] = 1-0+2 = 3\end{aligned}$$

8. $\lim_{t\to 2} \dfrac{t^2-4}{t^3-8} = \lim_{t\to 2} \dfrac{(t+2)(t-2)}{(t-2)(t^2+2t+4)} = \lim_{t\to 2} \dfrac{t+2}{t^2+2t+4} = \dfrac{2+2}{4+4+4} = \dfrac{4}{12} = \dfrac{1}{3}$

9. $\lim_{r\to 9} \dfrac{\sqrt{r}}{(r-9)^4} = \infty$ since $(r-9)^4 \to 0$ as $r \to 9$ and $\dfrac{\sqrt{r}}{(r-9)^4} > 0$ for $r \neq 9$.

10. $\lim_{v\to 4^+} \dfrac{4-v}{|4-v|} = \lim_{v\to 4^+} \dfrac{4-v}{-(4-v)} = \lim_{v\to 4^+} \dfrac{1}{-1} = -1$

11. $\lim_{u\to 1} \dfrac{u^4-1}{u^3+5u^2-6u} = \lim_{u\to 1} \dfrac{(u^2+1)(u^2-1)}{u(u^2+5u-6)} = \lim_{u\to 1} \dfrac{(u^2+1)(u+1)(u-1)}{u(u+6)(u-1)} = \lim_{u\to 1} \dfrac{(u^2+1)(u+1)}{u(u+6)} = \dfrac{2(2)}{1(7)} = \dfrac{4}{7}$

12.
$$\begin{aligned}\lim_{x\to 3} \frac{\sqrt{x+6}-x}{x^3-3x^2} &= \lim_{x\to 3} \left[\frac{\sqrt{x+6}-x}{x^2(x-3)} \cdot \frac{\sqrt{x+6}+x}{\sqrt{x+6}+x}\right] = \lim_{x\to 3} \frac{(\sqrt{x+6})^2-x^2}{x^2(x-3)\left(\sqrt{x+6}+x\right)} \\ &= \lim_{x\to 3} \frac{x+6-x^2}{x^2(x-3)\left(\sqrt{x+6}+x\right)} = \lim_{x\to 3} \frac{-(x^2-x-6)}{x^2(x-3)\left(\sqrt{x+6}+x\right)} = \lim_{x\to 3} \frac{-(x-3)(x+2)}{x^2(x-3)\left(\sqrt{x+6}+x\right)} \\ &= \lim_{x\to 3} \frac{-(x+2)}{x^2\left(\sqrt{x+6}+x\right)} = -\frac{5}{9(3+3)} = -\frac{5}{54}\end{aligned}$$

13. Let $t = \sin x$. Then as $x \to \pi^-$, $\sin x \to 0^+$, so $t \to 0^+$. Thus, $\lim_{x\to\pi^-} \ln(\sin x) = \lim_{t\to 0^+} \ln t = -\infty$.

14. $\lim_{x\to-\infty} \dfrac{1-2x^2-x^4}{5+x-3x^4} = \lim_{x\to-\infty} \dfrac{(1-2x^2-x^4)/x^4}{(5+x-3x^4)/x^4} = \lim_{x\to-\infty} \dfrac{1/x^4-2/x^2-1}{5/x^4+1/x^3-3} = \dfrac{0-0-1}{0+0-3} = \dfrac{-1}{-3} = \dfrac{1}{3}$

15. Since x is positive, $\sqrt{x^2} = |x| = x$. Thus,

$$\lim_{x\to\infty} \frac{\sqrt{x^2-9}}{2x-6} = \lim_{x\to\infty} \frac{\sqrt{x^2-9}/\sqrt{x^2}}{(2x-6)/x} = \lim_{x\to\infty} \frac{\sqrt{1-9/x^2}}{2-6/x} = \frac{\sqrt{1-0}}{2-0} = \frac{1}{2}$$

16. Let $t = x - x^2 = x(1-x)$. Then as $x \to \infty$, $t \to -\infty$, and $\lim\limits_{x\to\infty} e^{x-x^2} = \lim\limits_{t\to-\infty} e^t = 0$.

17. $$\lim_{x\to\infty} \left(\sqrt{x^2+4x+1} - x\right) = \lim_{x\to\infty} \left[\frac{\sqrt{x^2+4x+1}-x}{1} \cdot \frac{\sqrt{x^2+4x+1}+x}{\sqrt{x^2+4x+1}+x}\right] = \lim_{x\to\infty} \frac{(x^2+4x+1)-x^2}{\sqrt{x^2+4x+1}+x}$$

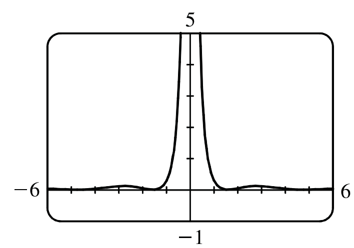
$$= \lim_{x\to\infty} \frac{(4x+1)/x}{\left(\sqrt{x^2+4x+1}+x\right)/x} \qquad \left[\text{divide by } x = \sqrt{x^2} \text{ for } x > 0\right]$$

$$= \lim_{x\to\infty} \frac{4+1/x}{\sqrt{1+4/x+1/x^2}+1} = \frac{4+0}{\sqrt{1+0+0}+1} = \frac{4}{2} = 2$$

18. $$\lim_{x\to1} \left(\frac{1}{x-1} + \frac{1}{x^2-3x+2}\right) = \lim_{x\to1} \left[\frac{1}{x-1} + \frac{1}{(x-1)(x-2)}\right] = \lim_{x\to1} \left[\frac{x-2}{(x-1)(x-2)} + \frac{1}{(x-1)(x-2)}\right]$$

$$= \lim_{x\to1} \left[\frac{x-1}{(x-1)(x-2)}\right] = \lim_{x\to1} \frac{1}{x-2} = \frac{1}{1-2} = -1$$

19. From the graph of $y = \left(\cos^2 x\right)/x^2$, it appears that $y = 0$ is the horizontal asymptote and $x = 0$ is the vertical asymptote. Now $0 \le (\cos x)^2 \le 1 \;\Rightarrow\; \dfrac{0}{x^2} \le \dfrac{\cos^2 x}{x^2} \le \dfrac{1}{x^2} \;\Rightarrow\; 0 \le \dfrac{\cos^2 x}{x^2} \le \dfrac{1}{x^2}$. But $\lim\limits_{x\to\pm\infty} 0 = 0$ and $\lim\limits_{x\to\pm\infty} \dfrac{1}{x^2} = 0$, so by the Squeeze Theorem, $\lim\limits_{x\to\pm\infty} \dfrac{\cos^2 x}{x^2} = 0$.



Thus, $y = 0$ is the horizontal asymptote. $\lim\limits_{x\to0} \dfrac{\cos^2 x}{x^2} = \infty$ because $\cos^2 x \to 1$ and $x^2 \to 0$ as $x \to 0$, so $x = 0$ is the vertical asymptote.

20. From the graph of $y = f(x) = \sqrt{x^2+x+1} - \sqrt{x^2-x}$, it appears that there are 2 horizontal asymptotes and possibly 2 vertical asymptotes. To obtain a different form for f, let's multiply and divide it by its conjugate.

$$f_1(x) = \left(\sqrt{x^2+x+1} - \sqrt{x^2-x}\right) \frac{\sqrt{x^2+x+1}+\sqrt{x^2-x}}{\sqrt{x^2+x+1}+\sqrt{x^2-x}} = \frac{(x^2+x+1)-(x^2-x)}{\sqrt{x^2+x+1}+\sqrt{x^2-x}}$$

$$= \frac{2x+1}{\sqrt{x^2+x+1}+\sqrt{x^2-x}}$$

Now

$$\lim_{x\to\infty} f_1(x) = \lim_{x\to\infty} \frac{2x+1}{\sqrt{x^2+x+1}+\sqrt{x^2-x}}$$

$$= \lim_{x\to\infty} \frac{2+(1/x)}{\sqrt{1+(1/x)+(1/x^2)}+\sqrt{1-(1/x)}} \qquad \left[\text{since } \sqrt{x^2} = x \text{ for } x > 0\right]$$

$$= \frac{2}{1+1} = 1,$$

so $y = 1$ is a horizontal asymptote. For $x < 0$, we have $\sqrt{x^2} = |x| = -x$, so when we divide the denominator by x, with $x < 0$, we get

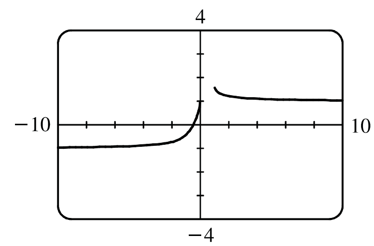
$$\frac{\sqrt{x^2+x+1}+\sqrt{x^2-x}}{x} = -\frac{\sqrt{x^2+x+1}+\sqrt{x^2-x}}{\sqrt{x^2}} = -\left[\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+\sqrt{1-\frac{1}{x}}\right]$$

Therefore,

$$\lim_{x\to-\infty} f_1(x) = \lim_{x\to-\infty}\frac{2x+1}{\sqrt{x^2+x+1}+\sqrt{x^2-x}} = \lim_{x\to\infty}\frac{2+(1/x)}{-\left[\sqrt{1+(1/x)+(1/x^2)}+\sqrt{1-(1/x)}\right]}$$
$$= \frac{2}{-(1+1)} = -1,$$

so $y=-1$ is a horizontal asymptote.

The domain of f is $(-\infty,0]\cup[1,\infty)$. As $x\to 0^-$, $f(x)\to 1$, so $x=0$ is *not* a vertical asymptote. As $x\to 1^+$, $f(x)\to\sqrt{3}$, so $x=1$ is *not* a vertical asymptote and hence there are no vertical asymptotes.



21. Since $2x-1\le f(x)\le x^2$ for $0<x<3$ and $\lim_{x\to1}(2x-1)=1=\lim_{x\to1}x^2$, we have $\lim_{x\to1}f(x)=1$ by the Squeeze Theorem.

22. Let $f(x)=-x^2$, $g(x)=x^2\cos(1/x^2)$ and $h(x)=x^2$. Then since $\left|\cos(1/x^2)\right|\le 1$ for $x\ne 0$, we have $f(x)\le g(x)\le h(x)$ for $x\ne0$, and so $\lim_{x\to0}f(x)=\lim_{x\to0}h(x)=0 \quad\Rightarrow\quad \lim_{x\to0}g(x)=0$ by the Squeeze Theorem.

23. (a) $f(x)=\sqrt{-x}$ if $x<0$, $f(x)=3-x$ if $0\le x<3$, $f(x)=(x-3)^2$ if $x>3$.

(i) $\lim_{x\to0^+}f(x)=\lim_{x\to0^+}(3-x)=3$

(ii) $\lim_{x\to0^-}f(x)=\lim_{x\to0^-}\sqrt{-x}=0$

(iii) Because of (i) and (ii), $\lim_{x\to0}f(x)$ does not exist.

(iv) $\lim_{x\to3^-}f(x)=\lim_{x\to3^-}(3-x)=0$

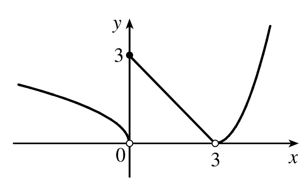
(v) $\lim_{x\to3^+}f(x)=\lim_{x\to3^+}(x-3)^2=0$

(vi) Because of (iv) and (v), $\lim_{x\to3}f(x)=0$.

(b) f is discontinuous at 0 since $\lim_{x\to0}f(x)$ does not exist.

f is discontinuous at 3 since $f(3)$ does not exist.

(c)



24. (a) x^2-9 is continuous on $\mathbb{R}$ since it is a polynomial and $\sqrt{x}$ is continuous on $[0,\infty)$, so the composition $\sqrt{x^2-9}$ is continuous on $\{x \mid x^2-9\ge0\}=(-\infty,-3]\cup[3,\infty)$. Note that $x^2-2\ne0$ on this set and so the quotient function $g(x)=\dfrac{\sqrt{x^2-9}}{x^2-2}$ is continuous on its domain, $(-\infty,-3]\cup[3,\infty)$.

(b) $\sin x$ is continuous on $\mathbb{R}$ by Theorem 7 in Section 2.5. Since e^x is continuous on $\mathbb{R}$, $e^{\sin x}$ is continuous on $\mathbb{R}$ by Theorem 9 in Section 2.5. Lastly, x is continuous on $\mathbb{R}$ since it's a polynomial and the product $xe^{\sin x}$ is continuous on its domain $\mathbb{R}$ by Theorem 4 in Section 2.5.

25. $f(x) = 2x^3 + x^2 + 2$ is a polynomial, so it is continuous on $[-2, -1]$ and $f(-2) = -10 < 0 < 1 = f(-1)$. So by the Intermediate Value Theorem there is a number c in $(-2, -1)$ such that $f(c) = 0$, that is, the equation $2x^3 + x^2 + 2 = 0$ has a root in $(-2, -1)$.

26. $f(x) = e^{-x^2} - x$ is continuous on $\mathbb{R}$ so it is continuous on $[0, 1]$. $f(0) = 1 > 0 > 1/e - 1 = f(1)$. So by the Intermediate Value Theorem, there is a number c in $(0, 1)$ such that $f(c) = 0$. Thus, $e^{-x^2} - x = 0$, or $e^{-x^2} = x$, has a root in $(0, 1)$.

27. (a) $s = s(t) = 1 + 2t + t^2/4$. The average velocity over the time interval $[1, 1+h]$ is

$$v_{\text{ave}} = \frac{s(1+h) - s(1)}{(1+h) - 1} = \frac{1 + 2(1+h) + (1+h)^2/4 - 13/4}{h} = \frac{10h + h^2}{4h} = \frac{10+h}{4}$$

So for the following intervals the average velocities are:

(i) $[1, 3]$: $h = 2$, $v_{\text{ave}} = (10 + 2)/4 = 3$ m/s
(ii) $[1, 2]$: $h = 1$, $v_{\text{ave}} = (10 + 1)/4 = 2.75$ m/s
(iii) $[1, 1.5]$: $h = 0.5$, $v_{\text{ave}} = (10 + 0.5)/4 = 2.625$ m/s
(iv) $[1, 1.1]$: $h = 0.1$, $v_{\text{ave}} = (10 + 0.1)/4 = 2.525$ m/s

(b) When $t = 1$, the instantaneous velocity is $\displaystyle\lim_{h\to 0} \frac{s(1+h) - s(1)}{h} = \lim_{h\to 0} \frac{10+h}{4} = \frac{10}{4} = 2.5$ m/s.

28. (a) When V increases from 200 in^3 to 250 in^3, we have $\Delta V = 250 - 200 = 50$ in^3, and since $P = 800/V$,

$\Delta P = P(250) - P(200) = \dfrac{800}{250} - \dfrac{800}{200} = 3.2 - 4 = -0.8$ lb/in^2. So the average rate of change

is $\dfrac{\Delta P}{\Delta V} = \dfrac{-0.8}{50} = -0.016 \dfrac{\text{lb/in}^2}{\text{in}^3}$.

(b) Since $V = 800/P$, the instantaneous rate of change of V with respect to P is

$$\lim_{h\to 0} \frac{\Delta V}{\Delta P} = \lim_{h\to 0} \frac{V(P+h) - V(P)}{h} = \lim_{h\to 0} \frac{800/(P+h) - 800/P}{h} = \lim_{h\to 0} \frac{800\,[P - (P+h)]}{h(P+h)P}$$
$$= \lim_{h\to 0} \frac{-800}{(P+h)P} = -\frac{800}{P^2}$$

which is inversely proportional to the square of P.

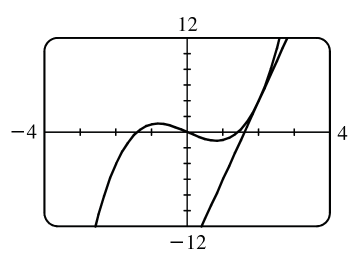
29. Estimating the slopes of the tangent lines at $x = 2$, 3, and 5, we obtain approximate values 0.4, 2, and 0.1. Since the graph is concave downward at $x = 5$, $f''(5)$ is negative. Arranging the numbers in increasing order, we have: $f''(5) < 0 < f'(5) < f'(2) < 1 < f'(3)$.

30. (a) $\displaystyle f'(2) = \lim_{x\to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x\to 2} \frac{x^3 - 2x - 4}{x - 2}$

$$= \lim_{x\to 2} \frac{(x-2)(x^2 + 2x + 2)}{x - 2} = \lim_{x\to 2} (x^2 + 2x + 2) = 10$$

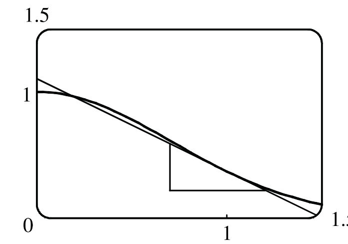
(b) $y - 4 = 10(x - 2)$ or $y = 10x - 16$

(c)



31. (a) Estimating $f'(1)$ from the triangle in the graph, we get $\dfrac{\Delta y}{\Delta x} \approx \dfrac{-0.37}{0.50} = -0.74$.

To estimate $f'(1)$ numerically, we have



$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{e^{-(1+h)^2} - e^{-1}}{h} = y$$

From the table, we have $f'(1) \approx -0.736$.

h	y
0.01	−0.732
0.001	−0.735
0.0001	−0.736
−0.01	−0.739
−0.001	−0.736
−0.0001	−0.736

(b) $y - e^{-1} \approx -0.736(x - 1)$ or $y \approx -0.736x + 1.104$

(c) See the graph in part (a).

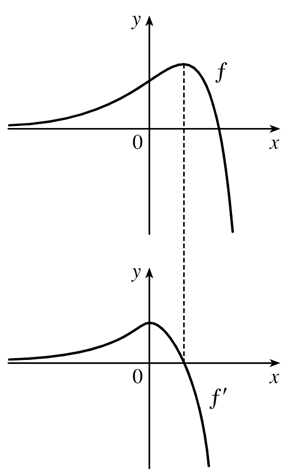
32. $2^6 = 64$, so $f(x) = x^6$ and $a = 2$.

33. (a) $f'(r)$ is the rate at which the total cost changes with respect to the interest rate. Its units are dollars/(percent per year).

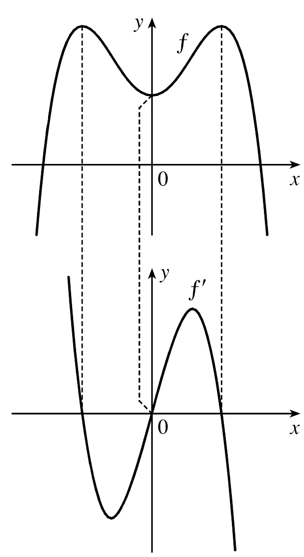
(b) The total cost of paying off the loan is increasing by \$1200/(percent per year) as the interest rate reaches 10%. So if the interest rate goes up from 10% to 11%, the cost goes up approximately \$1200.

(c) As r increases, C increases. So $f'(r)$ will always be positive.

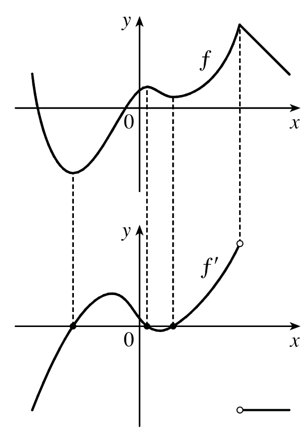
34.



