Chapter 2 Solutions

Problem 2.1

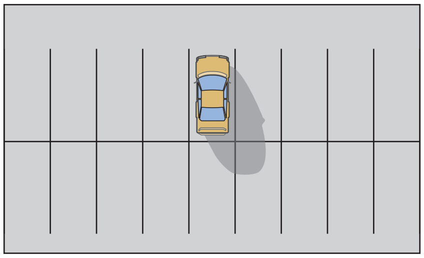
If $\vec{v}_{\text{avg}}$ is the average velocity of a point P over a given time interval, is $|\vec{v}_{\text{avg}}|$, the magnitude of the average velocity, equal to the average speed of P over the time interval in question?

Solution

In general, $|\vec{v}_{\text{avg}}|$ is not equal to v_{avg}. To see this, consider a car that drives along a loop of length ΔL over a time interval Δt such that the departure and arrival points coincide. Since the departure and arrival positions coincide, $\vec{v}_{\text{avg}}$ is equal to zero. This, implies that $|\vec{v}_{\text{avg}}|$ is also equal to zero. By contrast, the average speed will be different from zero because it is equal to the ratio $\Delta L/\Delta t$.

Problem 2.2

A car is seen parked in a given parking space at 8:00 A.M. on a Monday morning and is then seen parked in the same spot the next morning at the same time. What is the displacement of the car between the two observations? What is the distance traveled by the car during the two observations?

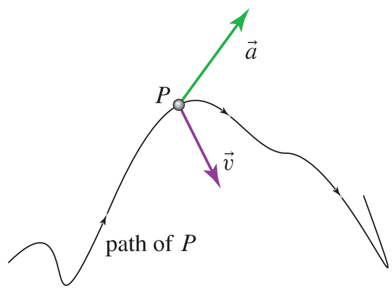


Solution

The displacement is equal to zero because the difference in position over the time interval considered is equal to zero. As far as the distance traveled is concerned, we cannot determine it from the information given. To determine the distance traveled we would need to know the position of the car at every time instant during the time interval considered instead of just at the beginning and end of the time interval in question.

Problem 2.3

Is it possible for the vector $\vec{v}$ shown to represent the velocity of the point P?

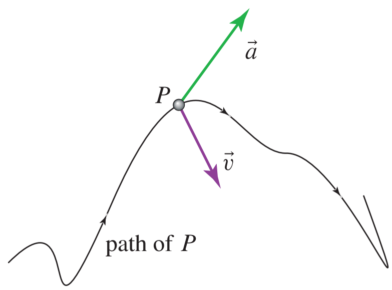


Solution

No, because the vector $\vec{v}$ shown is not tangent to the path at point P, which it must.

Problem 2.4

Is it possible for the vector $\vec{a}$ shown to be the acceleration of the point P?



Solution

No, because $\vec{a}$ does not point toward the concave side of the trajectory of P, which it must.

Problem 2.5

Two points P and Q happen to go by the same location in space (though at different times).

(a) What must the paths of P and Q have in common if, at the location in question, P and Q have identical speeds?

(b) What must the paths of P and Q have in common if, at the location in question, P and Q have identical velocities?

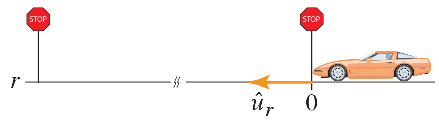
Solution

Part (a) In the first case, what we can expect the paths to share is that point in space which, at different instants, is occupied by P and Q.

Part (b) In the second case, the paths in question will not only share a point, like in the previous case, but will also have the same tangent line at that point, since the velocity vector is *always* tangent to the path.

Problem 2.6

The position of a car traveling between two stop signs along a straight city block is given by $r = [9t - (45/2)\sin(2t/5)]$ m, where t denotes time (in seconds), and where the argument of the sine function is measured in radians. Compute the displacement of the car between 2.1 and 3.7 s, as well as between 11.1 and 12.7 s. For each of these time intervals compute the average velocity.



Solution

We denote the quantities computed between 2.1 and 3.7 s by subscript 1, and between 11.1 and 12.7 s by subscript 2.

Using the definition of displacement, we have

$$\Delta\vec{r}_1 = [r(3.7\,\text{s}) - r(2.1\,\text{s})]\,\hat{u}_r \quad \text{and} \quad \Delta\vec{r}_2 = [r(12.7\,\text{s}) - r(11.1\,\text{s})]\,\hat{u}_r. \tag{1}$$

Applying the definition of average velocity we have

$$(\vec{v}_{\text{avg}})_1 = \frac{r(3.7\,\text{s}) - r(2.1\,\text{s})}{3.7\,\text{s} - 2.1\,\text{s}}\,\hat{u}_r \quad \text{and} \quad (\vec{v}_{\text{avg}})_2 = \frac{r(12.7\,\text{s}) - r(11.1\,\text{s})}{12.7\,\text{s} - 11.1\,\text{s}}\,\hat{u}_r. \tag{2}$$

Using the expression for $r(t)$ in the problem statement, the expressions in Eqs. (1) and (2) can be evaluated to obtain

$$\boxed{\Delta\vec{r}_1 = 8.747\,\hat{u}_r\,\text{m} \quad \text{and} \quad \Delta\vec{r}_2 = 13.73\,\hat{u}_r\,\text{m},}$$

and

$$\boxed{(\vec{v}_{\text{avg}})_1 = 5.467\,\hat{u}_r\,\text{m/s} \quad \text{and} \quad (\vec{v}_{\text{avg}})_2 = 8.579\,\hat{u}_r\,\text{m/s}.}$$

Problem 2.7

A city bus covers a 15 km route in 45 min. If the initial departure and final arrival points coincide, determine the average velocity and the average speed of the bus over the entire duration of the ride. Express the answers in m/s.

Solution

Since the departure and arrival points coincide, the displacement vector over the duration of the ride is equal to zero. This implies that the average velocity of the bus over the duration of the ride is equal to zero:

$$\boxed{\vec{v}_{\text{avg}} = \vec{0}.}$$

Letting d denote the total distance traveled by the bus and letting Δt denote the time to travel the distance d, the average speed over the duration of the ride is

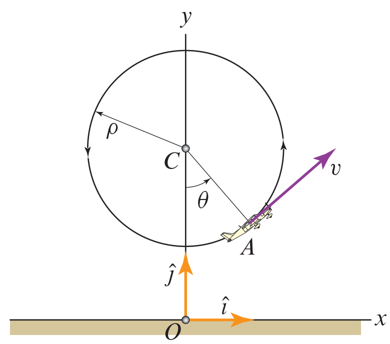
$$v_{\text{avg}} = \frac{d}{\Delta t}. \tag{1}$$

Since $d = 15\,\text{km} = 15\times10^3\,\text{m}$ and $\Delta t = 45\,\text{min} = 2700\,\text{s}$, we can evaluate the above expression to obtain

$$\boxed{v_{\text{avg}} = 5.556\,\text{m/s}.}$$

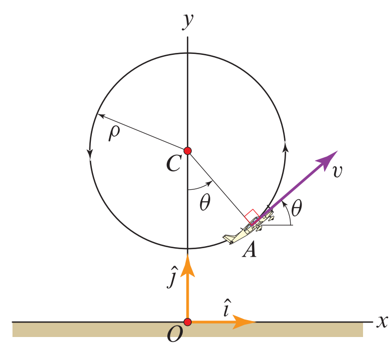
Problem 2.8

An airplane A is performing a loop with constant radius ρ. When $\theta = 120°$, the speed of the airplane is $v_0 = 210$ mph. Modeling the airplane as a point, find the velocity of the airplane at this instant using the component system shown. Express your answer in ft/s.



Solution

The velocity of the airplane is tangent to the path. In this case the path is a circle centered at C. Referring to the figure at the right, we see that the tangent to the path at A is perpendicular to the radial line connecting C to A. In turn, this means that, for a generic value of θ, the velocity vector forms an angle θ with the horizontal direction and can be represented as



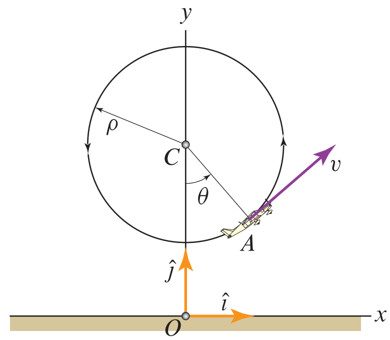
$$\vec{v} = v(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}), \tag{1}$$

where $v = |\vec{v}|$ is the speed. For $\theta = 120°$ we have $v = v_0 = 210\,\text{mph} = 210\,\frac{5280}{3600}$ ft/s. Therefore, for $\theta = 120°$ we can evaluate Eq. (1) to obtain

$$\vec{v} = (-154.0\,\hat{\imath} + 266.7\,\hat{\jmath})\ \text{ft/s}.$$

Problem 2.9

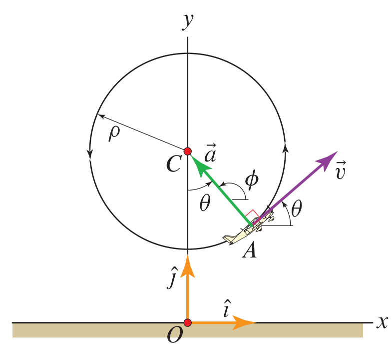
An airplane A is performing a loop with constant radius $\rho = 300\,\text{m}$. From elementary physics, we know that the acceleration of a point in uniform circular motion (i.e., circular motion at constant speed) is directed toward the center of the circle and has magnitude equal to v^2/ρ, where v is the speed. Assuming that A can maintain its speed constant and using the component system shown, provide the expressions of the velocity and acceleration of A when $\theta = 40°$ and $|\vec{a}| = 3g$, where $\vec{a}$ is the acceleration of A and g is the acceleration due to gravity.



Solution

From the problem statement, the relation between the speed and the magnitude of the acceleration in this problem is

$$|\vec{a}| = \frac{v^2}{\rho}. \tag{1}$$



Setting the left-hand side of Eq. (1) equal to $3g$ and solving for v, we have

$$v = \sqrt{3g\rho}. \tag{2}$$

The velocity of the airplane is tangent to the path. In this case the path is a circle centered at C. Referring to the figure at the right, we see that the tangent to the path at A is perpendicular to the radial line connecting C to A. This means that, for a generic value of θ, the velocity vector forms an angle θ with the horizontal direction and can be represented as

$$\vec{v} = v(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}) \quad \Rightarrow \quad \vec{v} = \sqrt{3g\rho}\,(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}), \tag{3}$$

where we have used the result in Eq. (2).

Letting ϕ denote the angle formed by the acceleration and the horizontal direction, since the acceleration is directed toward the center of the loop, we have that

$$\phi = \theta + 90°. \tag{4}$$

Hence, recalling that the magnitude of the acceleration is $3g$, the acceleration can be written as follows:

$$\vec{a} = 3g(\cos\phi\,\hat{\imath} + \sin\phi\,\hat{\jmath}) \quad \Rightarrow \quad \vec{a} = 3g(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}), \tag{5}$$

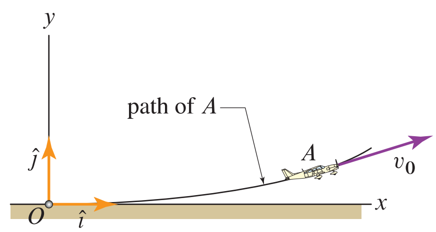
where we have used Eq. (4) and the trigonometric identities $\cos(\theta+90°) = -\sin\theta$ and $\sin(\theta+90°) = \cos\theta$.

Since $\theta = 40°$, $\rho = 300\,\text{m}$, and $g = 9.81\,\text{m/s}^2$, we can evaluate the second of Eqs. (3) and the second of Eqs. (5) to obtain

$$\vec{v} = (71.98\,\hat{\imath} + 60.40\,\hat{\jmath})\,\text{m/s} \quad \text{and} \quad \vec{a} = (-18.92\,\hat{\imath} + 22.54\,\hat{\jmath})\,\text{m/s}^2.$$

Problem 2.10

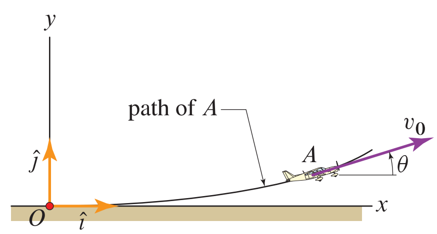
An airplane takes off as shown following a trajectory described by equation $y = \kappa x^2$, where $\kappa = 2 \times 10^{-4}\,\text{ft}^{-1}$. When $x = 1200\,\text{ft}$, the speed of the plane is $v_0 = 110\,\text{mph}$. Using the component system shown, provide the expression for the velocity of the airplane when $x = 1200\,\text{ft}$. Express your answer in ft/s.



Solution

Referring to the figure at the right, we denote by θ the angle formed by the velocity with the horizontal direction. We write the velocity as follows:

$$\vec{v} = v_0(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}). \tag{1}$$



Since the velocity is tangent to the path and since the trajectory of the airplane is known as a function of x, namely $y = \kappa x^2$, the tangent of θ coincides with the derivative of the trajectory with respect to x:

$$\tan\theta = \frac{dy}{dx} = 2\kappa x \quad \Rightarrow \quad \cos\theta = \frac{1}{\sqrt{1+4\kappa^2x^2}} \quad \text{and} \quad \sin\theta = \frac{2\kappa x}{\sqrt{1+4\kappa^2x^2}}, \tag{2}$$

where we have used the trigonometric identities $\cos\theta = 1/\sqrt{1+\tan^2\theta}$ (for $0 \le \theta < 90^\circ$) and $\sin\theta = \tan\theta\cos\theta$. Substituting the last two of Eqs. (2) into Eq. (1), we then have

$$\vec{v} = \frac{v_0}{\sqrt{1+4\kappa^2x^2}}\,\hat{\imath} + \frac{2\kappa x v_0}{\sqrt{1+4\kappa^2x^2}}\,\hat{\jmath}. \tag{3}$$

For $v_0 = 110\,\text{mph} = 110\,\frac{5280}{3600}\,\text{ft/s}$, $\kappa = 2\times10^{-4}\,\text{ft}^{-1}$, and $x = 1200\,\text{ft}$, we evaluate Eq. (3) to obtain

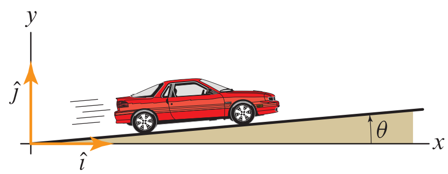
$$\boxed{\vec{v} = (145.4\,\hat{\imath} + 69.81\,\hat{\jmath})\,\text{ft/s}.}$$

Problem 2.11

The position of a car as a function of time t, with $t > 0$ and expressed in seconds, is

$$\vec{r}(t) = [(5.98t^2 + 0.139t^3 - 0.0149t^4)\,\hat{\imath} + (0.523t^2 + 0.0122t^3 - 0.00131t^4)\,\hat{\jmath}]\,\text{ft}.$$

Determine the velocity, speed, and acceleration of the car for $t = 15\,\text{s}$.



Solution

The velocity is obtained by taking the derivative of the position with respect to time. This gives

$$\vec{v} = [(11.96t + 0.4170t^2 - 0.05960t^3)\,\hat{\imath} + (1.046t + 0.03660t^2 - 0.005240t^3)\,\hat{\jmath}]\,\text{ft/s}. \tag{1}$$

The speed is the magnitude of the velocity. Using Eq. (1), we have

$$v = \sqrt{(11.96t + 0.4170t^2 - 0.05960t^3)^2 + (1.046t + 0.03660t^2 - 0.005240t^3)^2}\,\text{ft/s}, \tag{2}$$

which can be simplified to

$$v = t\sqrt{144.1 + 10.05t - 1.261t^2 - 0.05009t^3 + 0.003580t^4}\,\text{ft/s}. \tag{3}$$

The acceleration is computed by taking the derivative of the velocity with respect to time. Using Eq. (1), we have

$$\vec{a} = [(11.96 + 0.8340t - 0.1788t^2)\,\hat{\imath} + (1.046 + 0.07320t - 0.01572t^2)\,\hat{\jmath}]\,\text{ft/s}^2. \tag{4}$$

Evaluating Eqs. (1), (3), and (4) for $t = 15\,\text{s}$, we have

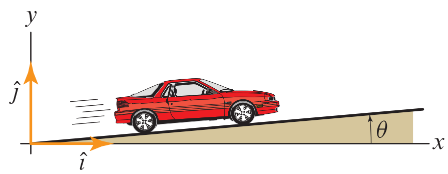
$$\vec{v}(15\,\text{s}) = (72.08\,\hat{\imath} + 6.240\,\hat{\jmath})\,\text{ft/s}, \quad v(15\,\text{s}) = 72.34\,\text{ft/s}, \quad \vec{a}(15\,\text{s}) = -(15.76\,\hat{\imath} + 1.393\,\hat{\jmath})\,\text{ft/s}^2.$$

Problem 2.12

The position of a car as a function of time t, with $t > 0$ and expressed in seconds, is

$$\vec{r}(t) = [12.3(t + 1.54e^{-0.65t})\,\hat{\imath} + 2.17(t + 1.54e^{-0.65t})\,\hat{\jmath}]\,\mathrm{m}.$$

Find the difference between the average velocity over the time interval $0 \le t \le 2\,\mathrm{s}$ and the true velocity computed at the midpoint of the interval, i.e., at $t = 1\,\mathrm{s}$. Repeat the calculation for the time interval $8\,\mathrm{s} \le t \le 10\,\mathrm{s}$. Explain why the difference between the average velocity and the true velocity over the time interval $0 \le t \le 2\,\mathrm{s}$ is not equal to that over $8\,\mathrm{s} \le t \le 10\,\mathrm{s}$.



Solution

The velocity is obtained by taking the derivative of the position with respect to time. This gives

$$\vec{v} = [12.30(1 - 1.001e^{-0.6500t})\,\hat{\imath} + 2.170(1 - 1.001e^{-0.6500t})\,\hat{\jmath}]\,\mathrm{m/s}. \tag{1}$$

Using Eq. (1), for $t = 1\,\mathrm{s}$ we have

$$\vec{v}(1\,\mathrm{s}) = (5.872\,\hat{\imath} + 1.036\,\hat{\jmath})\,\mathrm{m/s}. \tag{2}$$

The average velocity over the time interval $0 \le t \le 2\,\mathrm{s}$, which we will denote by $(\vec{v}_{\mathrm{avg}})_1$, is

$$(\vec{v}_{\mathrm{avg}})_1 = \frac{\vec{r}(2\,\mathrm{s}) - \vec{r}(0)}{2\,\mathrm{s}} = (5.410\,\hat{\imath} + 0.9545\,\hat{\jmath})\,\mathrm{m/s}. \tag{3}$$

Letting $\Delta\vec{v}_1 = (\vec{v}_{\mathrm{avg}})_1 - \vec{v}(1\,\mathrm{s})$, using the results in Eqs. (2) and (3), we have

$$\boxed{\Delta\vec{v}_1 = -(0.4623\,\hat{\imath} + 0.08155\,\hat{\jmath})\,\mathrm{m/s}.}$$

Using Eq. (1), for $t = 9\,\mathrm{s}$ we have

$$\vec{v}(9\,\mathrm{s}) = (12.26\,\hat{\imath} + 2.164\,\hat{\jmath})\,\mathrm{m/s}. \tag{4}$$

The average velocity over the time interval $8\,\mathrm{s} \le t \le 10\,\mathrm{s}$, which we will denote by $(\vec{v}_{\mathrm{avg}})_2$, is

$$(\vec{v}_{\mathrm{avg}})_2 = \frac{\vec{r}(10\,\mathrm{s}) - \vec{r}(8\,\mathrm{s})}{2\,\mathrm{s}} = (12.26\,\hat{\imath} + 2.163\,\hat{\jmath})\,\mathrm{m/s}. \tag{5}$$

Letting $\Delta\vec{v}_2 = (\vec{v}_{\mathrm{avg}})_2 - \vec{v}(9\,\mathrm{s})$, using the results in Eqs. (4) and (5), we have

$$\boxed{\Delta\vec{v}_2 = -(0.002550\,\hat{\imath} + 0.0004499\,\hat{\jmath})\,\mathrm{m/s}.}$$

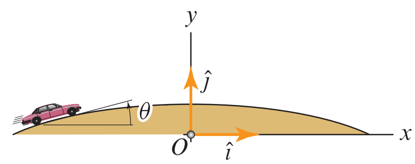
We observe that $\Delta\vec{v}_1 \ne \Delta\vec{v}_2$. This is due to the fact that, in general, the approximation of the true velocity by the average velocity over a given time interval is a function of the time interval in question, i.e., $\Delta\vec{v}$ changes depending on the interval on which it is computed.

Problem 2.13

The position of a car as a function of time t, with $t > 0$ and expressed in seconds, is

$$\vec{r}(t) = [(66t - 120)\,\hat{\imath} + (1.2 + 31.7t - 8.71t^2)\,\hat{\jmath}]\,\text{ft}.$$

If the speed limit is 55 mph, determine the time at which the car will exceed this limit.



Solution

In order to solve the problem we need to determine the speed of the car. So, we first determine the velocity of the car and then we compute its magnitude.

The velocity is found by taking the time derivative of the position. This yields,

$$\vec{v} = [66\,\hat{\imath} + (31.7 - 17.42t)\,\hat{\jmath}]\,\text{ft/s}. \tag{1}$$

The speed is the magnitude of the velocity. Using Eq. (1), we have

$$v = \sqrt{66^2 + (31.7 - 17.42t)^2}\,\text{ft/s}, \tag{2}$$

which can be simplified to

$$v = \sqrt{5361 - 1104t + 303.5t^2}\,\text{ft/s}. \tag{3}$$

Setting the speed in Eq. (3) equal to the speed limit 55 mph = 80.67 ft/s, and solving for t we have

$$\sqrt{5361 - 1104t + 303.5t^2}\,\text{ft/s} = 80.67\,\text{ft/s} \quad \Rightarrow \quad 5361 - 1104t + 303.5t^2 = (80.67\,\text{ft/s})^2. \tag{4}$$

The second of Eqs. (4) is a second order algebraic equation in t with the following two roots:

$$t = -0.8427\,\text{s} \quad \text{and} \quad t = 4.482\,\text{s}. \tag{5}$$

Since $t > 0$ we can only accept the second of the two roots in Eq. (5). Therefore, we conclude that the car will exceed the given speed limit at

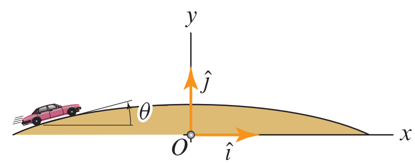
$$\boxed{t = 4.482\,\text{s}.}$$

Problem 2.14

The position of a car as a function of time t, with $t > 0$ and expressed in seconds, is

$$\vec{r}(t) = [(66t - 120)\,\hat{\imath} + (1.2 + 31.7t - 8.71t^2)\,\hat{\jmath}]\,\text{ft}.$$

Determine the slope θ of the trajectory of the car for $t_1 = 1$ s and $t_2 = 3$ s. In addition, find the angle ϕ between velocity and acceleration for $t_1 = 1$ s and $t_2 = 3$ s. Based on the values of ϕ at t_1 and t_2, argue whether the speed of the car is increasing or decreasing at t_1 and t_2.



Solution

Since the velocity is always tangent to the path, the angle θ can be computed by finding the velocity and then determining the orientation of the velocity relative to the horizontal direction. The velocity is the time derivative of the position. Differentiating the given expression for the position with respect to time, we have

$$\vec{v} = [66\,\hat{\imath} + (31.7 - 17.42t)\,\hat{\jmath}]\,\text{ft/s}. \tag{1}$$

The orientation θ of the velocity vector can be computed as:

$$\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right), \tag{2}$$

where, referring to Eq. (1),

$$v_x = 66\,\text{ft/s} \quad \text{and} \quad v_y = (31.7 - 17.42t)\,\text{ft/s}. \tag{3}$$

Substituting Eqs. (3) into Eq. (2) and evaluating the corresponding expression for $t = t_1 = 1$ s and $t = t_2 = 3$ s, we have

$$\boxed{\theta_1 = 12.21° \quad \text{and} \quad \theta_2 = -17.30°,} \tag{4}$$

where θ_1 and θ_2 are the values of θ at t_1 and t_2, respectively.

To determine the angle ϕ, we first determine the acceleration as the time derivative of the velocity. Differentiating Eq. (1) with respect to time gives

$$\vec{a} = -17.42\,\hat{\jmath}\,\text{ft/s}^2. \tag{5}$$

The angle ϕ is obtained as

$$\phi = \cos^{-1}\left(\frac{\vec{v}\cdot\vec{a}}{|\vec{v}||\vec{a}|}\right) = \cos^{-1}\left[\frac{17.42(17.42t - 31.7)}{17.42\sqrt{66^2 + (31.7 - 17.42t)^2}}\right], \tag{6}$$

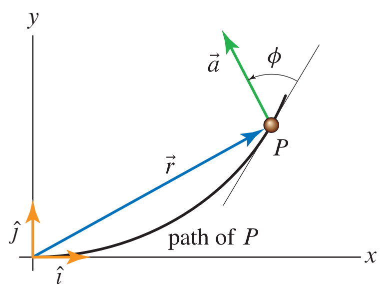
where we have used Eqs. (1) and (5). Denoting by ϕ_1 and ϕ_2 the values of ϕ at times t_1 and t_2, respectively, Eq. (6) gives

$$\boxed{\phi_1 = 102.2° \quad \text{and} \quad \phi_2 = 72.70°.} \tag{7}$$

Since $\phi_1 > 90°$, at t_1 the acceleration has a component opposite to the velocity. This indicates that the speed of the car is decreasing at t_1. Since $\phi_2 < 90°$, at t_2 the acceleration has a component pointing in the same direction as the velocity. This indicates that the speed of the car is increasing at t_2.

Problem 2.15

Let $\vec{r} = [t\,\hat{\imath} + (2 + 3t + 2t^2)\,\hat{\jmath}]$ m describe the motion of the point P relative to the Cartesian frame of reference shown. Determine an analytic expression of the type $y = y(x)$ for the trajectory of P for $0 \le t \le 5$ s.



Solution

The position of P is given as

$$\vec{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath}, \tag{1}$$

where

$$x(t) = t\ \text{m} \quad \text{and} \quad y(t) = \left(2 + 3t + 2t^2\right)\text{m}. \tag{2}$$

Solving the first of Eqs. (2) with respect to time, we have

$$t = x. \tag{3}$$

Substituting Eq. (3) into the second of Eqs. (2), we obtain:

$$y(x) = \left(2 + 3x + 2x^2\right)\text{m}. \tag{4}$$

Now we observe that the coordinate x is an increasing function of time. Therefore, the range of x covered for $0 \le t \le 5$ s is determined by computing the value of x corresponding to $t = 0$ and $t = 5$ s. Using the first of Eqs. (2), we have

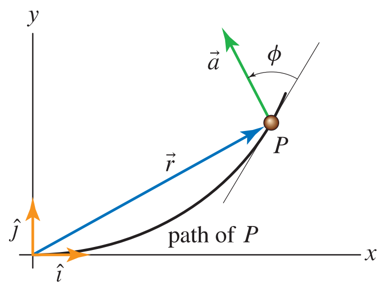
$$x(0) = 0 \quad \text{and} \quad x(5\,\text{s}) = 5\,\text{m}. \tag{5}$$

So, the trajectory of P for $0 \le t \le 5$ s is given by

$$\boxed{y(x) = \left(2 + 3x + 2x^2\right)\text{m} \quad \text{for} \quad 0 \le x \le 5\,\text{m}.}$$

Problem 2.16

Let $\vec{r} = [t\,\hat{\imath} + (2 + 3t + 2t^2)\,\hat{\jmath}]$ ft describe the motion of a point P relative to the Cartesian frame of reference shown. Recalling that for any two vectors $\vec{p}$ and $\vec{q}$ we have that $\vec{p} \cdot \vec{q} = |\vec{p}|\,|\vec{q}| \cos\beta$, where β is the angle formed by $\vec{p}$ and $\vec{q}$, and recalling that the velocity vector is *always* tangent to the trajectory, determine the function $\phi(x)$ describing the angle between the acceleration vector and the tangent to the path of P.



Solution

The velocity vector is the time derivative of the position vector:

$$\vec{v} = [1\,\hat{\imath} + (3 + 4t)\,\hat{\jmath}]\ \text{ft/s}. \tag{1}$$

The acceleration vector is the time derivative of the velocity vector. Therefore, differentiating both sides of Eq. (1) with respect to time, we have

$$\vec{a} = 4\,\hat{\jmath}\ \text{ft/s}^2. \tag{2}$$

As stated in the problem, ϕ is the angle between the acceleration vector and the tangent to the path of P. Since $\vec{v}$ is always tangent to the path, ϕ can be computed as the angle formed by the vectors $\vec{a}$ and $\vec{v}$, which is

$$\phi = \cos^{-1}\left(\frac{\vec{v} \cdot \vec{a}}{|\vec{v}||\vec{a}|}\right). \tag{3}$$

From Eqs. (1) and (2) we have that

$$|\vec{v}| = \left(\sqrt{1^2 + (3 + 4t)^2}\right)\text{ft/s} = \left(\sqrt{10 + 24t + 16t^2}\right)\text{ft/s} \quad \text{and} \quad |\vec{a}| = 4\,\text{ft/s}^2. \tag{4}$$

Substituting the expressions for $\vec{v}$, $\vec{a}$, $|\vec{v}|$, and $|\vec{a}|$ into Eq. (3), we have

$$\phi = \cos^{-1}\left(\frac{3 + 4t}{\sqrt{10 + 24t + 16t^2}}\right). \tag{5}$$

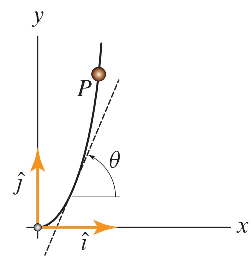
Since we have that $r_x = x = t$ ft, we can replace t with x in the last of Eqs. (5) to obtain:

$$\boxed{\phi(x) = \cos^{-1}\left(\frac{3 + 4x}{\sqrt{10 + 24x + 16x^2}}\right).}$$

Problem 2.17

The motion of a point P with respect to a Cartesian coordinate system is described by $\vec{r} = \big[2\sqrt{t}\,\hat{\imath} + \big(4\ln(t+1) + 2t^2\big)\,\hat{\jmath}\big]$ ft, where t denotes time, $t > 0$, and is expressed in seconds.

Determine the angle θ formed by the tangent to the path and the horizontal direction at $t = 3$ s.



Solution

Since the velocity is always tangent to the path, we can find the angle θ by determining the angle formed by the velocity vector and the horizontal direction. The velocity is the time derivative of the position. Hence, differentiating the position with respect to time, we have

$$\vec{v} = \left[\frac{1}{\sqrt{t}}\,\hat{\imath} + 4\left(\frac{1}{t+1} + t\right)\hat{\jmath}\right] \text{ft/s}. \tag{1}$$

Evaluating Eq. (1) at $t = 3$ s, we have

$$\vec{v}(3\,\text{s}) = (0.5774\,\hat{\imath} + 13.00\,\hat{\jmath})\ \text{ft/s}. \tag{2}$$

Since both components of the velocity are positive, at $t = 3$ s, the angle θ can be computed as follows

$$\theta = \tan^{-1}\left[\frac{v_y(3\,\text{s})}{v_x(3\,\text{s})}\right], \tag{3}$$

where, referring to Eq. (2),

$$v_x(3\,\text{s}) = 0.5774\,\text{ft/s} \quad \text{and} \quad v_y(3\,\text{s}) = 13.00\,\text{ft/s}. \tag{4}$$

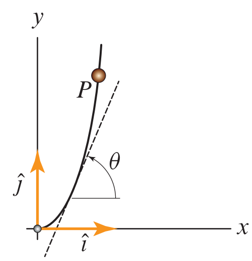
Substituting Eqs. (4) into Eq. (3), we have

$$\boxed{\theta = 87.46^\circ.}$$

Problem 2.18

The motion of a point P with respect to a Cartesian coordinate system is described by $\vec{r} = \bigl[2\sqrt{t}\,\hat{\imath} + \bigl(4\ln(t+1) + 2t^2\bigr)\,\hat{\jmath}\bigr]$ ft, where t denotes time, $t > 0$, and is expressed in seconds.

Determine the average acceleration of P between times $t_1 = 4$ s and $t_2 = 6$ s and find the difference between it and the true acceleration of P at $t = 5$ s.



Solution

By definition, the average acceleration over a time interval $t_1 \le t \le t_2$ is

$$\vec{a}_{\text{avg}} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \vec{a}\,dt. \tag{1}$$

Recalling that $\vec{a} = d\vec{v}/dt$, we can write $\vec{a}\,dt = d\vec{v}$. Substituting this expression into Eq. (1), gives

$$\vec{a}_{\text{avg}} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1}, \tag{2}$$

where $\vec{v}_1$ and $\vec{v}_2$ are the values of $\vec{v}$ at times t_1 and t_2, respectively.

We now proceed to determine the velocity as the time derivative of the position. This gives

$$\vec{v} = \left[\frac{1}{\sqrt{t}}\,\hat{\imath} + 4\left(\frac{1}{t+1} + t\right)\hat{\jmath}\right] \text{ft/s}. \tag{3}$$

Evaluating the expression in Eq. (3) at $t = t_1 = 4$ s and $t = t_2 = 6$ s, and using the results to evaluate Eq. (2), we have

$$\boxed{\vec{a}_{\text{avg}} = (-0.04588\,\hat{\imath} + 3.886\,\hat{\jmath})\ \text{ft/s}^2.} \tag{4}$$

We now determine the (true) acceleration as the time derivative of the velocity. Using Eq. (3), this gives

$$\vec{a} = \left\{-\frac{1}{2t^{3/2}}\,\hat{\imath} + 4\left[1 - \frac{1}{(t+1)^2}\right]\hat{\jmath}\right\} \text{ft/s}^2. \tag{5}$$

At $t = 5$ s, Eq. (5) gives

$$\vec{a}(5\,\text{s}) = (-0.04472\,\hat{\imath} + 3.889\,\hat{\jmath})\ \text{ft/s}^2. \tag{6}$$

Subtracting Eq. (6) from Eq. (4) side by side, we have

$$\boxed{\vec{a}_{\text{avg}} - \vec{a}(5\,\text{s}) = (-0.001154\,\hat{\imath} - 0.003175\,\hat{\jmath})\ \text{ft/s}^2.}$$

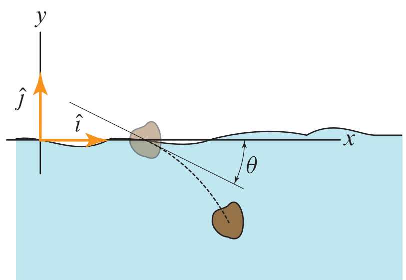
The above results allows one to measure the error made in approximating the true acceleration at $t = s$ s with $\vec{a}_{\text{avg}}$.

Problem 2.19

The motion of a stone thrown into a pond is described by

$$\vec{r}(t) = \left[\left(1.5 - 0.3e^{-13.6t}\right)\hat{\imath} + \left(0.094e^{-13.6t} - 0.094 - 0.72t\right)\hat{\jmath}\right]\,\mathrm{m},$$

where t is time expressed in seconds, and $t = 0\,\mathrm{s}$ is the time when the stone first hits the water. Determine the stone's velocity and acceleration. In addition, find the initial angle of impact θ of the stone with the water, i.e., the angle formed by the stone's trajectory and the horizontal direction at $t = 0$.



Solution

The velocity of the stone is found by differentiating the position vector with respect to time. This gives

$$\vec{v}(t) = \left[\left(4.080e^{-13.60t}\right)\hat{\imath} - \left(1.278e^{-13.60t} + 0.7200\right)\hat{\jmath}\right]\,\mathrm{m/s}. \tag{1}$$

The acceleration vector is found by differentiating the velocity vector in Eq. (1) with respect to time:

$$\vec{a}(t) = \left[\left(-55.49e^{-13.60t}\right)\hat{\imath} + \left(17.39e^{-13.60t}\right)\hat{\jmath}\right]\,\mathrm{m/s^2}. \tag{2}$$

The impact angle θ is the slope of the stone's trajectory at the time that the stone enters the water. Then, recalling that the velocity is always tangent to the trajectory, we can compute θ using the components of $\vec{v}$ at time $t = 0$. Using Eq. (1) to evaluate the velocity components, and observing that at time $t = 0$ we have $v_y(0) < 0$, θ is given by

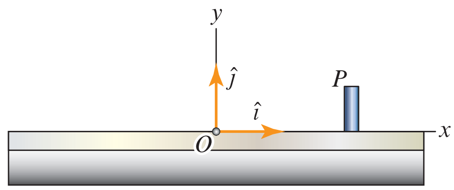
$$\theta = \tan^{-1}\left(\frac{-v_y(0)}{v_x(0)}\right) \quad \Rightarrow \quad \boxed{\theta = 26.10^\circ.}$$

Problem 2.20

As part of a mechanism, a peg P is made to slide within a rectilinear guide with the following prescribed motion:

$$\vec{r}(t) = x_0\big[\sin(2\pi\omega t) - 3\sin(\pi\omega t)\big]\,\hat{\imath},$$

where t denotes time in seconds, $x_0 = 1.2$ in., and $\omega = 0.5$ rad/s. Determine the displacement and the distance traveled over the time interval $0 \le t \le 4$ s. In addition, determine the corresponding average velocity and average speed. Express displacement and distance traveled in ft, and express velocity and speed in ft/s. You may find useful the following trigonometric identity: $\cos(2\beta) = 2\cos^2\beta - 1$.



Solution

The function that describes $\vec{r}(t)$ is the sum of two periodic functions. The period of the function $\sin(2\pi\omega t)$ is half the period of the function $\sin(\pi\omega t)$. Hence, the overall period p of $\vec{r}(t)$ coincides with the period of $\sin(\pi\omega t)$. We determine p as follows:

$$\pi\omega p = 2\pi \quad \Rightarrow \quad p = 2/\omega = 4\,\text{s}, \tag{1}$$

where we have used the fact that $\omega = 0.5$ rad/s. Therefore, letting $t_i = 0$ and $t_f = 4\,\text{s} = p$, we see that t_i and t_f are exactly one period apart. This implies that the position of the peg at times t_i and t_f is identical. These considerations tell us that the displacement over the given time interval is equal to zero:

$$\boxed{\Delta\vec{r} = \vec{0},}$$

where $\Delta\vec{r} = \vec{r}(t_f) - \vec{r}(t_i)$. This implies that the corresponding average velocity is also equal to zero:

$$\boxed{\vec{v}_{\text{avg}} = \vec{0}.}$$

To determine the distance traveled, we begin by observing that the peg starts its motion at the origin, i.e., $\vec{r}(0) = \vec{0}$. Next we determine the velocity of the peg as the time derivative of the position:

$$\vec{v} = \pi\omega x_0\big[2\cos(2\pi\omega t) - 3\cos(\pi\omega t)\big]\,\hat{\imath}. \tag{2}$$

Using the expression in Eq. (2), at time $t = 0$, we have $\vec{v}(0) = -\pi\omega x_0\,\hat{\imath}$. This result implies that, at $t = 0$, the peg is moving to the left. In order to come back to its initial position, the peg must reverse the direction of motion. This observation is important in that it leads us to a strategy to determine the distance traveled. Let the total time interval starting at t_i and ending at t_f be subdivided as follows:

$$0 = t_i < t_1 < t_2 < \cdots < t_n < t_f = 4\,\text{s}, \tag{3}$$

where $t_1, t_2, \ldots,$ and t_n, are the times at which the peg changes direction of motion. If we can determine the n times $t_1, t_2, \ldots t_n$, then the total distance traveled is the sum of the length of each segment traveled between the time instants in Eq. (3), that is,

$$d = |x(t_i) - x(t_1)| + |x(t_1) - x(t_2)| + \cdots + |x(t_n) - x(t_f)|, \tag{4}$$

where, referring to the problem statement,

$$x(t) = x_0\big[\sin(2\pi\omega t) - 3\sin(\pi\omega t)\big]. \tag{5}$$

To determine the times at which the peg changes the direction of motion, we need to determine the times at which the x component of velocity changes sign, which corresponds to the times at which the velocity equals zero. Referring to Eq. (2), this requires that we solve the equation

$$2\cos(2\pi\omega t) - 3\cos(\pi\omega t) = 0. \tag{6}$$

Using the trigonometric identity provided in the problem statement, Eq. (6) can be rewritten as

$$4\cos^2(\pi\omega t) - 3\cos(\pi\omega t) - 2 = 0, \tag{7}$$

which is a quadratic equation in $\cos(\pi\omega t)$ whose solution is

$$\cos(\pi\omega t) = \frac{3 \pm \sqrt{41}}{8} \quad \Rightarrow \quad \cos(\pi\omega t) = 1.176 \quad \text{and} \quad \cos(\pi\omega t) = -0.4254. \tag{8}$$

The first root is not acceptable because the cosine function cannot take on values larger than one. Hence, the times at which the x component of the velocity is equal to zero are given by the following sequence of time values:

$$t = \frac{1}{\pi\omega}\cos^{-1}\left(\frac{3-\sqrt{41}}{8}\right) \pm 2\pi n, \quad n = 0, 1, 2, \ldots \tag{9}$$

Since the time values we are interested in must be between $t_i = 0$ and $t_f = 4$ s, then the only acceptable solutions are

$$t_1 = 1.280\,\text{s} \quad \text{and} \quad t_2 = 2.720\,\text{s}. \tag{10}$$

Now that the times at which the peg changes direction of motion are known, referring to Eq. (4), we can then apply the formula giving the distance traveled:

$$d = |x(0) - x(t_1)| + |x(t_1) - x(t_2)| + |x(t_2) - x(4\,\text{s})|. \tag{11}$$

Using Eqs.(5) and (10), we have

$$x(0) = 0, \quad x(t_1) = -0.3485\,\text{ft}, \quad x(t_2) = 0.3485\,\text{ft}, \quad x(4\,\text{s}) = 0. \tag{12}$$

Using the (full precision values of the) results in Eq. (12), we can evaluate Eq. (11) to obtain

$$\boxed{d = 1.394\,\text{ft}.} \tag{13}$$

Now that the distance traveled is known, the corresponding average speed is obtained by dividing the distance traveled by the length of the time interval considered:

$$v_{\text{avg}} = \frac{d}{4\,\text{s}}. \tag{14}$$

Using the (full precision value of the) result for d in Eq. (13) we have

$$\boxed{v_{\text{avg}} = 0.3485\,\text{ft/s}.}$$

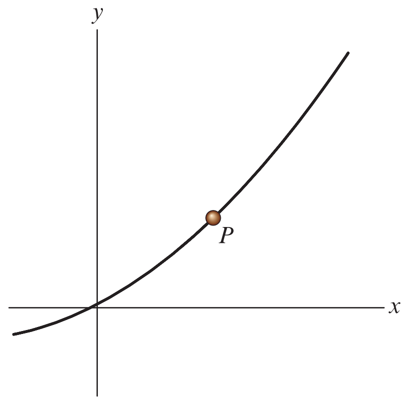
Problem 2.21

The position of point P as a function of time t, $t \geq 0$ and expressed in seconds, is

$$\vec{r}(t) = 2.0\,[0.5 + \sin(\omega t)]\,\hat{\imath} + \big[9.5 + 10.5\sin(\omega t) + 4.0\sin^2(\omega t)\big]\,\hat{\jmath},$$

where $\omega = 1.3\,\text{rad/s}$ and the position is measured in meters.

Find the trajectory of P in Cartesian components and then, using the x component of $\vec{r}(t)$, find the maximum and minimum values of x reached by P. The equation for the trajectory is valid for all values of x, yet the maximum and minimum values of x as given by the x component of $\vec{r}(t)$ are finite. What is the origin of this discrepancy?



Solution

We begin by writing the position of P as

$$\vec{r} = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath}, \tag{1}$$

where

$$x(t) = [2.0(0.5 + \sin\omega t)]\,\text{m} \quad \text{and} \quad y(t) = (9.5 + 10.5\sin\omega t + 4.0\sin^2\omega t)\,\text{m}. \tag{2}$$

Next, we solve the first of Eqs. (2) for $\sin\omega t$ as a function of x:

$$\sin\omega t = \frac{x-1}{2}. \tag{3}$$

Substituting Eq. (3) into the second of Eqs. (2) we have

$$y(x) = \left[9.5 + 10.5\left(\frac{x-1}{2}\right) + 4.000\left(\frac{x-1}{2}\right)^2\right]\text{m}$$

$$\Rightarrow \quad \boxed{y(x) = \big(5.250 + 3.250x + 1.000x^2\big)\,\text{m}.} \tag{4}$$

We now need to determine the range of x covered by the motion of P. To do so, referring to the first of Eqs. (2), we observe that the minimum and maximum values of x are achieved when the function $\sin\omega t$ achieves its minimum and maximum values, respectively, which are the values -1 and $+1$, respectively. Substituting the values -1 and $+1$ for $\sin\omega t$ in the first of Eqs. (2), we have, respectively,

$$\boxed{x_{\text{min}} = -1\,\text{m} \quad \text{and} \quad x_{\text{max}} = 3\,\text{m}.}$$

The maximum and minimum values of the x coordinate occur due to the presence of a periodic function in the equation for this component. Therefore, the trajectory, which is valid for all times t, is constrained to oscillate between -1 m and 3 m on the x axis.

Problem 2.22

The position of point P as a function of time t, $t \geq 0$ and expressed in seconds, is

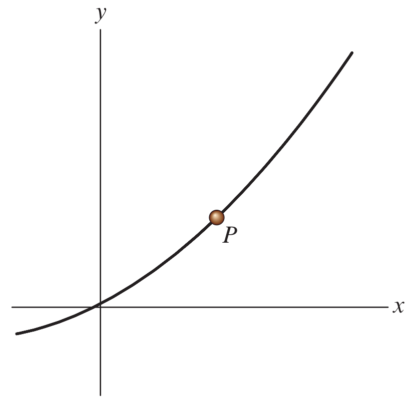
$$\vec{r}(t) = 2.0\,[0.5 + \sin(\omega t)]\,\hat{\imath} + \big[9.5 + 10.5\sin(\omega t) + 4.0\sin^2(\omega t)\big]\,\hat{\jmath},$$

where $\omega = 1.3\,\text{rad/s}$ and the position is measured in meters.

(a) Plot the trajectory of P for $0 \leq t \leq 0.6\,\text{s}$, $0 \leq t \leq 1.4\,\text{s}$, $0 \leq t \leq 2.3\,\text{s}$, and $0 \leq t \leq 5\,\text{s}$.

(b) Plot the $y(x)$ trajectory for $-10\,\text{m} \leq x \leq 10\,\text{m}$.

(c) You will notice that the trajectory found in (b) does not agree with any of those found in (a). Explain this discrepancy by analytically determining the minimum and maximum values of x reached by P. As you look at this sequence of plots, why does the trajectory change between some times and not others?



Solution

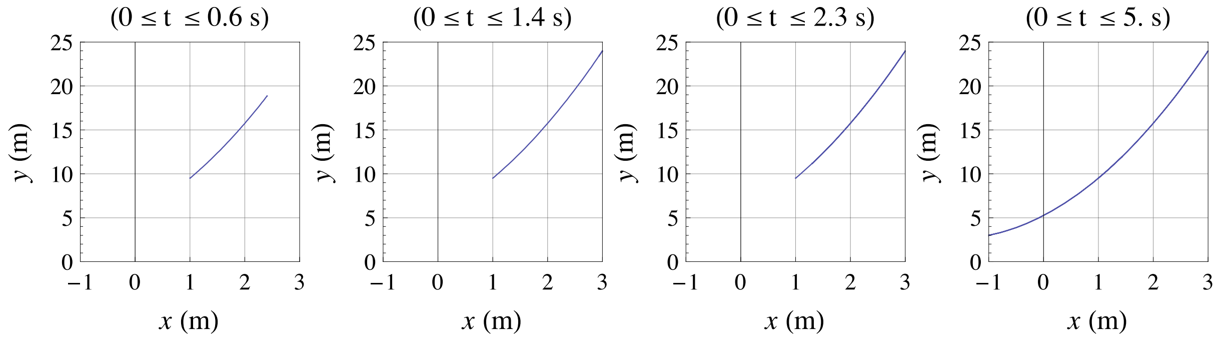
Part (a). Since $\omega = 1.3\,\text{rad/s}$, we write the x and y coordinates of point P as

$$x(t) = \{2.0[0.5 + \sin(1.3t)]\}\,\text{m} \quad \text{and} \quad y(t) = [9.5 + 10.5\sin(1.3t) + 4.0\sin^2(1.3t)]\,\text{m}. \tag{1}$$

One strategy to plot the trajectory of point P is to plot the line that connects the points of coordinates $[x(t), y(t)]$ as time t varies within a given time interval. This way of plotting the trajectory *does not* involve finding y as a function of x. Rather, it consists of generating a list of (x, y) values, each of which is computed by first assigning a specific value of time. This procedure is called a *parametric plot*, where the parameter used to generate the plotted points is time and does not appear directly on the plot (i.e., the plot uses x and y axes, but it does not show the time values corresponding to the points on the plot). Parametric plots can be generated using any appropriate numerical software such as MATLAB or *Mathematica*.

The parametric plots of the trajectory of P shown below were generated in *Mathematica* with the following code:

```
x = 2.0 (0.5 + Sin[1.3 t]);
y = 9.5 + 10.5 Sin[1.3 t] + 4 (Sin[1.3 t])^2;
tf = {0.6, 1.4, 2.3, 5.0};
Table[ParametricPlot[{x, y}, {t, 0, tf[[i]]}, PlotRange → {{-1, 3}, {0, 25}}, Frame → True,
  FrameTicks → {{Automatic, None}, {{-1, 0, 1, 2, 3}, None}}, AspectRatio → 1,
  GridLines → Automatic, FrameLabel → {"x  (m)", "y  (m)"},
  PlotLabel → StringJoin["(0 ≤ t ≤ ", ToString[tf[[i]]], " s)"]], {i, 1, Length[tf]}]
```



Part (b). In this part of the problem we first need to write the trajectory in the form $y = y(x)$. To do so, we start with solving the first of Eq. (1) for $\sin(1.3t)$ as a function of x. This gives

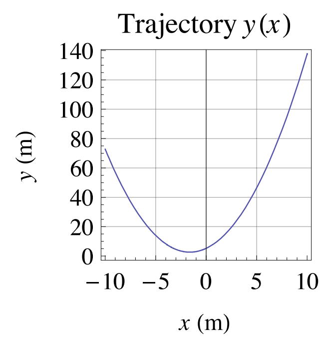
$$\sin(1.3t) = \frac{x - 1.000\,\mathrm{m}}{2.000\,\mathrm{m}}, \tag{2}$$

which can then be substituted into the second of Eqs. (1) to obtain

$$y(x) = \left[9.500 + 10.50\left(\frac{x - 1.000}{2.000}\right) + 4.000\left(\frac{x - 1.000}{2.000}\right)^2\right]\mathrm{m}$$
$$\Rightarrow \quad y(x) = \left(5.250 + 3.250x + 1.000x^2\right)\mathrm{m}. \tag{3}$$

Now that we have the trajectory in the form $y(x)$, we can plot it over the given interval $-10\,\mathrm{m} \le x \le 10\,\mathrm{m}$ as shown on the right. This plot was generated using *Mathematica* with the following code:

```
Plot[5.250 + 3.250 x + 1.00 x^2, {x, -10.0, 10.0}, Frame → True,
 FrameTicks → {{Automatic, None}, {Automatic, None}}, AspectRatio → 1,
 GridLines → Automatic, ImageSize → 170, FrameLabel → {"x (m) ", "y (m) "},
 PlotLabel → "Trajectory y(x) "]
```



Part (c). Referring to Eq. (1), for $x_{\max}$, $\sin 1.3t = +1 \Rightarrow t = \frac{\pi}{2(1.3)}$ s and for $x_{\min}$, $\sin 1.3t = -1 \Rightarrow t = \frac{3\pi}{2(1.3)}$ s. Thus, the minimum and maximum values are, respectively,

$$x_{\min} = -1.000\,\mathrm{m} \quad \text{and} \quad x_{\max} = 3.000\,\mathrm{m}. \tag{4}$$

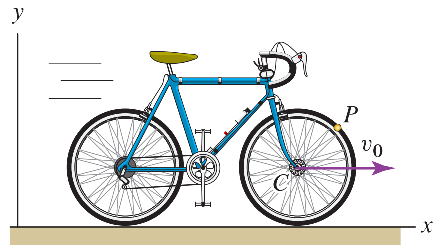
The minimum and maximum values of the x coordinate occur due to the presence of a periodic function in the equation for this component. Therefore, the trajectory, which is valid for all times t, is constrained to oscillate between -1.000 m and 3.000 m on the x axis. This fact explains why in the plot sequence in Part (a) the trajectory seems not to change after a while: point P keeps tracing the same curve segment again and again. The periodicity of the motion of P also explains the discrepancy between the trajectory obtained in Part (a) and that obtained in Part (b). In fact, since the plot generated in Part (a) was based on a direct application of Eqs. (1), we see that the plot in question correctly reflects the periodic time dependence the coordinates of point P. By contrast, the trajectory computed in Part (b) no longer carries any direct relationship with time. The trajectory plotted in Part (b) does contain the trajectory plotted in Part (a) as a sub-curve. The problem with the trajectory in Part (b) is that we have no direct way of knowing what part of the entire curve actually pertains to the motion of point P.

Problem 2.23

A bicycle is moving to the right at a speed $v_0 = 20\,\text{mph}$ on a horizontal and straight road. The radius of the bicycle's wheels is $R = 1.15\,\text{ft}$. Let P be a point on the periphery of the front wheel. One can show that the x and y coordinates of P are described by the following functions of time:

$$x(t) = v_0 t + R\sin(v_0 t/R) \quad \text{and} \quad y(t) = R\big[1 + \cos(v_0 t/R)\big].$$

Determine the expressions for the velocity, speed, and acceleration of P as functions of time.



Solution

The velocity of P is the time derivative of P's position, which, in the coordinate system shown, is

$$\vec{r} = \big[v_0 t + R\sin(v_0 t/R)\big]\,\hat{\imath} + R\big[1 + \cos(v_0 t/R)\big]\,\hat{\jmath}. \tag{1}$$

Differentiating the above expression with respect to time, we have

$$\vec{v} = v_0\left[1 + \cos\left(\frac{v_0 t}{R}\right)\right]\hat{\imath} - v_0 \sin\left(\frac{v_0 t}{R}\right)\hat{\jmath}. \tag{2}$$

Since $v_0 = 20\,\text{mph} = 20\frac{5280}{3600}\,\text{ft/s}$ and $R = 1.15\,\text{ft}$, we have

$$\boxed{\vec{v} = (29.33\,\text{ft/s})[1 + \cos[(25.51\,\text{rad/s})t]\,\hat{\imath} - (29.33\,\text{ft/s})\sin[(25.51\,\text{rad/s})t]\,\hat{\jmath}.}$$

The speed is the magnitude of the velocity vector:

$$v = \sqrt{v_x^2(t) + v_y^2(t)} = v_0\sqrt{2 + 2\cos\left(\frac{v_0 t}{R}\right)} \quad \Rightarrow \quad \boxed{v = (29.33\,\text{ft/s})\sqrt{2 + 2\cos[(25.51\,\text{rad/s})t]}.} \tag{3}$$

The acceleration of P is the time derivative of P's velocity. From Eq. (2), we have

$$\vec{a} = -\frac{v_0^2}{R}\sin\left(\frac{v_0 t}{R}\right)\hat{\imath} - \frac{v_0^2}{R}\cos\left(\frac{v_0 t}{R}\right)\hat{\jmath}. \tag{4}$$

Since $v_0 = 20\,\text{mph} = 20\,\frac{5280}{3600}\,\text{ft/s}$ and $R = 1.15\,\text{ft}$, we have

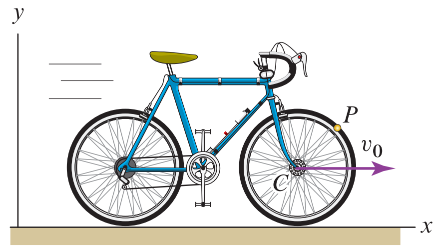
$$\boxed{\vec{a} = -\big(748.2\,\text{ft/s}^2\big)\sin[(25.51\,\text{rad/s})t]\,\hat{\imath} - \big(748.2\,\text{ft/s}^2\big)\cos[(25.51\,\text{rad/s})t]\,\hat{\jmath}.}$$

Problem 2.24

A bicycle is moving to the right at a speed $v_0 = 20\,\text{mph}$ on a horizontal and straight road. The radius of the bicycle's wheels is $R = 1.15\,\text{ft}$. Let P be a point on the periphery of the front wheel. One can show that the x and y coordinates of P are described by the following functions of time:

$$x(t) = v_0 t + R\sin(v_0 t/R) \quad \text{and} \quad y(t) = R\big[1 + \cos(v_0 t/R)\big].$$

Determine the maximum and minimum speed achieved by P, as well as the y coordinate of P when the maximum and minimum speeds are achieved. Finally, compute the acceleration of P when P achieves its maximum and minimum speeds.



Solution

The speed of P is the magnitude of the velocity of P. Hence, we first compute the velocity of P, which is the time derivative of P's position. In the coordinate system shown, the position of P is

$$\vec{r} = \big[v_0 t + R\sin(v_0 t/R)\big]\,\hat{\imath} + R\big[1 + \cos(v_0 t/R)\big]\,\hat{\jmath}. \tag{1}$$

Differentiating the above expression with respect to time, we have

$$\vec{v} = v_0\left[1 + \cos\left(\frac{v_0 t}{R}\right)\right]\hat{\imath} - v_0\sin\left(\frac{v_0 t}{R}\right)\hat{\jmath}. \tag{2}$$

Hence, the speed is

$$v = \sqrt{v_x^2(t) + v_y^2(t)} = v_0\sqrt{2 + 2\cos\left(\frac{v_0 t}{R}\right)}, \tag{3}$$

which implies that v is maximum when $\cos(v_0 t/R) = 1$ and minimum when $\cos(v_0 t/R) = -1$, i.e.,

$$\boxed{v_{\text{max}} = 2v_0 = 58.67\,\text{ft/s} \quad \text{and} \quad v_{\text{min}} = 0\,\text{ft/s},} \tag{4}$$

where we have used the fact that $v_0 = 20\,\text{mph} = 20\,\frac{5280}{3600}\,\text{ft/s}$.

Since $v = v_{\text{min}}$ when $\cos(v_0 t/R) = -1$ and $v = v_{\text{max}}$ when $\cos(v_0 t/R) = 1$, using the expression for the y component of the position of P in Eq. (1), we have

$$\boxed{y_{v_{\text{min}}} = 0\,\text{ft} \quad \text{and} \quad y_{v_{\text{max}}} = 2R = 2.300\,\text{ft},}$$

where we have used the fact that $R = 1.15\,\text{ft}$.

To determine the acceleration corresponding to v_{min} and v_{max}, we first determine the acceleration of P by differentiating with respect to time the expression in Eq. (2). This gives

$$\vec{a} = -\frac{v_0^2}{R}\sin\left(\frac{v_0 t}{R}\right)\hat{\imath} - \frac{v_0^2}{R}\cos\left(\frac{v_0 t}{R}\right)\hat{\jmath}. \tag{5}$$

Now, recall that for $v = v_{\text{min}}$ we have $\cos(v_0 t/R) = -1$ and for $v = v_{\text{max}}$ we have $\cos(v_0 t/R) = 1$. In both cases, we have $\sin(v_0 t/R) = 0$. Using these considerations along with Eq. (5), we have

$$\boxed{\vec{a}_{v_{\text{min}}} = \frac{v_0^2}{R}\,\hat{\jmath} = (748.2\,\text{ft/s}^2)\,\hat{\jmath} \quad \text{and} \quad \vec{a}_{v_{\text{max}}} = -\frac{v_0^2}{R}\,\hat{\jmath} = (-748.2\,\text{ft/s}^2)\,\hat{\jmath},}$$

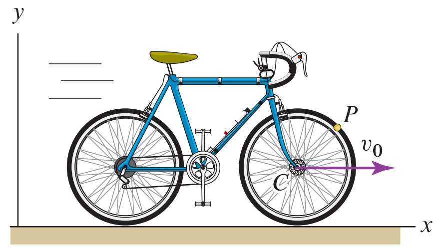
where we have used the following numerical data: $v_0 = 20\,\text{mph} = 20(5280/3600)\,\text{ft/s}$ and $R = 1.15\,\text{ft}$.

Problem 2.25

A bicycle is moving to the right at a speed $v_0 = 20\,\text{mph}$ on a horizontal and straight road. The radius of the bicycle's wheels is $R = 1.15\,\text{ft}$. Let P be a point on the periphery of the front wheel. One can show that the x and y coordinates of P are described by the following functions of time:

$$x(t) = v_0 t + R\sin(v_0 t/R) \quad \text{and} \quad y(t) = R\big[1 + \cos(v_0 t/R)\big].$$

Plot the trajectory of P for $0 \le t \le 1\,\text{s}$. For the same time interval, plot the speed as a function of time, as well as the components of the velocity and acceleration of P.



Solution

The velocity of P is the time derivative of P's position, which, in the coordinate system shown, is

$$\vec{r} = \big[v_0 t + R\sin(v_0 t/R)\big]\,\hat{\imath} + R\big[1 + \cos(v_0 t/R)\big]\,\hat{\jmath}. \tag{1}$$

Differentiating Eq. (1) with respect to time, we have

$$\vec{v} = v_0\left[1 + \cos\left(\frac{v_0 t}{R}\right)\right]\hat{\imath} - v_0 \sin\left(\frac{v_0 t}{R}\right)\hat{\jmath}. \tag{2}$$

The speed is now found by taking the magnitude of the velocity vector. Hence, we have

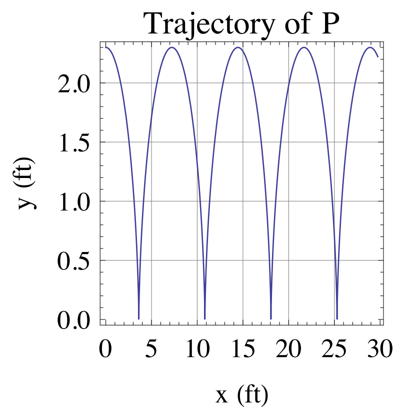
$$v = \sqrt{v_x^2(t) + v_y^2(t)} = v_0\sqrt{2 + 2\cos\left(\frac{v_0 t}{R}\right)}. \tag{3}$$

The acceleration of P is the time derivative of P's velocity. Hence, from Eq. (2), we have

$$\vec{a} = -\frac{v_0^2}{R}\sin\left(\frac{v_0 t}{R}\right)\hat{\imath} - \frac{v_0^2}{R}\cos\left(\frac{v_0 t}{R}\right)\hat{\jmath}. \tag{4}$$

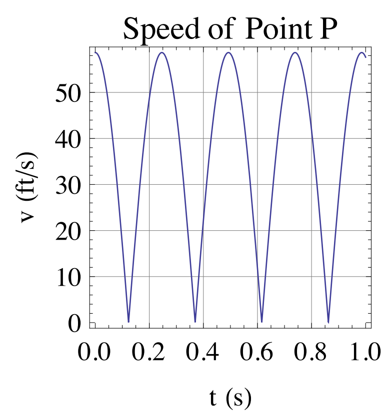
Trajectory of P. We can plot the trajectory of P for $0 \le t \le 1\,\text{s}$ by plotting the line connecting the x and y coordinates of P computed as functions of the parameter t. The x and y coordinates of P are the component of $\vec{r}$ (in Eq. (1)) in the x and y direction, respectively. This can be done with a variety of pieces of numerical software. Since $v_0 = 20\,\text{mph} = 29.33\,\text{ft/s}$ and $R = 1.15\,\text{ft}$, the plot presented below has been obtained using *Mathematica* with the following code

```
Parameters = {v0 → 29.33, R → 1.15};
x = v0 t + R Sin[v0 t / R]; y = R (1 + Cos[v0 t / R]);
ParametricPlot[{x, y} /. Parameters, {t, 0, 1}, Frame → True, GridLines → Automatic,
 FrameLabel → {"x (ft)", "y (ft)"}, PlotLabel → "Trajectory of P", AspectRatio → 1]
```



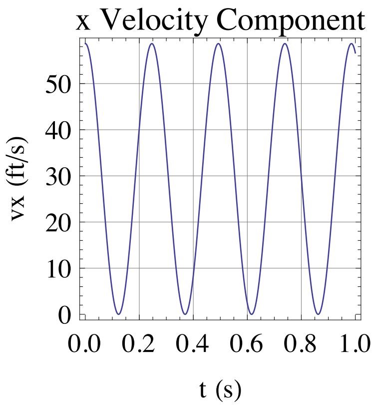
Speed of P. The speed of P can be plotted for $0 \leq t \leq 1$ s using Eq. (3), with $v_0 = 20$ mph $= 29.33$ ft/s and $R = 1.15$ ft. The plot shown below was generated using *Mathematica* with the following code:

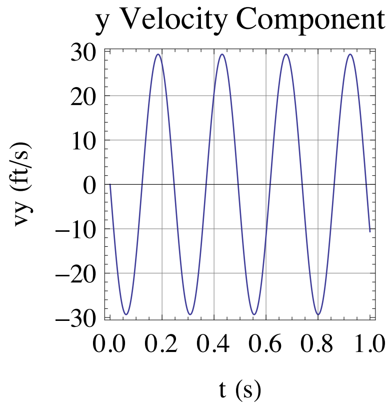
```
Parameters = {v0 → 29.33, R → 1.15};
v = v0 √(2 + 2 Cos[v0 t / R]) ;
Plot[v /. Parameters, {t, 0, 1}, Frame → True, GridLines → Automatic,
 FrameLabel → {"t (s)", "v (ft/s)"}, AspectRatio → 1, PlotLabel → "Speed of Point P"]
```



Velocity Components The components of the velocity of P can be plotted for $0 \leq t \leq 1$ s using the expressions in Eq. (2) with $v_0 = 20$ mph $= 29.33$ ft/s and $R = 1.15$ ft. The plot shown below was generated using *Mathematica* with the following code:

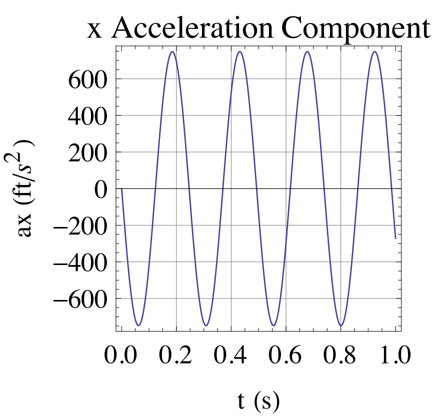
```
Parameters = {v0 → 29.33, R → 1.15};
vx = v0 (1 + Cos[v0 t / R]); vy = -v0 Sin[v0 t / R];
Plot[vx /. Parameters, {t, 0, 1}, Frame → True, GridLines → Automatic,
 FrameLabel → {"t (s)", "vx (ft/s)"}, PlotLabel → "x Velocity Component", AspectRatio → 1]
Plot[vy /. Parameters, {t, 0, 1}, Frame → True, GridLines → Automatic,
 FrameLabel → {"t (s)", "vy (ft/s)"}, PlotLabel → "y Velocity Component",
 AspectRatio → 1]
```

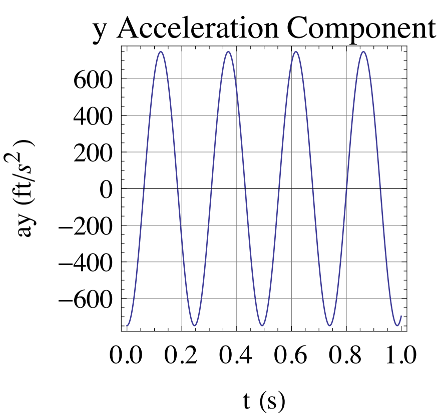




Acceleration Components The components of the acceleration of P can be plotted for $0 \leq t \leq 1$ s using the expressions in Eq. (4) with $v_0 = 20\,\text{mph} = 29.33\,\text{ft/s}$ and $R = 1.15\,\text{ft}$. The plot shown below was generated using *Mathematica* with the following code:

```
Parameters = {v0 → 29.33, R → 1.15};
ax = -(v0^2 / R) Sin[v0 t / R]; ay = -(v0^2 / R) Cos[v0 t / R];
Plot[ax /. Parameters, {t, 0, 1}, Frame → True, GridLines → Automatic,
 FrameLabel → {"t (s)", "ax (ft/s^2)"}, PlotLabel → "x Acceleration Component",
 AspectRatio → 1]
Plot[ay /. Parameters, {t, 0, 1}, Frame → True, GridLines → Automatic,
 FrameLabel → {"t (s)", "ay (ft/s^2)"}, PlotLabel → "y Acceleration Component",
 AspectRatio → 1]
```





Problem 2.26

Find the x and y components of the acceleration in Example 2.3 (except for the plots) by simply differentiating Eqs. (4) and (5) with respect to time. Verify that you get the results given in Example 2.3.

Solution

Referring to Eqs. (4) and (5) of Example 2.3 on p. 38 of the textbook, we recall that the x and y components of the velocity are, respectively,

$$\dot{x} = \frac{v_0 y}{\sqrt{y^2 + 4a^2}} \quad \text{and} \quad \dot{y} = \frac{2v_0 a}{\sqrt{y^2 + 4a^2}}. \tag{1}$$

To determine $\ddot{x}$, we differentiate $\dot{x}$ with respect to time with the help of the chain rule:

$$\ddot{x} = \frac{d\dot{x}}{dt} = \frac{d\dot{x}}{dy}\frac{dy}{dt} = \dot{y}\frac{d\dot{x}}{dy}. \tag{2}$$

Differentiating the first of Eqs. (1) with respect to y and substituting the result along with the second of Eqs. (1) into Eq. (2) we then have

$$\ddot{x} = -\frac{v_0 y^2 \dot{y}}{\left(y^2 + 4a^2\right)^{3/2}} + \frac{v_0 \dot{y}}{\sqrt{y^2 + 4a^2}} \quad \Rightarrow \quad \boxed{\ddot{x} = \frac{8v_0^2 a^3}{\left(y^2 + 4a^2\right)^2}.} \tag{3}$$

To determine $\ddot{y}$ we differentiate $\dot{y}$ with respect to time with the help of the chain rule. This gives

$$\ddot{y} = \frac{d\dot{y}}{dt} = \frac{d\dot{y}}{dy}\frac{dy}{dt} = \dot{y}\frac{d\dot{y}}{dy}. \tag{4}$$

Substituting the second of Eqs. (1) into Eq. (4) and simplifying, we have

$$\boxed{\ddot{y} = \frac{-4v_0^2 a^2 y}{\left(y^2 + 4a^2\right)^2}.}$$

Our results match those in Example 2.3.

Problem 2.27

Find the x and y components of the acceleration in Example 2.3 (except for the plots) by differentiating the first of Eqs. (3) and the last of Eqs. (1) with respect to time and then solving the resulting two equations for $\ddot{x}$ and $\ddot{y}$. Verify that you get the results given in Example 2.3.

Solution

We recall that the first of Eqs. (3) and the last of Eqs. (1) in Example 2.3 on p. 38 are, respectively,

$$v_0^2 = \dot{x}^2 + \dot{y}^2 \quad \text{and} \quad 2y\dot{y} = 4a\dot{x}. \tag{1}$$

Recalling that v_0 and a are constants, differentiating Eqs. (1) with respect to time gives

$$0 = 2\dot{x}\ddot{x} + 2\dot{y}\ddot{y} \quad \text{and} \quad 2a\ddot{x} = \dot{y}^2 + y\ddot{y}. \tag{2}$$

We now view Eqs. (1) as a system of two equations in the two unknowns $\dot{x}$ and $\dot{y}$. Solving the system in question gives

$$\dot{x} = \frac{yv_0}{\sqrt{y^2 + 4a^2}} \quad \text{and} \quad \dot{y} = \frac{2av_0}{\sqrt{y^2 + 4a^2}}, \tag{3}$$

where, similarly to what was done in Example 2.3 on p. 38 of the textbook, we have enforced the condition that $\dot{y} > 0$. Next, we view Eqs. (2) as a system of two equations in the two unknowns $\ddot{x}$ and $\ddot{y}$ whose solution is

$$\ddot{x} = \frac{\dot{y}^3}{y\dot{x} + 2a\dot{y}} \quad \text{and} \quad \ddot{y} = \frac{-\dot{x}\dot{y}^2}{y\dot{x} + 2a\dot{y}}. \tag{4}$$

Substituting Eqs. (3) into Eqs. (4) and simplifying, we have

$$\boxed{\ddot{x} = \frac{8v_0^2a^3}{\left(4a^2 + y^2\right)^2} \quad \text{and} \quad \ddot{y} = \frac{-4v_0^2a^2y}{\left(4a^2 + y^2\right)^2}.}$$

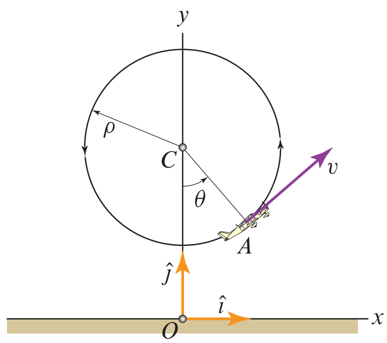
Our results match those in Example 2.3.

Problem 2.28

Airplane A is performing a loop with constant radius $\rho = 1000$ ft. The equation describing the loop is as follows:

$$(x - x_C)^2 + (y - y_C)^2 = \rho^2,$$

where $x_C = 0$ and $y_C = 1500$ ft are the coordinates of the center of the loop. If the plane were capable of maintaining its speed constant and equal to $v_0 = 160$ mph, determine the velocity and acceleration of the plane for $\theta = 30°$.



Solution

We begin by expressing the position as follows:

$$\vec{r} = x\,\hat{\imath} + y\,\hat{\jmath}. \tag{1}$$

For $\theta = 30°$, the plane is in the lower right quadrant of the loop and the expression for the path of the airplane can be written as

$$y = y_C - \sqrt{\rho^2 - x^2}, \tag{2}$$

where we have accounted for the fact that $x_C = 0$. We observe that

$$x = \rho \sin\theta. \tag{3}$$

Equations (1) and (2) combined indicate that we can regard position as being a known function of x. With this in mind, we can write an expression for the velocity of the airplane by differentiating Eq. (1) with respect to time with the help of the chain rule. This gives

$$\vec{v} = \dot{x}\,\hat{\imath} + \frac{dy}{dx}\dot{x}\,\hat{\jmath} \quad \Rightarrow \quad \vec{v} = \dot{x}\,\hat{\imath} + \frac{x\dot{x}}{\sqrt{\rho^2 - x^2}}\,\hat{\jmath}. \tag{4}$$

The quantity $\dot{x}$ in the last of Eqs. (4) is unknown, but it can be found by enforcing the condition that the magnitude of the velocity is equal to v_0. Doing so gives

$$v^2 = \dot{x}^2 + \frac{x^2\dot{x}^2}{\rho^2 - x^2} = v_0^2 \quad \Rightarrow \quad \dot{x} = \frac{v_0}{\rho}\sqrt{\rho^2 - x^2}, \tag{5}$$

where we have accounted for the fact that, at $\theta = 30°$, $\dot{x} > 0$. Substituting the last of Eqs. (5) into the last of Eqs. (4), we have

$$\vec{v} = \frac{v_0}{\rho}\sqrt{\rho^2 - x^2}\,\hat{\imath} + \frac{v_0 x}{\rho}\,\hat{\jmath}. \tag{6}$$

Since $v_0 = 160\,\text{mph} = 160\frac{5280}{3600}\,\text{ft/s}$, $\rho = 1000\,\text{ft}$, and using Eq.(3), we can evaluate Eq. (6) to obtain

$$\boxed{\vec{v} = (203.2\,\hat{\imath} + 117.3\,\hat{\jmath})\,\text{ft/s}.}$$

To determine the acceleration we differentiate the expression for the velocity in Eq. (6) with respect to time, which gives

$$\vec{a} = -\frac{v_0 x\dot{x}}{\rho\sqrt{\rho^2 - x^2}}\,\hat{\imath} + \frac{v_0}{\rho}\dot{x}\,\hat{\jmath}. \tag{7}$$

Substituting the last of Eqs. (5) into Eq. (7) and simplifying, we have

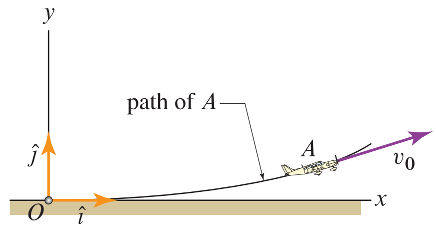
$$\vec{a} = -\frac{v_0^2 x}{\rho^2}\,\hat{\imath} + \frac{v_0^2}{\rho^2}\sqrt{\rho^2 - x^2}\,\hat{\jmath}. \tag{8}$$

Recalling that $v_0 = 160\,\text{mph} = 160\frac{5280}{3600}\,\text{ft/s}$, and using Eq.(3), we can evaluate Eq. (8) to obtain

$$\boxed{\vec{a} = (-27.53\,\hat{\imath} + 47.69\,\hat{\jmath})\,\text{ft/s}^2.}$$

Problem 2.29

An airplane A takes off as shown with a constant speed equal to $v_0 = 160\,\mathrm{km/h}$. The path of the airplane is described by the equation $y = \kappa x^2$, where $\kappa = 6\times10^{-4}\,\mathrm{m}^{-1}$. Using the component system shown, provide the expression for the velocity and acceleration of the airplane when $x = 400\,\mathrm{m}$. Express the velocity in m/s and the acceleration in $\mathrm{m/s^2}$.



Solution

The position of the airplane can be described as

$$\vec{r} = x\,\hat{\imath} + y\,\hat{\jmath}. \tag{1}$$

Since $y = \kappa x^2$, Eq. (1) becomes

$$\vec{r} = x\,\hat{\imath} + \kappa x^2\,\hat{\jmath}. \tag{2}$$

Using the chain rule to differentiate Eq. (2) with respect to time, the velocity of the airplane is

$$\vec{v} = \dot{x}\,\hat{\imath} + 2\kappa x\dot{x}\,\hat{\jmath}. \tag{3}$$

The term $\dot{x}$ in Eq. (3) can be determined by enforcing the condition that the magnitude of the velocity is equal to v_0. This gives

$$\dot{x}^2 + 4\kappa^2x^2\dot{x}^2 = v_0^2 \quad\Rightarrow\quad \dot{x} = \frac{v_0}{\sqrt{1+4\kappa^2x^2}}, \tag{4}$$

where we have chosen $\dot{x} > 0$ because the airplane is moving in the positive x direction. Substituting the last of Eqs. (4) into Eq. (3) gives

$$\vec{v} = \frac{v_0}{\sqrt{1+4\kappa^2x^2}}\,\hat{\imath} + \frac{2\kappa v_0 x}{\sqrt{1+4\kappa^2x^2}}. \tag{5}$$

For $v_0 = 160\,\mathrm{km/h} = 160\frac{1000}{3600}\,\mathrm{m/s}$, $\kappa = 6\times10^{-4}\,\mathrm{m}^{-1}$, and $x = 400\,\mathrm{m}$, we can evaluate Eq. (5) to obtain

$$\boxed{\vec{v} = (40.07\,\hat{\imath} + 19.23\,\hat{\jmath})\,\mathrm{m/s}.}$$

The acceleration is the time derivative of the velocity. Using Eq. (3) and the chain rule, we have

$$\vec{a} = \frac{d\dot{x}}{dx}\dot{x}\,\hat{\imath} + \left(2\kappa\dot{x}^2 + 2\kappa x\frac{d\dot{x}}{dx}\dot{x}\right)\hat{\jmath}. \tag{6}$$

Differentiating the last of Eqs. (4) with respect to x, we have

$$\frac{d\dot{x}}{dx} = -\frac{4\kappa^2 v_0 x}{\left(1+4\kappa^2x^2\right)^{3/2}}. \tag{7}$$

Substituting the last of Eqs. (4) and Eq. (7) into Eq. (6), after simplification, gives

$$\vec{a} = -\frac{4\kappa^2v_0^2x}{\left(1+4\kappa^2x^2\right)^2}\,\hat{\imath} + \frac{2\kappa v_0^2}{\left(1+4\kappa^2x^2\right)^2}\,\hat{\jmath}. \tag{8}$$

For $v_0 = 160\,\mathrm{km/h} = 160\frac{1000}{3600}\,\mathrm{m/s}$, $\kappa = 6\times10^{-4}\,\mathrm{m}^{-1}$, and $x = 400\,\mathrm{m}$, Eq. (8) gives

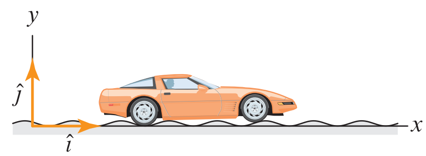
$$\boxed{\vec{a} = (-0.7516\,\hat{\imath} + 1.566\,\hat{\jmath})\,\mathrm{m/s^2}.}$$

Problem 2.30

A test track for automobiles has a portion with a specific profile described by:

$$y = h\big[1 - \sin(x/w)\big],$$

where $h = 0.5$ ft and $w = 8$ ft, and where the argument of the sine function is understood to be in radians. A car travels in the positive x direction such that the horizontal component of velocity remains constant and equal to 55 mph. Modeling the car as a point moving along the given profile, determine the maximum speed of the car. Express your answer in ft/s.



Solution

Letting x and y represent the coordinates of the car, the position of the car is

$$\vec{r} = x\,\hat{\imath} + h\big[1 - \sin(x/w)\big]\,\hat{\jmath}. \tag{1}$$

The velocity is the time derivative of the position. Using Eq. (1) and the chain rule, we have

$$\vec{v} = \dot{x}\,\hat{\imath} - \frac{h}{w}\cos\left(\frac{x}{w}\right)\dot{x}\,\hat{\jmath}. \tag{2}$$

To determine the maximum speed, we first determine the speed, which is the magnitude of the velocity. From Eq. (2), we have

$$v = |\dot{x}|\sqrt{1 + \frac{h^2}{w^2}\cos^2\left(\frac{x}{w}\right)}. \tag{3}$$

From Eq. (3) we see that the speed varies because of the presence of the cosine function, whose maximum possible value is equal to one. We conclude that the maximum value of the speed is

$$v_{\max} = |\dot{x}|\sqrt{1 + \frac{h^2}{w^2}}. \tag{4}$$

For $\dot{x} = 55$ mph $= 55\frac{5280}{3600}$ ft/s, $h = 0.5$ ft, and $w = 8$ ft, Eq. (4) gives

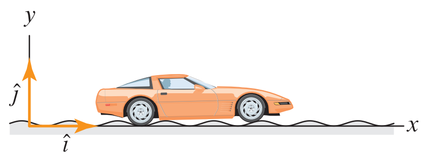
$$\boxed{v_{\max} = 80.82\,\text{ft/s}.}$$

Problem 2.31

A test track for automobiles has a portion with a specific profile described by:

$$y = h\big[1 - \cos(x/w)\big],$$

where $h = 0.20\,\mathrm{m}$ and $w = 2\,\mathrm{m}$, and where the argument of the cosine function is understood to be in radians. A car travels in the positive x direction with a constant x component of velocity equal to $100\,\mathrm{km/h}$. Modeling the car as a point moving along the given profile, determine the velocity and acceleration (expressed in m/s and m/s^2, respectively) of the car for $x = 24\,\mathrm{m}$.



Solution

Letting x and y represent the coordinates of the car, the position of the car is $\vec{r} = x\,\hat{\imath} + y\,\hat{\jmath}$, i.e.,

$$\vec{r} = x\,\hat{\imath} + h\big[1 - \cos(x/w)\big]\,\hat{\jmath}. \tag{1}$$

The velocity is the time derivative of the position. Using Eq. (1) and the chain rule, we have

$$\vec{v} = \dot{x}\,\hat{\imath} + \frac{h}{w}\sin\left(\frac{x}{w}\right)\dot{x}\,\hat{\jmath}. \tag{2}$$

The acceleration is the time derivative of the velocity. Recalling that $\dot{x}$ is constant, differentiating Eq. (2) with respect to time and using the chain rule, we have

$$\vec{a} = \frac{h}{w^2}\cos\left(\frac{x}{w}\right)\dot{x}^2\,\hat{\jmath}. \tag{3}$$

For $\dot{x} = 100\,\mathrm{km/h} = 100\frac{1000}{3600}\,\mathrm{m/s}$, $h = 0.20\,\mathrm{m}$, $w = 2\,\mathrm{m}$, and $x = 24\,\mathrm{m}$, we can evaluate the expressions in Eqs. (2) and (3) to obtain

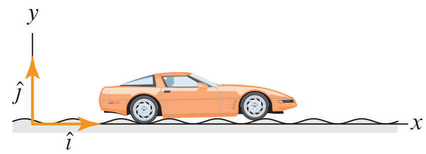
$$\vec{v} = (27.78\,\hat{\imath} - 1.490\,\hat{\jmath})\,\mathrm{m/s} \quad \text{and} \quad \vec{a} = 32.56\,\hat{\jmath}\,\mathrm{m/s^2}.$$

Problem 2.32

A test track for automobiles has a portion with a specific profile described by:

$$y = h\big[1 - \cos(x/w)\big],$$

where $h = 0.75\,\text{ft}$ and $w = 10\,\text{ft}$, and where the argument of the cosine function is understood to be in radians. A car drives at a constant speed $v_0 = 35\,\text{mph}$. Modeling the car as a point moving along the given profile, find the velocity and acceleration of the car for $x = 97\,\text{ft}$. Express velocity in ft/s and acceleration in ft/s^2.



Solution

Letting x and y represent the coordinates of the car, the position of the car is $bvr = x\,\hat{\imath} + y\,\hat{\jmath}$, i.e.,

$$\vec{r} = x\,\hat{\imath} + h\big[1 - \cos(x/w)\big]\,\hat{\jmath}. \tag{1}$$

The velocity is the time derivative of the position. Using Eq. (1) and the chain rule, we have

$$\vec{v} = \dot{x}\,\hat{\imath} + \frac{h}{w}\sin\left(\frac{x}{w}\right)\dot{x}\,\hat{\jmath}. \tag{2}$$

The quantity $\dot{x}$ is currently unknown but it can be determined by enforcing the condition that the speed is equal to v_0. Recalling that $v = \sqrt{\dot{x}^2 + \dot{y}^2}$, from Eq. (2), setting the speed equal to v_0 gives

$$\dot{x}^2 + \frac{h^2}{w^2}\sin^2\left(\frac{x}{w}\right)\dot{x}^2 = v_0^2 \quad \Rightarrow \quad \dot{x} = \frac{w v_0}{\sqrt{w^2 + h^2\sin^2(x/w)}}, \tag{3}$$

where we have taken $\dot{x} > 0$ because the car moves in the positive x direction. Substituting the last of Eqs. (3) into Eq. (2), we have

$$\vec{v} = \frac{w v_0}{\sqrt{w^2 + h^2\sin^2(x/w)}}\,\hat{\imath} + \frac{h v_0 \sin(x/w)}{\sqrt{w^2 + h^2\sin^2(x/w)}}\,\hat{\jmath}. \tag{4}$$

For $v_0 = 35\,\text{mph} = 35\frac{5280}{3600}\,\text{ft/s}$, $h = 0.75\,\text{ft}$, $w = 10\,\text{ft}$, and $x = 97\,\text{ft}$, Eq. (4) gives

$$\boxed{\vec{v} = (51.32\,\hat{\imath} - 1.046\,\hat{\jmath})\,\text{ft/s}.}$$

The acceleration is the time derivative of the velocity. Differentiating Eq. (2) with respect to time and using the chain rule, we have

$$\vec{a} = \frac{d\dot{x}}{dx}\dot{x}\,\hat{\imath} + \frac{h}{w}\left[\frac{\dot{x}^2}{w}\cos\left(\frac{x}{w}\right) + \sin\left(\frac{x}{w}\right)\frac{d\dot{x}}{dx}\dot{x}\right]\hat{\jmath}. \tag{5}$$

Differentiating the last of Eqs. (3) with respect to x we have

$$\frac{d\dot{x}}{dx} = -\frac{h^2 v_0 \sin(2x/w)}{2\big[w^2 + h^2 \sin^2(x/w)\big]^{3/2}}, \tag{6}$$

where we have used the trigonometric identity $2\sin\alpha\cos\alpha = \sin(2\alpha)$. Substituting the last of Eqs. (3) and Eq. (6) into Eq. (5) and simplifying, we have

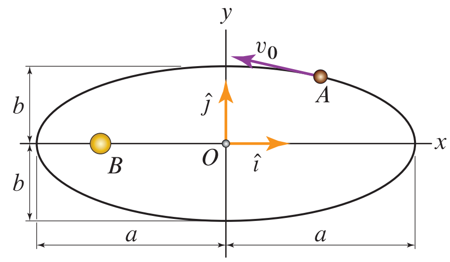
$$\vec{a} = -\frac{h^2 w v_0^2 \sin(2x/w)}{2\big[w^2 + h^2 \sin^2(x/w)\big]^2}\,\hat{\imath} + \frac{h w^2 v_0^2 \cos(x/w)}{\big[w^2 + h^2 \sin^2(x/w)\big]^2}\,\hat{\jmath}, \tag{7}$$

where, again, we have used the trigonometric identity $2\sin\alpha\cos\alpha = \sin(2\alpha)$. For $v_0 = 35\,\text{mph} = 35\frac{5280}{3600}\,\text{ft/s}$, $h = 0.75\,\text{ft}$, $w = 10\,\text{ft}$, and $x = 97\,\text{ft}$, Eq. (7) gives

$$\boxed{\vec{a} = -(0.3873\,\hat{\imath} + 19.00\,\hat{\jmath})\,\text{ft/s}^2.}$$

Problem 2.33

The orbit of a satellite A around planet B is the ellipse shown and is described by the equation $(x/a)^2 + (y/b)^2 = 1$, where a and b are the semimajor and semiminor axes of the ellipse, respectively. When $x = a/2$ and $y > 0$, the satellite is moving with a speed v_0 as shown. Determine the expression for the satellite's velocity $\vec{v}$ in terms of v_0, a, and b for $x = a/2$ and $y > 0$.



Solution

We begin by identifying the value of the y coordinate of the satellite corresponding to $x = a/2$ and the condition $y > 0$. Setting $x = a/2$ in the equation describing the path of the satellite, we have

$$\frac{1}{4} + \frac{y^2}{b^2} = 1 \quad \Rightarrow \quad y = \frac{\sqrt{3}}{2}b, \tag{1}$$

where we have selected the positive root to satisfy the requirement that $y > 0$. Next, differentiating the relation describing the path of the satellite with respect to time, we have

$$\frac{x\dot{x}}{a^2} + \frac{y\dot{y}}{b^2} = 0. \tag{2}$$

Substituting $x = a/2$ and the last of Eqs. (1) in Eq. (2), after simplifying, we have

$$\dot{x} + \frac{a\sqrt{3}\dot{y}}{b} = 0. \tag{3}$$

We also know that

$$\dot{x}^2 + \dot{y}^2 = v_0^2, \tag{4}$$

where we treat the speed v_0 as a known quantity. Equations (3) and (4) form a system of two second order algebraic equations in the two unknowns $\dot{x}$ and $\dot{y}$. This system has two solutions. However, as shown in the figure, at the instant considered A is moving upward and to the left. Hence, we have that $\dot{x} < 0$, which is sufficient to allow us to determine the following unique solution:

$$\dot{x} = \frac{-\sqrt{3}av_0}{\sqrt{3a^2 + b^2}} \quad \text{and} \quad \dot{y} = \frac{v_0 b}{\sqrt{3a^2 + b^2}}. \tag{5}$$

Recalling that the velocity is given by $\vec{v} = \dot{x}\,\hat{\imath} + \dot{y}\,\hat{\jmath}$, using Eq. (5), we can express the velocity as

$$\boxed{\vec{v} = \frac{v_0}{\sqrt{3a^2 + b^2}}\left(-\sqrt{3}a\,\hat{\imath} + b\,\hat{\jmath}\right).}$$

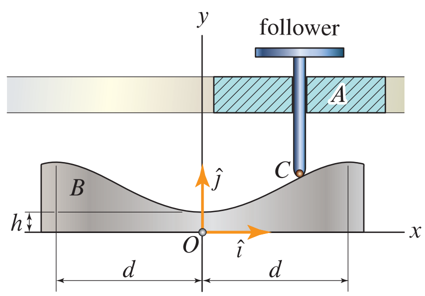
Problem 2.34

In the mechanism shown, block B is fixed and has a profile described by the following relation:

$$y = h\left[1 + \frac{1}{2}\left(\frac{x}{d}\right)^2 - \frac{1}{4}\left(\frac{x}{d}\right)^4\right].$$

The follower moves with the shuttle A, and the tip C of the follower remains in contact with B.

Assume that $h = 0.25$ in., $d = 1$ in., and the horizontal position of C is $x = d\sin(\omega t)$, where $\omega = 2\pi$ rad/s, and t is time in seconds. Determine an analytical expression for the speed of C as a function of x and the parameters d, h, and ω. Then, evaluate the speed of C for $x = 0$, $x = 0.5$ in., and $x = 1$ in. Express your answers in ft/s.



Solution

The position of C is

$$\vec{r} = x\,\hat{\imath} + y\,\hat{\jmath}. \tag{1}$$

Since C remains in contact with B, then the vertical position of C, namely y, is given as a function of x in the problem statement. Hence, we can rewrite Eq. (1) as follows:

$$\vec{r} = x\,\hat{\imath} + h\left[1 + \frac{1}{2}\left(\frac{x}{d}\right)^2 - \frac{1}{4}\left(\frac{x}{d}\right)^4\right]\hat{\jmath}. \tag{2}$$

The velocity of C is obtained by differentiating Eq. (2) with respect to time. Using the chain rule, this gives

$$\vec{v} = \dot{x}\,\hat{\imath} + h\left(\frac{x}{d^2} - \frac{x^3}{d^4}\right)\dot{x}\,\hat{\jmath}. \tag{3}$$

Recalling that the speed is the magnitude of the velocity, using Eq. (3) we have

$$v = |\dot{x}|\sqrt{1 + h^2\left(\frac{x}{d^2} - \frac{x^3}{d^4}\right)^2}. \tag{4}$$

Since $x = d\sin(\omega t)$, differentiating this expression with respect to time, we have

$$\dot{x} = d\omega\cos(\omega t). \tag{5}$$

Equation (5) implies that

$$|\dot{x}| = d\omega|\cos(\omega t)| \quad\Rightarrow\quad |\dot{x}| = d\omega\sqrt{1 - \sin^2(\omega t)} \quad\Rightarrow\quad |\dot{x}| = d\omega\sqrt{1 - (x/d)^2}, \tag{6}$$

where, in obtaining the last of Eqs. (6), we have used the fact that $x = d \sin(\omega t)$. Substituting the last of Eqs. (6) into Eq. (4), we have the expression for the speed requested by the problem statement:

$$v = \omega\sqrt{d^2 - x^2}\sqrt{1 + h^2\left(\frac{x}{d^2} - \frac{x^3}{d^4}\right)^2}. \tag{7}$$

Recalling that $h = 0.25\,\text{in.} = \frac{0.25}{12}\,\text{ft}$, $d = 1\,\text{in.} = \frac{1}{12}\,\text{ft}$, and $\omega = 2\pi\,\text{rad/s}$, we can evaluate Eq. (7) for $x = 0$, $x = 0.5\,\text{in.} = \frac{0.5}{12}\,\text{ft}$, and $x = 1\,\text{in.} = \frac{1}{12}\,\text{ft}$ to obtain, respectively,

$$v = 0.5236\,\text{ft/s}, \quad v = 0.4554\,\text{ft/s}, \quad \text{and} \quad v = 0.$$

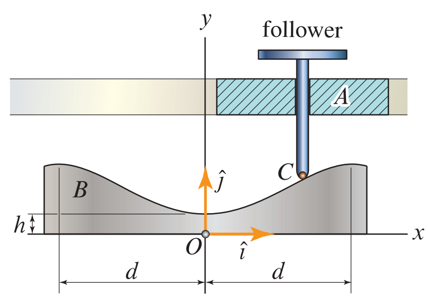
Problem 2.35

In the mechanism shown, block B is fixed and has a profile described by the following relation:

$$y = h\left[1 + \frac{1}{2}\left(\frac{x}{d}\right)^2 - \frac{1}{4}\left(\frac{x}{d}\right)^4\right].$$

The follower moves with the shuttle A, and the tip C of the follower remains in contact with B.

Assume that $h = 2\,\text{mm}$, $d = 20\,\text{mm}$, and A is made to move from $x = -d$ to $x = d$ with a constant speed $v_0 = 0.1\,\text{m/s}$. Determine the acceleration of C for $x = 15\,\text{mm}$. Express your answer in m/s^2.



Solution

The position of C is

$$\vec{r} = x\,\hat{\imath} + y\,\hat{\jmath}. \tag{1}$$

Since C remains in contact with B, then the vertical position of C, namely y, is given as a function of x in the problem statement. Hence, we can rewrite Eq. (1) as follows:

$$\vec{r} = x\,\hat{\imath} + h\left[1 + \frac{1}{2}\left(\frac{x}{d}\right)^2 - \frac{1}{4}\left(\frac{x}{d}\right)^4\right]\hat{\jmath}. \tag{2}$$

The velocity of C is obtained by differentiating Eq. (2) with respect to time. Using the chain rule, this gives

$$\vec{v} = \dot{x}\,\hat{\imath} + h\left(\frac{x}{d^2} - \frac{x^3}{d^4}\right)\dot{x}\,\hat{\jmath}. \tag{3}$$

Since the follower moves with A, and since A moves to the right with the constant speed v_0, we then have

$$\dot{x} = v_0. \tag{4}$$

Substituting Eq. (4) into Eq. (3) we then have that the velocity of C is

$$\vec{v} = v_0\,\hat{\imath} + v_0 h\left(\frac{x}{d^2} - \frac{x^3}{d^4}\right)\hat{\jmath}. \tag{5}$$

The acceleration of C is obtained by differentiating the expression of the velocity in Eq. (5) with respect to time. Using the chain rule, this gives

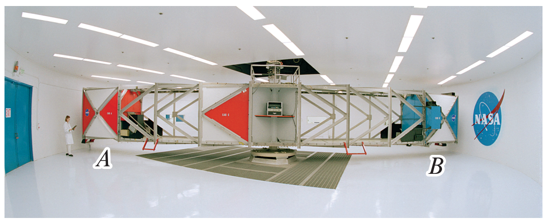
$$\vec{a} = \frac{h v_0^2}{d^2}\left(1 - 3\frac{x^2}{d^2}\right)\hat{\jmath}, \tag{6}$$

where we have accounted for the fact that v_0 is constant. Recalling that $h = 2\,\text{mm} = \frac{2}{1000}\,\text{m}$, $d = 20\,\text{mm} = \frac{20}{1000}\,\text{m}$, $v_0 = 0.1\,\text{m/s}$, we can evaluate the expression in Eq. (6) for $x = 15\,\text{mm} = \frac{15}{1000}\,\text{m}$ to obtain

$$\boxed{\vec{a} = -0.03438\,\hat{\jmath}\,\text{m/s}^2.}$$

Problem 2.36

The Center for Gravitational Biology Research at NASA's Ames Research Center runs a large centrifuge capable of $20g$ of acceleration, where g is the acceleration due to gravity ($12.5g$ is the maximum for human subjects). The distance from the axis of rotation to the cab at either A or B is $R = 25\,\text{ft}$. The trajectory of A is described by $y_A = \sqrt{R^2 - x_A^2}$ for $y_A \geq 0$ and by $y_A = -\sqrt{R^2 - x_A^2}$ for $y_A < 0$. If A moves at the constant speed $v_A = 120\,\text{ft/s}$, determine the velocity and acceleration of A when $x_A = -20\,\text{ft}$ and $y_A > 0$.



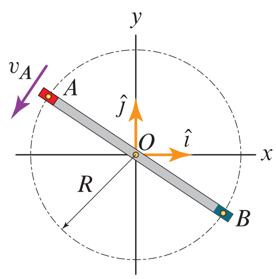


Photo credit: NASA

Solution

Starting with the equation of the trajectory for $y_A > 0$, and differentiating it with respect to time, we have

$$\dot{y}_A = \frac{-x_A \dot{x}_A}{\sqrt{R^2 - x_A^2}}. \tag{1}$$

We now recall that the speed of A can be computed as $v_A = \sqrt{\dot{x}_A^2 + \dot{y}_A^2}$, which implies

$$\dot{x}_A^2 + \dot{y}_A^2 = v_A^2 \quad \Rightarrow \quad \dot{x}_A^2 + \frac{x_A^2 \dot{x}_A^2}{R^2 - x_A^2} = v_A^2, \tag{2}$$

where the last of Eqs. (2) was obtained by using Eq. (1). The last of Eqs. (2) can be solved for $\dot{x}_A$ to obtain

$$\dot{x}_A = -\frac{v_A}{R}\sqrt{R^2 - x_A^2}, \tag{3}$$

where, referring to the figure in the problem statement, we have chosen the root with $\dot{x}_A < 0$ since, for $x_A = -20\,\text{ft}$, A is moving down and to the left. Substituting Eq. (3) into Eq. (1), we have

$$\dot{y}_A = \frac{v_A x_A}{R}. \tag{4}$$

Since $\vec{v}_A = \dot{x}_A\,\hat{\imath} + \dot{y}_A\,\hat{\jmath}$, using Eqs. (3) and (4), we can now express the velocity of A as

$$\vec{v}_A = -\frac{v_A}{R}\sqrt{R^2 - x_A^2}\,\hat{\imath} + \frac{v_A x_A}{R}\,\hat{\jmath}. \tag{5}$$

For $v_A = 120\,\text{ft/s}$, $R = 25\,\text{ft}$, and $x_A = -20\,\text{ft}$, we can evaluate Eq. (5) to obtain

$$\boxed{\vec{v}_A = -(72.00\,\hat{\imath} + 96.00\,\hat{\jmath})\,\text{ft/s}.}$$

Since the acceleration is the time derivative of the velocity, we can determine the acceleration of A by differentiating Eq. (5) with respect to time. Using the chain rule, this gives

$$\vec{a}_A = \frac{v_A}{R} \frac{x_A \dot{x}_A}{\sqrt{R^2 - x_A^2}}\,\hat{\imath} + \frac{v_A}{R}\dot{x}_A\,\hat{\jmath}. \tag{6}$$

Substituting Eq. (3) into Eq. (6) and simplifying, we have

$$\vec{a}_A = -\frac{v_A^2 x_A}{R^2}\,\hat{\imath} - \frac{v_A^2}{R^2}\sqrt{R^2 - x_A^2}\,\hat{\jmath}. \tag{7}$$

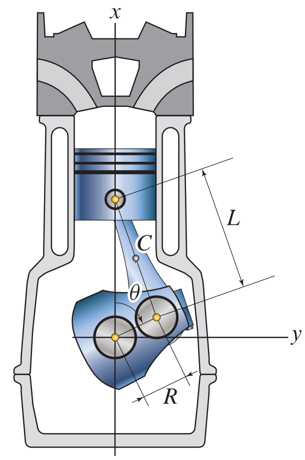
For $v_A = 120\,\text{ft/s}$, $R = 25\,\text{ft}$, and $x_A = -20\,\text{ft}$, we can evaluate Eq. (7) to obtain

$$\boxed{\vec{a}_A = (460.8\,\hat{\imath} - 345.6\,\hat{\jmath})\,\text{ft/s}^2.}$$

Problem 2.37

Point C is a point on the connecting rod of a mechanism called a *slider-crank*. The x and y coordinates of C can be expressed as follows: $x_C = R\cos\theta + \frac{1}{2}\sqrt{L^2 - R^2\sin^2\theta}$ and $y_C = (R/2)\sin\theta$, where θ describes the position of the crank. The crank rotates at a constant rate such that $\theta = \omega t$, where t is time.

Find expressions for the velocity, speed, and acceleration of C as functions of the angle θ and the parameters, R, L, and ω.



Solution

Using the coordinate system and expressions given in the problem statement, the position of point C can be expressed as as a function of θ as follows:

$$\vec{r}_C = x_C\,\hat{\imath} + y_C\,\hat{\jmath} = \left(R\cos\theta + \tfrac{1}{2}\sqrt{L^2 - R^2\sin^2\theta}\right)\hat{\imath} + \left(\tfrac{1}{2}R\sin\theta\right)\hat{\jmath}. \tag{1}$$

The velocity is the time derivative of the position. Hence, differentiating Eq. (1) with respect to time and using the chain rule, we have

$$\vec{v}_C = \frac{-\dot{\theta}R}{2}\left(2\sin\theta + \frac{R\sin\theta\cos\theta}{\sqrt{L^2 - R^2\sin^2\theta}}\right)\hat{\imath} + \frac{\dot{\theta}R}{2}\cos\theta\,\hat{\jmath}. \tag{2}$$

Since $\theta = \omega t$ and that therefore $\dot{\theta} = \omega$, we can rewrite Eq. (2) as

$$\boxed{\vec{v}_C = \frac{-\omega R}{2}\left(2\sin\theta + \frac{R\sin\theta\cos\theta}{\sqrt{L^2 - R^2\sin^2\theta}}\right)\hat{\imath} + \frac{\omega R}{2}\cos\theta\,\hat{\jmath}.} \tag{3}$$

The speed is now found by taking the magnitude of the velocity vector. Using Eq. (3), this gives

$$\boxed{v_C = \frac{\omega R}{2}\sqrt{4\sin^2\theta + \frac{4R\sin^2\theta\cos\theta}{\sqrt{L^2 - R^2\sin^2\theta}} + \frac{R^2\sin^2\theta\cos^2\theta}{L^2 - R^2\sin^2\theta} + \cos^2\theta}.} \tag{4}$$

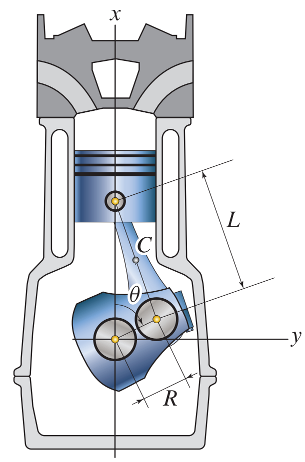
The acceleration is found by taking the derivative of the velocity. Hence, differentiating Eq. (3) with respect to time, using the chain rule and recalling that $\dot{\theta} = \omega$, we have

$$\boxed{\vec{a}_C = \frac{-\omega^2 R}{2}\left[2\cos\theta + \frac{R(\cos^2\theta - \sin^2\theta)}{\sqrt{L^2 - R^2\sin^2\theta}} + \frac{R^3\cos^2\theta\sin^2\theta}{(L^2 - R^2\sin^2\theta)^{3/2}}\right]\hat{\imath} - \frac{\omega^2 R}{2}\sin\theta\,\hat{\jmath}.} \tag{5}$$

Problem 2.38

Point C is a point on the connecting rod of a mechanism called a *slider-crank*. The x and y coordinates of C can be expressed as follows: $x_C = R\cos\theta + \frac{1}{2}\sqrt{L^2 - R^2\sin^2\theta}$ and $y_C = (R/2)\sin\theta$, where θ describes the position of the crank. The crank rotates at a constant rate such that $\theta = \omega t$, where t is time.

Let t be expressed in seconds, $R = 0.1$ m, $L = 0.25$ m, and $\omega = 250$ rad/s. Plot the trajectory of point C for $0 \le t \le 0.025$ s. For the same interval of time, plot the speed as a function of time, as well as the components of the velocity and acceleration of C.



Solution

The velocity of point C is the time derivative of the position of C. Using the coordinate system shown and since $\theta = \omega t$, can be written as

$$\vec{r}_C = x_C\,\hat{\imath} + y_C\,\hat{\jmath} = \left(R\cos\omega t + \tfrac{1}{2}\sqrt{L^2 - R^2\sin^2\omega t}\right)\hat{\imath} + \left(\tfrac{1}{2}R\sin\omega t\right)\hat{\jmath}. \tag{1}$$

Hence, differentiating the above expression with respect to time and simplifying, we have

$$\vec{v}_C = \frac{-\omega R}{2}\left(2\sin\omega t + \frac{R\sin\omega t\cos\omega t}{\sqrt{L^2 - R^2\sin^2\omega t}}\right)\hat{\imath} + \frac{\omega R}{2}\cos\omega t\,\hat{\jmath}. \tag{2}$$

The speed is now found by taking the magnitude of the velocity vector:

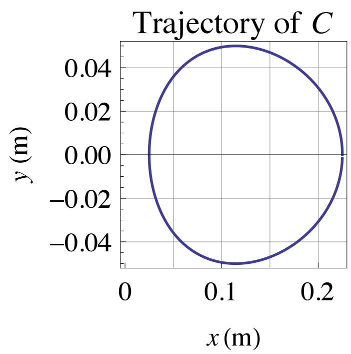
$$v_C = \frac{\omega R}{2}\sqrt{4\sin^2\omega t + \frac{4R\sin^2\omega t\cos\omega t}{\sqrt{L^2 - R^2\sin^2\omega t}} + \frac{R^2\sin^2\omega t\cos^2\omega t}{L^2 - R^2\sin^2\omega t} + \cos^2\omega t}. \tag{3}$$

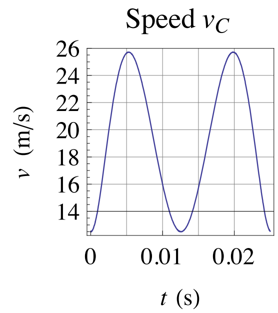
The acceleration is found by taking the derivative of the velocity. Hence, differentiating Eq. (2) with respect to time, we have

$$\vec{a}_C = \frac{-\omega^2 R}{2}\left[2\cos\omega t + \frac{R(\cos^2\omega t - \sin^2\omega t)}{\sqrt{L^2 - R^2\sin^2\omega t}} + \frac{R^3\cos^2\theta\sin^2\omega t}{(L^2 - R^2\sin^2\omega t)^{3/2}}\right]\hat{\imath} - \frac{\omega^2 R}{2}\sin\omega t\,\hat{\jmath}. \tag{4}$$

Plot of the trajectory and speed of C. Plots of the trajectory and speed of C for $0 < t < 0.025$ s can be generated with appropriate numerical software. The plots presented below were generated using *Mathematica* with the following code:

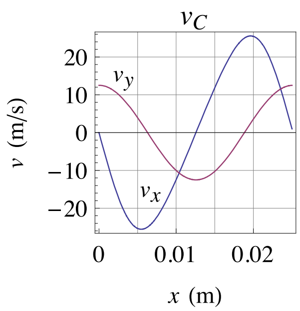
```
Parameters = {ω → 250., L -> 0.25, R → 0.1};
xC = R Cos[ω t] + 1/2 Sqrt[L^2 - R^2 Sin[ω t]^2]; yC = R/2 Sin[ω t];
vC = (ω R)/2 Sqrt[4 Sin[ω t]^2 + (4 R Sin[ω t]^2 Cos[ω t])/Sqrt[L^2 - R^2 Sin[ω t]^2] + (R^2 Sin[ω t]^2 Cos[ω t]^2)/(L^2 - R^2 Sin[ω t]^2) + Cos[ω t]^2];
ParametricPlot[{xC, yC} /. Parameters, {t, 0, 0.025}, Frame → True,
 FrameTicks → {{Automatic, None}, {{0, 0.1, 0.2}, None}}, GridLines → Automatic,
 AspectRatio → 1, FrameLabel → {"x (m)", "y (m)"}, PlotLabel → "Trajectory of C"]
Plot[vC /. Parameters, {t, 0, 0.025}, Frame → True,
 FrameTicks → {{Automatic, None}, {{0, 0.01, 0.02, 0.04}, None}}, GridLines → Automatic,
 AspectRatio → 1, FrameLabel → {"t  (s)", "v  (m/s)"}, PlotLabel → "Speed v_C"]
```





Plots of the components of $\vec{v}_C$ and $\vec{a}_C$. The components of point C's velocity and acceleration are shown below in the plots to the left and right, respectively. These plots were obtained using the following code:

```
Parameters = {ω → 250., L -> 0.25, R → 0.1};
vCx = (-ω R)/2 (2 Sin[ω t] + (R Sin[ω t] Cos[ω t])/Sqrt[L^2 - R^2 Sin[ω t]^2]); vCy = (ω R)/2 Cos[ω t];
aCx = (-ω^2 R)/2 (2 Cos[ω t] + (R (Cos[ω t]^2 - Sin[ω t]^2))/Sqrt[L^2 - R^2 Sin[ω t]^2] + (R^3 Cos[ω t]^2 Sin[ω t]^2)/(L^2 - R^2 Sin[ω t]^2)^(3/2));
aCy = -(ω^2 R)/2 Sin[ω t];
Plot[{vCx, vCy} /. Parameters, {t, 0, 0.025}, Frame → True,
 FrameTicks → {{Automatic, None}, {{0, 0.01, 0.02, 0.04}, None}}, GridLines → Automatic,
 AspectRatio → 1, FrameLabel → {"x  (m)", "v  (m/s)"}, PlotLabel → "v_C"]
Plot[{aCx, aCy} /. Parameters, {t, 0, 0.025}, Frame → True,
 FrameTicks → {{Automatic, None}, {{0, 0.01, 0.02, 0.04}, None}}, GridLines → Automatic,
 AspectRatio → 1, FrameLabel → {"t  (s)", "a  (m/s^2)"}, PlotLabel → "a_C"]
```


v_C
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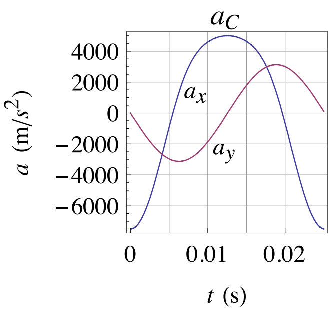
10
0
−10
−20
v_y
v_x
v (m/s)
0
0.01
0.02
x (m)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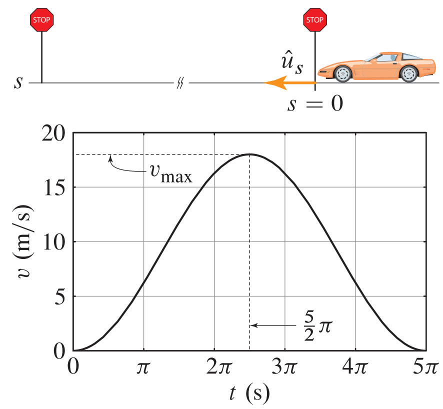
a_y
a (m/s^2)
0
0.01
0.02
t (s)

Problem 2.39

The following four problems refer to a car traveling between two stop signs, in which the car's velocity is assumed to be given by $v(t) = [9 - 9\cos(2t/5)]$ m/s for $0 \le t \le 5\pi$ s.

Determine $v_{\max}$, the maximum velocity reached by the car. Furthermore, determine the position $s_{v_{\max}}$ and the time $t_{v_{\max}}$ at which $v_{\max}$ occurs.



Solution

The expression for $v(t)$ consists of the difference between the constant 9 m/s and a cosine function multiplied by the value 9 m/s. The cosine function can only range between −1 and 1. The corresponding range of v is therefore $v = 18$ m/s, when $\cos(2t/5) = -1$, and $v = 0$, when $\cos(2t/5) = 1$. Hence, the maximum value of $v(t)$ is

$$\boxed{v_{\max} = 18.00\,\text{m/s}.} \tag{1}$$

Since $v = v_{\max}$ when the cosine function is equal to −1, we have

$$\cos(2t_{v_{\max}}/5) = -1 \quad \Rightarrow \quad 2t_{v_{\max}}/5 = \pi\ \text{s} \quad \Rightarrow \quad t_{v_{\max}} = (5\pi/2)\ \text{s}, \tag{2}$$

where the last of Eqs. (2) is the only admissible solution with $0 \le t \le 5\pi$ s. Evaluating the last of Eqs. (2) to four significant figures, we have

$$\boxed{t_{v_{\max}} = 7.854\,\text{s}.}$$

To find the position $s_{v_{\max}}$ at which $v_{\max}$ is achieved, we first determine $s(t)$. Recalling that $v = ds/dt$, we can write

$$ds = v(t)\,dt \quad \Rightarrow \quad \int_0^s ds = \int_0^t v(t)\,dt, \tag{3}$$

where the limits of integration reflect the fact that $s = 0$ for $t = 0$. Integrating $v(t)$, we have

$$s = \left[9t - \frac{45}{2}\sin\!\left(\frac{2t}{5}\right)\right]\text{m}. \tag{4}$$

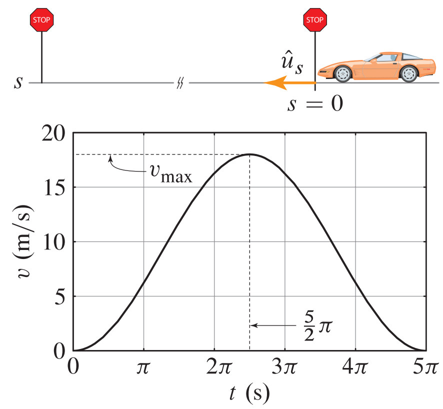
Given that we have already determined $t_{v_{\max}}$, we can find $s_{v_{\max}}$ by letting $t = t_{v_{\max}}$ in Eq. (4). This gives $s_{v_{\max}} = (45\pi/2)$ m, which, to four significant figures, gives

$$\boxed{s_{v_{\max}} = 70.69\,\text{m}.} \tag{5}$$

Problem 2.40

The following four problems refer to a car traveling between two stop signs, in which the car's velocity is assumed to be given by $v(t) = [9 - 9\cos(2t/5)]$ m/s for $0 \le t \le 5\pi$ s.

Determine the time at which the brakes are applied and the car starts to slow down.



Solution

Applying the brakes causes the acceleration $a = dv/dt$ to go from a positive value to a negative value. The instant $t = t_{braking}$ at which the brakes are applied corresponds to the time instant at which $a = 0$. Differentiating the given expression for $v(t)$ with respect to time and setting the result equal to zero, we have

$$\left[\frac{18}{5}\sin\left(\frac{2t_{braking}}{5}\right)\right] = 0 \quad \Rightarrow \quad t_{braking} = \frac{5\pi}{2}\,\text{s}, \tag{1}$$

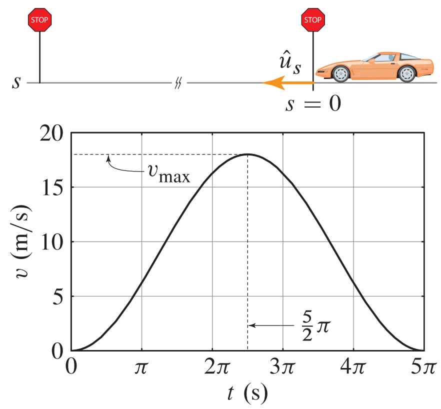
where we have selected the only solution in the range $0 \le t \le 5\pi$ s. Expressing the above result to four siginificant figures, we have

$$\boxed{t_{braking} = 7.854\,\text{s}.} \tag{2}$$

Problem 2.41

The following four problems refer to a car traveling between two stop signs, in which the car's velocity is assumed to be given by $v(t) = [9 - 9\cos(2t/5)]$ m/s for $0 \le t \le 5\pi$ s.

Determine the average velocity of the car between the two stop signs.



Solution

By definition, the average velocity is the change in position divided by the time it takes for the change in position to occur. We denote by t_i and t_f the times at which the car starts moving and comes to a stop, respectively (the subscripts i and f stand for initial and final, respectively). The car starts moving at time $t_i = 0$. For $t = t_f$, $v = 0$, so that we can find t_f by solving the equation $v(t_f) = 0$, i.e.,

$$[9 - 9\cos(2t_f/5)]\,\text{m/s} = 0 \quad \Rightarrow \quad \cos(2t_f/5) = 1 \quad \Rightarrow \quad t_f = 5\pi\,\text{s}. \tag{1}$$

To compute the change in position, we first determine $s(t)$. Recalling that $v = ds/dt$, we can write

$$ds = v(t)\,dt \quad \Rightarrow \quad \int_0^s ds = \int_0^t v(t)\,dt, \tag{2}$$

where the limits of integration reflect the fact that $s = 0$ for $t = 0$. Integrating $v(t)$, we have

$$s = \left[9t - \frac{45}{2}\sin\left(\frac{2t}{5}\right)\right]\text{m}. \tag{3}$$

Evaluating Eq. (3) with the value of the last of Eqs. (1), we have

$$s(t_f) = 45\pi\,\text{m}. \tag{4}$$

As stated earlier,

$$v_{avg} = \frac{s(t_f) - s(t_i)}{t_f - t_i}. \tag{5}$$

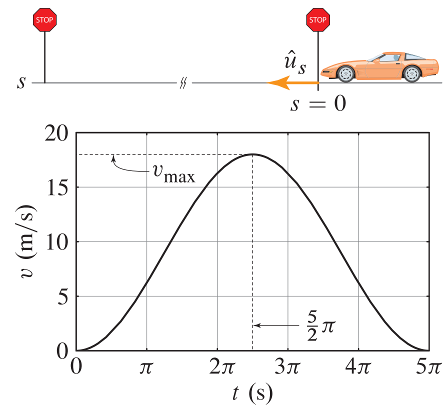
Recalling that $t_i = 0$, $s(t_i) = 0$, and using the results in Eqs. (1) and (4), we can evaluate v_{avg} to obtain

$$\boxed{v_{avg} = 9.000\,\text{m/s}.}$$

Problem 2.42

The following four problems refer to a car traveling between two stop signs, in which the car's velocity is assumed to be given by $v(t) = [9 - 9\cos(2t/5)]$ m/s for $0 \le t \le 5\pi$ s.

Determine $|a|_{max}$, the maximum of the magnitude of the acceleration reached by the car, and determine the position(s) at which $|a|_{max}$ occurs.



Solution

The acceleration is the time derivative of the velocity:

$$a = \frac{dv}{dt} = \frac{18}{5}\sin\left(\frac{2t}{5}\right)\,\mathrm{m/s^2} \quad\Rightarrow\quad |a| = \frac{18}{5}\left|\sin\left(\frac{2t}{5}\right)\right|\,\mathrm{m/s^2}. \tag{1}$$

The maximum value of $|a|$ corresponds to the maximum value of $|\sin(2t/5)|$, which is equal to one. Therefore $|a|_{max} = (18/5)\,\mathrm{m/s^2}$, which, expressed to four significant figures, is

$$\boxed{|a|_{max} = 3.600\,\mathrm{m/s^2}.}$$

Letting $t_{|a|_{max}}$ denote the time at which $|a|_{max}$ is achieved, we have that $\sin(2t_{|a|_{max}}/5) = \pm 1$, i.e.,

$$\left(t_{|a_{max}|}\right)_1 = (5\pi/4)\,\mathrm{s} \quad\text{and}\quad \left(t_{|a_{max}|}\right)_2 = (15\pi/4)\,\mathrm{s}, \tag{2}$$

where we have considered the only solutions in the time interval $0 \le t \le 5\pi$ s. Expressed to four significant figures, times $\left(t_{|a_{max}|}\right)_1$ and $\left(t_{|a_{max}|}\right)_2$ are

$$\boxed{\left(t_{|a_{max}|}\right)_1 = 3.927\,\mathrm{s} \quad\text{and}\quad \left(t_{|a_{max}|}\right)_2 = 11.78\,\mathrm{s}.}$$

To find the positions at which $|a|_{max}$ is achieved, we first determine $s(t)$. Recalling that $v = ds/dt$, we can write

$$ds = v(t)\,dt \quad\Rightarrow\quad \int_0^s ds = \int_0^t v(t)\,dt, \tag{3}$$

where the limits of integration reflect the fact that $s = 0$ for $t = 0$. Integrating $v(t)$, we have

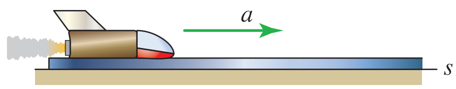
$$s = \left[9t - \frac{45}{2}\sin\left(\frac{2t}{5}\right)\right]\,\mathrm{m}. \tag{4}$$

Evaluating s in Eq. (4) at the times in Eqs. (2), we have

$$\boxed{\left(s_{|a_{max}|}\right)_1 = 12.84\,\mathrm{m} \quad\text{and}\quad \left(s_{|a_{max}|}\right)_2 = 128.5\,\mathrm{m}.}$$

Problem 2.43

The acceleration of a sled is prescribed to have the following form: $a = \beta\sqrt{t}$, where t is time expressed in seconds, and β is a constant. The sled starts from rest at $t = 0$. Determine β in such a way that the distance traveled after 1 s is 25 ft.



Solution

The acceleration is $a = dv/dt$. Since a is given as a function of time, we can separate the velocity and time variable as follows:

$$\frac{dv}{dt} = a(t) \quad \Rightarrow \quad dv = a(t)\,dt. \tag{1}$$

Recalling that $a(t) = \beta\sqrt{t}$ and that the velocity is equal to zero for $t = 0$, we integrate the last of Eqs. (1) as follows:

$$\int_0^v dv = \int_0^t \beta\sqrt{t}\,dt \quad \Rightarrow \quad v(t) = \tfrac{2}{3}\beta t^{3/2}. \tag{2}$$

We recall that the velocity is ds/dt, so that we can write

$$\frac{ds}{dt} = v(t) \quad \Rightarrow \quad ds = v(t)\,dt. \tag{3}$$

Letting s_0 denote the value of s for $t = 0$ and using the expression for $v(t)$ in the last of Eqs. (2), we can integrate the last of Eqs. (3):

$$\int_{s_0}^s ds = \int_0^t \tfrac{2}{3}\beta t^{3/2}\,dt \quad \Rightarrow \quad s - s_0 = \tfrac{4}{15}\beta t^{5/2}. \tag{4}$$

Letting d denote the distance traveled during $\Delta t = 1$ s, we can rewrite the last of Eqs. (4) as follows:

$$d = \tfrac{4}{15}\beta\Delta t^{5/2}, \tag{5}$$

which can be solved for β to obtain

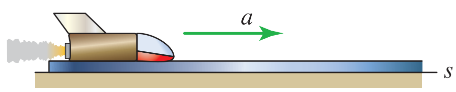
$$\beta = \frac{15d}{4\Delta t^{5/2}}. \tag{6}$$

Recalling that $d = 25$ ft and that $\Delta t = 1$ s, we can evaluate Eq. (6) to obtain

$$\boxed{\beta = 93.75\ \text{ft/s}^{5/2}.}$$

Problem 2.44

The acceleration of a sled can be prescribed to have one of the following forms: $a = \beta_1\sqrt{t}$, $a = \beta_2 t$, and $a = \beta_3 t^2$, where t is time expressed in seconds, $\beta_1 = 1\,\text{m/s}^{5/2}$, $\beta_2 = 1\,\text{m/s}^3$, and $\beta_3 = 1\,\text{m/s}^4$. The sled starts from rest at $t = 0$. Determine which of the three cases allows the sled to cover the largest distance in 1 s. In addition, determine the distance covered for the case in question.



Solution

We will determine how position depends on time for each of the three cases. We observe that the acceleration is given as a function of time in each of three cases. Recalling that $dv/dt = a$, we can separate velocity and time by writing

$$\frac{dv}{dt} = a(t) \quad \Rightarrow \quad dv = a(t)\,dt. \tag{1}$$

Recalling that the velocity is equal to zero for $t = 0$, we can integrate the last of Eqs. (1) as follows:

$$\int_0^v dv = \int_0^t a(t)\,dt. \tag{2}$$

Using the expression for $a(t)$ for the three given cases, we have, respectively,

$$v = \tfrac{2}{3}\beta_1 t^{3/2}, \quad v = \tfrac{1}{2}\beta_2 t^2, \quad \text{and} \quad v = \tfrac{1}{3}\beta_3 t^3. \tag{3}$$

We recall that the velocity is ds/dt. Since the velocity is now known as a function of time, we can separate the position and time by writing

$$\frac{ds}{dt} = v(t) \quad \Rightarrow \quad ds = v(t)\,dt. \tag{4}$$

Letting s_0 denote the value of s for $t = 0$ and using the expression for $v(t)$, we can integrate the last of Eqs. (4) as follows:

$$\int_{s_0}^s ds = \int_0^t v(t)\,dt \quad \Rightarrow \quad s - s_0 = \int_0^t v(t)\,dt. \tag{5}$$

Substituting into the last of Eqs. (5) the expressions for $v(t)$ given in Eqs. (3), we have

$$d = \tfrac{4}{15}\beta_1 t^{5/2}, \quad d = \tfrac{1}{6}\beta_2 t^3, \quad \text{and} \quad d = \tfrac{1}{12}\beta_3 t^4, \tag{6}$$

where we have denoted by d the distance $s - s_0$ traveled by the sled as a function of time. Recalling that $\beta_1 = 1\,\text{m/s}^{5/2}$, $\beta_2 = 1\,\text{m/s}^3$, $\beta_3 = 1\,\text{m/s}^4$, we can evaluate Eqs. (6) for $t = 1$ s to obtain, respectively,

$$d = 0.2667\,\text{m}, \quad d = 0.1667\,\text{m}, \quad \text{and} \quad d = 0.08333\,\text{m}. \tag{7}$$

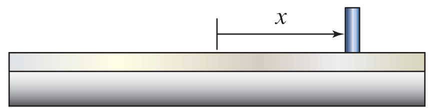
Comparing the three values of d we conclude that

Largest distance traveled in 1 s is $d = 0.2667$ m, corresponding to $a = \beta_1\sqrt{t}$.

Problem 2.45

A peg is constrained to move in a rectilinear guide and is given the following acceleration: $a = a_0 \sin \omega t$, where $a_0 = 20\,\text{ft/s}^2$, $\omega = 250\,\text{rad/s}$, and t is time expressed in seconds.

If $x = 0$ and $v = 0$ for $t = 0$, determine the position of the peg at $t = 4\,\text{s}$.



Solution

Recall that $a = dv/dt$. Since a is given as a function of time, we can separate the variables v and t by writing

$$dv = a(t)\,dt \quad \Rightarrow \quad dv = a_0 \sin \omega t\,dt. \tag{1}$$

Since $v = 0$ for $t = 0$, we can integrate the last of Eqs. (1) as follows:

$$\int_0^v dv = \int_0^t a_0 \sin \omega t\,dt \quad \Rightarrow \quad v = \frac{a_0}{\omega}(1 - \cos \omega t). \tag{2}$$

Next recall that $dx/dt = v$. Since the velocity is now a known function of time, applying separation of variables for this case, we have

$$dx = v(t)\,dt \quad \Rightarrow \quad dx = \frac{a_0}{\omega}(1 - \cos \omega t)\,dt, \tag{3}$$

where we have made use of the last of Eqs. (2). Recalling that $x = 0$ for $t = 0$, we can integrate the last of Eqs. (3) as follows:

$$\int_0^x dx = \int_0^t \frac{a_0}{\omega}(1 - \cos \omega t)\,dt \quad \Rightarrow \quad x = \frac{a_0}{\omega}\left(t - \frac{1}{\omega}\sin \omega t\right). \tag{4}$$

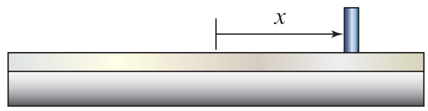
Recalling that $a_0 = 20\,\text{ft/s}^2$ and $\omega = 250\,\text{rad/s}$, at $t = 4\,\text{s}$, we have

$$x(4\,\text{s}) = 0.3197\,\text{ft}.$$

Problem 2.46

A peg is constrained to move in a rectilinear guide and is given the following acceleration: $a = a_0 \sin \omega t$, where $a_0 = 20\,\text{ft/s}^2$, $\omega = 250\,\text{rad/s}$, and t is time expressed in seconds.

Determine the value of the velocity of the peg at $t = 0$ so that $x(t)$ is periodic.



Solution

Recall that the acceleration is dv/dt. Hence, applying separation of variables for the case in which acceleration is given as a function of time, we can write

$$dv = a(t)\,dt \quad \Rightarrow \quad dv = a_0 \sin \omega t\,dt. \tag{1}$$

Letting v_0 be the value of the velocity for $t = 0$, we can integrate the last of Eqs. (1) as follows:

$$\int_{v_0}^{v} dv = \int_0^t a_0 \sin \omega t\,dt \quad \Rightarrow \quad v = v_0 + \frac{a_0}{\omega}(1 - \cos \omega t). \tag{2}$$

Next recall that $dx/dt = v$. Since the velocity is now a known function of time, applying separation of variables for this case, we have

$$dx = v(t)\,dt \quad \Rightarrow \quad dx = \left[v_0 + \frac{a_0}{\omega}(1 - \cos \omega t)\right] dt, \tag{3}$$

where we have made use of the last of Eqs. (2). Letting x_0 be the value of x for $t = 0$, we can integrate the last of Eqs. (3) as follows:

$$\int_0^x dx = \left[v_0 + \frac{a_0}{\omega}(1 - \cos \omega t)\right] dt \quad \Rightarrow \quad x = x_0 + v_0 t + \frac{a_0}{\omega}\left(t - \frac{1}{\omega}\sin \omega t\right). \tag{4}$$

To answer the question in the problem statement, we rewrite x as follows:

$$x = x_0 + \left(v_0 + \frac{a_0}{\omega}\right)t - \frac{a_0}{\omega^2}\sin \omega t. \tag{5}$$

We now observe that the first term on the right-hand side of Eq. (5) is a constant and therefore is a special case of a periodic function. The second term on the right-hand side of Eq. (5) is linear in t and therefore is not periodic. Finally, the third term term on the right-hand side of Eq. (5), is a periodic function of time with period $p = 2\pi\,\text{rad}/\omega$. We therefore conclude that for the motion to be periodic, we must require that

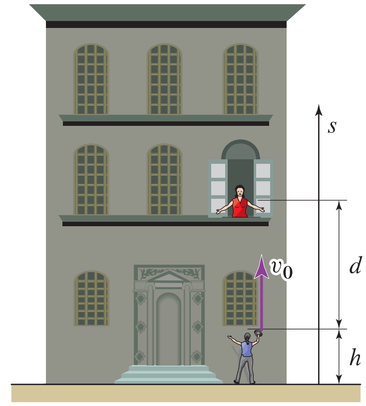
$$v_0 + \frac{a_0}{\omega} = 0 \quad \Rightarrow \quad v_0 = -\frac{a_0}{\omega}. \tag{6}$$

Recalling that $v_0 = v(0)$, $a_0 = 20\,\text{ft/s}^2$, and $\omega = 250\,\text{rad/s}$, we can evaluate the last of Eqs. (6) to obtain

$$\boxed{v(0) = -0.08000\,\text{ft/s}.}$$

Problem 2.47

A ring is thrown straight upward from a height $h = 2.5$ m off the ground and with an initial velocity $v_0 = 3.45$ m/s. Gravity causes the ring to have a constant downward acceleration $g = 9.81$ m/s^2. Determine h_{max}, the maximum height reached by the ring.



Solution

We note that the motion of the ring occurs with constant acceleration equal to g and directed downward. Denoting the release height of the ring by s_0, and using the constant acceleration equation relating position and velocity, we have

$$v^2 = v_0^2 - 2g(s - s_0), \tag{1}$$

where v_0 is the value of the speed corresponding to the position value s_0. The maximum height is the value of s corresponding to $v = 0$. Hence, setting $s = h_{max}$ and $v = 0$ in the above equation, and solving for h_{max}, we have

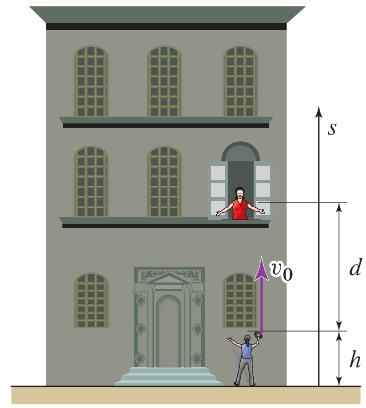
$$h_{max} = \frac{v_0^2}{2g} + s_0. \tag{2}$$

Recalling that $v_0 = 3.45$ m/s, $s_0 = 2.5$ m, and $g = 9.81$ m/s^2, we can evaluate Eq. (2) to obtain

$$\boxed{h_{max} = 3.107 \text{ m.}}$$

Problem 2.48

A ring is thrown straight upward from a height $h = 2.5$ m off the ground. Gravity causes the ring to have a constant downward acceleration $g = 9.81\,\mathrm{m/s^2}$. Letting $d = 5.2$ m, if the person at the window is to receive the ring in the gentlest possible manner, determine the initial velocity v_0 the ring must be given when first released.



Solution

We note that the motion of the ring occurs with constant acceleration equal to g and directed downward. Denoting the release height of the ring by s_0, and using the constant acceleration equation relating position to velocity, we have

$$v^2 = v_0^2 - 2g(s - s_0), \tag{1}$$

where v_0 is the value of the speed corresponding to the position value s_0.

The person receives the ring in the gentlest possible manner when the ring reaches the height $h + d$ with velocity equal zero. Hence, substituting $s_0 = h$, $s = (h + d)$, and $v = 0$ into Eq. (1) and setting the outcome equal to zero, gives

$$0 = v_0^2 - 2gd, \tag{2}$$

which is an equation that can be solved for v_0 to obtain

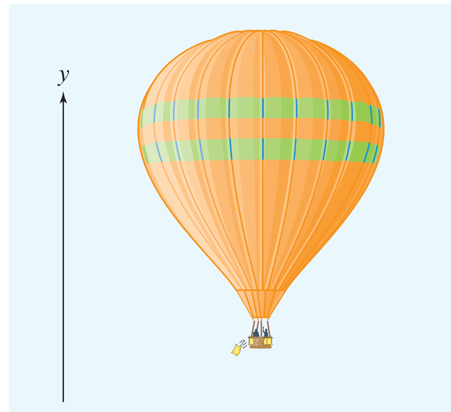
$$v_0 = \sqrt{2gd}, \tag{3}$$

where we have selected the positive root since the ring is thrown in the positive s direction. Recalling that $g = 9.81\,\mathrm{m/s^2}$ and $d = 5.2$ m, we can evaluate Eq. (3) to obtain

$$\boxed{v_0 = 10.10\,\mathrm{m/s}.}$$

Problem 2.49

A hot air balloon is climbing with a velocity of $7\,\mathrm{m/s}$ when a sandbag (used as ballast) is released at an altitude of $305\,\mathrm{m}$. Assuming that the sandbag is subject only to gravity and that therefore its acceleration is given by $\ddot{y} = -g$, g being the acceleration due to gravity, determine how long the sandbag takes to hit the ground and its impact velocity.



Solution

Since the acceleration of the sandbag is constant and equal to $-g$, position and time are related by the following equation

$$y = y_0 + \dot{y}_0 t - \tfrac{1}{2} g t^2, \tag{1}$$

where y_0 and $\dot{y}_0$ are the position and velocity at time $t = 0$. Letting t_{impact} denote the time at which the sandbag hits the ground, at $t = t_{\text{impact}}$ we must have

$$0 = y_0 + \dot{y}_0 t_{\text{impact}} - \tfrac{1}{2} g t_{\text{impact}}^2. \tag{2}$$

This is a second order algebraic equation for t_{impact} whose only physically admissible solution is

$$t_{\text{impact}} = \frac{1}{g}\left(\dot{y}_0 + \sqrt{\dot{y}_0^2 + 2 g y_0}\right). \tag{3}$$

Recalling that $g = 9.81\,\mathrm{m/s^2}$, $\dot{y}_0 = 7\,\mathrm{m/s}$, and $y_0 = 305\,\mathrm{m}$, we can evaluate Eq. (3) to obtain

$$\boxed{t_{\text{impact}} = 8.631\,\mathrm{s}.}$$

The expression of the velocity can be obtained by differentiating Eq. (1) with respect to time. This gives

$$\dot{y} = \dot{y}_0 - g t. \tag{4}$$

The impact velocity can be found by substituting t_{impact} from Eq. (3) into the (constant acceleration) Eq. (4), which, after simplification, gives

$$\dot{y}_{\text{impact}} = -\sqrt{\dot{y}_0^2 + 2 g y_0}. \tag{5}$$

Recalling again that $g = 9.81\,\mathrm{m/s^2}$, $\dot{y}_0 = 7\,\mathrm{m/s}$, and $y_0 = 305\,\mathrm{m}$, we can evaluate Eq. (5) to obtain

$$\boxed{\dot{y}_{\text{impact}} = -77.67\,\mathrm{m/s}.}$$

Problem 2.50

Approximately 1 h 15 min into the movie *King Kong* (the one directed by Peter Jackson), there is a scene in which Kong is holding Ann Darrow (played by the actress Naomi Watts) in his hand while swinging his arm in anger. A quick analysis of the movie indicates that at a particular moment Kong displaces Ann from rest by roughly 10 ft in a span of four frames. Knowing that the DVD plays at 24 frames per second and assuming that Kong subjects Ann to a constant acceleration, determine the acceleration Ann experiences in the scene in question. Express your answer in terms of the acceleration due to gravity g. Comment on what would happen to a person *really* subjected to this acceleration.

Solution

Let Δt denote the time it takes to play four frames at 24 frames per second. Hence we have

$$\Delta t = \frac{4\,\text{frames}}{24\,\text{frames/s}} = \frac{1}{6}\,\text{s}. \tag{1}$$

Δt is also the time it takes for King Kong to displace Ann by a distance $\Delta s = 10\,\text{ft}$. Since we are assuming that the acceleration is constant, the motion is rectilinear and we can apply the constant acceleration equation relating position to time:

$$s = s_0 + v_0(t - t_0) + \frac{1}{2}a_c(t - t_0)^2, \tag{2}$$

where s is the position at time t, and where s_0 and v_0 are the position and velocity at time $t = t_0$, respectively. Letting $\Delta t = t - t_0$ and $\Delta s = s - s_0$, solving Eq. (2) for a_c gives

$$a_c = 2\frac{\Delta s - v_0 \Delta t}{\Delta t^2}. \tag{3}$$

Since $\Delta t = \frac{1}{6}$ s, $\Delta s = 10\,\text{ft}$, and $v_0 = 0$, we can evaluate a_c/g, with $g = 32.2\,\text{ft/s}^2$, to obtain

$$a_c/g = 22.36. \tag{4}$$

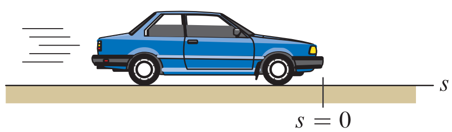
Finally, in terms of g we can then say that the acceleration to which Ann is subject is

$$\boxed{a_c = 22.36\,g.}$$

Since the human body cannot withstand much more that 10–15g of acceleration, an acceleration of more than 22g would likely kill Ann.

Problem 2.51

A car travels on a rectilinear stretch of road at a constant speed $v_0 = 65\,\text{mph}$. At $s = 0$ the driver applies the brakes hard enough to cause the car to skid. Assume that the car keeps sliding until it stops, and assume that throughout this process the car's acceleration is given by $\ddot{s} = -\mu_k g$, where $\mu_k = 0.76$ is the kinetic friction coefficient and g is the acceleration of gravity. Compute the car's stopping distance and time.



Solution

To determine the stopping distance, we recall that

$$\ddot{s} = \frac{d\dot{s}}{dt} \quad \Rightarrow \quad \ddot{s} = \frac{d\dot{s}}{ds}\frac{ds}{dt} \quad \Rightarrow \quad \ddot{s} = \dot{s}\frac{d\dot{s}}{ds}, \tag{1}$$

where we have used the chain rule and the definition of velocity to obtain the second and the third of Eqs. (1), respectively. Recalling that $\ddot{s} = -\mu_k g$, using the last of Eqs. (1), we can write

$$-\mu_k g\, ds = \dot{s}\, d\dot{s} \quad \Rightarrow \quad \int_0^s -\mu_k g\, ds = \int_{\dot{s}_0}^{\dot{s}} \dot{s}\, d\dot{s}, \tag{2}$$

where $s = 0$ is the position at which the brakes are applied, and where $\dot{s}_0$ is the velocity of the car for $s = 0$. Carrying out the integration in the last of Eqs. (2), we obtain

$$-\mu_k g s = \tfrac{1}{2}(\dot{s}^2 - \dot{s}_0^2). \tag{3}$$

Letting s_{stop} denote the stopping distance, we have that $\dot{s} = 0$ for $s = s_{\text{stop}}$. Enforcing this condition in Eq. (3) and solving for s_{stop}, we have

$$s_{\text{stop}} = \frac{\dot{s}_0^2}{2\mu_k g}. \tag{4}$$

Recalling that $\dot{s}_0 = v_0 = 65\,\text{mph} = 65\frac{5280}{3600}\,\text{ft/s}$, $\mu_k = 0.76$, and $g = 32.2\,\text{ft/s}^2$, we can evaluate Eq. (4) to obtain

$$\boxed{s_{\text{stop}} = 185.7\,\text{ft}.}$$

To determine the stopping time, we recall that $\ddot{s} = d\dot{s}/dt$. Also, recalling that $\ddot{s} = -\mu_k g$, we can then write $-\mu_k g\, dt = d\dot{s}$ and integrate as follows:

$$\int_0^{t_{\text{stop}}} -\mu_k g\, dt = \int_{\dot{s}_0}^0 d\dot{s} \quad \Rightarrow \quad -\mu_k g t_{\text{stop}} = -\dot{s}_0 \quad \Rightarrow \quad t_{\text{stop}} = \dot{s}_0/(\mu_k g), \tag{5}$$

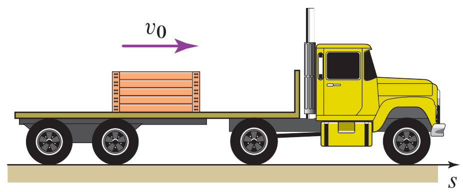
where $t = 0$ is the time at which the brakes are applied and t_{stop} denotes the stopping time. Recalling that $\dot{s}_0 = v_0 = 65\,\text{mph} = 65\frac{5280}{3600}\,\text{ft/s}$, $\mu_k = 0.76$, and $g = 32.2\,\text{ft/s}^2$, we can evaluate the last of Eqs. (5) to obtain

$$\boxed{t_{\text{stop}} = 3.896\,\text{s}.}$$

Problem 2.52

If the truck brakes and the crate slides to the right relative to the truck, the horizontal acceleration of the crate is given by $\ddot{s} = -g\mu_k$, where g is the acceleration of gravity, $\mu_k = 0.87$ is the kinetic friction coefficient, and s is the position of the crate relative to a coordinate system attached to the ground (rather than the truck).

Assuming that the crate slides without hitting the right end of the truck bed, determine the time it takes to stop if its velocity at the start of the sliding motion is $v_0 = 55$ mph.



Solution

To determine the stopping time, we recall that $\ddot{s} = d\dot{s}/dt$. Since $\ddot{s} = -\mu_k g$, we can then write $-\mu_k g\, dt = d\dot{s}$ and integrate as follows:

$$\int_0^{t_{\text{stop}}} -\mu_k g\, dt = \int_{\dot{s}_0}^0 d\dot{s} \quad\Rightarrow\quad -\mu_k g t_{\text{stop}} = -\dot{s}_0 \quad\Rightarrow\quad t_{\text{stop}} = \frac{\dot{s}_0}{\mu_k g}, \tag{1}$$

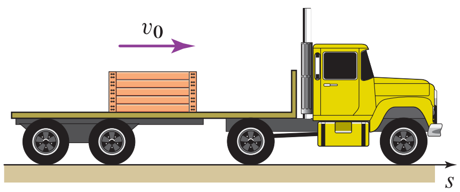
where $t = 0$ is the time at which the crate starts sliding and t_{stop} denotes the stopping time. Recalling that $\dot{s}_0 = v_0 = 55\,\text{mph} = 55\frac{5280}{3600}\,\text{ft/s}$, $\mu_k = 0.87$, and $g = 32.2\,\text{ft/s}^2$, we can evaluate the last of Eqs. (1) to obtain

$$t_{\text{stop}} = 2.880\,\text{s}.$$

Problem 2.53

If the truck brakes and the crate slides to the right relative to the truck, the horizontal acceleration of the crate is given by $\ddot{s} = -g\mu_k$, where g is the acceleration of gravity, $\mu_k = 0.87$ is the kinetic friction coefficient, and s is the position of the crate relative to a coordinate system attached to the ground (rather than the truck).

Assuming that the crate slides without hitting the right end of the truck bed, determine the distance it takes to stop if its velocity at the start of the sliding motion is $v_0 = 75\,\text{km/h}$.



Solution

To determine the stopping distance, we recall that

$$\ddot{s} = \frac{d\dot{s}}{dt} \quad \Rightarrow \quad \ddot{s} = \frac{d\dot{s}}{ds}\frac{ds}{dt} \quad \Rightarrow \quad \ddot{s} = \dot{s}\frac{d\dot{s}}{ds}, \tag{1}$$

where we have used the chain rule and the definition of velocity to obtain the second and the third of Eqs. (1), respectively. Recalling that $\ddot{s} = -\mu_k g$, using the last of Eqs. (1), we can write

$$-\mu_k g\, ds = \dot{s}\, d\dot{s} \quad \Rightarrow \quad \int_0^s -\mu_k g\, ds = \int_{\dot{s}_0}^{\dot{s}} \dot{s}\, d\dot{s}, \tag{2}$$

where $s = 0$ is the position at which the brakes are applied and $\dot{s}_0$ is the velocity of the crate at $s = 0$. Carrying out the integration in the last of Eqs. (2), we obtain

$$-\mu_k g s = \tfrac{1}{2}(\dot{s}^2 - \dot{s}_0^2). \tag{3}$$

Letting s_{stop} denote the stopping distance, we have that $\dot{s} = 0$ for $s = s_{\text{stop}}$. Enforcing this condition in Eq. (3) and solving for s_{stop}, we have

$$s_{\text{stop}} = \frac{\dot{s}_0^2}{2\mu_k g}. \tag{4}$$

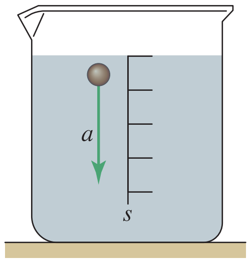
Recalling that $\dot{s}_0 = v_0 = 75\,\text{km/h} = 75\frac{1000}{3600}\,\text{m/s}$, $\mu_k = 0.87$, and $g = 9.81\,\text{m/s}^2$, we can evaluate Eq. (4) to obtain

$$\boxed{s_{\text{stop}} = 25.43\,\text{m}.}$$

Problem 2.54

A sphere is dropped from rest at the free surface of a thick polymer fluid. The acceleration of the sphere has the form $a = g - \eta v$, where g is the acceleration due to gravity, η is a constant, and v is the sphere's velocity.

The sphere is observed to reach a constant sinking velocity equal to 0.1 m/s. Determine η.



Solution

If the sphere achieves what appears to be a constant velocity, then the corresponding acceleration is equal to zero, i.e.,

$$g - \eta v = 0 \quad \Rightarrow \quad \eta = \frac{g}{v}. \tag{1}$$

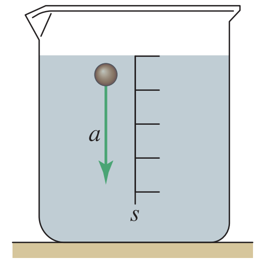
Recalling that $g = 9.81\,\text{m/s}^2$ and $v = 0.1\,\text{m/s}$, we can evaluate the second of Eqs. (1) to obtain

$$\boxed{\eta = 98.10\,\text{s}^{-1}.}$$

Problem 2.55

A sphere is dropped from rest at the free surface of a thick polymer fluid. The acceleration of the sphere has the form $a = g - \eta v$, where g is the acceleration due to gravity, η is a constant, and v is the sphere's velocity.

If $\eta = 50\,\mathrm{s}^{-1}$ determine the velocity of the sphere after 0.02 s. Express the result in feet per second.