35.



36.



37. (a) $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0} \dfrac{\sqrt{3 - 5(x+h)} - \sqrt{3 - 5x}}{h} \dfrac{\sqrt{3 - 5(x+h)} + \sqrt{3 - 5x}}{\sqrt{3 - 5(x+h)} + \sqrt{3 - 5x}}$

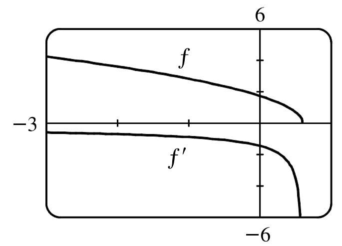
$$= \lim_{h \to 0} \frac{[3 - 5(x+h)] - (3 - 5x)}{h\left(\sqrt{3 - 5(x+h)} + \sqrt{3 - 5x}\right)} = \lim_{h \to 0} \frac{-5}{\sqrt{3 - 5(x+h)} + \sqrt{3 - 5x}} = \frac{-5}{2\sqrt{3 - 5x}}$$

(b) Domain of f: (the radicand must be nonnegative) $3 - 5x \geq 0 \Rightarrow$

$5x \leq 3 \Rightarrow x \in \left(-\infty, \frac{3}{5}\right]$

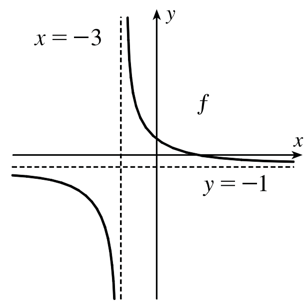
Domain of f': exclude $\frac{3}{5}$ because it makes the denominator zero;

$x \in \left(-\infty, \frac{3}{5}\right)$

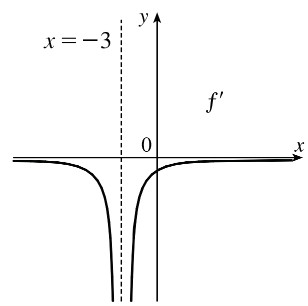


(c) Our answer to part (a) is reasonable because $f'(x)$ is always negative and f is always decreasing.

38. (a) As $x \to \pm\infty$, $f(x) = (4 - x)/(3 + x) \to -1$, so there is a horizontal asymptote at $y = -1$. As $x \to -3^+$, $f(x) \to \infty$, and as $x \to -3^-$, $f(x) \to -\infty$. Thus, there is a vertical asymptote at $x = -3$.



(b) Note that f is decreasing on $(-\infty, -3)$ and $(-3, \infty)$, so f' is negative on those intervals. As $x \to \pm\infty$, $f' \to 0$. As $x \to -3^-$ and as $x \to -3^+$, $f' \to -\infty$.



(c) $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\dfrac{4 - (x+h)}{3 + (x+h)} - \dfrac{4 - x}{3 + x}}{h} = \lim_{h \to 0} \frac{(3 + x)\,[4 - (x + h)] - (4 - x)\,[3 + (x + h)]}{h\,[3 + (x + h)]\,(3 + x)}$$

$$= \lim_{h \to 0} \frac{(12 - 3x - 3h + 4x - x^2 - hx) - (12 + 4x + 4h - 3x - x^2 - hx)}{h[3 + (x + h)](3 + x)}$$

$$= \lim_{h \to 0} \frac{-7h}{h\,[3 + (x + h)]\,(3 + x)} = \lim_{h \to 0} \frac{-7}{[3 + (x + h)]\,(3 + x)} = -\frac{7}{(3 + x)^2}$$

(d) The graphing device confirms our graph in part (b).

39. f is not differentiable: at $x = -4$ because f is not continuous, at $x = -1$ because f has a corner, at $x = 2$ because f is not continuous, and at $x = 5$ because f has a vertical tangent.

40. The graph of a has tangent lines with positive slope for $x < 0$ and negative slope for $x > 0$, and the values of c fit this pattern, so c must be the graph of the derivative of the function for a. The graph of c has horizontal tangent lines to the left and right of the x-axis and b has zeros at these points. Hence, b is the graph of the derivative of the function for c. Therefore, a is the graph of f, c is the graph of f', and b is the graph of f''.

41. $C'(1990)$ is the rate at which the total value of US currency in circulation is changing in billions of dollars per year. To estimate the value of $C'(1990)$, we will average the difference quotients obtained using the times $t = 1985$ and $t = 1995$.

Let $A = \dfrac{C(1985) - C(1990)}{1985 - 1990} = \dfrac{187.3 - 271.9}{-5} = \dfrac{-84.6}{-5} = 16.92$ and

$B = \dfrac{C(1995) - C(1990)}{1995 - 1990} = \dfrac{409.3 - 271.9}{5} = \dfrac{137.4}{5} = 27.48$. Then

$C'(1990) = \displaystyle\lim_{t \to 1990} \frac{C(t) - C(1990)}{t - 1990} \approx \frac{A + B}{2} = \frac{16.92 + 27.48}{2} = \frac{44.4}{2} = 22.2$ billion dollars/year.

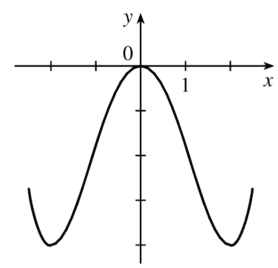
42. Let $C(t)$ be the function that denotes the cost of living in terms of time t. $C(t)$ is an increasing function, so $C'(t) > 0$. Since the cost of living is rising at a slower rate, the slopes of the tangent lines are positive but decreasing as t increases. Hence, $C''(t) < 0$.

43. (a) $f'(x) > 0$ on $(-2, 0)$ and $(2, \infty)$ $\Rightarrow$ f is increasing on those intervals. $f'(x) < 0$ on $(-\infty, -2)$ and $(0, 2)$ $\Rightarrow$ f is decreasing on those intervals.

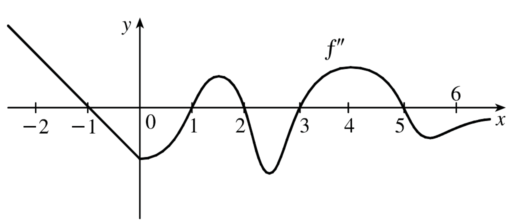
(b) $f'(x) = 0$ at $x = -2$, 0, and 2, so these are where local maxima or minima will occur. At $x = \pm 2$, f' changes from negative to positive, so f has local minima at those values. At $x = 0$, f' changes from positive to negative, so f has a local maximum there.

(c) f' is increasing on $(-\infty, -1)$ and $(1, \infty)$ $\Rightarrow$ $f'' > 0$ and f is concave upward on those intervals. f' is decreasing on $(-1, 1)$ $\Rightarrow$ $f'' < 0$ and f is concave downward on this interval.

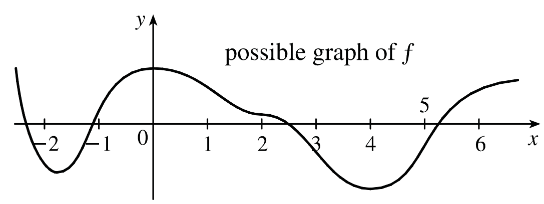
(d)



44. (a)



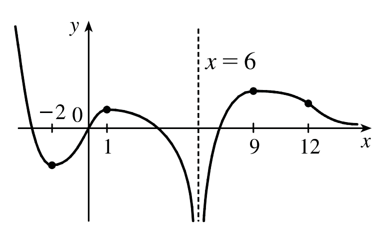
(b)



45. $f(0) = 0$, $f'(-2) = f'(1) = f'(9) = 0$, $\displaystyle\lim_{x \to \infty} f(x) = 0$, $\displaystyle\lim_{x \to 6} f(x) = -\infty$,

$f'(x) < 0$ on $(-\infty, -2)$, $(1, 6)$, and $(9, \infty)$, $f'(x) > 0$ on $(-2, 1)$ and $(6, 9)$,

$f''(x) > 0$ on $(-\infty, 0)$ and $(12, \infty)$, $f''(x) < 0$ on $(0, 6)$ and $(6, 12)$



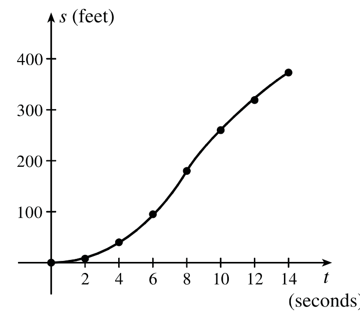
46. (a) Drawing slope triangles, we obtain the following estimates: $F'(1950) \approx \frac{1.1}{10} = 0.11$, $F'(1965) \approx \frac{-1.6}{10} = -0.16$, and $F'(1987) \approx \frac{0.2}{10} = 0.02$.

(b) The rate of change of the average number of children born to each woman was increasing by 0.11 in 1950, decreasing by 0.16 in 1965, and increasing by 0.02 in 1987.

(c) There are many possible reasons:

- In the baby-boom era (post-WWII), there was optimism about the economy and family size was rising.
- In the baby-bust era, there was less economic optimism, and it was considered less socially responsible to have a large family.
- In the baby-boomlet era, there was increased economic optimism and a return to more conservative attitudes.

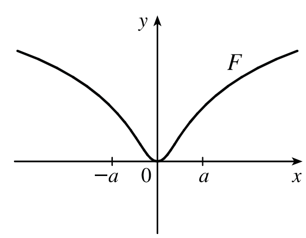
47. (a) Using the data closest to $t = 6$, we have $\dfrac{s(8) - s(6)}{8 - 6} = \dfrac{180 - 95}{2} = 42.5$ and $\dfrac{s(4) - s(6)}{4 - 6} = \dfrac{40 - 95}{-2} = 27.5$. Averaging these two values gives us $\dfrac{42.5 + 27.5}{2} = 35$ ft/s as an estimate for the speed of the car after 6 seconds.



(b) From the graph, it appears that the inflection point is at $(8, 180)$.

(c) The velocity of the car is at a maximum at the inflection point.

48. Let f be the function shown. Since f is negative for $x < 0$ and positive for $x > 0$, F is decreasing for $x < 0$ and increasing for $x > 0$. f is increasing on $(-a, a)$ (from the low point to the high point) so its derivative f' (the second derivative of F) is positive, making F concave upward on $(-a, a)$. f is decreasing elsewhere, so its derivative f' is negative and F is concave downward on $(-\infty, -a)$ and (a, ∞).



FOCUS ON PROBLEM SOLVING

1. Let $t = \sqrt[6]{x}$, so $x = t^6$. Then $t \to 1$ as $x \to 1$, so

$$\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{\sqrt{x}-1} = \lim_{t\to 1}\frac{t^2-1}{t^3-1} = \lim_{t\to 1}\frac{(t-1)(t+1)}{(t-1)\left(t^2+t+1\right)} = \lim_{t\to 1}\frac{t+1}{t^2+t+1} = \frac{1+1}{1^2+1+1} = \frac{2}{3}.$$

Another method: Multiply both the numerator and the denominator by $(\sqrt{x}+1)\left(\sqrt[3]{x^2}+\sqrt[3]{x}+1\right)$.

2. First rationalize the numerator: $\lim\limits_{x\to 0}\dfrac{\sqrt{ax+b}-2}{x}\cdot\dfrac{\sqrt{ax+b}+2}{\sqrt{ax+b}+2} = \lim\limits_{x\to 0}\dfrac{ax+b-4}{x\left(\sqrt{ax+b}+2\right)}$. Now since the denominator approaches 0 as $x \to 0$, the limit will exist only if the numerator also approaches 0 as $x \to 0$. So we require that $a(0)+b-4=0 \;\Rightarrow\; b=4$. So the equation becomes $\lim\limits_{x\to 0}\dfrac{a}{\sqrt{ax+4}+2} = 1 \;\Rightarrow\; \dfrac{a}{\sqrt{4}+2} = 1 \;\Rightarrow\; a = 4$.
Therefore, $a = b = 4$.

3. For $-\frac{1}{2} < x < \frac{1}{2}$, we have $2x-1<0$ and $2x+1>0$, so $|2x-1| = -(2x-1)$ and $|2x+1| = 2x+1$.
Therefore, $\lim\limits_{x\to 0}\dfrac{|2x-1|-|2x+1|}{x} = \lim\limits_{x\to 0}\dfrac{-(2x-1)-(2x+1)}{x} = \lim\limits_{x\to 0}\dfrac{-4x}{x} = \lim\limits_{x\to 0}(-4) = -4$.

4. Let R be the midpoint of OP, so the coordinates of R are $\left(\frac{1}{2}x, \frac{1}{2}x^2\right)$ since the coordinates of P are $\left(x, x^2\right)$. Let $Q = (0, a)$.
Since the slope $m_{OP} = \dfrac{x^2}{x} = x$, $m_{QR} = -\dfrac{1}{x}$ (negative reciprocal). But $m_{QR} = \dfrac{\frac{1}{2}x^2 - a}{\frac{1}{2}x - 0} = \dfrac{x^2-2a}{x}$, so we conclude that $-1 = x^2 - 2a \;\Rightarrow\; 2a = x^2+1 \;\Rightarrow\; a = \frac{1}{2}x^2 + \frac{1}{2}$. As $x \to 0$, $a \to \frac{1}{2}$, and the limiting position of Q is $\left(0, \frac{1}{2}\right)$.

5. Since $[\![x]\!] \le x < [\![x]\!]+1$, we have $\dfrac{[\![x]\!]}{[\![x]\!]} \le \dfrac{x}{[\![x]\!]} < \dfrac{[\![x]\!]+1}{[\![x]\!]} \;\Rightarrow\; 1 \le \dfrac{x}{[\![x]\!]} < 1 + \dfrac{1}{[\![x]\!]}$ for $x \ge 1$. As $x \to \infty$, $[\![x]\!] \to \infty$, so $\dfrac{1}{[\![x]\!]} \to 0$ and $1 + \dfrac{1}{[\![x]\!]} \to 1$. Thus, $\lim\limits_{x\to\infty}\dfrac{x}{[\![x]\!]} = 1$ by the Squeeze Theorem.

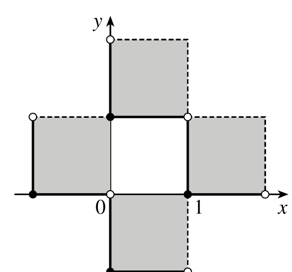
6. (a) $[\![x]\!]^2 + [\![y]\!]^2 = 1$. Since $[\![x]\!]^2$ and $[\![y]\!]^2$ are positive integers or 0, there are only 4 cases:

Case (i): $[\![x]\!] = 1, [\![y]\!] = 0 \;\Rightarrow 1 \le x < 2$ and $0 \le y < 1$

Case (ii): $[\![x]\!] = -1, [\![y]\!] = 0 \Rightarrow -1 \le x < 0$ and $0 \le y < 1$

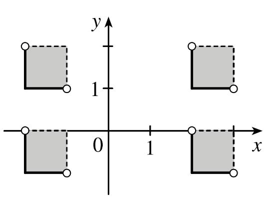
Case (iii): $[\![x]\!] = 0, [\![y]\!] = 1 \;\Rightarrow 0 \le x < 1$ and $1 \le y < 2$

Case (iv): $[\![x]\!] = 0, [\![y]\!] = -1 \Rightarrow 0 \le x < 1$ and $-1 \le y < 0$

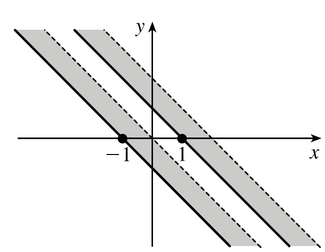


(b) $[\![x]\!]^2 - [\![y]\!]^2 = 3$. The only integral solution of $n^2 - m^2 = 3$ is $n = \pm 2$ and $m = \pm 1$. So the graph is

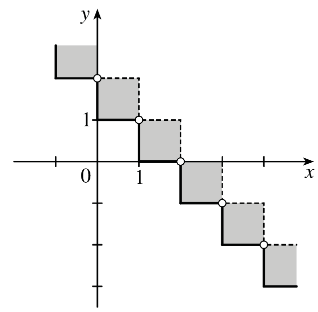
$$\{(x,y) \mid [\![x]\!] = \pm 2,\ [\![y]\!] = \pm 1\} = \left\{(x,y)\ \middle|\ \begin{array}{l} 2 \le x \le 3 \text{ or } -2 \le x < 1, \\ 1 \le y < 2 \text{ or } -1 \le y < 0 \end{array}\right\}.$$



(c) $[\![x+y]\!]^2 = 1 \;\Rightarrow\; [\![x+y]\!] = \pm 1 \;\Rightarrow\; 1 \le x+y < 2$ or $-1 \le x+y < 0$

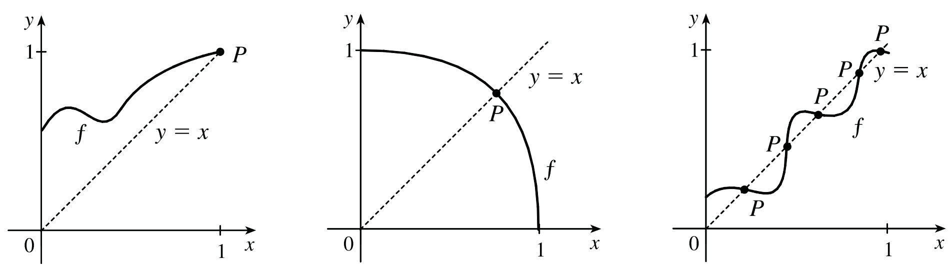


(d) For $n \le x < n+1$, $[\![x]\!] = n$. Then $[\![x]\!] + [\![y]\!] = 1 \Rightarrow [\![y]\!] = 1 - n \Rightarrow$ $1 - n \le y < 2 - n$. Choosing integer values for n produces the graph.



7. f is continuous on $(-\infty, a)$ and (a, ∞). To make f continuous on $\mathbb{R}$, we must have continuity at a. Thus, $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) \Rightarrow \lim_{x\to a^+} x^2 = \lim_{x\to a^-} (x+1) \Rightarrow a^2 = a + 1 \Rightarrow a^2 - a - 1 = 0 \Rightarrow$ [by the quadratic formula] $a = \left(1 \pm \sqrt{5}\right)/2 \approx 1.618$ or -0.618.

8. (a) Here are a few possibilities:



(b) The "obstacle" is the line $x = y$ (see diagram). Any intersection of the graph of f with the line $y = x$ constitutes a fixed point, and if the graph of the function does not cross the line somewhere in $(0, 1)$, then it must either start at $(0, 0)$ (in which case 0 is a fixed point) or finish at $(1, 1)$ (in which case 1 is a fixed point).

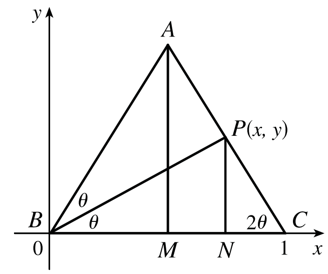
(c) Consider the function $F(x) = f(x) - x$, where f is any continuous function with domain $[0, 1]$ and range in $[0, 1]$. We shall prove that f has a fixed point. Now if $f(0) = 0$ then we are done: f has a fixed point (the number 0), which is what we are trying to prove. So assume $f(0) \ne 0$. For the same reason we can assume that $f(1) \ne 1$. Then $F(0) = f(0) > 0$ and $F(1) = f(1) - 1 < 0$. So by the Intermediate Value Theorem, there exists some number c in the interval $(0, 1)$ such that $F(c) = f(c) - c = 0$. So $f(c) = c$, and therefore f has a fixed point.

9. (a) Consider $G(x) = T(x + 180^\circ) - T(x)$. Fix any number a. If $G(a) = 0$, we are done: Temperature at $a =$ Temperature at $a + 180^\circ$. If $G(a) > 0$, then $G(a + 180^\circ) = T(a + 360^\circ) - T(a + 180^\circ) = T(a) - T(a + 180^\circ) = -G(a) < 0$. Also, G is continuous since temperature varies continuously. So, by the Intermediate Value Theorem, G has a zero on the interval $[a, a + 180^\circ]$. If $G(a) < 0$, then a similar argument applies.

(b) Yes. The same argument applies.

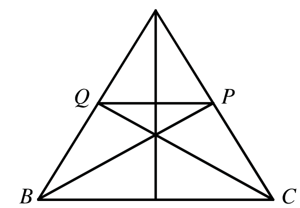
(c) The same argument applies for quantities that vary continuously, such as barometric pressure. But one could argue that altitude above sea level is sometimes discontinuous, so the result might not always hold for that quantity.

10. (a) *Solution 1:* We introduce a coordinate system and drop a perpendicular from P, as shown. We see from $\angle NCP$ that $\tan 2\theta = \dfrac{y}{1-x}$, and from $\angle NBP$ that $\tan\theta = y/x$. Using the double-angle formula for tangents, we get $\dfrac{y}{1-x} = \tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta} = \dfrac{2(y/x)}{1-(y/x)^2}$. After a bit of simplification, this becomes $\dfrac{1}{1-x} = \dfrac{2x}{x^2-y^2} \quad\Leftrightarrow\quad y^2 = x(3x-2)$.

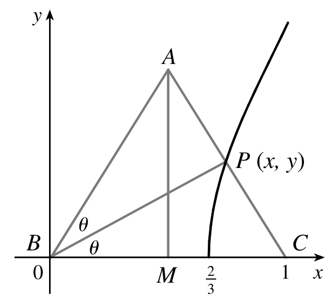


As the altitude AM decreases in length, the point P will approach the x-axis, that is, $y \to 0$, so the limiting location of P must be one of the roots of the equation $x(3x-2) = 0$. Obviously it is not $x = 0$ (the point P can never be to the left of the altitude AM, which it would have to be in order to approach 0) so it must be $3x - 2 = 0$, that is, $x = \frac{2}{3}$.

Solution 2: We add a few lines to the original diagram, as shown. Now note that $\angle BPQ = \angle PBC$ (alternate angles; $QP \parallel BC$ by symmetry) and similarly $\angle CQP = \angle QCB$. So ΔBPQ and ΔCQP are isosceles, and the line segments BQ, QP and PC are all of equal length. As $|AM| \to 0$, P and Q approach points on the base, and the point P is seen to approach a position two-thirds of the way between B and C, as above.



(b) The equation $y^2 = x(3x-2)$ calculated in part (a) is the equation of the curve traced out by P. Now as $|AM| \to \infty$, $2\theta \to \frac{\pi}{2}$, $\theta \to \frac{\pi}{4}$, $x \to 1$, and since $\tan\theta = y/x$, $y \to 1$. Thus, P only traces out the part of the curve with $0 \le y < 1$.



11. Let a be the x-coordinate of Q. Since the derivative of $y = 1 - x^2$ is $y' = -2x$, the slope at Q is $-2a$. But since the triangle is equilateral, $\overline{AO}/\overline{OC} = \sqrt{3}/1$, so the slope at Q is $-\sqrt{3}$. Therefore, we must have that $-2a = -\sqrt{3} \quad\Rightarrow\quad a = \frac{\sqrt{3}}{2}$. Thus, the point Q has coordinates $\left(\frac{\sqrt{3}}{2}, 1 - \left(\frac{\sqrt{3}}{2}\right)^2\right) = \left(\frac{\sqrt{3}}{2}, \frac{1}{4}\right)$ and by symmetry, P has coordinates $\left(-\frac{\sqrt{3}}{2}, \frac{1}{4}\right)$.

12. (a) $V'(t)$ is the rate of change of the volume of the water with respect to time. $H'(t)$ is the rate of change of the height of the water with respect to time. Since the volume and the height are increasing, $V'(t)$ and $H'(t)$ are positive.

(b) $V'(t)$ is constant, so $V''(t)$ is zero (the slope of a constant function is 0).

(c) At first, the height H of the water increases quickly because the tank is narrow. But as the sphere widens, the rate of increase of the height slows down, reaching a minimum at $t = t_2$. Thus, the height is increasing at a decreasing rate on $(0, t_2)$, so its graph is concave downward and $H''(t_1) < 0$. As the sphere narrows for $t > t_2$, the rate of increase of the height begins to increase, and the graph of H is concave upward. Therefore, $H''(t_2) = 0$ and $H''(t_3) > 0$.

13. (a) Put $x = 0$ and $y = 0$ in the equation: $f(0+0) = f(0) + f(0) + 0^2 \cdot 0 + 0 \cdot 0^2 \quad\Rightarrow\quad f(0) = 2f(0)$. Subtracting $f(0)$ from each side of this equation gives $f(0) = 0$.

(b) $f'(0) = \lim_{h \to 0} \dfrac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \dfrac{\left[f(0) + f(h) + 0^2 h + 0h^2\right] - f(0)}{h} = \lim_{h \to 0} \dfrac{f(h)}{h} = \lim_{x \to 0} \dfrac{f(x)}{x} = 1$

(c) $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \dfrac{\left[f(x) + f(h) + x^2 h + xh^2\right] - f(x)}{h} = \lim_{h \to 0} \dfrac{f(h) + x^2 h + xh^2}{h}$

$= \lim_{h \to 0} \left[\dfrac{f(h)}{h} + x^2 + xh\right] = 1 + x^2$

14. We find the equation of the parabola by substituting the point $(-100, 100)$, at which the car is situated, into the general equation $y = ax^2$: $100 = a(-100)^2 \;\Rightarrow\; a = \frac{1}{100}$. Now we find the equation of a tangent to the parabola at the point (x_0, y_0). We can show that $y' = a(2x) = \frac{1}{100}(2x) = \frac{1}{50}x$, so an equation of the tangent is $y - y_0 = \frac{1}{50}x_0(x - x_0)$. Since the point (x_0, y_0) is on the parabola, we must have $y_0 = \frac{1}{100}x_0^2$, so our equation of the tangent can be simplified to $y = \frac{1}{100}x_0^2 + \frac{1}{50}x_0(x - x_0)$. We want the statue to be located on the tangent line, so we substitute its coordinates $(100, 50)$ into this equation: $50 = \frac{1}{100}x_0^2 + \frac{1}{50}x_0(100 - x_0) \;\Rightarrow\; x_0^2 - 200x_0 + 5000 = 0 \;\Rightarrow$ $x_0 = \frac{1}{2}\left[200 \pm \sqrt{200^2 - 4(5000)}\right] \;\Rightarrow\; x_0 = 100 \pm 50\sqrt{2}$. But $x_0 < 100$, so the car's headlights illuminate the statue when it is located at the point $\left(100 - 50\sqrt{2}, 150 - 100\sqrt{2}\right) \approx (29.3, 8.6)$, that is, about 29.3 m east and 8.6 m north of the origin.

15. $\lim_{x \to a} f(x) = \lim_{x \to a} \left(\frac{1}{2}[f(x) + g(x)] + \frac{1}{2}[f(x) - g(x)]\right) = \frac{1}{2} \lim_{x \to a} [f(x) + g(x)] + \frac{1}{2} \lim_{x \to a} [f(x) - g(x)]$

$= \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 1 = \frac{3}{2}$,

and $\lim_{x \to a} g(x) = \lim_{x \to a} \Big([f(x) + g(x)] - f(x)\Big) = \lim_{x \to a}[f(x) + g(x)] - \lim_{x \to a} f(x) = 2 - \frac{3}{2} = \frac{1}{2}$.

So $\lim_{x \to a} [f(x)g(x)] = \left[\lim_{x \to a} f(x)\right]\left[\lim_{x \to a} g(x)\right] = \frac{3}{2} \cdot \frac{1}{2} = \frac{3}{4}$.

Another solution: Since $\lim_{x \to a} [f(x) + g(x)]$ and $\lim_{x \to a} [f(x) - g(x)]$ exist, we must have

$\lim_{x \to a} [f(x) + g(x)]^2 = \left(\lim_{x \to a} [f(x) + g(x)]\right)^2$ and $\lim_{x \to a} [f(x) - g(x)]^2 = \left(\lim_{x \to a} [f(x) - g(x)]\right)^2$, so

$\lim_{x \to a} [f(x)\,g(x)] = \lim_{x \to a} \frac{1}{4}\left([f(x) + g(x)]^2 - [f(x) - g(x)]^2\right)$ [because all of the f^2 and g^2 cancel]

$= \frac{1}{4}\left(\lim_{x \to a} [f(x) + g(x)]^2 - \lim_{x \to a} [f(x) - g(x)]^2\right) = \frac{1}{4}\left(2^2 - 1^2\right) = \frac{3}{4}$.

16. $g'(x) = \lim_{h \to 0} \dfrac{g(x+h) - g(x)}{h} = \lim_{h \to 0} \dfrac{(x+h)f(x+h) - xf(x)}{h} = \lim_{h \to 0} \left[\dfrac{xf(x+h) - xf(x)}{h} + \dfrac{hf(x+h)}{h}\right]$

$= x \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h} + \lim_{h \to 0} f(x+h) = xf'(x) + f(x)$

because f is differentiable and therefore continuous.

17. We are given that $|f(x)| \le x^2$ for all x. In particular, $|f(0)| \le 0$, but $|a| \ge 0$ for all a. The only conclusion is that $f(0) = 0$. Now $\left|\dfrac{f(x) - f(0)}{x - 0}\right| = \left|\dfrac{f(x)}{x}\right| = \dfrac{|f(x)|}{|x|} \le \dfrac{x^2}{|x|} = \dfrac{|x^2|}{|x|} = |x| \;\Rightarrow\; -|x| \le \dfrac{f(x) - f(0)}{x - 0} \le |x|$.

But $\lim_{x \to 0} (-|x|) = 0 = \lim_{x \to 0} |x|$, so by the Squeeze Theorem, $\lim_{x \to 0} \dfrac{f(x) - f(0)}{x - 0} = 0$. So by the definition of a derivative, f is differentiable at 0 and, furthermore, $f'(0) = 0$.

3 ☐ DIFFERENTIATION RULES

3.1 Derivatives of Polynomials and Exponential Functions

1. (a) e is the number such that $\lim_{h \to 0} \dfrac{e^h - 1}{h} = 1$.

(b)

x	$\dfrac{2.7^x - 1}{x}$
-0.001	0.9928
-0.0001	0.9932
0.001	0.9937
0.0001	0.9933

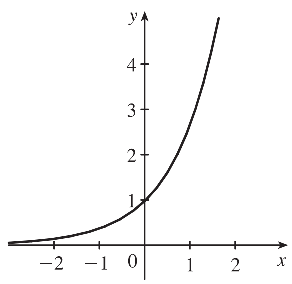
x	$\dfrac{2.8^x - 1}{x}$
-0.001	1.0291
-0.0001	1.0296
0.001	1.0301
0.0001	1.0297

From the tables (to two decimal places),

$\lim_{h \to 0} \dfrac{2.7^h - 1}{h} = 0.99$ and $\lim_{h \to 0} \dfrac{2.8^h - 1}{h} = 1.03$.

Since $0.99 < 1 < 1.03$, $2.7 < e < 2.8$.

2. (a)



The function value at $x = 0$ is 1 and the slope at $x = 0$ is 1.

(b) $f(x) = e^x$ is an exponential function and $g(x) = x^e$ is a power function. $\dfrac{d}{dx}(e^x) = e^x$ and $\dfrac{d}{dx}(x^e) = ex^{e-1}$.

(c) $f(x) = e^x$ grows more rapidly than $g(x) = x^e$ when x is large.

3. $f(x) = 186.5$ is a constant function, so its derivative is 0, that is, $f'(x) = 0$.

4. $f(x) = \sqrt{30}$ is a constant function, so its derivative is 0, that is, $f'(x) = 0$.

5. $f(t) = 2 - \frac{2}{3}t \quad \Rightarrow \quad f'(t) = 0 - \frac{2}{3} = -\frac{2}{3}$

6. $F(x) = \frac{3}{4}x^8 \quad \Rightarrow \quad F'(x) = \frac{3}{4}(8x^7) = 6x^7$

7. $f(x) = x^3 - 4x + 6 \quad \Rightarrow \quad f'(x) = 3x^2 - 4(1) + 0 = 3x^2 - 4$

8. $f(t) = \frac{1}{2}t^6 - 3t^4 + t \quad \Rightarrow \quad f'(t) = \frac{1}{2}(6t^5) - 3(4t^3) + 1 = 3t^5 - 12t^3 + 1$

9. $f(t) = \frac{1}{4}(t^4 + 8) \quad \Rightarrow \quad f'(t) = \frac{1}{4}(t^4 + 8)' = \frac{1}{4}(4t^{4-1} + 0) = t^3$

10. $h(x) = (x - 2)(2x + 3) = 2x^2 - x - 6 \quad \Rightarrow \quad h'(x) = 2(2x) - 1 - 0 = 4x - 1$

11. $A(s) = -\dfrac{12}{s^5} = -12s^{-5} \quad \Rightarrow \quad A'(s) = -12(-5s^{-6}) = 60s^{-6}$ or $60/s^6$

12. $B(y) = cy^{-6} \quad \Rightarrow \quad B'(y) = c(-6y^{-7}) = -6cy^{-7}$

13. $g(t) = 2t^{-3/4} \quad \Rightarrow \quad g'(t) = 2(-\frac{3}{4}t^{-7/4}) = -\frac{3}{2}t^{-7/4}$

14. $h(t) = \sqrt[4]{t} - 4e^t = t^{1/4} - 4e^t \quad \Rightarrow \quad h'(t) = \frac{1}{4}t^{-3/4} - 4(e^t) = \frac{1}{4}t^{-3/4} - 4e^t$

15. $y = 3e^x + \dfrac{4}{\sqrt[3]{x}} = 3e^x + 4x^{-1/3} \quad \Rightarrow \quad y' = 3(e^x) + 4(-\frac{1}{3})x^{-4/3} = 3e^x - \frac{4}{3}x^{-4/3}$

16. $y = \sqrt{x}\,(x-1) = x^{3/2} - x^{1/2} \quad\Rightarrow\quad y' = \frac{3}{2}x^{1/2} - \frac{1}{2}x^{-1/2} = \frac{1}{2}x^{-1/2}(3x-1)$ [factor out $\frac{1}{2}x^{-1/2}$]

or $\quad y' = \dfrac{3x-1}{2\sqrt{x}}$.

17. $F(x) = \left(\frac{1}{2}x\right)^5 = \left(\frac{1}{2}\right)^5 x^5 = \frac{1}{32}x^5 \quad\Rightarrow\quad F'(x) = \frac{1}{32}(5x^4) = \frac{5}{32}x^4$

18. $f(x) = \dfrac{x^2 - 3x + 1}{x^2} = 1 - \dfrac{3}{x} + \dfrac{1}{x^2} = 1 - 3x^{-1} + x^{-2} \quad\Rightarrow$

$f'(x) = 0 - 3(-1)x^{-2} + (-2)x^{-3} = 3x^{-2} - 2x^{-3}$ or $\dfrac{3}{x^2} - \dfrac{2}{x^3}$ or $\dfrac{3x-2}{x^3}$

19. $y = \dfrac{x^2 + 4x + 3}{\sqrt{x}} = x^{3/2} + 4x^{1/2} + 3x^{-1/2} \quad\Rightarrow$

$y' = \frac{3}{2}x^{1/2} + 4\left(\frac{1}{2}\right)x^{-1/2} + 3\left(-\frac{1}{2}\right)x^{-3/2} = \frac{3}{2}\sqrt{x} + \dfrac{2}{\sqrt{x}} - \dfrac{3}{2x\sqrt{x}}$ $\left[\text{note that } x^{3/2} = x^{2/2} \cdot x^{1/2} = x\sqrt{x}\right]$

The last expression can be written as $\dfrac{3x^2}{2x\sqrt{x}} + \dfrac{4x}{2x\sqrt{x}} - \dfrac{3}{2x\sqrt{x}} = \dfrac{3x^2 + 4x - 3}{2x\sqrt{x}}$.

20. $g(u) = \sqrt{2}\,u + \sqrt{3u} = \sqrt{2}\,u + \sqrt{3}\,\sqrt{u} \quad\Rightarrow\quad g'(u) = \sqrt{2}\,(1) + \sqrt{3}\left(\frac{1}{2}u^{-1/2}\right) = \sqrt{2} + \frac{\sqrt{3}}{2\sqrt{u}}$

21. $y = 4\pi^2 \quad\Rightarrow\quad y' = 0$ since $4\pi^2$ is a constant.

22. $y = ae^v + \dfrac{b}{v} + \dfrac{c}{v^2} = ae^v + bv^{-1} + cv^{-2} \quad\Rightarrow\quad y' = ae^v - bv^{-2} - 2cv^{-3} = ae^v - \dfrac{b}{v^2} - \dfrac{2c}{v^3}$

23. $u = \sqrt[5]{t} + 4\sqrt{t^5} = t^{1/5} + 4t^{5/2} \quad\Rightarrow\quad u' = \frac{1}{5}t^{-4/5} + 4\left(\frac{5}{2}t^{3/2}\right) = \frac{1}{5}t^{-4/5} + 10t^{3/2}$ or $1/\left(5\sqrt[5]{t^4}\right) + 10\sqrt{t^3}$

24. $v = \left(\sqrt{x} + \dfrac{1}{\sqrt[3]{x}}\right)^2 = \left(\sqrt{x}\right)^2 + 2\sqrt{x} \cdot \dfrac{1}{\sqrt[3]{x}} + \left(\dfrac{1}{\sqrt[3]{x}}\right)^2 = x + 2x^{1/2-1/3} + 1/x^{2/3} = x + 2x^{1/6} + x^{-2/3} \quad\Rightarrow$

$v' = 1 + 2\left(\frac{1}{6}x^{-5/6}\right) - \frac{2}{3}x^{-5/3} = 1 + \frac{1}{3}x^{-5/6} - \frac{2}{3}x^{-5/3}$ or $1 + \dfrac{1}{3\sqrt[6]{x^5}} - \dfrac{2}{3\sqrt[3]{x^5}}$

25. $z = \dfrac{A}{y^{10}} + Be^y = Ay^{-10} + Be^y \quad\Rightarrow\quad z' = -10Ay^{-11} + Be^y = -\dfrac{10A}{y^{11}} + Be^y$

26. $y = e^{x+1} + 1 = e^x e^1 + 1 = e \cdot e^x + 1 \quad\Rightarrow\quad y' = e \cdot e^x = e^{x+1}$

27. $y = \sqrt[4]{x} = x^{1/4} \quad\Rightarrow\quad y' = \frac{1}{4}x^{-3/4} = \dfrac{1}{4\sqrt[4]{x^3}}$. At $(1,1)$, $y' = \frac{1}{4}$ and an equation of the tangent line is

$y - 1 = \frac{1}{4}(x-1)$ or $y = \frac{1}{4}x + \frac{3}{4}$.