Solution

We recall that $a = dv/dt$. Here the acceleration is given as a function of velocity, i.e., $a = a(v)$. Therefore we have $a(v) = dv/dt$ so that we can separate the v and t variables by writing

$$dt = \frac{dv}{a(v)} \quad \Rightarrow \quad dt = \frac{dv}{g - \eta v}. \tag{1}$$

Let $t_f = 0.02\,\mathrm{s}$ and $v_f = v(t_f)$. Since the sphere is released from rest at time $t = 0$, we can integrate the last of Eqs. (1) as follows:

$$\int_0^{t_f} dt = \int_0^{v_f} \frac{dt}{g - \eta v} \quad \Rightarrow \quad t_f = \left[-\frac{1}{\eta}\ln(g - \eta v)\right]_0^{v_f} \quad \Rightarrow \quad t_f = -\frac{1}{\eta}\ln\left(\frac{g - \eta v_f}{g}\right), \tag{2}$$

where we have used the logarithm property according to which $\ln a - \ln b = \ln(a/b)$. The last of Eqs. (2) can be solved for v_f. To do so, we first isolate the logarithmic term and then take the exponential of both sides of the resulting equation:

$$-\eta t_f = \ln\left(\frac{g - \eta v_f}{g}\right) \quad \Rightarrow \quad e^{-\eta t_f} = \frac{g - \eta v_f}{g} \quad \Rightarrow \quad v_f = \frac{g}{\eta}\left(1 - e^{-\eta t_f}\right). \tag{3}$$

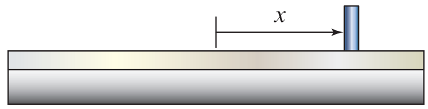
Recalling that $v_f = v(0.02\,\mathrm{s})$, $g = 32.2\,\mathrm{ft/s^2}$, $\eta = 50\,\mathrm{s}^{-1}$, and $t_f = 0.02\,\mathrm{s}$, we can evaluate the last of Eqs. (3) to obtain

$$\boxed{v(0.02\,\mathrm{s}) = 0.4071\,\mathrm{ft/s}.}$$

Problem 2.56

The motion of a peg sliding within a rectilinear guide is controlled by an actuator in such a way that the peg's acceleration takes on the form $\ddot{x} = a_0(2\cos 2\omega t - \beta \sin \omega t)$, where t is time, $a_0 = 3.5\,\text{m/s}^2$, $\omega = 0.5\,\text{rad/s}$, and $\beta = 1.5$.

Determine the expressions for the velocity and the position of the peg as functions of time if $\dot{x}(0) = 0\,\text{m/s}$ and $x(0) = 0\,\text{m}$.



Solution

Since $\ddot{x} = d\dot{x}/dt = a_0(2\cos 2\omega t - \beta \sin \omega t)$, we can write

$$a_0(2\cos 2\omega t - \beta \sin \omega t)\,dt = d\dot{x} \quad \Rightarrow \quad \int_0^t a_0(2\cos 2\omega t - \beta \sin \omega t)\,dt = \int_0^{\dot{x}} d\dot{x}, \tag{1}$$

where, in choosing the limits of integration, we accounted for the fact that $\dot{x} = 0$ for $t = 0$. Carrying out the integration in the second of Eqs. (1) and solving for $\dot{x}$, we have

$$\dot{x} = \frac{a_0}{\omega}(\sin 2\omega t + \beta \cos \omega t - \beta). \tag{2}$$

Recalling that $a_0 = 3.5\,\text{m/s}^2$, $\omega = 0.5\,\text{rad/s}$, and $\beta = 1.5$, we can express Eq. (2) as

$$\boxed{\dot{x} = \big\{7.000 \sin[(1.000\,\text{rad/s})t] + 10.50 \cos[(0.5000\,\text{rad/s})t] - 10.50\big\}\,\text{m/s},}$$

where t is in seconds.

Recalling that $\dot{x} = dx/dt$ so that we have $dx = \dot{x}\,dt$ and using the expression of $\dot{x}$ in Eq. (2), we can write

$$\frac{a_0}{\omega}(\sin 2\omega t + \beta \cos \omega t - \beta)\,dt = dx \quad \Rightarrow \quad \int_0^t \frac{a_0}{\omega}(\sin 2\omega t + \beta \cos \omega t - \beta)\,dt = \int_0^x dx, \tag{3}$$

where, in choosing the limits of integration, we accounted for the fact that $x = 0$ for $t = 0$. Carrying out the integration in the second of Eqs. (3) and solving for x, we have

$$x = \frac{a_0}{2\omega^2}(1 - \cos 2\omega t + 2\beta \sin \omega t - 2\beta\omega t). \tag{4}$$

Recalling that $a_0 = 3.5\,\text{m/s}^2$, $\omega = 0.5\,\text{rad/s}$, and $\beta = 1.5$, we can express Eq. (4) as

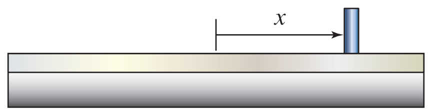
$$\boxed{x = \big\{7.000 - 7.000 \cos[(1.000\,\text{rad/s})t] + 21.00 \sin[(0.5000\,\text{rad/s})t] - (10.50\,\text{s}^{-1})t\big\}\,\text{m/s}^2,}$$

where, again, t is in seconds.

Problem 2.57

The motion of a peg sliding within a rectilinear guide is controlled by an actuator in such a way that the peg's acceleration takes on the form $\ddot{x} = a_0(2\cos 2\omega t - \beta \sin \omega t)$, where t is time, $a_0 = 3.5\,\text{m/s}^2$, $\omega = 0.5\,\text{rad/s}$, and $\beta = 1.5$.

Determine the total distance traveled by the peg during the time interval $0\,\text{s} \le t \le 5\,\text{s}$ if $\dot{x}(0) = a_0\beta/\omega$.



Solution

Since $\ddot{x} = d\dot{x}/dt = a_0(2\cos 2\omega t - \beta \sin \omega t)$, we can write

$$a_0(2\cos 2\omega t - \beta \sin \omega t)\,dt = d\dot{x} \quad \Rightarrow \quad \int_0^t a_0(2\cos 2\omega t - \beta \sin \omega t)\,dt = \int_{\frac{a_0\beta}{\omega}}^{\dot{x}} d\dot{x}, \tag{1}$$

where, in choosing the limits of integration, we accounted for the fact that $\dot{x} = a_0\beta/\omega$ at $t = 0$. Carrying out the integration in the second of Eqs. (1) and solving for $\dot{x}$, we have

$$\dot{x} = \frac{a_0}{\omega}(\sin 2\omega t + \beta \cos \omega t). \tag{2}$$

To find the total distance traveled we must first determine when $\dot{x}(t)$ changes sign during the specified time interval. To do so, we first rewrite Eq. (2) as follows:

$$\dot{x}(t) = \frac{a_0}{\omega}\cos \omega t\big(2\sin \omega t + \beta\big), \tag{3}$$

where we have used the trigonometric identity $\sin 2\omega t = 2\sin \omega t \cos \omega t$ and then factored the $\cos \omega t$ term. Then, referring to Eq. (3), and keeping in mind that we are only interested in the peg's motion for $0 \le t \le 5\,\text{s}$, we see that

$$\cos \omega t > 0 \quad \text{for} \quad 0 < \omega t < \frac{\pi}{2}\,\text{rad} \quad \Rightarrow \quad 0 < t < \frac{\pi\,\text{rad}}{2\omega} = \pi\,\text{s} < 5\,\text{s}, \tag{4}$$

where we have used the fact that $\omega = 0.5\,\text{rad/s}$. In addition, we have that

$$2\sin \omega t + \beta > 0 \quad \text{for all} \quad 0 < t < 5\,\text{s}. \tag{5}$$

Since, for $0 \le t \le 5\,\text{s}$, $\dot{x}$ changes sign when $t = \pi\,\text{s}$, the distance traveled must be computed by integrating the velocity as follows:

$$\begin{aligned} d &= \int_{0\,\text{s}}^{\pi\,\text{s}} \frac{a_0}{\omega}\big[\cos \omega t(2\sin \omega t + \beta)\big]\,dt - \int_{\pi\,\text{s}}^{5\,\text{s}} \frac{a_0}{\omega}\big[\cos \omega t(2\sin \omega t + \beta)\big]\,dt \\ &= \frac{a_0}{2\omega^2}\Big[-\cos 2\omega t + 2\beta \sin \omega t\Big]_{0\,\text{s}}^{\pi\,\text{s}} - \frac{a_0}{2\omega^2}\Big[-\cos 2\omega t + 2\beta \sin \omega t\Big]_{\pi\,\text{s}}^{5\,\text{s}} \\ &= \frac{a_0}{2\omega^2}\Big\{-2\cos[2\omega(\pi\,\text{s})] + 4\beta \sin[\omega(\pi\,\text{s})] + 1 + \cos[2\omega(5\,\text{s})] - 2\beta \sin[\omega(5\,\text{s})]\Big\}. \end{aligned} \tag{6}$$

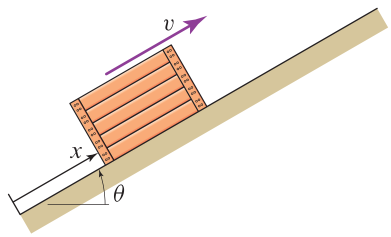
Recalling $a_0 = 3.5\,\text{m/s}^2$, $\omega = 0.5\,\text{rad/s}$, and $\beta = 1.5$, we can evaluate the result in Eq. (6) to obtain

$$\boxed{d = 52.42\,\text{m}.}$$

Problem 2.58

A package is pushed up an incline at $x = 0$ with an initial speed v_0. The incline is coated with a thin viscous layer so that the acceleration of the package is given by $a = -(g \sin\theta + \eta v)$, where g is the acceleration due to gravity, η is a constant, and v is the velocity of the package.

If $\theta = 30°$, $v_0 = 10\,\text{ft/s}$, and $\eta = 8\,\text{s}^{-1}$, determine the time it takes for the package to come to a stop.



Solution

We recall that $a = dv/dt$ and since the acceleration is given as a function of velocity, we can separate the variables v and t by writing

$$dt = \frac{dv}{a(v)} \quad \Rightarrow \quad dt = -\frac{dv}{g\sin\theta + \eta v}. \tag{1}$$

Letting t_{stop} be the time at which $v = 0$ and given that $v = v_0$ at $t = 0$, we can integrate the second of Eqs. (1) as follows:

$$\int_0^{t_{\text{stop}}} dt = \int_{v_0}^{0} -\frac{dv}{g\sin\theta + \eta v}$$

$$\Rightarrow \quad t_{\text{stop}} = -\frac{1}{\eta}\ln(g\sin\theta + \eta v)\Big|_{v_0}^{0}$$

$$\Rightarrow \quad t_{\text{stop}} = \frac{1}{\eta}\ln\left(\frac{g\sin\theta + \eta v_0}{g\sin\theta}\right), \tag{2}$$

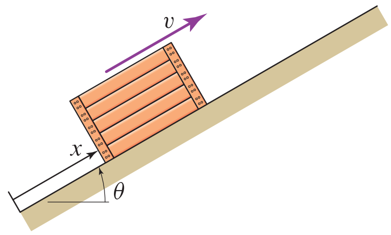
where we have used the logarithm property $\ln a - \ln b = \ln(a/b)$. Given that $\eta = 8\,\text{s}^{-1}$, $g = 32.2\,\text{ft/s}^2$, $\theta = 30°$, and $v_0 = 10\,\text{ft/s}$, we can evaluate t_{stop} to obtain

$$\boxed{t_{\text{stop}} = 0.2233\,\text{s}.}$$

Problem 2.59

A package is pushed up an incline at $x = 0$ with an initial speed v_0. The incline is coated with a thin viscous layer so that the acceleration of the package is given by $a = -(g \sin\theta + \eta v)$, where g is the acceleration due to gravity, η is a constant, and v is the velocity of the package.

If $\theta = 25°$, $v_0 = 7\,\text{m/s}$, and $\eta = 8\,\text{s}^{-1}$, determine the distance d traveled by the package before it comes to a stop.



Solution

We recall that $a = dv/dt$ and that, since we need to relate a change in position to a corresponding change in velocity, the chain rule allows us to write $a = v\,dv/dx$. Then, since the acceleration is given as a function of velocity, we can separate the x and v variables as follows:

$$dx = \frac{v\,dv}{a(v)} \quad \Rightarrow \quad dx = -\frac{v}{g\sin\theta + \eta v}\,dv. \tag{1}$$

Since $v = v_0$ for $x = 0$ and $v = 0$, for $x = d$, we can integrate the second of Eqs. (1) as follows:

$$\int_0^d dx = \int_{v_0}^0 -\frac{v}{g\sin\theta + \eta v}\,dv \quad \Rightarrow \quad d = \int_{v_0}^0 -\frac{v}{g\sin\theta + \eta v}\,dv. \tag{2}$$

To facilitate the integration of the right-hand side of the last of Eqs. (2), we observe that the integrand can be rewritten as follows:

$$-\frac{v}{g\sin\theta + \eta v} = -\frac{1}{\eta} + \frac{g\sin\theta}{\eta(g\sin\theta + \eta v)}. \tag{3}$$

Substituting Eq. (3) into the last of Eqs. (2) and carrying out the integration, we have

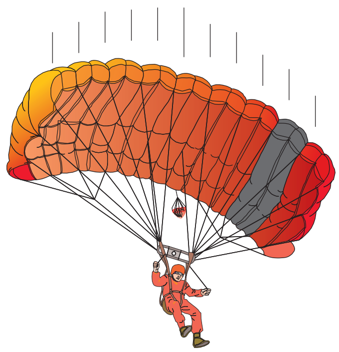
$$d = \frac{v_0}{\eta} + \frac{g\sin\theta}{\eta^2}\ln(g\sin\theta + \eta v)\Big|_{v_0}^{0} \quad \Rightarrow \quad d = \frac{v_0}{\eta} + \frac{g\sin\theta}{\eta^2}\ln\left(\frac{g\sin\theta}{g\sin\theta + \eta v_0}\right), \tag{4}$$

where we have used the logarithm property $\ln a - \ln b = \ln(a/b)$. Given that $v_0 = 7\,\text{m/s}$, $\eta = 8\,\text{s}^{-1}$, $g = 9.81\,\text{m/s}^2$, $\theta = 25°$, we can evaluate the last of Eqs. (4) to obtain

$$\boxed{d = 0.7017\,\text{m}.}$$

Problem 2.60

Referring to Example 2.8 on p. 56, and defining *terminal velocity* as the velocity at which a falling object stops accelerating, determine the skydiver's terminal velocity without performing any integrations.



Solution

From Example 2.8 on p. 56, we have that the acceleration of the skydiver is

$$a = g - \frac{C_d}{m} v^2. \tag{1}$$

Denoting the terminal velocity by v_{term}, we have that $a = 0$ for $v = v_{\text{term}}$. Enforcing this condition in Eq. (1) and solving for v_{term}, we have

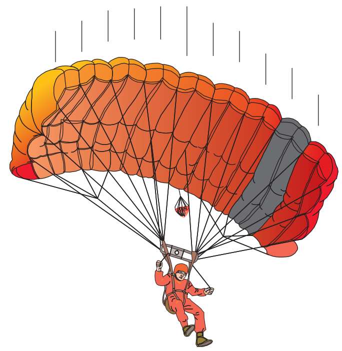
$$v_{\text{term}} = \sqrt{\frac{mg}{C_d}}. \tag{2}$$

From Example 2.8, we have that $C_d = 43.2\,\text{kg/m}$, $m = 110\,\text{kg}$, and $g = 9.81\,\text{m/s}^2$, so that we can evaluate Eq. (2) to obtain

$$\boxed{v_{\text{term}} = 4.998\,\text{m/s}.}$$

Problem 2.61

Referring to Example 2.8 on p. 56, determine the distance d traveled by the skydiver from the instant the parachute is deployed until the difference between the velocity and the terminal velocity is 10% of the terminal velocity.



Solution

The acceleration can be related to the velocity and position as follows:

$$a = \frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = v\frac{dv}{ds}. \tag{1}$$

From Example 2.8 on p. 56 we have that $a = g - C_d v^2/m$. Hence, substituting this expression into Eq. (1), we can separate the variables s and v as follows:

$$ds = \frac{v}{g - C_d v^2/m}\,dv. \tag{2}$$

From Example 2.8, we know that when the parachute is deployed, the velocity of the skydiver is $v_0 = 44.5\,\text{m/s}$. In addition, we know that the terminal velocity is $v_{\text{term}} = \sqrt{mg/C_d}$, where $m = 110\,\text{kg}$, $g = 9.81\,\text{m/s}^2$, and $C_d = 43.2\,\text{kg/m}$. Letting $v_{\text{qt}} = 1.1 v_{\text{term}}$ denote the value of the velocity that is 10% away from that of the terminal velocity (where the subscript 'qt' stands for quasi-terminal), and letting d be the distance traveled to achieve v_{qt} starting from v_0, we can integrate Eq. (2) as follows:

$$\int_0^d ds = \int_{v_0}^{v_{\text{qt}}} \frac{v}{g - (C_d/m)v^2}\,dv. \tag{3}$$

Carrying out the above integrations, we have

$$d = -\frac{m}{2C_d}\ln\left[\frac{g - (C_d/m)v_{\text{qt}}^2}{g - (C_d/m)v_0^2}\right] = -\frac{m}{2C_d}\ln\left[\frac{g(1 - 1.1^2)}{g - (C_d/m)v_0^2}\right], \tag{4}$$

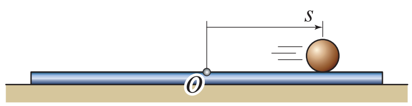
where we have used the expression of v_{qt} to obtain this last expression. So, recalling that $m = 110\,\text{kg}$, $v_0 = 44.5\,\text{m/s}$, $C_d = 43.2\,\text{kg/m}$, and $g = 9.81\,\text{m/s}^2$, we can evaluate the above expression to obtain

$$\boxed{d = 7.538\,\text{m}.}$$

Problem 2.62

In a physics experiment, a sphere with a given electric charge is constrained to move along a rectilinear guide with the following acceleration: $a = a_0 \sin(2\pi s/\lambda)$, where $a_0 = 8\,\mathrm{m/s^2}$, π is measured in radians, s is the position of the sphere measured in meters, $-\lambda \le s \le \lambda$, and $\lambda = 0.25\,\mathrm{m}$.

If the sphere is placed at rest at $s = 0$ and then gently nudged away from this position, what is the maximum speed that the sphere could achieve, and where would this maximum occur?



Solution

Recall that $a = dv/dt$. In this problem, the acceleration is given as a function of position. Hence, to relate a change in velocity to a correspondent change in position, we start by rewriting the acceleration via the chain rule: $a = v\,dv/ds$. This allows us to write

$$v\,dv = a(s)\,ds \quad \Rightarrow \quad v\,dv = a_0 \sin(2\pi s/\lambda)\,ds, \tag{1}$$

where we have separated the variables v and s and used the given expression for the acceleration. Recalling that $v = 0$ for $s = 0$, we can now integrate the last of Eqs. (1) as follows:

$$\int_0^v v\,dv = \int_0^s a_0 \sin(2\pi s/\lambda)\,ds \quad \Rightarrow \quad v^2 = \frac{a_0\lambda}{\pi}[1 - \cos(2\pi s/\lambda)]. \tag{2}$$

The speed is the magnitude of the velocity, namely, $|v|$. Since $|v| = \sqrt{v^2}$, we can solve the second of Eqs. (2) for $|v|$ to obtain

$$|v| = \sqrt{\frac{a_0\lambda}{\pi}[1 - \cos(2\pi s/\lambda)]}. \tag{3}$$

To determine the maximum possible value of the speed, we observe that the cosine function under the square root in Eq. (3) can vary only between the values -1 and 1. Hence, the maximum possible value of the speed is achieved where the cosine function takes on the value -1, which gives

$$|v|_{\max} = \sqrt{\frac{2a_0\lambda}{\pi}}. \tag{4}$$

Recalling that $a_0 = 8\,\mathrm{m/s^2}$ and $\lambda = 0.25\,\mathrm{m}$, we can evaluate Eq. (4) to obtain

$$\boxed{|v|_{\max} = 1.128\,\mathrm{m/s}.}$$

We have already argued that $|v|_{\max}$ occurs where the cosine function under the square root in Eq. (3) achieves the value -1. In turn, this implies that

$$\frac{2\pi s_{|v|_{\max}}}{\lambda} = \pi + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots \quad \Rightarrow \quad s_{|v|_{\max}} = \tfrac{1}{2}\lambda(1 + 2n), \quad n = 0, \pm 1, \pm 2, \ldots, \tag{5}$$

where $s_{|v|_{\max}}$ denotes the value of s for which $|v| = |v|_{\max}$. Recalling that $\lambda = 0.15\,\mathrm{m}$ and that $-\lambda \le s \le \lambda$, we can evaluate the second of Eqs. (5) to obtain following two values for $s_{|v|_{\max}}$ that are within the admissible range for s:

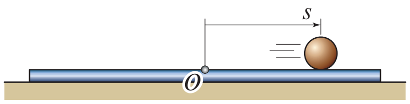
$$\boxed{s_{|v|_{\max}} = 0.1250\,\mathrm{m} \quad \text{and} \quad s_{|v|_{\max}} = -0.1250\,\mathrm{m},}$$

which correspond to $n = 0$ and $n = -1$, respectively.

Problem 2.63

In a physics experiment, a sphere with a given electric charge is constrained to move along a rectilinear guide with the following acceleration: $a = a_0 \sin(2\pi s/\lambda)$, where $a_0 = 8\,\mathrm{m/s^2}$, π is measured in radians, s is the position of the sphere measured in meters, $-\lambda \le s \le \lambda$, and $\lambda = 0.25\,\mathrm{m}$.

Suppose that the velocity of the sphere is equal to zero for $s = \lambda/4$. Determine the range of motion of the sphere, that is, the interval along the s axis within which the sphere moves. *Hint:* Determine the speed of the sphere and the interval along the s axis within which the speed has admissible values.



Solution

Recall that $a = dv/dt$. Since the acceleration is given as a function of position, to relate a change in velocity to a correspondent change in position, we start by rewriting $a = dv/dt$ via the chain rule: $a = v\,dv/ds$. This allows us to write

$$v\,dv = a(s)\,ds \quad \Rightarrow \quad v\,dv = a_0 \sin(2\pi s/\lambda)\,ds, \tag{1}$$

where we have separated the variables v and s and used the given expression for the acceleration. Recalling that $v = 0$ for $s = \lambda/4$, we can now integrate the last of Eqs. (1) as follows:

$$\int_0^v v\,dv = \int_{\frac{\lambda}{4}}^s a_0 \sin(2\pi s/\lambda)\,ds \quad \Rightarrow \quad v^2 = -\frac{a_0\lambda}{\pi}\cos(2\pi s/\lambda). \tag{2}$$

The speed is the magnitude of the velocity, namely, $|v|$. Since $|v| = \sqrt{v^2}$, the second of Eqs. (2) implies

$$|v| = \sqrt{-\frac{a_0\lambda}{\pi}\cos(2\pi s/\lambda)}. \tag{3}$$

The result in Eq. (3) is acceptable if $\cos(2\pi s/\lambda) \le 0$. From a mathematical viewpoint, this implies that

$$\tfrac{1}{2}\pi + 2\pi n \le \frac{2\pi s}{\lambda} \le \tfrac{3}{2}\pi + 2\pi n, \quad n = 0, \pm1, \pm2, \ldots$$
$$\Rightarrow \quad \tfrac{1}{4}\lambda(1+4n) \le s \le \tfrac{3}{4}\lambda(1+4n) \quad n = 0, \pm1, \pm2, \ldots \tag{4}$$

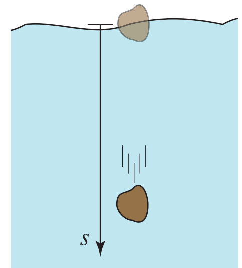
However, from a physical viewpoint, we do not expect the sphere to "jump" from an admissible range of motion to another. Observing that $a = a_0 > 0$ for $s = \lambda/4$, the sphere will move to the right when it is released from rest at $\lambda/4$. Therefore, referring to the last of Eqs. (4), the only acceptable range of motion is $\frac{1}{4}\lambda \le s \le \frac{3}{4}\lambda$, corresponding to $n = 0$. Since $\lambda = 0.25\,\mathrm{m}$, we have that the range of motion is

$$\boxed{0.06250\,\mathrm{m} \le s \le 0.1875\,\mathrm{m}.}$$

Problem 2.64

The acceleration of an object in rectilinear free fall while immersed in a linear viscous fluid is $a = g - C_d v/m$, where g is the acceleration of gravity, C_d is a constant drag coefficient, v is the object's velocity, and m is the object's mass.

Letting $t_0 = 0$ and $v_0 = 0$, find the velocity as a function of time and find the terminal velocity.



Solution

Acceleration is given as a function of velocity, so we first find time as a function of velocity and invert that result to determine the velocity as a function of time. Recalling that $a = dv/dt$, we can write

$$dt = \frac{dv}{a} \quad \Rightarrow \quad \int_0^t dt = \int_0^v \frac{dv}{g - (C_d/m)v}$$

$$\Rightarrow \quad t = -\frac{m}{C_d}\ln\bigl[g - (C_d/m)v\bigr]\Big|_0^v = -\frac{m}{C_d}\left\{\ln\bigl[g - (C_d/m)v\bigr] - \ln g\right\}$$

$$= -\frac{m}{C_d}\ln\left[\frac{g - (C_d/m)v}{g}\right] = \frac{-m}{C_d}\ln\left(1 - \frac{C_d}{mg}v\right),$$

where we have used $t_0 = 0$ and $v_0 = 0$ to obtain the lower limits on the two definite integrals. Solving this result for t as a function of v to find $v(t)$, we obtain

$$\boxed{v(t) = \frac{mg}{C_d}\left(1 - e^{-C_d t/m}\right).}$$

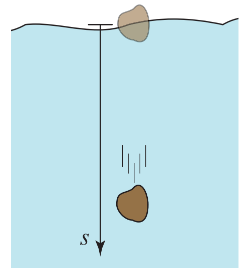
To find the terminal velocity, we can either take the limit as $t \to \infty$ of $v(t)$ or we can determine the velocity at which $a = 0$ in the given expression for the acceleration. Doing the latter, we obtain

$$0 = g - \frac{C_d v_{\text{term}}}{m} \quad \Rightarrow \quad \boxed{v_{\text{term}} = \frac{mg}{C_d}.}$$

Problem 2.65

The acceleration of an object in rectilinear free fall while immersed in a linear viscous fluid is $a = g - C_d v/m$, where g is the acceleration of gravity, C_d is a constant drag coefficient, v is the object's velocity, and m is the object's mass.

Letting $s_0 = 0$ and $v_0 = 0$, find the position as a function of velocity.



Solution

Recall that acceleration, velocity, and position can be related as follows:

$$a = v\frac{dv}{ds} \quad \Rightarrow \quad ds = \frac{v\,dv}{a}. \tag{1}$$

Since the acceleration is given as a function of the velocity, we can determine the position as a function of the velocity as follows:

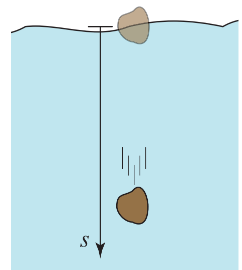
$$s(v) = \int_0^v \frac{v}{g - (C_d/m)v}\,dv = \int_0^v \frac{(C_d/m)v + g - g}{(C_d/m)[g - (C_d/m)v]}\,dv = \int_0^v \left[\frac{mg/C_d}{g - (C_d/m)v} - \frac{m}{C_d}\right]dv, \tag{2}$$

which can be evaluated to obtain

$$s(v) = \frac{-mg}{C_d}\frac{m}{C_d}\ln\left(\frac{g - (C_d/m)v}{g}\right) - \frac{m}{C_d}v \quad \Rightarrow \quad \boxed{s(v) = -\frac{m}{C_d}\left[v + \frac{mg}{C_d}\ln\left(1 - \frac{C_d v}{mg}\right)\right].} \tag{3}$$

Problem 2.66

A 1.5 kg rock is released from rest at the surface of a calm lake. If the resistance offered by the water as the rock falls is directly proportional to the rock's velocity, the rock's acceleration is $a = g - C_d v/m$, where g is the acceleration of gravity, C_d is a constant drag coefficient, v is the rock's velocity, and m is the rock's mass. Letting $C_d = 4.1\,\text{kg/s}$, determine the rock's velocity after 1.8 s.



Solution

We recall that $a = dv/dt$. Using the given expression for a we can write

$$g - \frac{C_d}{m} v = \frac{dv}{dt} \quad \Rightarrow \quad dt = \frac{dv}{g - (C_d/m)v}, \tag{1}$$

where, in writing the second of Eqs. (1), we have separated the variables v and t. Observing that $v = 0$ at $t = 0$, and letting v_f (f stands for final) denote the value of v at $t = t_f = 1.8\,\text{s}$, we can integrate the last of Eqs. (1) as follows:

$$\int_0^{t_f} dt = \int_0^{v_f} \frac{dv}{g - (C_d/m)v} \quad \Rightarrow \quad t_f = -\frac{m}{C_d} \ln\left[\frac{g - (C_d/m)v_f}{g}\right]$$

$$\Rightarrow \quad t_f = -\frac{m}{C_d} \ln\left(1 - \frac{C_d}{mg} v_f\right). \tag{2}$$

Solving the last of Eqs. (2) for v_f, we have

$$v_f = \frac{mg}{C_d}\left(1 - e^{-C_d t_f/m}\right). \tag{3}$$

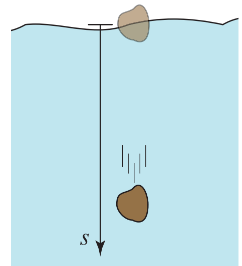
Given that $m = 1.5\,\text{kg}$, $g = 9.81\,\text{m/s}^2$, $C_d = 4.1\,\text{kg/s}$, and $t_f = 1.8\,\text{s}$, we can evaluate v_f to obtain

$$\boxed{v_f = 3.563\,\text{m/s}.}$$

Problem 2.67

A 3.1 lb rock is released from rest at the surface of a calm lake, and its acceleration is $a = g - C_d v/m$, where g is the acceleration of gravity, $C_d = 0.27\,\text{lb}\cdot\text{s/ft}$ is a constant drag coefficient, v is the rock's velocity, and m is the rock's mass.

Determine the depth to which the rock will have sunk when the rock achieves 99% of its terminal velocity.



Solution

We begin by determining the expression of the terminal velocity, which we denote by v_{term}. This is the velocity at which the acceleration is equal to zero. Using the given expression for the acceleration, we have

$$0 = g - \frac{C_d}{m} v_{\text{term}} \quad \Rightarrow \quad v_{\text{term}} = \frac{mg}{C_d}. \tag{1}$$

We denote by v_{qt} (qt stads for quasi-terminal) the value of v corresponding to 99% of v_{term}, i.e.,

$$v_{\text{qt}} = \frac{99}{100} \frac{mg}{C_d}. \tag{2}$$

Next, we recall that $a = dv/dt$. To relate the acceleration to position, we can use the chain rule and write $a = (dv/ds)(ds/dt) = v\,dv/ds$. Using this expression and the given expression for a, we can then write

$$g - \frac{C_d}{m} v = v \frac{dv}{ds} \quad \Rightarrow \quad ds = \frac{v}{g - (C_d/m)v}\, dv, \tag{3}$$

where, in writing the second of Eqs. (3) we have separated the variables v and s. We now observe that $v = 0$ for $s = 0$ and letting d be the value of s corresponding to $v = v_{\text{qt}}$, we can integrate the second of Eqs. (3) as follows:

$$\int_0^d ds = \int_0^{v_{\text{qt}}} \frac{v}{g - (C_d/m)v}\, dv \quad \Rightarrow \quad d = \int_0^{v_{\text{qt}}} \frac{(C_d/m)v + g - g}{(C_d/m)[g - (C_d/m)v]}\, dv$$
$$\Rightarrow \quad d = \int_0^{v_{\text{qt}}} \left[\frac{mg/C_d}{g - (C_d/m)v} - \frac{m}{C_d} \right] dv, \tag{4}$$

which can be evaluated to obtain

$$d = -\frac{m^2 g}{C_d^2} \ln\left(1 - \frac{C_d}{mg} v_{\text{qt}}\right) - \frac{m}{C_d} v_{\text{qt}} \quad \Rightarrow \quad d = \frac{m^2 g}{C_d^2} \left[\ln(100) - \frac{99}{100} \right], \tag{5}$$

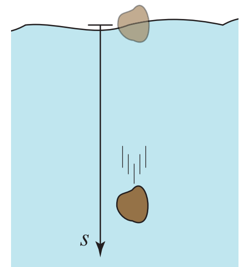
where, in writing the last of Eqs. (5), we have used Eq. (2). Recalling that $m = 3.1\,\text{lb}/g$, $g = 32.2\,\text{ft/s}^2$, and $C_d = 0.27\,\text{lb}\cdot\text{s/ft}$, we can evaluate the last of Eqs. (5) to obtain

$$\boxed{d = 14.80\,\text{ft}.}$$

Problem 2.68

A 3.1 lb rock is released from rest at the surface of a calm lake, and its acceleration is $a = g - C_d v/m$, where g is the acceleration of gravity, $C_d = 0.27\,\text{lb}\cdot\text{s/ft}$ is a constant drag coefficient, v is the rock's velocity, and m is the rock's mass.

Determine the rock's velocity after it drops 5 ft.



Solution

We recall that $a = dv/dt$. To relate the acceleration to position, we can use the chain rule and write $a = (dv/ds)(ds/dt) = v\,dv/ds$. Using this expression and the given expression for a, we can then write

$$g - \frac{C_d}{m}v = v\frac{dv}{ds} \quad\Rightarrow\quad ds = \frac{v}{g - (C_d/m)v}\,dv, \tag{1}$$

where, in writing the second of Eqs. (1) we have separated the variables v and s. We now observe that $v = 0$ for $s = 0$ so that we can integrate the second of Eqs. (1) as follows:

$$\int_0^s ds = \int_0^v \frac{v}{g - (C_d/m)v}\,dv \quad\Rightarrow\quad s = \int_0^v \frac{(C_d/m)v + g - g}{(C_d/m)[g - (C_d/m)v]}\,dv$$
$$\Rightarrow\quad s = \int_0^v \left[\frac{mg/C_d}{g - (C_d/m)v} - \frac{m}{C_d}\right] dv, \tag{2}$$

which can be evaluated to obtain

$$s = -\frac{m^2 g}{C_d^2}\ln\left(1 - \frac{C_d}{mg}v\right) - \frac{m}{C_d}v. \tag{3}$$

We now need to solve the above equation for v after setting $s = 5\,\text{ft}$. Since this cannot be done analytically, we will need to do it numerically. Given that $m = 3.1\,\text{lb}/g$, $g = 32.2\,\text{ft/s}^2$, and $C_d = 0.27\,\text{lb}\cdot\text{s/ft}$, the solution presented below was obtained using *Mathematica* via the following code:

```
Parameters = {m → 3.1/32.2, Cd → 0.27, g → 32.2, s → 5.};

FindRoot[(s == -(m^2 g)/(Cd^2) Log[1 - Cd/(m g) v] - m/Cd v /. Parameters), {v, 1.}]
```

where we note that, as required by most root finding algorithms, one needs to specify an initial guess for the solution (we have used $v = 1\,\text{ft/s}$). The execution of the above code, gives the following result (expressed to 4 significant figures)

$$\boxed{v(5\,\text{ft}) = 10.07\,\text{ft/s}.}$$

Problem 2.69

Suppose that the acceleration of an object of mass m along a straight line is $a = g - C_d v/m$, where the constants g and C_d are given and v is the object's velocity. If $v(t)$ is unknown and $v(0)$ is given, can you determine the object's velocity with the following integral?

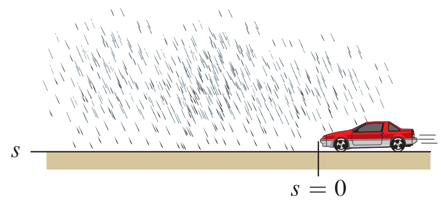
$$v(t) = v(0) + \int_0^t \left(g - \frac{C_d}{m} v \right) dt.$$

Solution

No, because the integrand is not an explicit function of the variable of integration, which is t. Clearly, if $v(t)$ is provided as an explicit function of time then one could integrate. However, if $v(t)$ were given it would seem superfluous to perform the integration to obtain it a second time.

Problem 2.70

Heavy rains cause a particular stretch of road to have a coefficient of friction that changes as a function of location. Specifically, measurements indicate that the friction coefficient has a 3% decrease per meter. Under these conditions the acceleration of a car skidding while trying to stop can be approximated by $\ddot{s} = -(\mu_k - cs)g$ (the 3% decrease in friction was used in deriving this equation for acceleration), where μ_k is the friction coefficient under dry conditions, g is the acceleration of gravity, and c, with units of m^{-1}, describes the rate of friction decrement. Let $\mu_k = 0.5$, $c = 0.015\,\mathrm{m}^{-1}$, and $v_0 = 45\,\mathrm{km/h}$, where v_0 is the initial velocity of the car. Determine the distance it will take the car to stop and the percentage of increase in stopping distance with respect to dry conditions, i.e., when $c = 0$.



Solution

We recall that the acceleration is $a = dv/dt$. The acceleration can be related to the position and velocity using the chain rule: $a = (dv/ds)(ds/dt) = v\,dv/ds$, where we have also used the fact that $ds/dt = v$. Hence, observing that $a = \ddot{s} = -(\mu_k - cs)g$, we can write

$$-(\mu_k - cs)g = v\frac{dv}{ds} \quad\Rightarrow\quad -(\mu_k - cs)g\,ds = v\,dv, \tag{1}$$

where we have separated the variables s and v. We now observe that $v = v_0$ for $s = 0$. In addition, letting s_{wet} denote the stopping distance over wet ground, we have that $v = 0$ for $s = s_{\mathrm{wet}}$. Therefore, the last of Eqs. (1) can be integrated as follows:

$$\int_0^{s_{\mathrm{wet}}} -(\mu_k - cs)g\,ds = \int_{v_0}^{0} v\,dv \quad\Rightarrow\quad \bigl(-\mu_k s_{\mathrm{wet}} + \tfrac{1}{2}cs_{\mathrm{wet}}^2\bigr)g = -\tfrac{1}{2}v_0^2. \tag{2}$$

Dividing the last of Eqs. (2) by g, multiplying by 2, and rearranging terms, we have

$$cs_{\mathrm{wet}}^2 - 2\mu_k s_{\mathrm{wet}} + v_0^2/g = 0 \quad\Rightarrow\quad s_{\mathrm{wet}} = \frac{1}{c}\left[\mu_k \pm \sqrt{\mu_k^2 - (cv_0^2)/g}\right]. \tag{3}$$

Given that $c = 0.015\,\mathrm{m}^{-1}$, $\mu_k = 0.5$, $v_0 = 45\,\mathrm{km/h} = 45(1000/3600)\,\mathrm{m/s}$, and $g = 9.81\,\mathrm{m/s^2}$, we can evaluate the last of Eqs. (3) to obtain the following two values of s_{wet}:

$$(s_{\mathrm{wet}})_1 = \frac{1}{c}\left[\mu_k - \sqrt{\mu_k^2 - (cv_0^2)/g}\right] = 26.31\,\mathrm{m} \quad\text{and}\quad (s_{\mathrm{wet}})_2 = \frac{1}{c}\left[\mu_k + \sqrt{\mu_k^2 - (cv_0^2)/g}\right] = 40.35\,\mathrm{m}. \tag{4}$$

Only the solution $s_{\mathrm{wet}} = (s_{\mathrm{wet}})_1$ is meaningful because for the is not assumed to keep moving after it comes to a stop at $s_{\mathrm{wet}} = (s_{\mathrm{wet}})_1$. Hence, we have

$$\boxed{s_{\mathrm{wet}} = 26.31\,\mathrm{m}.}$$

For dry conditions, $c = 0$ so that $\ddot{s} = -\mu_k g$, which implies that the acceleration is constant and we can use the equation $v^2 = v_0^2 + 2a_c(s - s_0)$ to determine the stopping distance. Denoting by s_{dry} the position s at which $v = 0$ under dry conditions, we have

$$0 = v_0^2 - 2\mu_k g(s_{\mathrm{dry}} - s_0) \quad\Rightarrow\quad s_{\mathrm{dry}} = \frac{v_0^2}{2\mu_k g}. \quad\Rightarrow\quad s_{\mathrm{dry}} = 15.93\,\mathrm{m}, \tag{5}$$

where we have set $s_0 = 0$, and where we have used the following numerical values: $\mu_k = 0.5$, $v_0 = 12.50\,\mathrm{m/s}$, and $g = 9.81\,\mathrm{m/s^2}$.

Using the values of s_{wet} and s_{dry} in the first of Eqs. (4) and the last of Eqs. (5), respectively, the percentage increase in stopping distance is calculated as follows:

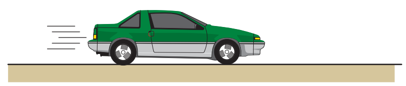
$$\frac{(s_{\text{wet}} - s_{\text{dry}})}{s_{\text{dry}}}(100\%) = \frac{2\mu_k g}{v_0^2}\left\{\frac{1}{c}\left[\mu_k - \sqrt{\mu_k^2 - (c v_0^2)/g}\right] - \frac{v_0^2}{2\mu_k g})\right\}(100\%). \tag{6}$$

Recalling that $c = 0.015\,\mathrm{m^{-1}}$, $\mu_k = 0.5$, $v_0 = 45\,\mathrm{km/h} = 45(1000/3600)\,\mathrm{m/s}$, and $g = 9.81\,\mathrm{m/s^2}$, we can evaluate the above expression to obtain

$$\boxed{\frac{(s_{\text{wet}} - s_{\text{dry}})}{s_{\text{dry}}}(100\%) = 65.21\%.}$$

Problem 2.71

A car stops 4 s after the application of the brakes while covering a rectilinear stretch 337 ft long. If the motion occurred with a constant acceleration a_c, determine the initial speed v_0 of the car and the acceleration a_c. Express v_0 in mph and a_c in terms of g, the acceleration of gravity.



Solution

Recall that $a = dv/dt$. Since $a = a_c$ is constant, we can separate the v and t variables by writing $dv = a_c\,dt$. Letting $t = 0$ be the time at which the brakes are applied and for which $v = v_0$, we can integrate the expression $dv = a_c\,dt$ as follows

$$\int_{v_0}^{v} dv = \int_0^t a_c\,dt \quad \Rightarrow \quad v = v_0 + a_c t. \tag{1}$$

Recalling that $v = ds/dt$, we can write $ds = v\,dt$ and use the expression in the last of Eqs. (1) to write

$$\int_0^s ds = \int_0^t (v_0 + a_c t)\,dt \quad \Rightarrow \quad s = v_0 t + \tfrac{1}{2} a_c t^2, \tag{2}$$

where we have chosen $s = 0$ to be the value of s when the brakes are applied. Letting t_s and d_s be stopping time and distance, respectively, and using the last of Eqs. (1) and of Eqs. (2), we must have

$$0 = v_0 + a_c t_s \quad \text{and} \quad d_s = v_0 t_s + \tfrac{1}{2} a_c t_s^2. \tag{3}$$

This is a system of two equations in the two unknowns v_0 and a_c whose solution is

$$v_0 = 2d_s/t_s \quad \text{and} \quad a_c = -2d_s/t_s^2. \tag{4}$$

Recalling that $t_s = 4$ s and $d_s = 337$ ft we can evaluate the above results to obtain

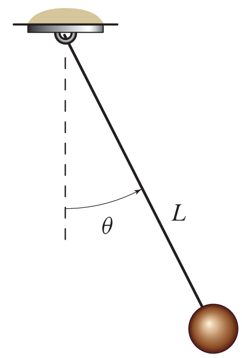
$$\boxed{v_0 = 114.9\,\text{mph} \quad \text{and} \quad a_c = -1.308g,}$$

where we have expressed the acceleration in term of $g = 32.2\,\text{ft/s}^2$, the acceleration of gravity.

Problem 2.72

As you will learn in Chapter 3, the angular acceleration of a simple pendulum is given by $\ddot{\theta} = -(g/L)\sin\theta$, where g is the acceleration of gravity and L is the length of the pendulum cord.

Derive the expression of the angular velocity $\dot{\theta}$ as a function of the angular coordinate θ. The initial conditions are $\theta(0) = \theta_0$ and $\dot{\theta}(0) = \dot{\theta}_0$.



Solution

Recall that $\ddot{\theta} = d\dot{\theta}/dt$. Applying the chain rule, we write

$$\ddot{\theta} = \frac{d\dot{\theta}}{dt} = \frac{d\dot{\theta}}{d\theta}\frac{d\theta}{dt} = \dot{\theta}\frac{d\dot{\theta}}{d\theta} \quad \Rightarrow \quad \dot{\theta}\,d\dot{\theta} = \ddot{\theta}\,d\theta. \tag{1}$$

Substituting the given expression for $\ddot{\theta}$ into the last of Eqs. (1), we can then write

$$\dot{\theta}\,d\dot{\theta} = -(g/L)\sin\theta\,d\theta, \tag{2}$$

where we have separated the variables $\dot{\theta}$ and θ. Since $\dot{\theta} = \dot{\theta}_0$ for $\theta = \theta_0$, we can integrate Eq. (2) as follows:

$$\int_{\dot{\theta}_0}^{\dot{\theta}} \dot{\theta}\,d\dot{\theta} = \int_{\theta_0}^{\theta} -(g/L)\sin\theta\,d\theta \quad \Rightarrow \quad \tfrac{1}{2}\big(\dot{\theta}^2 - \dot{\theta}_0\big) = \frac{g}{L}(\cos\theta - \cos\theta_0). \tag{3}$$

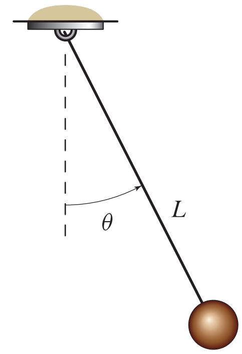
Solving the last of Eqs. (3) for $\dot{\theta}$ as a function of θ, we have

$$\dot{\theta}(\theta) = \pm\sqrt{\dot{\theta}_0^2 + 2\frac{g}{L}\big(\cos\theta - \cos\theta_0\big)}.$$

Problem 2.73

As you will learn in Chapter 3, the angular acceleration of a simple pendulum is given by $\ddot{\theta} = -(g/L)\sin\theta$, where g is the acceleration of gravity and L is the length of the pendulum cord.

Let the length of the pendulum cord be $L = 1.5$ m. If $\dot{\theta} = 3.7$ rad/s when $\theta = 14°$, determine the maximum value of θ achieved by the pendulum.



Solution

Let $\theta_{\max}$ be the maximum value of θ. This value is achieved when $\dot{\theta} = 0$. To find $\theta_{\max}$ we need to relate velocity with position. To do so, we recall that $\ddot{\theta} = d\dot{\theta}/dt$ and, applying the chain rule, we write

$$\ddot{\theta} = \frac{d\dot{\theta}}{dt} = \frac{d\dot{\theta}}{d\theta}\frac{d\theta}{dt} = \dot{\theta}\frac{d\dot{\theta}}{d\theta} \quad \Rightarrow \quad \dot{\theta}\,d\dot{\theta} = \ddot{\theta}\,d\theta. \tag{1}$$

Substituting the given expression for $\ddot{\theta}$ into the last of Eqs. (1), we can then write

$$\dot{\theta}\,d\dot{\theta} = -(g/L)\sin\theta\,d\theta, \tag{2}$$

where we have separated the variables $\dot{\theta}$ and θ. Letting $\dot{\theta}_i = 3.7$ rad/s and $\theta_i = 14°$ be the initial angular velocity and the initial angle, respectively, we can then integrate Eq. (2) as follows:

$$\int_{\dot{\theta}_i}^{0} \dot{\theta}\,d\dot{\theta} = \int_{\theta_i}^{\theta_{\max}} -(g/L)\sin\theta\,d\theta \quad \Rightarrow \quad -\tfrac{1}{2}\dot{\theta}_i^2 = \frac{g}{L}(\cos\theta_{\max} - \cos\theta_i). \tag{3}$$

The last of Eqs. (3) can be solved for $\theta_{\max}$ to obtain

$$\theta_{\max} = \cos^{-1}\left(\cos\theta_i - \frac{L\dot{\theta}_i^2}{2g}\right). \tag{4}$$

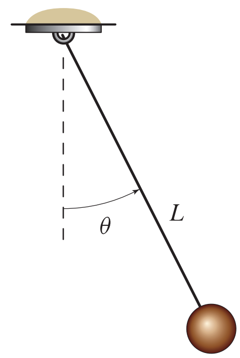
Recalling that $L = 1.5$ m, $\theta_i = 14°$, $\dot{\theta}_i = 3.7$ rad/s, and $g = 9.81\ \text{m/s}^2$

$$\boxed{\theta_{\max} = 94.38°.}$$

Problem 2.74

As you will learn in Chapter 3, the angular acceleration of a simple pendulum is given by $\ddot{\theta} = -(g/L)\sin\theta$, where g is the acceleration of gravity and L is the length of the pendulum cord.

The given angular acceleration remains valid even if the pendulum cord is replaced by a massless rigid bar. For this case, let $L = 5.3\,\text{ft}$ and assume that the pendulum is placed in motion at $\theta = 0°$. What is the minimum angular velocity at this position for the pendulum to swing through a full circle?



Solution

The minimum value of $\dot{\theta}$ at $\theta = 0$ for which the pendulum swings through a full circle is the value that allows the pendulum to reach the angle $\theta = \pi$ rad with $\dot{\theta} = 0$. This means that we need to relate velocity with position. To do so, we recall that $\ddot{\theta} = d\dot{\theta}/dt$ and, applying the chain rule, we write

$$\ddot{\theta} = \frac{d\dot{\theta}}{dt} = \frac{d\dot{\theta}}{d\theta}\frac{d\theta}{dt} = \dot{\theta}\frac{d\dot{\theta}}{d\theta} \quad \Rightarrow \quad \dot{\theta}\,d\dot{\theta} = \ddot{\theta}\,d\theta. \tag{1}$$

Substituting the given expression for $\ddot{\theta}$ into the last of Eqs. (1), we can then write

$$\dot{\theta}\,d\dot{\theta} = -(g/L)\sin\theta\,d\theta, \tag{2}$$

where we have separated the variables $\dot{\theta}$ and θ. Then, using the above observation concerning the value of $\dot{\theta}$ for $\theta = \pi$ rad, we can integrate Eq. (2) as follows:

$$\int_{\dot{\theta}_{\text{min}}}^{0} \dot{\theta}\,d\dot{\theta} = \int_{0}^{\pi\,\text{rad}} -(g/L)\sin\theta\,d\theta \quad \Rightarrow \quad -\tfrac{1}{2}\dot{\theta}_{\text{min}}^2 = -2\frac{g}{L}. \tag{3}$$

The last of Eqs. (3) can be solved for $\dot{\theta}_{\text{min}}$ to obtain

$$\dot{\theta}_{\text{min}} = \pm 2\sqrt{\frac{g}{L}}. \tag{4}$$

Recalling that $g = 32.2\,\text{ft/s}^2$ and $L = 5.3\,\text{ft}$, the result in Eq. (4) can be evaluated to obtain

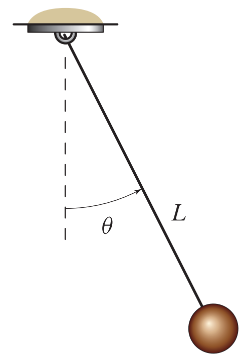
$$\boxed{\dot{\theta}_{\text{min}} = 4.930\,\text{rad/s},}$$

where we have considered only the positive value of $\dot{\theta}_{\text{min}}$ because in going from 0 to π rad the pendulum bob moves counterclockwise.

Problem 2.75

As you will learn in Chapter 3, the angular acceleration of a simple pendulum is given by $\ddot{\theta} = -(g/L)\sin\theta$, where g is the acceleration of gravity and L is the length of the pendulum cord.

Let $L = 3.5$ ft and suppose that at $t = 0$ s the pendulum's position is $\theta(0) = 32°$ with $\dot{\theta}(0) = 0$ rad/s. Determine the pendulum's period of oscillation, i.e., from its initial position back to this position.



Solution

To determine the period of the pendulum using the given initial conditions, we need to establish a relationship between the angle θ and time. To do so, we begin by establishing a relation between the angular velocity $\dot{\theta}$ and and swing angle θ, and then we integrate that result to determine $\theta(t)$. To find $\dot{\theta}(\theta)$, we begin by applying the chain rule as follows

$$\ddot{\theta} = \frac{d\dot{\theta}}{dt} = \frac{d\dot{\theta}}{d\theta}\frac{d\theta}{dt} = \dot{\theta}\frac{d\dot{\theta}}{d\theta} \quad \Rightarrow \quad \dot{\theta}\,d\dot{\theta} = \ddot{\theta}\,d\theta \quad \Rightarrow \quad \dot{\theta}\,d\dot{\theta} = \ddot{\theta}\,d\theta. \tag{1}$$

Substituting the given expression for $\ddot{\theta}$ into the last of Eqs. (1), we can then write

$$\dot{\theta}\,d\dot{\theta} = -(g/L)\sin\theta\,d\theta. \tag{2}$$

Since $\dot{\theta} = 0$ when $\theta = \theta_0 = 32°$, we can integrate Eq. (2) as follows:

$$\int_0^{\dot{\theta}} \dot{\theta}\,d\dot{\theta} = \int_{\theta_0}^{\theta} -(g/L)\sin\theta\,d\theta \quad \Rightarrow \quad \tfrac{1}{2}\dot{\theta}^2 = \frac{g}{L}(\cos\theta - \cos\theta_0). \tag{3}$$

Solving for $\dot{\theta}$, we have

$$\dot{\theta} = \pm\sqrt{\frac{2g}{L}(\cos\theta - \cos\theta_0)}. \tag{4}$$

Once the pendulum is released from $\theta_0 = 32°$, the angle θ will decrease until it becomes $-32°$. Then the pendulum will then swing back to the original angle. The time taken to go from 32° to −32° is equal to the time taken to swing back from −32° to 32°. Hence, the period of oscillation, which we will denote by p, is twice the time that the pendulum takes to go from 32° to −32°. With this in mind, referring to Eq. (4), and because θ will initially decrease after release, the expression for $\dot{\theta}$ to use when θ goes from 32° to −32° is

$$\dot{\theta} = -\sqrt{\frac{2g}{L}(\cos\theta - \cos\theta_0)}, \tag{5}$$

Now, recalling that $\dot{\theta} = d\theta/dt$, we can write $dt = d\theta/\dot{\theta}$, which, because of Eq. (5), we can write

$$dt = -\sqrt{\frac{L}{2g}}\frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}}. \tag{6}$$

We now integrate over one half of a complete swing to obtain

$$\int_0^{p/2} dt = \int_{32°}^{-32°} -\sqrt{\frac{L}{2g}} \frac{d\theta}{\sqrt{\cos\theta - \cos 32°}} \quad \Rightarrow \quad p = \int_{32°}^{-32°} -\sqrt{\frac{2L}{g}} \frac{d\theta}{\sqrt{\cos\theta - \cos 32°}}.$$

The above integral can be evaluated numerically. Recalling that $L = 3.5\,\text{ft}$ and $g = 32.2\,\text{ft/s}^2$, we have used *Mathematica* with the following code:

```
NIntegrate[-Sqrt[2·3.5/32.2] 1/Sqrt[Cos[θ] - Cos[32. Degree]], {θ, 32. Degree, -32. Degree}]
```

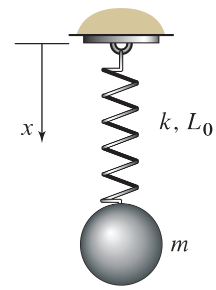
which yields

$$\boxed{p = 2.113\,\text{s}.}$$

Problem 2.76

As we will see in Chapter 3, the acceleration of a particle of mass m suspended by a linear spring with spring constant k and unstretched length L_0 (when the spring length is equal to L_0, the spring exerts no force on the particle) is given by $\ddot{x} = g - (k/m)(x - L_0)$.

Derive the expression for the particle's velocity $\dot{x}$ as a function of position x. Assume that at $t = 0$, the particle's velocity is v_0 and its position is x_0.



Solution

The acceleration can be related to the position and the velocity as follows: $\ddot{x} = \dot{x}\,d\dot{x}/dx$, which can then be rewritten as $\dot{x}\,d\dot{x} = \ddot{x}\,dx$. This latter expression can be integrated as follows:

$$\int_{v_0}^{\dot{x}} \dot{x}\,d\dot{x} = \int_{x_0}^{x} \left[g - \left(\frac{k}{m}\right)(x - L_0) \right] dx, \tag{1}$$

where, as indicated in the problem statement, v_0 is the value of $\dot{x}$ for $x = x_0$. Evaluating the integral gives us the velocity as a function of x.

$$\tfrac{1}{2}\dot{x}^2 - \tfrac{1}{2}v_0^2 = g(x - x_0) - \frac{k}{2m}\left(x^2 - x_0^2\right) + \frac{kL_0}{m}(x - x_0). \tag{2}$$

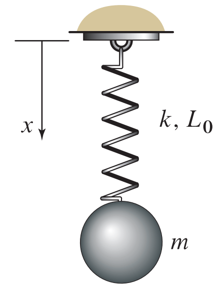
Solving for $\dot{x}$, we have

$$\boxed{\dot{x} = \pm\sqrt{v_0^2 + 2\left(g + \frac{kL_0}{m}\right)(x - x_0) - \frac{k}{m}\left(x^2 - x_0^2\right)}.} \tag{3}$$

Problem 2.77

As we will see in Chapter 3, the acceleration of a particle of mass m suspended by a linear spring with spring constant k and unstretched length L_0 (when the spring length is equal to L_0, the spring exerts no force on the particle) is given by $\ddot{x} = g - (k/m)(x - L_0)$.

Let $k = 100\,\text{N/m}$, $m = 0.7\,\text{kg}$, and $L_0 = 0.75\,\text{m}$. If the particle is released from rest at $x = 0\,\text{m}$, determine the maximum length achieved by the spring.



Solution

The acceleration can be related to the position and the velocity as follows: $\ddot{x} = \dot{x}\,d\dot{x}/dx$, which can then be rewritten as $\dot{x}\,d\dot{x} = \ddot{x}\,dx$. This latter expression can be integrated as follows:

$$\int_{v_0}^{\dot{x}} \dot{x}\,d\dot{x} = \int_{x_0}^{x} \left[g - \left(\frac{k}{m}\right)(x - L_0)\right] dx, \tag{1}$$

where as indicated in the problem statement, v_0 is the value of $\dot{x}$ for $x = x_0$. Now, in this particular problem, since the particle starts from rest at $x = 0$, we set $v_0 = 0$ and $x_0 = 0$. We now observe that the coordinate x (when positive) measures the length of the spring. In addition, denoting the maximum length by x_{max}, we observe that x_{max} is achieved when $\dot{x} = 0$, i.e., when the spring has stretched to the point that its velocity is equal to zero (before recoiling back). Using these considerations, we can rewrite Eq. (1) as

$$\int_{0}^{0} \dot{x}\,d\dot{x} = \int_{0}^{x_{\text{max}}} \left[g - \left(\frac{k}{m}\right)(x - L_0)\right] dx \quad \Rightarrow \quad 0 = g x_{\text{max}} - \frac{k}{2m} x_{\text{max}}^2 + \frac{k L_0}{m} x_{\text{max}}. \tag{2}$$

Solving the last of the above equations for x_{max}, we obtain

$$x_{\text{max}} = \frac{2(mg + kL_0)}{k}. \tag{3}$$

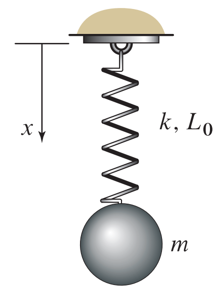
Recalling that $m = 0.7\,\text{kg}$, $g = 9.81\,\text{m/s}^2$, $k = 100\,\text{N/m}$, and $L_0 = 0.75\,\text{m}$, we can evaluate Eq. (3) to obtain

$$\boxed{x_{\text{max}} = 1.637\,\text{m}.}$$

Problem 2.78

As we will see in Chapter 3, the acceleration of a particle of mass m suspended by a linear spring with spring constant k and unstretched length L_0 (when the spring length is equal to L_0, the spring exerts no force on the particle) is given by $\ddot{x} = g - (k/m)(x - L_0)$.

Let $k = 8\,\text{lb/ft}$, $m = 0.048\,\text{slug}$, and $L_0 = 2.5\,\text{ft}$. If the particle is released from rest at $x = 0\,\text{ft}$, determine how long it takes for the spring to achieve its maximum length. *Hint:* A good table of integrals will come in handy.



Solution

The acceleration can be related to the position and the velocity as follows: $\ddot{x} = \dot{x}\,d\dot{x}/dx$, which can then be rewritten as $\dot{x}\,d\dot{x} = \ddot{x}\,dx$. This latter expression can be integrated as follows:

$$\int_{v_0}^{\dot{x}} \dot{x}\,d\dot{x} = \int_{x_0}^{x} \left[g - \left(\frac{k}{m}\right)(x - L_0)\right] dx, \tag{1}$$

where as indicated in the problem statement, v_0 is the value of $\dot{x}$ for $x = x_0$. Evaluating the integral gives us the velocity as a function of x.

$$\tfrac{1}{2}\dot{x}^2 - \tfrac{1}{2}v_0^2 = g(x - x_0) - \frac{k}{2m}\left(x^2 - x_0^2\right) + \frac{kL_0}{m}(x - x_0). \tag{2}$$

Then, keeping in mind that we are interested in the motion of the spring for $\dot{x} \geq 0$ (i.e., we are not interested in the recoiling motion of the spring after it has stretched to its maximum length), and solving for $\dot{x}$, we have

$$\dot{x} = \sqrt{v_0^2 + 2\left(g + \frac{kL_0}{m}\right)(x - x_0) - \frac{k}{m}\left(x^2 - x_0^2\right)}. \tag{3}$$

Since the particle is released from rest at $x = 0$, we have $v_0 = 0$. Consequently, the above equation can be simplified to obtain

$$\dot{x} = \sqrt{\frac{2(mg + kL_0)}{m}x - \frac{k}{m}x^2}. \tag{4}$$

Next, we determine the maximum length of the spring, which is achieved when $\dot{x} = 0$. Hence, setting $\dot{x} = 0$ and $x = x_{max}$ in Eq. (4) and solving for x_{max}, we have

$$\frac{2(mg + kL_0)}{m}x_{max} - \frac{k}{m}x_{max}^2 = 0 \quad\Rightarrow\quad x_{max} = \frac{2(mg + kL_0)}{k} \quad\Rightarrow\quad 2(mg + kL_0) = x_{max}k. \tag{5}$$

Substituting the last of Eqs. (5) into Eq. (4), we have

$$\dot{x} = \sqrt{\frac{k}{m}x x_{max} - \frac{k}{m}x^2} \quad\Rightarrow\quad \dot{x} = \sqrt{\frac{k}{m}}\sqrt{x\left(x_{max} - x\right)}. \tag{6}$$

Now we recall that $\dot{x} = dx/dt$. Therefore, we can rearrange the terms in the last of Eqs. (4) to integrate as follows:

$$\dot{x} = \frac{dx}{dt} \quad\Rightarrow\quad \int_0^{t_{x_{max}}} dt = \int_0^{x_{max}} \frac{dx}{\dot{x}} \quad\Rightarrow\quad t_{x_{max}} = \sqrt{\frac{m}{k}}\int_0^{x_{max}} \frac{dx}{\sqrt{x\,(x_{max} - x)}}. \tag{7}$$

The integral on the right-hand side of the last of Eqs. (7) can be carried out by substitution, or by consulting a table of integrals, or by using a symbolic mathematical software. Regardless of the method, we have

$$\int \frac{dx}{\sqrt{x\,(x_{\max} - x)}} = 2\sin^{-1}\left(\sqrt{\frac{x}{x_{\max}}}\right) + C, \tag{8}$$

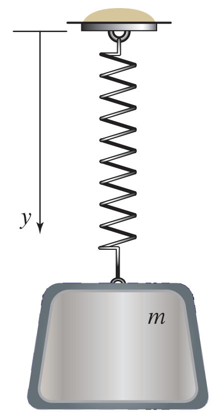
where C is a constant of integration. Then using the above result we have that the last of Eqs. (7) gives

$$t_{x_{\max}} = \sqrt{\frac{m}{k}}\,2[\sin^{-1}(1) - \sin^{-1}(0)] = \pi\sqrt{\frac{m}{k}} \quad \Rightarrow \quad \boxed{t_{x_{\max}} = 0.2433\,\mathrm{s},} \tag{9}$$

where we have used the following numerical data: $m = 0.048$ slug and $k = 8\,\mathrm{lb/ft}$.

Problem 2.79

A weight A with mass $m = 18\,\text{kg}$ is attached to the free end of a nonlinear spring such that the acceleration of A is $a = g - (\gamma/m)(y - L_0)^3$, where g is the acceleration due to gravity, γ is a constant, and $L_0 = 0.5\,\text{m}$. Determine γ such that A does not fall below $y = 1\,\text{m}$ when released from rest at $y = L_0$.



Solution

Recall that $a = dv/dt$. In this problem, the acceleration is given as a function of position. Hence, to relate, a change in velocity to a correspondent change in position, we start by rewriting the acceleration via the chain rule: $a = v\,dv/dy$. This allows us to write

$$v\,dv = a(y)\,dy \quad \Rightarrow \quad v\,dv = \left[g - \frac{\gamma}{m}(y - L_0)^3\right] dy, \tag{1}$$

where we have separated the variables v and s and used the given expression for the acceleration. Recalling that $v = 0$ for $y = L_0$, we can now integrate the last of Eqs. (1) as follows:

$$\int_0^v v\,dv = \int_{L_0}^y \left[g - \frac{\gamma}{m}(y - L_0)^3\right] dy \quad \Rightarrow \quad v^2 = 2g(y - L_0) - \frac{\gamma}{2m}(y - L_0)^4. \tag{2}$$

Let $\tilde{y} = 1\,\text{m}$. In order for the weight not to fall below $\tilde{y}$, the speed must become equal to zero at $y = \tilde{y}$. Hence, letting $v = 0$ for $y = \tilde{y}$, the last of Eqs. (2) gives

$$0 = 2g(\tilde{y} - L_0) - \frac{\gamma}{2m}(\tilde{y} - L_0)^4, \tag{3}$$

which is an equation in γ whose solution is

$$\gamma = \frac{4mg}{(\tilde{y} - L_0)^3}. \tag{4}$$

Recalling that $m = 18\,\text{kg}$, $g = 9.81\,\text{m/s}^2$, $L_0 = 0.5\,\text{m}$, and $\tilde{y} = 1\,\text{m}$, we can evaluate Eq. (4) to obtain

$$\boxed{\gamma = 5651\,\text{kg}/(\text{m}^2{\cdot}\text{s}^2).}$$

Problem 2.80

Two masses m_A and m_B are placed at a distance r_0 from one another. Because of their mutual gravitational attraction, the acceleration of sphere B as seen from sphere A is given by

$$\ddot{r} = -G\left(\frac{m_A + m_B}{r^2}\right),$$

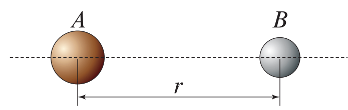
where $G = 6.674\times10^{-11}\,\text{m}^3/(\text{kg}\cdot\text{s}^2) = 3.439\times10^{-8}\,\text{ft}^3/(\text{slug}\cdot\text{s}^2)$ is the universal gravitational constant. If the spheres are released from rest, determine

(a) The velocity of B (as seen by A) as a function of the distance r.

(b) The velocity of B (as seen by A) at impact if $r_0 = 7\,\text{ft}$, the weight of A is 2.1 lb, the weight of B is 0.7 lb, and

  (i) The diameters of A and B are $d_A = 1.5\,\text{ft}$ and $d_B = 1.2\,\text{ft}$, respectively.

  (ii) The diameters of A and B are infinitesimally small.



Solution

Part (a). Recalling that we can relate the acceleration to the velocity and position as $\ddot{r} = \dot{r}\,d\dot{r}/dt$, we can then write

$$\ddot{r} = \dot{r}\frac{d\dot{r}}{dr} \quad\Rightarrow\quad \int_0^{\dot{r}} \dot{r}\,dr = -G\,(m_A + m_B)\int_{r_0}^{r} \frac{1}{r^2}\,dr,$$

which can be evaluated to obtain

$$\tfrac{1}{2}\dot{r}^2 = G\big(m_A + m_B\big)\left(\frac{1}{r} - \frac{1}{r_0}\right) \quad\Rightarrow\quad \boxed{\dot{r} = -\sqrt{2G\,(m_A + m_B)}\sqrt{\frac{r_0 - r}{r r_0}},} \tag{1}$$

where we have chosen the negative root because the masses are moving toward each other.

Part (b). Now that we have the velocity as a function of position, we can proceed to answer the questions posed in Part (b) of the problem. For question (i), when the masses touch $r = r_A + r_B = (d_A + d_B)/2$, so that

$$\dot{r} = -\sqrt{2G\,(m_A + m_B)}\sqrt{\frac{2}{d_A + d_B} - \frac{1}{r_0}}. \tag{2}$$

Recalling that $G = 3.439\times10^{-8}\,\text{ft}^3/\text{slug}\cdot\text{s}^2$, $m_A = 2.1\,\text{lb}/g$, $m_B = 0.7\,\text{lb}/g$, $g = 32.2\,\text{ft/s}^2$, $d_A = 1.5\,\text{ft}$, $d_B = 1.2\,\text{ft}$, and $r_0 = 7\,\text{ft}$, we can evaluate the above expression to obtain

$$\boxed{\dot{r} = -5.980\times10^{-5}\,\text{ft/s}.} \tag{3}$$

For part (ii), we take the limit of Eq. (2) as $r \to 0$ to obtain

$$\boxed{\dot{r} \to -\infty.}$$

Problem 2.81

Two masses m_A and m_B are placed at a distance r_0 from one another. Because of their mutual gravitational attraction, the acceleration of sphere B as seen from sphere A is given by

$$\ddot{r} = -G\left(\frac{m_A + m_B}{r^2}\right),$$

where $G = 6.674\times10^{-11}\,\mathrm{m^3/(kg\cdot s^2)} = 3.439\times10^{-8}\,\mathrm{ft^3/(slug\cdot s^2)}$ is the universal gravitational constant.
Assume that the particles are released from rest at $r = r_0$.

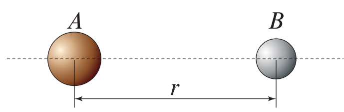
(a) Determine the expression relating their relative position r and time. *Hint:*

$$\int \sqrt{x/(1-x)}\,dx = \sin^{-1}\bigl(\sqrt{x}\bigr) - \sqrt{x(1-x)}.$$

(b) Determine the time it takes for the objects to come into contact if $r_0 = 3\,\mathrm{m}$, A and B have masses of 1.1 and 2.3 kg, respectively, and

(i) The diameters of A and B are $d_A = 22\,\mathrm{cm}$ and $d_B = 15\,\mathrm{cm}$, respectively.

(ii) The diameters of A and B are infinitesimally small.



Solution

Part (a). To find the relation between position and velocity, we observe that we can relate the acceleration to the velocity and position as $\ddot{r} = \dot{r}\,d\dot{r}/dt$. Hence, we can then write

$$\ddot{r} = \dot{r}\frac{d\dot{r}}{dr} \quad\Rightarrow\quad \int_0^{\dot{r}} \dot{r}\,dr = -G\,(m_A + m_B)\int_{r_0}^{r}\frac{1}{r^2}\,dr,$$

where we have used the fact that the spheres are released from rest so $\dot{r} = 0$ and have let their initial separation distance be r_0. These integrals can be evaluated to obtain

$$\tfrac{1}{2}\dot{r}^2 = G\bigl(m_A + m_B\bigr)\left(\frac{1}{r} - \frac{1}{r_0}\right) \quad\Rightarrow\quad \dot{r} = -\sqrt{2G\,(m_A + m_B)}\sqrt{\frac{r_0 - r}{r r_0}}, \tag{1}$$

where we have chosen the negative root because the masses are moving toward each other and so r is decreasing.

Next, we observe that $\dot{r} = dr/dt$ and we write $dt = dr/\dot{r}$. Using this expression and the expression for $\dot{r}$ in the last of Eqs. (1), we can then write

$$\int_0^t dt = -\frac{1}{\sqrt{2G\,(m_A + m_B)}}\int_{r_0}^{r}\sqrt{\frac{r}{1 - r/r_0}}\,dr,$$

where we have divided both the numerator and the denominator of the fraction under the square root by r_0 and we have used the fact that $t_0 = 0$. Making the substitution $x = r/r_0$ so that $dr = r_0\,dx$, and evaluating the integral on the left-hand side, we obtain

$$t = -\frac{r_0^{3/2}}{\sqrt{2G\,(m_A + m_B)}}\int_1^{r/r_0}\sqrt{\frac{x}{1-x}}\,dx.$$

Using the hint given in the problem statement, this becomes becomes

$$t = -\frac{r_0^{3/2}}{\sqrt{2G\,(m_A + m_B)}}\left[\sin^{-1}\sqrt{x} - \sqrt{x\,(1-x)}\right]\Big|_1^{r/r_0},$$

$$\Rightarrow \quad \boxed{t = \frac{-r_0^{3/2}}{\sqrt{2G\,(m_A + m_B)}}\left[\sin^{-1}\sqrt{\frac{r}{r_0}} - \sqrt{\frac{r}{r_0}\left(1-\frac{r}{r_0}\right)} - \frac{\pi}{2}\right].} \qquad (2)$$

Part (b). We are given that $r_0 = 3\,\text{m}$, $m_A = 1.1\,\text{kg}$, and $m_B = 2.3\,\text{kg}$, and we know that $G = 6.674 \times 10^{-11}\,\text{m}^3/\text{kg}\cdot\text{s}^2$.

(i) Using $d_A = 22\,\text{cm} = 0.2200\,\text{m}$, $d_B = 15\,\text{cm} = 0.1500\,\text{m}$, and $r = (d_A + d_B)/2 = 0.1850\,\text{m}$ in Eq. (2), we obtain

$$\boxed{t = 380{,}600\,\text{s}.}$$

(ii) If the diameters are infinitesimally small, $r \to 0$. Hence, from Eq. (2) we obtain

$$t = \frac{(\pi/2)\,r_0^{3/2}}{\sqrt{2G(m_A + m_B)}} \quad \Rightarrow \quad \boxed{t = 383{,}100\,\text{s}.}$$

Problem 2.82

Suppose that the acceleration $\ddot{r}$ of an object moving along a straight line takes on the form

$$\ddot{r} = -G\left(\frac{m_A + m_B}{r^2}\right),$$

where the constants G, m_A, and m_B are known. If $\dot{r}(0)$ is given, under what conditions can you determine $\dot{r}(t)$ via the following integral?

$$\dot{r}(t) = \dot{r}(0) - \int_0^t G\,\frac{m_A + m_B}{r^2}\,dt.$$

Solution

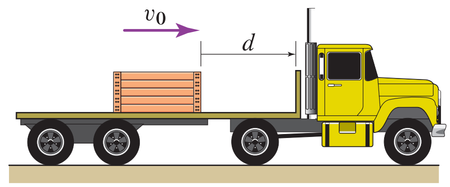
$\dot{r}(t)$ can be determined if the position r of the object is known as a function of time t and if $r(t) \neq 0$ during the time interval of interest.

Problem 2.83

If the truck brakes hard enough that the crate slides to the right relative to the truck, the distance d between the crate and the front of the trailer changes according to the relation

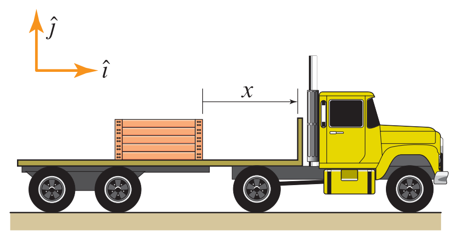
$$\ddot{d} = \begin{cases} \mu_k g + a_T & \text{for } t < t_s, \\ \mu_k g & \text{for } t > t_s, \end{cases}$$

where t_s is the time it takes the truck to stop, a_T is the acceleration of the truck, g is the acceleration of gravity, and μ_k is the kinetic friction coefficient between the truck and the crate. Suppose that the truck and the crate are initially traveling to the right at $v_0 = 60\,\text{mph}$ and the brakes are applied so that $a_T = -10.0\,\text{ft/s}^2$. Determine the minimum value of μ_k so that the crate does not hit the right end of the truck bed if the initial distance d is 12 ft. *Hint:* The truck stops *before* the crate stops.



Solution

Referring to figure on the right, the acceleration of the truck relative to the crate is given by



$$\ddot{d} = \begin{cases} \ddot{x}_1 = \mu_k g + a_T & \text{for } t < t_s, \\ \ddot{x}_2 = \mu_k g & \text{for } t > t_s, \end{cases} \tag{1}$$

where t_s is the time at which the truck comes to a stop, and where the subscripts 1 and 2 are used to distinguish expressions corresponding to $t < t_s$ from those for $t > t_s$, respectively. Using the constant acceleration equation of the type $v = v_0 + a_c(t - t_0)$, the time t_s at which the truck stops is

$$0 = v_0 + a_T t_s \quad \Rightarrow \quad t_s = -\frac{v_0}{a_T}, \tag{2}$$

where v_0 is common the initial speed of the truck and crate. Letting x_1 be the position of the truck relative to the crate at the time the truck comes to a stop, using the constant acceleration equation of the type $s = s_0 + v_0 t + \frac{1}{2} a_c t^2$, we have

$$\begin{aligned} x_1 &= x_0 + \dot{x}_0 t_s + \tfrac{1}{2}\ddot{x}_1 t_s^2 \\ &= d + \tfrac{1}{2}\left(\mu_k g + a_T\right) t_s^2, \end{aligned} \tag{3}$$

where $x_0 = d$ is the initial position of the truck relative to the crate and $\dot{x}_0 = 0$ is the initial velocity of the truck relative to the crate.