28. $y = x^4 + 2x^2 - x \quad\Rightarrow\quad y' = 4x^3 + 4x - 1$. At $(1,2)$, $y' = 7$ and an equation of the tangent line is

$y - 2 = 7(x-1)$ or $y = 7x - 5$.

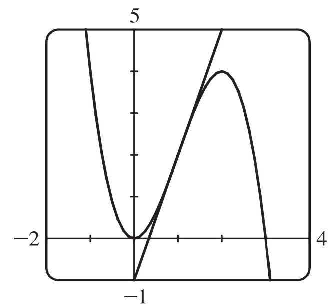
29. $y = x^4 + 2e^x \quad\Rightarrow\quad y' = 4x^3 + 2e^x$. At $(0,2)$, $y' = 2$ and an equation of the tangent line is $y - 2 = 2(x-0)$ or $y = 2x + 2$. The slope of the normal line is $-\frac{1}{2}$ (the negative reciprocal of 2) and an equation of the normal line is

$y - 2 = -\frac{1}{2}(x-0)$ or $y = -\frac{1}{2}x + 2$.

30. $y = (1 + 2x)^2 = 1 + 4x + 4x^2 \;\Rightarrow\; y' = 4 + 8x$. At $(1, 9)$, $y' = 12$ and an equation of the tangent line is $y - 9 = 12(x - 1)$ or $y = 12x - 3$. The slope of the normal line is $-\frac{1}{12}$ (the negative reciprocal of 12) and an equation of the normal line is $y - 9 = -\frac{1}{12}(x - 1)$ or $y = -\frac{1}{12}x + \frac{109}{12}$.

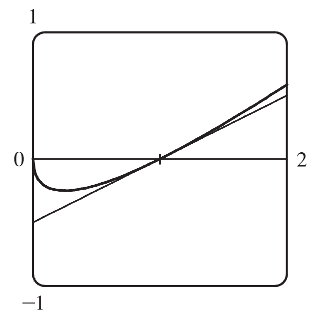
31. $y = 3x^2 - x^3 \;\Rightarrow\; y' = 6x - 3x^2$.

At $(1, 2)$, $y' = 6 - 3 = 3$, so an equation of the tangent line is $y - 2 = 3(x - 1)$ or $y = 3x - 1$.



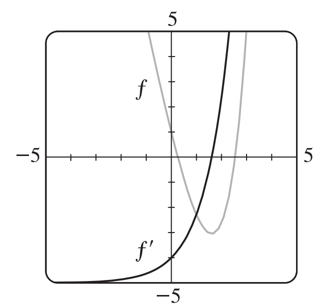
32. $y = x - \sqrt{x} \;\Rightarrow\; y' = 1 - \frac{1}{2}x^{-1/2} = 1 - \dfrac{1}{2\sqrt{x}}$.

At $(1, 0)$, $y' = \frac{1}{2}$, so an equation of the tangent line is $y - 0 = \frac{1}{2}(x - 1)$ or $y = \frac{1}{2}x - \frac{1}{2}$.



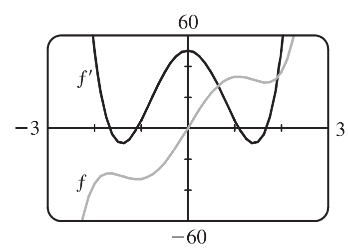
33. $f(x) = e^x - 5x \;\Rightarrow\; f'(x) = e^x - 5$.

Notice that $f'(x) = 0$ when f has a horizontal tangent, f' is positive when f is increasing, and f' is negative when f is decreasing.



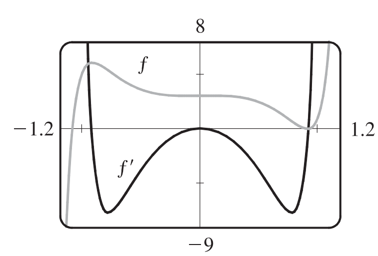
34. $f(x) = 3x^5 - 20x^3 + 50x \;\Rightarrow\; f'(x) = 15x^4 - 60x^2 + 50$.

Notice that $f'(x) = 0$ when f has a horizontal tangent and that f' is an even function while f is an odd function.



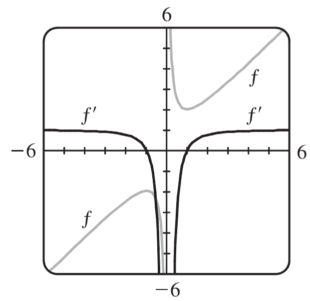
35. $f(x) = 3x^{15} - 5x^3 + 3 \;\Rightarrow\; f'(x) = 45x^{14} - 15x^2$.

Notice that $f'(x) = 0$ when f has a horizontal tangent, f' is positive when f is increasing, and f' is negative when f is decreasing.



36. $f(x) = x + 1/x = x + x^{-1} \quad \Rightarrow \quad f'(x) = 1 - x^{-2} = 1 - 1/x^2$.

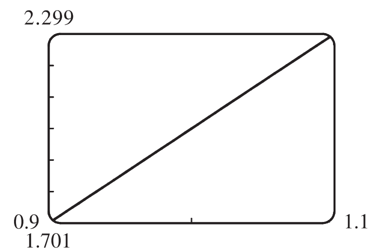
Notice that $f'(x) = 0$ when f has a horizontal tangent, f' is positive when f is increasing, and f' is negative when f is decreasing.



37. To graphically estimate the value of $f'(1)$ for $f(x) = 3x^2 - x^3$, we'll graph f in the viewing rectangle $[1 - 0.1, 1 + 0.1]$ by $[f(0.9), f(1.1)]$, as shown in the figure. [When assigning values to the window variables, it is convenient to use $\text{Y}_1(0.9)$ for Y_{min} and $\text{Y}_1(1.1)$ for Y_{max}.] If we have sufficiently zoomed in on the graph of f, we should obtain a graph that looks like a diagonal line; if not, graph again with $1 - 0.01$ and $1 + 0.01$, etc.

Estimated value:
$f'(1) \approx \dfrac{2.299 - 1.701}{1.1 - 0.9} = \dfrac{0.589}{0.2} = 2.99.$

Exact value: $f(x) = 3x^2 - x^3 \quad \Rightarrow \quad f'(x) = 6x - 3x^2$,
so $f'(1) = 6 - 3 = 3$.

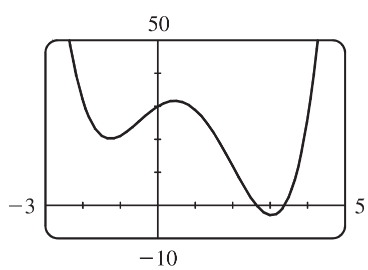


38. See the previous exercise. Since f is a decreasing function, assign $\text{Y}_1(3.9)$ to Y_{max} and $\text{Y}_1(4.1)$ to Y_{min}.

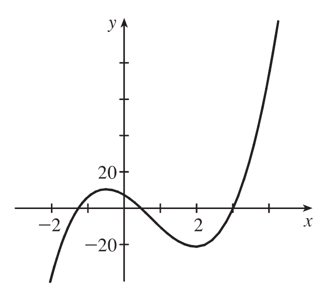
Estimated value: $f'(4) \approx \dfrac{0.49386 - 0.50637}{4.1 - 3.9} = \dfrac{-0.01251}{0.2} = -0.06255.$

Exact value: $f(x) = x^{-1/2} \quad \Rightarrow \quad f'(x) = -\frac{1}{2}x^{-3/2}$, so $f'(4) = -\frac{1}{2}(4^{-3/2}) = -\frac{1}{2}\left(\frac{1}{8}\right) = -\frac{1}{16} = -0.0625.$

39. (a)

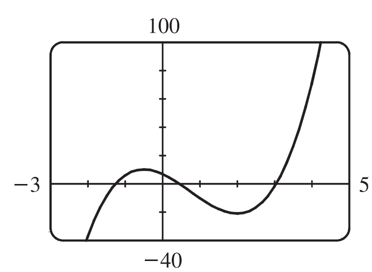


(b) From the graph in part (a), it appears that f' is zero at $x_1 \approx -1.25$, $x_2 \approx 0.5$, and $x_3 \approx 3$. The slopes are negative (so f' is negative) on $(-\infty, x_1)$ and (x_2, x_3). The slopes are positive (so f' is positive) on (x_1, x_2) and (x_3, ∞).

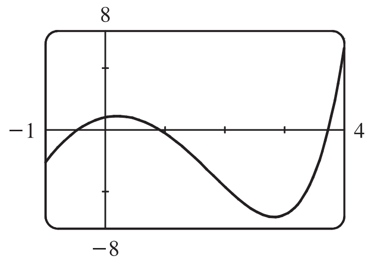


(c) $f(x) = x^4 - 3x^3 - 6x^2 + 7x + 30 \quad \Rightarrow$

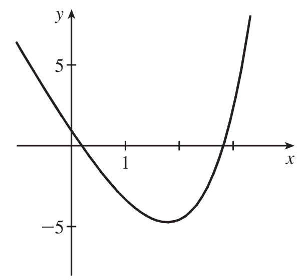
$f'(x) = 4x^3 - 9x^2 - 12x + 7$



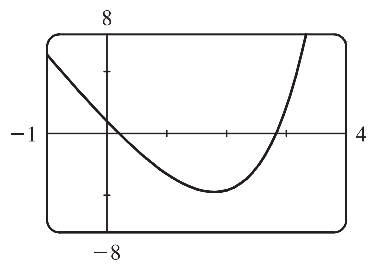
40. (a)



(b) From the graph in part (a), it appears that f' is zero at $x_1 \approx 0.2$ and $x_2 \approx 2.8$. The slopes are positive (so f' is positive) on $(-\infty, x_1)$ and (x_2, ∞). The slopes are negative (so f' is negative) on (x_1, x_2).



(c) $g(x) = e^x - 3x^2 \quad \Rightarrow \quad g'(x) = e^x - 6x$

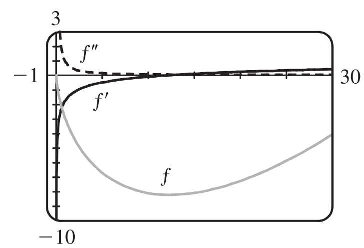


41. $f(x) = 10x^{10} + 5x^5 - x \quad \Rightarrow \quad f'(x) = 100x^9 + 25x^4 - 1 \quad \Rightarrow \quad f''(x) = 900x^8 + 100x^3$

42. $G(r) = \sqrt{r} + \sqrt[3]{r} \quad \Rightarrow \quad G'(r) = \frac{1}{2}r^{-1/2} + \frac{1}{3}r^{-2/3} \quad \Rightarrow \quad G''(r) = -\frac{1}{4}r^{-3/2} - \frac{2}{9}r^{-5/3}$

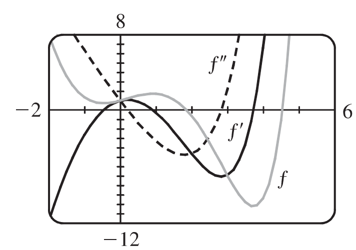
43. $f(x) = 2x - 5x^{3/4} \quad \Rightarrow \quad f'(x) = 2 - \frac{15}{4}x^{-1/4} \quad \Rightarrow \quad f''(x) = \frac{15}{16}x^{-5/4}$

Note that f' is negative when f is decreasing and positive when f is increasing. f'' is always positive since f' is always increasing.



44. $f(x) = e^x - x^3 \quad \Rightarrow \quad f'(x) = e^x - 3x^2 \quad \Rightarrow \quad f''(x) = e^x - 6x$

Note that $f'(x) = 0$ when f has a horizontal tangent and that $f''(x) = 0$ when f' has a horizontal tangent.



45. (a) $s = t^3 - 3t \quad \Rightarrow \quad v(t) = s'(t) = 3t^2 - 3 \quad \Rightarrow \quad a(t) = v'(t) = 6t$

(b) $a(2) = 6(2) = 12 \text{ m/s}^2$

(c) $v(t) = 3t^2 - 3 = 0$ when $t^2 = 1$, that is, $t = 1$ and $a(1) = 6 \text{ m/s}^2$.

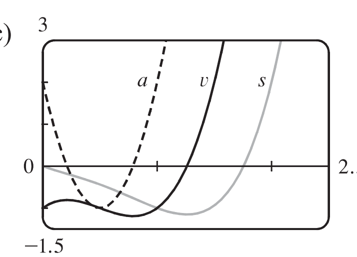
46. (a) $s = t^4 - 2t^3 + t^2 - t \quad \Rightarrow$

$v(t) = s'(t) = 4t^3 - 6t^2 + 2t - 1 \quad \Rightarrow$

$a(t) = v'(t) = 12t^2 - 12t + 2$

(b) $a(1) = 12(1)^2 - 12(1) + 2 = 2 \text{ m/s}^2$

(c)



47. $f(x) = 5x - e^x \quad \Rightarrow \quad f'(x) = 5 - e^x. \quad f'(x) > 0 \quad \Rightarrow \quad 5 - e^x > 0 \quad \Rightarrow \quad e^x < 5 \quad \Rightarrow \quad x < \ln 5 \approx 1.61.$

f is increasing when f' is positive; that is, on $(-\infty, \ln 5)$.

48. $f(x) = x^3 - 4x^2 + 5x \quad \Rightarrow \quad f'(x) = 3x^2 - 8x + 5 \quad \Rightarrow \quad f''(x) = 6x - 8.$

$f''(x) > 0 \quad \Rightarrow \quad 6x - 8 > 0 \quad \Rightarrow \quad x > \frac{4}{3}$. f is concave upward when $f''(x) > 0$; that is, on $\left(\frac{4}{3}, \infty\right)$.

49. The curve $y = 2x^3 + 3x^2 - 12x + 1$ has a horizontal tangent when $y' = 6x^2 + 6x - 12 = 0 \quad \Leftrightarrow \quad 6(x^2 + x - 2) = 0 \quad \Leftrightarrow$ $6(x + 2)(x - 1) = 0 \quad \Leftrightarrow \quad x = -2$ or $x = 1$. The points on the curve are $(-2, 21)$ and $(1, -6)$.

50. $f(x) = x^3 + 3x^2 + x + 3$ has a horizontal tangent when $f'(x) = 3x^2 + 6x + 1 = 0 \quad \Leftrightarrow$

$$x = \frac{-6 \pm \sqrt{36 - 12}}{6} = -1 \pm \tfrac{1}{3}\sqrt{6}.$$

51. $y = 6x^3 + 5x - 3 \quad \Rightarrow \quad m = y' = 18x^2 + 5$, but $x^2 \geq 0$ for all x, so $m \geq 5$ for all x.

52. $y = x\sqrt{x} = x^{3/2} \quad \Rightarrow \quad y' = \frac{3}{2}x^{1/2}$. The slope of the line $y = 1 + 3x$ is 3, so the slope of any line parallel to it is also 3. Thus, $y' = 3 \quad \Rightarrow \quad \frac{3}{2}x^{1/2} = 3 \quad \Rightarrow \quad \sqrt{x} = 2 \quad \Rightarrow \quad x = 4$, which is the x-coordinate of the point on the curve at which the slope is 3. The y-coordinate is $y = 4\sqrt{4} = 8$, so an equation of the tangent line is $y - 8 = 3(x - 4)$ or $y = 3x - 4$.

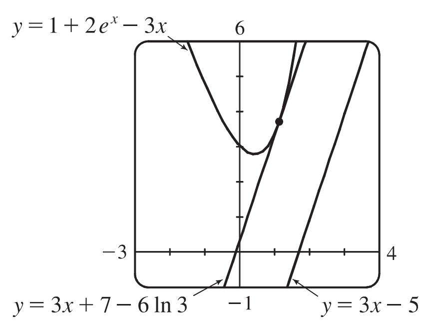
53. The slope of the line $12x - y = 1$ (or $y = 12x - 1$) is 12, so the slope of both lines tangent to the curve is 12. $y = 1 + x^3 \quad \Rightarrow \quad y' = 3x^2$. Thus, $3x^2 = 12 \quad \Rightarrow \quad x^2 = 4 \quad \Rightarrow \quad x = \pm 2$, which are the x-coordinates at which the tangent lines have slope 12. The points on the curve are $(2, 9)$ and $(-2, -7)$, so the tangent line equations are $y - 9 = 12(x - 2)$ or $y = 12x - 15$ and $y + 7 = 12(x + 2)$ or $y = 12x + 17$.

54. The slope of $y = 1 + 2e^x - 3x$ is given by $m = y' = 2e^x - 3$.

The slope of $3x - y = 5 \quad \Leftrightarrow \quad y = 3x - 5$ is 3.

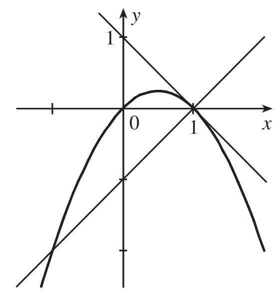
$m = 3 \quad \Rightarrow \quad 2e^x - 3 = 3 \quad \Rightarrow \quad e^x = 3 \quad \Rightarrow \quad x = \ln 3.$

This occurs at the point $(\ln 3, 7 - 3\ln 3) \approx (1.1, 3.7)$.



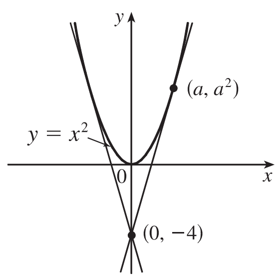
55. The slope of $y = x^2 - 5x + 4$ is given by $m = y' = 2x - 5$. The slope of $x - 3y = 5 \quad \Leftrightarrow \quad y = \frac{1}{3}x - \frac{5}{3}$ is $\frac{1}{3}$, so the desired normal line must have slope $\frac{1}{3}$, and hence, the tangent line to the parabola must have slope -3. This occurs if $2x - 5 = -3 \quad \Rightarrow \quad 2x = 2 \quad \Rightarrow \quad x = 1$. When $x = 1$, $y = 1^2 - 5(1) + 4 = 0$, and an equation of the normal line is $y - 0 = \frac{1}{3}(x - 1)$ or $y = \frac{1}{3}x - \frac{1}{3}$.

56. $y = f(x) = x - x^2 \quad \Rightarrow \quad f'(x) = 1 - 2x$.



So $f'(1) = -1$, and the slope of the normal line is the negative reciprocal of that of the tangent line, that is, $-1/(-1) = 1$. So the equation of the normal line at $(1, 0)$ is $y - 0 = 1(x - 1) \quad \Leftrightarrow \quad y = x - 1$. Substituting this into the equation of the parabola, we obtain $x - 1 = x - x^2 \quad \Leftrightarrow \quad x = \pm 1$. The solution $x = -1$ is the one we require. Substituting $x = -1$ into the equation of the parabola to find the y-coordinate, we have $y = -2$. So the point of intersection is $(-1, -2)$, as shown in the sketch.

57.