After the truck comes to a stop, the truck continues to slide relative to the crate with the acceleration $\ddot{x}_2$ in Eq. (1). Using this acceleration, the distance the truck moves relative to the crate after the truck comes to a stop can be found using

$$\dot{x}_f^2 = \dot{x}_1^2 + 2\ddot{x}_2\left(x_f - x_1\right), \tag{4}$$

where $\dot{x}_f = 0$ is the final velocity of the truck relative to the crate, $\dot{x}_1$ is the velocity of the truck relative to the crate at time t_s, $x_f = 0$ is final position of the truck relative to the crate, and x_1 is the position of the truck relative to the crate at time t_s and is given by Eq. (3). We now need to find $\dot{x}_1$, which can be done using:

$$\dot{x}_1 = \dot{x}_0 + \ddot{x}_1 t_s \quad \Rightarrow \quad \dot{x}_1 = (\mu_k g + a_T)\, t_s, \tag{5}$$

where we have again used the fact that $\dot{x}_0 = 0$. Substituting Eqs. (1), (3) and (5) into Eq. (4), we obtain

$$0 = [(\mu_k g + a_T)\, t_s]^2 + 2(\mu_k g)\left[-d - \tfrac{1}{2}(\mu_k g + a_T)\, t_s^2\right]. \tag{6}$$

Finally, substituting in t_s from Eq. (2), we get the final equation for μ_k:

$$0 = \left[(\mu_k g + a_T)\left(-\frac{v_0}{a_T}\right)\right]^2 + 2(\mu_k g)\left[-d - \tfrac{1}{2}(\mu_k g + a_T)\left(-\frac{v_0}{a_T}\right)^2\right], \tag{7}$$

which, after simplification, becomes

$$0 = \frac{v_0^2}{a_T}(\mu_k g + a_T) - 2d\mu_k g. \tag{8}$$

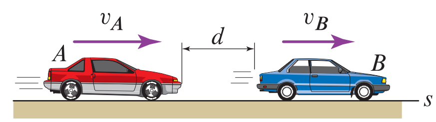
Solving for μ_k we get

$$\mu_k = \frac{v_0^2 a_T}{g\left[2d a_T - v_0^2\right]} \quad \Rightarrow \quad \boxed{\mu_k = 0.3012,} \tag{9}$$

where we have used $v_0 = 60\,\text{mph} = 60\left(\frac{5280}{3600}\right)\text{ft/s}$, $a_T = -10\,\text{ft/s}^2$, $g = 32.2\,\text{ft/s}^2$, and $d = 12\,\text{ft}$ to obtain the numerical result.

Problem 2.84

Cars A and B are traveling at $v_A = 72\,\text{mph}$ and $v_B = 67\,\text{mph}$, respectively, when the driver of car B applies the brakes abruptly, causing the car to slide to a stop. The driver of car A takes 1.5 s to react to the situation and applies the brakes in turn, causing car A to slide as well. If A and B slide with equal accelerations, i.e., $\ddot{s}_A = \ddot{s}_B = -\mu_k g$, where $\mu_k = 0.83$ is the kinetic friction coefficient and g is the acceleration of gravity, compute the minimum distance d between A and B at the time B starts sliding to avoid a collision.



Solution

We will denote by d_A and d_B the stopping distances of cars A and B, respectively. The stopping distance of car B is completely determined by the application of the brakes. The stopping distance of car A is determined by both the reaction time and the subsequent application of the brakes, that is,

$$d_A = d_{A_r} + d_{A_b}, \tag{1}$$

where d_{A_r} is the distance traveled by A during the reaction time and d_{A_b} is the distance traveled by A during the application of the brakes. During the reaction time, the speed of A remains constant. Therefore,

$$d_{A_r} = v_A t_r, \tag{2}$$

where $t_r = 1.5\,\text{s}$ is the reaction time of the driver of car A. After both cars apply the brakes, for both cars we can use the constant acceleration equation $v^2 = v_0^2 + 2a_c(s - s_0)$ to relate the speeds of the cars to their acceleration and stopping distances. Setting to zero the final velocities of both cars, we have

$$0 = v_A^2 - 2\mu_k g d_{A_b} \quad \Rightarrow \quad d_{A_b} = \frac{v_A^2}{2\mu_k g}, \tag{3}$$

$$0 = v_B^2 - 2\mu_k g d_B \quad \Rightarrow \quad d_B = \frac{v_B^2}{2\mu_k g}. \tag{4}$$

From Eqs. (1)–(4), we have

$$d_A = v_A t_r + \frac{v_A^2}{2\mu_k g} \quad \text{and} \quad d_B = \frac{v_B^2}{2\mu_k g}. \tag{5}$$

We observe that, in order to avoid a collision, the separation d between A and B at the moment that B applies the brakes must be such that $d_A = d_B + d$. Then, using Eqs. (5), we have

$$d = d_A - d_B = \frac{1}{2\mu_k g}\left(v_A^2 - v_B^2\right) + v_A t_r. \tag{6}$$

Recalling that $\mu_k = 0.83$, $g = 32.2\,\text{ft/s}^2$, $v_A = 72\,\text{mph} = 72\frac{5280}{3600}\,\text{ft/s}$, $v_B = 67\,\text{mph} = 67\frac{5280}{3600}\,\text{ft/s}$, and $t_r = 1.5\,\text{s}$, we can evaluate d to obtain

$$\boxed{d = 186.4\,\text{ft}.}$$

Problem 2.85

The spool of paper used in a printing process is unrolled with velocity v_p and acceleration a_p. The thickness of the paper is h, and the outer radius of the spool at any instant is r.

If the velocity at which the paper is unrolled is *constant*, determine the angular acceleration α_s of the spool as a function of r, h, and v_p. Evaluate your answer for $h = 0.0048$ in., for $v_p = 1000$ ft/min, and two values of r, that is, $r_1 = 25$ in. and $r_2 = 10$ in.

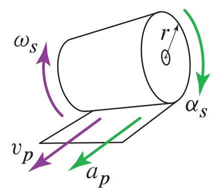


Photo credit: © David Lees/CORBIS

Solution

The radius decreases by the paper thickness h for every one revolution. Hence, letting θ be the angle measuring the angular position of a fixed radial line on the spool (θ increases when the spool turns clockwise), then we have

$$\frac{\Delta r}{\Delta\theta} = \frac{-h}{2\pi}. \tag{1}$$

Assuming that the decrease in radius can be viewed as occurring continuously, we can change the above relation into a relation in terms of differentials. That is we can write

$$\frac{dr}{d\theta} = \frac{-h}{2\pi}. \tag{2}$$

Next, observing that the angular velocity of the spool is $\omega_s = d\theta/dt$, we can then use the above equation to relate the time rate of change of r to ω_s by applying the chain rule as follows:

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} \quad\Rightarrow\quad \dot{r} = \frac{-h}{2\pi}\omega_s \quad\Rightarrow\quad \omega_s = \frac{-2\pi}{h}\dot{r}. \tag{3}$$

Recalling that the linear velocity of the paper is related to the angular velocity of the spool as $v_p = r\omega_s$, we can use Eq. (3) to relate v_p to $\dot{r}$ as follows:

$$v_p = r\omega_s \quad\Rightarrow\quad v_p = \frac{-2\pi}{h}r\dot{r} \quad\Rightarrow\quad \dot{r} = \frac{-hv_p}{2\pi r}. \tag{4}$$

Taking the time derivative of v_p in the second of Eqs. (4), accounting for the fact that v_p is constant (i.e., $a_p = 0$), and using the last of Eqs. (4), we have

$$a_p = 0 = \frac{-2\pi}{h}\left(\dot{r}^2 + r\ddot{r}\right) \quad\Rightarrow\quad 0 = \frac{-2\pi}{h}\left(\frac{h^2v_p^2}{4\pi^2r^2} + r\ddot{r}\right). \tag{5}$$

Taking the time derivative of the last of Eqs. (3) and solving the last of Eqs. (5) to find expressions for α_s and $\ddot{r}$, respectively, we have

$$\alpha_s = \frac{-2\pi}{h}\ddot{r} \quad\text{and}\quad \ddot{r} = \frac{-h^2v_p^2}{4\pi^2r^3}, \tag{6}$$

which, when combined, imply that

$$\alpha_s = \frac{h v_p^2}{2\pi r^3}.$$

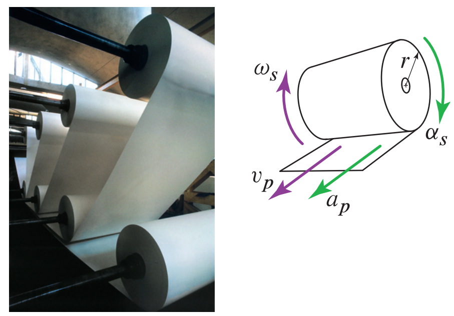
Evaluating the expression above for $h = 0.0048\,\text{in.} = (0.0048/12)\,\text{ft}$, $v_p = 1000\,\text{ft/min} = (1000/60)\,\text{ft/s}$, $r_1 = 25\,\text{in.} = (25/12)\,\text{ft}$, and $r_2 = 10\,\text{in.} = (10/12)\,\text{ft}$, we have

$$\alpha_s\big|_{r=r_1} = 0.001956\,\text{rad/s}^2 \quad \text{and} \quad \alpha_s\big|_{r=r_2} = 0.03056\,\text{rad/s}^2.$$

Problem 2.86

The spool of paper used in a printing process is unrolled with velocity v_p and acceleration a_p. The thickness of the paper is h, and the outer radius of the spool at any instant is r.

If the velocity at which the paper is unrolled is *not constant*, determine the angular acceleration α_s of the spool as a function of r, h, v_p, and a_p. Evaluate your answer for $h = 0.0048$ in., $v_p = 1000$ ft/min, $a_p = 3\,\text{ft/s}^2$, and two values of r, that is, $r_1 = 25$ in. and $r_2 = 10$ in.



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Solution

The radius decreases by the paper thickness h for every one revolution. Hence, letting θ be the angle (in radians) measuring the angular position of a fixed radial line on the spool (θ increases when the spool turns clockwise), then we have

$$\frac{\Delta r}{\Delta \theta} = \frac{-h}{2\pi}. \tag{1}$$

Assuming that the decrease in radius can be viewed as occurring continuously, we can change the above relation into a relation in terms of differentials. That is we can write

$$\frac{dr}{d\theta} = \frac{-h}{2\pi}. \tag{2}$$

Next, observing that the angular velocity of the spool is $\omega_s = d\theta/dt$, we can then use the above equation to relate the time rate of change of r to ω_s by applying the chain rule as follows:

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} \quad \Rightarrow \quad \dot{r} = \frac{-h}{2\pi}\omega_s \quad \Rightarrow \quad \omega_s = \frac{-2\pi}{h}\dot{r}. \tag{3}$$

Recalling that the linear velocity of the paper is related to the angular velocity of the spool as $v_p = r\omega_s$, we can use Eq. (3) to relate v_p to $\dot{r}$ as follows:

$$v_p = r\omega_s \quad \Rightarrow \quad v_p = \frac{-2\pi}{h} r\dot{r} \quad \Rightarrow \quad \dot{r} = \frac{-h v_p}{2\pi r}. \tag{4}$$

Taking the time derivative of the second of Eqs. (4) and then using the last of Eqs. (4) to substitute for the term $\dot{r}$, we have

$$\dot{v}_p = a_p = \frac{-2\pi}{h}\left(\frac{h^2 v_p^2}{4\pi^2 r^2} + r\ddot{r}\right). \tag{5}$$

To find an expression for the term $\ddot{r}$, we take the time derivative of $\dot{r}$ in Eq. (3) and obtain

$$\ddot{r} = \frac{-h}{2\pi}\alpha_s. \tag{6}$$

Substituting the result from Eq. (6) into Eq. (5), we have

$$a_p = \frac{-2\pi}{h}\left(\frac{h^2 v_p^2}{4\pi^2 r^2} - \frac{hr}{2\pi}\alpha_s\right). \tag{7}$$

Then, solving for α_s and simplifying, we obtain

$$\boxed{\alpha_s = \frac{a_p}{r} + \frac{h v_p^2}{2\pi r^3}.}$$

Evaluating the expression above for $h = 0.0048\,\text{in.} = (0.0048/12)\,\text{ft}$, $v_p = 1000\,\text{ft/min} = (1000/60)\,\text{ft/s}$, $a_p = 3\,\text{ft/s}^2$, $r_1 = 25\,\text{in.} = (25/12)\,\text{ft}$, and $r_2 = 10\,\text{in.} = (10/12)\,\text{ft}$, we have

$$\boxed{\alpha_s\big|_{r=r_1} = 1.442\,\text{rad/s}^2 \quad \text{and} \quad \alpha_s\big|_{r=r_2} = 3.631\,\text{rad/s}^2.}$$

Problem 2.87

The spool of paper used in a printing process is unrolled with velocity v_p and acceleration a_p. The thickness of the paper is h, and the outer radius of the spool at any instant is r.

If the velocity at which the paper is unrolled is *constant*, determine the angular acceleration α_s of the spool as a function of r, h, and v_p. Plot your answer for $h = 0.0048$ in. and $v_p = 1000$ ft/min as a function of r for 1 in. $\leq r \leq 25$ in. Over what range does α_s vary?

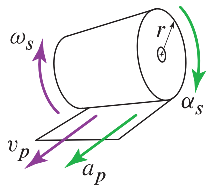



Photo credit: © David Lees/CORBIS

Solution

The radius decreases by the paper thickness h for every one revolution. Hence, letting θ be the angle measuring the angular position of a fixed radial line on the spool (θ increases when the spool turns clockwise), then we have

$$\frac{\Delta r}{\Delta\theta} = \frac{-h}{2\pi}. \tag{1}$$

Assuming that the decrease in radius can be viewed as occurring continuously, we can change the above relation into a relation in terms of differentials. That is we can write

$$\frac{dr}{d\theta} = \frac{-h}{2\pi}. \tag{2}$$

Next, observing that the angular velocity of the spool is $\omega_s = d\theta/dt$, we can then use the above equation to relate the time rate of change of r to ω_s by applying the chain rule as follows:

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} \quad\Rightarrow\quad \dot{r} = \frac{-h}{2\pi}\omega_s \quad\Rightarrow\quad \omega_s = \frac{-2\pi}{h}\dot{r}. \tag{3}$$

Recalling that the linear velocity of the paper is related to the angular velocity of the spool as $v_p = r\omega_s$, we can use Eq. (3) to relate v_p to $\dot{r}$ as follows:

$$v_p = r\omega_s \quad\Rightarrow\quad v_p = \frac{-2\pi}{h} r\dot{r} \quad\Rightarrow\quad \dot{r} = \frac{-hv_p}{2\pi r}. \tag{4}$$

Taking the time derivative of v_p in the second of Eqs. (4), accounting for the fact that v_p is constant (i.e., $a_p = 0$), and using the last of Eqs. (4), we have

$$a_p = 0 = \frac{-2\pi}{h}\left(\dot{r}^2 + r\ddot{r}\right) \quad\Rightarrow\quad 0 = \frac{-2\pi}{h}\left(\frac{h^2 v_p^2}{4\pi^2 r^2} + r\ddot{r}\right). \tag{5}$$

Taking the time derivative of the last of Eqs. (3) and the last of Eqs. (4) to find expressions for α_s and $\ddot{r}$, we can write

$$\alpha_s = \frac{-2\pi}{h}\ddot{r} \quad\text{and}\quad \ddot{r} = \frac{-h^2 v_p^2}{4\pi^2 r^3}, \tag{6}$$

from which we have

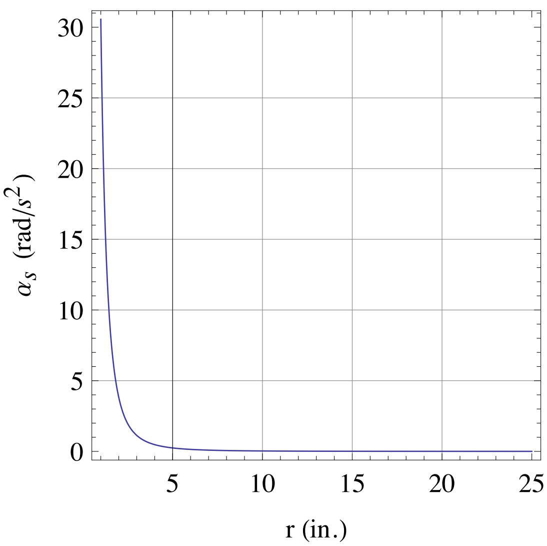
$$\alpha_s = \frac{h v_p^2}{2\pi r^3}.$$

To plot the above function, we first substitute the values of the known coefficients, i.e., we recall that $h = 0.0048\,\text{in.}$, and $v_p = 1000\,\text{ft/min} = 200.0\,\text{in./s}$ and rewrite α_s as

$$\alpha_s = \frac{30.56\,\text{in.}^3/\text{s}^2}{r^3}. \tag{7}$$

The above function can now be plotted. The plot below was generated using*Mathematica* with the following code:

```
Plot[30.56/r^3, {r, 1, 25}, PlotRange → All, Frame → True, GridLines → Automatic,
 AspectRatio → 1, FrameLabel → {"r (in.)", "αs (rad/s^2)"}]
```



The quantity α_s appears to vary from $30.5\,\text{rad/s}^2$ to close to zero as r varies from 1 in. to 4 in.

Problem 2.88

Derive the constant acceleration relation in Eq. (2.32), starting from Eq. (2.24). State what assumption you need to make about the acceleration a to complete the derivation. Finally, use Eq. (2.27), along with the result of your derivation, to derive Eq. (2.33). Be careful to do the integral in Eq. (2.27) before substituting your result for $v(t)$ (try it without doing so, to see what happens). After completing this problem, notice that Eqs. (2.32) and (2.33) are *not* subject to the same assumption you needed to make to solve both parts of this problem.

Solution

Assuming that the acceleration is not equal to 0 and integrating Eq. (2.24), we have

$$t(v) = t_0 + \frac{1}{a_c}\int_{v_0}^{v} dv \quad \Rightarrow \quad t(v) = t_0 + \frac{1}{a_c}(v - v_0) \quad \Rightarrow \quad \boxed{v = v_0 + a_c(t - t_0).} \tag{1}$$

Integrating Eq. (2.27) we have

$$s = s_0 + \frac{1}{a_c}\int_{v_0}^{v} v\,dv \quad \Rightarrow \quad s = s_0 + \frac{1}{2a_c}\left(v^2 - v_0^2\right). \tag{2}$$

Substituting for v from Eq. (1), we obtain

$$s = s_0 + \frac{1}{2a_c}\left[a_c^2(t - t_0)^2 + 2v_0 a_c\,(t - t_0)\right], \tag{3}$$

which can be simplified to obtain

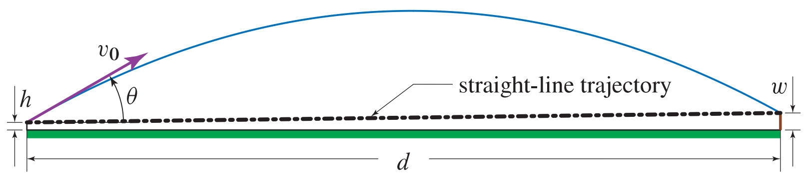
$$\boxed{s = s_0 + v_0(t - t_0) + \tfrac{1}{2}a_c(t - t_0)^2.} \tag{4}$$

Problem 2.89

The discussion in Example 2.12 revealed that the angle θ had to be greater than $\theta_{\mathrm{min}} = 0.716°$. Find an analytical expression for θ_{min} in terms of h, w, and d.

Solution

The smallest possible angle (with respect to the horizontal) corresponds to the straight-line trajectory that going from the point at which the ball is hit to the top of the center field wall.



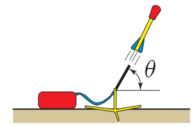
Using elementary trigonometry, we have that the analytical expression of the slope of the straight-line trajectory is

$$\theta = \tan^{-1}\left(\frac{w-h}{d}\right).$$

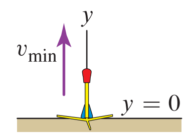
To achieve this trajectory the ball would need to be imparted an infinite speed. That is, the straight-line trajectory cannot be achieved in practice.

Problem 2.90

A stomp rocket is a toy consisting of a hose connected to a "blast pad" (i.e., an air bladder) at one end and to a short pipe mounted on a tripod at the other end. A rocket with a hollow body is mounted onto the pipe and is propelled into the air by "stomping" on the blast pad. Some manufacturers claim that one can shoot a rocket over 200 ft in the air. Neglecting air resistance, determine the rocket's minimum initial speed such that it reaches a maximum flight height of 200 ft.



Solution



The maximum height depends on the vertical component of the launch velocity. The higher this component the higher the height. Therefore, the minimum value of the speed needed to reach the desired height is found by launching the rocket purely in the vertical direction. Referring to the figure at the right, we consider the case in which the motion is completely in the y direction. Since the positive y direction is opposite to gravity, we have that the acceleration of the rocket is $\ddot{y} = -g =$ constant. We can relate velocity to position using the following constant acceleration equation:

$$\dot{y}^2 = \dot{y}_0^2 - 2g(y - y_0), \tag{1}$$

where $\dot{y}_0$ is the velocity of the rocket for $y = y_0$, and where we choose y_0 to denote the launch position of the rocket. Setting $y_0 = 0$ and recalling that the maximum height is achieved when $\dot{y} = 0$, for $y = h_{max}$ Eq. (1) becomes

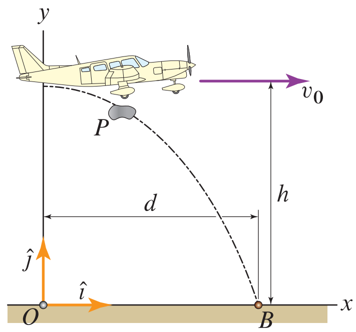
$$0 = \dot{y}_0^2 - 2gh_{max} \quad \Rightarrow \quad \dot{y}_0 = \sqrt{2gh_{max}}, \tag{2}$$

where we have chosen the positive root since the rocket is initially launched upward. Recalling that $h_{max} = 200$ ft and $g = 32.2\,\text{ft/s}^2$, and observing that the initial speed coincides with $\dot{y}_0$, we can evaluate the last of Eqs. (2) to obtain

$$\boxed{v_{min} = 113.5\,\text{ft/s}.}$$

Problem 2.91

An airplane flying horizontally at elevation $h = 150$ ft and at a constant speed $v_0 = 80$ mph drops a package P when passing over point O. Determine the horizontal distance d between the drop point and point B at which the package hits the ground.



Solution

We model the motion of the package as projectile motion. Observing that the positive direction of the y axis shown is opposite to that of gravity, we have that the components of the *constant* acceleration of the package P are

$$\ddot{x} = 0 \quad \text{and} \quad \ddot{y} = -g. \tag{1}$$

Letting t denote time and $t = 0$ denote the instant at which the package is released, Eqs. (1), along with Eq. (2.33) on p. 49, tell us that the x and y coordinates of the package as functions of time are

$$x(t) = x(0) + \dot{x}(0)t \quad \text{and} \quad y(t) = y(0) + \dot{y}(0)t - \tfrac{1}{2}gt^2, \tag{2}$$

where $\dot{x}(t)$ and $\dot{y}(t)$ are the x and y components of the package's velocity. The package is released at a height h over point O. Also, the velocity of the package at the instant of release is equal to the velocity of the plane. Hence, at the instant of release, we have

$$x(0) = 0, \quad y(0) = h, \quad \dot{x}(0) = v_0, \quad \text{and} \quad \dot{y}(0) = 0. \tag{3}$$

Substituting Eqs. (3) into Eqs. (2), $x(t)$ and $y(t)$ become:

$$x(t) = v_0 t \quad \text{and} \quad y(t) = h - \tfrac{1}{2}gt^2. \tag{4}$$

Let t_i denote the time at which the package impacts the ground. Since $y(t_i) = 0$, referring to the second of Eqs. (4), we have

$$h - \tfrac{1}{2}gt_i^2 = 0 \quad \Rightarrow \quad t_i = \sqrt{\frac{2h}{g}}. \tag{5}$$

We now observe that $d = x(t_i)$. Substituting the second of Eqs. (5) into the first of Eqs. (4), we have

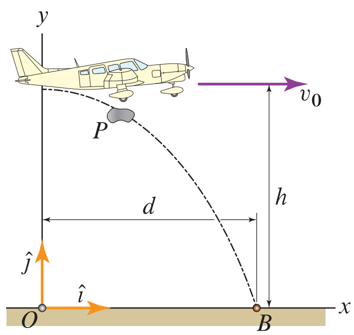
$$d = v_0\sqrt{\frac{2h}{g}}. \tag{6}$$

Recalling that $v_0 = 80\,\text{mph} = 80(5280/3600)\,\text{ft/s}$, $h = 150\,\text{ft}$, and $g = 32.2\,\text{ft/s}^2$, we can evaluate Eq. (6) to obtain

$$\boxed{d = 358.1\,\text{ft.}}$$

Problem 2.92

An airplane flying horizontally at elevation $h = 60\,\text{m}$ and at a constant speed $v_0 = 120\,\text{km/h}$ drops a package P when passing over point O. Determine the time it takes for the package to hit the ground at point B. In addition, determine the velocity of the package at B.



Solution

We model the motion of the package as projectile motion. Observing that the positive direction of the y axis shown is opposite to that of gravity, the components of the *constant* acceleration of the package P are

$$\ddot{x} = 0 \quad \text{and} \quad \ddot{y} = -g. \tag{1}$$

Letting t denote time and $t = 0$ denote the instant of release, Eqs. (1), along with Eq. (2.32) on p. 49 and Eq. (2.33) on p. 49, tell us that the x and y components of the velocity of the package, along with the x and y coordinates of the package as functions of time are

$$\dot{x}(t) = \dot{x}(0), \quad \dot{y}(t) = \dot{y}(0) - gt, \quad x(t) = x(0) + \dot{x}(0)t, \quad \text{and} \quad y(t) = y(0) + \dot{y}(0)t - \tfrac{1}{2}gt^2, \tag{2}$$

where $\dot{x}(t)$ and $\dot{y}(t)$ are the x and y components of the package's velocity. Since the package is released at a height h over point O while traveling with the airplane, at the instant of release we have

$$x(0) = 0, \quad y(0) = h, \quad \dot{x}(0) = v_0, \quad \text{and} \quad \dot{y}(0) = 0. \tag{3}$$

Substituting Eqs. (3) into Eqs. (2), we have

$$\dot{x}(t) = v_0, \quad \dot{y}(t) = -gt, \quad x(t) = v_0 t, \quad \text{and} \quad y(t) = h - \tfrac{1}{2}gt^2. \tag{4}$$

Let t_i be the time at which P impacts the ground. Since $y(t_i) = 0$, referring to the last of Eqs. (4), we have

$$h - \tfrac{1}{2}gt_i^2 = 0 \quad \Rightarrow \quad t_i = \sqrt{2h/g}. \tag{5}$$

Recalling that $h = 60\,\text{m}$ and $g = 9.81\,\text{m/s}^2$, we can evaluate the last of Eqs. (5) to obtain

$$\boxed{t_i = 3.497\,\text{s}.}$$

Denoting the velocity of P at B by $\vec{v}_B$, we now observe that the velocity at B is $\vec{v}_B = \dot{x}(t_i)\,\hat{\imath} + \dot{y}(t_i)\,\hat{\jmath}$. Hence, substituting the second of Eqs. (5) into the first two of Eqs. (4), we have

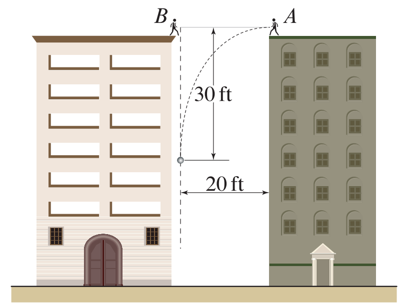
$$\vec{v}_B = v_0\,\hat{\imath} - g\sqrt{2h/g}\,\hat{\jmath}. \tag{6}$$

Recalling that $v_0 = 120\,\text{km/h} = 120(1000/3600)\,\text{m/s}$, $h = 60\,\text{m}$, and $g = 9.81\,\text{m/s}^2$, Eq. (6) gives

$$\boxed{\vec{v}_B = (33.33\,\hat{\imath} - 34.31\,\hat{\jmath})\,\text{m/s}.}$$

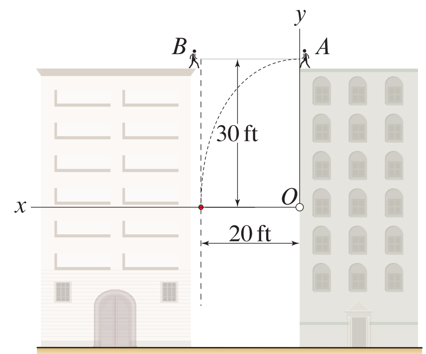
Problem 2.93

Stuntmen A and B are shooting a movie scene in which A needs to pass a gun to B. Stuntman B is supposed to start falling vertically precisely when A throws the gun to B. Treating the gun and the stuntman B as particles, find the velocity of the gun as it leaves A's hand so that B will catch it after falling 30 ft.



Solution

The gun and B drop at the same time from the same height, and are assumed to be both subject to the same acceleration, i.e., gravity. In order for the gun and B to the same vertical position at the time that B grasps the gun, it is necessary for the gun and B to fall with equal vertical velocities. This can only be achieved if the initial vertical velocity of the gun is equal to that of B, namely, zero. Hence, the rest of the problem is devoted to finding the horizontal component of the velocity of the gun at the time the gun is thrown. To do so, we start by finding the time that B takes to fall the distance $h = 30$ ft. Using the coordinate system shown at the right, and using constant acceleration equations, we have



$$y_B = y_{B0} + \dot{y}_{B0}(t - t_0) - \tfrac{1}{2}g(t - t_0)^2, \tag{1}$$

where y_{B0} and $\dot{y}_{B0}$ are the position and the vertical velocity of B at time t_0, respectively. Setting $t_0 = 0$ and recalling that B drops from rest a distance h above the origin of the y axis, we can rewrite the above equation in the following form:

$$y_B = h - \tfrac{1}{2}gt^2. \tag{2}$$

Due to our choice of origin, B will grasp the gun at $y_B = 0$. Letting t_f (the subscript f stands for final) denote the time at which B grasps the gun, from Eq. (2) we then have

$$t_f = \sqrt{2h/g}. \tag{3}$$

We now observe that the motion of the gun in the horizontal direction is also a constant acceleration motion with acceleration equal to zero. Hence, using the subscript A to refer to the gun (as opposed to the stuntman who initially threw it) we have

$$x_A = x_{A0} + \dot{x}_{A0}(t - t_0), \tag{4}$$

where x_{A0} and $\dot{x}_{A0}$ are the gun's position and velocity in the x direction at time t_0. Having already set $t_0 = 0$ and observing that $x_{A0} = 0$, and that for $t = t_f$ we must have $x_A = d$, from the above equation we have

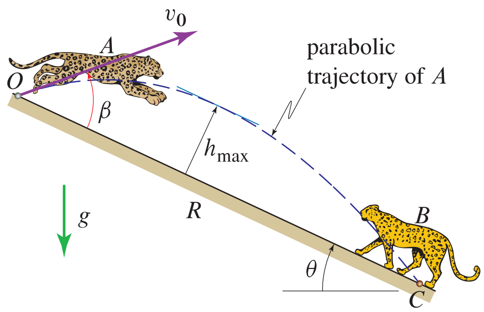
$$d = \dot{x}_{A0}\sqrt{\frac{2h}{g}} \quad \Rightarrow \quad \dot{x}_{A0} = d\sqrt{\frac{g}{2h}} = 14.65\,\text{ft/s}, \tag{5}$$

where we have used the following numerical data: $d = 20$ ft, $h = 30$ ft, and $g = 32.2\,\text{ft/s}^2$. In summary, expressing our answer in vector form, we have

$$\vec{v}_{\text{gun}} = (14.65\,\text{ft/s})\,\hat{\imath}.$$

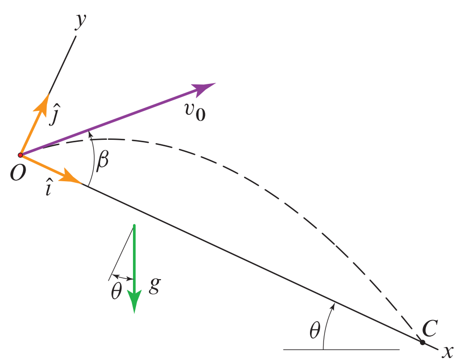
Problem 2.94

The jaguar A leaps from O at speed $v_0 = 6\,\mathrm{m/s}$ and angle $\beta = 35°$ relative to the incline to try to intercept the panther B at C. Determine the distance R that the jaguar jumps from O to C (i.e., R is the distance between the two points of the trajectory that intersect the incline), given that the angle of the incline is $\theta = 25°$.



Solution

The acceleration of A is completely in the vertical direction. Hence, referring to the figure at the right, the components of the acceleration of A in the x and y directions are $\ddot{x} = g\sin\theta$ and $\ddot{y} = -g\cos\theta$. The components of the initial velocity of A are $v_{x0} = v_0\cos\beta$ and $v_{y0} = v_0\sin\beta$. Observing that the x and y components of acceleration of A are both constants, we can use constant acceleration equations to write



$$x = v_0(\cos\beta)t + \tfrac{1}{2}g(\sin\theta)t^2 = t\big[v_0\cos\beta + \tfrac{1}{2}g(\sin\theta)t\big], \quad (1)$$

$$y = v_0(\sin\beta)t - \tfrac{1}{2}g(\cos\theta)t^2 = t\big[v_0\sin\beta - \tfrac{1}{2}g(\cos\theta)t\big], \quad (2)$$

where we have accounted for the fact that, at time $t = 0$, A is at the origin of the chosen coordinate system. Denoting by t_C the time at which A reaches C, we observe that for $t = t_C$, $y = 0$. Therefore, from Eq. (2),

$$0 = v_0\sin\beta - \tfrac{1}{2}g(\cos\theta)t_C \quad \Rightarrow \quad t_C = \frac{2v_0\sin\beta}{g\cos\theta}. \tag{3}$$

Observing that, for $t = t_C$, $x(t_C) = R$, substituting the last of Eqs. (3) into Eq. (1), we have

$$R = \frac{2v_0\sin\beta}{g\cos\theta}\left[v_0\cos\beta + \tfrac{1}{2}g(\sin\theta)\frac{2v_0\sin\beta}{g\cos\theta}\right], \tag{4}$$

which can be simplified to

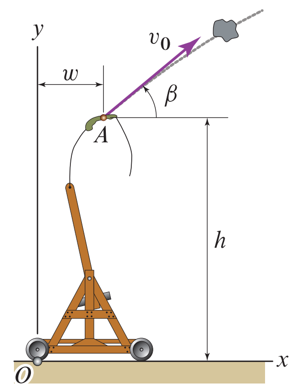
$$R = \frac{2v_0^2\sin\beta}{g\cos\theta}\big(\cos\beta + \tan\theta\sin\beta\big). \tag{5}$$

Recalling that $v_0 = 6\,\mathrm{m/s}$, $\beta = 35°$, $g = 9.81\,\mathrm{m/s^2}$, and $\theta = 25°$, we can evaluate the above result to obtain

$$\boxed{R = 5.047\,\mathrm{m}.}$$

Problem 2.95

If the projectile is released at A with initial speed v_0 and angle β, derive the projectile's trajectory, using the coordinate system shown. Neglect air resistance.



Solution

Using the coordinate system indicated in the problem statement, we set up the following constant acceleration equations for both the x and y coordinates of the projectile:

$$x = x_A + (v_0 \cos\beta)t, \tag{1}$$

$$y = y_A + (v_0 \sin\beta)t - \tfrac{1}{2}gt^2, \tag{2}$$

where x_A and y_A are the coordinates of the fixed point A at which the projectile is released. From Eq. (1) we have that $t = (x - x_A)/(v_0 \cos\beta)$. Substituting this result into Eq. (2), we obtain

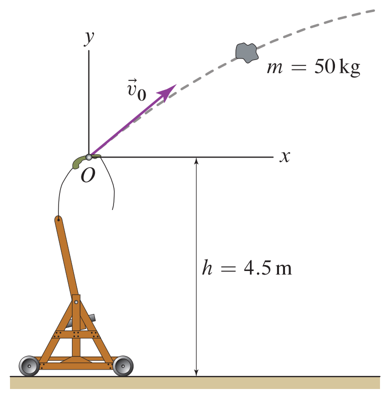
$$y - y_A = \tan\beta(x - x_A) - \frac{g}{2v_0^2\cos^2\beta}(x - x_A)^2. \tag{3}$$

Observing that $x_A = w$ and $y_A = h$, the above equation becomes

$$y = h + \tan\beta(x - w) - \frac{g}{2v_0^2\cos^2\beta}(x - w)^2.$$

Problem 2.96

A trebuchet releases a rock with mass $m = 50\,\text{kg}$ at point O. The initial velocity of the projectile is $\vec{v}_0 = (45\,\hat{\imath} + 30\,\hat{\jmath})\,\text{m/s}$. Neglecting aerodynamic effects, determine where the rock will land and its time of flight.



Solution

Referring to the coordinate system defined in the problem statement, we see that y_{land}, the y coordinate of the rock when it lands on the ground, is $-h$. With this in mind, we can write the following constant acceleration equation for the y coordinate of the rock:

$$y = v_{0y}t - \tfrac{1}{2}gt^2,$$

where it is understood that $t = 0$ is the time of release and $y = 0$ and v_{0y} are the vertical position and the vertical component of velocity of the rock at time $t = 0$, respectively. Denoting by t_{flight} the time at which the rock impacts the ground, we have

$$-h = v_{0y}t_{\text{flight}} - \tfrac{1}{2}gt_{\text{flight}}^2 \quad \Rightarrow \quad gt_{\text{flight}}^2 - 2v_{0y}t_{\text{flight}} - 2h = 0 \quad \Rightarrow \quad t_{\text{flight}} = \frac{v_{0y} \pm \sqrt{v_{0y}^2 + 2gh}}{g}. \tag{1}$$

The only physically meaningful solution for t_{flight} is that corresponding to the $+$ sign in front of the square root, that is,

$$t_{\text{flight}} = \frac{1}{g}\left(v_{y0} + \sqrt{v_{y0}^2 + 2gh}\right). \tag{2}$$

Recalling that $g = 9.81\,\text{m/s}^2$, $v_{0y} = 30\,\text{m/s}$, and $h = 4.5\,\text{m}$, we can evaluate the expression above to obtain

$$\boxed{t_{\text{flight}} = 6.263\,\text{s}.}$$

Next observing that the motion is in the x direction is a constant acceleration motion with acceleration equal to zero, the x coordinate of the rock is described by the following (constant acceleration) equation:

$$x = v_{0x}t, \tag{3}$$

where we have accounted for the fact that $x = 0$ for $t = 0$, and where v_{0x} is the x component of the velocity of the rock for $t = 0$. Substituting Eq. (2) into Eq. (3), for $t = t_{\text{flight}}$ we have

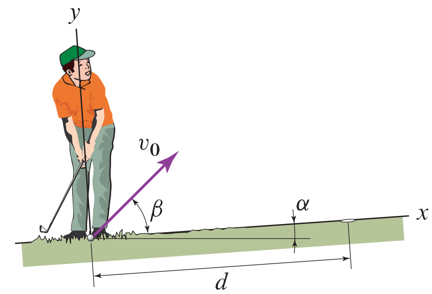
$$x_{\text{land}} = \frac{v_{0x}}{g}\left(v_{y0} + \sqrt{v_{y0}^2 + 2gh}\right). \tag{4}$$

The position of the rock when the rock hits the ground is $\vec{r}_{\text{land}} = x_{\text{land}}\,\hat{\imath} - h\,\hat{\jmath}$. Therefore, recalling that $g = 9.81\,\text{m/s}^2$, $v_{0x} = 45\,\text{m/s}$, $v_{0y} = 30\,\text{m/s}$, and $h = 4.5\,\text{m}$, and using Eq. (4) to evaluate x_{land} we have

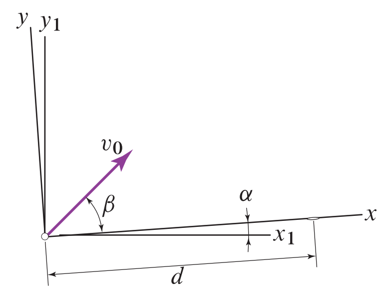
$$\boxed{\vec{r}_{\text{land}} = (281.8\,\hat{\imath} - 4.500\,\hat{\jmath})\,\text{m}.}$$

Problem 2.97

A golfer chips the ball into the hole on the fly from the rough at the edge of the green. Letting $\alpha = 4°$ and $d = 2.4$ m, verify that the golfer will place the ball within 10 mm of the center of the hole if the ball leaves the rough with a speed $v_0 = 5.03$ m/s and an angle $\beta = 41°$.



Solution



Referring to the figure at the right, we will use the coordinate system with axes x_1 and y_1, which are horizontal and vertical, respectively. The acceleration of the ball in this coordinate system has components

$$\ddot{x}_1 = 0 \quad \text{and} \quad \ddot{y}_1 = -g. \tag{1}$$

Letting $t = 0$ be the initial time, and using constant acceleration equations, we have

$$\dot{x}_1 = v_0 \cos(\alpha + \beta) \quad \text{and} \quad \dot{y}_1 = v_0 \sin(\alpha + \beta) - gt, \tag{2}$$

where the have used the fact that the initial velocity of the ball is $\vec{v}(0) = v_0(\cos(\alpha + \beta)\,\hat{\imath} + \sin(\alpha + \beta)\,\hat{\jmath})$. Integrating Eqs. (2) with respect to time, and enforcing the fact that $x_1 = 0$ and $y_1 = 0$ for $t = 0$, we have

$$x_1 = v_0 \cos(\alpha + \beta)t \quad \text{and} \quad y_1 = v_0 \sin(\alpha + \beta)t - \tfrac{1}{2}gt^2. \tag{3}$$

From the first of Eqs. (3) we have $t = x_1/[v_0 \cos(\alpha + \beta)]$. Substituting this result into the second of Eqs. (3), we have

$$y_1 = \tan(\alpha + \beta)x_1 - \left[\frac{g \sec^2(\alpha + \beta)}{2v_0^2}\right]x_1^2. \tag{4}$$

Recalling that $\alpha = 4°$ and $\beta = 41°$, so that $\alpha + \beta = 45°$, $\tan(\alpha + \beta) = 1$ and $\sec^2(\alpha + \beta) = 2$, so that Eq. (4) simplifies to:

$$y_1 = x_1 - \frac{g}{v_0^2}x_1^2. \tag{5}$$

The x_1 and y_1 coordinates of the point at which the ball lands must satisfy the condition $x_1 \tan\alpha = y_1$. Combining this requirement with Eq. (5) we have

$$x_1 \tan\alpha = x_1 - \frac{g}{v_0^2}x_1^2 \quad \Rightarrow \quad x_1 = \frac{v_0^2}{g}\big(1 - \tan\alpha\big) = 2.399\,\text{m}, \tag{6}$$

where we have used the following numerical values: $v_0 = 5.03$ m, $g = 9.81\,\text{m/s}^2$, and $\alpha = 4°$. The value of x_1 in Eq. (6) is the x_1 coordinate of the ball's landing spot. With this information, letting d_L denote the distance between the ball's landing spot and the the golfer, we can determine d_L using trigonometry as follows:

$$d_L = \frac{x_1}{\cos\alpha} = 2.405\,\text{m} \quad \Rightarrow \quad d - d_L = 0.005\,\text{m} < 10\,\text{mm}, \tag{7}$$

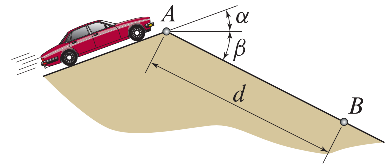
where we have used the fact that $\alpha = 4°$ and $d = 2.4$ m. We can then conclude that

The golfer's chip shot is successful.

Problem 2.98

In a movie scene involving a car chase, a car goes over the top of a ramp at A and lands at B below.

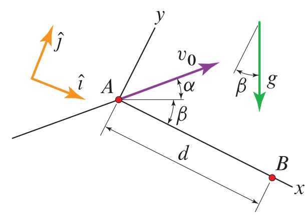
If $\alpha = 20°$ and $\beta = 23°$, determine the distance d covered by the car if the car's speed at A is 45 km/h. Neglect aerodynamic effects.



Solution

The acceleration of the car in the xy coordinate system shown is

$$\vec{a} = g \sin\beta\, \hat{\imath} - g \cos\beta\, \hat{\jmath} \quad \Rightarrow \quad \ddot{x} = g \sin\beta \quad \text{and} \quad \ddot{y} = -g\cos\beta. \tag{1}$$



We set $t = 0$ to be the time at which the car jumps off at the origin A. Hence, $y = 0$ at $t = 0$. Also, at $t = 0$, the velocity components of the cars are

$$v_x(0) = v_0 \cos(\alpha + \beta) \quad \text{and} \quad v_y(0) = v_0 \sin(\alpha + \beta). \tag{2}$$

Hence, using the above considerations along with constant acceleration equations, the y coordinate of the car as a function of time is given by

$$y = v_0 \sin(\alpha + \beta)t - \tfrac{1}{2} g(\cos\beta)t^2. \tag{3}$$

Letting t_B denote the time at which the car lands at B, since $y_B = 0$, from Eq. (3) we have

$$v_0 \sin(\alpha + \beta)t_B - \tfrac{1}{2}(g\cos\beta)t_B^2 = 0 \quad \Rightarrow \quad t_B = \frac{2v_0 \sin(\alpha + \beta)}{g\cos\beta}. \tag{4}$$

Next observing that the x component of the acceleration in the second of Eqs. (1) is also constant, using constant acceleration equations, the x coordinate of the car as a function of time is given by

$$x = v_0 \cos(\alpha + \beta)t + \tfrac{1}{2}(g\sin\beta)t^2. \tag{5}$$

Substituting the last of Eqs. (4) into Eq. (5) and simplifying, we have

$$x_B = d = \frac{2v_0^2 \sin(\alpha + \beta)}{g\cos\beta}[\cos(\alpha + \beta) + \tan\beta \sin(\alpha + \beta)]. \tag{6}$$

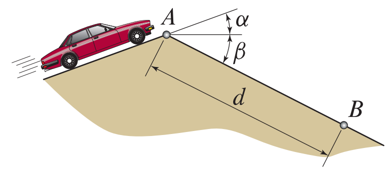
Recalling that $v_0 = 45\,\text{km/h} = 45(1000/3600)\,\text{m/s}$, $\alpha = 20°$, $\beta = 23°$, and $g = 9.81\,\text{m/s}^2$, we can evaluate the above result to obtain

$$\boxed{d = 24.09\,\text{m}.} \tag{7}$$

Problem 2.99

In a movie scene involving a car chase, a car goes over the top of a ramp at A and lands at B below.

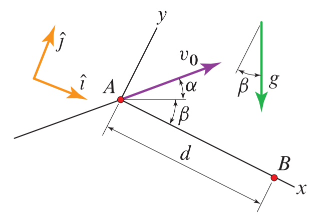
Determine the speed of the car at A if the car is to cover distance $d = 150$ ft for $\alpha = 20°$ and $\beta = 27°$. Neglect aerodynamic effects.



Solution

The acceleration vector of the car in the xy coordinate system shown is

$$\vec{a} = g\sin\beta\,\hat{\imath} - g\cos\beta\,\hat{\jmath} \quad \Rightarrow \quad \ddot{x} = g\sin\beta \quad \text{and} \quad \ddot{y} = -g\cos\beta. \tag{1}$$



We set $t = 0$ to be the time at which the car jumps off at the origin A. Hence, $y = 0$ at $t = 0$. Also, at $t = 0$, the velocity components of the cars are

$$v_x(0) = v_0\cos(\alpha+\beta) \quad \text{and} \quad v_y(0) = v_0\sin(\alpha+\beta). \tag{2}$$

Hence, using the above considerations along with constant acceleration equations, the y coordinate of the car as a function of time is given by

$$y = v_0\sin(\alpha+\beta)t - \tfrac{1}{2}g(\cos\beta)t^2. \tag{3}$$

Letting t_B denote the time at which the car lands at B, since $y_B = 0$, from Eq. (3) we have

$$v_0\sin(\alpha+\beta)t_B - \tfrac{1}{2}(g\cos\beta)t_B^2 = 0 \quad \Rightarrow \quad t_B = \frac{2v_0\sin(\alpha+\beta)}{g\cos\beta}. \tag{4}$$

Next observing that the x component of the acceleration in the second of Eqs. (1) is also constant, using constant acceleration equations, the x coordinate of the car as a function of time is given by

$$x = v_0\cos(\alpha+\beta)t + \tfrac{1}{2}(g\sin\beta)t^2. \tag{5}$$

Substituting the last of Eqs. (4) into Eq. (5) and simplifying, we have

$$x_B = d = \frac{2v_0^2\sin(\alpha+\beta)}{g\cos\beta}[\cos(\alpha+\beta) + \tan\beta\sin(\alpha+\beta)], \tag{6}$$

which can be solved for v_0 to obtain

$$v_0 = \sqrt{\frac{dg\cos\beta}{2\sin(\alpha+\beta)[\cos(\alpha+\beta)+\tan\beta\sin(\alpha+\beta)]}}. \tag{7}$$

Recalling that $d = 150$ ft, $\alpha = 20°$, $\beta = 27°$, and $g = 32.2\,\text{ft/s}^2$, Eq. (7) can be evaluated to obtain

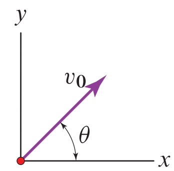
$$\boxed{v_0 = 52.82\,\text{ft/s}.}$$

Problem 2.100

The M777 lightweight 155 mm howitzer is a piece of artillery whose rounds are ejected from the gun with a speed of 829 m/s. Assuming that the gun is fired over a flat battlefield and ignoring aerodynamic effects, determine (*a*) the elevation angle needed to achieve the maximum range, (*b*) the maximum possible range of the gun, and (*c*) the time it would take a projectile to cover the maximum range. Express the result for the range as a percentage of the actual maximum range of this weapon, which is 30 km for unassisted ammunition.

Photo credit: U.S. Army Photo

Solution



We will solve the problem using the xy coordinate system shown in the figure at the right. The x axis is assumed to lie on the ground. We assume that the projectile is launched at $t = 0$ from the origin. The acceleration of the projectile is constant and has horizontal and vertical components $\ddot{x} = 0$ and $\ddot{y} = -g$, respectively. Then, letting θ be the elevation angle and v_0 the intial speed of the projectile, and using constant acceleration equations, the x and y coordinates of the projectile are given by

$$x = (v_0 \cos\theta)t \quad \text{and} \quad y = (v_0 \sin\theta)t - \tfrac{1}{2}gt^2. \tag{1}$$

Denoting the time of flight by t_f, then $y = 0$ for $t = t_f$. Hence, from the second of Eqs. (1), we have

$$t_f = 2v_0 \sin\theta/g. \tag{2}$$

The range R is given by the value of x for $t = t_f$. Substituting Eq. (2) into the first of Eqs. (1), and using the trigonometric identity $2\sin\theta\cos\theta = \sin 2\theta$, we have

$$R = (v_0^2/g)\sin 2\theta. \tag{3}$$

Part (a). From Eq. (3), $R_{\max}$ occurs when $\sin 2\theta = 1$. This equation has infinitely many solutions for θ, but the only meaningful solution is

$$2\theta_{R_{\max}} = \frac{\pi}{2}\,\text{rad} \quad \Rightarrow \quad \boxed{\theta_{R_{\max}} = \frac{\pi}{4}\,\text{rad} = 45^\circ.} \tag{4}$$

Part (b). Substituting the last of Eqs. (4) into Eq. (3), we have that $R_{\max}$ is

$$R_{\max} = v_0^2/g. \tag{5}$$

Recalling that $v_0 = 829\,\text{m/s}$ and $g = 9.81\,\text{m/s}^2$, and expressing $R_{\max}$ as a percentage of the actual range of 30 km, we can evaluate the above expression to obtain

$$\boxed{R_{\max} = 233.5\% \text{ of the actual maximum range.}}$$

Part (c). Denoting the time the projectile takes to cover $R_{\max}$ by $t_{R_{\max}}$, we have that $t_{R_{\max}}$ is equal to t_f for $\theta = \theta_{R_{\max}}$. Hence, substituting the last of Eqs. (4) into Eq. (2), we have

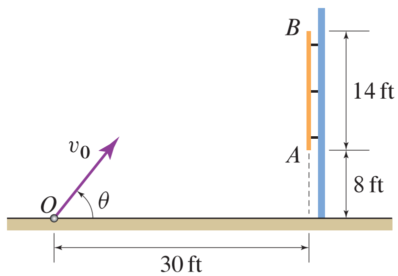
$$t_{R_{\max}} = \sqrt{2}v_0/g. \tag{6}$$

Recalling that $v_0 = 829\,\text{m/s}$ and $g = 9.81\,\text{m/s}^2$, we can evaluate the above result to obtain

$$\boxed{t_{R_{\max}} = 119.5\,\text{s}.}$$

Problem 2.101

You want to throw a rock from point O to hit the vertical advertising sign AB, which is $R = 30$ ft away. You can throw a rock at the speed $v_0 = 45$ ft/s. The bottom of the sign is 8 ft off the ground and the sign is 14 ft tall. Determine the range of angles at which the projectile can be thrown in order to hit the target, and compare this with the angle subtended by the target as seen from an observer at point O. Compare your results with those found in Example 2.11.



Solution

This problem can be solved as illustrated in Example 2.11 on p. 70 of the textbook. We recall Eq. (7) in Example 2.11 on p. 70 of the textbook:

$$\tan\theta = \frac{v_0^2 \pm \sqrt{v_0^4 - g(gR^2 + 2yv_0^2)}}{gR}. \tag{1}$$

Observe that we are given all of the data needed to use the above equation. Namely, we have $v_0 = 45$ ft/s, $g = 32.2$ ft/s^2, $R = 30$ ft, so that substituting in the above equation $y_A = 8$ ft and $y_B = 22$ ft we have

$$y = y_A = 8\,\text{ft} \quad \Rightarrow \quad \begin{cases} \theta_1 = 30.43°, \\ \theta_2 = 74.50°, \end{cases} \tag{2}$$

$$y = y_B = 22\,\text{ft} \quad \Rightarrow \quad \begin{cases} \theta_1 = 56.84°, \\ \theta_2 = 69.41°. \end{cases} \tag{3}$$

Following the same logic as in Example 2.11, we obtain the two ranges of firing angles as

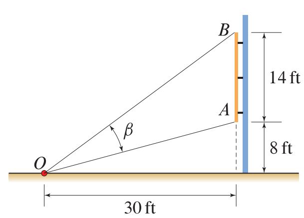
$$\boxed{30.43° \le \theta \le 56.84° \quad \text{and} \quad 69.41° \le \theta \le 74.50°.}$$

The sizes of these intervals are, respectively,

$$\boxed{\Delta\theta_1 = 26.41° \quad \text{and} \quad \Delta\theta_2 = 5.087°.}$$

The angle subtended by the target as seen from an observer at point O is

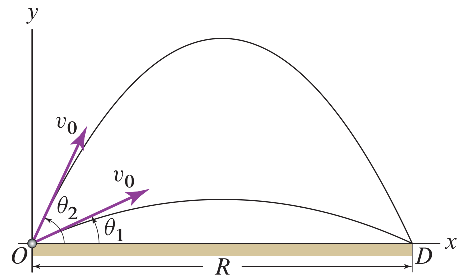
$$\beta = \tan^{-1}\left(\frac{22}{30}\right) - \tan^{-1}\left(\frac{8}{30}\right) \quad \Rightarrow \quad \boxed{\beta = 21.32°.} \tag{4}$$



Unlike Example 2.11, the difference between the angle subtended by the target and $\Delta\theta_1$ or $\Delta\theta_2$ is significant. In addition, we see that the value of $\Delta\theta_1$ is much closer to β than $\Delta\theta_2$.

Problem 2.102

Suppose that you can throw a projectile at a large enough v_0 so that it can hit a target a distance R downrange. Given that you know v_0 and R, determine the general expressions for the *two* distinct launch angles θ_1 and θ_2 that will allow the projectile to hit D. For $v_0 = 30\,\text{m/s}$ and $R = 70\,\text{m}$, determine numerical values for θ_1 and θ_2.



Solution

Using the axes in the figure, the components of the acceleration of the projectile are $\ddot{x} = 0$ and $\ddot{y} = -g$, where g is the acceleration due to gravity. Therefore, the acceleration of the projectile is constant and applying the constant acceleration relation in Eq. (2.33) on p. 49 of the textbook, we have

$$x = x_0 + (v_0 \cos\theta)t, \quad \text{and} \quad y = y_0 + (v_0 \sin\theta)t - \tfrac{1}{2}gt^2, \tag{1}$$

where we have accounted for the fact that the projectile is at O for $t = 0$, and where we have denoted by θ the orientation of the initial velocity of the projectile. For $x = R$ we have that $y = 0$. Enforcing this condition, Eqs. (1) give

$$R = x_0 + (v_0 \cos\theta)t_D, \quad \text{and} \quad 0 = y_0 + (v_0 \sin\theta)t_D - \tfrac{1}{2}gt_D^2, \tag{2}$$

where t_D is the time the projectile takes to go from O to D. Eliminating t_D from Eqs. (2), with $x_0 = 0$ and $y_0 = 0$, we have

$$gR - 2v_0^2 \sin\theta\cos\theta = 0 \quad \Rightarrow \quad \sin(2\theta) = gR/v_0^2, \tag{3}$$

where, in writing the second of Eqs. (3), we have used the trigonometric identity $\sin(2\theta) = 2\sin\theta\cos\theta$. As long as the values of R and v_0 are such that $gR/v_0^2 \le 1$, and observing that the physically acceptable values of θ lie in the range $0 \le \theta \le 90°$, we have that the last of Eqs. (3) admits the following two solutions:

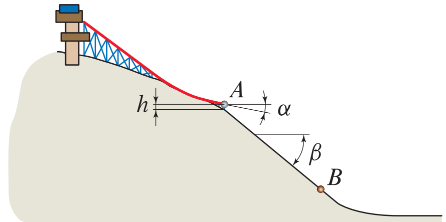
$$\theta_1 = \tfrac{1}{2}\sin^{-1}(gR/v_0^2) \quad \text{and} \quad \theta_2 = 90° - \tfrac{1}{2}\sin^{-1}(gR/v_0^2).$$

For the given values of $R = 70\,\text{m}$ and $v_0 = 30\,\text{m/s}$, and recalling that $g = 9.81\,\text{m/s}^2$, we can evaluate θ_1 and θ_2 to obtain

$$\theta_1 = 24.86° \quad \text{and} \quad \theta_2 = 65.14°.$$

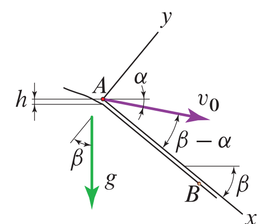
Problem 2.103

An alpine ski jumper can fly distances in excess of 100 m by using his or her body and skis as a "wing" and therefore, taking advantage of aerodynamic effects. With this in mind and assuming that a ski jumper could survive the jump, determine the distance the jumper could "fly" without aerodynamic effects, i.e., if the jumper were in free fall after clearing the ramp. For the purpose of your calculation, use the following typical data: $\alpha = 11°$ (slope of ramp at takeoff point A), $\beta = 36°$ (average slope of the hill), $v_0 = 86\,\mathrm{km/h}$ (speed at A), $h = 3\,\mathrm{m}$ (height of takeoff point with respect to the hill). Finally, for simplicity, let the jump distance be the distance between the takeoff point A and the landing point B.



Solution

We will solve the problem using a Cartesian coordinate system with origin at A and axes x and y oriented such that the x axis is parallel to the hill (see figure at the right). In the chosen coordinate system, the velocity of the jumper at A is



$$\vec{v}_A = v_0 \cos(\beta - \alpha)\,\hat{\imath} + v_0 \sin(\beta - \alpha)\,\hat{\jmath}. \tag{1}$$

Once airborne, the acceleration of the jumper is

$$\vec{a} = g \sin\beta\,\hat{\imath} - g\cos\beta\,\hat{\jmath}. \tag{2}$$

Using constant acceleration equations, we then have that

$$x = v_0\cos(\beta-\alpha)t + \tfrac{1}{2}(g\sin\beta)t^2 \quad \text{and} \quad y = v_0\sin(\beta-\alpha)t - \tfrac{1}{2}(g\cos\beta)t^2, \tag{3}$$

where we have set $t = 0$ to be the time at which the jumper takes off at A, and where we have accounted for the fact that, at $t = 0$, the velocity of the jumper is that in Eq. (1). Letting t_B denote the time at which the jumper lands at B, we can replace t with t_B in the second of Eqs. (3) and enforce the condition that $y_B = -h\cos\beta$. This gives

$$-h\cos\beta = v_0\sin(\beta-\alpha)t_B - \tfrac{1}{2}(g\cos\beta)t_B^2$$

$$\Rightarrow \quad t_B = \frac{v_0\sin(\beta-\alpha) \pm \sqrt{v_0^2\sin^2(\beta-\alpha) + 2hg\cos^2\beta}}{g\cos\beta}. \tag{4}$$

The solution for t_B corresponding to the minus sign in front of the square root is negative. Hence, the only acceptable value for t_B is that with the + sign. Recalling that $v_0 = 86\,\mathrm{km/h} = 86(1000/3600)\,\mathrm{m/s}$, $\beta = 36°$, $\alpha = 11°$, $h = 3\,\mathrm{m}$, and $g = 9.81\,\mathrm{m/s^2}$, we then have

$$t_B = 2.765\,\mathrm{s}. \tag{5}$$

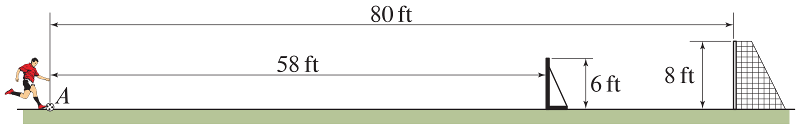
Using the data listed right above Eq. (5) and substituting t_B into the first of Eqs. (3), we have $x_B = 81.92\,\mathrm{m}$. Then, recalling that $y_B = -h\cos\beta = -2.427\,\mathrm{m}$, we have that the distance between points A and B can be calculated using the Pythagorean theorem, i.e., $d_{AB} = \sqrt{x_B^2 + y_B^2}$ which gives

$$\boxed{d_{AB} = 81.96\,\mathrm{m}.} \tag{6}$$

Problem 2.104

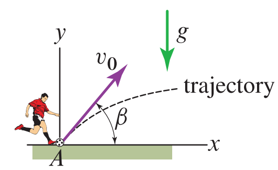
A soccer player practices kicking a ball from A directly into the goal (i.e., the ball does not bounce first) while clearing a 6 ft tall fixed barrier.

Determine the minimum speed that the player needs to give the ball to accomplish the task. *Hint:* Consider the equation for the projectile's trajectory of the form $y = C_0 + C_1 x + C_2 x^2$, with the y axis parallel to the direction of gravity, for the case in which the ball reaches the goal at its base. Solve this equation for the initial speed v_0 as a function of the initial angle θ, and finally find $(v_0)_{\text{min}}$ as you learned in calculus. Don't forget to check whether or not the ball clears the barrier.



Solution

The coordinate system shown at the right has origin at A, the position of the ball at $t = 0$. The trajectory of the ball has the form $y = C_0 + C_1 x + C_2 x^2$. To find C_0, C_1 and C_2 we proceed as follows. First, we observe that $y = 0$ for $x = 0$, which implies that $C_0 = 0$. Second, recall that the velocity is always tangent to the trajectory. Therefore, given that β is the orientation of the velocity at $t = 0$, the slope of the trajectory at $x = 0$ must be equal to $\tan\beta$, i.e., $C_1 = (dy/dx)_{x=0} = \tan\beta$. We know that the trajectory has the form $y = (\tan\beta)x + C_2 x^2$. To find C_2, we now recall that $\ddot{y} = -g$. Using the chain rule to differentiate the trajectory with respect to time, we have



$$\dot{y} = (\tan\beta)\dot{x} + 2C_2 x\dot{x} \quad\Rightarrow\quad \ddot{y} = (\tan\beta)\ddot{x} + 2C_2\dot{x}^2 + 2C_2 x\ddot{x}. \tag{1}$$

Since $\ddot{x} = 0$, $\dot{x}$ is constant and therefore equal to its initial value, i.e., $\dot{x} = v_0\cos\beta$. Substituting this condition into the last of Eqs. (1) along with $\ddot{y} = -g$, we have

$$-g = 2C_2(v_0\cos\beta)^2 \quad\Rightarrow\quad C_2 = -g\sec^2\beta/(2v_0^2). \tag{2}$$

In summary, the trajectory of the ball is given by

$$y = (\tan\beta)x - [g\sec^2\beta/(2v_0^2)]x^2. \tag{3}$$

Let $x_G = 80\,\text{ft}$ and $y_G = 0$ be the coordinates of the base of the goal. For the ball to land at the base of the goal, we have

$$0 = (\tan\beta)x_G - [g\sec^2\beta/(2v_0^2)]x_G^2 \quad\Rightarrow\quad v_0 = \sqrt{g x_G/\sin 2\beta}. \tag{4}$$

Minimizing v_0 with respect to β requires making the denominator of the fraction under the square root of the last of Eqs. (4) as large as possible. The maximum value of the sine function is 1, which is achieved when $2\beta = (\pi/2)$ rad. Hence, we have

$$\beta = (\pi/4)\,\text{rad} \quad\Rightarrow\quad (v_0)_{\text{min}} = \sqrt{g x_G} \quad\Rightarrow\quad \boxed{(v_0)_{\text{min}} = 50.75\,\text{ft/s},} \tag{5}$$

where we have used the fact that $g = 32.2\,\text{ft/s}^2$ and $x_G = 80\,\text{ft}$. Substituting $\beta = (\pi/4)$ rad and $v_0 = (v_0)_{\text{min}}$ into Eq. (3), and computing the value of y corresponding to $x = 58\,\text{ft}$ (which is the x coordinate of the barrier), we have

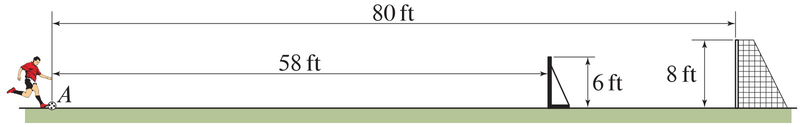
$$y(58\,\text{ft}) = 15.94\,\text{ft} > 6\,\text{ft},$$

that is, the ball clears the obstacle in front of the goal.

Problem 2.105

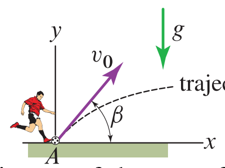
A soccer player practices kicking a ball from A directly into the goal (i.e., the ball does not bounce first) while clearing a 6 ft tall fixed barrier.

Find the initial speed and angle that allow the ball to barely clear the barrier while barely reaching the goal at its base. *Hint:* A projectile's trajectory can be given the form $y = C_1 x - C_2 x^2$, where the coefficients C_1 and C_2 can be found by forcing the parabola to go through two given points.



Solution

As explained in the hint, the trajectory can be given the form



$$y = C_1 x - C_2 x^2, \tag{1}$$

where, referring to the figure at the right, it is understood that the origin of the coordinate system used is at point A. Let $x_B = 58$ ft and $y_B = 6$ ft be the coordinates of the top of the barrier. Also, let $x_G = 80$ ft and $y_G = 0$ be the coordinates of the base of the goal. The ball must barely clear the barrier and then it must barely reach the goal. Hence, we have

$$y_B = C_1 x_B - C_2 x_B^2 \quad \text{and} \quad 0 = C_1 x_G - C_2 x_G^2. \tag{2}$$

The above equations form a system of two equations in the two unknowns C_1 and C_2 whose solution is

$$C_1 = \frac{x_G y_B}{x_B(x_G - x_B)} \quad \text{and} \quad C_2 = \frac{y_B}{x_B(x_G - x_B)}. \tag{3}$$

We now need to relate the coefficients C_1 and C_2 to the initial speed and angle of the ball. To do so, we begin with noticing that since the velocity is tangent to the trajectory, and since the initial orientation of the velocity is the angle β, the slope of the trajectory at $x = 0$ must be equal to $\tan\beta$. That is

$$(dy/dx)_{x=0} = C_1 = \tan\beta \quad \Rightarrow \quad \beta = \tan^{-1}\left[\frac{x_G y_B}{x_B(x_G - x_B)}\right] \quad \Rightarrow \quad \boxed{\beta = 20.62^\circ,} \tag{4}$$

where we have used the fact that $x_G = 80$ ft, $y_B = 6$ ft, and $x_B = 58$ ft. Next we recall that $\ddot{y} = -g$. Using the chain rule to differentiate Eq. (1) with respect to time, we have

$$\dot{y} = C_1\dot{x} - 2C_2 x\dot{x} \quad \Rightarrow \quad \ddot{y} = C_1\ddot{x} - 2C_2\dot{x}^2 - 2C_2 x\ddot{x}. \tag{5}$$

We now observe that we have $\ddot{x} = 0$. This also implies that $\dot{x}$ is constant and therefore equal to its initial value, i.e., $\dot{x} = v_0\cos\beta$. Enforcing these conditions, along with $\ddot{y} = -g$, in the last of Eqs. (5), we have

$$-g = -2C_2(v_0\cos\beta)^2 \quad \Rightarrow \quad v_0 = \sqrt{g/(2C_2)}\sec\beta. \tag{6}$$

Now that β is known, recalling again that $x_G = 80$ ft, $y_B = 6$ ft, and $x_B = 58$ ft and using the second of Eqs. (3) to evaluate C_2, we can evaluate v_0 to obtain

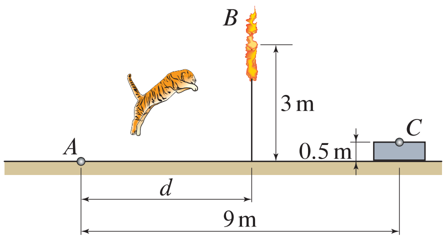
$$\boxed{v_0 = 62.52\ \text{ft/s},}$$

where we have also used the fact that $g = 32.2\ \text{ft/s}^2$.

Problem 2.106

In a circus act a tiger is required to jump from point A to point C so that it goes through the ring of fire at B. *Hint:* A projectile's trajectory can be given the form $y = C_1 x - C_2 x^2$, where the coefficients C_1 and C_2 can be found by forcing the parabola to go through two given points.

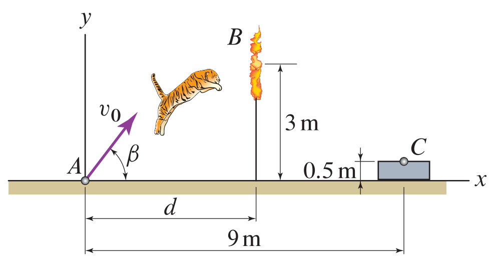
Determine the tiger's initial velocity if the ring of fire is placed at a distance $d = 5.5$ m from A. Furthermore, determine the slope of the tiger's trajectory as the tiger goes through the ring of fire.



Solution

Referring to the figure at the right, we adopt a Cartesian coordinate system with origin at A. The trajectory of the tiger is of the form:

$$y = C_1 x - C_2 x^2. \tag{1}$$



We find C_1 and C_2 by observing that the tiger passes through point B of coordinates $(x_B, y_B) = (5.5, 3)$ m, and lands at C of coordinates $(x_C, y_C) = (9, 0.5)$ m. Using Eq. (1) to enforce these conditions, we have

$$y_B = C_1 x_B - C_2 x_B^2 \quad \text{and} \quad y_C = C_1 x_C - C_2 x_C^2, \tag{2}$$

which is a system of two equations in the two unknowns C_1 and C_2 whose solution is

$$C_1 = \frac{x_C^2 y_B - x_B^2 y_C}{x_C x_B (x_C - x_B)} \quad \text{and} \quad C_2 = \frac{x_C y_B - x_B y_C}{x_C x_B (x_C - x_B)}. \tag{3}$$

We now need to relate the C_1 and C_2 to the initial speed and angle of the tiger. To do so, we notice that since the velocity is tangent to the trajectory, and since the initial orientation of the velocity is the angle β, the slope of the trajectory at $x = 0$ must be equal to $\tan\beta$. That is

$$(dy/dx)_{x=0} = C_1 = \tan\beta \quad \Rightarrow \quad \beta = \tan^{-1}\left[\frac{x_C^2 y_B - x_B^2 y_C}{x_C x_B (x_C - x_B)}\right] = 52.75°, \tag{4}$$

where we have used the fact that $(x_B, y_B) = (5.5, 3)$ m and $(x_C, y_C) = (9, 0.5)$ m. Next we recall that $\ddot{y} = -g$. Using the chain rule to differentiate Eq. (1) with respect to time, we have

$$\dot{y} = C_1 \dot{x} - 2C_2 x \dot{x} \quad \Rightarrow \quad \ddot{y} = C_1 \ddot{x} - 2C_2 \dot{x}^2 - 2C_2 x \ddot{x}. \tag{5}$$

We now observe that we have $\ddot{x} = 0$. This also implies that $\dot{x}$ is constant and therefore equal to its initial value, i.e., $\dot{x} = v_0 \cos\beta$. Enforcing these conditions, along with $\ddot{y} = -g$, the last of Eqs. (5) gives

$$-g = -2C_2 (v_0 \cos\beta)^2 \quad \Rightarrow \quad v_0 = \sqrt{g/(2C_2)}\sec\beta \quad \Rightarrow \quad v_0 = 9.781\ \text{m/s}, \tag{6}$$

where, again recalling that $(x_B, y_B) = (5.5, 3)$ m and $(x_C, y_C) = (9, 0.5)$ m, we have evaluated v_0 by first evaluating C_2 in the last of Eqs. (3) and then the angle β in the last of Eqs. (4). Now that v_0 and β are known, observing that $\vec{v}_{\text{initial}} = v_0 \cos\beta\,\hat{\imath} + v_0 \sin\beta\,\hat{\jmath}$, we can evaluate $\vec{v}_{\text{initial}}$ to obtain

$$\boxed{\vec{v}_{\text{initial}} = (5.920\,\hat{\imath} + 7.786\,\hat{\jmath})\ \text{m/s}.}$$

The slope of the trajectory is obtained by differentiating Eq. (1) with respect to x:

$$dy/dx = C_1 - 2C_2 x. \tag{7}$$

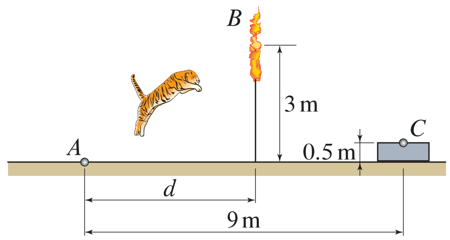
Recalling that $(x_B, y_B) = (5.5, 3)$ m and $(x_C, y_C) = (9, 0.5)$ m, and evaluating C_1 and C_2 in Eqs. (3), we can evaluate to slope at $x = x_B$ to obtain

$$\boxed{\left.\frac{dy}{dx}\right|_{x=x_B} = -0.2244.}$$

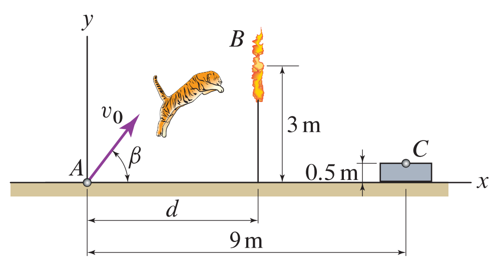
Problem 2.107

In a circus act a tiger is required to jump from point A to point C so that it goes through the ring of fire at B. *Hint:* A projectile's trajectory can be given the form $y = C_1 x - C_2 x^2$, where the coefficients C_1 and C_2 can be found by forcing the parabola to go through two given points.

Determine the tiger's initial velocity, as well as the distance d so that the slope of the tiger's trajectory as the tiger goes through the ring of fire is completely horizontal.



Solution



Referring to the figure at the right, we will be using a Cartesian coordinate system with origin at A. The trajectory of the tiger is

$$y = C_1 x - C_2 x^2, \tag{1}$$

where C_1 and C_2 are constants to be determined by making sure that the tiger passes through point B of coordinates $(x_B, y_B) = (d, 3\,\text{m})$ with zero slope, and then lands on point C of coordinates $(x_C, y_C) = (9\,\text{m}, 0.5\,\text{m})$. Using Eq. (1) to enforce these conditions, we have

$$y_B = C_1 x_B - C_2 x_B^2 \quad \Rightarrow \quad 3\,\text{m} = C_1 d - C_2 d^2, \tag{2}$$

$$\left.\frac{dy}{dx}\right|_{x=d} = 0 \quad \Rightarrow \quad 0 = C_1 - 2 d C_2 \tag{3}$$

$$y_C = C_1 x_C - C_2 x_C^2 \quad \Rightarrow \quad 0.5\,\text{m} = (9\,\text{m}) C_1 - (81.00\,\text{m}^2) C_2. \tag{4}$$

The last of Eqs. (2)–(4) form a system of three equations in the three unknowns d, C_1, and C_2, which can be solved numerically. For example we have used *Mathematica* with the following code

```
NSolve[{3. == C1 d - C2 d^2, 0. == C1 - 2 C2 d, 0.5 == C1 9. - C2 81.}, {C1, C2, d}]
```

which yields the following two solutions:

$$C_1 = 0.05809 \qquad C_2 = 0.0002812\,\text{m}^{-1} \qquad d = 103.3\,\text{m}, \tag{5}$$

$$C_1 = 1.275 \qquad C_2 = 0.1355\,\text{m}^{-1} \qquad d = 4.705\,\text{m}. \tag{6}$$

Because the first solution implies that $d > x_C$, the solution in question is not acceptable and therefore we have that the only acceptable solution is

$$C_1 = 1.275, \quad C_2 = 0.1355\,\text{m}^{-1}, \quad \boxed{d = 4.705\,\text{m}.} \tag{7}$$

Now we turn to the determination of the initial velocity of the tiger. To do so, we notice that since the velocity is tangent to the trajectory, and since the initial orientation of the velocity is the angle β, the slope of the trajectory at $x = 0$ must be equal to $\tan\beta$. That is

$$(dy/dx)_{x=0} = C_1 = \tan\beta \quad \Rightarrow \quad \beta = \tan^{-1}(C_1) = 51.89^\circ, \tag{8}$$

where we have used the numerical solution for C_1. Next we recall that $\ddot{y} = -g$. Differentiating Eq. (1) with respect to time, we have

$$\dot{y} = C_1\dot{x} - 2C_2 x\dot{x} \quad \Rightarrow \quad \ddot{y} = C_1\ddot{x} - 2C_2\dot{x}^2 - 2C_2 x\ddot{x}. \tag{9}$$

We now observe that we have $\ddot{x} = 0$. This also implies that $\dot{x}$ is constant and therefore equal to its initial value, i.e., $\dot{x} = v_0\cos\beta$. Enforcing these conditions, along with $\ddot{y} = -g$, the last of Eqs. (9) gives

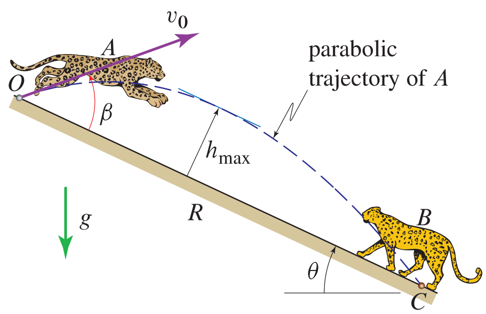
$$-g = -2C_2(v_0\cos\beta)^2 \quad \Rightarrow \quad v_0 = \sqrt{g/(2C_2)}\,\sec\beta \quad \Rightarrow \quad v_0 = 9.749\,\text{m/s}, \tag{10}$$

where we have used the fact that $g = 9.81\,\text{m/s}^2$, and, again, we have used the numerical solution for C_2 and β. Now that v_0 and β are known, observing that $\vec{v}_{\text{initial}} = v_0\cos\beta\,\hat{\imath} + v_0\sin\beta\,\hat{\jmath}$, we can evaluate $\vec{v}_{\text{initial}}$ to obtain

$$\vec{v}_{\text{initial}} = (6.017\,\hat{\imath} + 7.671\,\hat{\jmath})\,\text{m/s}.$$

Problem 2.108

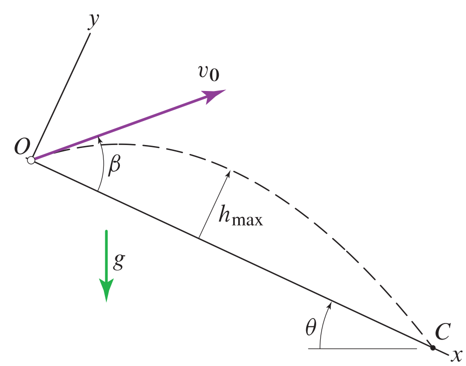
A jaguar A leaps from O at speed v_0 and angle β relative to the incline to attack a panther B at C. Determine an expression for the maximum *perpendicular* height h_{max} above the incline achieved by the leaping jaguar, given that the angle of the incline is θ.



Solution

We will use a Cartesian coordinate system aligned with the incline as shown at the right. The acceleration vector is then given by

$$\vec{a} = g\sin\theta\,\hat{\imath} - g\cos\theta\,\hat{\jmath} \quad \Rightarrow \quad a_y = -g\cos\theta.$$



Applying the constant acceleration equation in the y direction, we have

$$v_y^2 = v_{0y}^2 + 2a_y(y - y_0). \tag{1}$$

At the h_{max} position the y component of velocity must be equal to zero. Enforcing this condition, we have

$$0 = (v_0\sin\beta)^2 - 2g\cos\theta(h_{max} - 0). \tag{2}$$

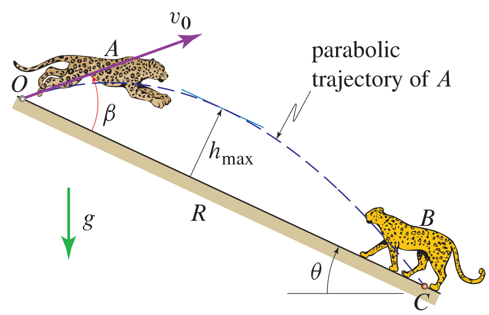
Solving Eq. (2) for h_{max}, we obtain

$$h_{max} = \frac{v_0^2\sin^2\beta}{2g\cos\theta}.$$

Problem 2.109

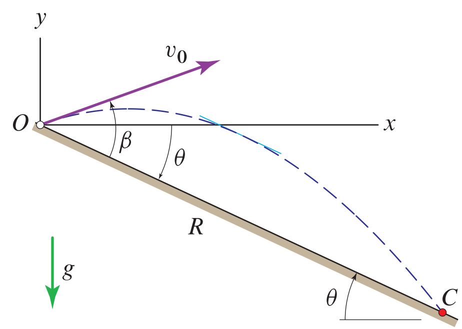
The jaguar A leaps from O at speed v_0 and angle β relative to the incline to intercept the panther B at C. The distance along the incline from O to C is R, and the angle of the incline with respect to the horizontal is θ.

Determine an expression for v_0 as a function of β for A to be able to get from O to C.



Solution

Using the xy coordinate system shown at the right, we write the following two constant acceleration equations describing the x and y coordinates of jaguar A as a function of time:



$$x = x_0 + v_{0x}t, \tag{1}$$
$$y = y_0 + v_{0y}t - \tfrac{1}{2}gt^2, \tag{2}$$

where it is understood that the jaguar leaps at time $t = 0$, and where (x_0, y_0) are the coordinates of the point from which the jaguar leaps. Since the jaguar leaps from the origin of the chosen coordinate system, letting t_C denote the time at which jaguar A arrives at C, we have

$$R\cos\theta = [v_0\cos(\beta-\theta)]\,t_C, \tag{3}$$
$$-R\sin\theta = [v_0\sin(\beta-\theta)]\,t_C - \tfrac{1}{2}gt_C^2, \tag{4}$$

where R is distance from O to C. Eliminating t_C from Eqs. (3) and (4) gives

$$-\sin\theta = \cos\theta\tan(\beta-\theta) - \frac{gR}{2v_0^2}\frac{\cos^2\theta}{\cos^2(\beta-\theta)}. \tag{5}$$

Multiplying all terms in the above equation by $cos(\beta-\theta)$, we have

$$-\sin\theta\cos(\beta-\theta) = \cos\theta\sin(\beta-\theta) - \frac{gR}{2v_0^2}\frac{\cos^2\theta}{\cos(\beta-\theta)}$$
$$\Rightarrow\quad \sin\theta\cos(\beta-\theta) + \cos\theta\sin(\beta-\theta) = \frac{gR}{2v_0^2}\frac{\cos^2\theta}{\cos(\beta-\theta)} \quad\Rightarrow\quad \sin\beta = \frac{gR}{2v_0^2}\frac{\cos^2\theta}{\cos(\beta-\theta)}, \tag{6}$$

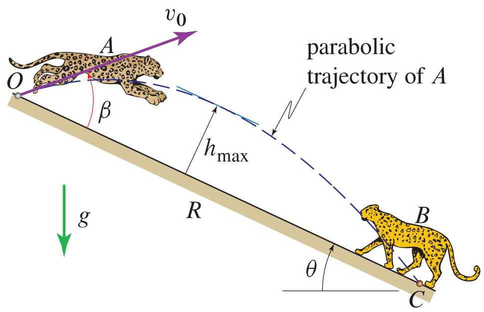
where we have used the identity $\sin(A+B) = \sin A\cos B + \cos A\sin B$. Solving the last of Eqs. (6) for v_0, we have

$$v_0 = \sqrt{\frac{gR}{2}}\frac{\cos\theta}{\sqrt{\sin\beta\cos(\beta-\theta)}}.$$

Problem 2.110

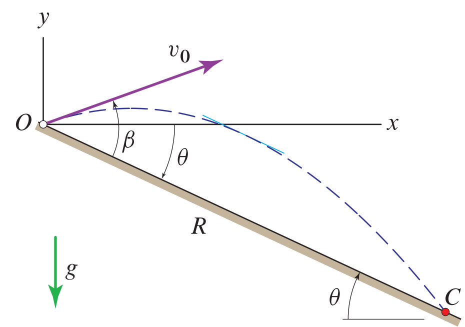
The jaguar A leaps from O at speed v_0 and angle β relative to the incline to intercept the panther B at C. The distance along the incline from O to C is R, and the angle of the incline with respect to the horizontal is θ.

Derive v_0 as a function of β to leap a given distance R along with the optimal value of launch angle β, i.e., the value of β necessary to leap a given distance R with the minimum v_0. Then plot v_0 as a function of β for $g = 9.81\,\text{m/s}^2$, $R = 7\,\text{m}$, and $\theta = 25°$, and find a numerical value of the optimal β and the corresponding value of v_0 for the given set of parameters.



Solution

Using the xy coordinate system shown at the right, we write the following two constant acceleration equations describing the x and y coordinates of jaguar A as a function of time:



$$x = x_0 + v_{0x}t, \tag{1}$$

$$y = y_0 + v_{0y}t - \tfrac{1}{2}gt^2, \tag{2}$$

where it is understood that the jaguar leaps at time $t = 0$, and where (x_0, y_0) are the coordinates of the point from which of the jaguar leaps. Since the jaguar leaps from the origin of the chosen coordinate system, letting t_C denote the time at which jaguar A arrives at C, we have

$$R\cos\theta = [v_0\cos(\beta-\theta)]\,t_C, \tag{3}$$

$$-R\sin\theta = [v_0\sin(\beta-\theta)]\,t_C - \tfrac{1}{2}gt_C^2, \tag{4}$$

where R is distance from O to C. Eliminating t_C from Eqs. (3) and (4) gives

$$-\sin\theta = \cos\theta\tan(\beta-\theta) - \frac{gR}{2v_0^2}\frac{\cos^2\theta}{\cos^2(\beta-\theta)}. \tag{5}$$

Multiplying all terms in the above equation by $cos(\beta-\theta)$, we have

$$-\sin\theta\cos(\beta-\theta) = \cos\theta\sin(\beta-\theta) - \frac{gR}{2v_0^2}\frac{\cos^2\theta}{\cos(\beta-\theta)}$$

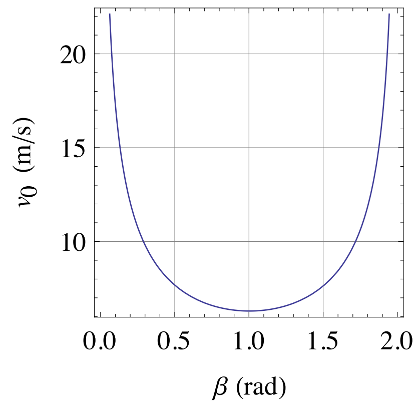
$$\Rightarrow\quad \sin\theta\cos(\beta-\theta) + \cos\theta\sin(\beta-\theta) = \frac{gR}{2v_0^2}\frac{\cos^2\theta}{\cos(\beta-\theta)} \quad\Rightarrow\quad \sin\beta = \frac{gR}{2v_0^2}\frac{\cos^2\theta}{\cos(\beta-\theta)}, \tag{6}$$

where we have used the identity $\sin(A+B) = \sin A\cos B + \cos A\sin B$. Solving the last of Eqs. (6) for v_0, we have

$$v_0 = \sqrt{\frac{gR}{2}}\frac{\cos\theta}{\sqrt{\sin\beta\cos(\beta-\theta)}}. \tag{7}$$

Recalling that $g = 9.81\,\text{m/s}^2$, $R = 7\,\text{m}$, and $\theta = 25°$, we can plot the above function with any appropriate mathematical software. The plot shown below was obtained using *Mathematica* with the following code:

```
Parameters = {g → 9.81, θ → 25. Degree, R → 7.};
Plot[Sqrt[g R/2] Cos[θ]/Sqrt[Sin[β] Cos[β - θ]] /. Parameters, {β, 0, 2}, Frame → True,
 GridLines → Automatic, AspectRatio → 1, FrameLabel → {"β (rad)", "v0 (m/s)"}]
```



We can find the optimal value of β to reach a distance of $R = 7$ m by differentiating v_0 with respect to β and setting it equal to zero. Recalling that $\theta = 25°$, this gives

$$\frac{dv_0}{d\beta} = -\frac{1}{2}\sqrt{\frac{gR}{2}}\cos\theta\,\frac{\cos\beta\cos(\beta-\theta) - \sin\beta\sin(\beta-\theta)}{[\sin\beta\cos(\beta-\theta)]^{3/2}} = 0$$
$$\Rightarrow \quad \cos\beta\cos(\beta - 25°) - \sin\beta\sin(\beta - 25°) = 0. \quad (8)$$

The above equation is a transcendental equation that we will solve numerically. Again, this can be done with any appropriate mathematical software. We have used *Mathematica* with the following code

```
FindRoot[Cos[β] Cos[β - 25. Degree] - Sin[β] Sin[β - 25. Degree] == 0, {β, 25. Degree}]
```

Note that the use of root finding algorithms generally requires the user to provide a guess of the value of the solution. As can be seen in the above code (see information provided at the end of the code line), we have provided a guess of 25°. The outcome of this calculation gives

$$\beta_{\text{optimal}} = 57.52°.$$

Then, using the above value of β along with $g = 9.81\,\text{m/s}^2$, $R = 7$ m, and $\theta = 25°$, from Eq. (7) we have that the corresponding value of v_0 is

$$(v_0)_{\text{optimal}} = 6.297\,\text{m/s}.$$

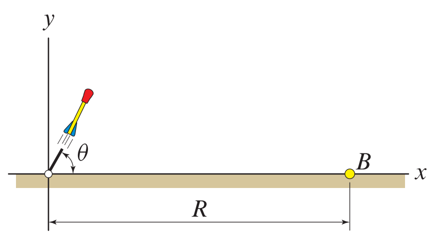
Problem 2.111

A stomp rocket is a toy consisting of a hose connected to a blast pad (i.e., an air bladder) at one end and to a short pipe mounted on a tripod at the other end. A rocket with a hollow body is mounted onto the pipe and is propelled into the air by stomping on the blast pad.

If the rocket can be imparted an initial speed $v_0 = 120\,\text{ft/s}$, and if the rocket's landing spot at B is at the same elevation as the launch point, i.e., $h = 0\,\text{ft}$, neglect air resistance and determine the rocket's launch angle θ such that the rocket achieves the maximum possible range. In addition, compute R, the rocket's maximum range, and t_f, the corresponding flight time.

Solution

Referring to the figure at the right, we will use an xy coordinate system with origin at the launch point of the rocket. Let θ be the elevation angle and $v_0 = 120\,\text{ft/s}$ be the intial speed. The acceleration of the rocket is equal to $-g$ in the y direction and zero in the x direction. Hence, we can use the constant acceleration equation $s = v_0 t + \frac{1}{2}a_c t^2$, to express the x and y coordinates of the rocket as a function of time. This gives



$$x = (v_0 \cos\theta)t, \tag{1}$$

$$y = (v_0 \sin\theta)t - \tfrac{1}{2}gt^2. \tag{2}$$

Since in this problem point B lies on the line $y = 0$, the time taken by the rocket to arrive at B can be obtained by equating Eq. (2) to zero. This gives

$$t_B = \frac{2v_0 \sin\theta}{g}. \tag{3}$$

Substituting Eq. (3) in place of t into Eq. (1), using the trigonometric identity $2\sin\theta\cos\theta = \sin 2\theta$, and observing that $x_B = R$, we get

$$R = \frac{v_0^2 \sin 2\theta}{g}, \tag{4}$$

where R is the range of the rocket. The maximum value of R occurs when $\sin 2\theta = 1$, i.e.,

$$2\theta = \frac{\pi}{2}\,\text{rad} \quad \Rightarrow \quad \boxed{\theta_{R_{\max}} = \frac{\pi}{4}\,\text{rad} = 45.00^\circ.} \tag{5}$$

Substituting $\theta_{R_{\max}} = 45.00^\circ$ into Eq. (4) with $g = 32.2\,\text{ft/s}^2$ gives

$$\boxed{R_{\max} = 447.2\,\text{ft}.}$$

Observing that the time of flight is the same as t_B given in Eq. (3), for $\theta = \theta_{R_{\max}}$, we have $t_f = 2v_0 \sin\theta_{R_{\max}}/g$, which can be evaluated to obtain

$$\boxed{t_f = 5.270\,\text{s}.}$$

Problem 2.112

A stomp rocket is a toy consisting of a hose connected to a blast pad (i.e., an air bladder) at one end and to a short pipe mounted on a tripod at the other end. A rocket with a hollow body is mounted onto the pipe and is propelled into the air by stomping on the blast pad.

Assuming the rocket can be given an initial speed $v_0 = 120\,\text{ft/s}$, the rocket's landing spot at B is 10 ft higher than the launch point, i.e., $h = 10\,\text{ft}$, and neglecting air resistance, find the rocket's launch angle θ such that the rocket achieves the maximum possible range. In addition, as part of the solution, compute the corresponding maximum range and flight time. To do this:

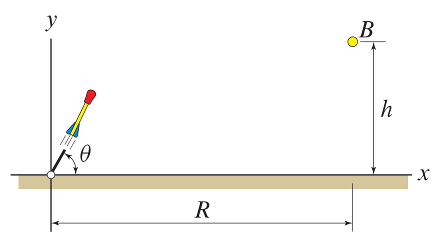
(a) Determine the range R as a function of time.

(b) Take the expression for R found in (a), square it, and then differentiate it with respect to time to find the flight time that corresponds to the maximum range, and then find that maximum range.

(c) Use the time found in (b) to then find the angle required to achieve the maximum range.

Solution

Referring to the figure on the right, we will use an xy coordinate system with origin at the launch point of the rocket. Let θ be the elevation angle and $v_0 = 120\,\text{ft/s}$ be the intial speed. The acceleration of the rocket is equal to $-g$ in the y direction and zero in the x direction. Hence, we can use the constant acceleration equation $s = v_0 t + \frac{1}{2}a_c t^2$, to express the x and y coordinates of the rocket as a function of time. This gives



$$x = (v_0 \cos\theta)t, \tag{1}$$
$$y = (v_0 \sin\theta)t - \tfrac{1}{2}gt^2. \tag{2}$$

Letting t_f denote the time of flight, for $t = t_f$ the rocket is at B, so that we must have

$$R = v_0(\cos\theta)t_f \quad \Rightarrow \quad \cos\theta = \frac{R}{v_0 t_f}. \tag{3}$$
$$h = v_0(\sin\theta)t_f - \tfrac{1}{2}gt_f^2 \quad \Rightarrow \quad \sin\theta = \frac{2h + gt_f^2}{2v_0 t_f}, \tag{4}$$

Next, recalling that $\sin^2\theta + \cos^2\theta = 1$, using the last of Eqs. (3) and (4), we have

$$\left(\frac{2h + gt_f^2}{2v_0 t_f}\right)^2 + \left(\frac{R}{v_0 t_f}\right)^2 = 1, \tag{5}$$

which, for convenience, we view as an equation for R^2 whose solution is

$$R^2 = v_0^2 t_f^2 - \left(h + \tfrac{1}{2}gt_f^2\right)^2. \tag{6}$$

Now we maximize R with respect to t_f. Since the value of t_f for which R is maximum coincides with the value of t_f for which R^2 is maximum, we can find the value of t_f in question by differentiating Eq. (6) with respect to t_f and then setting the results to 0. This gives

$$\frac{dR^2}{dt_f} = 0 = 2v_0 t_{f_{R_{\max}}} - 2\big(h + \tfrac{1}{2} g t_{f_{R_{\max}}}^2\big)(g t_{f_{R_{\max}}}) \quad \Rightarrow \quad t_{f_{R_{\max}}} = \sqrt{\frac{2}{g^2}(v_0^2 - hg)}. \tag{7}$$

Recalling that $g = 32.2\,\text{ft/s}^2$, $v_0 = 120\,\text{ft/s}$, and $h = 10\,\text{ft}$, we can evaluate $t_{f_{R_{\max}}}$ to obtain

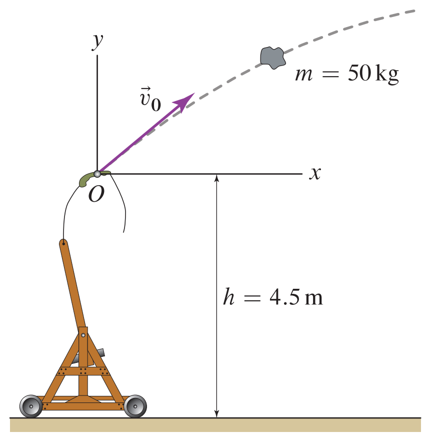
$$\boxed{t_{f_{R_{\max}}} = 5.211\,\text{s}.}$$

To find $R_{\max}$, we substitute the expression of $t_{f_{R_{\max}}}$ from the last of Eqs. (7) into Eq. (6) and then we take a square root. To find the corresponding value of θ we substitute the value of $t_{f_{R_{\max}}}$ from the last of Eqs. (7) into the last of Eqs. (4) and solve for θ. Recalling that we have $g = 32.2\,\text{ft/s}^2$, $v_0 = 120\,\text{ft/s}$, $h = 10\,\text{ft}$, these operations yield the following results:

$$\boxed{R_{\max} = 437.1\,\text{ft} \quad \text{and} \quad \theta_{R_{\max}} = 45.66^\circ.}$$

Problem 2.113

A trebuchet releases a rock with mass $m = 50\,\mathrm{kg}$ at the point O. The initial velocity of the projectile is $\vec{v}_0 = (45\,\hat{\imath} + 30\,\hat{\jmath})\,\mathrm{m/s}$. If one were to model the effects of air resistance via a drag force directly proportional to the projectile's velocity, the resulting accelerations in the x and y directions would be $\ddot{x} = -(\eta/m)\dot{x}$ and $\ddot{y} = -g - (\eta/m)\dot{y}$, respectively, where g is the acceleration of gravity and $\eta = 0.64\,\mathrm{kg/s}$ is a viscous drag coefficient. Find an expression for the trajectory of the projectile.



Solution

We can integrate the x and the y components of acceleration to obtain the x and y displacement as a function of time. The problem states that $\ddot{x} = -(\eta/m)\dot{x}$. Then, recalling that $\ddot{x} = \frac{d\dot{x}}{dt}$, we can write

$$-\frac{\eta}{m}\dot{x} = \frac{d\dot{x}}{dt} \quad\Rightarrow\quad -\frac{\eta}{m}\,dt = \frac{d\dot{x}}{\dot{x}} \quad\Rightarrow\quad \int_{(v_0)_x}^{\dot{x}} \frac{d\dot{x}}{\dot{x}} = -\int_0^t \frac{\eta}{m}\,dt \quad\Rightarrow\quad \dot{x} = (v_0)_x e^{-\frac{\eta}{m}t}, \tag{1}$$

where $(v_0)_x$ is the x component of the velocity of the projectile at $t = 0$. Next, we recall that $\dot{x} = dx/dt$. So, using the last of Eqs. (1) we have

$$\frac{dx}{dt} = (v_0)_x e^{-\frac{\eta}{m}t} \quad\Rightarrow\quad dx = (v_0)_x e^{-\frac{\eta}{m}t}\,dt$$

$$\Rightarrow\quad \int_0^x dx = (v_0)_x \int_0^t e^{-\frac{\eta}{m}t}\,dt \quad\Rightarrow\quad x = \frac{m(v_0)_x}{\eta}\left(1 - e^{-\frac{\eta}{m}t}\right). \tag{2}$$

We can now repeat these steps starting with the acceleration in the y direction. Doing so, we have

$$-\int_{(v_0)_y}^{\dot{y}} \frac{d\dot{y}}{g + (\eta/m)\,\dot{y}} = \int_0^t dt \quad\Rightarrow\quad \dot{y} = \frac{mg}{\eta}\left(e^{-\frac{\eta}{m}t} - 1\right) + (v_0)_y e^{-\frac{\eta}{m}t} \tag{3}$$

where $(v_0)_y$ is the y component of the velocity of the projectile at $t = 0$. Integrating Eq. (3) again with respect to time, we obtain

$$y = \left(\frac{m^2 g}{\eta^2} - \frac{mgt}{\eta} + \frac{m}{\eta}(v_0)_y\right) - \left(\frac{m^2 g}{\eta^2} + \frac{m}{\eta}(v_0)_y\right) e^{-\frac{\eta}{m}t}. \tag{4}$$

From Eq. (2) we find

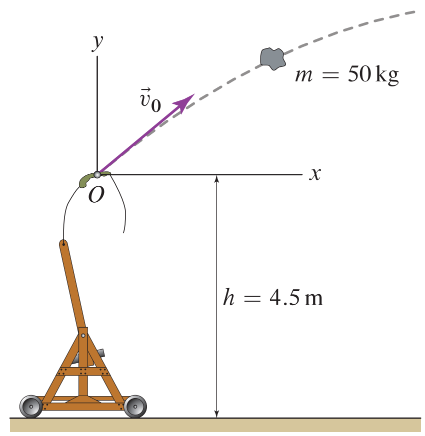
$$e^{-\frac{\eta}{m}t} = \left(1 - \frac{\eta x}{m(v_0)_x}\right) \quad\Rightarrow\quad t = -\frac{m}{\eta}\ln\left(1 - \frac{\eta x}{m(v_0)_x}\right). \tag{5}$$

Substituting the last of Eqs. (5) into Eq. (4) and recalling that $m = 50\,\mathrm{kg}$, $(v_0)_x = 45\,\mathrm{m/s}$, $(v_0)_y = 30\,\mathrm{m/s}$ and $\eta = 0.64\,\mathrm{kg/s}$, we obtain

$$\boxed{y = \left[59.88\times10^3 \ln\left(1 - 2.844\times10^{-4}x\right) + 17.70x\right]\,\mathrm{m}.} \tag{6}$$

Problem 2.114

Continue Prob. 2.113 and, for the case where $\eta = 0.64\,\mathrm{kg/s}$, determine the maximum height from the ground reached by the projectile and the time it takes to achieve it. Compare the result with what you would obtain in the absence of air resistance.



Solution

When the projectile is at its maximum height, the y component of its velocity must be zero. Therefore, we first need to find $\dot{y}$ as a function of time. Using $\ddot{y} = d\dot{y}/dt$ and $\ddot{y} = -g - (\eta/m)\dot{y}$, we obtain

$$\frac{d\dot{y}}{dt} = -[g + (\eta/m)\dot{y}] \quad \Rightarrow \quad -\int_{v_{0y}}^{\dot{y}} \frac{d\dot{y}}{g + (\eta/m)\,\dot{y}} = \int_0^t dt \quad \Rightarrow \quad -\frac{m}{\eta}\ln\left(g + \frac{\eta}{m}\dot{y}\right)\Big|_{v_{0y}}^{\dot{y}} = t$$

$$\Rightarrow \quad \ln\left(g + \frac{\eta}{m}\dot{y}\right) - \ln\left(g + \frac{\eta}{m}v_{0y}\right) = -\eta t/m \quad \Rightarrow \quad \ln\left(\frac{g + \frac{\eta}{m}\dot{y}}{g + \frac{\eta}{m}v_{0y}}\right) = -\eta t/m$$

$$\Rightarrow \quad \frac{\eta}{m}\dot{y} = \left(g + \frac{\eta}{m}v_{0y}\right)e^{-\eta t/m} - g \quad \Rightarrow \quad \dot{y} = \left(\frac{mg}{\eta} + v_{0y}\right)e^{-\eta t/m} - \frac{mg}{\eta}, \qquad (1)$$

where we have used $t_0 = 0$ and have let the y component of the initial velocity be v_{0y}.

Now that we have an expression for $\dot{y}(t)$, we again observe that the maximum height is reached when $\dot{y} = 0$. Letting $H_{\max}$ denote the maximum height achieved by the projectile and $t_{\max}$ the time at which this height is achieved, setting the last of Eqs. (1) to zero, we obtain

$$\left(\frac{mg}{\eta} + v_{0y}\right)e^{-\eta t_{\max}/m} - \frac{mg}{\eta} = 0 \quad \Rightarrow \quad -\eta t_{\max}/m = \ln\left(\frac{mg}{mg + \eta v_{0y}}\right)$$

$$\Rightarrow \quad t_{\max} = \frac{m}{\eta}\ln\left(\frac{mg + \eta v_{0y}}{mg}\right) \quad \Rightarrow \quad \boxed{t_{\max} = 3.000\,\mathrm{s},} \qquad (2)$$

where we have used $m = 50\,\mathrm{kg}$, $g = 9.81\,\mathrm{m/s^2}$, $v_{0y} = 30\,\mathrm{m/s}$, and $\eta = 0.64\,\mathrm{kg/s}$ to obtain the numerical result.

The maximum height in the presence of air resistance can be calculated as $H_{\max} = y_{\max} + h$, where h is given as $4.5\,\mathrm{m}$ and $y_{\max}$ is the value of y when $t = t_{\max}$. Therefore, we now need to find $y(t)$ by integrating Eq. (1) one more time using

$$\dot{y} = \frac{dy}{dt} = \left(\frac{mg}{\eta} + v_{0y}\right)e^{-\eta t/m} - \frac{mg}{\eta} \quad \Rightarrow \quad \int_0^y dy = \int_0^t \left[\left(\frac{mg}{\eta} + v_{0y}\right)e^{-\eta t/m} - \frac{mg}{\eta}\right]dt$$

$$\Rightarrow \quad y = -\frac{m}{\eta}\left(\frac{mg}{\eta} + v_{0y}\right)\left(e^{-\eta t/m} - 1\right) - \frac{mg}{\eta}t, \qquad (3)$$

where we have used $y_0 = 0$ and $t_0 = 0$. Substituting $t = t_{max} = 3.000\,\mathrm{s}$ from Eq. (2), $m = 50\,\mathrm{kg}$, $g = 9.81\,\mathrm{m/s^2}$, $v_{0y} = 30\,\mathrm{m/s}$, and $\eta = 0.64\,\mathrm{kg/s}$ into Eq. (3), we obtain $y_{max} = 44.71\,\mathrm{m}$. Therefore, H_{max} is

$$H_{max} = y_{max} + h = 49.21\,\mathrm{m} \quad \Rightarrow \quad \boxed{H_{max} = 49.21\,\mathrm{m},}$$

where $h = 4.5\,\mathrm{m}$. In the absence of air resistance, the maximum height is given by $(H_{max})_{\text{no air}} = (y_{max})_{\text{no air}} + h$, where $(y_{max})_{\text{no air}}$ can be calculated by letting $\dot{y} = 0$ in the projectile motion equations with constant gravity. Therefore,

$$\dot{y}^2 = \dot{y}_0^2 - 2g(y_{max})_{\text{no air}} = 0 \quad \Rightarrow \quad (y_{max})_{\text{no air}} = \frac{\dot{y}_0^2}{2g} = 45.87\,\mathrm{m}, \tag{4}$$

where we have used the fact that $y_0 = 0$, $\dot{y}_0 = 30\,\mathrm{m/s}$, and $g = 9.81\,\mathrm{m/s^2}$. Hence, the maximum height in the absence of air resistance $(H_{max})_{\text{no air}}$ is given by

$$(H_{max})_{\text{no air}} = (y_{max})_{\text{no air}} + h = 50.37\,\mathrm{m}, \tag{5}$$

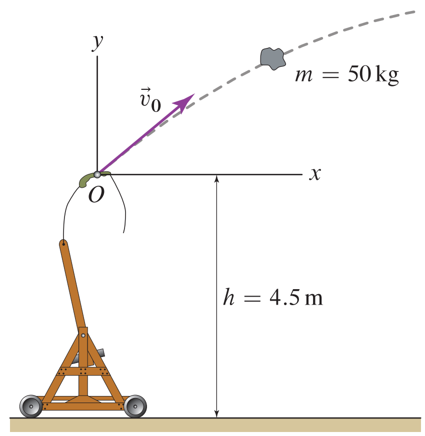
where $h = 4.5\,\mathrm{m}$. Therefore, the percent increase in height with no air resistance is

$$\text{percent increase in height with no air resistance} = \frac{50.37 - 49.21}{49.21} \times 100\%$$

$$\Rightarrow \quad \boxed{\text{percent increase in height with no air resistance} = 2.363\%.}$$

Problem 2.115

Continue Prob. 2.113 and, for the case where $\eta = 0.64\,\text{kg/s}$, determine t_I and x_I, the value of t, and the x position corresponding to the projectile's impact with the ground.



Solution

We begin by working part of the solution to Prob. 2.113. Specifically, we can integrate the x and the y components of acceleration to get the x and y displacement as a function of time. The problem states that $\ddot{x} = -(\eta/m)\dot{x}$. Then, recalling that $\ddot{x} = \frac{d\dot{x}}{dt}$, we can write

$$-\frac{\eta}{m}\dot{x} = \frac{d\dot{x}}{dt} \quad \Rightarrow \quad -\frac{\eta}{m}\,dt = \frac{d\dot{x}}{\dot{x}} \quad \Rightarrow \quad \int_{(v_0)_x}^{\dot{x}} \frac{d\dot{x}}{\dot{x}} = -\int_0^t \frac{\eta}{m}\,dt \quad \Rightarrow \quad \dot{x} = (v_0)_x e^{-\frac{\eta}{m}t} \tag{1}$$

where $(v_0)_x$ is the velocity component of the projectile. Integrating Eq. (1) again with respect to time, we obtain

$$\int_0^x dx = (v_0)_x \int_0^t e^{-\frac{\eta}{m}t}\,dt \quad \Rightarrow \quad x = \frac{mv_{x0}}{\eta}\left(1 - e^{-\frac{\eta}{m}t}\right). \tag{2}$$

Proceeding similarly to obtain the expression of y as a function of time, we have

$$-\int_{(v_0)_y}^{\dot{y}} \frac{d\dot{y}}{g + (\eta/m)\,\dot{y}} = \int_0^t dt \quad \Rightarrow \quad \dot{y} = \frac{mg}{\eta}\left(e^{-\frac{\eta}{m}t} - 1\right) + (v_0)_y e^{-\frac{\eta}{m}t}. \tag{3}$$

Integrating Eq. (3) again with respect to time, we obtain

$$y = \left(\frac{m^2 g}{\eta^2} - \frac{mgt}{\eta} + \frac{m}{\eta}(v_0)_y\right) - \left(\frac{m^2 g}{\eta^2} + \frac{m}{\eta}(v_0)_y\right) e^{-\frac{\eta}{m}t}. \tag{4}$$

Next, from Eq. (2) we find

$$e^{-\frac{\eta}{m}t} = \left(1 - \frac{\eta x}{m(v_0)_x}\right) \quad \Rightarrow \quad t = -\frac{m}{\eta}\ln\left(1 - \frac{\eta x}{m(v_0)_x}\right). \tag{5}$$

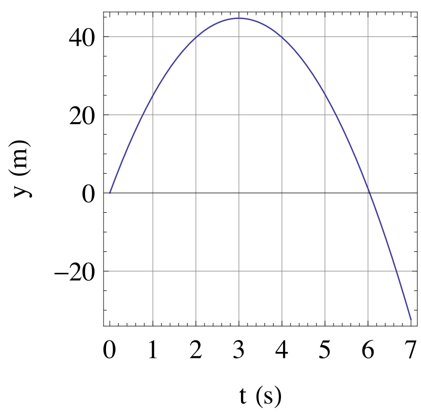
Substituting the last of Eqs. (5) into Eq. (4), we have

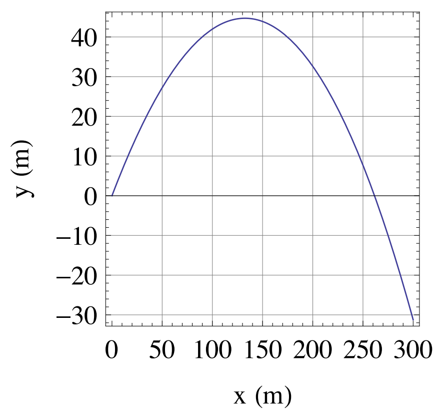
$$y = \left[\frac{m^2 g}{\eta^2} + \frac{m^2 g}{\eta^2}\ln\left(1 - \frac{\eta x}{m(v_0)_x}\right) + \frac{m}{\eta}(v_0)_y\right] - \left(\frac{m^2 g}{\eta^2} + \frac{m}{\eta}(v_0)_y\right)\left(1 - \frac{\eta x}{m(v_0)_x}\right). \tag{6}$$

To find the time of impact t_I and the location x_I of the impact we observe that the impact is characterized by the condition $y = -h = -4.5\,\text{m}$. Hence we can use a numerical root finding method to find the value of t in Eq. (4) for which the condition $y = -h$ is satisfied. Similarly, we can use a numerical root finding method to find the the value of x in Eq. (6) for which the condition $y = -h$ is satisfied. Because the majority of root finding methods require us to provide a guess of the solution, before using any such methods, we proceed to plot $y(t)$ as given in Eq. (4) and $y(x)$ as given in Eq. (6). Recalling that we are given $m = 50\,\text{kg}$, $(v_0)_x = 45\,\text{m/s}$, $(v_0)_y = 30\,\text{m/s}$ and $\eta = 0.64\,\text{kg/s}$, the plots shown below were obtained using *Mathematica* with the following code:

```
Parameters = {m → 50., g → 9.81, h → 4.5, v0x → 45., v0y → 30., η → 0.64};
yt = (m^2 g/η^2 - m g t/η + m/η v0y) - (m^2 g/η^2 + m/η v0y) ⅇ^(-t η/m);
yx = (m^2 g/η^2 + m^2 g/η^2 Log[1 - η x/(m v0x)] + m/η v0y) - (m^2 g/η^2 + m/η v0y) (1 - η x/(m v0x));
Plot[yt /. Parameters, {t, 0, 7}, Frame → True, GridLines → Automatic,
 FrameLabel → {"t (s)", "y (m)"}, AspectRatio → 1]
Plot[yx /. Parameters, {x, 0, 300}, Frame → True, GridLines → Automatic,
 FrameLabel → {"x (m)", "y (m)"}, AspectRatio → 1]
```

which gives





From the above two plots, we see that t_I is close to $6\,\text{s}$ and x_I is close to $250\,\text{m}$. Hence, we will use the values just listed as guesses in an appropriate root finding numerical method to find more accurate values to the quantities t_I and x_I. For example, this can be done using *Mathematica* with the following code:

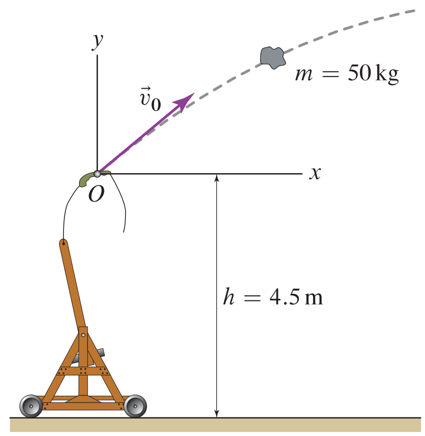
```
Parameters = {m → 50., g → 9.81, h → 4.5, v0x → 45., v0y → 30., η → 0.64};
yt = (m^2 g/η^2 - m g t/η + m/η v0y) - (m^2 g/η^2 + m/η v0y) ⅇ^(-t η/m);
yx = (m^2 g/η^2 + m^2 g/η^2 Log[1 - η x/(m v0x)] + m/η v0y) - (m^2 g/η^2 + m/η v0y) (1 - η x/(m v0x));
FindRoot[yt == -h /. Parameters, {t, 6}]
FindRoot[yx == -h /. Parameters, {x, 250}]
```

Evaluating the outcome of the above code, we have

$$t_I = 6.189\,\text{s} \quad \text{and} \quad x_I = 267.7\,\text{m}.$$

Problem 2.116

With reference to Probs. 2.113 and 2.115, assume that an experiment is conducted so that the measured value of x_I is 10% smaller than what is predicted in the absence of viscous drag. Find the value of η that would be required for the theory in Prob. 2.113 to match the experiment.



Solution

For the case of no air drag, the time of impact can be calculated by equating the constant acceleration equation for the y coordinate of the projectile to $-h$. Letting t_I denote the time of impact, we would have

$$y = (v_0)_y t - \tfrac{1}{2} g t^2 \quad \Rightarrow \quad (v_0)_y t_I - \tfrac{1}{2} g t_I^2 = -h \quad \Rightarrow \quad g t_I^2 - 2(v_0)_y t_I - 2h = 0$$

$$\Rightarrow \quad t_I = (1/g)\left[(v_0)_y \pm \sqrt{v_{0y}^2 + 2gh}\right] \quad \Rightarrow \quad t_I = 6.263\,\text{s}, \tag{1}$$

where we have discarded the solution with the negative square root because it yields a negative time value, and where we have used the following numerical data: $(v_0)_y = 30\,\text{m/s}$, $g = 9.81\,\text{m/s}^2$, and $h = 4.5\,\text{m}$. The impact distance x_I for the case of no air drag can be calculated again by using constant acceleration equations (with $a_x = 0$). This gives

$$x = (v_0)_x t \quad \Rightarrow \quad x_I = (v_0)_x t_I = 281.8\,\text{m}, \tag{2}$$

where $(v_0)_x = 45\,\text{m/s}$ and we used the expression for t_I in Eq. (1). The problem statement indicates that the x position of the rock in the presence of air drag is: $(x_I)_{\text{air}} = 0.9 x_I$, i.e.,

$$(x_I)_{\text{air}} = 253.6\,\text{m}. \tag{3}$$

To be able to use these results, we first determine the trajectory of the projectile in the presence of air resistance. We begin by working part of the solution to Prob. 2.113. Specifically, we can integrate the x and the y components of acceleration to get the x and y displacements as a function of time. Starting with the given acceleration components, using $\ddot{x} = \frac{d\dot{x}}{dt}$, we can integrate the expression for the x component of acceleration to get $\dot{x}$.

$$\int_{(v_0)_x}^{\dot{x}} \frac{d\dot{x}}{\dot{x}} = -\int_0^t \frac{\eta}{m}\,dt \quad \Rightarrow \quad \dot{x} = (v_0)_x e^{-\frac{\eta}{m}t} \tag{4}$$

where $(v_0)_x$ is the velocity component of the projectile. Integrating Eq. (4) with respect to time, we obtain

$$\int_0^x dx = (v_0)_x \int_0^t e^{-\frac{\eta}{m}t}\,dt \quad \Rightarrow \quad x = \frac{m v_{x0}}{\eta}\left(1 - e^{-\frac{\eta}{m}t}\right). \tag{5}$$

Using $\ddot{y} = \frac{d\dot{y}}{dt}$, we can integrate the expression for the y component of acceleration to get $\dot{y}$.

$$-\int_{(v_0)_y}^{\dot{y}} \frac{d\dot{y}}{g + (\eta/m)\,\dot{y}} = \int_0^t dt \quad \Rightarrow \quad \dot{y} = \frac{mg}{\eta}\left(e^{-\frac{\eta}{m}t} - 1\right) + (v_0)_y e^{-\frac{\eta}{m}t}. \tag{6}$$

Integrating Eq. (6) with respect to time, we obtain

$$y = \left(\frac{m^2 g}{\eta^2} - \frac{mgt}{\eta} + \frac{m}{\eta}(v_0)_y\right) - \left(\frac{m^2 g}{\eta^2} + \frac{m}{\eta}(v_0)_y\right) e^{-\frac{\eta}{m}t}. \tag{7}$$

Next, from Eq. (5) we find

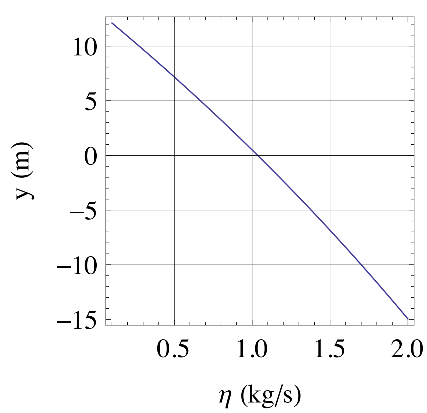
$$e^{-\frac{\eta}{m}t} = \left(1 - \frac{\eta x}{m(v_0)_x}\right) \quad \Rightarrow \quad t = -\frac{m}{\eta}\ln\left(1 - \frac{\eta x}{m(v_0)_x}\right). \tag{8}$$

Substituting the last of Eqs. (8) into Eq. (7), we have

$$y = \left[\frac{m^2 g}{\eta^2} + \frac{m^2 g}{\eta^2}\ln\left(1 - \frac{\eta x}{m(v_0)_x}\right) + \frac{m}{\eta}(v_0)_y\right] - \left(\frac{m^2 g}{\eta^2} + \frac{m}{\eta}(v_0)_y\right)\left(1 - \frac{\eta x}{m(v_0)_x}\right). \tag{9}$$

We now observe that at impact the x and y coordinates of the rock are $(x_I)_{\text{air}}$ and $-h$. By enforcing this condition in Eq. (9) we obtain an equation in η that can be solved numerically. Because most root finding algorithms require the user to supply a guess of the solution, we begin by plotting the value of y for $x = (x_I)_{\text{air}} = 253.6\,\text{m}$ (see Eq. (3)) as a function of η. The plot presented below was obtained in *Mathematica* using the following code:

```
Parameters = {m → 50., g → 9.81, h → 4.5, v0x → 45., v0y → 30., xAir → 253.6};
yx = (m^2 g/η^2 + m^2 g/η^2 Log[1 - η x/(m v0x)] + m/η v0y) - (m^2 g/η^2 + m/η v0y) (1 - η x/(m v0x));
Plot[yx /. x → xAir /. Parameters, {η, 0.1, 2}, Frame → True, GridLines → Automatic,
 FrameLabel → {"η (kg/s)", "y (m)"}, AspectRatio → 1]
```



From the plot above, we see that $y = 0$ for $x = (x_I)_{\text{air}}$ when η is a bit greater than $1\,\text{kg/s}$. Hence, we will use the value $\eta = 1\,\text{kg/s}$ as the guess for a root finding algorithm in order to find a more accurate value of the value of η. When using Mathematica, this can be done with the following code

```
Parameters = {m → 50., g → 9.81, h → 4.5, v0x → 45., v0y → 30., xAir → 253.6};
yx = (m^2 g/η^2 + m^2 g/η^2 Log[1 - η x/(m v0x)] + m/η v0y) - (m^2 g/η^2 + m/η v0y) (1 - η x/(m v0x));
FindRoot[yx == -h /. x → xAir /. Parameters, {η, 1}]
```

which yields the following solution

$$\boxed{\eta = 1.345\,\text{kg/s}.}$$

Problem 2.117

Consider the vectors $\vec{a} = 2\,\hat{\imath} + 1\,\hat{\jmath} + 7\,\hat{k}$ and $\vec{b} = 1\,\hat{\imath} + 2\,\hat{\jmath} + 3\,\hat{k}$. Compute the following quantities.

(a) $\vec{a} \times \vec{b}$

(b) $\vec{b} \times \vec{a}$

(c) $\vec{a} \times \vec{b} + \vec{b} \times \vec{a}$

(d) $\vec{a} \times \vec{a}$

(e) $(\vec{a} \times \vec{a}) \times \vec{b}$

(f) $\vec{a} \times (\vec{a} \times \vec{b})$

Parts (a)–(d) of this problem are meant to be a reminder that the cross product is an *anticommutative* operation, while Parts (e) and (f) are meant to be a reminder that the cross product is an operation that is *not associative*.

Solution

Using the vectors given in the problem statement, various properties of the cross-product are illustrated through a few simple exercises.

Part (a) The commutative relationship for the cross-product is demonstrated by first evaluating

$$\vec{a} \times \vec{b} = \det \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 1 & 7 \\ 1 & 2 & 3 \end{vmatrix} \quad \Rightarrow \quad \boxed{\vec{a} \times \vec{b} = \left(-11\,\hat{\imath} + 1\,\hat{\jmath} + 3\,\hat{k}\right).} \tag{1}$$

Part (b) The cross-product is again evaluated, but this time in the opposite order such that

$$\vec{b} \times \vec{a} = \det \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 3 \\ 2 & 1 & 7 \end{vmatrix} \quad \Rightarrow \quad \boxed{\vec{b} \times \vec{a} = \left(11\,\hat{\imath} - 1\,\hat{\jmath} - 3\,\hat{k}\right).} \tag{2}$$

Thus the cross-product is *anti-commutative* because the results are equal in magnitude, but opposite in direction (sign).

Part (c) The fact that the cross-product relation between two vectors is anti-commutative is also demonstrated through the equation below, where

$$\vec{a} \times \vec{b} + \vec{b} \times \vec{a} = \det \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 1 & 7 \\ 1 & 2 & 3 \end{vmatrix} + \det \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 3 \\ 2 & 1 & 7 \end{vmatrix} = \left(-11\,\hat{\imath} + 1\,\hat{\jmath} + 3\,\hat{k}\right) + \left(11\,\hat{\imath} - 1\,\hat{\jmath} - 3\,\hat{k}\right), \tag{3}$$

$$\Rightarrow \quad \boxed{\vec{a} \times \vec{b} + \vec{b} \times \vec{a} = \vec{0}.} \tag{4}$$

Part (d) The cross-product of a vector with itself, such as

$$\vec{a} \times \vec{a} = \det \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 1 & 7 \\ 2 & 1 & 7 \end{vmatrix} = \vec{0}, \tag{5}$$

will always yield the zero vector.

Part (e) Demonstrating the non-associative nature of the cross-product, the example shows one possible way to calculate the product of three vectors, where

$$(\vec{a} \times \vec{a}) \times \vec{b} = \vec{0} \times \vec{b} = \vec{0}. \tag{6}$$

Part (f) The associative property does not hold for cross-products since the result of part (e) is not equal to the result of

$$\vec{a} \times \left(\vec{a} \times \vec{b}\right) = \vec{a} \times \det \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 1 & 7 \\ 1 & 2 & 3 \end{vmatrix} = \det \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 1 & 7 \\ -11 & 1 & 3 \end{vmatrix},$$

$$\Rightarrow \vec{a} \times \left(\vec{a} \times \vec{b}\right) = \left(-4\,\hat{\imath} - 83\,\hat{\jmath} + 13\,\hat{k}\right). \tag{7}$$

Problem 2.118

Consider two vectors $\vec{a} = 1\,\hat{\imath} + 2\,\hat{\jmath} + 3\,\hat{k}$ and $\vec{b} = -6\,\hat{\imath} + 3\,\hat{\jmath}$.

(a) Verify that $\vec{a}$ and $\vec{b}$ are perpendicular to one another.

(b) Compute the vector triple product $\vec{a} \times (\vec{a} \times \vec{b})$.

(c) Compare the result from calculating $\vec{a} \times (\vec{a} \times \vec{b})$ with the vector $-|\vec{a}|^2\,\vec{b}$.

The purpose of this exercise is to show that as long as $\vec{a}$ and $\vec{b}$ are perpendicular to one another, you can always write $\vec{a} \times (\vec{a} \times \vec{b}) = -|\vec{a}|^2\,\vec{b}$. This identity turns out to be very useful in the study of the planar motion of rigid bodies.

Solution

Part (a) Two vectors are perpendicular if their dot product is zero. Thus,

$$\boxed{\vec{a} \cdot \vec{b} = 1 \cdot -6 + 2 \cdot 3 + 3 \cdot 0 = 0.} \tag{1}$$

Part (b) The triple product is evaluated by first calculating the cross-product of $\vec{a}$ and $\vec{b}$ and then taking the cross-product of $\vec{a}$ with the cross-product of $\vec{a}$ and $\vec{b}$. The calculation proceeds by

$$\vec{a} \times \left(\vec{a} \times \vec{b}\right) = \vec{a} \times \det \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 3 \\ -6 & 3 & 0 \end{vmatrix} = \vec{a} \times \left(-9\,\hat{\imath} - 18\,\hat{\jmath} + 15\,\hat{k}\right) = \det \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 3 \\ -9 & -18 & 15 \end{vmatrix}, \tag{2}$$

$$\Rightarrow \quad \boxed{\vec{a} \times \left(\vec{a} \times \vec{b}\right) = \left(84\,\hat{\imath} - 42\,\hat{\jmath} + 0\,\hat{k}\right).} \tag{3}$$

Part (c) The vector expression is evaluated as

$$-|\vec{a}|^2\vec{b} = -\left(\sqrt{1^2 + 2^2 + 3^2}\right)^2 \left(-6\,\hat{\imath} + 3\,\hat{\jmath} + 0\,\hat{k}\right), \tag{4}$$

$$= \left(84\,\hat{\imath} - 42\,\hat{\jmath} + 0\,\hat{k}\right), \tag{5}$$

which is the same as $\vec{a} \times \left(\vec{a} \times \vec{b}\right)$.

Problem 2.119

Let $\vec{r}$ be the position vector of a point P with respect to a Cartesian coordinate system with axes x, y, and z. Let the motion of P be confined to the xy plane, so that $\vec{r} = r_x\,\hat{\imath} + r_y\,\hat{\jmath}$ (i.e., $\vec{r}\cdot\hat{k} = 0$). Also, let $\vec{\omega}_r = \omega_r\,\hat{k}$ be the angular velocity vector of the vector $\vec{r}$. Compute the outcome of the products $\vec{\omega}_r \times (\vec{\omega}_r \times \vec{r})$ and $\vec{\omega}_r \times (\vec{r} \times \vec{\omega}_r)$.

Solution

Use the property verified in part (c) of the solution to Problem 2.118 :

$$\vec{\omega}_r \times \left(\vec{\omega}_r \times \vec{r}\right) = -|\vec{\omega}_r|^2\vec{r} = -\omega_r^2\left(r_x\,\hat{\imath} + r_y\,\hat{\jmath}\right). \tag{1}$$

The cross-product is anti-commutative, as verified in part (b) of the solution to Problem 2.117. Therefore the triple cross-product $\vec{\omega}_r \times \left(\vec{r} \times \vec{\omega}_r\right)$ is Eq. (1) multiplied by -1.

$$\vec{\omega}_r \times \left(\vec{r} \times \vec{\omega}_r\right) = \vec{\omega}_r \times \left[-\left(\vec{\omega}_r \times \vec{r}\right)\right] = \omega_r^2\left(r_x\,\hat{\imath} + r_y\,\hat{\jmath}\right). \tag{2}$$

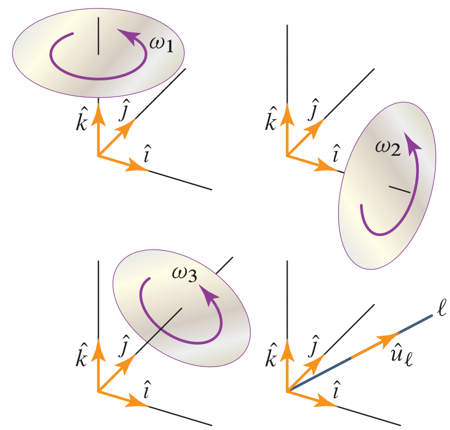
Problem 2.120

The three propellers shown are all rotating with the same *angular speed* of 1000 rpm about different coordinate axes.

(a) Provide the proper vector expressions for the *angular velocity* of each of the three propellers.

(b) Suppose that an identical propeller rotates at 1000 rpm about the axis ℓ oriented by the unit vector $\hat{u}_\ell$. Let any point P on ℓ have coordinates such that $x_P = y_P = z_P$. Find the vector representation of the angular velocity of this fourth propeller.

Express the answers using units of radians per second.



Solution

Part (a) 1000 rpm $= (100\pi/3)$ rad/s $= 104.7$ rad/s. The angular velocity vectors can be written as

$$\vec{\omega}_1 = \left(104.7\,\hat{k}\right)\text{rad/s}, \quad \vec{\omega}_2 = \left(104.7\,\hat{\imath}\right)\text{rad/s} \quad \text{and} \quad \vec{\omega}_3 = \left(-104.7\,\hat{\jmath}\right)\text{rad/s}. \tag{1}$$

Part (b) The unit vector in the ℓ direction is

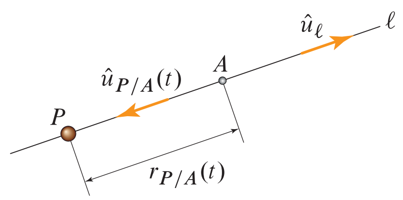
$$\hat{u}_\ell = \frac{1}{\sqrt{3}}\left(\hat{\imath} + \hat{\jmath} + \hat{k}\right). \tag{2}$$

Therefore, the angular velocity is expressed as

$$\vec{\omega}_\ell = \frac{100\pi}{3\sqrt{3}}\left(\hat{\imath} + \hat{\jmath} + \hat{k}\right)\text{rad/s} = 60.46\left(\hat{\imath} + \hat{\jmath} + \hat{k}\right)\text{rad/s}. \tag{3}$$

Problem 2.121

Point P is constrained to move along a straight line ℓ whose positive orientation is described by the unit vector $\hat{u}_\ell$. Point A is a fixed reference point on ℓ. Let the vector $\vec{r}_{P/A}$ denote the position of P relative to A and let $\hat{u}_{P/A}$ be a unit vector pointing from A to P. Use the concept of time derivative of a vector to describe the velocity and acceleration of P. In addition, comment on what happens to the description of the velocity and acceleration when P happens to coincide with the fixed point A.



Solution

The motion of P is rectilinear. Using the unit vector $\hat{u}_{P/A}$, we write the position vector of P as

$$\vec{r}_{P/A} = r_{P/A}\,\hat{u}_{P/A}, \tag{1}$$

where $r_{P/A}$ is the distance between P and A. We can calculate the velocity of P using Eq. (2.48) on p. 81 of the textbook:

$$\vec{v}_P = \dot{r}_{P/A}\,\hat{u}_{P/A} + \vec{\omega}_r \times \vec{r}_{P/A}, \tag{2}$$

where $\vec{\omega}_r$ is the angular velocity vector of $\vec{r}_{P/A}$, which is the same as that of the unit vector $\hat{u}_{P/A}$, and by Eq. (2.46) on p. 81 of the textbook, we can write

$$\dot{\hat{u}}_{P/A} = \vec{\omega}_r \times \hat{u}_{P/A}. \tag{3}$$

Since the motion of P is rectilinear, $\hat{u}_{P/A}$ is essentially a constant (see discussion at the end of the solution). Referring to Eq. (3) (and since $\hat{u}_{P/A}$ can never be zero), this means that, for rectilinear motions,

$$\vec{\omega}_r = \vec{0}. \tag{4}$$

Hence the expression for the velocity of P is

$$\boxed{\vec{v}_P = \dot{r}_{P/A}\,\hat{u}_{P/A}.} \tag{5}$$

To derive the acceleration vector of P, we can proceed in the same manner as for the velocity to obtain

$$\vec{a}_P = \dot{\vec{v}}_P = \ddot{r}_{P/A}\,\hat{u}_{P/A} + \vec{\omega}_v \times \vec{v}_P, \tag{6}$$

where $\vec{\omega}_v$ is the angular velocity of the vector $\vec{v}_P$. As with $\vec{r}_{P/A}$, the angular velocity of $\vec{v}_P$ is the same as that of $\hat{u}_{P/A}$, which, by Eq. (4), is zero. Therefore, the acceleration of P is

$$\boxed{\vec{a}_P = \ddot{r}_{P/A}\,\hat{u}_{P/A}.} \tag{7}$$

We stated that $\hat{u}_{P/A}$ is essentially a constant. Referring to the problem's figure, we have

$$\hat{u}_{P/A}(t) = \begin{cases} \hat{u}_\ell & \text{when } P \text{ follows } A, \\ -\hat{u}_\ell & \text{when } P \text{ precedes } A. \end{cases} \tag{8}$$

When P coincides with A, then $\hat{u}_{P/A}$ is undefined. Hence we can say that $\hat{u}_{P/A}$ is constant except when P coincides with A, at which case $\hat{u}_{P/A}$ is undefined and therefore cannot be used to describe the position, and consequently velocity and acceleration, of P relative to A.

Problem 2.122

Starting with Eq. (2.48), show that the second derivative with respect to time of an arbitrary vector $\vec{A}$ is given by

$$\ddot{\vec{A}} = \ddot{A}\,\hat{u}_A + 2\vec{\omega}_A \times \dot{A}\,\hat{u}_A + \dot{\vec{\omega}}_A \times \vec{A} + \vec{\omega}_A \times (\vec{\omega}_A \times \vec{A}).$$

Keep the answer in pure vector form, and do not resort to using components in any component system.

Solution

Beginning with Eq. (2.48),

$$\dot{\vec{A}} = \dot{A}\,\hat{u}_A + \vec{\omega}_A \times \vec{A}. \tag{1}$$

and taking the derivative with respect to time, gives

$$\ddot{\vec{A}} = \ddot{A}\,\hat{u}_A + \dot{A}\,\dot{\hat{u}}_A + \dot{\vec{\omega}}_A \times \vec{A} + \vec{\omega}_A \times \dot{\vec{A}}.$$

But, $\dot{\hat{u}}_A = \vec{\omega}_A \times \hat{u}_A$ and $\dot{\vec{A}}$ is given by Eq. (1), so this expression becomes

$$\ddot{\vec{A}} = \ddot{A}\,\hat{u}_A + \dot{A}(\vec{\omega}_A \times \hat{u}_A) + \dot{\vec{\omega}}_A \times \vec{A} + \vec{\omega}_A \times (\dot{A}\,\hat{u}_A + \vec{\omega}_A \times \vec{A}),$$

Noting that $\dot{A}$ is a scalar and so it can be moved inside the cross product $\vec{\omega}_A \times \hat{u}_A$ and distributing the cross product in the last term yields

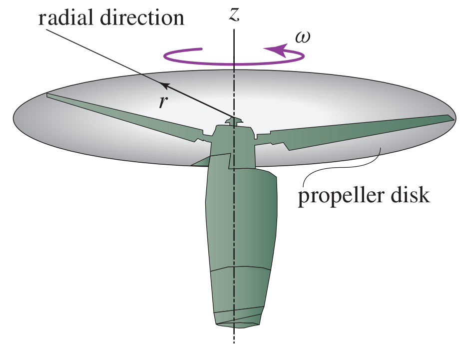
$$\ddot{\vec{A}} = \ddot{A}\,\hat{u}_A + \vec{\omega}_A \times \dot{A}\,\hat{u}_A + \dot{\vec{\omega}}_A \times \vec{A} + \vec{\omega}_A \times \dot{A}\,\hat{u}_A + \vec{\omega}_A \times (\vec{\omega} \times \vec{A}).$$

Combining the two terms $\vec{\omega}_A \times \dot{A}\,\hat{u}_A$, we obtain the desired result, that is,

$$\ddot{\vec{A}} = \ddot{A}\,\hat{u}_A + 2\vec{\omega}_A \times \dot{A}\,\hat{u}_A + \dot{\vec{\omega}}_A \times \vec{A} + \vec{\omega}_A \times (\vec{\omega}_A \times \vec{A}).$$

Problem 2.123

The propeller shown has a diameter of 38 ft and is rotating with a constant angular speed of 400 rpm. At a given instant, a point P on the propeller is at $\vec{r}_P = (12.5\,\hat{\imath} + 14.3\,\hat{\jmath})$ ft. Use Eq. (2.48) and the equation derived in Prob. 2.122 to compute the velocity and acceleration of P, respectively.



Solution

The position of point P is

$$\vec{r}_P = r_{Px}\,\hat{\imath} + r_{Py}\,\hat{\jmath}, \tag{1}$$

where $r_{Px} = 12.5$ ft and $r_{Py} = 14.3$ ft. Applying Eqs. (2.48) on p. 81 of the textbook, we have

$$\vec{v}_P = \dot{\vec{r}}_P = \frac{d|\vec{r}_P|}{dt}\,\hat{u}_{r_P} + \vec{\omega}_{r_P} \times \vec{r}_P, \tag{2}$$

where $\hat{u}_{r_P} = \vec{r}_P/|\vec{r}_P|$ and where $\vec{\omega}_{r_P}$ is the angular velocity of $\vec{r}_P$ and coincides with the angular velocity of the propeller so that

$$\vec{\omega}_{r_P} = \omega_{\text{prop}}\,\hat{k}, \tag{3}$$

where $\omega_{\text{prop}} = 400$ rpm and $\hat{k}$ is a unit vector pointing in the positive z direction. We observe that $d|\vec{r}_P|/dt = 0$ because P does not change its distance from the axis of rotation. Hence, substituting Eqs. (1) and (3) into Eq. (2), and carrying out the cross product, we have

$$\vec{v}_P = \omega_{\text{prop}}(-r_{Py}\,\hat{\imath} + r_{Px}\,\hat{\jmath}) \quad \Rightarrow \quad \boxed{\vec{v}_P = (-599.0\,\hat{\imath} + 523.6\,\hat{\jmath})\text{ ft/s},} \tag{4}$$

where we have used the fact that $r_{Px} = 12.5$ ft, $r_{Py} = 14.3$ ft, and $\omega_{\text{prop}} = 400\text{ rpm} = 400\frac{2\pi}{60}\text{ rad/s}$. Next, applying the equation derived in Problem 2.122, we have that

$$\vec{a}_P = \frac{d^2|\vec{r}_P|}{dt^2} + 2\vec{\omega}_{r_P} \times \frac{d|\vec{r}_P|}{dt}\,\hat{u}_{\vec{r}_P} + \dot{\vec{\omega}}_{r_P} \times \vec{r}_P + \vec{\omega}_{r_P} \times (\vec{\omega}_{r_P} \times \vec{r}_P). \tag{5}$$

Because $|\vec{r}_P|$ and $\vec{\omega}_{r_P}$ are constant, Eq. (5) reduces to

$$\vec{a}_P = \vec{\omega}_{r_P} \times (\vec{\omega}_{r_P} \times \vec{r}_P). \tag{6}$$

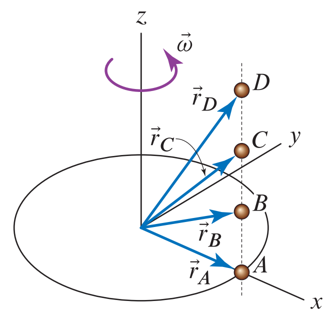
Substituting Eqs. (1) and (3) into Eq. (6), and carrying out the cross products, we have

$$\vec{a}_P = -\omega_{\text{prop}}^2(r_{Px}\,\hat{\imath} + r_{Py}\,\hat{\jmath}) \quad \Rightarrow \quad \boxed{\vec{a}_P = -(21{,}930\,\hat{\imath} + 25{,}090\,\hat{\jmath})\text{ ft/s}^2,} \tag{7}$$

where, again, we have used the fact that $r_{Px} = 12.5$ ft, $r_{Py} = 14.3$ ft, and $\omega_{\text{prop}} = 400\text{ rpm} = 400\frac{2\pi}{60}\text{ rad/s}$.

Problem 2.124

Consider the four points whose positions are given by the vectors $\vec{r}_A = \left(2\,\hat{\imath} + 0\,\hat{k}\right)$ m, $\vec{r}_B = \left(2\,\hat{\imath} + 1\,\hat{k}\right)$ m, $\vec{r}_C = \left(2\,\hat{\imath} + 2\,\hat{k}\right)$ m, and $\vec{r}_D = \left(2\,\hat{\imath} + 3\,\hat{k}\right)$ m. Knowing that the magnitude of these vectors is constant and that the angular velocity of these vectors at a given instant is $\vec{\omega} = 5\,\hat{k}$ rad/s, apply Eq. (2.48) to find the velocities $\vec{v}_A$, $\vec{v}_B$, $\vec{v}_C$, and $\vec{v}_D$. Explain why all the velocity vectors are the same even though the position vectors are not.



Solution

Using Eq. (2.48), the velocities of points A, B, C, and D are

$$\vec{v}_A = \frac{d|\vec{r}_A|}{dt}\,\hat{u}_{r_A} + \vec{\omega}_{r_A} \times \vec{r}_A, \qquad \vec{v}_B = \frac{d|\vec{r}_B|}{dt}\,\hat{u}_{r_B} + \vec{\omega}_{r_B} \times \vec{r}_B, \tag{1}$$

$$\vec{v}_C = \frac{d|\vec{r}_C|}{dt}\,\hat{u}_{r_C} + \vec{\omega}_{r_C} \times \vec{r}_C, \qquad \vec{v}_D = \frac{d|\vec{r}_D|}{dt}\,\hat{u}_{r_D} + \vec{\omega}_{r_D} \times \vec{r}_D, \tag{2}$$

where $\hat{u}_{r_A} = \vec{r}_A/|\vec{r}_A|$, $\hat{u}_{r_B} = \vec{r}_B/|\vec{r}_B|$, $\hat{u}_{r_C} = \vec{r}_C/|\vec{r}_C|$, $\hat{u}_{r_D} = \vec{r}_D/|\vec{r}_D|$, and where, following the problem statement

$$\vec{\omega}_{r_A} = \vec{\omega}_{r_B} = \vec{\omega}_{r_C} = \vec{\omega}_{r_D} = \vec{\omega} = 5\,\hat{k}\ \text{rad/s}. \tag{3}$$

Since the magnitudes of $\vec{r}_A$, $\vec{r}_B$, $\vec{r}_C$, and $\vec{r}_D$ are constant, and in view of Eq. (3), Eqs. (1) and (2) reduce to

$$\vec{v}_A = \vec{\omega} \times \vec{r}_A, \quad \vec{v}_B = \vec{\omega} \times \vec{r}_B, \tag{4}$$

$$\vec{v}_C = \vec{\omega} \times \vec{r}_C, \quad \vec{v}_D = \vec{\omega} \times \vec{r}_D. \tag{5}$$

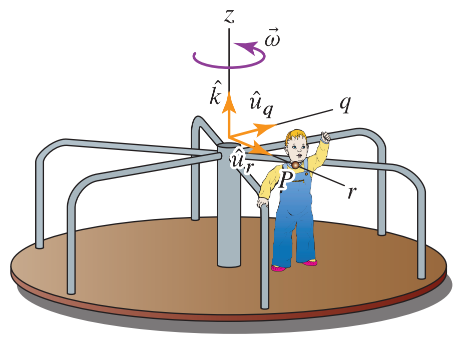
Recalling that $\vec{r}_A = \left(2\,\hat{\imath} + 0\,\hat{k}\right)$ m, $\vec{r}_B = \left(2\,\hat{\imath} + 1\,\hat{k}\right)$ m, $\vec{r}_C = \left(2\,\hat{\imath} + 2\,\hat{k}\right)$ m, $\vec{r}_D = \left(2\,\hat{\imath} + 3\,\hat{k}\right)$ m, and $\vec{\omega} = 5\,\hat{k}$ rad/s, we can evaluate Eqs. (4) and (5) to obtain

$$\boxed{\vec{v}_A = 10.00\,\hat{\jmath}\ \text{m/s}, \quad \vec{v}_B = 10.00\,\hat{\jmath}\ \text{m/s}, \quad \vec{v}_C = 10.00\,\hat{\jmath}\ \text{m/s}, \quad \text{and} \quad \vec{v}_D = 10.00\,\hat{\jmath}\ \text{m/s}.}$$

The velocities are the same because the given position vectors all have the same $\hat{\imath}$ component and because the points in question have the same distance from the axis of rotation.

Problem 2.125

A child on a merry-go-round is moving radially outward at a constant rate of 4 ft/s. If the merry-go-round is spinning at 30 rpm, determine the velocity and acceleration of point P on the child when the child is 0.5 and 2.3 ft from the spin axis. Express the answers using the component system shown.



Solution

Let $\vec{r}_P$ denote the position of P relative to the origin of the rqz coordinate system shown in the problem's figure. The origin in question is on the z axis, which is fixed. Therefore the velocity and acceleration of P are the first and second time derivatives of $\vec{r}_P$, respectively. Using the component system shown, we have

$$\vec{r}_P = r\,\hat{u}_r, \tag{1}$$

where r is the distance of P from the z axis. The unit vector $\hat{u}_r$ rotates with the merry-go-round. Applying Eq. (2.48) on p. 81 of the textbook, we have

$$\vec{v}_P = \dot{r}\,\hat{u}_r + r\vec{\omega}_r \times \hat{u}_r, \tag{2}$$

where $\vec{\omega}_r$ is the angular velocity of $\hat{u}_r$ and coincides with the angular velocity of the merry-go-round, i.e.,

$$\vec{\omega}_r = \omega\,\hat{k}, \tag{3}$$

with $\omega = 30\,\text{rpm} = \pi\,\text{rad/s}$. Substituting Eq. (3) into Eq. (2) and observing that $\hat{k} \times \hat{u}_r = \hat{u}_q$, we have

$$\vec{v}_P = \dot{r}\,\hat{u}_r + \omega r\,\hat{u}_q. \tag{4}$$

Recalling that $\dot{r} = 4\,\text{ft/s}$ and $\omega = \pi\,\text{rad/s}$, we can evaluate Eq. (4) for $r = 0.5\,\text{ft}$ and $r = 2.3\,\text{ft}$ to obtain

$$\vec{v}_P = \begin{cases} (4.000\,\hat{u}_r + 1.571\,\hat{u}_q)\,\text{ft/s} & \text{for } r = 0.5\,\text{ft}, \\ (4.000\,\hat{u}_r + 7.226\,\hat{u}_q)\,\text{ft/s} & \text{for } r = 2.3\,\text{ft}. \end{cases}$$

Differentiating Eq. (2) with respect to time we have $\vec{a}_P = \ddot{r}\,\hat{u}_r + \dot{r}\,\dot{\hat{u}}_r + \dot{r}\vec{\omega}_r \times \hat{u}_r + r\dot{\vec{\omega}}_r \times \hat{u}_r + r\vec{\omega}_r \times \dot{\hat{u}}_r$. Using Eq. (2.46) on p. 81, we have that $\dot{\hat{u}}_r = \vec{\omega}_r \times \hat{u}_r$, so that $\vec{a}_P$ can be written as

$$\vec{a}_P = \ddot{r}\,\hat{u}_r + 2\vec{\omega}_r \times \dot{r}\,\hat{u}_r + \dot{\vec{\omega}}_r \times r\,\hat{u}_r + \vec{\omega}_r \times (\vec{\omega}_r \times r\,\hat{u}_r). \tag{5}$$

Recalling that $\dot{r}$ and ω are constant, and using Eq. (3), Eq. (5) gives

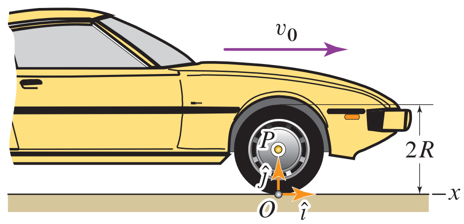
$$\vec{a}_P = 2\omega\,\hat{k} \times \dot{r}\,\hat{u}_r + \omega\,\hat{k} \times (\omega\,\hat{k} \times r\,\hat{u}_r) \quad \Rightarrow \quad \vec{a}_P = -r\omega^2\,\hat{u}_r + 2\omega\dot{r}\,\hat{u}_q. \tag{6}$$

Finally, recalling that $\dot{r} = 4\,\text{ft/s}$ and $\omega = \pi\,\text{rad/s}$, we can evaluate the last of Eqs. (6) for $r = 0.5\,\text{ft}$ and $r = 2.3\,\text{ft}$ to obtain

$$\vec{a}_P = \begin{cases} (-4.935\,\hat{u}_r + 25.13\,\hat{u}_q)\,\text{ft/s}^2 & \text{for } r = 0.5\,\text{ft}, \\ (-22.70\,\hat{u}_r + 25.13\,\hat{u}_q)\,\text{ft/s}^2 & \text{for } r = 2.3\,\text{ft}. \end{cases}$$

Problem 2.126

When a wheel rolls without slipping on a stationary surface, the point O on the wheel that is in contact with the rolling surface has zero velocity. With this in mind, consider a nondeformable wheel rolling without slip on a flat stationary surface. The center of the wheel P is traveling to the right with a constant speed $v_0 = 23\,\mathrm{m/s}$. Letting $R = 0.35\,\mathrm{m}$, determine the angular velocity of the wheel, using the stationary component system shown.