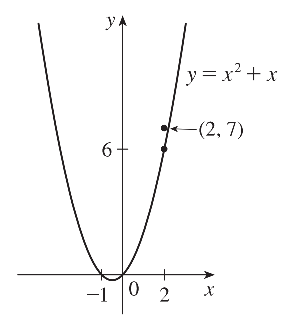
Let (a, a^2) be a point on the parabola at which the tangent line passes through the point $(0, -4)$. The tangent line has slope $2a$ and equation $y - (-4) = 2a(x - 0) \quad \Leftrightarrow \quad y = 2ax - 4$. Since (a, a^2) also lies on the line, $a^2 = 2a(a) - 4$, or $a^2 = 4$. So $a = \pm 2$ and the points are $(2, 4)$ and $(-2, 4)$.

58. (a) If $y = x^2 + x$, then $y' = 2x + 1$. If the point at which a tangent meets the parabola is $(a, a^2 + a)$, then the slope of the tangent is $2a + 1$. But since it passes through $(2, -3)$, the slope must also be $\dfrac{\Delta y}{\Delta x} = \dfrac{a^2 + a + 3}{a - 2}$.

Therefore, $2a + 1 = \dfrac{a^2 + a + 3}{a - 2}$. Solving this equation for a we get $a^2 + a + 3 = 2a^2 - 3a - 2 \quad \Leftrightarrow$ $a^2 - 4a - 5 = (a - 5)(a + 1) = 0 \quad \Leftrightarrow \quad a = 5$ or -1. If $a = -1$, the point is $(-1, 0)$ and the slope is -1, so the equation is $y - 0 = (-1)(x + 1)$ or $y = -x - 1$. If $a = 5$, the point is $(5, 30)$ and the slope is 11, so the equation is $y - 30 = 11(x - 5)$ or $y = 11x - 25$.

(b) As in part (a), but using the point $(2, 7)$, we get the equation

$$2a + 1 = \frac{a^2 + a - 7}{a - 2} \quad \Rightarrow \quad 2a^2 - 3a - 2 = a^2 + a - 7 \quad \Leftrightarrow \quad a^2 - 4a + 5 = 0.$$



The last equation has no real solution (discriminant $= -16 < 0$), so there is no line through the point $(2, 7)$ that is tangent to the parabola. The diagram shows that the point $(2, 7)$ is "inside" the parabola, but tangent lines to the parabola do not pass through points inside the parabola.

59. $f'(x) = \displaystyle\lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{\dfrac{1}{x + h} - \dfrac{1}{x}}{h} = \lim_{h \to 0} \frac{x - (x + h)}{hx(x + h)} = \lim_{h \to 0} \frac{-h}{hx(x + h)} = \lim_{h \to 0} \frac{-1}{x(x + h)} = -\frac{1}{x^2}$

60. (a) $f(x) = x^n \quad \Rightarrow \quad f'(x) = nx^{n-1} \quad \Rightarrow \quad f''(x) = n\,(n - 1)\,x^{n-2} \quad \Rightarrow \quad \cdots \quad \Rightarrow$

$f^{(n)}(x) = n(n - 1)(n - 2) \cdots 2 \cdot 1 x^{n-n} = n!$

(b) $f(x) = x^{-1} \Rightarrow f'(x) = (-1)x^{-2} \Rightarrow f''(x) = (-1)(-2)x^{-3} \Rightarrow \cdots \Rightarrow$

$f^{(n)}(x) = (-1)(-2)(-3)\cdots(-n)x^{-(n+1)} = (-1)^n n! x^{-(n+1)}$ or $\dfrac{(-1)^n\, n!}{x^{n+1}}$

61. Let $P(x) = ax^2 + bx + c$. Then $P'(x) = 2ax + b$ and $P''(x) = 2a$. $P''(2) = 2 \Rightarrow 2a = 2 \Rightarrow a = 1$.

$P'(2) = 3 \Rightarrow 2(1)(2) + b = 3 \Rightarrow 4 + b = 3 \Rightarrow b = -1$.

$P(2) = 5 \Rightarrow 1(2)^2 + (-1)(2) + c = 5 \Rightarrow 2 + c = 5 \Rightarrow c = 3$. So $P(x) = x^2 - x + 3$.

62. $y = Ax^2 + Bx + C \Rightarrow y' = 2Ax + B \Rightarrow y'' = 2A$. We substitute these expressions into the equation

$y'' + y' - 2y = x^2$ to get

$$(2A) + (2Ax + B) - 2(Ax^2 + Bx + C) = x^2$$
$$2A + 2Ax + B - 2Ax^2 - 2Bx - 2C = x^2$$
$$(-2A)x^2 + (2A - 2B)x + (2A + B - 2C) = (1)x^2 + (0)x + (0)$$

The coefficients of x^2 on each side must be equal, so $-2A = 1 \Rightarrow A = -\frac{1}{2}$. Similarly, $2A - 2B = 0 \Rightarrow$

$A = B = -\frac{1}{2}$ and $2A + B - 2C = 0 \Rightarrow -1 - \frac{1}{2} - 2C = 0 \Rightarrow C = -\frac{3}{4}$.

63. (a) At this stage, we would guess that an antiderivative of x^2 must have x^3 in it. Differentiating x^3 gives us $3x^2$, so we know that we must divide x^3 by 3. That gives us $F(x) = \frac{1}{3}x^3$. Checking, we have $F'(x) = \frac{1}{3}(3x^2) = x^2 = f(x)$. Because we can add an arbitrary constant C to F without changing its derivative, we have an infinite number of antiderivatives of the form $F(x) = \frac{1}{3}x^3 + C$.

(b) As in part (a), antiderivatives of $f(x) = x^3$ and $f(x) = x^4$ are $F(x) = \frac{1}{4}x^4 + C$ and $F(x) = \frac{1}{5}x^5 + C$.

(c) Similarly, an antiderivative for $f(x) = x^n$ is $F(x) = \dfrac{1}{n+1}x^{n+1} + C$, since then

$F'(x) = \dfrac{1}{n+1}\left[(n+1)x^n\right] = x^n = f(x)$ for $n \neq -1$.

64. (a) $f(x) = \sqrt{x} = x^{1/2} \Rightarrow F(x) = \frac{1}{(1/2)+1}x^{(1/2)+1} + C = \frac{2}{3}x^{3/2} + C$

(b) $f(x) = e^x + 8x^3 \Rightarrow F(x) = e^x + 8 \cdot \frac{1}{3+1}x^{3+1} + C = e^x + 2x^4 + C$

65. Substituting $x = 1$ and $y = 1$ into $y = ax^2 + bx$ gives us $a + b = 1$ **(1)**. The slope of the tangent line $y = 3x - 2$ is 3 and the slope of the tangent to the parabola at (x, y) is $y' = 2ax + b$. At $x = 1$, $y' = 3 \Rightarrow 3 = 2a + b$ **(2)**. Subtracting **(1)** from **(2)** gives us $2 = a$ and it follows that $b = -1$. The parabola has equation $y = 2x^2 - x$.

66. $y = x^4 + ax^3 + bx^2 + cx + d \Rightarrow y(0) = d$. Since the tangent line $y = 2x + 1$ is equal to 1 at $x = 0$, we must have $d = 1$. $y' = 4x^3 + 3ax^2 + 2bx + c \Rightarrow y'(0) = c$. Since the slope of the tangent line $y = 2x + 1$ at $x = 0$ is 2, we must have $c = 2$. Now $y(1) = 1 + a + b + c + d = a + b + 4$ and the tangent line $y = 2 - 3x$ at $x = 1$ has y-coordinate -1, so $a + b + 4 = -1$ or $a + b = -5$ **(1)**. Also, $y'(1) = 4 + 3a + 2b + c = 3a + 2b + 6$ and the slope of the tangent line $y = 2 - 3x$ at $x = 1$ is -3, so $3a + 2b + 6 = -3$ or $3a + 2b = -9$ **(2)**. Adding -2 times **(1)** to **(2)** gives us $a = 1$ and hence, $b = -6$. The curve has equation $y = x^4 + x^3 - 6x^2 + 2x + 1$.

67. $y = f(x) = ax^3 + bx^2 + cx + d \quad \Rightarrow \quad f'(x) = 3ax^2 + 2bx + c$. The point $(-2, 6)$ is on f, so $f(-2) = 6 \quad \Rightarrow$ $-8a + 4b - 2c + d = 6$ **(1)**. The point $(2, 0)$ is on f, so $f(2) = 0 \quad \Rightarrow \quad 8a + 4b + 2c + d = 0$ **(2)**. Since there are horizontal tangents at $(-2, 6)$ and $(2, 0)$, $f'(\pm 2) = 0$. $f'(-2) = 0 \quad \Rightarrow \quad 12a - 4b + c = 0$ **(3)** and $f'(2) = 0 \quad \Rightarrow$ $12a + 4b + c = 0$ **(4)**. Subtracting equation **(3)** from **(4)** gives $8b = 0 \quad \Rightarrow \quad b = 0$. Adding **(1)** and **(2)** gives $8b + 2d = 6$, so $d = 3$ since $b = 0$. From **(3)** we have $c = -12a$, so **(2)** becomes $8a + 4(0) + 2(-12a) + 3 = 0 \quad \Rightarrow \quad 3 = 16a \quad \Rightarrow$ $a = \frac{3}{16}$. Now $c = -12a = -12\left(\frac{3}{16}\right) = -\frac{9}{4}$ and the desired cubic function is $y = \frac{3}{16}x^3 - \frac{9}{4}x + 3$.

68. The slope of the curve $y = c\sqrt{x}$ is $y' = \dfrac{c}{2\sqrt{x}}$ and the slope of the tangent line $y = \frac{3}{2}x + 6$ is $\frac{3}{2}$. These must be equal at the point of tangency $\left(a, c\sqrt{a}\right)$, so $\dfrac{c}{2\sqrt{a}} = \dfrac{3}{2} \quad \Rightarrow \quad c = 3\sqrt{a}$. The y-coordinates must be equal at $x = a$, so $c\sqrt{a} = \frac{3}{2}a + 6 \quad \Rightarrow \quad \left(3\sqrt{a}\right)\sqrt{a} = \frac{3}{2}a + 6 \quad \Rightarrow \quad 3a = \frac{3}{2}a + 6 \quad \Rightarrow \quad \frac{3}{2}a = 6 \quad \Rightarrow \quad a = 4$. Since $c = 3\sqrt{a}$, we have $c = 3\sqrt{4} = 6$.

69. $y = f(x) = ax^2 \quad \Rightarrow \quad f'(x) = 2ax$. So the slope of the tangent to the parabola at $x = 2$ is $m = 2a(2) = 4a$. The slope of the given line, $2x + y = b \quad \Leftrightarrow \quad y = -2x + b$, is seen to be -2, so we must have $4a = -2 \quad \Leftrightarrow \quad a = -\frac{1}{2}$. So when $x = 2$, the point in question has y-coordinate $-\frac{1}{2} \cdot 2^2 = -2$. Now we simply require that the given line, whose equation is $2x + y = b$, pass through the point $(2, -2)$: $2(2) + (-2) = b \quad \Leftrightarrow \quad b = 2$. So we must have $a = -\frac{1}{2}$ and $b = 2$.

70. (a) $xy = c \quad \Rightarrow \quad y = \dfrac{c}{x}$. Let $P = \left(a, \dfrac{c}{a}\right)$. The slope of the tangent line at $x = a$ is $y'(a) = -\dfrac{c}{a^2}$. Its equation is $y - \dfrac{c}{a} = -\dfrac{c}{a^2}(x - a)$ or $y = -\dfrac{c}{a^2}x + \dfrac{2c}{a}$, so its y-intercept is $\dfrac{2c}{a}$. Setting $y = 0$ gives $x = 2a$, so the x-intercept is $2a$. The midpoint of the line segment joining $\left(0, \dfrac{2c}{a}\right)$ and $(2a, 0)$ is $\left(a, \dfrac{c}{a}\right) = P$.

(b) We know the x- and y-intercepts of the tangent line from part (a), so the area of the triangle bounded by the axes and the tangent is $\frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}xy = \frac{1}{2}(2a)(2c/a) = 2c$, a constant.

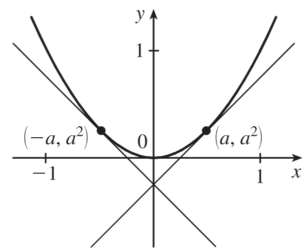
71. *Solution 1:* Let $f(x) = x^{1000}$. Then, by the definition of a derivative, $f'(1) = \displaystyle\lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{x^{1000} - 1}{x - 1}$. But this is just the limit we want to find, and we know (from the Power Rule) that $f'(x) = 1000x^{999}$, so $f'(1) = 1000(1)^{999} = 1000$. So $\displaystyle\lim_{x \to 1} \frac{x^{1000} - 1}{x - 1} = 1000$.

[continued]

Solution 2: Note that $(x^{1000}-1)=(x-1)(x^{999}+x^{998}+x^{997}+\cdots+x^2+x+1)$. So

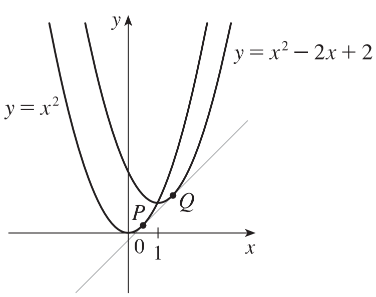
$$\lim_{x\to 1}\frac{x^{1000}-1}{x-1}=\lim_{x\to 1}\frac{(x-1)(x^{999}+x^{998}+x^{997}+\cdots+x^2+x+1)}{x-1}=\lim_{x\to 1}(x^{999}+x^{998}+x^{997}+\cdots+x^2+x+1)$$
$$=\underbrace{1+1+1+\cdots+1+1+1}_{1000\text{ ones}}=1000\text{, as above.}$$

72. In order for the two tangents to intersect on the y-axis, the points of tangency must be at equal distances from the y-axis, since the parabola $y=x^2$ is symmetric about the y-axis. Say the points of tangency are (a,a^2) and $(-a,a^2)$, for some $a>0$. Then since the derivative of $y=x^2$ is $dy/dx=2x$, the left-hand tangent has slope $-2a$ and equation $y-a^2=-2a(x+a)$, or $y=-2ax-a^2$, and similarly the right-hand tangent line has equation $y-a^2=2a(x-a)$, or $y=2ax-a^2$. So the two lines intersect at $(0,-a^2)$. Now if the lines are perpendicular, then the product of their slopes is -1, so $(-2a)(2a)=-1 \iff a^2=\frac{1}{4} \iff a=\frac{1}{2}$. So the lines intersect at $\left(0,-\frac{1}{4}\right)$.



73. $y=x^2 \Rightarrow y'=2x$, so the slope of a tangent line at the point (a,a^2) is $y'=2a$ and the slope of a normal line is $-1/(2a)$, for $a\neq 0$. The slope of the normal line through the points (a,a^2) and $(0,c)$ is $\dfrac{a^2-c}{a-0}$, so $\dfrac{a^2-c}{a}=-\dfrac{1}{2a} \Rightarrow$ $a^2-c=-\frac{1}{2} \Rightarrow a^2=c-\frac{1}{2}$. The last equation has two solutions if $c>\frac{1}{2}$, one solution if $c=\frac{1}{2}$, and no solution if $c<\frac{1}{2}$. Since the y-axis is normal to $y=x^2$ regardless of the value of c (this is the case for $a=0$), we have three normal lines if $c>\frac{1}{2}$ and one normal line if $c\leq\frac{1}{2}$.

74.



From the sketch, it appears that there may be a line that is tangent to both curves. The slope of the line through the points $P(a,a^2)$ and $Q(b,b^2-2b+2)$ is $\dfrac{b^2-2b+2-a^2}{b-a}$. The slope of the tangent line at P is $2a$ $[y'=2x]$ and at Q is $2b-2$ $[y'=2x-2]$. All three slopes are equal, so $2a=2b-2 \iff a=b-1$.

Also, $2b-2=\dfrac{b^2-2b+2-a^2}{b-a} \Rightarrow 2b-2=\dfrac{b^2-2b+2-(b-1)^2}{b-(b-1)} \Rightarrow 2b-2=b^2-2b+2-b^2+2b-1 \Rightarrow$ $2b=3 \Rightarrow b=\frac{3}{2}$ and $a=\frac{3}{2}-1=\frac{1}{2}$. Thus, an equation of the tangent line at P is $y-\left(\frac{1}{2}\right)^2=2\left(\frac{1}{2}\right)\left(x-\frac{1}{2}\right)$ or $y=x-\frac{1}{4}$.

APPLIED PROJECT Building a Better Roller Coaster

1. (a) $f(x) = ax^2 + bx + c \quad \Rightarrow \quad f'(x) = 2ax + b$.

The origin is at P:	$f(0) = 0$	$\Rightarrow$	$c = 0$
The slope of the ascent is 0.8:	$f'(0) = 0.8$	$\Rightarrow$	$b = 0.8$
The slope of the drop is -1.6:	$f'(100) = -1.6$	$\Rightarrow$	$200a + b = -1.6$

(b) $b = 0.8$, so $200a + b = -1.6 \quad \Rightarrow \quad 200a + 0.8 = -1.6 \quad \Rightarrow \quad 200a = -2.4 \quad \Rightarrow \quad a = -\dfrac{2.4}{200} = -0.012$.

Thus, $f(x) = -0.012x^2 + 0.8x$.

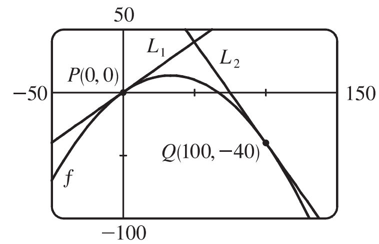
(c) Since L_1 passes through the origin with slope 0.8, it has equation $y = 0.8x$.

The horizontal distance between P and Q is 100, so the y-coordinate at Q is

$f(100) = -0.012(100)^2 + 0.8(100) = -40$. Since L_2 passes through the

point $(100, -40)$ and has slope -1.6, it has equation $y + 40 = -1.6(x - 100)$

or $y = -1.6x + 120$.



(d) The difference in elevation between $P(0, 0)$ and $Q(100, -40)$ is $0 - (-40) = 40$ feet.

2. (a)

Interval	Function	First Derivative	Second Derivative
$(-\infty, 0)$	$L_1(x) = 0.8x$	$L_1'(x) = 0.8$	$L_1''(x) = 0$
$[0, 10)$	$g(x) = kx^3 + lx^2 + mx + n$	$g'(x) = 3kx^2 + 2lx + m$	$g''(x) = 6kx + 2l$
$[10, 90]$	$q(x) = ax^2 + bx + c$	$q'(x) = 2ax + b$	$q''(x) = 2a$
$(90, 100]$	$h(x) = px^3 + qx^2 + rx + s$	$h'(x) = 3px^2 + 2qx + r$	$h''(x) = 6px + 2q$
$(100, \infty)$	$L_2(x) = -1.6x + 120$	$L_2'(x) = -1.6$	$L_2''(x) = 0$

There are 4 values of x (0, 10, 90, and 100) for which we must make sure the function values are equal, the first derivative values are equal, and the second derivative values are equal. The third column in the following table contains the value of each side of the condition — these are found after solving the system in part (b).