Solution

Since P moves parallel to the ground, we have that the velocity of P can be expressed as follows:

$$\vec{v}_P = v_0\,\hat{\imath}. \tag{1}$$

Letting $\vec{r}_P$ denote the position of P relative to O, since the velocity of O is equal to zero at the instant considered, we also know that $\vec{v}_P = \dot{\vec{r}}_P$. Hence, applying Eq. (2.48) on p. 81 of the textbook, we can write

$$\vec{v}_P = \frac{d|\vec{r}_P|}{dt}\,\hat{u}_{r_P} + \vec{\omega}_{r_P} \times \vec{r}_P, \tag{2}$$

where $\vec{\omega}_{r_P}$ is the angular velocity of the vector $\vec{r}_P$. As both points O and P are on the wheel, we have that

$$|\vec{r}_P| = R = \text{constant} \quad \text{and} \quad \vec{\omega}_{r_P} = \vec{\omega}_{\text{wheel}} = \omega_{\text{wheel}}\,\hat{k}. \tag{3}$$

In addition, at the instant shown, we also have

$$\vec{r}_P = R\,\hat{\jmath}. \tag{4}$$

Substituting Eqs. (3) and (4) into Eq. (2), carrying out the cross product and simplifying, we have

$$\vec{v}_P = -\omega_{\text{wheel}} R\,\hat{\imath}. \tag{5}$$

Comparing Eqs. (1) and (5), we then conclude that

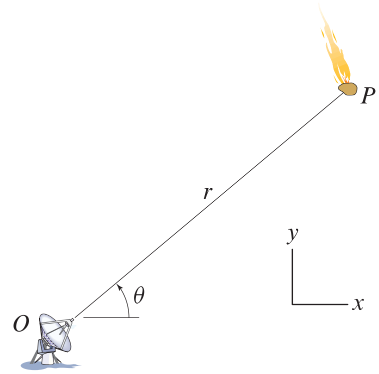
$$\omega_{\text{wheel}} = -\frac{v_0}{R} \quad \Rightarrow \quad \vec{\omega}_{\text{wheel}} = -\frac{v_0}{R}\,\hat{k}. \tag{6}$$

Recalling that $v_0 = 23\,\mathrm{m/s}$ and $R = 0.35\,\mathrm{m}$, we can evaluate the last of Eqs. (6) to obtain

$$\boxed{\vec{\omega}_{\text{wheel}} = -65.71\,\hat{k}\ \mathrm{rad/s}.}$$

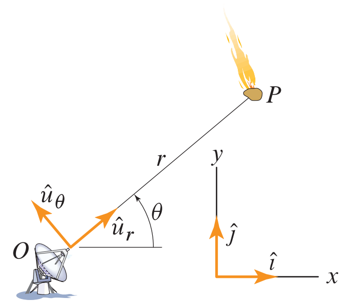
Problem 2.127

The radar station at O is tracking the meteor P as it moves through the atmosphere. At the instant shown, the station measures the following data for the motion of the meteor: $r = 21{,}000$ ft, $\theta = 40°$, $\dot{r} = -22{,}440$ ft/s, and $\dot{\theta} = -2.935$ rad/s. Use Eq. (2.48) to determine the magnitude and direction (relative to the xy coordinate system shown) of the velocity vector at this instant.



Solution

Referring to the figure at the right, the unit vector $\hat{u}_r$ always points toward P. The unit vector $\hat{u}_\theta$ is perpendicular to $\hat{u}_r$ and points in the direction of increasing θ. Then, letting r denote the distance between P and the fixed point O, we have that the position of P is described by $\vec{r}_P = r\,\hat{u}_r$. Applying Eq. (2.48) on p. 81 of the textbook, we have



$$\vec{v}_P = \dot{r}\,\hat{u}_r + r\vec{\omega}_r \times \hat{u}_r, \tag{1}$$

where $\vec{\omega}_r$ is the angular velocity of the unit vector $\hat{u}_r$. Since the angle θ describes the orientation of $\vec{r}_P$, we have that

$$\vec{\omega}_r = \dot{\theta}\,\hat{k}, \tag{2}$$

where $\hat{k} = \hat{u}_r \times \hat{u}_\theta$. Substituting Eq. (2) into Eq. (1) gives

$$\vec{v}_P = \dot{r}\,\hat{u}_r + r\dot{\theta}\,\hat{u}_\theta. \tag{3}$$

Finding the magnitude of the vector, we then have that

$$|\vec{v}_P| = \sqrt{\dot{r}^2 + r^2\dot{\theta}^2} \quad \Rightarrow \quad \boxed{|\vec{v}_P| = 65{,}590\ \text{ft/s},} \tag{4}$$

where we recalled that $r = 21{,}000$ ft, $\dot{r} = -22{,}440$ ft/s, and $\dot{\theta} = -2.935$ rad/s. We now observe that

$$\hat{u}_r = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath} \quad \text{and} \quad \hat{u}_\theta = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}, \tag{5}$$

so that Eq. (3) can be rewritten as

$$\vec{v}_P = \underbrace{(\dot{r}\cos\theta - r\dot{\theta}\sin\theta)}_{v_x}\,\hat{\imath} + \underbrace{(\dot{r}\sin\theta + r\dot{\theta}\cos\theta)}_{v_{Py}}\,\hat{\jmath} = (22{,}430\,\hat{\imath} - 61{,}640\,\hat{\jmath})\ \text{ft/s}. \tag{6}$$

Since $\vec{v}_P$ is directed downward and to the right, the orientation of $\vec{v}_P$ is $-\tan^{-1}(|v_{Py}/v_{P_x}|)$:

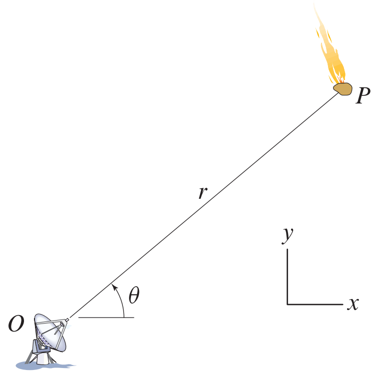
$$\text{Orientation of } \vec{v}_P \text{ from } x \text{ axis} = -\tan^{-1}\left(\left|\frac{\dot{r}\sin\theta + r\dot{\theta}\cos\theta}{\dot{r}\cos\theta - r\dot{\theta}\sin\theta}\right|\right) = -70.01°$$

$$\Rightarrow \quad \boxed{\text{Orientation of } \vec{v} \text{ from } x \text{ axis} = 70.01°\ (\text{cw}),} \tag{7}$$

where, again, $r = 21{,}000$ ft, $\dot{r} = -22{,}440$ ft/s, and $\dot{\theta} = -2.935$ rad/s.

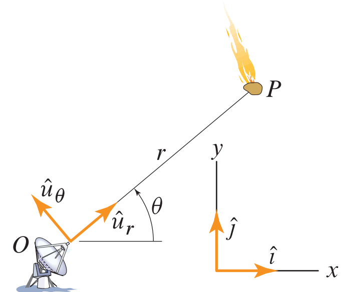
Problem 2.128

The radar station at O is tracking the meteor P as it moves through the atmosphere. At the instant shown, the station measures the following data for the motion of the meteor: $r = 21{,}000$ ft, $\theta = 40°$, $\dot{r} = -22{,}440$ ft/s, $\dot{\theta} = -2.935$ rad/s, $\ddot{r} = 187{,}500$ ft/s^2, and $\ddot{\theta} = -5.409$ rad/s^2. Use the equation derived in Prob. 2.122 to determine the magnitude and direction (relative to the xy coordinate system shown) of the acceleration vector at this instant.



Solution

Referring to the figure at the right, the unit vector $\hat{u}_r$ always points toward P. The unit vector $\hat{u}_\theta$ is perpendicular to $\hat{u}_r$ and points in the direction of increasing θ. The distance between P and O is r so the position of P is $\vec{r}_P = r\,\hat{u}_r$. Applying the equation derived in Problem 2.122, we can write



$$\vec{a}_P = \ddot{\vec{r}}_P = \ddot{r}\,\hat{u}_r + 2\vec{\omega}_r \times \dot{r}\,\hat{u}_r + \dot{\vec{\omega}}_r \times \vec{r}_P + \vec{\omega}_r \times \left(\vec{\omega}_r \times \vec{r}_P\right), \qquad (1)$$

where $\vec{\omega}_r$ is the angular velocity of $\vec{r}_P$. Since the angle θ describes the orientation of $\vec{r}_P$, we have that

$$\vec{\omega}_r = \dot{\theta}\,\hat{k}, \qquad (2)$$

where $\hat{k} = \hat{u}_r \times \hat{u}_\theta$. Because the direction of $\hat{k}$ is fixed, $\dot{\vec{\omega}}_r = \ddot{\theta}\,\hat{k}$. Hence, substituting $\vec{r}_P = r\,\hat{u}_r$ and Eq. (2) into Eq. (1) and simplifying, we have

$$\vec{a}_P = (\ddot{r} - r\dot{\theta}^2)\,\hat{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{u}_\theta. \qquad (3)$$

Finding the magnitude of the vector, we then have that

$$|\vec{a}_P| = \sqrt{\left(\ddot{r} - r\dot{\theta}^2\right)^2 + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)^2} \quad \Rightarrow \quad \boxed{|\vec{a}_P| = 19{,}300\ \text{ft/s}^2,} \qquad (4)$$

where $r = 21{,}000$ ft, $\dot{r} = -22{,}440$ ft/s, $\dot{\theta} = -2.935$ rad/s, $\ddot{r} = 187{,}500$ ft/s^2, and $\ddot{\theta} = -5.409$ rad/s^2. Noting that $\hat{u}_r = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$ and $\hat{u}_\theta = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}$, Eq. (3) becomes

$$\vec{a}_P = \underbrace{\left[\left(\ddot{r} - r\dot{\theta}^2\right)\cos\theta - \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\sin\theta\right]}_{a_{Px}}\,\hat{\imath} + \underbrace{\left[\left(\ddot{r} - r\dot{\theta}^2\right)\sin\theta + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\cos\theta\right]}_{a_{Py}}\,\hat{\jmath}$$

$$\Rightarrow \quad \vec{a}_P = (-6599\,\hat{\imath} + 18{,}130\,\hat{\jmath})\ \text{ft/s}^2, \qquad (5)$$

Since $\vec{a}_P$ is directed upward and to the left, the orientation of $\vec{a}_P$ is given by $180° - \tan^{-1}(|a_{Py}/a_{Px}|)$, i.e.,

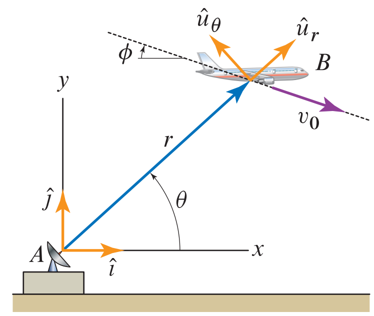
$$\text{Orientation of } \vec{a}_P \text{ from } x \text{ axis} = 180° - \tan^{-1}\left[\left|\frac{\left(\ddot{r} - r\dot{\theta}^2\right)\sin\theta + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\cos\theta}{\left(\ddot{r} - r\dot{\theta}^2\right)\cos\theta - \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\sin\theta}\right|\right]$$

$$\Rightarrow \quad \boxed{\text{Orientation of } \vec{a}_P \text{ from } x \text{ axis} = 110.0°\ \text{(ccw)}.}$$

where $\theta = 40°$, $r = 21{,}000$ ft, $\dot{r} = -22{,}440$ ft/s, $\dot{\theta} = -2.935$ rad/s, $\ddot{r} = 187{,}500$ ft/s^2, and $\ddot{\theta} = -5.409$ rad/s^2.

Problem 2.129

A plane B is approaching a runway along the trajectory shown while the radar antenna A is monitoring the distance r between A and B, as well as the angle θ. If the plane has a constant approach speed v_0 as shown, use Eq. (2.48) to determine the expressions for $\dot{r}$ and $\dot{\theta}$ in terms of r, θ, v_0, and ϕ.



Solution

The position of the airplane B can be written as $\vec{r} = r\,\hat{u}_r$. Differentiating $\vec{r}$ with respect to time according to Eq. (2.48) on p. 81 of the textbook, we obtain the velocity of the airplane as

$$\vec{v} = \dot{r}\,\hat{u}_r + \vec{\omega}_r \times r\,\hat{u}_r, \tag{1}$$

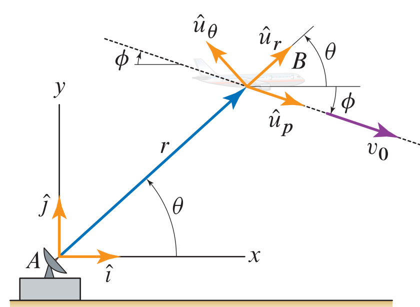
where $\vec{\omega}_r$ is the angular velocity of the vector $\vec{r}$. Letting $\hat{k} = \hat{u}_r \times \hat{u}_\theta$, since θ describes the orientation of the vector $\vec{r}$, we have

$$\vec{\omega}_r = \dot{\theta}\,\hat{k}. \tag{2}$$

Substituting Eq. (2) into Eq. (1) and carrying out the cross-product, we have

$$\vec{v} = \dot{r}\,\hat{u}_r + \dot{\theta} r\,\hat{u}_\theta. \tag{3}$$

Referring to the figure on the right, we observe that the velocity of the airplane can also be expressed as $\vec{v} = v_0\,\hat{u}_p$, where $\hat{u}_p$ is the unit vector describing the orientation of the airplane's landing path. Observing that the angle between $\hat{u}_p$ and $\hat{u}_r$ is equal to $\phi + \theta$, we can express $\hat{u}_p$ as follows:



$$\hat{u}_p = \cos(\phi + \theta)\,\hat{u}_r - \sin(\phi + \theta)\,\hat{u}_\theta. \tag{4}$$

Therefore, $\vec{v}_B$ can be expressed as follows:

$$\vec{v} = v_0 \cos(\phi + \theta)\,\hat{u}_r - v_0 \sin(\phi + \theta)\,\hat{u}_\theta. \tag{5}$$

Equating Eqs. (3) and (5) component by component, we obtain

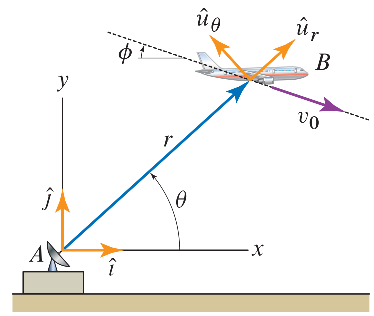
$$\dot{r} = v_0 \cos(\phi + \theta) \quad \text{and} \quad \dot{\theta} r = -v_0 \sin(\phi + \theta), \tag{6}$$

which we can rewrite as

$$\boxed{\dot{r} = v_0 \cos(\phi + \theta) \quad \text{and} \quad \dot{\theta} = -\frac{v_0}{r} \sin(\phi + \theta).}$$

Problem 2.130

A plane B is approaching a runway along the trajectory shown with $\phi = 15°$, while the radar antenna A is monitoring the distance r between A and B, as well as the angle θ. The plane has a constant approach speed v_0. In addition, when $\theta = 20°$, it is known that $\dot{r} = 216\,\text{ft/s}$ and $\dot{\theta} = -0.022\,\text{rad/s}$. Use Eq. (2.48) to determine the corresponding values of v_0 and of the distance between the plane and the radar antenna.



Solution

The position of the airplane B can be written as $\vec{r} = r\,\hat{u}_r$. Differentiating $\vec{r}$ with respect to time according to Eq. (2.48) on p. 81 of the textbook, we obtain the velocity of the airplane as

$$\vec{v} = \dot{r}\,\hat{u}_r + \vec{\omega}_r \times r\,\hat{u}_r, \tag{1}$$

where $\vec{\omega}_r$ is the angular velocity of the vector $\vec{r}$. Letting $\hat{k} = \hat{u}_r \times \hat{u}_\theta$, since θ describes the orientation of the vector $\vec{r}$, we have

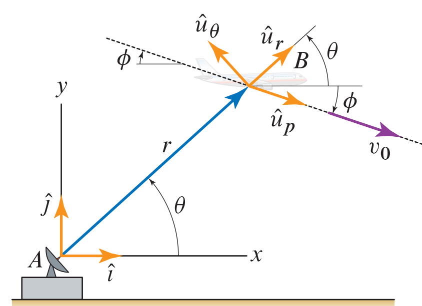
$$\vec{\omega}_r = \dot{\theta}\,\hat{k}. \tag{2}$$

Substituting Eq. (2) into Eq. (1) and carrying out the cross-product, we have

$$\vec{v} = \dot{r}\,\hat{u}_r + \dot{\theta} r\,\hat{u}_\theta. \tag{3}$$

Referring to the figure on the right, we observe that the velocity of the airplane can also be expressed as $\vec{v} = v_0\,\hat{u}_p$, where $\hat{u}_p$ is the unit vector describing the orientation of the airplane's landing path. Observing that the angle between $\hat{u}_p$ and $\hat{u}_r$ is equal to $\phi + \theta$, we can express $\hat{u}_p$ as follows:

$$\hat{u}_p = \cos(\phi + \theta)\,\hat{u}_r - \sin(\phi + \theta)\,\hat{u}_\theta. \tag{4}$$



Therefore, $\vec{v}_B$ can be expressed as follows:

$$\vec{v} = v_0 \cos(\phi + \theta)\,\hat{u}_r - v_0 \sin(\phi + \theta)\,\hat{u}_\theta. \tag{5}$$

Equating Eqs. (3) and (5) component by component, we obtain

$$\dot{r} = v_0 \cos(\phi + \theta) \quad \text{and} \quad \dot{\theta} r = -v_0 \sin(\phi + \theta). \tag{6}$$

Equations (6) form a system of two equations in the two unknowns v_0 and r whose solution is

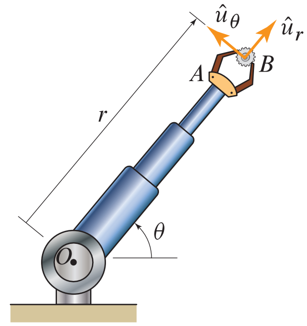
$$v_0 = \frac{\dot{r}}{\cos(\phi + \theta)} \quad \text{and} \quad r = -\frac{\dot{r}}{\dot{\theta}} \tan(\phi + \theta). \tag{7}$$

Recalling that $\dot{r} = 216\,\text{ft/s}$, $\phi = 15°$, $\theta = 20°$, and $\dot{\theta} = -0.022\,\text{rad/s}$, we can evaluate Eqs. (7) to obtain

$$\boxed{v_0 = 263.7\,\text{ft/s} \quad \text{and} \quad r = 6875\,\text{ft}.}$$

Problem 2.131

The end B of a robot arm is being extended with the constant rate $\dot{r} = 4\,\text{ft/s}$. Knowing that $\dot{\theta} = 0.4\,\text{rad/s}$ and is constant, use Eq. (2.48) and the equation derived in Prob. 2.122 to determine the velocity and acceleration of B when $r = 2\,\text{ft}$. Express your answer using the component system shown.



Solution

The position of point B relative to the fixed point O, can be expressed as $\vec{r} = r\,\hat{u}_r$. Differentiating $\vec{r}$ with respect to time according to Eq. (2.48) on p. 81 of the textbook, gives the velocity of B as follows:

$$\vec{v}_B = \dot{r}\,\hat{u}_r + \vec{\omega}_r \times r\,\hat{u}_r, \tag{1}$$

where $\vec{\omega}_r$ is the angular velocity of the vector $\vec{r}$. Letting $\hat{k}$ denote the unit vector perpendicular to the plane of motion such that $\hat{k} = \hat{u}_r \times \hat{u}_\theta$, and observing that the angle θ describes the orientation of the vector $\vec{r}$, we have

$$\vec{\omega}_r = \dot{\theta}\,\hat{k}. \tag{2}$$

Substituting Eq. (2) into Eq. (1) and carrying out the cross-product, we have

$$\vec{v}_B = \dot{r}\,\hat{u}_r + \dot{\theta}\,\hat{k} \times r\,\hat{u}_r \quad \Rightarrow \quad \vec{v}_B = \dot{r}\,\hat{u}_r + \dot{\theta} r\,\hat{u}_\theta. \tag{3}$$

Given that $r = 2\,\text{ft}$, $\dot{r} = 4\,\text{ft/s}$, and $\dot{\theta} = 0.4\,\text{rad/s}$, we can evaluate the last of Eqs. (3) to obtain

$$\boxed{\vec{v}_B = (4.000\,\hat{u}_r + 0.8000\,\hat{u}_\theta)\,\text{ft/s}.}$$

To obtain the acceleration of B, we compute the second time derivative of the position vector $\vec{r} = r\,\hat{u}_r$ using the equation derived in Problem 2.122. This gives

$$\vec{a}_B = \ddot{r}\,\hat{u}_r + 2\vec{\omega}_r \times \dot{r}\,\hat{u}_r + \dot{\vec{\omega}}_r \times r\,\hat{u}_r + \vec{\omega}_r \times \left(\vec{\omega}_r \times r\,\hat{u}_r\right). \tag{4}$$

Using Eq. (2), we have

$$\dot{\vec{\omega}}_r = \ddot{\theta}\,\hat{k} + \dot{\theta}\,\dot{\hat{k}} = \ddot{\theta}\,\hat{k}, \tag{5}$$

where $\dot{\hat{k}} = \vec{0}$ because the motion is planar. Since $\dot{r}$ and $\dot{\theta}$ are constant,

$$\ddot{r} = 0, \quad \text{and} \quad \ddot{\theta} = 0. \tag{6}$$

Using Eqs. (2), (5), and (6) we can simplify Eq. (4) to read

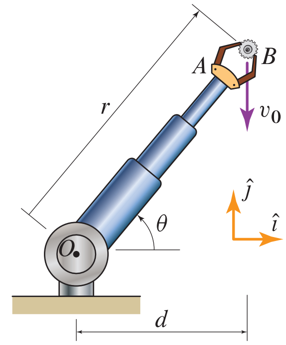
$$\vec{a}_B = 2\dot{\theta}\,\hat{k} \times \dot{r}\,\hat{u}_r + \dot{\theta}\,\hat{k} \times \left(\dot{\theta}\,\hat{k} \times r\,\hat{u}_r\right) \quad \Rightarrow \quad \vec{a}_r = -r\dot{\theta}^2\,\hat{u}_r + 2\dot{\theta}\dot{r}\,\hat{u}_\theta. \tag{7}$$

Again, since $r = 2\,\text{ft}$, $\dot{r} = 4\,\text{ft/s}$, and $\dot{\theta} = 0.4\,\text{rad/s}$, we can evaluate the last of Eqs. (7) for to obtain

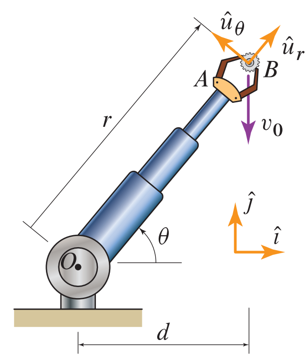
$$\boxed{\vec{a}_B = (-0.3200\,\hat{u}_r + 3.200\,\hat{u}_\theta)\,\text{ft/s}^2.}$$

Problem 2.132

The end B of a robot arm is moving vertically down with a constant speed $v_0 = 2\,\text{m/s}$. Letting $d = 1.5\,\text{m}$, apply Eq. (2.48) to determine the rate at which r and θ are changing when $\theta = 37°$.



Solution



Referring to the figure on the right, we begin by describing the position of point B relative to O using the $(\hat{u}_r, \hat{u}_\theta)$ component system:

$$\vec{r}_B = r\,\hat{u}_r. \tag{1}$$

The velocity of B is $\vec{v}_B = \dot{\vec{r}}_B$. Then, using Eq. (2.48) on p. 81 of the textbook, we have

$$\vec{v}_B = \dot{r}\,\hat{u}_r + \vec{\omega}_r \times r\,\hat{u}_r, \tag{2}$$

where $\vec{\omega}_r$ is the angular velocity of the vector $\vec{r}_B$. Since the vector $\vec{r}_B$ rotates with the robotic arm, we have

$$\vec{\omega}_r = \dot{\theta}\,\hat{k}. \tag{3}$$

Substituting Eq. (3) into Eq. (2) we have

$$\vec{v}_B = \dot{r}\,\hat{u}_r + \dot{\theta}\,\hat{k} \times r\,\hat{u}_r \quad \Rightarrow \quad \vec{v}_B = \dot{r}\,\hat{u}_r + \dot{\theta} r\,\hat{u}_\theta. \tag{4}$$

Since point B is moving downward along a vertical line with speed v_0, using the $(\hat{\imath}, \hat{\jmath})$ component system, the velocity of B can also be described as follows:

$$\vec{v}_B = -v_0\,\hat{\jmath}. \tag{5}$$

We now observe that

$$\hat{\jmath} = \sin\theta\,\hat{u}_r + \cos\theta\,\hat{u}_\theta. \tag{6}$$

Therefore, Eq. (5) can be rewritten as

$$\vec{v}_B = -v_0(\sin\theta\,\hat{u}_r + \cos\theta\,\hat{u}_\theta). \tag{7}$$

Equating the second of Eqs. (4) and Eq. (5) component by component, we have

$$\dot{r} = -v_0 \sin\theta \quad \text{and} \quad \dot{\theta} r = -v_0 \cos\theta. \tag{8}$$

Recognizing that $r\cos\theta = d$, i.e., $r = d/\cos\theta$, we can solve Eqs. (8) for $\dot{r}$ and $\dot{\theta}$ to obtain

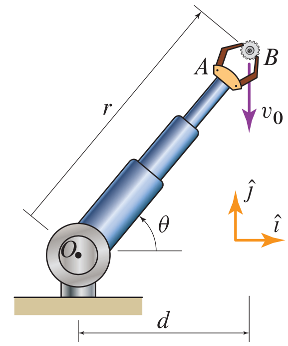
$$\dot{r} = -v_0 \sin\theta \quad \text{and} \quad \dot{\theta} = -\frac{v_0 \cos^2\theta}{d}. \tag{9}$$

Finally, recalling that $v_0 = 2\,\text{m/s}$, $\theta = 37°$, and $d = 1.5\,\text{m}$, Eqs. (9) can be evaluated to obtain

$$\boxed{\dot{r} = -1.204\,\text{m/s} \quad \text{and} \quad \dot{\theta} = -0.8504\,\text{rad/s}.}$$

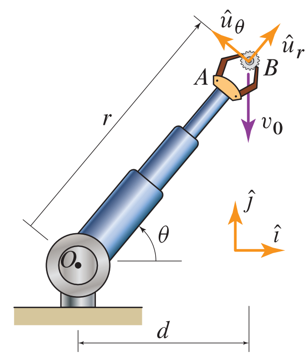
Problem 2.133

The end B of a robot arm is moving vertically down with a constant speed $v_0 = 6\,\text{ft/s}$. Letting $d = 4\,\text{ft}$, use Eq. (2.48) and the equation derived in Prob. 2.122 to determine $\dot{r}$, $\dot{\theta}$, $\ddot{r}$, and $\ddot{\theta}$ when $\theta = 0°$.



Solution

Referring to the figure on the right, the velocity is expressed both in terms of the $(\hat{\imath}, \hat{\jmath})$ component system and using the component system $(\hat{u}_r, \hat{u}_\theta)$ along with Eq. (2.48) on p. 81 of the textbook. This gives



$$\vec{v} = -v_0\,\hat{\jmath} \quad \text{and} \quad \vec{v} = \dot{r}\,\hat{u}_r + \dot{\theta}\,\hat{k} \times r\,\hat{u}_r = \dot{r}\,\hat{u}_r + \dot{\theta} r\,\hat{u}_\theta. \tag{1}$$

Expressing $(\hat{u}_r, \hat{u}_\theta)$ in terms of $(\hat{\imath}, \hat{\jmath})$, we have

$$\hat{u}_r = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath} \quad \text{and} \quad \hat{u}_\theta = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}. \tag{2}$$

Substituting Eqs. (2) into the last of Eqs. (1) and collecting the $\hat{\imath}$ and $\hat{\jmath}$ terms, we have

$$\vec{v} = \left(\dot{r}\cos\theta - r\dot{\theta}\sin\theta\right)\hat{\imath} + \left(\dot{r}\sin\theta + r\dot{\theta}\cos\theta\right)\hat{\jmath}. \tag{3}$$

Equating the $\hat{\imath}$ and $\hat{\jmath}$ components of velocity given by the first of Eqs. (1) and Eq. (3), and keeping in mind that $r = d/\cos\theta$, we have

$$\dot{r}\cos\theta - r\dot{\theta}\sin\theta = 0 \quad \Rightarrow \quad \dot{r}\cos\theta - \dot{\theta} d\tan\theta = 0 \tag{4}$$

$$\dot{r}\sin\theta + r\dot{\theta}\cos\theta = -v_0 \quad \Rightarrow \quad \dot{r}\sin\theta + \dot{\theta} d = -v_0. \tag{5}$$

Substituting $\theta = 0$ into Eqs. (4) and (5), we have

$$\boxed{\dot{r} = 0 \quad \text{and} \quad \dot{\theta} = \frac{-v_0}{d} = -1.500\,\text{rad/s},} \tag{6}$$

where we have used the following numerical data: $v_0 = 6\,\text{ft/s}$ and $d = 4\,\text{ft}$. Using the equation derived in Problem 2.122, the acceleration expressed in the $(\hat{u}_r, \hat{u}_\theta)$ component system is:

$$\vec{a} = \ddot{r}\,\hat{u}_r + 2\dot{\theta}\,\hat{k} \times \dot{r}\,\hat{u}_r + \ddot{\theta}\,\hat{k} \times r\,\hat{u}_r + \dot{\theta}\,\hat{k} \times (\dot{\theta}\,\hat{k} \times r\,\hat{u}_r)$$

$$\Rightarrow \quad \vec{a} = \ddot{r}\,\hat{u}_r + 2\dot{\theta}\dot{r}\,\hat{u}_\theta + \ddot{\theta} r\,\hat{u}_\theta - \dot{\theta}^2 r\,\hat{u}_r.$$

Alternatively, differentiating Eq. (3) with respect to time and rearranging terms, the acceleration expressed in the $(\hat{\imath}, \hat{\jmath})$ component system is:

$$\vec{a} = \ddot{r}(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}) + 2\dot{\theta}\dot{r}(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath})$$

$$+ \ddot{\theta} r(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}) - \dot{\theta}^2 r(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}). \quad (7)$$

Collecting $\hat{\imath}$ and $\hat{\jmath}$ terms, then substituting $r = d$, $\theta = 0$, and the expressions in Eqs. (6), we have

$$\vec{a}\big|_{\theta=0^\circ} = \left(\ddot{r} - \frac{v_0^2}{d}\right)\hat{\imath} + \left(\ddot{\theta} d\right)\hat{\jmath}.$$

Since $\vec{v}$ is constant, $\vec{a} = \vec{0}$, so that

$$\boxed{\ddot{r} = \frac{v_0^2}{d} = 9.000\,\text{ft/s}^2 \quad \text{and} \quad \ddot{\theta} = 0,}$$

where, again we have used the fact that $v_0 = 6\,\text{ft/s}$ and $d = 4\,\text{ft}$.

Problem 2.134

A micro spiral pump consists of a spiral channel attached to a stationary plate. This plate has two ports, one for fluid inlet and the other for outlet, the outlet being farther from the center of the plate than the inlet. The system is capped by a rotating disk. The fluid trapped between the rotating disk and stationary plate is put in motion by the rotation of the top disk, which pulls the fluid through the spiral channel. With this in mind, consider a channel with geometry given by the equation $r = \eta\theta + r_0$, where $\eta = 12\,\mu$m is called the polar slope, $r_0 = 146\,\mu$m is the radius at the inlet, r is the distance from the spin axis, and θ, measured in radians, is the angular position of a point in the spiral channel. If the top disk rotates with a constant angular speed $\omega = 30{,}000$ rpm, and assuming that the fluid particles in contact with the rotating disk are essentially stuck to it, determine the velocity and acceleration of one such fluid particle when it is at $r = 170\,\mu$m. Express the answer using the component system shown (which rotates with the top disk).

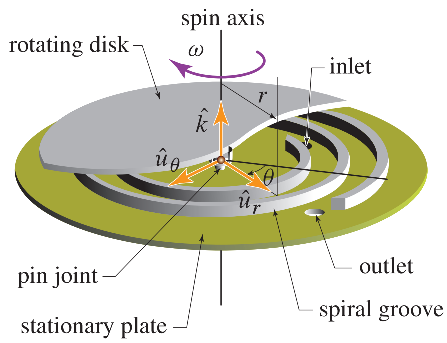


Photo credit: "Design and Analysis of a Surface Micromachined Spiral-Channel Viscous Pump," by M. I. Kilani, P. C. Galambos, Y. S. Haik, C. H. Chen, *Journal of Fluids Engineering*, Vol. 125, pp. 339–344, 2003.

Solution

Referring to the problem's figure, we focus our attention on a single fluid particle moving along the channel. The unit vector $\hat{u}_r$ always points from the origin of the coordinate system to the particle so that the position of the particle is $\vec{r} = r\,\hat{u}_r$, where r is the distance of the particle from the spin axis. Using Eq. (2.48) on p. 81 of the textbook, we can express the velocity of a particle as

$$\vec{v} = \dot{r}\,\hat{u}_r + \vec{\omega} \times \vec{r}, \tag{1}$$

where $\vec{\omega} = -\dot{\theta}\,\hat{k}$, with $\dot{\theta} = 30{,}000\frac{2\pi}{60}$ rad/s, is the angular velocity of $\vec{r}$. Recalling that $r = \eta\theta + r_0$, Eq. (1) can be rewritten as

$$\vec{v} = \eta\dot{\theta}\,\hat{u}_r - \dot{\theta}\,\hat{k} \times (\eta\theta + r_0)\,\hat{u}_r \quad \Rightarrow \quad \vec{v} = \eta\dot{\theta}\,\hat{u}_r + \dot{\theta}(\eta\theta + r_0)\,\hat{u}_\theta, \tag{2}$$

where we have used the fact that $\hat{k} \times \hat{u}_r = -\hat{u}_\theta$. Solving $r = \eta\theta + r_0$ for θ we find that $\theta = (r - r_0)/\eta$. Recalling that $r_0 = 146\,\mu$m and $\eta = 12\,\mu$m, for $r = 170\,\mu$m, $\theta = 2.000$ rad. Hence, given that $\dot{\theta} = 1000\pi$ rad/s, we can evaluate $\vec{v}$ in the last of Eqs. (2) to obtain

$$\boxed{\vec{v} = (0.03770\,\hat{u}_r + 0.5341\,\hat{u}_\theta)\ \text{m/s}.}$$

Differentiating the last of Eqs. (2) with respect to time, we have

$$\vec{a} = \eta\ddot{\theta}\,\hat{u}_r + \eta\dot{\theta}\,\dot{\hat{u}}_r + \ddot{\theta}(\eta\theta + r_0)\,\hat{u}_\theta + \eta\dot{\theta}^2\,\hat{u}_\theta + \dot{\theta}(\eta\theta + r_0)\,\dot{\hat{u}}_\theta. \tag{3}$$

Observing that both $\hat{u}_r$ and $\hat{u}_\theta$ rotate with angular velocity $\vec{\omega} = -\dot{\theta}\,\hat{k}$, applying Eq. (2.46) on p. 81 of the textbook, we have

$$\dot{\hat{u}}_r = -\dot{\theta}\,\hat{k} \times \hat{u}_r = \dot{\theta}\,\hat{u}_\theta \quad \text{and} \quad \dot{\hat{u}}_\theta = -\dot{\theta}\,\hat{k} \times \hat{u}_\theta = -\dot{\theta}\,\hat{u}_r. \tag{4}$$

Substituting Eqs. (4) into Eq. (3), with $\ddot{\theta} = 0$ because $\dot{\theta}$ is constant, we have

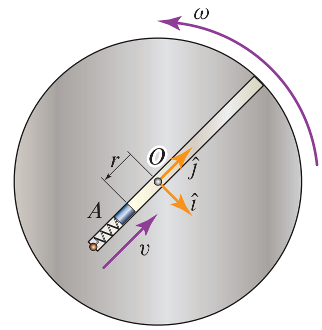
$$\vec{a} = -\dot{\theta}^2(\eta\theta + r_0)\,\hat{u}_r + 2\eta\dot{\theta}^2\,\hat{u}_\theta. \tag{5}$$

Recalling that, for $r = 170\,\mu\text{m}$, $\theta = 2.000\,\text{rad}$, and recalling that $r_0 = 146\,\mu\text{m}$, $\eta = 12\,\mu\text{m}$, and $\dot{\theta} = 30{,}000\frac{2\pi}{60}\,\text{rad/s}$, we can evaluate $\vec{a}$ to obtain

$$\boxed{\vec{a} = \left(-1678\,\hat{u}_r + 236.9\,\hat{u}_\theta\right)\,\text{m/s}^2.}$$

Problem 2.135

A disk rotates about its center, which is the fixed point O. The disk has a straight channel whose centerline passes by O and within which a collar A is allowed to slide. If, when A passes by O, the speed of A relative to the channel is $v = 14\,\mathrm{m/s}$ and is increasing in the direction shown with a rate of $5\,\mathrm{m/s^2}$, determine the acceleration of A given that $\omega = 4\,\mathrm{rad/s}$ and is constant. Express the answer using the component system shown, which rotates with the disk. *Hint:* Apply the equation derived in Prob. 2.122 to the vector describing the position of A relative to O and then let $r = 0$.



Solution

Let $\vec{r}_A$ be the position of A relative to the fixed point O. Using the $(\hat{\imath}, \hat{\jmath})$ component system, $\vec{r}$ can be written as

$$\vec{r}_A = -r\,\hat{\jmath}. \tag{1}$$

Applying the equation derived in Problem 2.122, the acceleration of A is

$$\vec{a}_A = -\ddot{r}\,\hat{\jmath} - 2\vec{\omega}_r \times \dot{r}\,\hat{\jmath} + \dot{\vec{\omega}}_r \times \vec{r}_A + \vec{\omega}_r \times \big(\vec{\omega}_r \times \vec{r}_A\big), \tag{2}$$

where $\vec{\omega}_r$ is the angular velocity of the vector $\vec{r}$. When A is at O, $\vec{r} = \vec{0}$ so that Eq. (2) can be simplified to

$$\vec{a}_A\big|_{r=0} = -\ddot{r}\,\hat{\jmath} - 2\vec{\omega}_r \times \dot{r}\,\hat{\jmath}. \tag{3}$$

Now, we observe that

$$\vec{\omega}_r = \omega\,\hat{k}. \tag{4}$$

Substituting Eq. (4) in Eq. (3), we have

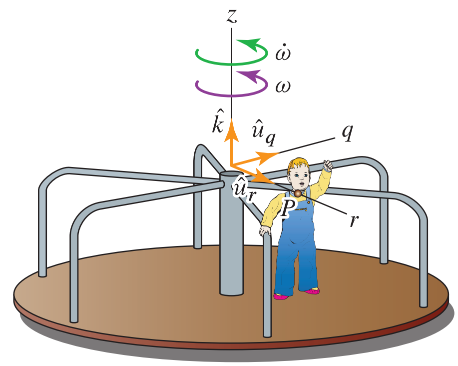
$$\vec{a}_A\big|_{r=0} = -\ddot{r}\,\hat{\jmath} - 2\omega\,\hat{k} \times \dot{r}\,\hat{\jmath} \quad \Rightarrow \quad \vec{a}_A\big|_{r=0} = 2\omega\dot{r}\,\hat{\imath} - \ddot{r}\,\hat{\jmath}. \tag{5}$$

Recalling that $\dot{r} = -v = -14\,\mathrm{m/s}$, $\ddot{r} = -5\,\mathrm{m/s^2}$, and $\omega = 4\,\mathrm{rad/s}$, we can evaluate the last of Eqs. (5) to obtain

$$\boxed{\vec{a}_A\big|_{r=0} = (-112.0\,\hat{\imath} + 5.000\,\hat{\jmath})\,\mathrm{m/s^2}.}$$

Problem 2.136

At the instant shown, the angular velocity and acceleration of the merry-go-round are as indicated in the figure. The distance of the child from the spin axis is r_P, so his acceleration is $\vec{a}_P = \ddot{r}_P\,\hat{u}_r + \dot{r}_P\,\dot{\hat{u}}_r + \dot{\vec{\omega}} \times r_P\,\hat{u}_r + \vec{\omega} \times \dot{r}_P\,\hat{u}_r + \vec{\omega} \times r_P\,\dot{\hat{u}}_r$. Assuming that the child is walking along a radial line, should the child walk outward or inward to make sure that he does not experience any sideways acceleration (i.e., in the direction of $\hat{u}_q$)?



Solution

Using the component system shown, the position of the child is $\vec{r}_P = r\,\hat{u}_r$, where r is the distance from the spin axis and $\hat{u}_r$ is the unit vector always pointing from the origin of the system toward P. Applying Eq. (2.48) on p. (2.48) of the textbook, we know that

$$\vec{v}_P = \dot{r}\,\hat{u}_r + \vec{\omega}_r \times r\,\hat{u}_r, \tag{1}$$

where $\vec{\omega}_r$ is the angular velocity of the vector $\vec{r}_P$, and where this angular velocity coincides with the angular velocity of the merry-go-round. Differentiating the above expression with respect to time, and replacing $\vec{\omega}_r$ with $\vec{\omega}$, we have

$$\vec{a}_P = \ddot{r}\,\hat{u}_r + \dot{r}\,\dot{\hat{u}}_r + \dot{\vec{\omega}} \times r\,\hat{u}_r + \vec{\omega} \times \dot{r}\,\hat{u}_r + \vec{\omega} \times r\,\dot{\hat{u}}_r. \tag{2}$$

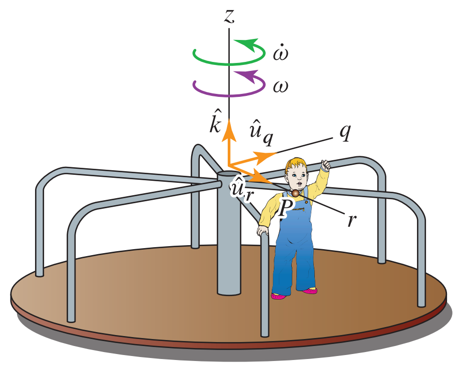
Using Eq. (2.46) on p. 81 of the textbook, we have that $\dot{\hat{u}}_r = \vec{\omega} \times \hat{u}_r$. This allows us to rewrite the acceleration of the child as

$$\vec{a}_P = \ddot{r}\,\hat{u}_r + 2\vec{\omega} \times \dot{r}\,\hat{u}_r + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times \big(\vec{\omega} \times \vec{r}\,\big). \tag{3}$$

From the expression above, we see that the terms that contribute to the acceleration in the direction of $\hat{u}_q$ are $2\vec{\omega} \times \dot{r}\,\hat{u}_r$ and $\dot{\vec{\omega}} \times \vec{r}$. Since $\vec{\omega}$ and $\dot{\vec{\omega}}$ are in the same direction then we must have $\dot{r} < 0$ if we hope that the terms will cancel. Hence, the child should move inward.

Problem 2.137

Assuming that the child shown is moving on the merry-go-round along a radial line, use the equation derived in Prob. 2.122 to determine the relation that ω, $\dot{\omega}$, r, and $\dot{r}$ must satisfy so that the child will not experience any sideways acceleration.



Solution

Using the component system shown, the position of the child is $\vec{r} = r\,\hat{u}_r$, where r is the distance from the spin axis and $\hat{u}_r$ is the unit vector always pointing from the origin of the system toward the child. Using the equation derived in Problem 2.122, we can express the acceleration of the child takes in the following form:

$$\vec{a} = \ddot{r}\,\hat{u}_r + 2\vec{\omega} \times \dot{r}\,\hat{u}_r + \dot{\vec{\omega}} \times \vec{r} - \vec{\omega} \times (\vec{\omega} \times \vec{r}), \tag{1}$$

where we have recognized that the angular velocity of the vector $\vec{r}$ is $\vec{\omega}$, which is the angular velocity of the merry-go-round. Using the component system shown in the figure, we have that

$$\vec{\omega} = \omega\,\hat{k} \quad \text{and} \quad \dot{\vec{\omega}} = \dot{\omega}\,\hat{k}, \tag{2}$$

since the direction of the unit vector $\hat{k}$ remains fixed. Recalling that $\vec{r} = r\,\hat{u}_r$, and substituting Eqs. (2) into Eq. (1), we have

$$\vec{a} = (\ddot{r} - \omega^2 r)\,\hat{u}_r + (2\omega\dot{r} + r\dot{\omega})\,\hat{u}_q. \tag{3}$$

Equation (3) shows that the component of the acceleration in the direction of $\hat{u}_q$ is

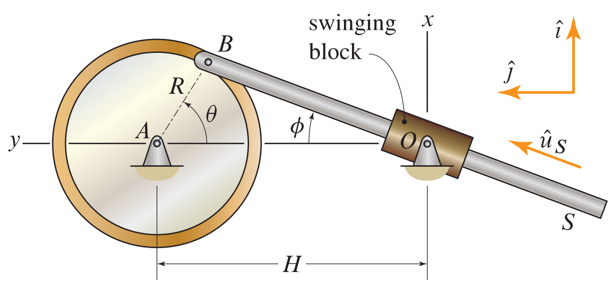
$$a_q = (2\omega\dot{r} + \dot{\omega}r). \tag{4}$$

Hence, in order for the child not to experience sideways acceleration, we must have

$$\boxed{2\omega\dot{r} + \dot{\omega}r = 0.}$$

Problem 2.138

The mechanism shown is called a *swinging block* slider crank. First used in various steam locomotive engines in the 1800s, this mechanism is often found in door-closing systems. If the disk is rotating with a constant angular velocity $\dot{\theta} = 60\,\text{rpm}$, $H = 4\,\text{ft}$, $R = 1.5\,\text{ft}$, and r is the distance between B and O, compute $\dot{r}$ and $\dot{\phi}$ when $\theta = 90°$. *Hint:* Apply Eq. (2.48) to the vector describing the position of B relative to O.



Solution

We can express the velocity of B in two ways. First, as the time derivative of the position vector $\vec{r}_{B/A}$ and second as the time derivative of the position vector $\vec{r}_{B/O}$. Referring to the figure in the problem statement, we can express these two position vectors as follows:

$$\vec{r}_{B/A} = R\,\hat{u}_{B/A} \quad \text{and} \quad \vec{r}_{B/O} = r\,\hat{u}_S, \tag{1}$$

where, as given the problem statement, r is the distance between B and O, and where we observe that the angular velocities of the unit vector in the above equations are

$$\vec{\omega}_{\hat{u}_{B/A}} = \dot{\theta}\,\hat{k} \quad \text{and} \quad \vec{\omega}_{\hat{u}_S} = -\dot{\phi}\,\hat{k}. \tag{2}$$

Hence, observing that $\dot{R} = 0$ since R is a constant, using Eq. (2.48) on p. 81 of the textbook, we have

$$\vec{v}_B = \dot{\vec{r}}_{B/A} = \dot{\theta}\,\hat{k} \times R\,\hat{u}_{B/A} \quad \text{and} \quad \vec{v}_B = \dot{\vec{r}}_{B/O} = \dot{r}\,\hat{u}_S - \dot{\phi}\,\hat{k} \times r\,\hat{u}_S. \tag{3}$$

Next, we observe that, for $\theta = 90°$, we have

$$\hat{u}_{B/A} = \hat{\imath}, \quad r = \sqrt{H^2 + R^2}, \quad \text{and} \quad \hat{u}_S = \frac{1}{\sqrt{R^2 + H^2}}(R\,\hat{\imath} + H\,\hat{\jmath}). \tag{4}$$

Substituting Eqs. (4) into Eqs. (3), for $\theta = 90°$, we have

$$\vec{v}_B\big|_{\theta=90°} = R\dot{\theta}\,\hat{\jmath} \quad \text{and} \quad \vec{v}_B\big|_{\theta=90°} = \frac{\dot{r}}{\sqrt{R^2 + H^2}}(R\,\hat{\imath} + H\,\hat{\jmath}) + H\dot{\phi}\,\hat{\imath} - R\dot{\phi}\,\hat{\jmath}. \tag{5}$$

Equating the two above expressions for $\vec{v}_B$ component by component, we have

$$\hat{\imath}: \quad \frac{\dot{r}R}{\sqrt{R^2 + H^2}} + H\dot{\phi} = 0, \tag{6}$$

$$\hat{\jmath}: \quad \frac{\dot{r}H}{\sqrt{R^2 + H^2}} - R\dot{\phi} = R\dot{\theta}. \tag{7}$$

Equations (6) and (7) form a system of two equations in the two unknowns $\dot{r}$ and $\dot{\phi}$ (at $\theta = 90°$) whose solution is

$$\dot{r}\big|_{\theta=90^\circ} = \frac{RH\dot{\theta}}{\sqrt{R^2 + H^2}} \quad \text{and} \quad \dot{\phi}\big|_{\theta=90^\circ} = -\frac{R^2\dot{\theta}}{R^2 + H^2}. \tag{8}$$

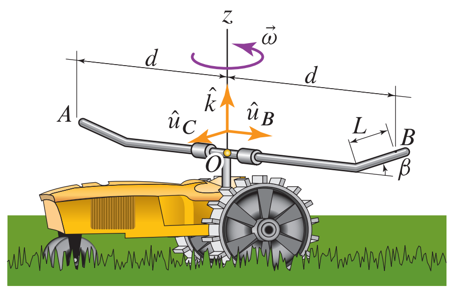
Recalling that we have $\dot{\theta} = 60\,\text{rpm} = 60(2\pi/60)\,\text{rad/s}$, $H = 4\,\text{ft}$, $R = 1.5\,\text{ft}$, we can evaluate the quantities in Eqs. (8) to obtain

$$\boxed{\dot{r}\big|_{\theta=90^\circ} = 8.825\,\text{ft/s} \quad \text{and} \quad \dot{\phi}\big|_{\theta=90^\circ} = -0.7746\,\text{rad/s}.}$$

Problem 2.139

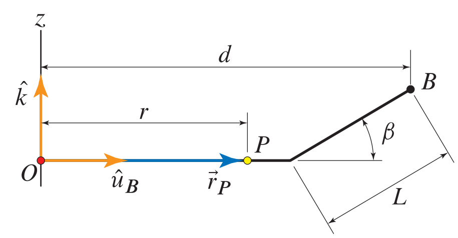
A sprinkler essentially consists of a pipe AB mounted on a hollow shaft. The water comes in the pipe at O and goes out the nozzles at A and B, causing the pipe to rotate. Assume that the particles of water move through the pipe at a constant rate *relative to the pipe* of 5 ft/s and that the pipe AB is rotating at a constant angular velocity of 250 rpm. In all cases, express the answers using the right-handed and orthogonal component system shown.

Determine the acceleration of the water particles when they are at $d/2$ from O (still within the horizontal portion of the pipe). Let $d = 7$ in.



Solution

Referring to the figure at the right, we consider a water particle P that is in the horizontal part of the tube. Using the component system shown, the position of P is



$$\vec{r}_P = r\,\hat{u}_B. \tag{1}$$

Then, applying Eq. (2.48) on p. 81 of the textbook, the velocity of P is

$$\vec{v}_P = \dot{r}\,\hat{u}_B + \vec{\omega} \times r\,\hat{u}_B, \tag{2}$$

where the angular velocity of the arm is also the angular velocity of the vector $\vec{r}_P$ as well as that of the unit vector $\hat{u}_B$. Differentiating Eq. (2) with respect to time, we have

$$\vec{a}_P = \ddot{r}\,\hat{u}_B + \dot{r}\,\dot{\hat{u}}_B + \dot{\vec{\omega}} \times r\,\hat{u}_B + \vec{\omega} \times \dot{r}\,\hat{u}_B + \vec{\omega} \times r\,\dot{\hat{u}}_B. \tag{3}$$

In Eqs. (2) and (3), $\dot{r}$ denotes the rate at which the water particles move relative to the arm. Therefore $\dot{r} = 5$ ft/s and $\ddot{r} = 0$. Also, $\vec{\omega} = \omega\,\hat{k}$, where $\omega = 250$ rpm is constant. In addition, using Eq. (2.46) on p. 81 of the textbook, we have that $\dot{\hat{u}}_B = \omega\,\hat{k} \times \hat{u}_B$. Therefore, Eq. (3) can be simplified to

$$\vec{a}_P = -2\dot{r}\omega\,\hat{u}_C - r\omega^2\,\hat{u}_B. \tag{4}$$

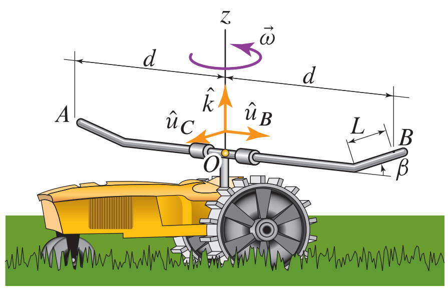
Recalling that $\dot{r} = 5$ ft/s and $\omega = 250$ rpm $= 250(2\pi/60)$ rad/s, for $r = d/2$, where $d = 7$ in. $= (7/12)$ ft, we can evaluate the above expression to obtain

$$\boxed{\vec{a}_P = (-261.8\,\hat{u}_C - 199.9\,\hat{u}_B)\ \text{ft/s}^2.}$$

Problem 2.140

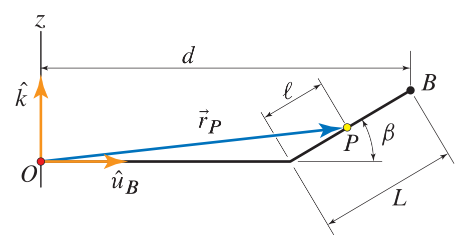
A sprinkler essentially consists of a pipe AB mounted on a hollow shaft. The water comes in the pipe at O and goes out the nozzles at A and B, causing the pipe to rotate. Assume that the particles of water move through the pipe at a constant rate *relative to the pipe* of 5 ft/s and that the pipe AB is rotating at a constant angular velocity of 250 rpm. In all cases, express the answers using the right-handed and orthogonal component system shown.

Determine the acceleration of the water particles right before they are expelled at B. Let $d = 7$ in., $\beta = 15°$, and $L = 2$ in. *Hint:* In this case, the vector describing the position of a water particle at B goes from O to B and is best written as $\vec{r} = r_B\,\hat{u}_B + r_z\,\hat{k}$.



Solution

Referring to the figure at the right, we consider a water particle P that is in the slanted part of the tube. Using the component system shown, the position of P is



$$\vec{r}_P = [d - (L - \ell)\cos\beta]\,\hat{u}_B + \ell \sin\beta\,\hat{k}. \tag{1}$$

Differentiating Eq. (1) with respect to time, we have

$$\vec{v}_P = \dot{\ell}\cos\beta\,\hat{u}_B + [d - (L - \ell)\cos\beta]\,\dot{\hat{u}}_B + \dot{\ell}\sin\beta\,\hat{k}, \tag{2}$$

where we have used the fact that d, L, and β are constant as well as the fact that $\hat{k}$ does not change direction. We observe that $\dot{\ell}$ is the rate at which the water particles move through the pipe, so that $\dot{\ell} = 5$ ft/s and $\ddot{\ell} = 0$ because $\dot{\ell}$ is constant. Using Eq. (2.46) on p. 81 of the textbook, we have that $\dot{\hat{u}}_B = \omega\,\hat{k} \times \hat{u}_B$, where we have recognized that the angular velocity of the arm, i.e., $\vec{\omega} = \omega\,\hat{k}$, is also the angular velocity of the unit vector $\hat{u}_B$. Hence, $\vec{a}_P$ can be rewritten as

$$\vec{v}_P = \dot{\ell}\cos\beta\,\hat{u}_B - \omega[d - (L - \ell)\cos\beta]\,\hat{u}_C + \dot{\ell}\sin\beta\,\hat{k}. \tag{3}$$

Differentiating Eq. (3) with respect to time, we have

$$\vec{a}_P = \dot{\ell}\cos\beta\,\dot{\hat{u}}_B - \omega\dot{\ell}\cos\beta]\,\hat{u}_C - \omega[d - (L - \ell)\cos\beta]\,\dot{\hat{u}}_C, \tag{4}$$

where we have used the fact that $\dot{\ell}$, ω, d, L, β, and $\hat{k}$ are constant. Using Eq. (2.46) on p. 81 of the textbook again to write $\dot{\hat{u}}_B = \omega\,\hat{k} \times \hat{u}_B = -\omega\,\hat{u}_C$ and $\dot{\hat{u}}_C = \omega\,\hat{k} \times \hat{u}_C = \omega\,\hat{u}_B$, we can rewrite $\vec{a}_P$ as

$$\vec{a}_P = -2\omega\dot{\ell}\cos\beta\,\hat{u}_C - \omega^2[d - (L - \ell)\cos\beta]\,\hat{u}_B. \tag{5}$$

Recalling that $\dot{\ell} = 5$ ft/s, $\omega = 250$ rpm $= 250(2\pi/60)$ rad/s, $\beta = 15°$, and $d = 7$ in. $= (7/12)$ ft, for $\ell = L = 2$ in. $= (2/12)$ ft, we can evaluate the above expression to obtain

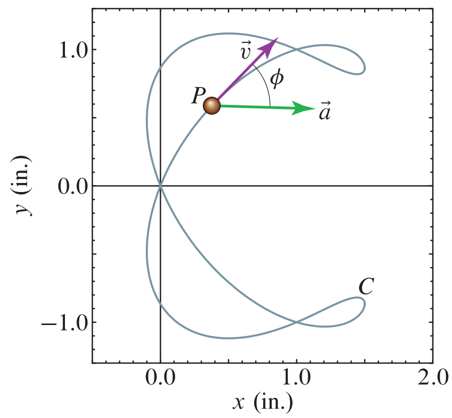
$$\boxed{\vec{a}_P = (-252.9\,\hat{u}_C - 399.8\,\hat{u}_B)\ \text{ft/s}^2.}$$

Problem 2.141

A particle P is moving along the curve C, whose equation is given by

$$\left(y^2 - x^2\right)(x-1)(2x-3) = 4\left(x^2 + y^2 - 2x\right)^2,$$

at a *constant* speed v_c. For any position on the curve C for which the radius of curvature is defined (i.e., *not* equal to infinity), what *must* be the angle ϕ between the velocity vector $\vec{v}$ and the acceleration vector $\vec{a}$?

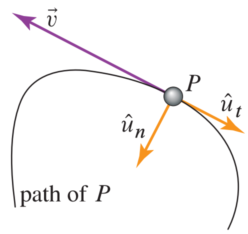


Solution

The speed is constant. This tells us that there is no component of acceleration in the direction of velocity. Therefore, the angle ϕ between $\vec{v}$ and $\vec{a}$ must be 90°.

Problem 2.142

A particle P is moving along a path with the velocity shown. Is the sketch of the normal-tangential component system at P correct?

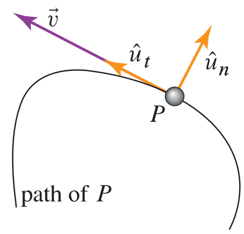


Solution

No, the unit vector $\hat{u}_t$ must point in the direction of $\vec{v}$.

Problem 2.143

A particle P is moving along a path with the velocity shown. Is the sketch of the normal-tangential component system at P correct?

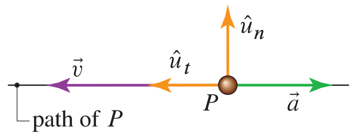


Solution

No, the unit vector $\hat{u}_n$ must point toward the concave side of the curve.

Problem 2.144

A particle P is moving along a straight line with the velocity and acceleration shown. What is wrong with the unit vectors shown in the figure?

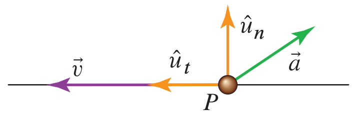


Solution

The unit vector $\hat{u}_n$ is not defined for a straight line.

Problem 2.145

A particle P is moving along some path with the velocity and acceleration shown. Can the path of P be the straight line shown?

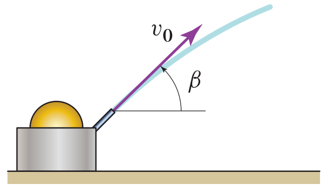


Solution

No, because the path is straight. It would need to be curved with a tangent at P coincident with $\hat{u}_t$ and concavity on the side of $\hat{u}_n$.

Problem 2.146

The water jet of a fountain is let out at a speed $v_0 = 80\,\text{ft/s}$ and at an angle $\beta = 60°$. Determine the radius of curvature of the jet at its highest point.



Solution

The tangent to the trajectory of the water jet at the highest point is horizontal. Therefore, the velocity at the highest point is completely horizontal and the normal direction coincides with the direction of gravity. We model the motion of the jet as projectile motion. This implies that the water particles have constant acceleration equal to the acceleration of gravity. In addition, the horizontal component of the velocity of the water particles is constant and therefore equal to the value it has when the jet is first emitted by the nozzle, namely, $v_0 \cos\beta$. Since at the highest point the acceleration, which is due to gravity, is along the normal direction, we have

$$a_n = \frac{v^2}{\rho} = g \quad \text{and} \quad a_t = 0, \tag{1}$$

where v is the speed of the water particles at the highest point on the trajectory. We have already argued that at the point in question the velocity vector is parallel to the horizontal direction. Therefore we must have

$$v = v_0 \cos\beta. \tag{2}$$

Substituting Eq. (2) into the first of Eqs. (1) and solving for ρ, we have

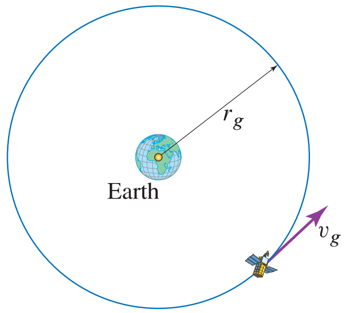
$$\rho = \frac{v_0^2 \cos^2\beta}{g}. \tag{3}$$

Recalling that $v_0 = 80\,\text{ft/s}$, $\beta = 60°$ and $g = 32.2\,\text{ft/s}^2$, we can evaluate Eq. (3) to obtain

$$\boxed{\rho = 49.69\,\text{ft.}}$$

Problem 2.147

A telecommunications satellite is made to orbit the Earth in such a way as to appear to hover in the same point in the sky as seen by a person standing on the surface of the Earth. Assuming that the satellite's orbit is circular with radius $r_g = 1.385 \times 10^8$ ft and knowing that the speed of the satellite is constant and equal to $v_g = 1.008 \times 10^4$ ft/s, determine the magnitude of the acceleration of the satellite.



Solution

Using normal-tangential components, the acceleration of the satellite is

$$\vec{a} = \dot{v}\,\hat{u}_t + \frac{v^2}{\rho}\,\hat{u}_n, \tag{1}$$

where v is the speed, ρ is the radius of curvature of the path, and where $\hat{u}_t$ and $\hat{u}_n$ are the unit vectors tangent and normal to the trajectory, respectively. We have

$$v = v_g = \text{constant}, \quad \dot{v} = 0, \quad \text{and} \quad \rho = r_g. \tag{2}$$

Substituting Eqs. (2) into Eq. (1), we have

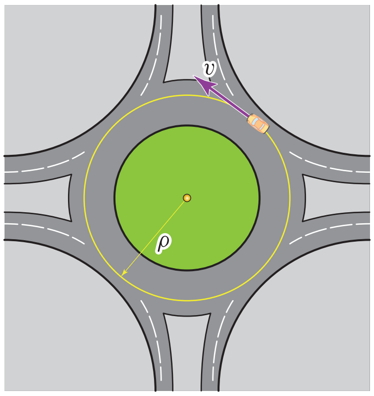
$$\vec{a} = \frac{v_g^2}{r_g}\,\hat{u}_n \quad \Rightarrow \quad |\vec{a}| = \frac{v_g^2}{r_g}. \tag{3}$$

Recalling that $v_g = 1.008 \times 10^4$ ft/s and $r_g = 1.385 \times 10^8$ ft, we can evaluate the last of Eqs. (3) to obtain

$$|\vec{a}| = 0.7336\,\text{ft/s}^2.$$

Problem 2.148

A car travels along a city roundabout with radius $\rho = 30\,\text{m}$. At the instant shown, the speed of the car is $v = 35\,\text{km/h}$ and the magnitude of the acceleration of the car is $4.5\,\text{m/s}^2$. If the car is increasing its speed, determine the time rate of change of the speed of the car at the instant shown.



Solution

Using normal-tangential components, the acceleration of the car can be expressed as

$$\vec{a} = \dot{v}\,\hat{u}_t + \frac{v^2}{\rho}\,\hat{u}_n, \tag{1}$$

where the unit vectors $\hat{u}_t$ and $\hat{u}_n$ are tangent and perpendicular to the path, respectively. Therefore, the magnitude of the acceleration is

$$|\vec{a}| = \sqrt{\dot{v}^2 + \left(\frac{v^2}{\rho}\right)^2}. \tag{2}$$

Solving the above equation for $\dot{v}$ and recalling that $\dot{v} > 0$ because the car is increasing its speed, we have

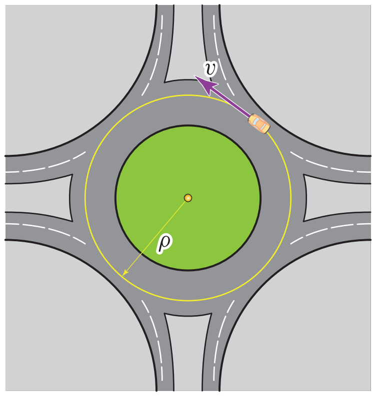
$$\dot{v} = \sqrt{|\vec{a}|^2 - \left(\frac{v^2}{\rho}\right)^2}. \tag{3}$$

Recalling that $|\vec{a}| = 4.5\,\text{m/s}^2$, $v = 35\,\text{km/h} = 35(1000/3600)\,\text{m/s}$, and $\rho = 30\,\text{m}$, we can evaluate Eq. (3) to obtain

$$\boxed{\dot{v} = 3.213\,\text{m/s}^2.}$$

Problem 2.149

A car travels along a city roundabout with radius $\rho = 100\,\text{ft}$. At the instant shown, the speed of the car is $v = 25\,\text{mph}$ and the speed is decreasing at the rate $8\,\text{ft/s}^2$. Determine the magnitude of the acceleration of the car at the instant shown.



Solution

Using normal-tangential components, the acceleration of the car can be expressed as

$$\vec{a} = \dot{v}\,\hat{u}_t + \frac{v^2}{\rho}\,\hat{u}_n, \tag{1}$$

where the unit vectors $\hat{u}_t$ and $\hat{u}_n$ are tangent and perpendicular to the path, respectively. Therefore, the magnitude of the acceleration is

$$|\vec{a}| = \sqrt{\dot{v}^2 + \left(\frac{v^2}{\rho}\right)^2}. \tag{2}$$

Recalling that $\dot{v} = -8\,\text{ft/s}^2$, $v = 25\,\text{mph} = 25(5280/3600)\,\text{ft/s}$, and $\rho = 100\,\text{ft}$, we can evaluate Eq. (2) to obtain

$$\boxed{|\vec{a}| = 15.64\,\text{ft/s}^2.}$$

Problem 2.150

Making the same assumptions stated in Example 2.15, consider the map of the Formula 1 circuit at Hockenheim in Germany and estimate the radius of curvature of the curves Südkurve and Nordkurve (at the locations indicated in gold).

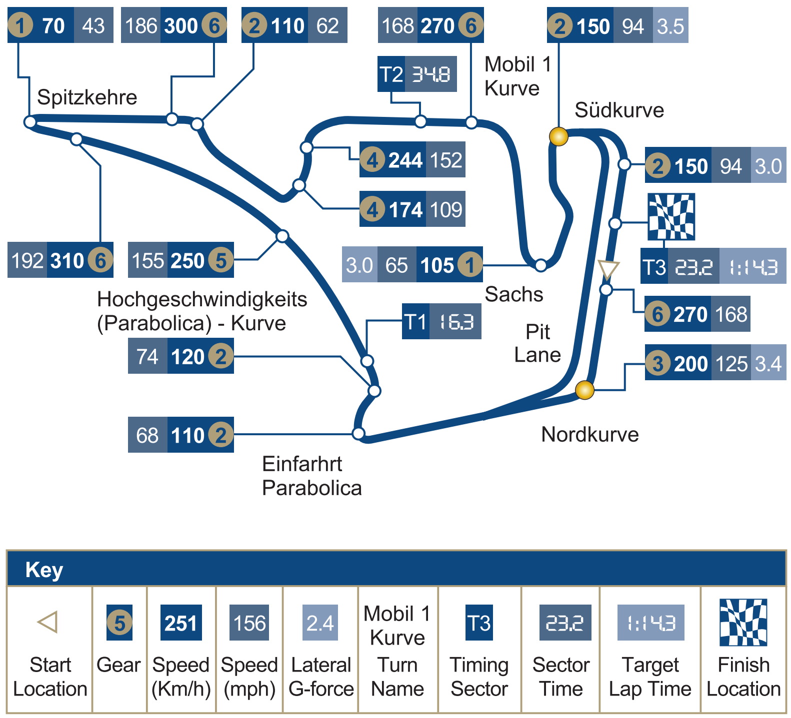


Photo credit: Courtesy of FIA

Solution

As was done in Example 2.15 on p. 95 of the textbook, we assume that by *lateral G-force* the Federation Internationale de l'Automobile (FiA) that compiled the map in the problem statement really meant to provide a measurement of the acceleration normal to the path of the racing cars expressed in "units of g," where g is the acceleration due to gravity. With this in mind, at the Südkurve the car is traveling at a speed of $150\,\mathrm{km/h}$ with an acceleration of $3.5g$. Therefore, denoting by $\rho_{\text{Südkurve}}$ the radius of curvature of the Südkurve, we must have

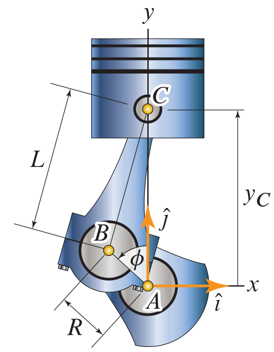
$$(a_n)_{\text{Südkurve}} = \frac{v^2_{\text{Südkurve}}}{\rho_{\text{Südkurve}}} \quad\Rightarrow\quad \rho_{\text{Südkurve}} = \frac{\left(150\frac{1000}{3600}\,\mathrm{m/s}\right)^2}{3.5(9.81\,\mathrm{m/s^2})} \quad\Rightarrow\quad \boxed{\rho_{\text{Südkurve}} = 50.56\,\mathrm{m}.}$$

Similarly, at the Nordkurve the car is traveling at a speed of $200\,\mathrm{km/h}$ with an acceleration of $3.4g$. Therefore, denoting by $\rho_{\text{Nordkurve}}$ the radius of curvature of the Nordkurve, we have

$$(a_n)_{\text{Nordkurve}} = \frac{v^2_{\text{Nordkurve}}}{\rho_{\text{Nordkurve}}} \quad\Rightarrow\quad \rho_{\text{Nordkurve}} = \frac{\left(200\frac{1000}{3600}\,\mathrm{m/s}\right)^2}{3.4(9.81\,\mathrm{m/s^2})} \quad\Rightarrow\quad \boxed{\rho_{\text{Nordkurve}} = 92.54\,\mathrm{m}.}$$

Problem 2.151

The position of the piston C, as a function of the crank angle ϕ and the lengths of the crank AB and connecting rod BC, is given by $y_C = R\cos\phi + L\sqrt{1-(R\sin\phi/L)^2}$ and $x_C = 0$. Using the component system shown, express $\hat{u}_t$, the unit vector tangent to the trajectory of C, as a function of the crank angle ϕ for $0 \le \phi \le 2\pi$ rad.

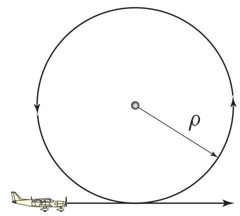


Solution

$\hat{u}_t = -\hat{\jmath}$ for $0 < \phi < \pi$ rad. $\hat{u}_t = \hat{\jmath}$ for $\pi < \phi < 2\pi$ rad. $\hat{u}_t$ is undefined at $\phi = 0$ and $\phi = 2\pi$ rad because the path of C folds back on itself at these two positions.

Problem 2.152

An aerobatics plane initiates the basic loop maneuver such that, at the bottom of the loop, the plane is going 140 mph, while subjecting the plane to approximately $4g$ of acceleration. Estimate the corresponding radius of the loop.



Solution

The acceleration of the airplane, expressed in normal tangential components, is

$$\vec{a} = \dot{v}\,\hat{u}_t + \frac{v^2}{\rho}\,\hat{u}_n, \tag{1}$$

where v is the speed of the airplane, $\hat{u}_t$ is the unit vector tangent to the path, $\hat{u}_n$ is the unit vector normal to the path, and ρ is the radius of curvature of the path. The change in speed as the airplane initiates the loop maneuver is negligible, so that, right at the beginning of the maneuver, we can simplify Eq. (1) to

$$\vec{a} = \frac{v^2}{\rho}\,\hat{u}_n. \tag{2}$$

Since $|\vec{a}| = 4g$, from Eq. (2) we have

$$4g = \frac{v^2}{\rho} \quad \Rightarrow \quad \rho = \frac{v^2}{4g}. \tag{3}$$

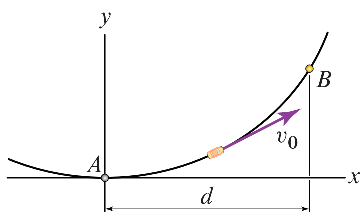
Recalling that $v = 140\,\text{mph} = 140(5280/3600)\,\text{ft/s}$ and $g = 32.2\,\text{ft/s}^2$, we can evaluate ρ to obtain

$$\boxed{\rho = 327.3\,\text{ft}.}$$

Problem 2.153

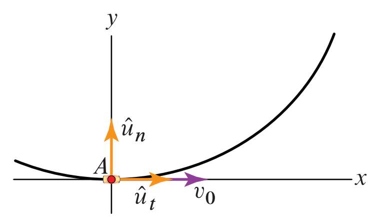
The portion of a race track between points A (corresponding to $x = 0$) and B is part of a parabolic curve described by the equation $y = \kappa x^2$, where κ is a constant. Let g denote the acceleration due to gravity.

Determine κ such that a car driving at constant speed $v_0 = 180$ mph experiences at A an acceleration with magnitude equal to $1.5g$.



Solution

Referring to the figure at the right, at point A the tangent and normal directions to the trajectory coincide with the x and the y axes, respectively. The expression of the acceleration in normal-tangential components is



$$\vec{a} = \dot{v}\,\hat{u}_t + \frac{v^2}{\rho}\,\hat{u}_n, \tag{1}$$

where v is the speed and ρ is the radius of curvature. Since the speed is constant and equal to v_0, we have that $\dot{v} = 0$ and Eq. (1) reduces to

$$\vec{a} = \frac{v_0^2}{\rho}\,\hat{u}_n. \tag{2}$$

We can determine ρ using Eq. (2.59) on p. 93 of the textbook, namely,

$$\rho = \frac{\left[1 + (dy/dx)^2\right]^{3/2}}{\left|d^2y/dx^2\right|} \quad\Rightarrow\quad \rho = \frac{\left[1 + (2\kappa x)^2\right]^{3/2}}{2\kappa}, \tag{3}$$

where, recalling that $y = \kappa x^2$, we have used the fact that $dy/dx = 2\kappa x$ and $d^2y/dx^2 = 2\kappa$. Substituting the last of Eqs. (3) into Eq. (2) gives

$$\vec{a} = \frac{2\kappa v_0^2}{\left[1 + (2\kappa x)^2\right]^{3/2}}\,\hat{u}_n. \tag{4}$$

Recalling that $|\vec{a}| = 1.5g$ for $x = 0$, from Eq. (4) we have

$$2\kappa v_0^2 = 1.5g = \tfrac{3}{2}g \quad\Rightarrow\quad \kappa = \frac{3g}{4v_0^2}. \tag{5}$$

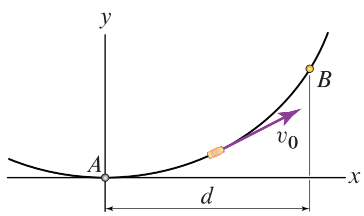
Recalling that $g = 32.2\,\text{ft/s}^2$ and $v_0 = 180\,\text{mph} = 180(5280/3600)\,\text{ft/s}$, we can evaluate the last of Eqs. (5) to obtain

$$\boxed{\kappa = 0.3465\times10^{-3}\,\text{ft}^{-1}.}$$

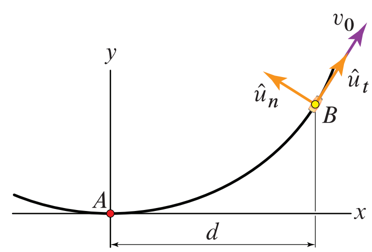
Problem 2.154

The portion of a race track between points A (corresponding to $x = 0$) and B is part of a parabolic curve described by the equation $y = \kappa x^2$, where κ is a constant. Let g denote the acceleration due to gravity.

If $\kappa = 0.4{\times}10^{-3}\,\text{ft}^{-1}$, determine d such that a car driving at constant speed $v_0 = 180\,\text{mph}$ experiences at B an acceleration with magnitude equal to g.



Solution



The expression of the acceleration in normal-tangential components is

$$\vec{a} = \dot{v}\,\hat{u}_t + \frac{v^2}{\rho}\,\hat{u}_n, \tag{1}$$

where v is the speed and ρ is the radius of curvature. Since the speed is constant and equal to v_0, $\dot{v} = 0$ and Eq. (1) reduces to

$$\vec{a} = \frac{v_0^2}{\rho}\,\hat{u}_n. \tag{2}$$

We can determine ρ using Eq. (2.59) on p. 93 of the textbook, namely,

$$\rho = \frac{\left[1 + (dy/dx)^2\right]^{3/2}}{\left|d^2y/dx^2\right|} \quad \Rightarrow \quad \rho = \frac{\left[1 + (2\kappa x)^2\right]^{3/2}}{2\kappa}, \tag{3}$$

where, recalling that $y = \kappa x^2$, we have used the fact that $dy/dx = 2\kappa x$ and $d^2y/dx^2 = 2\kappa$. Substituting the last of Eqs. (3) into Eq. (2), setting the magnitude of the result equal to the specified value of g, and letting $x = d$, we have

$$\frac{2\kappa v_0^2}{\left[1 + (2\kappa d)^2\right]^{3/2}} = g \quad \Rightarrow \quad d = \frac{1}{2\kappa}\sqrt{\left(\frac{2\kappa v_0^2}{g}\right)^{2/3} - 1}. \tag{4}$$

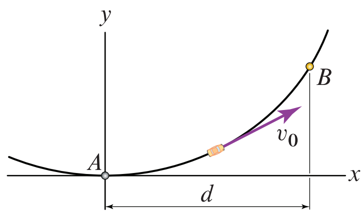
Recalling that $\kappa = 0.4\times10^{-3}\,\text{ft}^{-1}$, $v_0 = 180\,\text{mph} = 180(5280/3600)\,\text{ft/s}$, and $g = 32.2\,\text{ft/s}^2$, we can evaluate the last of Eqs. (4) to obtain

$$\boxed{d = 831.0\,\text{ft.}}$$

Problem 2.155

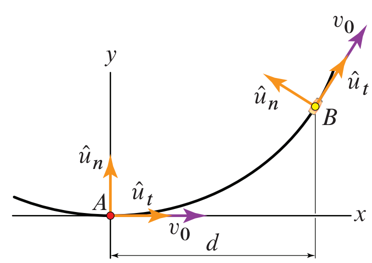
The portion of a race track between points A (corresponding to $x = 0$) and B is part of a parabolic curve described by the equation $y = \kappa x^2$, where κ is a constant. Let g denote the acceleration due to gravity.

Suppose a car travels from A to B with a constant speed $v_0 = 180\,\text{mph}$. Let $|\vec{a}|_{\text{min}}$ and $|\vec{a}|_{\text{max}}$ denote the minimum and maximum values of the magnitude of the acceleration, respectively. Determine $|\vec{a}|_{\text{min}}$ if $d = 1200\,\text{ft}$ and $|\vec{a}|_{\text{max}} = 1.5g$.



Solution

The expression of the acceleration in normal-tangential components is



$$\vec{a} = \dot{v}\,\hat{u}_t + \frac{v^2}{\rho}\,\hat{u}_n, \tag{1}$$

where v is the speed and ρ is the radius of curvature. Since the speed is constant and equal to v_0, $\dot{v} = 0$ and Eq. (1) reduces to

$$\vec{a} = \frac{v_0^2}{\rho}\,\hat{u}_n. \tag{2}$$

We can determine ρ using Eq. (2.59) on p. 93 of the textbook, namely,

$$\rho = \frac{\left[1 + (dy/dx)^2\right]^{3/2}}{\left|d^2y/dx^2\right|} \quad\Rightarrow\quad \rho = \frac{\left[1 + (2\kappa x)^2\right]^{3/2}}{2\kappa}, \tag{3}$$

where, recalling that $y = \kappa x^2$, we have used the fact that $dy/dx = 2\kappa x$ and $d^2y/dx^2 = 2\kappa$. Substituting Eq. (3) into Eq. (2), we have

$$\vec{a} = \frac{2\kappa v_0^2}{\left[1 + (2\kappa x)^2\right]^{3/2}}\,\hat{u}_n \quad\Rightarrow\quad |\vec{a}| = \frac{2\kappa v_0^2}{\left[1 + (2\kappa x)^2\right]^{3/2}}. \tag{4}$$

From the last of Eqs. (4) we see that the magnitude of the acceleration is maximum at for $x = 0$, i.e., at A and becomes smaller as x increases. This implies that, for $0 \le x \le d$,

$$|\vec{a}|_{\text{min}} = \frac{2\kappa v_0^2}{\left[1 + (2\kappa d)^2\right]^{3/2}} \quad\text{and}\quad |\vec{a}|_{\text{max}} = 2\kappa v_0^2. \tag{5}$$

Setting $|\vec{a}|_{\text{max}}$ from the last of Eqs. (5) equal to the specified maximum value of $1.5g$, we have

$$2\kappa v_0^2 = 1.5g = \tfrac{3}{2}g \quad\Rightarrow\quad \kappa = \frac{3g}{4v_0^2}. \tag{6}$$

Substituting the last of Eqs. (6) into the first of Eqs. (5), we then have that

$$|\vec{a}|_{\min} = \frac{3g}{2\left[1+\left(\frac{3gd}{2v_0^2}\right)^2\right]^{3/2}}, \tag{7}$$

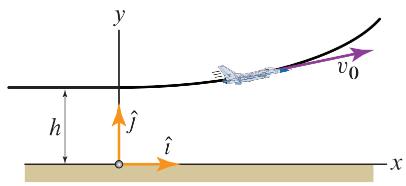
Recalling that $g = 32.2\,\text{ft/s}^2$, $d = 1200\,\text{ft}$, and $v_0 = 180\,\text{mph} = 180(5280/3600)\,\text{ft/s}$, we can evaluate Eq. (7) to obtain

$$\boxed{|\vec{a}|_{\min} = 21.95\,\text{ft/s}^2.}$$

Problem 2.156

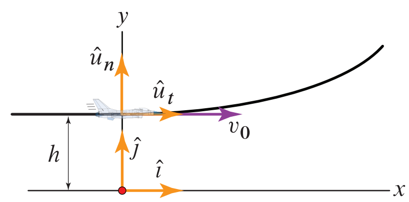
An airplane is flying straight and level at a constant speed v_0 when it starts climbing along a path described by the equation $y = h + \beta x^3$, where h and β are constants. Let g denote the acceleration due to gravity.

Determine the acceleration of the airplane at $x = 0$.



Solution

Referring to the figure on the right, at $x = 0$ the tangent and normal directions to the trajectory are parallel to the x and the y axes, respectively. The expression of the acceleration in normal-tangential components is



$$\vec{a} = \dot{v}\,\hat{u}_t + \frac{v^2}{\rho}\,\hat{u}_n, \tag{1}$$

where v is the speed and ρ is the radius of curvature. Since the speed is constant and equal to v_0, $\dot{v} = 0$ and Eq. (1) reduces to

$$\vec{a} = \frac{v_0^2}{\rho}\,\hat{u}_n. \tag{2}$$

We can determine ρ using Eq. (2.59) on p. 93 of the textbook, namely,

$$\rho = \frac{\left[1 + (dy/dx)^2\right]^{3/2}}{\left|d^2y/dx^2\right|} \quad \Rightarrow \quad \rho = \frac{\left[1 + \left(3\beta x^2\right)^2\right]^{3/2}}{6|\beta x|}, \tag{3}$$

where, recalling that $y = h + \beta x^3$, we have used the fact that $dy/dx = 3\beta x^2$ and $d^2y/dx^2 = 6\beta x$. Substituting the last of Eqs. (3) into Eq. (2), gives

$$\vec{a} = \frac{6|\beta x| v_0^2}{\left[1 + (3\beta x^2)^2\right]^{3/2}}\,\hat{u}_n. \tag{4}$$

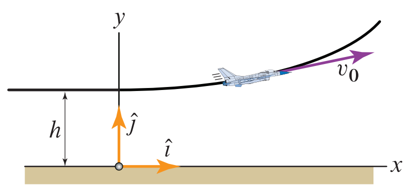
From Eq. (4), we see that for $x \to 0$ the acceleration vanishes. In addition, we observe that, before beginning to climb, the airplane is flying straight and level at constant speed so that its acceleration (right before beginning the climb) is equal to zero. Hence, the answer to this problem is

$$\boxed{\vec{a} = \vec{0}.}$$

Problem 2.157

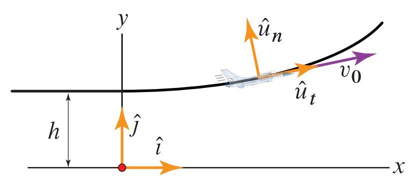
An airplane is flying straight and level at a constant speed v_0 when it starts climbing along a path described by the equation $y = h + \beta x^3$, where h and β are constants. Let g denote the acceleration due to gravity.

If $\beta = 0.05\times10^{-3}\,\text{m}^{-2}$ and v_0 remains constant, find v_0 such that the magnitude of the acceleration of the airplane is equal to $3g$ for $x = 300\,\text{m}$.



Solution

The expression of the acceleration in normal-tangential components is



$$\vec{a} = \dot{v}\,\hat{u}_t + \frac{v^2}{\rho}\,\hat{u}_n, \tag{1}$$

where v is the speed and ρ is the radius of curvature. Since the speed is constant and equal to v_0, $\dot{v} = 0$ and Eq. (1) reduces to

$$\vec{a} = \frac{v_0^2}{\rho}\,\hat{u}_n. \tag{2}$$

We can determine ρ using Eq. (2.59) on p. 93 of the textbook, namely,

$$\rho = \frac{\left[1 + (dy/dx)^2\right]^{3/2}}{\left|d^2y/dx^2\right|} \quad\Rightarrow\quad \rho = \frac{\left[1 + \left(3\beta x^2\right)^2\right]^{3/2}}{6|\beta x|}, \tag{3}$$

where, recalling that $y = h + \beta x^3$, we have used the fact that $dy/dx = 3\beta x^2$ and $d^2y/dx^2 = 6\beta x$. Substituting the last of Eqs. (3) into Eq. (2), gives

$$\vec{a} = \frac{6|\beta x| v_0^2}{\left[1 + (3\beta x^2)^2\right]^{3/2}}\,\hat{u}_n. \tag{4}$$

Setting the magnitude of the vector $\vec{a}$ in Eq. (4) equal to the specified value of $3g$, we have

$$\frac{6|\beta x| v_0^2}{\left[1 + (3\beta x^2)^2\right]^{3/2}} = 3g \quad\Rightarrow\quad v_0 = \sqrt{\frac{g\left[1 + (3\beta x^2)^2\right]^{3/2}}{2|\beta x|}}. \tag{5}$$

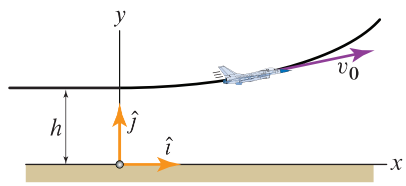
Recalling that $g = 9.81\,\text{m/s}^2$, $\beta = 0.05\times10^{-3}\,\text{m}^{-2}$, and $x = 300\,\text{m}$, we can evaluate the last of Eqs. (5) to obtain

$$\boxed{v_0 = 900.7\,\text{m/s}.}$$

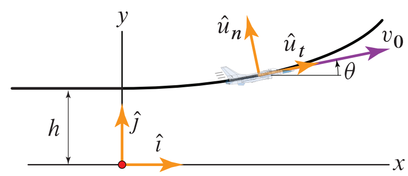
Problem 2.158

An airplane is flying straight and level at a constant speed v_0 when it starts climbing along a path described by the equation $y = h + \beta x^3$, where h and β are constants. Let g denote the acceleration due to gravity.

If $v_0 = 600\,\text{km/h}$ and $\beta = 0.025{\times}10^{-4}\,\text{m}^{-2}$, determine the acceleration of the airplane for $x = 350\,\text{m}$ and express it in the Cartesian component system shown.



Solution



The expression of the acceleration in normal-tangential components is

$$\vec{a} = \dot{v}\,\hat{u}_t + (v^2/\rho)\,\hat{u}_n, \tag{1}$$

where v is the speed and ρ is the radius of curvature. Since the speed is constant and equal to v_0, $\dot{v} = 0$ and Eq. (1) reduces to

$$\vec{a} = \frac{v_0^2}{\rho}\,\hat{u}_n. \tag{2}$$

We can determine ρ using Eq. (2.59) on p. 93 of the textbook, namely,

$$\rho = \frac{\left[1 + (dy/dx)^2\right]^{3/2}}{\left|d^2y/dx^2\right|} \quad\Rightarrow\quad \rho = \frac{\left[1 + \left(3\beta x^2\right)^2\right]^{3/2}}{6|\beta x|}, \tag{3}$$

where, since $y = h + \beta x^3$, $dy/dx = 3\beta x^2$ and $d^2y/dx^2 = 6\beta x$. Substituting the last of Eqs. (3) into Eq. (2) gives

$$\vec{a} = \frac{6|\beta x| v_0^2}{\left[1 + (3\beta x^2)^2\right]^{3/2}}\,\hat{u}_n. \tag{4}$$

To express $\vec{a}$ via $\hat{\imath}$ and $\hat{\jmath}$, we need to write $\hat{u}_n$ in terms of $\hat{\imath}$ and $\hat{\jmath}$. Referring to the figure at the right of Eq. (1), letting θ be the angle orienting the tangent to the trajectory relative to the x axis, we have

$$\hat{u}_n = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}, \tag{5}$$

where, recalling that $\beta = 0.025{\times}10^{-4}\,\text{m}^{-2}$ and $x = 350\,\text{m}$,

$$\theta = \tan^{-1}\!\left(\frac{dy}{dx}\right) = \tan^{-1}(3\beta x^2) \quad\Rightarrow\quad \theta = 42.58^\circ, \tag{6}$$

Substituting Eq. (5) into Eq. (4) we have

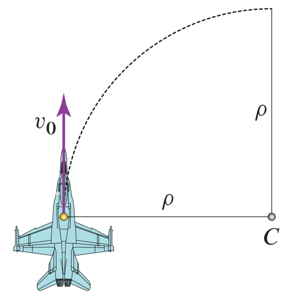
$$\vec{a} = \frac{6|\beta x| v_0^2}{\left[1 + (3\beta x^2)^2\right]^{3/2}}(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}). \tag{7}$$

Using the expression of θ in the first of Eqs. (6), and recalling that $\beta = 0.025{\times}10^{-3}\,\text{m}^{-2}$, $x = 350\,\text{m}$, and $v_0 = 600\,\text{km/h} = 600\frac{1000}{3600}\,\text{m/s}$, we can evaluate Eq. (7) to obtain

$$\boxed{\vec{a} = (-39.40\,\hat{\imath} + 42.88\,\hat{\jmath})\,\text{m/s}^2.}$$

Problem 2.159

A jet is flying at a constant speed $v_0 = 750\,\text{mph}$ while performing a constant speed circular turn. If the magnitude of the acceleration needs to remain constant and equal to $9g$, where g is the acceleration due to gravity, determine the radius of curvature of the turn.



Solution

Expressed in normal tangential components, the acceleration of the airplane is

$$\vec{a} = \dot{v}\,\hat{u}_t + \frac{v^2}{\rho}\,\hat{u}_n, \tag{1}$$

where v is the speed of the airplane, $\hat{u}_t$ is the unit vector tangent to the path in the direction of motion, $\hat{u}_n$ is the unit vector normal to the path and pointing toward the concave side of the path, and ρ is the radius of curvature of the path. Since the speed is constant, $\dot{v} = 0$ and Eq. (1) simplifies to

$$\vec{a} = \frac{v^2}{\rho}\,\hat{u}_n, \tag{2}$$

Recalling that $|\vec{a}| = 9g$, from the above equation we have that

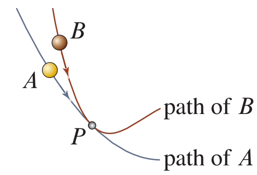
$$9g = \frac{v^2}{\rho} \quad \Rightarrow \quad \rho = \frac{v^2}{9g}. \tag{3}$$

Recalling that $v = v_0 = 750\,\text{mph} = 750(5280/3600)\,\text{ft/s}$ and $g = 32.2\,\text{ft/s}^2$, we have

$$\boxed{\rho = 4175\,\text{ft.}}$$

Problem 2.160

Particles A and B are moving in the plane with the same constant speed v, and their paths are tangent at P. Do these particles have zero acceleration at P? If not, do these particles have the same acceleration at P?



Solution

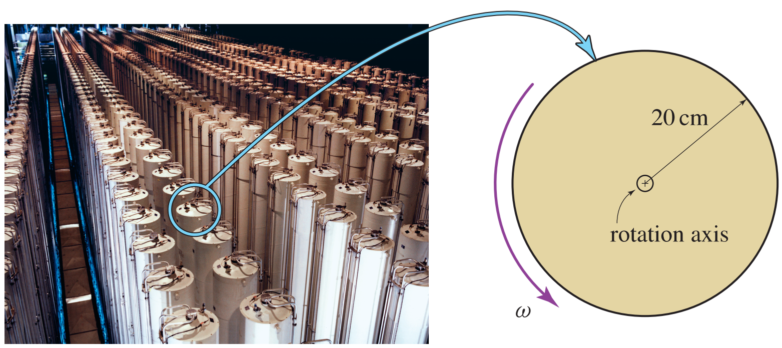
Without knowledge of the curvature for the two curves it is not possible to answer the question. If each curve had zero curvature at P, then the acceleration of the particles at P would be equal to zero because the particles are moving with constant speed. If the two curves at P had the same nonzero curvature, then the acceleration of the two particles at P would be the same. If the curves had different nonzero curvature at P, then the accelerations of the two particles would be different.

Problem 2.161

Uranium is used in light water reactors to produce a controlled nuclear reaction for the generation of power. When first mined, uranium comes out as the oxide U_3O_8, 0.7% of which is the isotope U-235 and 99.3% the isotope U-238. For it to be used in a nuclear reactor, the concentration of U-235 must be in the 3–5% range. The process of increasing the percentage of U-235 is called *enrichment*, and it is done in a number of ways. One method uses centrifuges, which spin at very high rates to create artificial gravity. In these centrifuges, the heavy U-238 atoms concentrate on the outside of the cylinder (where the acceleration is largest), and the lighter U-235 atoms concentrate near the spin axis. Before centrifuging, the uranium is processed into gaseous uranium hexafluoride or UF_6, which is then injected into the centrifuge. Assuming that the radius of the centrifuge is 20 cm and that it spins at 70,000 rpm, determine

(a) The velocity of the outer surface of the centrifuge.

(b) The acceleration in g experienced by an atom of uranium that is on the inside of the outer wall of the centrifuge.



array of enrichment centrifuges   centrifuge cross section

Photo credit: Courtesy of the Department of Energy

Solution

Part (a). Since the speed v and the angular speed ω are related as $v = \omega\rho$, we have that the speed of points on the outer surface of the centrifuge is

$$v = \rho\omega \quad \Rightarrow \quad \boxed{v = 1466\,\text{m/s},} \tag{1}$$

where we have used the fact that $\omega = 70{,}000\,\text{rpm} = 70{,}000(2\pi/60)\,\text{rad/s}$ and $\rho = 20\,\text{cm} = 0.2000\,\text{m}$.

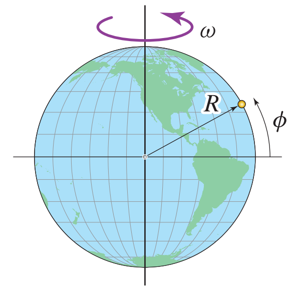
Part (b). Under the assumption that the centrifuge is spinning with a constant angular speed, the only component of acceleration of a point on the wall of the centrifuge will be the normal component. Hence, the acceleration experienced by an atom at the inside outer wall of the centrifuge is

$$a_n = \frac{v^2}{\rho} = \frac{(\omega\rho)^2}{g\rho}g = \frac{\omega^2\rho}{g}g \quad \Rightarrow \quad \boxed{\vec{a} = 1.096\times10^6 g\,\hat{u}_n,}$$

where the unit vector $\hat{u}_n$ always points from a point on the periphery of the centrifuge toward the center of the centrifuge, and where we have used the expression of v in Eq. (1) along with the fact that $\omega = 70{,}000\,\text{rpm} = 70{,}000(2\pi/60)\,\text{rad/s}$, $\rho = 20\,\text{cm} = 0.2000\,\text{m}$, and $g = 9.81\,\text{m/s}^2$.

Problem 2.162

Treating the center of the Earth as a fixed point, determine the magnitude of the acceleration of points on the surface of the earth as a function of the angle ϕ shown. Use $R = 6371$ km as the radius of the Earth.



Solution

Let ω denote the angular speed of the Earth. Since the Earth undergoes one full revolution per day, the angular speed ω is given by

$$\omega = \frac{1\,\text{rev}}{1\,\text{day}} = \frac{2\pi\,\text{rad}}{(24\,\text{h})(3600\,\text{s/h})} = \frac{\pi}{43{,}200}\,\text{rad/s}. \tag{1}$$

Let ρ denote the distance between the point indicated on the figure and the axis of rotation of the Earth, i.e., a point on the surface of the Earth characterized by the angle ϕ between the equator and the axis of rotation of the Earth. Then, we have we

$$\rho = R\cos\phi. \tag{2}$$

Next we observe that under the assumption that the angular speed of the Earth is constant, the only component of the acceleration of the point in question is the normal component. Hence, we must have

$$|\vec{a}| = a_n = \frac{v^2}{\rho} = \rho\omega^2 = R\omega^2\cos\phi. \tag{3}$$

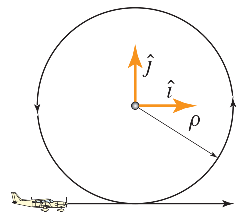
Recalling that $R = 6371\,\text{km} = 6{,}371{,}000\,\text{m}$, using the value of ω in Eq. (1), Eq. (3) can be evaluated to obtain

$$\boxed{\left|\vec{a}\right| = (33.69\times 10^{-3}\cos\phi)\,\text{m/s}^2.}$$

Problem 2.163

An airplane is flying straight and level at a speed $v_0 = 150\,\text{mph}$ and with a constant time rate of increase of speed $\dot{v} = 20\,\text{ft/s}^2$, when it starts to climb along a circular path with a radius of curvature $\rho = 2000\,\text{ft}$. The airplane maintains $\dot{v}$ constant for about 30 s.

Determine the acceleration of the airplane right at the start of the climb and express the result in the Cartesian component system shown.



Solution

Using normal-tangential components, the acceleration can be expressed as

$$\vec{a} = \dot{v}\,\hat{u}_t + \frac{v^2}{\rho}\,\hat{u}_n, \tag{1}$$

where the unit vectors $\hat{u}_t$ and $\hat{u}_n$ are parallel and normal to the path, respectively. At the bottom of the loop, we have

$$\hat{u}_t = \hat{\imath} \quad \text{and} \quad \hat{u}_n = \hat{\jmath}. \tag{2}$$

Substituting Eqs. (2) into Eq. (1), and recalling that at the start of the loop $v = v_0$, we have

$$\vec{a} = \dot{v}\,\hat{\imath} + \frac{v_0^2}{\rho}\,\hat{\jmath}. \tag{3}$$

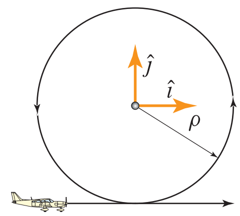
Recalling that $\dot{v} = 20\,\text{ft/s}^2$, $v_0 = 150\,\text{mph} = 150(5280/3600)\,\text{ft/s}$, and $\rho = 2000\,\text{ft}$, we can evaluate Eq. (3) to obtain

$$\boxed{\vec{a} = (20.00\,\hat{\imath} + 24.20\,\hat{\jmath})\,\text{ft/s}^2.}$$

Problem 2.164

An airplane is flying straight and level at a speed $v_0 = 150$ mph and with a constant time rate of increase of speed $\dot{v} = 20\,\text{ft/s}^2$, when it starts to climb along a circular path with a radius of curvature $\rho = 2000$ ft. The airplane maintains $\dot{v}$ constant for about 30 s.

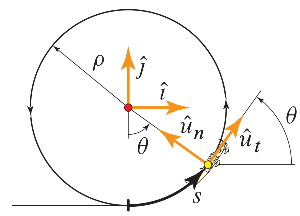
Determine the acceleration of the airplane 25 s after the start of the climb and express the result in the Cartesian component system shown.



Solution

Using normal-tangential components, the acceleration can be expressed as

$$\vec{a} = \dot{v}\,\hat{u}_t + \frac{v^2}{\rho}\,\hat{u}_n, \tag{1}$$



where the unit vectors $\hat{u}_t$ and $\hat{u}_n$ are parallel and normal to the path, respectively. Letting $t = 0$ be the time at which $v = v_0$ and recalling that $\dot{v}$ is constant, we can write

$$\dot{v} = \frac{dv}{dt} \quad\Rightarrow\quad dv = \dot{v}\,dt \quad\Rightarrow\quad \int_{v_0}^{v} dv = \int_0^t \dot{v}\,dt \quad\Rightarrow\quad v = v_0 + \dot{v}t. \tag{2}$$

Substituting the last of Eqs. (2) into Eq. (1), we have

$$\vec{a} = \dot{v}\,\hat{u}_t + \frac{(v_0 + \dot{v}t)^2}{\rho}\,\hat{u}_n. \tag{3}$$

The problem can be solved by evaluating Eq. (3) for $t = 25$ s as long as we are able to express $\hat{u}_t$ and $\hat{u}_n$ at $t = 25$ s in terms of $\hat{\imath}$ and $\hat{\jmath}$. To do so, referring to the figure at the right of Eq. (1), we have that

$$\hat{u}_t = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath} \quad\text{and}\quad \hat{u}_n = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}. \tag{4}$$

Substituting Eqs. (4) into Eq. (3) we have:

$$\vec{a} = \left[\dot{v}\cos\theta - \frac{(v_0 + \dot{v}t)^2}{\rho}\sin\theta\right]\hat{\imath} + \left[\dot{v}\sin\theta + \frac{(v_0 + \dot{v}t)^2}{\rho}\cos\theta\right]\hat{\jmath}. \tag{5}$$

Equation (5) implies that we will be able to provide the answer to the problem once we express θ as a function of time. To this end, again referring to the figure at the right of Eq. (1), we observe that

$$s = \theta\rho \quad\Rightarrow\quad \dot{s} = \dot{\theta}\rho \quad\Rightarrow\quad \frac{1}{\rho}(v_0 + \dot{v}t) = \frac{d\theta}{dt} \quad\Rightarrow\quad d\theta = \frac{1}{\rho}(v_0 + \dot{v}t)\,dt, \tag{6}$$

where we have used the fact that $\dot{s} = v$ and we have used the expression for v in the last of Eqs. (2). Now, observing that $\theta = 0$ for $t = 0$, we can integrate the last of Eqs. (6) as follows:

$$\int_0^\theta d\theta = \int_0^t \frac{1}{\rho}(v_0 + \dot{v}t)\,dt \quad\Rightarrow\quad \theta = \frac{1}{\rho}\big(v_0 t + \tfrac{1}{2}\dot{v}t^2\big). \tag{7}$$

Substituting the last of Eqs. (7) into Eq. (5) we have

$$\vec{a} = \left\{\dot{v}\cos\left[\frac{1}{\rho}\left(v_0 t + \tfrac{1}{2}\dot{v}t^2\right)\right] - \frac{(v_0 + \dot{v}t)^2}{\rho}\sin\left[\frac{1}{\rho}\left(v_0 t + \tfrac{1}{2}\dot{v}t^2\right)\right]\right\}\hat{\imath} + \left\{\dot{v}\sin\left[\frac{1}{\rho}\left(v_0 t + \tfrac{1}{2}\dot{v}t^2\right)\right] + \frac{(v_0 + \dot{v}t)^2}{\rho}\cos\left[\frac{1}{\rho}\left(v_0 t + \tfrac{1}{2}\dot{v}t^2\right)\right]\right\}\hat{\jmath}. \tag{8}$$

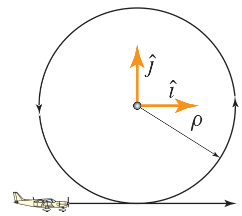
Recalling that $\dot{v} = 20\,\text{ft/s}^2$, $v_0 = 150\,\text{mph} = 150(5280/3600)\,\text{ft/s}$, and $\rho = 2000\,\text{ft}$, we can evaluate Eq. (8) for $t = 25\,\text{s}$ to obtain

$$\vec{a} = (121.2\,\hat{\imath} + 230.0\,\hat{\jmath})\,\text{ft/s}^2.$$

Problem 2.165

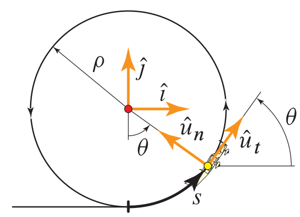
An airplane is flying straight and level at a speed $v_0 = 150\,\text{mph}$ and with a constant time rate of increase of speed $\dot{v} = 20\,\text{ft/s}^2$, when it starts to climb along a circular path with a radius of curvature $\rho = 2000\,\text{ft}$. The airplane maintains $\dot{v}$ constant for about 30 s.

Determine the acceleration of the airplane after it has traveled 150 ft along the path and express the result in the Cartesian component system shown.



Solution

Using normal-tangential components, the acceleration can be expressed as



$$\vec{a} = \dot{v}\,\hat{u}_t + \frac{v^2}{\rho}\,\hat{u}_n, \tag{1}$$

where the unit vectors $\hat{u}_t$ and $\hat{u}_n$ are parallel and normal to the path, respectively. Letting $s = 0$ correspond to $v = v_0$, recalling that $\dot{v}$ is constant, and using the chain rule, we can write

$$\dot{v} = v\frac{dv}{ds} \quad \Rightarrow \quad v\,dv = \dot{v}\,ds \quad \Rightarrow \quad \int_{v_0}^{v} v\,dv = \int_0^s \dot{v}\,ds \quad \Rightarrow \quad v^2 = v_0^2 + 2\dot{v}s. \tag{2}$$

Substituting the last of Eqs. (2) into Eq. (1), we have

$$\vec{a} = \dot{v}\,\hat{u}_t + \frac{v_0^2 + 2\dot{v}s}{\rho}\,\hat{u}_n, \tag{3}$$

Referring to the figure at the right of Eq. (1), we now observe that we can provide the solution to the problem by evaluating Eq. (3) for $s = 150\,\text{ft}$ as long as we are able to express $\hat{u}_t$ and $\hat{u}_n$ for $s = 150\,\text{ft}$ in terms of $\hat{\imath}$ and $\hat{\jmath}$. To do so, we observe that

$$\hat{u}_t = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath} \quad \text{and} \quad \hat{u}_n = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}. \tag{4}$$

Substituting Eqs. (4) into Eq. (3) we can then express the acceleration of the airplane as follows:

$$\vec{a} = \left[\dot{v}\cos\theta - \frac{v_0^2 + 2\dot{v}s}{\rho}\sin\theta\right]\hat{\imath} + \left[\dot{v}\sin\theta + \frac{v_0^2 + 2\dot{v}s}{\rho}\cos\theta\right]\hat{\jmath}. \tag{5}$$

Equation (5) implies that we will be able to provide the answer to the problem once we express θ as a function of the path coordinate s. To this end, we observe that

$$s = \theta\rho \quad \Rightarrow \quad \theta = \frac{s}{\rho}. \tag{6}$$

Substituting the last of Eqs. (6) into Eq. (5) we have

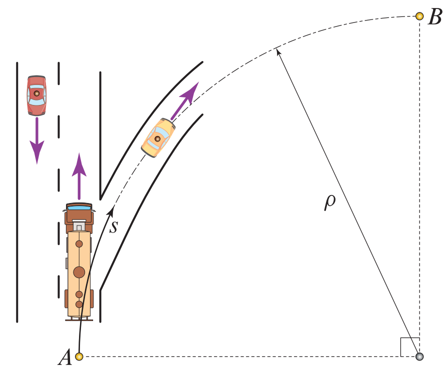
$$\vec{a} = \left[\dot{v}\cos\left(\frac{s}{\rho}\right) - \frac{v_0^2 + 2\dot{v}s}{\rho}\sin\left(\frac{s}{\rho}\right)\right]\hat{\imath} + \left[\dot{v}\sin\left(\frac{s}{\rho}\right) + \frac{v_0^2 + 2\dot{v}s}{\rho}\cos\left(\frac{s}{\rho}\right)\right]\hat{\jmath}. \tag{7}$$

Recalling that $\dot{v} = 20\,\mathrm{ft/s^2}$, $v_0 = 150\,\mathrm{mph} = 150(5280/3600)\,\mathrm{ft/s}$, and $\rho = 2000\,\mathrm{ft}$, we can evaluate Eq. (7) for $s = 150\,\mathrm{ft}$ to obtain

$$\boxed{\vec{a} = (17.91\,\hat{\imath} + 28.62\,\hat{\jmath})\,\mathrm{ft/s^2}.}$$

Problem 2.166

Suppose that a highway exit ramp is designed to be a circular segment of radius $\rho = 130$ ft. A car begins to exit the highway at A while traveling at a speed of 65 mph and goes by point B with a speed of 25 mph. Compute the acceleration vector of the car as a function of the arc length s, assuming that the tangential component of the acceleration is constant between points A and B.



Solution

Using normal tangential components, we have that the acceleration is given by

$$\vec{a} = a_t\,\hat{u}_t + a_n\,\hat{u}_n, \tag{1}$$

where $\hat{u}_t$ and $\hat{u}_n$ are the unit vectors tangent and normal to the path, respectively, and where, letting v denote the speed of the car,

$$a_t = \dot{v} \quad \text{and} \quad a_n = \frac{v^2}{\rho}, \tag{2}$$

where ρ is the radius of curvature of the path. The first of Eqs. (2) implies that the change in speed is completely due to the tangential acceleration. Since this component of acceleration is assumed to be constant, the speeds at A and B are related via the following constant acceleration equation:

$$v_B^2 - v_A^2 = 2a_t(s_B - s_A) \quad \Rightarrow \quad a_t = \frac{v_B^2 - v_A^2}{2(s_B - s_A)}. \tag{3}$$

The expression for the speed in terms of arc length s can be found again using constant acceleration equations, i.e.,

$$v^2(s) = v_A^2 + 2a_t(s - s_A) \quad \Rightarrow \quad v^2(s) = v_A^2 + \frac{v_B^2 - v_A^2}{s_B - s_A}(s - s_A). \tag{4}$$

Combining Eqs. (1)–(4), we can then write

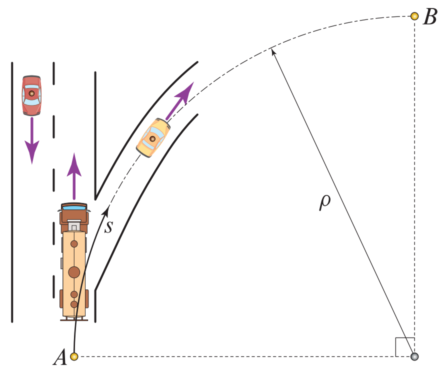
$$\vec{a} = \frac{v_B^2 - v_A^2}{2(s_B - s_A)}\,\hat{u}_t + \frac{1}{\rho}\left[v_A^2 + \frac{v_B^2 - v_A^2}{s_B - s_A}(s - s_A)\right]\hat{u}_n. \tag{5}$$

Recalling that $s_A = 0$, $v_A = 65\,\text{mph} = 65(5280/3600)\,\text{ft/s}$, $v_B = 25\,\text{mph} = 25(5280/3600)\,\text{ft/s}$, $\rho = 130\,\text{ft}$, and $s_B = \rho\frac{\pi}{2} = (65\pi)\,\text{ft}$, we can express Eq. (5) as follows:

$$\vec{a} = -\left(18.96\,\text{ft/s}^2\right)\hat{u}_t + \left[\left(69.91\,\text{ft/s}^2\right) - \left(0.2917\,\text{s}^{-2}\right)s\right]\hat{u}_n.$$

Problem 2.167

Suppose that a highway exit ramp is designed to be a circular segment of radius $\rho = 130$ ft. A car begins to exit the highway at A while traveling at a speed of 65 mph and goes by point B with a speed of 25 mph. Compute the acceleration vector of the car as a function of the arc length s, assuming that between A and B the speed was controlled so as to maintain constant the rate dv/ds.



Solution

Using normal-tangential components, the acceleration has the form

$$\vec{a} = \dot{v}\,\hat{u}_t + \frac{v^2}{\rho}\,\hat{u}_n, \tag{1}$$

where v is the speed of the car and ρ is the radius of curvature of the path. To solve the problem, we need to express both v and $\dot{v}$ as a function of s. We are told that the quantity dv/ds is constant. We begin by determining this constant, which we denote by K. Since $K = dv/ds$ we can separate the variables s and v as follows:

$$dv = K\,ds \quad \Rightarrow \quad \int_{v_A}^{v_B} dv = \int_{s_A}^{s_B} K\,ds \quad \Rightarrow \quad v_B - v_A = K(s_B - s_A) \quad \Rightarrow \quad K = \frac{v_B - v_A}{s_B - s_A}. \tag{2}$$

Now that K is known, we repeat the integration process using generic upper limits of integration, that is,

$$dv = K\,ds \quad \Rightarrow \quad \int_{v_A}^{v} dv = \int_{s_A}^{s} K\,ds \quad \Rightarrow \quad v - v_A = K(s - s_A). \tag{3}$$

Using the expression for K in the last of Eqs. (2), we have

$$v = v_A + \frac{v_B - v_A}{s_B - s_A}(s - s_A). \tag{4}$$

To determine $\dot{v}$ as a function of s, we use the chain rule to write

$$\dot{v} = \frac{dv}{dt} \quad \Rightarrow \quad \dot{v} = \frac{dv}{ds}\frac{ds}{dt} \quad \Rightarrow \quad \dot{v} = v\frac{dv}{ds}. \tag{5}$$

We know observe that v is given in Eq. (4) and $dv/ds = K$ is given in the last of Eqs. (2). Using these consideration, we can go back to Eq. (1) and rewrite it as follows:

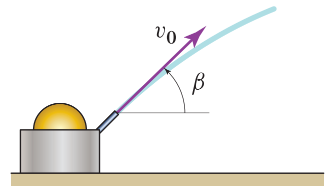
$$\vec{a} = \left[v_A + \frac{v_B - v_A}{s_B - s_A}(s - s_A)\right]\frac{v_B - v_A}{s_B - s_A}\,\hat{u}_t + \frac{1}{\rho}\left[v_A + \frac{v_B - v_A}{s_B - s_A}(s - s_A)\right]^2\hat{u}_n. \tag{6}$$

Recalling that $v_A = 65\,\text{mph} = 65(5280/3600)\,\text{ft/s}$, $v_A = 25\,\text{mph} = 25(5280/3600)\,\text{ft/s}$, $\rho = 130\,\text{ft}$, $s_A = 0$, $s_B = \rho\pi/2 = 65\pi$ ft, we can write Eq. (6) as follows:

$$\vec{a} = \big[\big(-27.39\,\text{ft/s}^2\big) + \big(0.08254\,\text{s}^{-2}\big)s\big]\,\hat{u}_t + \big[\big(69.91\,\text{ft/s}^2\big) - \big(0.4214\,\text{s}^{-2}\big)s + \big(0.0006349\,\text{ft}^{-1}{\cdot}\text{s}^{-2}\big)s^2\big]\,\hat{u}_n.$$

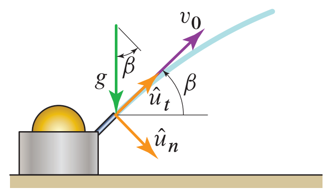
Problem 2.168

A water jet is ejected from the nozzle of a fountain with a speed $v_0 = 12\,\mathrm{m/s}$. Letting $\beta = 33°$, determine the rate of change of the speed of the water particles as soon as these are ejected as well as the corresponding radius of curvature of the water path.



Solution

Using the normal-tangential component system shown in the figure at the right, we can express the acceleration of a water particle as it is emitted from the nozzle as follows:



$$\vec{a} = \dot{v}\,\hat{u}_t + \frac{v^2}{\rho}\,\hat{u}_n, \tag{1}$$

where $v = v_0$ is the speed of the particle as it leaves the nozzle, $\hat{u}_t$ is the unit vector tangent to the path, $\hat{u}_n$ is the unit vector normal to the path, and ρ is the radius of curvature of the path. We assume that, as soon as a water particle leaves the nozzle, its motion is that of a projectile subjected only to the acceleration due to gravity. Therefore, the acceleration in question must be equal to

$$\vec{a} = g(-\sin\beta\,\hat{u}_t + \cos\beta\,\hat{u}_n). \tag{2}$$

Comparing Eqs. (1) and (2) we conclude that

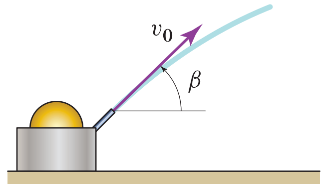
$$\dot{v} = -g\sin\beta \quad \text{and} \quad \rho = \frac{v_0^2}{g\cos\beta}, \tag{3}$$

where we have used the fact that $v = v_0$. Recalling that $v_0 = 12\,\mathrm{m/s}$, $g = 9.81\,\mathrm{m/s^2}$ and $\beta = 33°$, we can evaluate the results in Eqs. (3) to obtain

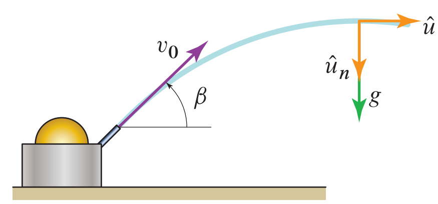
$$\boxed{\dot{v} = -5.343\,\mathrm{m/s^2} \quad \text{and} \quad \rho = 17.50\,\mathrm{m}.}$$

Problem 2.169

A water jet is ejected from the nozzle of a fountain with a speed v_0. Letting $\beta = 21°$, determine v_0 so that the radius of curvature at the highest point on the water arch is 10 ft.



Solution



The water particles are in projectile motion after they are emitted from the nozzle. Therefore, their acceleration is vertically downward and equal to the acceleration due to gravity. At the highest point on the water arch the vertical component of the velocity of the water particles is equal to zero. As is the case in projectile motion, the horizontal component is constant and therefore equal to the value at the beginning of the motion, namely, $v_0 \cos\beta$. Because this component of velocity is positive and it is the only nonzero component at the instant considered, $v_0 \cos\beta$ is also the value of the speed:

$$v = v_0 \cos\beta.$$

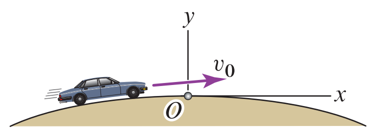
Recalling that speed and radius of curvature are related through the relation $a_n = v^2/\rho$ and observing that at the highest point on the water arch the normal direction coincides with the direction of gravity, we have that $a_n = g$, which implies

$$g = \frac{v^2}{\rho} \quad \Rightarrow \quad g = \frac{v_0^2 \cos^2\beta}{\rho} \quad \Rightarrow \quad v_0 = \sqrt{\frac{\rho g}{\cos^2\beta}} \quad \Rightarrow \quad \boxed{v_0 = 19.22\,\text{ft/s},}$$

where we have used $\beta = 21°$, $g = 32.2\,\text{ft/s}^2$, and $\rho = 10\,\text{ft}$.

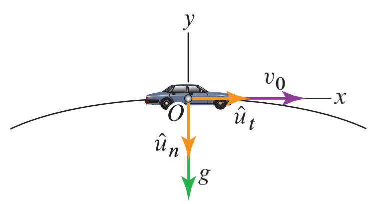
Problem 2.170

A car traveling with a speed $v_0 = 65$ mph almost loses contact with the ground when it reaches the top of the hill. Determine the radius of curvature of the hill at its top.



Solution

Using normal-tangential components, the acceleration of the car is



$$\vec{a} = \dot{v}\,\mathrm{t} + \frac{v^2}{\rho}\,\hat{u}_n, \tag{1}$$

where v is the speed of the car, ρ is the radius of curvature of the path, and $\hat{u}_t$ and $\hat{u}_n$ are the unit vectors tangent and normal to the path, respectively. The speed of the car is constant and equal to v_0. Therefore, Eq. (1) simplifies to

$$\vec{a} = \frac{v_0^2}{\rho}\,\hat{u}_n. \tag{2}$$

If the car were to lose contact with the ground, the car would be in projectile motion and its acceleration would be equal to that due to gravity. At the top of the hill the tangent to the path is horizontal and the normal direction coincides with that of gravity. Therefore, in view of Eq. (2), at the top of the hill we would have

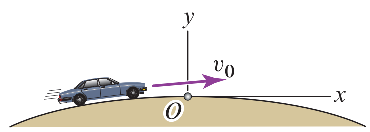
$$\vec{a} = g\,\hat{u}_n \quad \Rightarrow \quad g = \frac{v_0^2}{\rho} \quad \Rightarrow \quad \rho = \frac{v_0^2}{g}. \tag{3}$$

Recalling that $v_0 = 65\,\text{mph} = 65(5280/3600)\,\text{ft/s}$ and $g = 32.2\,\text{ft/s}^2$, we can evaluate ρ to obtain

$$\boxed{\rho = 282.2\,\text{ft.}}$$

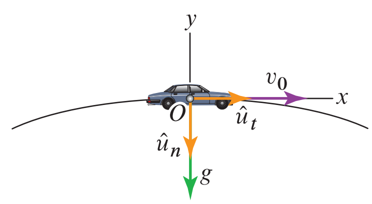
Problem 2.171

A car is traveling at a constant speed over a hill. If, using a Cartesian coordinate system with origin O at the top of the hill, the hill's profile is described by the function $y = -(0.003\,\mathrm{m}^{-1})x^2$, where x and y are in meters, determine the minimum speed at which the car would lose contact with the ground at the top of the hill. Express the answer in km/h.



Solution

Using normal-tangential components, the acceleration of the car is



$$\vec{a} = \dot{v}\,\mathrm{t} + \frac{v^2}{\rho}\,\hat{u}_n, \tag{1}$$

where v is the speed of the car, ρ is the radius of curvature of the path, and $\hat{u}_t$ and $\hat{u}_n$ are the unit vectors tangent and normal to the path, respectively. For a car moving at constant speed, Eq. (1) simplifies to

$$\vec{a} = \frac{v^2}{\rho}\,\hat{u}_n. \tag{2}$$

We will denote by v_{min} the minimum speed at which the car loses contact with the ground. If v were to exceed v_{min}, the car would be in projectile motion and its acceleration would be equal to that due to gravity. At the top of the hill the tangent to the path is horizontal and the normal direction coincides with that of gravity. Therefore, in view of Eq. (2), for $v = v_{\mathrm{min}}$, at the top of the hill, i.e., for $x = 0$, we would have

$$\vec{a} = g\,\hat{u}_n \quad \Rightarrow \quad g = \frac{v_{\mathrm{min}}^2}{\rho|_{x=0}} \quad \Rightarrow \quad v_{\mathrm{min}} = \sqrt{g\rho|_{x=0}}. \tag{3}$$

Recalling that the profile of the hill is $y = -(0.003\,\mathrm{m}^{-1})x^2$, and recalling that

$$\rho = \frac{\left[1 + (dy/dx)^2\right]^{3/2}}{\left|d^2y/dx^2\right|}, \tag{4}$$

we have

$$\rho(x) = \frac{\left[1 + \left(0.006000\,\mathrm{m}^{-1}\right)^2 x^2\right]^{3/2}}{0.006000\,\mathrm{m}^{-1}} \quad \Rightarrow \quad \rho(0) = \frac{1}{0.006000\,\mathrm{m}^{-1}}. \tag{5}$$

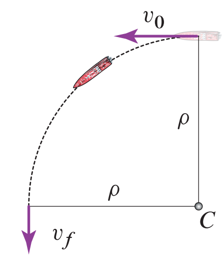
Substituting the last of Eqs. (5) into the last of Eqs. (3) and recalling that $g = 9.81\,\mathrm{m/s^2}$, we have

$$v_{\mathrm{min}} = 145.6\,\mathrm{km/h},$$

where, as requested in the problem statement we have expressed the final answer in km/h.

Problem 2.172

A race boat is traveling at a constant speed $v_0 = 130$ mph when it performs a turn with constant radius ρ to change its course by 90° as shown. The turn is performed while losing speed uniformly in time so that the boat's speed at the end of the turn is $v_f = 125$ mph. If the maximum allowed normal acceleration is equal to $2g$, where g is the acceleration due to gravity, determine the tightest radius of curvature possible and the time needed to complete the turn.



Solution

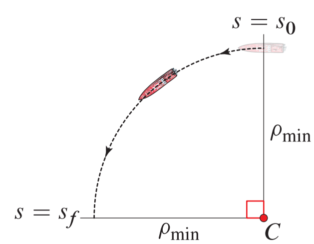
Let ρ_{min} denote the tightest radius of curvature. The normal acceleration is $a_n = v^2/\rho$. If the boat turns with a constant radius of curvature $\rho = \rho_{min}$, then the maximum normal acceleration occurs where the speed is maximum. Since $v_{max} = v_0$ (the boat's maximum speed is at the start of the turn), allowing a_n to take on its maximum allowed value of $2g$, we have

$$2g = v_0^2/\rho_{min} \quad \Rightarrow \quad \rho_{min} = v_0^2/(2g). \tag{1}$$

Recalling that $v_0 = 130\,\text{mph} = 130(5280/3600)\,\text{ft/s}$ and $g = 32.2\,\text{ft/s}^2$, from the last of Eqs. (1) we have

$$\boxed{\rho_{min} = 564.5\,\text{ft.}}$$

To determine the time needed to complete the turn, we begin by recalling that the speed decreases uniformly in time, i.e., $\dot{v} = a_t = \text{constant}$, where a_t is the tangential component of acceleration. Let s be the path coordinate, and let s_0 and s_f be the values of s at the beginning and the end of the turn, respectively. Since the boat travels over a quarter of a circle with radius ρ_{min}, we have



$$s_f - s_0 = \tfrac{1}{2}\pi\rho_{min} \quad \Rightarrow \quad s_f - s_0 = \pi v_0^2/(4g), \tag{2}$$

where we have used the last of Eqs. (1). Next, we can write

$$a_t = v\frac{dv}{ds} \quad \Rightarrow \quad a_t\,ds = v\,dv \quad \Rightarrow \quad \int_{s_0}^{s_f} a_t\,ds = \int_{v_0}^{v_f} v\,dv. \tag{3}$$

Recalling that a_t is constant and carrying out the integration in the last of Eqs. (3), we have

$$a_t(s_f - s_0) = \tfrac{1}{2}(v_f^2 - v_0^2) \quad \Rightarrow \quad a_t = \frac{2g(v_f^2 - v_0^2)}{\pi v_0^2}, \tag{4}$$

where we have used the expression for $s_f - s_0$ in the last of Eqs. (2). Now we denote by t_0 and t_f the time instants corresponding to $s = s_0$ and $s = s_f$, respectively. Then, recalling again that a_t is constant and using constant acceleration equations, we have

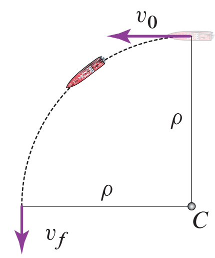
$$v_f = v_0 + a_t(t_f - t_0) \quad \Rightarrow \quad t_f - t_0 = \frac{v_f - v_0}{a_t} \quad \Rightarrow \quad t_f - t_0 = \frac{\pi v_0^2(v_f - v_0)}{2g(v_f^2 - v_0^2)} = \frac{\pi v_0^2}{2g(v_f + v_0)}. \tag{5}$$

Observing that $t_f - t_0$ is the time needed to perform the turn, recalling that $g = 32.2\,\text{ft/s}^2$, $v_0 = 130\,\text{mph} = 130(5280/3600)\,\text{ft/s}$, and $v_f = 125\,\text{mph} = 125(5280/3600)\,\text{ft/s}$, we can evaluate the last of Eqs. (5) to obtain

$$\boxed{t_f - t_0 = 4.742\,\text{s.}}$$

Problem 2.173

A race boat is traveling at a constant speed $v_0 = 130\,\text{mph}$ when it performs a turn with constant radius ρ to change its course by 90° as shown. The turn is performed while losing speed uniformly in time so that the boat's speed at the end of the turn is $v_f = 116\,\text{mph}$. If the magnitude of the acceleration is not allowed to exceed $2g$, where g is the acceleration due to gravity, determine the tightest radius of curvature possible and the time needed to complete the turn.



Solution

Using normal tangential components, the acceleration of the boat is given by

$$\vec{a} = a_t\,\hat{u}_t + a_n\,\hat{u}_n, \tag{1}$$

where $\hat{u}_t$ and $\hat{u}_n$ are unit vectors tangent and normal to the path. Denoting by ρ_{min} the minimum value of the radius of curvature, a_n is given by

$$a_n = \frac{v^2}{\rho_{\text{min}}}, \tag{2}$$

where v is the speed of the boat. The tangential acceleration a_t is constant. Therefore, applying constant acceleration equations, we have

$$v_f^2 - v_0^2 = 2a_t(s_f - s_0) \quad\Rightarrow\quad v_f^2 - v_0^2 = 2a_t\left(\frac{\pi}{2}\rho_{\text{min}}\right), \tag{3}$$

where $s_f - s_0 = (\pi/2)\rho_{\text{min}}$ is the distance covered by the boat along its path while performing the turn. Solving the above equation for a_t, we obtain

$$a_t = \frac{v_f^2 - v_0^2}{\pi\rho_{\text{min}}}. \tag{4}$$

The magnitude of the acceleration must not exceed the value $2g$. Recalling that $|\vec{a}| = \sqrt{a_t^2 + a_n^2}$, we can write $a_n^2 + a_t^2 = 4g^2$, which, using Eqs. (2) and (4), gives

$$\frac{v_0^4}{\rho_{\text{min}}^2} + \frac{\left(v_f^2 - v_0^2\right)^2}{\pi^2\rho_{\text{min}}^2} = 4g^2 \quad\Rightarrow\quad \rho_{\text{min}} = \frac{1}{2g}\sqrt{v_0^4 + \frac{1}{\pi^2}\left(v_f^2 - v_0^2\right)^2}. \tag{5}$$

Recalling that $v_0 = 130\,\text{mph} = 130(5280/3600)\,\text{ft/s}$, $v_f = 116\,\text{mph} = 116(5280/3600)\,\text{ft/s}$, and $g = 32.2\,\text{ft/s}^2$, we can evaluate the last of Eqs. (5) to obtain

$$\boxed{\rho_{\text{min}} = 565.7\,\text{ft}.}$$

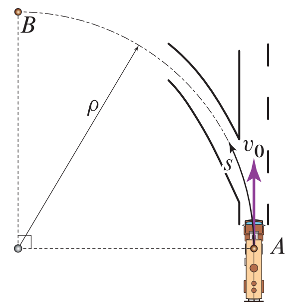
We denote the time needed to perform the turn by $t_f - t_0$. Because a_t is constant, we must have $v_f = v_0 + a_t(t_f - t_0)$, which, using Eq. (4) and the last of Eqs. (5), after simplification, gives

$$t_f - t_0 = \frac{\pi}{2g(v_f + v_0)}\sqrt{v_0^4 + \frac{1}{\pi^2}\left(v_f^2 - v_0^2\right)^2} \quad\Rightarrow\quad \boxed{t_f = 4.926\,\text{s},}$$

where we have used fact that $g = 32.2\,\text{ft/s}^2$, $v_0 = 130\,\text{mph} = 130(5280/3600)\,\text{ft/s}$, and $v_f = 116\,\text{mph} = 116(5280/3600)\,\text{ft/s}$.

Problem 2.174

A truck enters an exit ramp with an initial speed v_0. The ramp is a circular arc with radius ρ. Derive an expression for the magnitude of the acceleration of the truck as a function of the path coordinate s (and the parameters v_0 and ρ) if the truck stops at B and travels from A to B with a constant rate of change of the speed with respect to s.



Solution

Using normal-tangential components, the acceleration of the truck is

$$\vec{a} = \dot{v}\,\hat{u}_t + \frac{v^2}{\rho}\,\hat{u}_n, \tag{1}$$

where v is the speed of the truck, ρ is the radius of curvature of the path, and where $\hat{u}_t$ and $\hat{u}_n$ are the unit vectors tangent and normal to the path, respectively. The problem statement states that the quantity dv/ds is constant. Letting κ denote the constant in question, we can then write

$$\kappa = dv/ds \quad \Rightarrow \quad \kappa\,ds = dv. \tag{2}$$

Keeping in mind that the truck moves along a circular arc of radius ρ, let $s_A = 0$ and $s_B = \rho\pi/2$ be the values of the path coordinate s at A and B, respectively. Also, let $v_A = v_0$ and $v_B = 0$ be the values of speed for $s = s_A$ and $s = s_B$, respectively. Then, we can integrate the last of Eqs. (2) as follows:

$$\int_0^{\rho\pi/2} \kappa\,ds = \int_{v_0}^{0} dv \quad \Rightarrow \quad \tfrac{1}{2}\kappa\rho\pi = -v_0 \quad \Rightarrow \quad \kappa = -\frac{2v_0}{\pi\rho}. \tag{3}$$

Now let v be the value of speed corresponding to the generic value s of the path coordinate. Then, substituting the last of Eqs. (3) into the last of Eqs. (2), we can integrate again as follows:

$$\int_0^{s} -\frac{2v_0}{\pi\rho}\,ds = \int_{v_0}^{v} dv \quad \Rightarrow \quad -\frac{2v_0}{\pi\rho}s = v - v_0 \quad \Rightarrow \quad v = v_0 - \frac{2v_0}{\pi\rho}s. \tag{4}$$

Now we recall that, using the chain rule, we can write

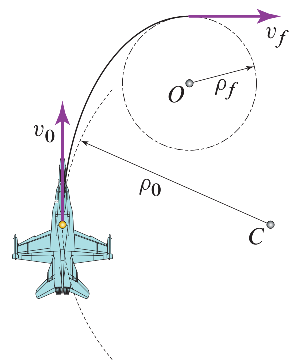
$$\dot{v} = \frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} \quad \Rightarrow \quad \dot{v} = \frac{dv}{ds}v \quad \Rightarrow \quad \dot{v} = -\frac{2v_0}{\pi\rho}\left(v_0 - \frac{2v_0}{\pi\rho}s\right), \tag{5}$$

where we have used the expression for v in the last of Eqs. (4), as well as the expression for $\kappa = dv/ds$ in the last of Eqs. (3). Referring to Eq. (1), we have that $|\vec{a}| = \sqrt{\dot{v}^2 + (v^2/\rho)^2}$. Therefore, using the last of Eqs. (4) and the last of Eqs.(5), after simplification, we have

$$|\vec{a}| = \frac{1}{\rho}\left(v_0 - \frac{2v_0}{\pi\rho}s\right)\sqrt{\frac{4v_0^2}{\pi^2} + \left(v_0 - \frac{2v_0}{\pi\rho}s\right)^2}.$$

Problem 2.175

A jet is flying straight and level at a speed $v_0 = 1100\,\mathrm{km/h}$ when it turns to change its course by 90° as shown. In an attempt to progressively tighten the turn, the speed of the plane is uniformly decreased in time while keeping the normal acceleration constant and equal to $8g$, where g is the acceleration due to gravity. At the end of the turn, the speed of the plane is $v_f = 800\,\mathrm{km/h}$. Determine the radius of curvature ρ_f at the end of the turn and the time t_f that the plane takes to complete its change in course.



Solution

Using normal-tangential components, the acceleration of the jet is

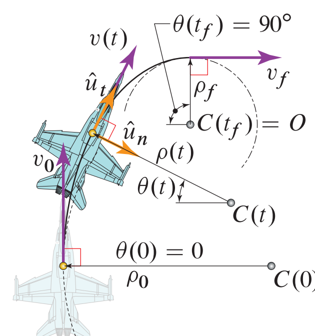
$$\vec{a} = \dot{v}\,\hat{u}_t + \frac{v^2}{\rho}\,\hat{u}_n, \tag{1}$$

where v is the speed of the jet, ρ is the radius of curvature of the path, and where $\hat{u}_t$ and $\hat{u}_n$ are the unit vectors tangent and normal to the path, respectively. The term v^2/ρ in Eq. (1) is the normal component of the acceleration. Since this component of the acceleration is constant and equal to $8g$, at the end of the turn we can write

$$v_f^2/\rho_f = 8g \quad \Rightarrow \quad \rho_f = v_f^2/(8g) \quad \Rightarrow \quad \boxed{\rho_f = 629.2\,\mathrm{m},} \tag{2}$$

where we have used the fact that $v_f = 800\,\mathrm{km/h} = 800(1000/3600)\,\mathrm{m/s}$ and $g = 9.81\,\mathrm{m/s^2}$.

Now, we need to relate the speed to the fact that the plane undergoes a change of course of 90°. Referring to figure at the right, let $C(t)$ denote the center of the circle tangent to the path at time t. Also, let $\theta(t)$ be the orientation of the line connecting $C(t)$ to the jet at time t. As the jet undergoes a change of course of 90°, θ also goes from 0 to 90°. In addition, the angular velocity $\dot{\theta}$ describes the time rate of change of the orientation of the jet, which is the angular velocity of the unit vectors $\hat{u}_t$ and $\hat{u}_n$. This implies that, for $0 \le t \le t_f$,



$$v = \dot{\theta}\rho. \tag{3}$$

Since $a_n = v^2/\rho = 8g$, we also have $\rho = v^2/(8g)$, which, combined with Eq. (3), gives

$$\dot{\theta} = \frac{8g}{v}. \tag{4}$$

Now, we enforce the condition that the speed is decreased uniformly in time. This implies that $\dot{v} = a_t =$ constant, so that, using constant acceleration equations, we write

$$v = v_0 + a_t t. \tag{5}$$

Substituting Eq. (5) into Eq. (4), and recalling that $\dot{\theta} = d\theta/dt$, we have

$$\frac{d\theta}{dt} = \frac{8g}{v_0 + a_t t} \quad \Rightarrow \quad d\theta = \frac{8g}{v_0 + a_t t}\,dt \quad \Rightarrow \quad \int_0^{t_f} \frac{8g}{v_0 + a_t t}\,dt = \int_0^{\pi/2} d\theta. \tag{6}$$

Carrying out the above integration, we have

$$\frac{8g}{a_t}\ln\left(1+\frac{a_t t_f}{v_0}\right)=\frac{\pi}{2}. \tag{7}$$

Because a_t is constant, we must have $a_t = (v_f - v_0)/t_f$, which, upon substituting into Eq. (7), gives

$$\frac{8g t_f}{v_f - v_0}\ln\left(1+\frac{v_f - v_0}{v_0}\right)=\frac{\pi}{2} \quad\Rightarrow\quad t_f = \frac{\pi(v_f - v_0)}{16g\ln(v_f/v_0)}. \tag{8}$$

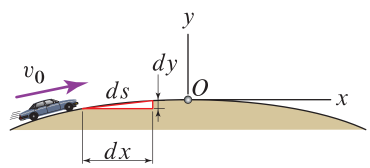
Recalling that $g = 9.81\,\mathrm{m/s^2}$, $v_f = 800\,\mathrm{km/h} = 800(1000/3600)\,\mathrm{m/s}$, and $v_0 = 1100\,\mathrm{km/h} = 1100(1000/3600)\,\mathrm{m/s}$, from the last of Eqs. (8) we have

$$\boxed{t_f = 5.238\,\mathrm{s}.}$$

Problem 2.176

A car is traveling over a hill with a constant speed $v_0 = 70\,\text{mph}$. Using the Cartesian coordinate system shown, the hill's profile is given by the function $y = -(0.0005\,\text{ft}^{-1})x^2$, where x and y are measured in feet. At $x = -300\,\text{ft}$, the driver applies the brakes, causing a constant time rate of change of speed $\dot{v} = -3\,\text{ft/s}^2$ until the car arrives at O. Determine the distance traveled while applying the brakes along with the time to cover this distance. *Hint:* To compute the distance traveled by the car along the car's path, observe that $ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + (dy/dx)^2}\,dx$, and that

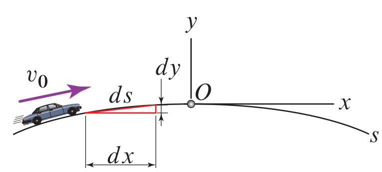
$$\int \sqrt{1 + C^2x^2}\,dx = \frac{x}{2}\sqrt{1 + C^2x^2} + \frac{1}{2C}\ln\Big(Cx + \sqrt{1 + C^2x^2}\Big).$$



Solution

Using normal-tangential components, the acceleration of the car is

$$\vec{a} = \dot{v}\,\hat{u}_t + \frac{v^2}{\rho}\,\hat{u}_n, \tag{1}$$



where v is the speed of the car, ρ is the radius of curvature of the path, and where $\hat{u}_t$ and $\hat{u}_n$ are the unit vectors tangent and normal to the path, respectively. Let $t_0 = 0$ the time at which $v = v_0$ and t_f the time at which the car arrives at O. Using the path coordinate s (see figure at the right), we denote by $d = s_f - s_0$ the distance traveled between t_0 and t_f. Since $a_t = \dot{v}$ is constant, we can apply constant acceleration equations to write

$$d = v_0 t_f + \tfrac{1}{2}a_t t_f^2 \quad\Rightarrow\quad t_f = -\frac{1}{a_t}\Big(v_0 \pm \sqrt{v_0^2 + 2a_t d}\Big). \tag{2}$$

Recalling that $a_t = -3\,\text{ft/s}^2 < 0$, we see that $v_0 > \sqrt{v_0^2 + 2a_t d}$. This implies that there are two positive real roots for t_f. The physically acceptable root is the smaller of the two, which corresponds to when the car arrives at O for the first time after applying the breaks. To interpret the second root we need to keep in mind that the car is still traveling to the right when it arrives at O for the first time. Hence, if $a_t < 0$ is maintained after the car arrives at O, the car would keep slowing down while traveling to the right until stopping and then to is would travel back and reach O a second time. For the purpose of the present calculation, we are only interested in the first time that the car arrives at O, which is

$$t_f = -\frac{1}{a_t}\Big(v_0 - \sqrt{v_0^2 + 2a_t d}\Big). \tag{3}$$

All the quantities in Eq. (3) are known except for d, which is one of the unknowns of the problem. Hence, we now determine d and then we will use Eq. (3) to determine t_f. One way to obtain d is to realize that

$$d = \int_{s_0}^{s_f} ds. \tag{4}$$

From the problem figure, we have that

$$ds = \sqrt{dx^2 + dy^2}. \tag{5}$$

Since the trajectory of the car is given in the form $y = y(x)$, we have $dy = (dy/dx)\,dx$, so that Eq. (5) implies that $ds = \sqrt{1 + (dy/dx)^2}\,dx$, and Eq. (4) can therefore be rewritten as

$$d = \int_{x_0}^{x_f} \sqrt{1 + (dy/dx)^2}\,dx, \tag{6}$$

where $x_0 = -300\,\text{ft}$ and $x_f = 0$. Now we recall that $y = -(0.0005\,\text{ft}^{-1})x^2$ so that we can write $dy/dx = -Cx$, where $C = 0.001000\,\text{ft}^{-1}$. Hence, we can write Eq. (6) as follows:

$$d = \int_{x_0}^{x_f} \sqrt{1 + C^2x^2}\,dx \quad \Rightarrow \quad d = \left[\frac{x}{2}\sqrt{1 + C^2x^2} + \frac{1}{2C}\ln\Bigl(Cx + \sqrt{1 + C^2x^2}\Bigr)\right]_{x_0}^{x_f}$$

$$\Rightarrow \quad d = -\frac{x_0}{2}\sqrt{1 + C^2x_0^2} - \frac{1}{2C}\ln\Bigl(Cx_0 + \sqrt{1 + C^2x_0^2}\Bigr), \tag{7}$$

where we have accounted for the fact that $x_f = 0$. Substituting the last of Eqs. (7) into Eq. (3), we have

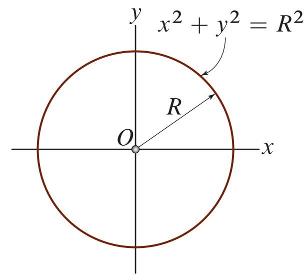
$$t_f = -\frac{1}{a_t}\left\{v_0 + \sqrt{v_0^2 - 2a_t\left[\frac{x_0}{2}\sqrt{1 + C^2x_0^2} + \frac{1}{2C}\ln\Bigl(Cx_0 + \sqrt{1 + C^2x_0^2}\Bigr)\right]}\right\}. \tag{8}$$

Recalling that $x_0 = -300\,\text{ft}$, $v_0 = 70\,\text{mph} = 70(5280/3600)\,\text{ft/s}$, $C = 0.001000\,\text{ft}^{-1}$, and $a_t = -3\,\text{ft/s}^2$, we can evaluate the last of Eqs. (7) and Eq. (8) to obtain

$$\boxed{d = 304.4\,\text{ft} \quad \text{and} \quad t_f = 3.106\,\text{s}.}$$

Problem 2.177

Recalling that a circle of radius R and center at the origin O of a Cartesian coordinate system with axes x and y can be expressed by the formula $x^2+y^2 = R^2$, use Eq. (2.59) to verify that the radius of curvature of this circle is equal to R.



Solution

Equation (2.59) on p. 93 of the textbook tells us that, given a curve of the form $y = (x)$, where x and y are Cartesian coordinates, then the radius of curvature of the curve as a function of x is

$$\rho = \frac{[1 + (dy/dx)^2]^{3/2}}{\left|d^2y/dx^2\right|}. \tag{1}$$

To solve the problem we need to determine the quantities dy/dx and d^2y/dx^2 for a circle. To do so, we start from the given equation for a circle with center at the origin and radius R, i.e., $x^2 + y^2 = R^2$. Differentiating this expression with respect to x, we have

$$2x + 2y\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{x}{y}. \tag{2}$$

Taking the derivative of Eq. (2) with respect to x, we have

$$\frac{d^2y}{dx^2} = -\frac{1}{y} + \frac{x}{y^2}\frac{dy}{dx} \quad \Rightarrow \quad \frac{d^2y}{dx^2} = -\frac{x^2+y^2}{y^3}, \tag{3}$$

where we have used the last of Eqs. (2) to obtain the last of Eqs. (3). Substituting the last of Eqs. (2) and the last of Eqs. (3) into Eq. (1), we have

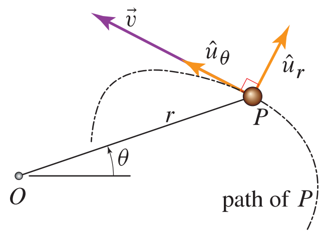
$$\rho = \frac{\left[1 + x^2/y^2\right]^{3/2}}{\left|x^2+y^2\right|}\left|y^3\right| \quad \Rightarrow \quad \rho = \frac{\left[x^2+y^2\right]^{3/2}}{\left|x^2+y^2\right|}\frac{\left|y^3\right|}{\left|y^3\right|} \quad \Rightarrow \quad \rho = \frac{\left[x^2+y^2\right]^{3/2}}{\left|x^2+y^2\right|}. \tag{4}$$

Recalling that $x^2 + y^2 = R^2$, the last of Eqs. (4) gives

$$\rho = \frac{R^3}{R^2} \quad \Rightarrow \quad \boxed{\rho = R.} \tag{5}$$

Problem 2.178

A particle P is moving along a path with the velocity shown. Discuss in detail whether or not there are incorrect elements in the sketch of the polar component system at P.

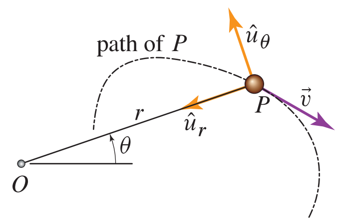


Solution

The unit vectors $\hat{u}_r$ and $\hat{u}_\theta$ are both incorrect. The unit vector $\hat{u}_r$ must be oriented along the radial line r and point away from the origin. The unit vector $\hat{u}_\theta$ must be oriented perpendicular to $\hat{u}_r$ and pointing in the direction of increasing θ.

Problem 2.179

A particle P is moving along a path with the velocity shown. Discuss in detail whether or not there are incorrect elements in the sketch of the polar component system at P.

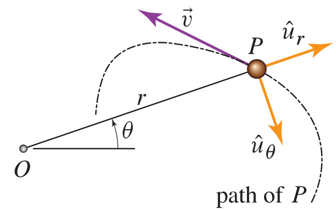


Solution

Since $\hat{u}_\theta$ is perpendicular to the radial line connecting P to O and it is directed in the direction of increasing θ, the unit vector $\hat{u}_\theta$ is oriented correctly. The unit vector $\hat{u}_r$ is parallel to the radial line connecting P to O, but it is pointing toward O and this is incorrect.

Problem 2.180

A particle P is moving along a path with the velocity shown. Discuss in detail whether or not there are incorrect elements in the sketch of the polar component system at P.

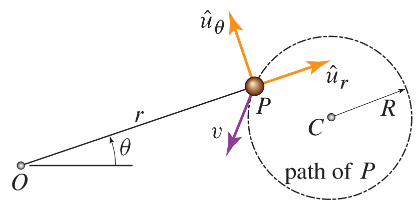


Solution

The unit vector $\hat{u}_r$ is oriented correctly, but $\hat{u}_\theta$ must be oriented opposite to the direction shown.

Problem 2.181

A particle P is moving along a circle with center C and radius R in the direction shown. Letting O be the origin of a polar coordinate system with the coordinates r and θ shown, discuss in detail whether or not there are incorrect elements in the sketch of the polar component system at P.

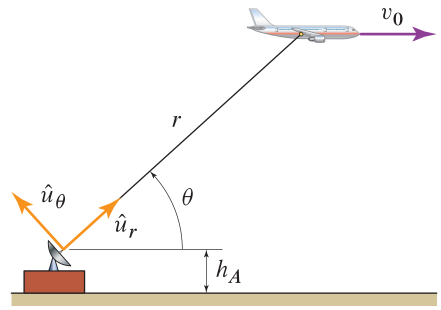


Solution

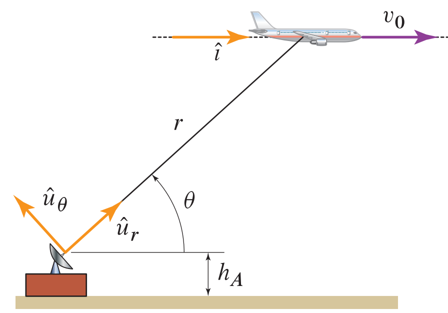
Both $\hat{u}_r$ and $\hat{u}_\theta$ are oriented as they should be.