At $x =$	Condition	Value	Resulting Equation
0	$g(0) = L_1(0)$	0	$n = 0$
	$g'(0) = L_1'(0)$	$\frac{4}{5}$	$m = 0.8$
	$g''(0) = L_1''(0)$	0	$2l = 0$
10	$g(10) = q(10)$	$\frac{68}{9}$	$1000k + 100l + 10m + n = 100a + 10b + c$
	$g'(10) = q'(10)$	$\frac{2}{3}$	$300k + 20l + m = 20a + b$
	$g''(10) = q''(10)$	$-\frac{2}{75}$	$60k + 2l = 2a$
90	$h(90) = q(90)$	$-\frac{220}{9}$	$729{,}000p + 8100q + 90r + s = 8100a + 90b + c$
	$h'(90) = q'(90)$	$-\frac{22}{15}$	$24{,}300p + 180q + r = 180a + b$
	$h''(90) = q''(90)$	$-\frac{2}{75}$	$540p + 2q = 2a$
100	$h(100) = L_2(100)$	-40	$1{,}000{,}000p + 10{,}000q + 100r + s = -40$
	$h'(100) = L_2'(100)$	$-\frac{8}{5}$	$30{,}000p + 200q + r = -1.6$
	$h''(100) = L_2''(100)$	0	$600p + 2q = 0$

(b) We can arrange our work in a 12×12 matrix as follows.

a	b	c	k	l	m	n	p	q	r	s	constant
0	0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0	0.8
0	0	0	0	2	0	0	0	0	0	0	0
−100	−10	−1	1000	100	10	1	0	0	0	0	0
−20	−1	0	300	20	1	0	0	0	0	0	0
−2	0	0	60	2	0	0	0	0	0	0	0
−8100	−90	−1	0	0	0	0	729,000	8100	90	1	0
−180	−1	0	0	0	0	0	24,300	180	1	0	0
−2	0	0	0	0	0	0	540	2	0	0	0
0	0	0	0	0	0	0	1,000,000	10,000	100	1	−40
0	0	0	0	0	0	0	30,000	200	1	0	−1.6
0	0	0	0	0	0	0	600	2	0	0	0

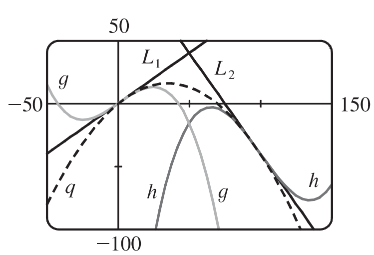
Solving the system gives us the formulas for q, g, and h.

$$\left.\begin{aligned} a &= -0.01\overline{3} = -\tfrac{1}{75} \\ b &= 0.9\overline{3} = \tfrac{14}{15} \\ c &= -0.\overline{4} = -\tfrac{4}{9} \end{aligned}\right\} q(x) = -\tfrac{1}{75}x^2 + \tfrac{14}{15}x - \tfrac{4}{9}$$

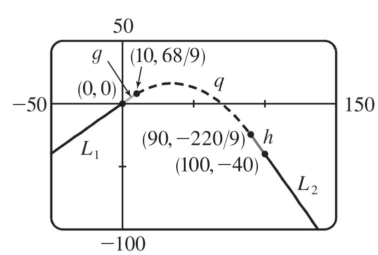
$$\left.\begin{aligned} k &= -0.000\overline{4} = -\tfrac{1}{2250} \\ l &= 0 \\ m &= 0.8 = \tfrac{4}{5} \\ n &= 0 \end{aligned}\right\} g(x) = -\tfrac{1}{2250}x^3 + \tfrac{4}{5}x$$

$$\left.\begin{aligned} p &= 0.000\overline{4} = \tfrac{1}{2250} \\ q &= -0.1\overline{3} = -\tfrac{2}{15} \\ r &= 11.7\overline{3} = \tfrac{176}{15} \\ s &= -324.\overline{4} = -\tfrac{2920}{9} \end{aligned}\right\} h(x) = \tfrac{1}{2250}x^3 - \tfrac{2}{15}x^2 + \tfrac{176}{15}x - \tfrac{2920}{9}$$

(c) Graph of L_1, q, g, h, and L_2:



The graph of the five functions as a piecewise-defined function:

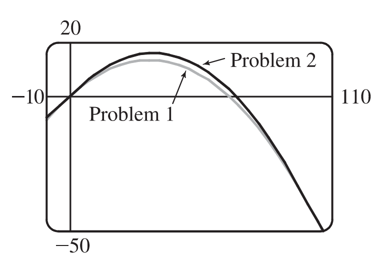


This is the piecewise-defined function assignment on a TI-83 Plus calculator, where $Y_2 = L_1$, $Y_6 = g$, $Y_5 = q$, $Y_7 = h$, and $Y_3 = L_2$.

```
Plot1 Plot2 Plot3
\Y8=Y2*(X<0)+Y6*
(X≥0 and X<10)+Y
5*(X≥10 and X≤90
)+Y7*(X>90 and X
≤100)+Y3*(X>100)

\Y9=
```

A comparison of the graphs in part 1(c) and part 2(c):



3.2 The Product and Quotient Rules

1. Product Rule: $f(x) = (1 + 2x^2)(x - x^2) \quad \Rightarrow$

$$f'(x) = (1 + 2x^2)(1 - 2x) + (x - x^2)(4x) = 1 - 2x + 2x^2 - 4x^3 + 4x^2 - 4x^3 = 1 - 2x + 6x^2 - 8x^3.$$

Multiplying first: $f(x) = (1 + 2x^2)(x - x^2) = x - x^2 + 2x^3 - 2x^4 \quad \Rightarrow \quad f'(x) = 1 - 2x + 6x^2 - 8x^3$ (equivalent).

2. Quotient Rule: $F(x) = \dfrac{x^4 - 5x^3 + \sqrt{x}}{x^2} = \dfrac{x^4 - 5x^3 + x^{1/2}}{x^2} \quad \Rightarrow$

$$F'(x) = \frac{x^2(4x^3 - 15x^2 + \frac{1}{2}x^{-1/2}) - (x^4 - 5x^3 + x^{1/2})(2x)}{(x^2)^2} = \frac{4x^5 - 15x^4 + \frac{1}{2}x^{3/2} - 2x^5 + 10x^4 - 2x^{3/2}}{x^4}$$

$$= \frac{2x^5 - 5x^4 - \frac{3}{2}x^{3/2}}{x^4} = 2x - 5 - \tfrac{3}{2}x^{-5/2}$$

Simplifying first: $F(x) = \dfrac{x^4 - 5x^3 + \sqrt{x}}{x^2} = x^2 - 5x + x^{-3/2} \quad \Rightarrow \quad F'(x) = 2x - 5 - \frac{3}{2}x^{-5/2}$ (equivalent).

For this problem, simplifying first seems to be the better method.

3. By the Product Rule, $f(x) = (x^3 + 2x)e^x \quad \Rightarrow$

$$f'(x) = (x^3 + 2x)(e^x)' + e^x(x^3 + 2x)' = (x^3 + 2x)e^x + e^x(3x^2 + 2)$$
$$= e^x[(x^3 + 2x) + (3x^2 + 2)] = e^x(x^3 + 3x^2 + 2x + 2)$$

4. By the Product Rule, $g(x) = \sqrt{x}\, e^x = x^{1/2}e^x \quad \Rightarrow \quad g'(x) = x^{1/2}(e^x) + e^x\left(\frac{1}{2}x^{-1/2}\right) = \frac{1}{2}x^{-1/2}e^x(2x + 1).$

5. By the Quotient Rule, $y = \dfrac{e^x}{x^2} \quad \Rightarrow \quad y' = \dfrac{x^2 \frac{d}{dx}(e^x) - e^x \frac{d}{dx}(x^2)}{(x^2)^2} = \dfrac{x^2(e^x) - e^x(2x)}{x^4} = \dfrac{xe^x(x - 2)}{x^4} = \dfrac{e^x(x - 2)}{x^3}.$

6. By the Quotient Rule, $y = \dfrac{e^x}{1 + x} \quad \Rightarrow \quad y' = \dfrac{(1 + x)\,e^x - e^x(1)}{(1 + x)^2} = \dfrac{e^x + xe^x - e^x}{(x + 1)^2} = \dfrac{xe^x}{(x + 1)^2}.$

The notations $\overset{\text{PR}}{\Rightarrow}$ and $\overset{\text{QR}}{\Rightarrow}$ indicate the use of the Product and Quotient Rules, respectively.

7. $g(x) = \dfrac{3x - 1}{2x + 1} \quad \overset{\text{QR}}{\Rightarrow} \quad g'(x) = \dfrac{(2x + 1)(3) - (3x - 1)(2)}{(2x + 1)^2} = \dfrac{6x + 3 - 6x + 2}{(2x + 1)^2} = \dfrac{5}{(2x + 1)^2}$

8. $f(t) = \dfrac{2t}{4 + t^2} \quad \overset{\text{QR}}{\Rightarrow} \quad f'(t) = \dfrac{(4 + t^2)(2) - (2t)(2t)}{(4 + t^2)^2} = \dfrac{8 + 2t^2 - 4t^2}{(4 + t^2)^2} = \dfrac{8 - 2t^2}{(4 + t^2)^2}$

9. $F(y) = \left(\dfrac{1}{y^2} - \dfrac{3}{y^4}\right)(y + 5y^3) = (y^{-2} - 3y^{-4})(y + 5y^3) \quad \overset{\text{PR}}{\Rightarrow}$

$$F'(y) = (y^{-2} - 3y^{-4})(1 + 15y^2) + (y + 5y^3)(-2y^{-3} + 12y^{-5})$$
$$= (y^{-2} + 15 - 3y^{-4} - 45y^{-2}) + (-2y^{-2} + 12y^{-4} - 10 + 60y^{-2})$$
$$= 5 + 14y^{-2} + 9y^{-4} \text{ or } 5 + 14/y^2 + 9/y^4$$

10. $R(t) = (t + e^t)(3 - \sqrt{t}) \overset{\text{PR}}{\Rightarrow}$

$$R'(t) = (t + e^t)\left(-\tfrac{1}{2}t^{-1/2}\right) + (3 - \sqrt{t})(1 + e^t)$$
$$= \left(-\tfrac{1}{2}t^{1/2} - \tfrac{1}{2}t^{-1/2}e^t\right) + (3 + 3e^t - \sqrt{t} - \sqrt{t}\,e^t) = 3 + 3e^t - \tfrac{3}{2}\sqrt{t} - \sqrt{t}\,e^t - e^t/(2\sqrt{t})$$

11. $y = \dfrac{x^3}{1 - x^2} \overset{\text{QR}}{\Rightarrow} y' = \dfrac{(1 - x^2)(3x^2) - x^3(-2x)}{(1 - x^2)^2} = \dfrac{x^2(3 - 3x^2 + 2x^2)}{(1 - x^2)^2} = \dfrac{x^2(3 - x^2)}{(1 - x^2)^2}$

12. $y = \dfrac{x + 1}{x^3 + x - 2} \overset{\text{QR}}{\Rightarrow}$

$$y' = \frac{(x^3 + x - 2)(1) - (x + 1)(3x^2 + 1)}{(x^3 + x - 2)^2} = \frac{x^3 + x - 2 - 3x^3 - 3x^2 - x - 1}{(x^3 + x - 2)^2} = \frac{-2x^3 - 3x^2 - 3}{(x^3 + x - 2)^2}$$

or $-\dfrac{2x^3 + 3x^2 + 3}{(x - 1)^2(x^2 + x + 2)^2}$

13. $y = \dfrac{t^2 + 2}{t^4 - 3t^2 + 1} \overset{\text{QR}}{\Rightarrow}$

$$y' = \frac{(t^4 - 3t^2 + 1)(2t) - (t^2 + 2)(4t^3 - 6t)}{(t^4 - 3t^2 + 1)^2} = \frac{2t[(t^4 - 3t^2 + 1) - (t^2 + 2)(2t^2 - 3)]}{(t^4 - 3t^2 + 1)^2}$$
$$= \frac{2t(t^4 - 3t^2 + 1 - 2t^4 - 4t^2 + 3t^2 + 6)}{(t^4 - 3t^2 + 1)^2} = \frac{2t(-t^4 - 4t^2 + 7)}{(t^4 - 3t^2 + 1)^2}$$

14. $y = \dfrac{t}{(t - 1)^2} = \dfrac{t}{t^2 - 2t + 1} \overset{\text{QR}}{\Rightarrow}$

$$y' = \frac{(t^2 - 2t + 1)(1) - t(2t - 2)}{[(t - 1)^2]^2} = \frac{(t - 1)^2 - 2t(t - 1)}{(t - 1)^4} = \frac{(t - 1)[(t - 1) - 2t]}{(t - 1)^4} = \frac{-t - 1}{(t - 1)^3}$$

15. $y = (r^2 - 2r)e^r \overset{\text{PR}}{\Rightarrow} y' = (r^2 - 2r)(e^r) + e^r(2r - 2) = e^r(r^2 - 2r + 2r - 2) = e^r(r^2 - 2)$

16. $y = \dfrac{1}{s + ke^s} \overset{\text{QR}}{\Rightarrow} y' = \dfrac{(s + ke^s)(0) - (1)(1 + ke^s)}{(s + ke^s)^2} = -\dfrac{1 + ke^s}{(s + ke^s)^2}$

17. $y = \dfrac{v^3 - 2v\sqrt{v}}{v} = v^2 - 2\sqrt{v} = v^2 - 2v^{1/2} \Rightarrow y' = 2v - 2\left(\tfrac{1}{2}\right)v^{-1/2} = 2v - v^{-1/2}$.

We can change the form of the answer as follows: $2v - v^{-1/2} = 2v - \dfrac{1}{\sqrt{v}} = \dfrac{2v\sqrt{v} - 1}{\sqrt{v}} = \dfrac{2v^{3/2} - 1}{\sqrt{v}}$

18. $z = w^{3/2}(w + ce^w) = w^{5/2} + cw^{3/2}e^w \Rightarrow z' = \tfrac{5}{2}w^{3/2} + c\left(w^{3/2} \cdot e^w + e^w \cdot \tfrac{3}{2}w^{1/2}\right) = \tfrac{5}{2}w^{3/2} + \tfrac{1}{2}cw^{1/2}e^w(2w + 3)$

19. $f(t) = \dfrac{2t}{2 + \sqrt{t}} \overset{\text{QR}}{\Rightarrow} f'(t) = \dfrac{(2 + t^{1/2})(2) - 2t\left(\frac{1}{2}t^{-1/2}\right)}{(2 + \sqrt{t})^2} = \dfrac{4 + 2t^{1/2} - t^{1/2}}{(2 + \sqrt{t})^2} = \dfrac{4 + t^{1/2}}{(2 + \sqrt{t})^2}$ or $\dfrac{4 + \sqrt{t}}{(2 + \sqrt{t})^2}$

20. $g(t) = \dfrac{t - \sqrt{t}}{t^{1/3}} = \dfrac{t}{t^{1/3}} - \dfrac{t^{1/2}}{t^{1/3}} = t^{2/3} - t^{1/6} \Rightarrow g'(t) = \tfrac{2}{3}t^{-1/3} - \tfrac{1}{6}t^{-5/6}$

21. $f(x) = \dfrac{A}{B + Ce^x} \overset{\text{QR}}{\Rightarrow} f'(x) = \dfrac{(B + Ce^x) \cdot 0 - A(Ce^x)}{(B + Ce^x)^2} = -\dfrac{ACe^x}{(B + Ce^x)^2}$

22. $f(x) = \dfrac{1 - xe^x}{x + e^x} \overset{\text{QR}}{\Rightarrow} f'(x) = \dfrac{(x + e^x)(-xe^x)' - (1 - xe^x)(1 + e^x)}{(x + e^x)^2}$

$$\overset{\text{PR}}{\Rightarrow} f'(x) = \frac{(x + e^x)[-(xe^x + e^x \cdot 1)] - (1 + e^x - xe^x - xe^{2x})}{(x + e^x)^2}$$

$$= \frac{-x^2e^x - xe^x - xe^{2x} - e^{2x} - 1 - e^x + xe^x + xe^{2x}}{(x + e^x)^2} = \frac{-x^2e^x - e^{2x} - e^x - 1}{(x + e^x)^2}$$

23. $f(x) = \dfrac{x}{x + c/x} \Rightarrow f'(x) = \dfrac{(x + c/x)(1) - x(1 - c/x^2)}{\left(x + \dfrac{c}{x}\right)^2} = \dfrac{x + c/x - x + c/x}{\left(\dfrac{x^2 + c}{x}\right)^2} = \dfrac{2c/x}{\dfrac{(x^2 + c)^2}{x^2}} \cdot \dfrac{x^2}{x^2} = \dfrac{2cx}{(x^2 + c)^2}$

24. $f(x) = \dfrac{ax + b}{cx + d} \Rightarrow f'(x) = \dfrac{(cx + d)(a) - (ax + b)(c)}{(cx + d)^2} = \dfrac{acx + ad - acx - bc}{(cx + d)^2} = \dfrac{ad - bc}{(cx + d)^2}$

25. $f(x) = x^4e^x \Rightarrow f'(x) = x^4e^x + e^x \cdot 4x^3 = (x^4 + 4x^3)e^x$ $\left[\text{or } x^3e^x(x + 4)\right] \Rightarrow$

$$f''(x) = (x^4 + 4x^3)e^x + e^x(4x^3 + 12x^2) = (x^4 + 4x^3 + 4x^3 + 12x^2)e^x$$
$$= (x^4 + 8x^3 + 12x^2)e^x \quad \left[\text{or } x^2e^x(x + 2)(x + 6)\right]$$

26. $f(x) = x^{5/2}e^x \Rightarrow f'(x) = x^{5/2}e^x + e^x \cdot \frac{5}{2}x^{3/2} = \left(x^{5/2} + \frac{5}{2}x^{3/2}\right)e^x$ $\left[\text{or } \frac{1}{2}x^{3/2}e^x(2x + 5)\right] \Rightarrow$

$$f''(x) = \left(x^{5/2} + \tfrac{5}{2}x^{3/2}\right)e^x + e^x\left(\tfrac{5}{2}x^{3/2} + \tfrac{15}{4}x^{1/2}\right) = \left(x^{5/2} + 5x^{3/2} + \tfrac{15}{4}x^{1/2}\right)e^x \quad \left[\text{or } \tfrac{1}{4}x^{1/2}e^x(4x^2 + 20x + 15)\right]$$

27. $f(x) = \dfrac{x^2}{1 + 2x} \Rightarrow f'(x) = \dfrac{(1 + 2x)(2x) - x^2(2)}{(1 + 2x)^2} = \dfrac{2x + 4x^2 - 2x^2}{(1 + 2x)^2} = \dfrac{2x^2 + 2x}{(1 + 2x)^2} \Rightarrow$

$$f''(x) = \frac{(1 + 2x)^2(4x + 2) - (2x^2 + 2x)(1 + 4x + 4x^2)'}{[(1 + 2x)^2]^2} = \frac{2(1 + 2x)^2(2x + 1) - 2x(x + 1)(4 + 8x)}{(1 + 2x)^4}$$
$$= \frac{2(1 + 2x)[(1 + 2x)^2 - 4x(x + 1)]}{(1 + 2x)^4} = \frac{2(1 + 4x + 4x^2 - 4x^2 - 4x)}{(1 + 2x)^3} = \frac{2}{(1 + 2x)^3}$$

28. $f(x) = \dfrac{x}{x^2 - 1} \Rightarrow f'(x) = \dfrac{(x^2 - 1)(1) - x(2x)}{(x^2 - 1)^2} = \dfrac{x^2 - 1 - 2x^2}{(x^2 - 1)^2} = \dfrac{-x^2 - 1}{(x^2 - 1)^2} \Rightarrow$

$$f''(x) = \frac{(x^2 - 1)^2(-2x) - (-x^2 - 1)(x^4 - 2x^2 + 1)'}{[(x^2 - 1)^2]^2} = \frac{(x^2 - 1)^2(-2x) + (x^2 + 1)(4x^3 - 4x)}{(x^2 - 1)^4}$$
$$= \frac{(x^2 - 1)^2(-2x) + (x^2 + 1)(4x)(x^2 - 1)}{(x^2 - 1)^4} = \frac{(x^2 - 1)[(x^2 - 1)(-2x) + (x^2 + 1)(4x)]}{(x^2 - 1)^4}$$
$$= \frac{-2x^3 + 2x + 4x^3 + 4x}{(x^2 - 1)^3} = \frac{2x^3 + 6x}{(x^2 - 1)^3}$$

29. $y = \dfrac{2x}{x + 1} \Rightarrow y' = \dfrac{(x + 1)(2) - (2x)(1)}{(x + 1)^2} = \dfrac{2}{(x + 1)^2}$.

At $(1, 1)$, $y' = \frac{1}{2}$, and an equation of the tangent line is $y - 1 = \frac{1}{2}(x - 1)$, or $y = \frac{1}{2}x + \frac{1}{2}$.

30. $y = \dfrac{e^x}{x} \Rightarrow y' = \dfrac{x \cdot e^x - e^x \cdot 1}{x^2} = \dfrac{e^x(x - 1)}{x^2}$.

At $(1, e)$, $y' = 0$, and an equation of the tangent line is $y - e = 0(x - 1)$, or $y = e$.

31. $y = 2xe^x \Rightarrow y' = 2(x \cdot e^x + e^x \cdot 1) = 2e^x(x+1)$.

At $(0,0)$, $y' = 2e^0(0+1) = 2 \cdot 1 \cdot 1 = 2$, and an equation of the tangent line is $y - 0 = 2(x - 0)$, or $y = 2x$. The slope of the normal line is $-\frac{1}{2}$, so an equation of the normal line is $y - 0 = -\frac{1}{2}(x-0)$, or $y = -\frac{1}{2}x$.

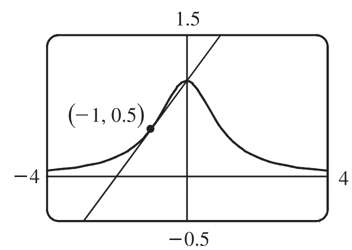
32. $y = \dfrac{\sqrt{x}}{x+1} \Rightarrow y' = \dfrac{(x+1)\left(\dfrac{1}{2\sqrt{x}}\right) - \sqrt{x}\,(1)}{(x+1)^2} = \dfrac{(x+1)-(2x)}{2\sqrt{x}\,(x+1)^2} = \dfrac{1-x}{2\sqrt{x}\,(x+1)^2}$.

At $(4, 0.4)$, $y' = \frac{-3}{100} = -0.03$, and an equation of the tangent line is $y - 0.4 = -0.03(x-4)$, or $y = -0.03x + 0.52$. The slope of the normal line is $\frac{100}{3}$, so an equation of the normal line is $y - 0.4 = \frac{100}{3}(x-4) \Leftrightarrow y = \frac{100}{3}x - \frac{400}{3} + \frac{2}{5} \Leftrightarrow$ $y = \frac{100}{3}x - \frac{1994}{15}$.

33. (a) $y = f(x) = \dfrac{1}{1+x^2} \Rightarrow$

$f'(x) = \dfrac{(1+x^2)(0) - 1(2x)}{(1+x^2)^2} = \dfrac{-2x}{(1+x^2)^2}$. So the slope of the tangent line at the point $\left(-1, \frac{1}{2}\right)$ is $f'(-1) = \dfrac{2}{2^2} = \frac{1}{2}$ and its equation is $y - \frac{1}{2} = \frac{1}{2}(x+1)$ or $y = \frac{1}{2}x + 1$.

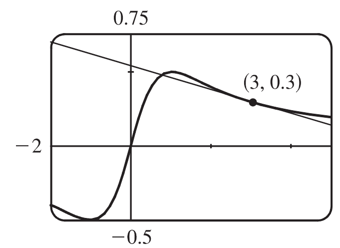
(b)



34. (a) $y = f(x) = \dfrac{x}{1+x^2} \Rightarrow$

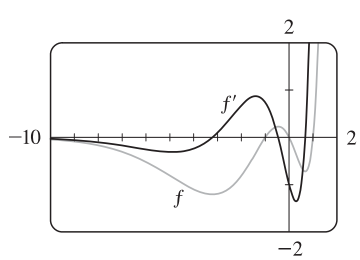
$f'(x) = \dfrac{(1+x^2)1 - x(2x)}{(1+x^2)^2} = \dfrac{1-x^2}{(1+x^2)^2}$. So the slope of the tangent line at the point $(3, 0.3)$ is $f'(3) = \frac{-8}{100}$ and its equation is $y - 0.3 = -0.08(x-3)$ or $y = -0.08x + 0.54$.

(b)



35. (a) $f(x) = (x^3 - x)e^x \Rightarrow f'(x) = (x^3-x)e^x + e^x(3x^2-1) = e^x(x^3 + 3x^2 - x - 1)$

(b)

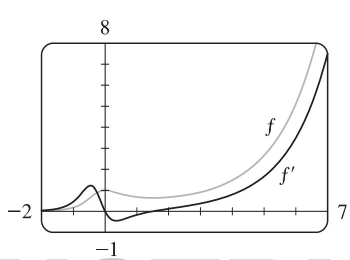


$f' = 0$ when f has a horizontal tangent line, f' is negative when f is decreasing, and f' is positive when f is increasing.

36. (a) $f(x) = \dfrac{e^x}{2x^2+x+1} \Rightarrow$

$$f'(x) = \frac{(2x^2+x+1)e^x - e^x(4x+1)}{(2x^2+x+1)^2} = \frac{e^x(2x^2+x+1-4x-1)}{(2x^2+x+1)^2} = \frac{e^x(2x^2-3x)}{(2x^2+x+1)^2}$$

(b)



$f' = 0$ when f has a horizontal tangent line, f' is negative when f is decreasing, and f' is positive when f is increasing.

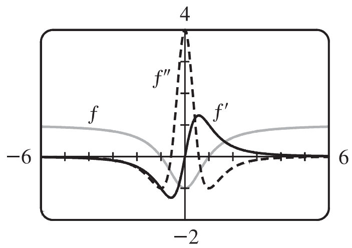
37. (a) $f(x) = \dfrac{x^2-1}{x^2+1} \quad\Rightarrow$

$$f'(x) = \frac{(x^2+1)(2x)-(x^2-1)(2x)}{(x^2+1)^2} = \frac{(2x)[(x^2+1)-(x^2-1)]}{(x^2+1)^2} = \frac{(2x)(2)}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2} \quad\Rightarrow$$

$$f''(x) = \frac{(x^2+1)^2(4)-4x(x^4+2x^2+1)'}{[(x^2+1)^2]^2} = \frac{4(x^2+1)^2-4x(4x^3+4x)}{(x^2+1)^4}$$

$$= \frac{4(x^2+1)^2-16x^2(x^2+1)}{(x^2+1)^4} = \frac{4(x^2+1)[(x^2+1)-4x^2]}{(x^2+1)^4} = \frac{4(1-3x^2)}{(x^2+1)^3}$$

(b)

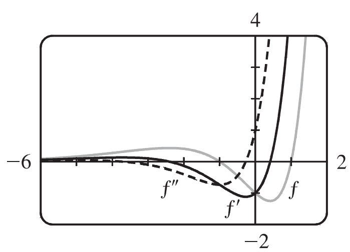


$f'=0$ when f has a horizontal tangent and $f''=0$ when f' has a horizontal tangent. f' is negative when f is decreasing and positive when f is increasing. f'' is negative when f' is decreasing and positive when f' is increasing. f'' is negative when f is concave down and positive when f is concave up.

38. (a) $f(x) = (x^2-1)e^x \quad\Rightarrow\quad f'(x) = (x^2-1)e^x + e^x(2x) = e^x(x^2+2x-1) \quad\Rightarrow$

$f''(x) = e^x(2x+2) + (x^2+2x-1)e^x = e^x(x^2+4x+1)$

(b)



We can see that our answers are plausible, since f has horizontal tangents where $f'(x)=0$, and f' has horizontal tangents where $f''(x)=0$.

39. $f(x) = \dfrac{x^2}{1+x} \quad\Rightarrow\quad f'(x) = \dfrac{(1+x)(2x)-x^2(1)}{(1+x)^2} = \dfrac{2x+2x^2-x^2}{(1+x)^2} = \dfrac{x^2+2x}{x^2+2x+1} \quad\Rightarrow$

$$f''(x) = \frac{(x^2+2x+1)(2x+2)-(x^2+2x)(2x+2)}{(x^2+2x+1)^2} = \frac{(2x+2)(x^2+2x+1-x^2-2x)}{[(x+1)^2]^2}$$

$$= \frac{2(x+1)(1)}{(x+1)^4} = \frac{2}{(x+1)^3},$$

so $f''(1) = \dfrac{2}{(1+1)^3} = \dfrac{2}{8} = \dfrac{1}{4}$.

40. $g(x) = \dfrac{x}{e^x} \quad\Rightarrow\quad g'(x) = \dfrac{e^x\cdot 1 - x\cdot e^x}{(e^x)^2} = \dfrac{e^x(1-x)}{(e^x)^2} = \dfrac{1-x}{e^x} \quad\Rightarrow$

$$g''(x) = \frac{e^x\cdot(-1)-(1-x)e^x}{(e^x)^2} = \frac{e^x[-1-(1-x)]}{(e^x)^2} = \frac{x-2}{e^x} \quad\Rightarrow$$

$$g'''(x) = \frac{e^x\cdot 1-(x-2)e^x}{(e^x)^2} = \frac{e^x[1-(x-2)]}{(e^x)^2} = \frac{3-x}{e^x} \quad\Rightarrow$$

$$g^{(4)}(x) = \frac{e^x\cdot(-1)-(3-x)e^x}{(e^x)^2} = \frac{e^x[-1-(3-x)]}{(e^x)^2} = \frac{x-4}{e^x}.$$

The pattern suggests that $g^{(n)}(x) = \dfrac{(x-n)(-1)^n}{e^x}$. (We could use mathematical induction to prove this formula.)

41. We are given that $f(5) = 1$, $f'(5) = 6$, $g(5) = -3$, and $g'(5) = 2$.

(a) $(fg)'(5) = f(5)g'(5) + g(5)f'(5) = (1)(2) + (-3)(6) = 2 - 18 = -16$

(b) $\left(\dfrac{f}{g}\right)'(5) = \dfrac{g(5)f'(5) - f(5)g'(5)}{[g(5)]^2} = \dfrac{(-3)(6) - (1)(2)}{(-3)^2} = -\dfrac{20}{9}$

(c) $\left(\dfrac{g}{f}\right)'(5) = \dfrac{f(5)g'(5) - g(5)f'(5)}{[f(5)]^2} = \dfrac{(1)(2) - (-3)(6)}{(1)^2} = 20$

42. We are given that $f(2) = -3$, $g(2) = 4$, $f'(2) = -2$, and $g'(2) = 7$.

(a) $h(x) = 5f(x) - 4g(x) \quad\Rightarrow\quad h'(x) = 5f'(x) - 4g'(x)$, so

$h'(2) = 5f'(2) - 4g'(2) = 5(-2) - 4(7) = -10 - 28 = -38.$

(b) $h(x) = f(x)g(x) \quad\Rightarrow\quad h'(x) = f(x)g'(x) + g(x)f'(x)$, so

$h'(2) = f(2)g'(2) + g(2)f'(2) = (-3)(7) + (4)(-2) = -21 - 8 = -29.$

(c) $h(x) = \dfrac{f(x)}{g(x)} \quad\Rightarrow\quad h'(x) = \dfrac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$, so

$h'(2) = \dfrac{g(2)f'(2) - f(2)g'(2)}{[g(2)]^2} = \dfrac{4(-2) - (-3)(7)}{4^2} = \dfrac{-8 + 21}{16} = \dfrac{13}{16}.$

(d) $h(x) = \dfrac{g(x)}{1 + f(x)} \quad\Rightarrow\quad h'(x) = \dfrac{[1 + f(x)]\,g'(x) - g(x)f'(x)}{[1 + f(x)]^2}$, so

$h'(2) = \dfrac{[1 + f(2)]\,g'(2) - g(2)\,f'(2)}{[1 + f(x)]^2} = \dfrac{[1 + (-3)](7) - 4(-2)}{[1 + (-3)]^2} = \dfrac{-14 + 8}{(-2)^2} = \dfrac{-6}{4} = -\dfrac{3}{2}.$

43. $f(x) = e^x g(x) \quad\Rightarrow\quad f'(x) = e^x g'(x) + g(x)e^x = e^x[g'(x) + g(x)]$. $f'(0) = e^0[g'(0) + g(0)] = 1(5 + 2) = 7$

44. $\dfrac{d}{dx}\left[\dfrac{h(x)}{x}\right] = \dfrac{xh'(x) - h(x)\cdot 1}{x^2} \quad\Rightarrow\quad \dfrac{d}{dx}\left[\dfrac{h(x)}{x}\right]_{x=2} = \dfrac{2h'(2) - h(2)}{2^2} = \dfrac{2(-3) - (4)}{4} = \dfrac{-10}{4} = -2.5$

45. (a) From the graphs of f and g, we obtain the following values: $f(1) = 2$ since the point $(1, 2)$ is on the graph of f; $g(1) = 1$ since the point $(1, 1)$ is on the graph of g; $f'(1) = 2$ since the slope of the line segment between $(0, 0)$ and $(2, 4)$ is $\dfrac{4 - 0}{2 - 0} = 2$; $g'(1) = -1$ since the slope of the line segment between $(-2, 4)$ and $(2, 0)$ is $\dfrac{0 - 4}{2 - (-2)} = -1$.

Now $u(x) = f(x)g(x)$, so $u'(1) = f(1)g'(1) + g(1)\,f'(1) = 2\cdot(-1) + 1\cdot 2 = 0$.

(b) $v(x) = f(x)/g(x)$, so $v'(5) = \dfrac{g(5)f'(5) - f(5)g'(5)}{[g(5)]^2} = \dfrac{2\left(-\frac{1}{3}\right) - 3\cdot\frac{2}{3}}{2^2} = \dfrac{-\frac{8}{3}}{4} = -\dfrac{2}{3}$

46. (a) $P(x) = F(x)\,G(x)$, so $P'(2) = F(2)\,G'(2) + G(2)\,F'(2) = 3\cdot\frac{2}{4} + 2\cdot 0 = \frac{3}{2}$.

(b) $Q(x) = F(x)/G(x)$, so $Q'(7) = \dfrac{G(7)\,F'(7) - F(7)\,G'(7)}{[G(7)]^2} = \dfrac{1\cdot\frac{1}{4} - 5\cdot\left(-\frac{2}{3}\right)}{1^2} = \dfrac{1}{4} + \dfrac{10}{3} = \dfrac{43}{12}$

47. (a) $y = xg(x) \quad\Rightarrow\quad y' = xg'(x) + g(x)\cdot 1 = xg'(x) + g(x)$

(b) $y = \dfrac{x}{g(x)} \quad\Rightarrow\quad y' = \dfrac{g(x)\cdot 1 - xg'(x)}{[g(x)]^2} = \dfrac{g(x) - xg'(x)}{[g(x)]^2}$

(c) $y = \dfrac{g(x)}{x} \quad\Rightarrow\quad y' = \dfrac{xg'(x) - g(x)\cdot 1}{(x)^2} = \dfrac{xg'(x) - g(x)}{x^2}$

48. (a) $y = x^2 f(x) \quad\Rightarrow\quad y' = x^2 f'(x) + f(x)(2x)$

(b) $y = \dfrac{f(x)}{x^2} \quad\Rightarrow\quad y' = \dfrac{x^2 f'(x) - f(x)(2x)}{(x^2)^2} = \dfrac{xf'(x) - 2f(x)}{x^3}$

(c) $y = \dfrac{x^2}{f(x)} \quad\Rightarrow\quad y' = \dfrac{f(x)(2x) - x^2 f'(x)}{[f(x)]^2}$

(d) $y = \dfrac{1 + xf(x)}{\sqrt{x}} \quad\Rightarrow$

$$y' = \frac{\sqrt{x}\,[xf'(x) + f(x)] - [1 + xf(x)]\,\dfrac{1}{2\sqrt{x}}}{\left(\sqrt{x}\,\right)^2}$$

$$= \frac{x^{3/2} f'(x) + x^{1/2} f(x) - \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{1/2} f(x)}{x} \cdot \frac{2x^{1/2}}{2x^{1/2}} = \frac{xf(x) + 2x^2 f'(x) - 1}{2x^{3/2}}$$

49. If $P(t)$ denotes the population at time t and $A(t)$ the average annual income, then $T(t) = P(t)A(t)$ is the total personal income. The rate at which $T(t)$ is rising is given by $T'(t) = P(t)A'(t) + A(t)P'(t) \quad\Rightarrow$

$$T'(1999) = P(1999)A'(1999) + A(1999)P'(1999) = (961{,}400)(\$1400/\text{yr}) + (\$30{,}593)(9200/\text{yr})$$
$$= \$1{,}345{,}960{,}000/\text{yr} + \$281{,}455{,}600/\text{yr} = \$1{,}627{,}415{,}600/\text{yr}$$

So the total personal income was rising by about \$1.627 billion per year in 1999.

The term $P(t)A'(t) \approx \$1.346$ billion represents the portion of the rate of change of total income due to the existing population's increasing income. The term $A(t)P'(t) \approx \$281$ million represents the portion of the rate of change of total income due to increasing population.

50. (a) $f(20) = 10{,}000$ means that when the price of the fabric is \$20/yard, 10,000 yards will be sold.

$f'(20) = -350$ means that as the price of the fabric increases past \$20/yard, the amount of fabric which will be sold is decreasing at a rate of 350 yards per (dollar per yard).

(b) $R(p) = pf(p) \quad\Rightarrow\quad R'(p) = pf'(p) + f(p)\cdot 1 \quad\Rightarrow\quad R'(20) = 20f'(20) + f(20)\cdot 1 = 20(-350) + 10{,}000 = 3000.$

This means that as the price of the fabric increases past \$20/yard, the total revenue is increasing at \$3000/(\$/yard). Note that the Product Rule indicates that we will lose \$7000/(\$/yard) due to selling less fabric, but this loss is more than made up for by the additional revenue due to the increase in price.

51. f is increasing when f' is positive. $f(x) = x^3 e^x \quad\Rightarrow\quad f'(x) = x^3 e^x + e^x(3x^2) = x^2 e^x(x+3)$. Now $x^2 \geq 0$ and $e^x > 0$ for all x, so $f'(x) > 0$ when $x + 3 > 0$ and $x \neq 0$; that is, when $x \in (-3, 0) \cup (0, \infty)$. So f is increasing on $(-3, \infty)$.

52. f is concave downward when f'' is negative. $f(x) = x^2 e^x \quad\Rightarrow\quad f'(x) = x^2 e^x + e^x(2x) \quad\Rightarrow$

$f''(x) = x^2 e^x + e^x(2x) + e^x(2) + (2x)e^x = e^x(x^2 + 2x + 2 + 2x) = e^x(x^2 + 4x + 2)$. Note that $e^x > 0$ for all x and $f''(x) = 0 \quad\Leftrightarrow\quad x = -2 \pm \sqrt{2}$. $f''(x) < 0$ when $x \in \left(-2 - \sqrt{2}, -2 + \sqrt{2}\,\right)$.

53. If $y = f(x) = \dfrac{x}{x+1}$, then $f'(x) = \dfrac{(x+1)(1) - x(1)}{(x+1)^2} = \dfrac{1}{(x+1)^2}$. When $x = a$, the equation of the tangent line is

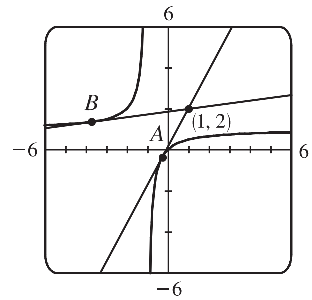
$y - \dfrac{a}{a+1} = \dfrac{1}{(a+1)^2}(x-a)$. This line passes through $(1,2)$ when $2 - \dfrac{a}{a+1} = \dfrac{1}{(a+1)^2}(1-a) \quad\Leftrightarrow$

$2(a+1)^2 - a(a+1) = 1 - a \quad\Leftrightarrow\quad 2a^2 + 4a + 2 - a^2 - a - 1 + a = 0 \quad\Leftrightarrow\quad a^2 + 4a + 1 = 0.$

The quadratic formula gives the roots of this equation as $a = \dfrac{-4 \pm \sqrt{4^2 - 4(1)(1)}}{2(1)} = \dfrac{-4 \pm \sqrt{12}}{2} = -2 \pm \sqrt{3}$,

so there are two such tangent lines. Since

$$f(-2 \pm \sqrt{3}) = \frac{-2 \pm \sqrt{3}}{-2 \pm \sqrt{3} + 1} = \frac{-2 \pm \sqrt{3}}{-1 \pm \sqrt{3}} \cdot \frac{-1 \mp \sqrt{3}}{-1 \mp \sqrt{3}}$$
$$= \frac{2 \pm 2\sqrt{3} \mp \sqrt{3} - 3}{1 - 3} = \frac{-1 \pm \sqrt{3}}{-2} = \frac{1 \mp \sqrt{3}}{2},$$



the lines touch the curve at $A\left(-2 + \sqrt{3}, \frac{1 - \sqrt{3}}{2}\right) \approx (-0.27, -0.37)$

and $B\left(-2 - \sqrt{3}, \frac{1 + \sqrt{3}}{2}\right) \approx (-3.73, 1.37)$.

54. $y = \dfrac{x-1}{x+1} \quad\Rightarrow\quad y' = \dfrac{(x+1)(1) - (x-1)(1)}{(x+1)^2} = \dfrac{2}{(x+1)^2}$. If the tangent intersects

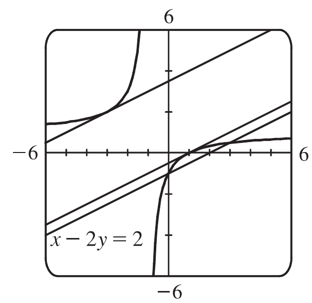
the curve when $x = a$, then its slope is $2/(a+1)^2$. But if the tangent is parallel to

$x - 2y = 2$, that is, $y = \frac{1}{2}x - 1$, then its slope is $\frac{1}{2}$. Thus, $\dfrac{2}{(a+1)^2} = \dfrac{1}{2} \quad\Rightarrow$

$(a+1)^2 = 4 \quad\Rightarrow\quad a + 1 = \pm 2 \quad\Rightarrow\quad a = 1$ or -3. When $a = 1$, $y = 0$ and the

equation of the tangent is $y - 0 = \frac{1}{2}(x-1)$ or $y = \frac{1}{2}x - \frac{1}{2}$. When $a = -3$, $y = 2$ and

the equation of the tangent is $y - 2 = \frac{1}{2}(x+3)$ or $y = \frac{1}{2}x + \frac{7}{2}$.



55. $R = \dfrac{f}{g} \quad\Rightarrow\quad R' = \dfrac{gf' - fg'}{g^2}$. For $f(x) = x - 3x^3 + 5x^5$, $f'(x) = 1 - 9x^2 + 25x^4$,

and for $g(x) = 1 + 3x^3 + 6x^6 + 9x^9$, $g'(x) = 9x^2 + 36x^5 + 81x^8$.

Thus, $R'(0) = \dfrac{g(0)f'(0) - f(0)g'(0)}{[g(0)]^2} = \dfrac{1 \cdot 1 - 0 \cdot 0}{1^2} = \dfrac{1}{1} = 1.$

56. $Q = \dfrac{f}{g} \quad\Rightarrow\quad Q' = \dfrac{gf' - fg'}{g^2}$. For $f(x) = 1 + x + x^2 + xe^x$, $f'(x) = 1 + 2x + xe^x + e^x$,

and for $g(x) = 1 - x + x^2 - xe^x$, $g'(x) = -1 + 2x - xe^x - e^x$.

Thus, $Q'(0) = \dfrac{g(0)f'(0) - f(0)g'(0)}{[g(0)]^2} = \dfrac{1 \cdot 2 - 1 \cdot (-2)}{1^2} = \dfrac{4}{1} = 4.$

57. (a) $(fgh)' = [(fg)h]' = (fg)'h + (fg)h' = (f'g + fg')h + (fg)h' = f'gh + fg'h + fgh'$

(b) Putting $f = g = h$ in part (a), we have $\dfrac{d}{dx}[f(x)]^3 = (fff)' = f'ff + ff'f + fff' = 3fff' = 3[f(x)]^2 f'(x)$.

(c) $\dfrac{d}{dx}(e^{3x}) = \dfrac{d}{dx}(e^x)^3 = 3(e^x)^2 e^x = 3e^{2x}e^x = 3e^{3x}$

58. (a) We use the Product Rule repeatedly: $F = fg \quad \Rightarrow \quad F' = f'g + fg' \quad \Rightarrow$
$F'' = (f''g + f'g') + (f'g' + fg'') = f''g + 2f'g' + fg''$.

(b) $F''' = f'''g + f''g' + 2\,(f''g' + f'g'') + f'g'' + fg''' = f'''g + 3f''g' + 3f'g'' + fg''' \quad \Rightarrow$
$F^{(4)} = f^{(4)}g + f'''g' + 3\,(f'''g' + f''g'') + 3\,(f''g'' + f'g''') + f'g''' + fg^{(4)}$
$\quad = f^{(4)}g + 4f'''g' + 6f''g'' + 4f'g''' + fg^{(4)}$

(c) By analogy with the Binomial Theorem, we make the guess:

$$F^{(n)} = f^{(n)}g + nf^{(n-1)}g' + \binom{n}{2}f^{(n-2)}g'' + \cdots + \binom{n}{k}f^{(n-k)}g^{(k)} + \cdots + nf'g^{(n-1)} + fg^{(n)},$$

where $\displaystyle\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$.

59. For $f(x) = x^2e^x$, $f'(x) = x^2e^x + e^x(2x) = e^x(x^2 + 2x)$. Similarly, we have

$$f''(x) = e^x(x^2 + 4x + 2)$$
$$f'''(x) = e^x(x^2 + 6x + 6)$$
$$f^{(4)}(x) = e^x(x^2 + 8x + 12)$$
$$f^{(5)}(x) = e^x(x^2 + 10x + 20)$$

It appears that the coefficient of x in the quadratic term increases by 2 with each differentiation. The pattern for the constant terms seems to be $0 = 1 \cdot 0$, $2 = 2 \cdot 1$, $6 = 3 \cdot 2$, $12 = 4 \cdot 3$, $20 = 5 \cdot 4$. So a reasonable guess is that $f^{(n)}(x) = e^x[x^2 + 2nx + n(n-1)]$.

Proof: Let S_n be the statement that $f^{(n)}(x) = e^x[x^2 + 2nx + n(n-1)]$.

1. S_1 is true because $f'(x) = e^x(x^2 + 2x)$.
2. Assume that S_k is true; that is, $f^{(k)}(x) = e^x[x^2 + 2kx + k(k-1)]$. Then

$$\begin{aligned} f^{(k+1)}(x) &= \frac{d}{dx}\left[f^{(k)}(x)\right] = e^x(2x + 2k) + [x^2 + 2kx + k(k-1)]e^x \\ &= e^x[x^2 + (2k+2)x + (k^2 + k)] = e^x[x^2 + 2(k+1)x + (k+1)k] \end{aligned}$$

   This shows that S_{k+1} is true.

3. Therefore, by mathematical induction, S_n is true for all n; that is, $f^{(n)}(x) = e^x[x^2 + 2nx + n(n-1)]$ for every positive integer n.

60. (a) $\displaystyle\frac{d}{dx}\left(\frac{1}{g(x)}\right) = \frac{g(x) \cdot \frac{d}{dx}(1) - 1 \cdot \frac{d}{dx}[g(x)]}{[g(x)]^2}$ [Quotient Rule] $\displaystyle = \frac{g(x) \cdot 0 - 1 \cdot g'(x)}{[g(x)]^2} = \frac{0 - g'(x)}{[g(x)]^2} = -\frac{g'(x)}{[g(x)]^2}$

(b) $\displaystyle y = \frac{1}{s + ke^s} \quad \Rightarrow \quad y' = -\frac{1 + ke^s}{(s + ke^s)^2}$

(c) $\displaystyle\frac{d}{dx}(x^{-n}) = \frac{d}{dx}\left(\frac{1}{x^n}\right) = -\frac{(x^n)'}{(x^n)^2}$ [by the Reciprocal Rule] $\displaystyle = -\frac{nx^{n-1}}{x^{2n}} = -nx^{n-1-2n} = -nx^{-n-1}$

3.3 Derivatives of Trigonometric Functions

1. $f(x) = 3x^2 - 2\cos x \quad \Rightarrow \quad f'(x) = 6x - 2(-\sin x) = 6x + 2\sin x$

2. $y = 2\csc x + 5\cos x \quad \Rightarrow \quad y' = -2\csc x\, \cot x - 5\sin x$

3. $f(x) = \sin x + \frac{1}{2}\cot x \quad \Rightarrow \quad f'(x) = \cos x - \frac{1}{2}\csc^2 x$

4. $f(x) = \sqrt{x}\sin x \quad \Rightarrow \quad f'(x) = \sqrt{x}\cos x + \sin x\left(\frac{1}{2}x^{-1/2}\right) = \sqrt{x}\cos x + \dfrac{\sin x}{2\sqrt{x}}$

5. $y = \sec\theta\,\tan\theta \quad \Rightarrow \quad y' = \sec\theta\,(\sec^2\theta) + \tan\theta\,(\sec\theta\tan\theta) = \sec\theta\,(\sec^2\theta + \tan^2\theta)$. Using the identity $1 + \tan^2\theta = \sec^2\theta$, we can write alternative forms of the answer as $\sec\theta\,(1 + 2\tan^2\theta)$ or $\sec\theta\,(2\sec^2\theta - 1)$.

6. $g(\theta) = e^\theta(\tan\theta - \theta) \quad \Rightarrow \quad g'(\theta) = e^\theta(\sec^2\theta - 1) + (\tan\theta - \theta)e^\theta = e^\theta(\sec^2\theta - 1 + \tan\theta - \theta)$

7. $y = c\cos t + t^2\sin t \quad \Rightarrow \quad y' = c(-\sin t) + t^2(\cos t) + \sin t\,(2t) = -c\sin t + t(t\cos t + 2\sin t)$

8. $f(t) = \dfrac{\cot t}{e^t} \quad \Rightarrow \quad f'(t) = \dfrac{e^t(-\csc^2 t) - (\cot t)e^t}{(e^t)^2} = \dfrac{e^t(-\csc^2 t - \cot t)}{(e^t)^2} = -\dfrac{\csc^2 t + \cot t}{e^t}$

9. $y = \dfrac{x}{2 - \tan x} \quad \Rightarrow \quad y' = \dfrac{(2 - \tan x)(1) - x(-\sec^2 x)}{(2 - \tan x)^2} = \dfrac{2 - \tan x + x\sec^2 x}{(2 - \tan x)^2}$

10. $y = \dfrac{1 + \sin x}{x + \cos x} \quad \Rightarrow$

$$\begin{aligned} y' &= \frac{(x + \cos x)(\cos x) - (1 + \sin x)(1 - \sin x)}{(x + \cos x)^2} = \frac{x\cos x + \cos^2 x - (1 - \sin^2 x)}{(x + \cos x)^2} \\ &= \frac{x\cos x + \cos^2 x - (\cos^2 x)}{(x + \cos x)^2} = \frac{x\cos x}{(x + \cos x)^2} \end{aligned}$$

11. $f(\theta) = \dfrac{\sec\theta}{1 + \sec\theta} \quad \Rightarrow$

$$f'(\theta) = \frac{(1 + \sec\theta)(\sec\theta\,\tan\theta) - (\sec\theta)(\sec\theta\,\tan\theta)}{(1 + \sec\theta)^2} = \frac{(\sec\theta\,\tan\theta)\,[(1 + \sec\theta) - \sec\theta]}{(1 + \sec\theta)^2} = \frac{\sec\theta\,\tan\theta}{(1 + \sec\theta)^2}$$

12. $y = \dfrac{1 - \sec x}{\tan x} \quad \Rightarrow$

$$y' = \frac{\tan x\,(-\sec x\,\tan x) - (1 - \sec x)(\sec^2 x)}{(\tan x)^2} = \frac{\sec x\left(-\tan^2 x - \sec x + \sec^2 x\right)}{\tan^2 x} = \frac{\sec x\,(1 - \sec x)}{\tan^2 x}$$

13. Using Exercise 3.2.57(a), $f(x) = xe^x\csc x \quad \Rightarrow$

$$\begin{aligned} f'(x) &= (x)'e^x\csc x + x(e^x)'\csc x + xe^x(\csc x)' = 1e^x\csc x + xe^x\csc x + xe^x(-\cot x\,\csc x) \\ &= e^x\csc x\,(1 + x - x\cot x) \end{aligned}$$

14. Using Exercise 3.2.57(a), $f(x) = x^2\sin x\,\tan x \quad \Rightarrow$

$$\begin{aligned} f'(x) &= (x^2)'\sin x\,\tan x + x^2(\sin x)'\tan x + x^2\sin x\,(\tan x)' = 2x\sin x\,\tan x + x^2\cos x\,\tan x + x^2\sin x\,\sec^2 x \\ &= 2x\sin x\,\tan x + x^2\sin x + x^2\sin x\,\sec^2 x = x\sin x\,(2\tan x + x + x\sec^2 x). \end{aligned}$$

15. $\dfrac{d}{dx}(\csc x) = \dfrac{d}{dx}\left(\dfrac{1}{\sin x}\right) = \dfrac{(\sin x)(0) - 1(\cos x)}{\sin^2 x} = \dfrac{-\cos x}{\sin^2 x} = -\dfrac{1}{\sin x} \cdot \dfrac{\cos x}{\sin x} = -\csc x\,\cot x$

16. $\dfrac{d}{dx}(\sec x) = \dfrac{d}{dx}\left(\dfrac{1}{\cos x}\right) = \dfrac{(\cos x)(0) - 1(-\sin x)}{\cos^2 x} = \dfrac{\sin x}{\cos^2 x} = \dfrac{1}{\cos x} \cdot \dfrac{\sin x}{\cos x} = \sec x\,\tan x$

17. $\dfrac{d}{dx}(\cot x) = \dfrac{d}{dx}\left(\dfrac{\cos x}{\sin x}\right) = \dfrac{(\sin x)(-\sin x) - (\cos x)(\cos x)}{\sin^2 x} = -\dfrac{\sin^2 x + \cos^2 x}{\sin^2 x} = -\dfrac{1}{\sin^2 x} = -\csc^2 x$

18. $f(x) = \cos x \quad \Rightarrow$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} = \lim_{h\to 0} \frac{\cos x\,\cos h - \sin x\,\sin h - \cos x}{h} \\ &= \lim_{h\to 0}\left(\cos x\,\frac{\cos h - 1}{h} - \sin x\,\frac{\sin h}{h}\right) = \cos x\,\lim_{h\to 0}\frac{\cos h - 1}{h} - \sin x\,\lim_{h\to 0}\frac{\sin h}{h} \\ &= (\cos x)(0) - (\sin x)(1) = -\sin x \end{aligned}$$

19. $y = \sec x \quad \Rightarrow \quad y' = \sec x\,\tan x$, so $y'\left(\frac{\pi}{3}\right) = \sec\frac{\pi}{3}\tan\frac{\pi}{3} = 2\sqrt{3}$. An equation of the tangent line to the curve $y = \sec x$ at the point $\left(\frac{\pi}{3}, 2\right)$ is $y - 2 = 2\sqrt{3}\left(x - \frac{\pi}{3}\right)$ or $y = 2\sqrt{3}\,x + 2 - \frac{2}{3}\sqrt{3}\,\pi$.

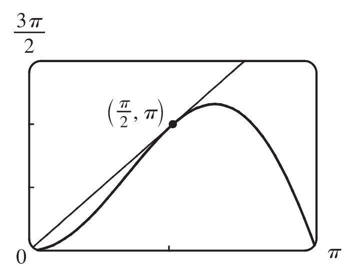
20. $y = e^x \cos x \quad \Rightarrow \quad y' = e^x(-\sin x) + (\cos x)e^x = e^x(\cos x - \sin x) \quad \Rightarrow$ the slope of the tangent line at $(0, 1)$ is $e^0(\cos 0 - \sin 0) = 1(1 - 0) = 1$ and an equation is $y - 1 = 1(x - 0)$ or $y = x + 1$.

21. $y = x + \cos x \quad \Rightarrow \quad y' = 1 - \sin x$. At $(0, 1)$, $y' = 1$, and an equation of the tangent line is $y - 1 = 1(x - 0)$, or $y = x + 1$.

22. $y = \dfrac{1}{\sin x + \cos x} \quad \Rightarrow \quad y' = -\dfrac{\cos x - \sin x}{(\sin x + \cos x)^2}$ [Reciprocal Rule]. At $(0, 1)$, $y' = -\dfrac{1 - 0}{(0 + 1)^2} = -1$, and an equation of the tangent line is $y - 1 = -1(x - 0)$, or $y = -x + 1$.

23. (a) $y = 2x\sin x \quad \Rightarrow \quad y' = 2(x\cos x + \sin x \cdot 1)$. At $\left(\frac{\pi}{2}, \pi\right)$, $y' = 2\left(\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\right) = 2(0 + 1) = 2$, and an equation of the tangent line is $y - \pi = 2\left(x - \frac{\pi}{2}\right)$, or $y = 2x$.

(b)



24. (a) $y = 3x + 6\cos x \quad \Rightarrow \quad y' = 3 - 6\sin x$. At $\left(\frac{\pi}{3}, \pi + 3\right)$, $y' = 3 - 6\sin\frac{\pi}{3} = 3 - 6\,\frac{\sqrt{3}}{2} = 3 - 3\sqrt{3}$, and an equation of the tangent line is $y - (\pi + 3) = \left(3 - 3\sqrt{3}\right)\left(x - \frac{\pi}{3}\right)$, or $y = \left(3 - 3\sqrt{3}\right) + 3 + \pi\sqrt{3}$.

(b)