Problem 2.182

A radar station is tracking a plane flying at a constant altitude with a constant speed $v_0 = 550$ mph. If at a given instant $r = 7$ mi and $\theta = 32°$, determine the corresponding values of $\dot{r}$, $\dot{\theta}$, $\ddot{r}$, and $\ddot{\theta}$.



Solution



Referring to the figure at the right, the unit vector $\hat{\imath}$ described the direction of the airplane. Hence, we can give the following form to the velocity $\vec{v}$ of the airplane:

$$\vec{v} = v_0\,\hat{\imath}. \tag{1}$$

Using the polar component system with unit vectors $\hat{u}_r$ and $\hat{u}_\theta$, the velocity of the airplane can also be written as

$$\vec{v} = \dot{r}\,\hat{u}_r + r\dot{\theta}\,\hat{u}_\theta. \tag{2}$$

In order to compare the expressions for $\vec{v}$ in Eqs. (1) and (2), we observe that the unit vector $\hat{\imath}$ can be written in terms of $\hat{u}_r$ and $\hat{u}_\theta$ as $\hat{\imath} = \cos\theta\,\hat{u}_r - \sin\theta\,\hat{u}_\theta$, so that Eq. (1) can be rewritten as

$$\vec{v} = v_0\cos\theta\,\hat{u}_r - v_0\sin\theta\,\hat{u}_\theta. \tag{3}$$

Equating Eq. (2) and Eq. (3) component by component, we have

$$\dot{r} = v_0\cos\theta \quad \text{and} \quad r\dot{\theta} = -v_0\sin\theta\,\hat{u}_\theta \quad \Rightarrow \quad \dot{r} = v_0\cos\theta \quad \text{and} \quad \dot{\theta} = -(v_0/r)\sin\theta. \tag{4}$$

Recalling that $v_0 = 550\,\text{mph} = 550(5280/3600)\,\text{ft/s}$, $\theta = 32°$, and $r = 7\,\text{mi} = 36{,}960\,\text{ft}$, we can evaluate the last two of Eqs. (4) to obtain

$$\boxed{\dot{r} = 684.1\,\text{ft/s} \quad \text{and} \quad \dot{\theta} = -0.01157\,\text{rad/s}.}$$

Denoting the acceleration of the airplane by $\vec{a}$, we have $\vec{a} = \vec{0}$ because the airplane is flying at constant speed on a straight path. Therefore, recalling the expression of the acceleration in polar coordinates, we have

$$(\ddot{r} - r\dot{\theta}^2)\,\hat{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{u}_\theta \quad \Rightarrow \quad \ddot{r} - r\dot{\theta}^2 = 0 \quad \text{and} \quad r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0. \tag{5}$$

Solving the last two of Eqs. (5) for $\ddot{r}$ and $\ddot{\theta}$ and using the expressions for $\dot{r}$ and $\dot{\theta}$ in the last two of Eqs. (4) we have

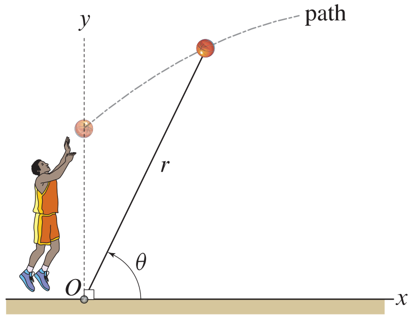
$$\ddot{r} = (v_0^2/r)\sin^2\theta \quad \text{and} \quad \ddot{\theta} = (v_0/r)^2\sin(2\theta), \tag{6}$$

where we have used the following trigonometric identity: $2\sin\theta\cos\theta = \sin(2\theta)$. Recalling again that $v_0 = 550\,\text{mph} = 550(5280/3600)\,\text{ft/s}$, $\theta = 32°$, and $r = 7\,\text{mi} = 36{,}960\,\text{ft}$, we can evaluate Eqs. (6) to obtain

$$\boxed{\ddot{r} = 4.944\,\text{ft/s}^2 \quad \text{and} \quad \ddot{\theta} = 0.0004281\,\text{rad/s}^2.}$$

Problem 2.183

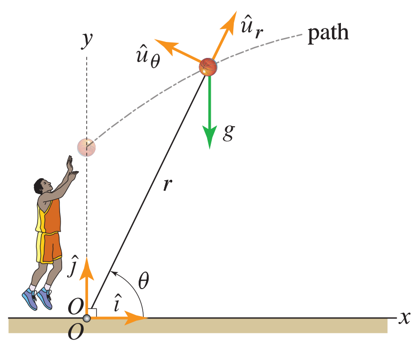
A basketball moves along the trajectory shown. Modeling the motion of the ball as a projectile motion, determine the radial and transverse components of the acceleration when $\theta = 65°$. Express your answer in SI units.



Solution

Modeling the motion of the ball as projectile motion, the acceleration of the ball is equal to the acceleration due to gravity g directed in the negative y direction. Referring to the figure at the right, we have

$$\vec{a} = -g\,\hat{\jmath}. \tag{1}$$



The unit vectors of the polar coordinate system given in the problem statement are

$$\hat{u}_r = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath} \quad \text{and} \quad \hat{u}_\theta = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}. \tag{2}$$

The radial and transverse components of the acceleration are given by

$$a_r = \vec{a}\cdot\hat{u}_r \quad \text{and} \quad a_\theta = \vec{a}\cdot\hat{u}_\theta. \tag{3}$$

Substituting Eqs. (1) and (2) into Eqs. (3), we have

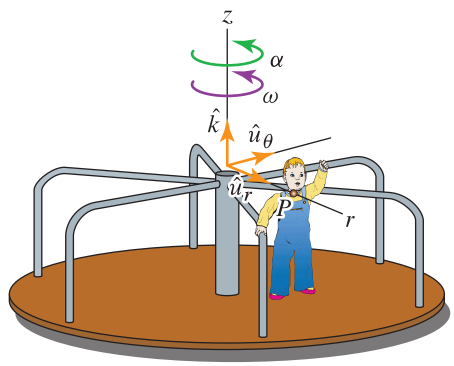
$$a_r = -g\sin\theta \quad \text{and} \quad a_\theta = -g\cos\theta. \tag{4}$$

Recalling that $g = 9.81\,\mathrm{m/s^2}$ and $\theta = 65°$, we can evaluate Eqs. (4) to obtain

$$\boxed{a_r = -8.891\,\mathrm{m/s^2} \quad \text{and} \quad a_\theta = -4.146\,\mathrm{m/s^2}.}$$

Problem 2.184

At a given instant, the merry-go-round is rotating with an angular velocity $\omega = 20\,\text{rpm}$ while the child is moving radially outward at a constant rate of $0.7\,\text{m/s}$. Assuming that the angular velocity of the merry-go-round remains constant, i.e., $\alpha = 0$, determine the magnitudes of the speed and of the acceleration of the child when he is $0.8\,\text{m}$ away from the spin axis.



Solution

The child's velocity is $\vec{v} = \dot{r}\,\hat{u}_r + r\dot{\theta}\,\hat{u}_\theta$, where $\dot{r} = 0.7\,\text{m/s}$ and $\dot{\theta} = \omega = 20\,\text{rpm} = 20(2\pi/60)\,\text{rad/s}$. Hence, for $r = 0.8\,\text{m}$, we have

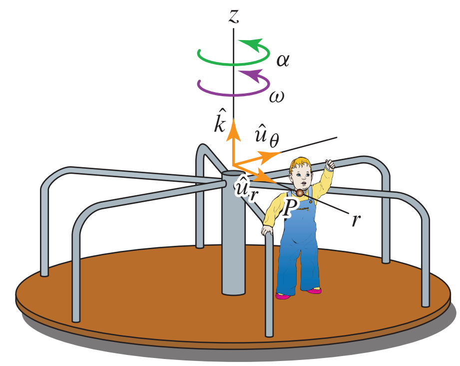
$$v = |\vec{v}| = \sqrt{\dot{r}^2 + r^2\dot{\theta}^2} \quad \Rightarrow \quad \boxed{v = 1.816\,\text{m/s.}} \tag{1}$$

The child's acceleration is $\vec{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{u}_\theta$. Since we have $\ddot{\theta} = \alpha = 0$ and $\ddot{r} = 0$, the magnitude of the acceleration is

$$|\vec{a}| = \sqrt{(-r\dot{\theta}^2)^2 + (2\dot{r}\dot{\theta})^2} \quad \Rightarrow \quad \boxed{|\vec{a}| = 4.573\,\text{m/s}^2.} \tag{2}$$

Problem 2.185

At a given instant, the merry-go-round is rotating with an angular velocity $\omega = 18$ rpm, and it is slowing down at a rate of $0.4\,\mathrm{rad/s^2}$. When the child is 2.5 ft away from the spin axis, determine the time rate of change of the child's distance from the spin axis so that the child experiences no transverse acceleration while moving along a radial line.



Solution

Using polar coordinates, the transverse component of acceleration is

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}. \tag{1}$$

Setting a_θ equal to zero and solving for $\dot{r}$, we have

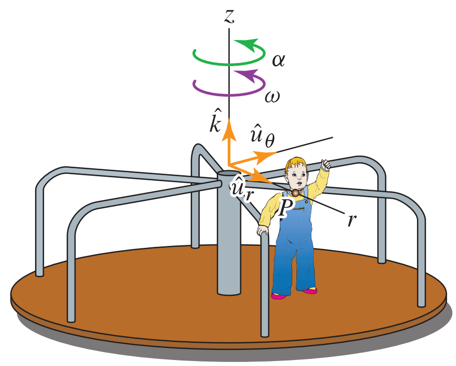
$$\dot{r} = -\frac{r\ddot{\theta}}{2\dot{\theta}}. \tag{2}$$

Recalling that $\dot{\theta} = \omega = 18\,\mathrm{rpm} = 18(2\pi/60)\,\mathrm{rad/s}$, and $\ddot{\theta} = \alpha = -0.4\,\mathrm{rad/s^2}$, we can evaluate $\dot{r}$ for $r = 2.5\,\mathrm{ft}$ to obtain

$$\boxed{\dot{r} = 0.2653\,\mathrm{ft/s}.}$$

Problem 2.186

At a given instant, the merry-go-round is rotating with an angular velocity $\omega = 18\,\text{rpm}$. When the child is 0.45 m away from the spin axis, determine the second derivative with respect to time of the child's distance from the spin axis so that the child experiences no radial acceleration.



Solution

Using polar coordinates, the transverse component of acceleration is

$$a_\theta = \ddot{r} - r\dot{\theta}^2. \tag{1}$$

Setting a_r equal to zero and solving for $\ddot{r}$, we have

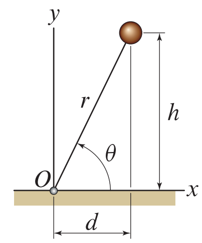
$$\ddot{r} = r\dot{\theta}^2. \tag{2}$$

Recalling that $\dot{\theta} = \omega = 18\,\text{rpm} = 18(2\pi/60)\,\text{rad/s}$, we can evaluate $\ddot{r}$ for $r = 0.45\,\text{m}$ to obtain

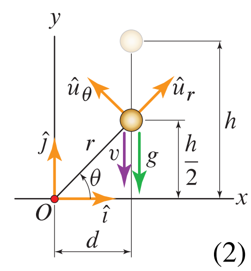
$$\boxed{\ddot{r} = 1.599\,\text{m/s}^2.}$$

Problem 2.187

A ball is dropped from rest from a height $h = 5$ ft. If the distance $d = 3$ ft, determine the radial and transverse components of the acceleration and the velocity of the ball when the ball has traveled a distance $h/2$ from its release position.



Solution



The acceleration of the ball is constant and equal to

$$\vec{a} = -g\,\hat{\jmath}. \tag{1}$$

Since the acceleration is constant and the motion of the ball is rectilinear, we can apply constant acceleration equations and obtain the velocity of the ball after the ball has dropped a distance $h/2$. This gives

$$\vec{v} = -\sqrt{gh}\,\hat{\jmath}. \tag{2}$$

To obtain the radial and transverse components of the velocity and acceleration of the ball we can now compute the dot-product of the vectors $\vec{v}$ and $\vec{a}$ with the unit vectors $\hat{u}_r$ and $\hat{u}_\theta$ of the polar coordinate system shown. The expressions for $\hat{u}_r$ and $\hat{u}_\theta$ in terms of the unit vectors $\hat{\imath}$ and $\hat{\jmath}$ are

$$\hat{u}_r = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath} \quad \text{and} \quad \hat{u}_\theta = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}. \tag{3}$$

Hence, the radial and transverse components of the velocity are

$$v_r = \vec{v}\cdot\hat{u}_r \quad \text{and} \quad v_\theta = \vec{v}\cdot\hat{u}_\theta. \tag{4}$$

Substituting Eqs. (2) and (3) into Eqs. (4) we have

$$v_r = -\sqrt{gh}\sin\theta \quad \text{and} \quad v_\theta = -\sqrt{gh}\cos\theta. \tag{5}$$

In order to evaluate the components we have just determined, we need to compute the angle θ at the location of interest, i.e., after the ball has dropped a distance $h/2$. Since $h/2$ is also the distance from the ground at the instant considered, then we have that

$$\theta = \tan^{-1}\left(\frac{h}{2d}\right). \tag{6}$$

For the acceleration, we can proceed in a similar manner. That is, we first observe that

$$a_r = \vec{a}\cdot\hat{u}_r \quad \text{and} \quad a_\theta = \vec{a}\cdot\hat{u}_\theta. \tag{7}$$

Then, substituting Eqs. (1) and (3) into Eqs. (7) we have

$$a_r = -g\sin\theta \quad \text{and} \quad a_\theta = -g\cos\theta. \tag{8}$$

Recalling that $g = 32.2\,\text{ft/s}^2$, $h = 5$ ft, and $d = 3$ ft, and recalling that θ is given by Eq. (6), we can evaluate Eqs. (5) and (8) to obtain

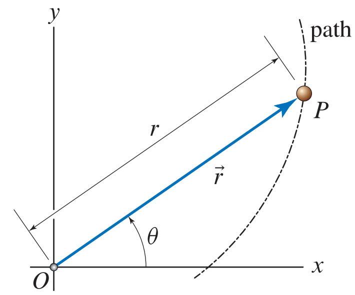
$$v_r = -8.123\,\text{ft/s}, \quad v_\theta = -9.748\,\text{ft/s}, \quad a_r = -20.61\,\text{ft/s}^2, \quad \text{and} \quad a_\theta = -24.74\,\text{ft/s}^2.$$

Problem 2.188

The polar coordinates of a particle are the following functions of time:

$$r = r_0 \sin(t^3/\tau^3) \quad \text{and} \quad \theta = \theta_0 \cos(t/\tau),$$

where r_0 and θ_0 are constants, $\tau = 1$ s, and where t is time in seconds. Determine r_0 and θ_0 such that the velocity of the particle is completely in the radial direction for $t = 15$ s and the corresponding speed is equal to 6 m/s.



Solution

In polar coordinates, the velocity vector is given by

$$\vec{v} = \dot{r}\,\hat{u}_r + r\dot{\theta}\,\hat{u}_\theta. \tag{1}$$

Using the expressions for r and θ given in the problem statement, we can rewrite Eq. (1) as follows:

$$\vec{v} = \frac{3r_0 t^2}{\tau^3}\cos\left(\frac{t^3}{\tau^3}\right)\hat{u}_r - \frac{r_0\theta_0}{\tau}\sin\left(\frac{t^3}{\tau^3}\right)\sin\left(\frac{t}{\tau}\right)\hat{u}_\theta. \tag{2}$$

Recalling that $\tau = 1$ s, for $t = 15$ s, Eq. (2) reduces to

$$\vec{v} = (403.8\,\mathrm{s}^{-1})r_0\,\hat{u}_r - (0.5211\,\mathrm{s}^{-1})r_0\theta_0\,\hat{u}_\theta. \tag{3}$$

From Eq. (3) we conclude that in order for the velocity to be completely in the radial direction, we must have

$$\boxed{\theta_0 = 0.} \tag{4}$$

Substituting the result in Eq. (4) into Eq. (2) we have

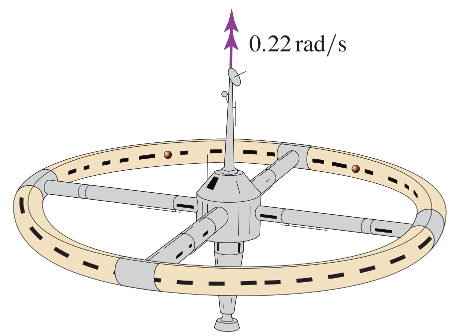
$$\vec{v} = \frac{3r_0 t^2}{\tau^3}\cos\left(\frac{t^3}{\tau^3}\right)\hat{u}_r \quad \Rightarrow \quad v = \left|\frac{3r_0 t^2}{\tau^3}\cos\left(\frac{t^3}{\tau^3}\right)\right| \quad \Rightarrow \quad r_0 = \frac{\tau^3 v}{3t^2\left|\cos(t^3/\tau^3)\right|}. \tag{5}$$

Enforcing the condition that $v = 6$ m/s for $t = 15$ s, and recalling that $\tau = 1$ s, we can evaluate the last of Eqs. (5) to obtain

$$\boxed{r_0 = 0.01486\,\mathrm{m}.}$$

Problem 2.189

A space station is rotating in the direction shown at a constant rate of 0.22 rad/s. A crew member travels from the periphery to the center of the station through one of the radial shafts at a constant rate of 1.3 m/s (relative to the shaft) while holding onto a handrail in the shaft. Taking $t = 0$ to be the instant at which travel through the shaft begins and knowing that the radius of the station is 200 m, determine the velocity and acceleration of the crew member as a function of *time*. Express your answer using a polar coordinate system with origin at the center of the station.



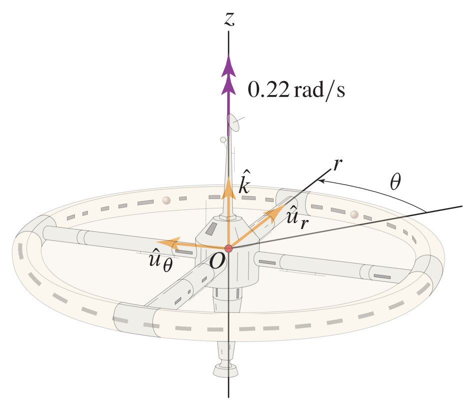
Solution

We will use a polar coordinate system with origin at O, the center of the station. The transverse coordinate θ is measured from a fixed direction. The astronaut moves in the $r\theta$ plane and r and θ are his coordinates. The velocity and acceleration of the astronaut are

$$\vec{v} = \dot{r}\,\hat{u}_r + r\dot{\theta}\,\hat{u}_\theta \tag{1}$$

and

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{u}_\theta. \tag{2}$$



Since $\dot{r} = -1.3\,\text{m/s} = \text{const.}$ and $\dot{\theta} = 0.22\,\text{rad/s} = \text{const.}$, Eq. (2) simplifies to

$$\vec{a} = -r\dot{\theta}^2\,\hat{u}_r + 2\dot{r}\dot{\theta}\,\hat{u}_\theta. \tag{3}$$

To evaluate the expressions for $\vec{v}$ and $\vec{a}$ we need r is as a function of time. Since $\dot{r}$ is constant, r as a function of time is

$$r = r_0 + \dot{r}t, \tag{4}$$

where $r_0 = 200\,\text{m}$ is the radial position of the astronaut at $t = 0$. Using Eq. (4), we can rewrite Eqs. (1) and (3) as follows:

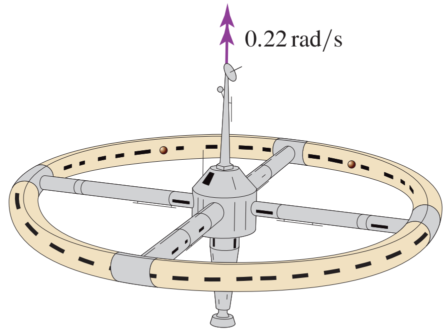
$$\vec{v} = \dot{r}\,\hat{u}_r + (r_0 + \dot{r}t)\dot{\theta}\,\hat{u}_\theta \quad \text{and} \quad \vec{a} = -(r_0 + \dot{r}t)\dot{\theta}^2\,\hat{u}_r + 2\dot{r}\dot{\theta}\,\hat{u}_\theta. \tag{5}$$

Recalling that $\dot{r} = -1.3\,\text{m/s}$, $\dot{\theta} = 0.22\,\text{rad/s}$, $r_0 = 200\,\text{m}$, we can express Eqs. (5) as

$$\vec{v} = (-1.3\,\text{m/s})\,\hat{u}_r + \big[44.00\,\text{m/s} - \big(0.2860\,\text{m/s}^2\big)t\big]\,\hat{u}_\theta,$$
$$\vec{a} = \big[-9.680\,\text{m/s}^2 + (0.06292\,\text{m/s}^3)t\big]\,\hat{u}_r - \big(0.5720\,\text{m/s}^2\big)\,\hat{u}_\theta.$$

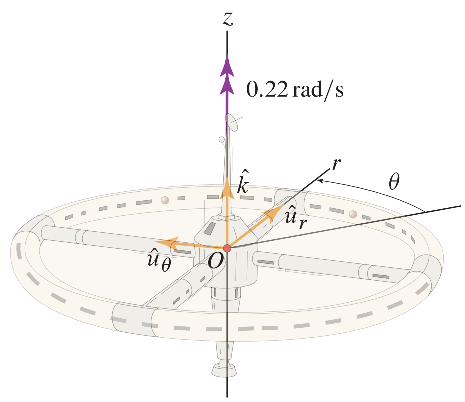
Problem 2.190

Solve Prob. 2.189 and express your answers as a function of *position* along the shaft traveled by the astronaut.



Solution

We will use a polar coordinate system with origin at O, the center of the station. The transverse coordinate θ is measured from a fixed direction. The astronaut moves in the $r\theta$ plane and r and θ are his coordinates. The velocity and acceleration of the astronaut are



$$\vec{v} = \dot{r}\,\hat{u}_r + r\dot{\theta}\,\hat{u}_\theta \tag{1}$$

and

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{u}_\theta. \tag{2}$$

Since $\dot{r} = -1.3\,\mathrm{m/s} = \text{const.}$ and $\dot{\theta} = 0.22\,\mathrm{rad/s} = \text{const.}$, Eq. (2) simplifies to

$$\vec{a} = -r\dot{\theta}^2\,\hat{u}_r + 2\dot{r}\dot{\theta}\,\hat{u}_\theta. \tag{3}$$

Recalling that $\dot{r} = -1.3\,\mathrm{m/s}$ and $\dot{\theta} = 0.22\,\mathrm{rad/s}$, we can express Eqs. (1) and (3) as

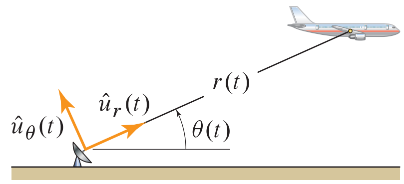
$$\vec{v} = (-1.3\,\mathrm{m/s})\,\hat{u}_r + \big(0.22\,\mathrm{s}^{-1}\big)r\,\hat{u}_\theta,$$
$$\vec{a} = -\big(0.04840\,\mathrm{s}^{-2}\big)r\,\hat{u}_r - \big(0.5720\,\mathrm{m/s^2}\big)\,\hat{u}_\theta.$$

Problem 2.191

During a given time interval, a radar station tracking an airplane records the readings

$$\dot{r}(t) = [449.8\cos\theta(t) + 11.78\sin\theta(t)]\,\text{mph},$$
$$r(t)\dot{\theta}(t) = [11.78\cos\theta(t) - 449.8\sin\theta(t)]\,\text{mph},$$

where t denotes time. Determine the speed of the plane. Furthermore, determine whether the plane being tracked is ascending or descending and the corresponding climbing rate (i.e., the rate of change of the plane's altitude) expressed in ft/s.



Solution

In polar coordinates, the radial and transverse components of the velocity are $v_r = \dot{r}$ and $v_\theta = r\dot{\theta}$, respectively. We observe that the problem statement provides v_r and v_θ as a function of time. With this in mind, we recall that the speed is given by $v = \sqrt{v_r^2 + v_\theta^2}$. Hence, using the information provided, we have

$$\begin{aligned}
v^2 &= v_r^2 + v_\theta^2 \\
&= \dot{r}^2 + (r\dot{\theta})^2 \\
&= \big[(449.8)^2\cos^2\theta + (11.78)^2\sin^2\theta + 2(449.8)(11.78)\sin\theta\cos\theta\big]\,\text{mph} \\
&\quad + \big[(11.78)^2\cos^2\theta + (449.8)^2\sin^2\theta - 2(449.8)(11.78)\sin\theta\cos\theta\big]\,\text{mph}.
\end{aligned} \tag{1}$$

Using the trigonometric identity $\sin^2\beta + \cos^2\beta = 1$, we can simplify the above expression to

$$v^2 = \big[(449.8)^2 + (11.78)^2\big]\,\text{mph} \quad \Rightarrow \quad \boxed{v = 450.0\,\text{mph.}} \tag{2}$$

Letting y denote the elevation of the airplane, we have that $y = r\sin\theta$. Taking the derivative of y with respect to time we have

$$\begin{aligned}
\dot{y} &= \dot{r}\sin\theta + r\dot{\theta}\cos\theta \\
&= [(449.8)\sin\theta\cos\theta + (11.78)\sin^2\theta + (11.78)\cos^2\theta - (449.8)\sin\theta\cos\theta]\,\text{mph} \\
&= 11.78\,\text{mph}.
\end{aligned}$$

Recalling that 1 mph = (5280/3600) ft/s, we have that

$$\boxed{\text{The airplane is ascending at a rate of 17.28 ft/s.}}$$

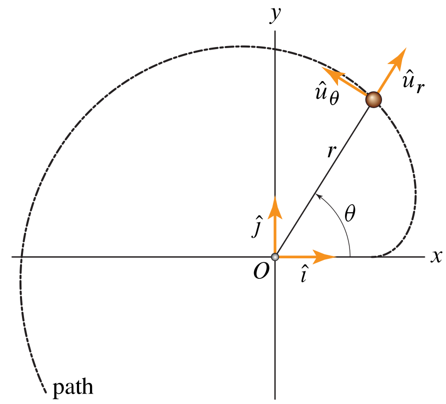
Problem 2.192

The polar coordinates of a particle are the following functions of time:

$$r = r_0\left(1 + \frac{t}{\tau}\right) \quad \text{and} \quad \theta = \theta_0 \frac{t^2}{\tau^2},$$

where $r_0 = 3$ ft, $\theta_0 = 1.2$ rad, $\tau = 20$ s, and t is time in seconds.

Determine the velocity and the acceleration of the particle for $t = 35$ s and express the result using the polar component system formed by the unit vectors $\hat{u}_r$ and $\hat{u}_\theta$ at $t = 35$ s.



Solution

In polar coordinates, the velocity is given by

$$\vec{v} = \dot{r}\,\hat{u}_r + r\dot{\theta}\,\hat{u}_\theta. \tag{1}$$

Using the expressions for r and θ given in the problem statement, we can rewrite Eq. (1) as follows:

$$\vec{v} = \frac{r_0}{\tau}\,\hat{u}_r + \frac{2r_0 t\theta_0(t+\tau)}{\tau^3}\,\hat{u}_\theta. \tag{2}$$

In polar coordinates, the acceleration is given by

$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{u}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{u}_\theta. \tag{3}$$

Again using the expressions for r and θ given in the problem statement, we can rewrite Eq. (1) as follows:

$$\vec{a} = -\frac{4r_0 t^2\theta_0^2}{\tau^5}(t+\tau)\,\hat{u}_r + \frac{2r_0\theta_0}{\tau^3}(3t+\tau)\,\hat{u}_\theta. \tag{4}$$

Recalling that $r_0 = 3$ ft, $\theta_0 = 1.2$ rad, and $\tau = 20$ s, for $t = 35$ s we can evaluate the expressions in Eqs. (2) and (4) to obtain

$$\vec{v} = (0.1500\,\hat{u}_r + 1.732\,\hat{u}_\theta)\text{ ft/s} \quad \text{and} \quad \vec{a} = (-0.3638\,\hat{u}_r + 0.1125\,\hat{u}_\theta)\text{ ft/s}^2.$$

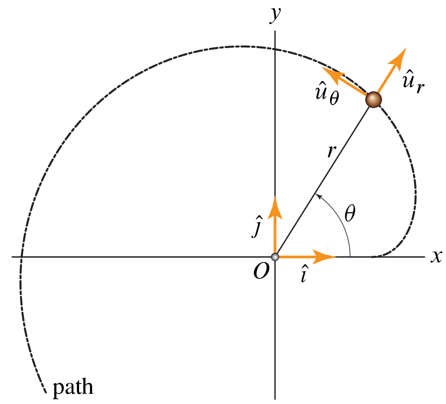
Problem 2.193

The polar coordinates of a particle are the following functions of time:

$$r = r_0\left(1 + \frac{t}{\tau}\right) \quad \text{and} \quad \theta = \theta_0 \frac{t^2}{\tau^2},$$

where $r_0 = 3$ ft, $\theta_0 = 1.2$ rad, $\tau = 20$ s, and t is time in seconds.

Determine the velocity and the acceleration of the particle for $t = 35$ s and express the result using the Cartesian component system formed by the unit vectors $\hat{\imath}$ and $\hat{\jmath}$.



Solution

In polar coordinates, the velocity is given by

$$\vec{v} = \dot{r}\,\hat{u}_r + r\dot{\theta}\,\hat{u}_\theta. \tag{1}$$

Using the expressions for r and θ given in the problem statement, we can rewrite Eq. (1) as follows:

$$\vec{v} = \frac{r_0}{\tau}\,\hat{u}_r + \frac{2r_0 t\theta_0(t+\tau)}{\tau^3}\,\hat{u}_\theta. \tag{2}$$

In polar coordinates, the acceleration is given by

$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{u}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{u}_\theta. \tag{3}$$

Again using the expressions for r and θ given in the problem statement, we can rewrite Eq. (1) as follows:

$$\vec{a} = -\frac{4r_0 t^2\theta_0^2}{\tau^5}(t+\tau)\,\hat{u}_r + \frac{2r_0\theta_0}{\tau^3}(3t+\tau)\,\hat{u}_\theta. \tag{4}$$

We now observe that the unit vectors $\hat{u}_r$ and $\hat{u}_\theta$ can be expressed in terms of the unit vectors $\hat{\imath}$ and $\hat{\jmath}$ as follows:

$$\hat{u}_r = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath} \quad \text{and} \quad \hat{u}_\theta = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}. \tag{5}$$

Substituting Eqs. (5) in Eq. (2), collecting terms, and using the given expression for θ as a function of time, the velocity takes on the form

$$\vec{v} = \frac{r_0}{\tau^3}\left[\tau^2\cos\left(\frac{t^2\theta_0}{\tau^2}\right) - 2t\theta_0(t+\tau)\sin\left(\frac{t^2\theta_0}{\tau^2}\right)\right]\hat{\imath} + \frac{r_0}{\tau^3}\left[2t\theta_0(t+\tau)\cos\left(\frac{t^2\theta_0}{\tau^2}\right) + \tau^2\sin\left(\frac{t^2\theta_0}{\tau^2}\right)\right]\hat{\jmath}. \tag{6}$$

Substituting Eqs. (5) in Eq. (4), collecting terms, and using the given expression for θ as a function of time, we that the acceleration takes on the form

$$\vec{a} = -\frac{2r_0\theta_0}{\tau^5}\left[2t^2\theta_0(t+\tau)\cos\left(\frac{t^2\theta_0}{\tau^2}\right) + \tau^2(3t+\tau)\sin\left(\frac{t^2\theta_0}{\tau^2}\right)\right]\hat{\imath} + \frac{2r_0\theta_0}{\tau^5}\left[\tau^2(3t+\tau)\cos\left(\frac{t^2\theta_0}{\tau^2}\right) - 2t^2\theta_0(t+\tau)\sin\left(\frac{t^2\theta_0}{\tau^2}\right)\right]\hat{\jmath}. \tag{7}$$

Recalling that $r_0 = 3$ ft, $\theta_0 = 1.2$ rad, and $\tau = 20$ s, for $t = 35$ s we can evaluate the expressions in Eqs. (6) and (7) to obtain

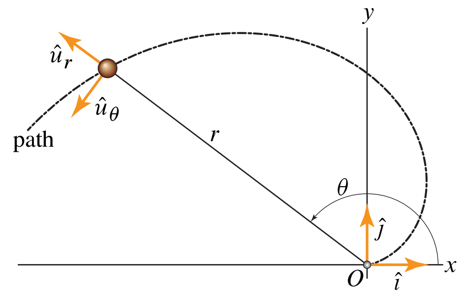
$$\vec{v} = (0.7518\,\hat{\imath} - 1.568\,\hat{\jmath})\text{ ft/s} \quad \text{and} \quad \vec{a} = (0.3705\,\hat{\imath} + 0.08812\,\hat{\jmath})\text{ ft/s}^2.$$

Problem 2.194

A particle is moving such that the time rate of change of its polar coordinates are

$$\dot{r} = \text{constant} = 3\,\text{ft/s} \quad \text{and} \quad \dot{\theta} = \text{constant} = 0.25\,\text{rad/s}.$$

Knowing that at time $t = 0$, a particle has polar coordinates $r_0 = 0.2$ ft and $\theta_0 = 15°$, determine the position, velocity, and acceleration of the particle for $t = 10$ s. Express your answers in the polar component system formed by the unit vectors $\hat{u}_r$ and $\hat{u}_\theta$ at $t = 10$ s.



Solution

Since $\dot{r}$ and $\dot{\theta}$ are constant, they can be integrated with respect to time to obtain

$$r = r_0 + \dot{r}t \quad \text{and} \quad \theta = \theta_0 + \dot{\theta}t, \tag{1}$$

where r_0 and θ_0 are the values of r and θ, respectively, for $t = 0$. The position vector is given by

$$\vec{r} = r\,\hat{u}_r \quad \Rightarrow \quad \vec{r} = (r_0 + \dot{r}t)\,\hat{u}_r, \tag{2}$$

where we have used the expression for r in the first of Eqs. (1). Recalling that $r_0 = 0.2$ ft and $\dot{r} = 3$ ft/s, for $t = 10$ s the position of the particle can be evaluated to obtain

$$\boxed{\vec{r} = 30.20\,\hat{u}_r\ \text{ft.}}$$

The velocity of the particle is given by

$$\vec{v} = \dot{r}\,\hat{u}_r + r\dot{\theta}\,\hat{u}_\theta. \tag{3}$$

Recalling that $\dot{r}$ and $\dot{\theta}$ are given and substituting the first of Eqs. (1) into Eq. (3), we have

$$\vec{v} = \dot{r}\,\hat{u}_r + (r_0 + \dot{r}t)\dot{\theta}\,\hat{u}_\theta. \tag{4}$$

Recalling that $r_0 = 0.2$ ft, $\dot{r} = 3$ ft/s, $\dot{\theta} = 0.25$ rad/s, for $t = 10$ s, Eq. (4) gives

$$\boxed{\vec{v} = (3\,\hat{u}_r + 7.550\,\hat{u}_\theta)\ \text{ft/s.}}$$

The acceleration of the particle is given by

$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{u}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{u}_\theta. \tag{5}$$

Substituting Eqs. (1) into Eq. (5) gives

$$\vec{a} = -(r_0 + \dot{r}t)\dot{\theta}^2\,\hat{u}_r + 2\dot{r}\dot{\theta}\,\hat{u}_\theta. \tag{6}$$

Recalling that $r_0 = 0.2$ ft, $\dot{r} = 3$ ft/s, $\dot{\theta} = 0.25$ rad/s, for $t = 10$ s the acceleration of the particle can be evaluated to obtain

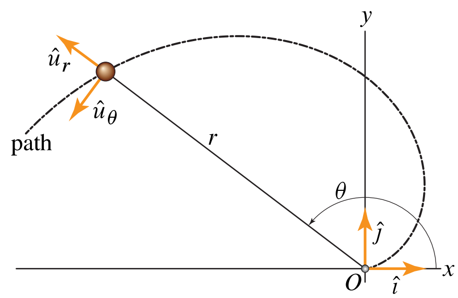
$$\boxed{\vec{a} = (-1.888\,\hat{u}_r + 1.500\,\hat{u}_\theta)\ \text{ft/s}^2.}$$

Problem 2.195

A particle is moving such that the time rate of change of its polar coordinates are

$$\dot{r} = \text{constant} = 3\,\text{ft/s} \quad \text{and} \quad \dot{\theta} = \text{constant} = 0.25\,\text{rad/s}.$$

Knowing that at time $t = 0$, a particle has polar coordinates $r_0 = 0.2$ ft and $\theta_0 = 15°$, determine the position, velocity, and acceleration of the particle for $t = 10$ s. Express your answers in the Cartesian component system formed by the unit vectors $\hat{\imath}$ and $\hat{\jmath}$.



Solution

Since $\dot{r}$ and $\dot{\theta}$ are constant, they can be integrated with respect to time to obtain

$$r = r_0 + \dot{r}t \quad \text{and} \quad \theta = \theta_0 + \dot{\theta}t, \tag{1}$$

where r_0 and θ_0 are the values of r and θ, respectively, for $t = 0$. The position vector is given by

$$\vec{r} = r\,\hat{u}_r \quad \Rightarrow \quad \vec{r} = (r_0 + \dot{r}t)\,\hat{u}_r, \tag{2}$$

where we have the expression for r given in the first of Eqs. (1). The velocity of the particle is given by

$$\vec{v} = \dot{r}\,\hat{u}_r + r\dot{\theta}\,\hat{u}_\theta. \tag{3}$$

Recalling that $\dot{r}$ and $\dot{\theta}$ are given and substituting the first of Eqs. (1) into Eq. (3), we have that the velocity becomes

$$\vec{v} = \dot{r}\,\hat{u}_r + (r_0 + \dot{r}t)\dot{\theta}\,\hat{u}_\theta. \tag{4}$$

The acceleration of the particle is given by

$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{u}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{u}_\theta. \tag{5}$$

Substituting Eqs. (1) into Eq. (5) gives

$$\vec{a} = -(r_0 + \dot{r}t)\dot{\theta}^2\,\hat{u}_r + 2\dot{r}\dot{\theta}\,\hat{u}_\theta. \tag{6}$$

We now observe that we can express the unit vectors $\hat{u}_r$ and $\hat{u}_\theta$ in terms of the unit vectors $\hat{\imath}$ and $\hat{\jmath}$ as follows:

$$\hat{u}_r = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath} \quad \text{and} \quad \hat{u}_\theta = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}. \tag{7}$$

Using Eqs. (1), substituting Eqs. (7) into the last of Eqs. (2), Eq. (4), and Eq. (6), we have

$$\vec{r} = (r_0 + \dot{r}t)\left[\cos\!\left(\theta_0 + \dot{\theta}t\right)\hat{\imath} + \sin\!\left(\theta_0 + \dot{\theta}t\right)\hat{\jmath}\right], \tag{8}$$

$$\begin{aligned}\vec{v} = {} & \left[\dot{r}\cos\!\left(\theta_0 + \dot{\theta}t\right) - (r_0 + \dot{r}t)\dot{\theta}\sin\!\left(\theta_0 + \dot{\theta}t\right)\right]\hat{\imath} \\ & + \left[(r_0 + \dot{r}t)\dot{\theta}\cos\!\left(\theta_0 + \dot{\theta}t\right) + \dot{r}\sin\!\left(\theta_0 + \dot{\theta}t\right)\right]\hat{\jmath},\end{aligned} \tag{9}$$

$$\begin{aligned}\vec{a} = {} & -\left[(r_0 + \dot{r}t)\dot{\theta}^2\cos\!\left(\theta_0 + \dot{\theta}t\right) - 2\dot{r}\dot{\theta}\sin\!\left(\theta_0 + \dot{\theta}t\right)\right]\hat{\imath} \\ & + \left[2\dot{r}\dot{\theta}\cos\!\left(\theta_0 + \dot{\theta}t\right) - (r_0 + \dot{r}t)\dot{\theta}^2\sin\!\left(\theta_0 + \dot{\theta}t\right)\right]\hat{\jmath}.\end{aligned} \tag{10}$$

Recalling that $r_0 = 0.2$ ft, $\dot{r} = 3$ ft/s, $\dot{\theta} = 0.25$ rad/s, and $\theta_0 = 15°$, for $t = 10$ s Eqs. (8)–(10) give

$$\vec{r} = (-28.05\,\hat{\imath} + 11.20\,\hat{\jmath})\,\text{ft}, \quad \vec{v} = (-5.585\,\hat{\imath} - 5.900\,\hat{\jmath})\,\text{ft/s}, \quad \text{and} \quad \vec{a} = (1.197\,\hat{\imath} - 2.093\,\hat{\jmath})\,\text{ft/s}^2.$$

Problem 2.196

A micro spiral pump consists of a spiral channel attached to a stationary plate. This plate has two ports, one for fluid inlet and another for outlet, the outlet being farther from the center of the plate than the inlet. The system is capped by a rotating disk. The fluid trapped between the rotating disk and the stationary plate is put in motion by the rotation of the top disk, which pulls the fluid through the spiral channel.

Consider a spiral channel with the geometry given by the equation $r = \eta\theta + r_0$, where $r_0 = 146\,\mu$m is the starting radius, r is the distance from the spin axis, and θ, measured in radians, is the angular position of a point in the spiral channel. Assume that the radius at the outlet is $r_{out} = 190\,\mu$m, that the top disk rotates with a constant angular speed ω, and that the fluid particles in contact with the rotating disk are essentially stuck to it. Determine the constant η and the value of ω (in rpm) such that after 1.25 rev of the top disk, the speed of the particles in contact with this disk is $v = 0.5$ m/s at the outlet.

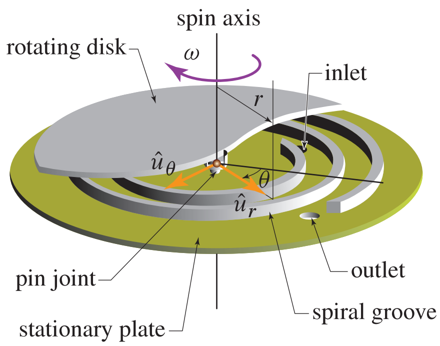


Photo credit: "Design and Analysis of a Surface Micromachined Spiral-Channel Viscous Pump," by M. I. Kilani, P. C. Galambos, Y. S. Haik, C. H. Chen, *Journal of Fluids Engineering*, Vol. 125, pp. 339–344, 2003.

Solution

We denote by θ_{out} the value of the coordinate θ corresponding to $r = r_{out}$, where r_{out} is the radial position of the outlet. Since $r = \eta\theta + r_0$, we have

$$r_{out} = \eta\theta_{out} + r_0 \quad \Rightarrow \quad \eta = \frac{r_{out} - r_0}{\theta_{out}} \quad \Rightarrow \quad \boxed{\eta = 5.602\,\mu\text{m},} \tag{1}$$

where we have used the fact that $r_{out} = 190\,\mu$m, $r_0 = 146\,\mu$m, and $\theta_{out} = 1.25\,\text{rev} = 1.25(2\pi)$ rad. Next, we recall that the velocity in polar coordinates is $\vec{v} = \dot{r}\,\hat{u}_r + r\dot{r}\,\hat{u}_\theta$, so that the speed is

$$v = \sqrt{\dot{r}^2 + \left(r\dot{\theta}\right)^2}. \tag{2}$$

We can obtain $\dot{r}$ by differentiating r with respect to time. This gives

$$\dot{r} = \eta\dot{\theta}. \tag{3}$$

Substituting Eq. (3) into Eq. (2) and solving for $\dot{\theta}$, we have

$$\dot{\theta} = \frac{v}{\sqrt{\eta^2 + r^2}} \quad \Rightarrow \quad \omega = \frac{v\theta_{out}}{\sqrt{(r_{out} - r_0)^2 + [(r_{out} - r_0)\theta + \theta_{out}r_0]^2}}, \tag{4}$$

where we have used the fact that $\omega = \dot{\theta}$, $r = \eta\theta + r_0$, and where we have used the expression for η in the second of Eqs. (1). Recalling that $r_{out} = 190\,\mu$m, $r_0 = 146\,\mu$m, and that for $\theta = \theta_{out} = 1.25\,\text{rev} = 1.25(2\pi)$ rad we must have $v = 0.5$ m/s, the last of Eqs. (4) gives

$$\boxed{\omega = 2630\,\text{rad/s} = 25{,}120\,\text{rpm}.}$$

Problem 2.197

A micro spiral pump consists of a spiral channel attached to a stationary plate. This plate has two ports, one for fluid inlet and another for outlet, the outlet being farther from the center of the plate than the inlet. The system is capped by a rotating disk. The fluid trapped between the rotating disk and the stationary plate is put in motion by the rotation of the top disk, which pulls the fluid through the spiral channel.

Consider a spiral channel with the geometry given by the equation $r = \eta\theta + r_0$, where $\eta = 12\,\mu$m is called the polar slope, $r_0 = 146\,\mu$m is the starting radius, r is the distance from the spin axis, and θ, measured in radians, is the angular position of a point in the spiral channel. If the top disk rotates with a constant angular speed $\omega = 30{,}000$ rpm, and assuming that the fluid particles in contact with the rotating disk are essentially stuck to it, use the polar coordinate system shown and determine the velocity and acceleration of one fluid particle when it is at $r = 170\,\mu$m.

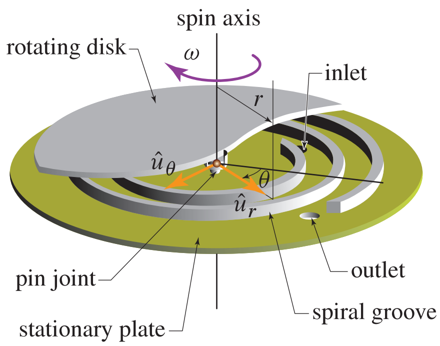


Photo credit: "Design and Analysis of a Surface Micromachined Spiral-Channel Viscous Pump," by M. I. Kilani, P. C. Galambos, Y. S. Haik, C. H. Chen, *Journal of Fluids Engineering*, Vol. 125, pp. 339–344, 2003.

Solution

In polar components, the velocity is given by

$$\vec{v} = \dot{r}\,\hat{u}_r + r\dot{\theta}\,\hat{u}_\theta. \tag{1}$$

Since $r = \eta\theta + r_0$ and $\dot{\theta} = \omega$, Eq. (1) becomes

$$\vec{v} = \eta\omega\,\hat{u}_r + r\omega\,\hat{u}_\theta. \tag{2}$$

Recalling that $\eta = 12\,\mu\text{m} = 12\times 10^{-6}$ m and $\omega = 30{,}000\,\text{rpm} = 30000\frac{2\pi}{60}$ rad/s, for $r = 170\,\mu\text{m} = 170\times 10^{-6}$ m, Eq. (2) gives

$$\boxed{\vec{v} = (0.03770\,\hat{u}_r + 0.5341\,\hat{u}_\theta)\ \text{m/s}.}$$

For the acceleration we have

$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{u}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{u}_\theta. \tag{3}$$

Recalling that $\dot{\theta} = \omega = $ const., and recalling that $r = \eta\theta + r_0$, we have

$$\dot{r} = \eta\omega, \quad \ddot{r} = 0, \quad \text{and} \quad \ddot{\theta} = 0. \tag{4}$$

Hence, Eq. (3) can be simplified to

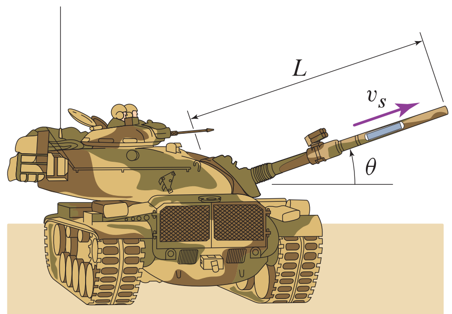
$$\vec{a} = -r\omega^2\,\hat{u}_r + 2\eta\omega^2\,\hat{u}_\theta. \tag{5}$$

Recalling that $\eta = 12\,\mu\text{m} = 12\times 10^{-6}$ m and $\omega = 30{,}000\,\text{rpm} = 30000\frac{2\pi}{60}$ rad/s, for $r = 170\,\mu\text{m} = 170\times 10^{-6}$ m, Eq. (5) gives

$$\boxed{\vec{a} = (-1678\,\hat{u}_r + 236.9\,\hat{u}_\theta)\ \text{m/s}^2.}$$

Problem 2.198

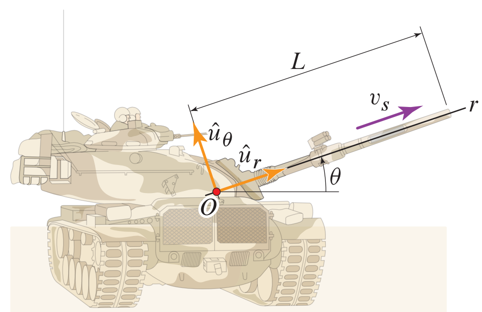
The cutaway of the gun barrel shows a projectile that, upon exit, moves with a speed $v_s = 5490\,\text{ft/s}$ relative to the gun barrel. The length of the gun barrel is $L = 15\,\text{ft}$. Assuming that the angle θ is increasing at a constant rate of $0.15\,\text{rad/s}$, determine the speed of the projectile right when it leaves the barrel. In addition, assuming that the projectile acceleration along the barrel is constant and that the projectile starts from rest, determine the magnitude of the acceleration upon exit.



Solution

Using the polar coordinate system shown at the right, the velocity of the projectile is

$$\vec{v} = \dot{r}\,\hat{u}_r + r\dot{\theta}\,\hat{u}_\theta.$$



Upon exit from the barrel, $r = L$ and $\dot{r} = v_s$, so the speed of the projectile is

$$v = |\vec{v}| = \sqrt{v_s^2 + (L\dot{\theta})^2}. \qquad (1)$$

Recalling that $v_s = 5490\,\text{ft/s}$, $L = 15\,\text{ft}$, and $\dot{\theta} = 0.15\,\text{rad/s}$, Eq. (1) gives

$$\boxed{v = 5490\,\text{ft/s},}$$

which shows that the contribution to v due to the angular velocity of the barrel is negligible compared to v_s.

The acceleration of the projectile along the gun is constant. So, using constant acceleration equations, we can compute $\ddot{r}$ by relating the initial and final speeds of the projectile (relative to the barrel) over the length of the barrel:

$$\dot{r}^2 = \dot{r}_0^2 + 2\ddot{r}(r - r_0) \quad\Rightarrow\quad v_s^2 = 2\ddot{r}L \quad\Rightarrow\quad \ddot{r} = \frac{v_s^2}{2L}. \qquad (2)$$

where we have used $r_0 = 0$, $\dot{r}_0 = 0$, (the projectile start from rest), $\dot{r} = v_s$, and $r = L$. Recalling that the acceleration of the projectile in polar coordinates is given by

$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{u}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{u}_\theta,$$

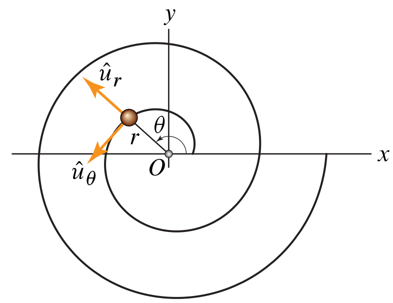
and that $\ddot{\theta} = 0$ (since $\dot{\theta} = 0$), using the last of Eqs. (2), for $r = L$ and $\dot{r} = v_s$, the magnitude of the acceleration becomes

$$|\vec{a}| = \sqrt{\left(\frac{v_s^2}{2L} - L\dot{\theta}^2\right)^2 + \left(2v_s\dot{\theta}\right)^2} \quad\Rightarrow\quad \boxed{|\vec{a}| = 1.005\times10^6\,\text{ft/s}^2,}$$

where we have used $v_s = 5490\,\text{ft/s}$, $L = 15\,\text{ft}$, and $\dot{\theta} = 0.15\,\text{rad/s}$.

Problem 2.199

A particle moves along a spiral described by the equation $r = r_0 + \kappa\theta$, where r_0 and κ are constants, and where θ is in radians. Assume that $\dot{\theta} = \alpha t$, where $\alpha = 0.15\,\text{rad/s}^2$ and t is time expressed in seconds. If $r = 0.25\,\text{m}$ and $\theta = 0$ for $t = 0$, determine κ such that, for $t = 10\,\text{s}$, the acceleration is completely in the radial direction. In addition, determine the value of the polar coordinates of the point for $t = 10\,\text{s}$.



Solution

Since $r = 0.25\,\text{m}$ and $\theta = 0$ for $t = 0$, using the equation of the spiral, we must have that

$$r_0 = 0.25\,\text{m}. \tag{1}$$

Next, recalling that the acceleration has the expression

$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{u}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{u}_\theta, \tag{2}$$

we proceed to determine expressions for the first and second time derivatives of the radial and transverse coordinates. The first time derivative of the transverse coordinate is given as αt. Hence, for this coordinate we have

$$\dot{\theta} = \alpha t \quad \Rightarrow \quad \ddot{\theta} = \alpha. \tag{3}$$

In addition, since we will need it to properly compute the radial coordinate, recalling that $\theta = 0$ for $t = 0$, we can integrate the first of Eqs. (3) with respect to time to obtain

$$\theta = \tfrac{1}{2}\alpha t^2. \tag{4}$$

To determine the time derivatives of the radial coordinate, we first substitute Eq. (4) into the equation of spiral and then differentiate with respect to time. This gives

$$r = r_0 + \tfrac{1}{2}\kappa\alpha t^2 \quad \Rightarrow \quad \dot{r} = \kappa\alpha t \quad \Rightarrow \quad \ddot{r} = \kappa\alpha. \tag{5}$$

Substituting Eqs. (3) and (5) into Eq. (2) gives

$$\vec{a} = \left(\kappa\alpha - r_0\alpha^2 t^2 - \tfrac{1}{2}\kappa\alpha^3 t^4\right)\hat{u}_r + \left(\alpha r_0 + \tfrac{5}{2}\kappa\alpha^2 t^2\right)\hat{u}_\theta. \tag{6}$$

For $t = t_f = 10\,\text{s}$ (the subscript f stands for "final") the transverse component of the acceleration vanishes, i.e.,

$$\alpha r_0 + \tfrac{5}{2}\kappa\alpha^2 t_f^2 = 0 \quad \Rightarrow \quad \kappa = -\frac{2r_0}{5\alpha t_f^2}. \tag{7}$$

Using Eq. (1), and recalling that $\alpha = 0.15\,\text{rad/s}^2$ and $t_f = 10\,\text{s}$, the last of Eqs. (7) gives

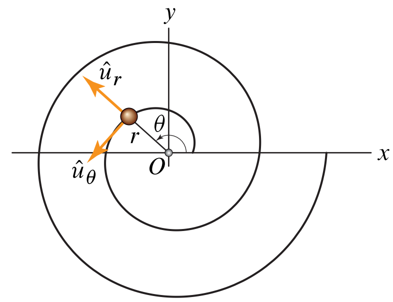
$$\boxed{\kappa = -0.006667\,\text{m}.} \tag{8}$$

Substituting the last of Eqs. (7) into the first of Eqs. (5), and recalling that θ is given by Eq. (4), we can evaluate r and θ for $t = 10\,\text{s}$ to obtain

$$\boxed{r = 0.2000\,\text{m} \quad \text{and} \quad \theta = 7.500\,\text{rad}.}$$

Problem 2.200

A point is moving counterclockwise at constant speed v_0 along a spiral described by the equation $r = r_0 + \kappa\theta$, where r_0 and κ are constants with dimensions of length. Determine the expressions of the velocity and the acceleration of the particle as a function of θ expressed in the polar component system shown.



Solution

In polar coordinates, the expressions for the velocity is

$$\vec{v} = \dot{r}\,\hat{u}_r + r\dot{\theta}\,\hat{u}_\theta. \tag{1}$$

observing that r is as a function of θ is given in the problem statement, we need to determine expression for $\dot{r}$ and $\dot{\theta}$ as functions of θ and the known quantities v_0, r_0, and κ. To do so, we begin with differentiating the given expression for r with respect to time:

$$\dot{r} = \kappa\dot{\theta}. \tag{2}$$

Substituting Eq. (2) into Eq. (1) and keeping in mind that $r = r_0 + \kappa\theta$, we enforce the condition that $|\vec{v}|$ is equal to v_0 as follows:

$$\kappa^2\dot{\theta}^2 + (r_0 + \kappa\theta)^2\dot{\theta}^2 = v_0^2 \quad \Rightarrow \quad \dot{\theta} = \frac{v_0}{\sqrt{\kappa^2 + (r_0 + \kappa\theta)^2}}, \tag{3}$$

where we have chosen the root with $\dot{\theta} > 0$ to be consistent with the fact that the point is moving counterclockwise. Using the last of Eqs. (3) along with Eqs. (1) and (2), we can rewrite the velocity as a function of θ and known constants:

$$\vec{v} = \frac{\kappa v_0}{\sqrt{\kappa^2 + (r_0 + \kappa\theta)^2}}\,\hat{u}_r + \frac{v_0(r_0 + \kappa\theta)}{\sqrt{\kappa^2 + (r_0 + \kappa\theta)^2}}\,\hat{u}_\theta.$$

To determine the acceleration, we recall that in polar coordinates the acceleration is given by

$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{u}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{u}_\theta. \tag{4}$$

We observe that we already have an expression for $\dot{\theta}$ and, through Eq. (2), a corresponding expression for $\dot{r}$. We now need to determine expressions for $\ddot{r}$ and $\ddot{\theta}$. For $\ddot{\theta}$ we can write

$$\ddot{\theta} = \frac{d\dot{\theta}}{dt} = \frac{d\dot{\theta}}{d\theta}\dot{\theta} \quad \Rightarrow \quad \ddot{\theta} = \frac{\kappa v_0^2(r_0 + \kappa\theta)}{\left[\kappa^2 + (r_0 + \kappa\theta)^2\right]^2}, \tag{5}$$

where we have used the last of Eqs. (3) to express $\dot{\theta}$ and to determine an expression for $d\dot{\theta}/d\theta$. Next, differentiating Eq. (2) with respect to time we have $\ddot{r} = \kappa\ddot{\theta}$ so that, using the last of Eqs. (5) we have

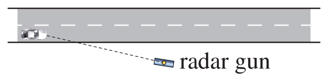
$$\ddot{r} = \frac{\kappa^2 v_0^2(r_0 + \kappa\theta)}{\left[\kappa^2 + (r_0 + \kappa\theta)^2\right]^2}. \tag{6}$$

Finally, recalling that $r = r_0 + \kappa\theta$, using Eq. (2), the last of Eqs.(3), the last of Eqs. (5), and Eq. (6), after simplification, we can rewrite Eq. (4) to obtain

$$\vec{a} = -\frac{v_0^2 (r_0 + \kappa\theta)^3}{\big[r_0^2 + 2r_0\kappa\theta + (1+\theta^2)\kappa^2\big]^2}\,\hat{u}_r + \frac{\kappa v_0^2\big\{(r_0+\kappa\theta)^2 + 2\big[\kappa^2 + (r_0+\kappa\theta)^2\big]\big\}}{\big[\kappa^2 + (r_0+\kappa\theta)^2\big]^2}\,\hat{u}_\theta.$$

Problem 2.201

A person driving along a rectilinear stretch of road is fined for speeding, having been clocked at 75 mph when the radar gun was pointing as shown. The driver claims that, because the radar gun is off to the side of the road instead of directly in front of his car, the radar gun overestimates his speed. Is he right or wrong and why?

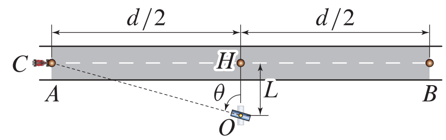


Solution

The driver is wrong. The speed recorded by the radar gun is the component (or projection) of the actual speed along the line connecting the radar gun and the moving car. As such, i.e., being a component, it can only be smaller than the true speed.

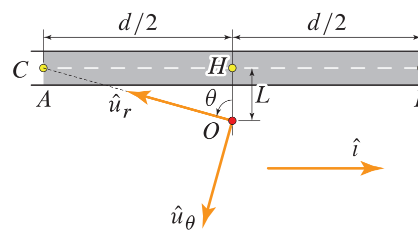
Problem 2.202

A motion tracking camera is placed along a rectilinear stretch of a racetrack (the figure is not to scale). A car C enters the stretch at A with a speed $v_A = 110$ mph and accelerates uniformly in time so that at B it has a speed $v_B = 175$ mph, where $d = 1$ mi. Letting the distance $L = 50$ ft, if the camera is to track C, determine the camera's angular velocity and the time rate of change of the angular velocity when the car is at A and at H.



Solution

Referring to figure at the right, the camera points from O to the car along the direction of $\hat{u}_r$. Hence, the angular velocity and acceleration of the camera are given by $\dot{\theta}$ and $\ddot{\theta}$. To determine these quantities at A and H, we first characterize the velocity and acceleration of the car using the unit vector $\hat{\imath}$ and then we find corresponding expressions using the polar component system with the unit vectors $\hat{u}_r$ and $\hat{u}_\theta$. We begin by observing that the motion of the car is rectilinear so that we have



$$\vec{v} = v\,\hat{\imath} \quad \text{and} \quad \vec{a} = a_c\,\hat{\imath}, \tag{1}$$

where v is the speed of the car and a_c is the magnitude of the (constant) acceleration of the car. We can determine a_c using the constant acceleration equation of the type $v^2 = v_0^2 + 2a_c(s - s_0)$:

$$v_B^2 = v_A^2 + 2a_c d \quad \Rightarrow \quad a_c = \frac{v_B^2 - v_A^2}{2d}. \tag{2}$$

Using the same constant acceleration equation, we can also determine the speed of the car at H:

$$v_H^2 = v_A^2 + 2a_c(d/2) \quad \Rightarrow \quad v_H^2 = v_A^2 + \tfrac{1}{2}\big(v_B^2 - v_A^2\big) \quad \Rightarrow \quad v_H = \sqrt{\frac{v_B^2 + v_A^2}{2}}, \tag{3}$$

where we have used the expression of a_c in the last of Eqs. (2). To relate v_A, v_H and a_c to $\dot{\theta}$ and $\ddot{\theta}$, we now observe that

$$\hat{\imath} = -\sin\theta\,\hat{u}_r - \cos\theta\,\hat{u}_\theta \quad \Rightarrow \quad \vec{v} = -v(\sin\theta\,\hat{u}_r + \cos\theta\,\hat{u}_\theta). \tag{4}$$

In polar coordinates we must have $\vec{v} = \dot{r}\,\hat{u}_r + r\dot{\theta}\,\hat{u}_\theta$, which, when compared to the last of Eqs. (4), implies

$$\dot{r} = -v\sin\theta \quad \text{and} \quad r\dot{\theta} = -v\cos\theta. \tag{5}$$

Solving the second of Eqs. (5) for $\dot{\theta}$, at A and H, we have

$$\dot{\theta}_A = -v_A\cos\theta_A/r_A \quad \text{and} \quad \dot{\theta}_H = -v_H\cos\theta_H/r_H. \tag{6}$$

Using trigonometry, we see that

$$r_A = \sqrt{(d/2)^2 + L^2}, \quad \cos\theta_A = L/\sqrt{(d/2)^2 + L^2}, \quad r_H = L, \quad \text{and} \quad \theta_H = 0. \tag{7}$$

Using Eqs. (7), (6), and the last of Eqs. (3), we have

$$\dot{\theta}_A = -\frac{v_A L}{(d/2)^2 + L^2} \quad \text{and} \quad \dot{\theta}_H = -\sqrt{\frac{v_B^2 + v_A^2}{2L^2}}. \tag{8}$$

Recalling that $v_A = 110\,\text{mph} = 110\big(\frac{5280}{3600}\big)\,\text{ft/s}$, $v_B = 175\,\text{mph} = 175(\frac{5280}{3600})\,\text{ft/s}$, $L = 50\,\text{ft}$, and $d = 1\,\text{mi} = 5280\,\text{ft}$, Eqs. (8) give

$$\boxed{\dot{\theta}_A = -0.001157\,\text{rad/s} \quad \text{and} \quad \dot{\theta}_H = -4.287\,\text{rad/s}.}$$

We now turn to the determination of $\ddot{\theta}$ at A and H. The acceleration of the car is

$$\vec{a} = -a_c(\sin\theta\,\hat{u}_r + \cos\theta\,\hat{u}_\theta). \tag{9}$$

In polar coordinates we have $\vec{a} = \big(\ddot{r} - r\dot{\theta}^2\big)\,\hat{u}_r + \big(r\ddot{\theta} + 2\dot{r}\dot{\theta}\big)\,\hat{u}_\theta$, which, when compared to Eq. (9), implies

$$-a_c\sin\theta = \ddot{r} - r\dot{\theta}^2 \quad \text{and} \quad -a_c\cos\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}. \tag{10}$$

Solving the second of Eqs. (10) for $\ddot{\theta}$, we have

$$\ddot{\theta} = -\frac{1}{r}\big(a_c\cos\theta - 2v\dot{\theta}\sin\theta\big), \tag{11}$$

where we have used the expression for $\dot{r}$ in the first of Eqs. (5). Hence, observing that $\sin\theta_A = d/(2r_A)$, using Eqs. (7), (8), the last of Eqs. (3), and the last of Eqs. (2), after simplification, we have

$$\ddot{\theta}_A = -\frac{dL}{(d/2)^2 + L^2}\left(\frac{v_B^2 - v_A^2}{2d^2} + \frac{v_A^2}{(d/2)^2 + L^2}\right) \quad \text{and} \quad \ddot{\theta}_H = -\frac{v_B^2 - v_A^2}{2dL}. \tag{12}$$

Recalling that $v_A = 110\,\text{mph} = 110\big(\frac{5280}{3600}\big)\,\text{ft/s}$, $v_B = 175\,\text{mph} = 175(\frac{5280}{3600})\,\text{ft/s}$, $L = 50\,\text{ft}$, and $d = 1\,\text{mi} = 5280\,\text{ft}$, Eqs. (12) give

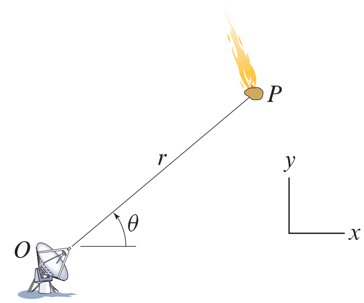
$$\boxed{\ddot{\theta}_A = -1.684\times10^{-4}\,\text{rad/s}^2 \quad \text{and} \quad \ddot{\theta}_H = -0.07547\,\text{rad/s}^2.}$$

Problem 2.203

The radar station at O is tracking a meteor P as it moves through the atmosphere. At the instant shown, the station measures the following data for the motion of the meteor: $r = 21{,}000$ ft, $\theta = 40°$, $\dot{r} = -22{,}440$ ft/s, $\dot{\theta} = -2.935$ rad/s, $\ddot{r} = 187{,}500$ ft/s^2, and $\ddot{\theta} = -5.409$ rad/s^2.

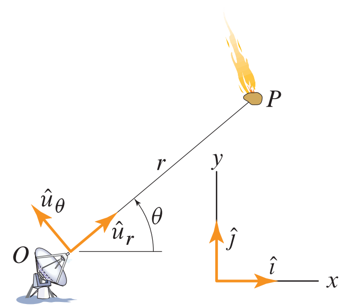
(a) Determine the magnitude and direction (relative to the xy coordinate system shown) of the velocity vector at this instant.

(b) Determine the magnitude and direction (relative to the xy coordinate system shown) of the acceleration vector at this instant.



Solution

Part (a). Referring to the figure at the right, the velocity of the meteor is



$$\vec{v} = \dot{r}\,\hat{u}_r + r\dot{\theta}\,\hat{u}_\theta. \tag{1}$$

Then the speed is

$$|\vec{v}| = \sqrt{(\dot{r})^2 + (r\dot{\theta})^2} \quad\Rightarrow\quad \boxed{|\vec{v}| = 65{,}590\,\text{ft/s},} \tag{2}$$

where we have used the following numerical data: $\dot{r} = -22{,}440$ ft/s, $r = 21{,}000$ ft, and $\dot{\theta} = -2.935$ rad/s. To find the orientation of $\vec{v}$ relative to the xy axes, we note that

$$\hat{u}_r = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath} \quad\text{and}\quad \hat{u}_\theta = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}, \tag{3}$$

so that Eq. (1) can be rewritten

$$\vec{v} = \underbrace{\left(\dot{r}\cos\theta - r\dot{\theta}\sin\theta\right)}_{v_x}\,\hat{\imath} + \underbrace{\left(\dot{r}\sin\theta + r\dot{\theta}\cos\theta\right)}_{v_y}\,\hat{\jmath} = (22{,}430\,\hat{\imath} - 61{,}640\,\hat{\imath})\,\text{ft/s}. \tag{4}$$

Since $\vec{v}$ points downward and to the right, its orientation from the x axis is

$$\text{Orientation of } \vec{v} \text{ from } x \text{ axis} = -\tan^{-1}\left(\left|\frac{v_y}{v_x}\right|\right) = -\tan^{-1}\left(\left|\frac{\dot{r}\sin\theta + r\dot{\theta}\cos\theta}{\dot{r}\cos\theta - r\dot{\theta}\sin\theta}\right|\right) = -70.01°, \tag{5}$$

where, again, we have used the fact that $\dot{r} = -22{,}440$ ft/s, $r = 21{,}000$ ft, and $\dot{\theta} = -2.935$ rad/s. Therefore, we have

$$\boxed{\text{Orientation of } \vec{v} \text{ from } x \text{ axis} = 70.01°\ (\text{cw}).}$$

Part (b). The acceleration of the meteor is

$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{u}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{u}_\theta. \tag{6}$$

Then the magnitude of $\vec{a}$ is

$$|\vec{a}| = \sqrt{\left(\ddot{r} - r\dot{\theta}^2\right)^2 + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)^2} \quad\Rightarrow\quad \boxed{|\vec{a}| = 19{,}300\,\text{ft/s}^2,} \tag{7}$$

where we have used the following numerical data: $\ddot{r} = 187{,}500\,\text{ft/s}^2$, $\dot{r} = -22{,}440\,\text{ft/s}$, $\dot{\theta} = -2.935\,\text{rad/s}$, $r = 21{,}000\,\text{ft}$, and $\ddot{\theta} = -5.409\,\text{rad/s}^2$.

To determine the orientation of $\vec{a}$ relative to the xy system, we use Eqs. (3) to rewrite Eq. (6) as

$$\vec{a} = \underbrace{\big[(\ddot{r} - r\dot{\theta}^2)\cos\theta - (r\ddot{\theta} + 2\dot{r}\dot{\theta})\sin\theta\big]}_{a_x}\,\hat{\imath} + \underbrace{\big[(\ddot{r} - r\dot{\theta}^2)\sin\theta + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\cos\theta\big]}_{a_y}\,\hat{\jmath}$$

$$= (-6599\,\hat{\imath} + 18{,}130\,\hat{\imath})\,\text{ft/s}. \quad (8)$$

Since $\vec{a}$ is directed upward and to the left, its orientation from the x axis is

$$\text{Orientation of } \vec{a} \text{ from } x \text{ axis} = 180^\circ - \tan^{-1}\left(\left|\frac{a_y}{a_x}\right|\right)$$

$$= 180^\circ - \tan^{-1}\left[\left|\frac{(\ddot{r} - r\dot{\theta}^2)\sin\theta + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\cos\theta}{(\ddot{r} - r\dot{\theta}^2)\cos\theta - (r\ddot{\theta} + 2\dot{r}\dot{\theta})\sin\theta}\right|\right] = 110.0^\circ, \quad (9)$$

where, again, we have used the fact that $\ddot{r} = 187{,}500\,\text{ft/s}^2$, $\dot{r} = -22{,}440\,\text{ft/s}$, $\dot{\theta} = -2.935\,\text{rad/s}$, $r = 21{,}000\,\text{ft}$, and $\ddot{\theta} = -5.409\,\text{rad/s}^2$. Hence, we have

$$\boxed{\text{Orientation of } \vec{a} \text{ from } x \text{ axis} = 110^\circ \text{ (ccw).}}$$

Problem 2.204

The time derivative of the acceleration, i.e., $\dot{\vec{a}}$, is usually referred to as the *jerk*. Starting from Eq. (2.70), compute the jerk in polar coordinates.

Solution

Equation (2.70) on p. 105 of the textbook states

$$\vec{a} = \big(\ddot{r} - r\dot{\theta}^2\big)\hat{u}_r + \big(r\ddot{\theta} + 2\dot{r}\dot{\theta}\big)\hat{u}_\theta = a_r\,\hat{u}_r + a_\theta\,\hat{u}_\theta, \tag{1}$$

where

$$a_r = \ddot{r} - r\dot{\theta}^2 \quad \text{and} \quad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}. \tag{2}$$

Differentiating with respect to time the second expression for $\vec{a}$ in Eq. (1), we have

$$\dot{\vec{a}} = \dot{a}_r\,\hat{u}_r + a_r\dot{\hat{u}}_r + \dot{a}_\theta\,\hat{u}_\theta + a_\theta\,\dot{\hat{u}}_\theta. \tag{3}$$

Recalling that

$$\dot{\hat{u}}_r = \dot{\theta}\,\hat{k}\times\hat{u}_r = \dot{\theta}\hat{u}_\theta \quad \text{and} \quad \dot{\hat{u}}_\theta = \dot{\theta}\,\hat{k}\times\hat{u}_\theta = -\dot{\theta}\hat{u}_r, \tag{4}$$

Eq. (3) can be rewritten as

$$\dot{\vec{a}} = \big(\dot{a}_r - \dot{\theta}a_\theta\big)\,\hat{u}_r + \big(\dot{a}_\theta + \dot{\theta}a_r\big)\,\hat{u}_\theta. \tag{5}$$

Differentiating with respect to time Eqs. (2) gives

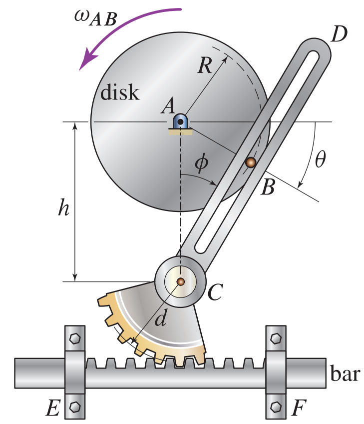
$$\dot{a}_r = \dddot{r} - \dot{r}\dot{\theta}^2 - 2r\dot{\theta}\ddot{\theta} \quad \text{and} \quad \dot{a}_\theta = \dot{r}\ddot{\theta} + r\dddot{\theta} + 2\ddot{r}\dot{\theta} + 2\dot{r}\ddot{\theta}. \tag{6}$$

Substituting Eqs. (2) and (6) into Eq. (5) and simplifying, we have

$$\boxed{\dot{\vec{a}} = \Big(\dddot{r} - 3r\dot{\theta}\ddot{\theta} - 3\dot{r}\dot{\theta}^2\Big)\,\hat{u}_r + \Big[r\Big(\dddot{\theta} - \dot{\theta}^3\Big) + 3\ddot{r}\dot{\theta} + 3\dot{r}\ddot{\theta}\Big]\,\hat{u}_\theta.}$$

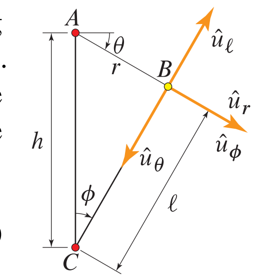
Problem 2.205

The *reciprocating rectilinear motion* mechanism shown consists of a disk pinned at its center at A that rotates with a constant angular velocity ω_{AB}, a slotted arm CD that is pinned at C, and a bar that can oscillate within the guides at E and F. As the disk rotates, the peg at B moves within the slotted arm, causing it to rock back and forth. As the arm rocks, it provides a slow advance and a quick return to the reciprocating bar due to the change in distance between C and B. Letting $\theta = 30°$, $\omega_{AB} = 50\,\text{rpm}$, $R = 0.3\,\text{ft}$, and $h = 0.6\,\text{ft}$, determine $\dot\phi$ and $\ddot\phi$, i.e., the angular velocity and angular acceleration of the slotted arm CD, respectively.



Solution

Referring to the diagram at the right, we define two component systems: one consisting of the unit vectors $\hat u_r$ and $\hat u_\theta$, and the other consisting of the unit vectors $\hat u_\ell$ and $\hat u_\phi$. Both these component systems are polar, the first with coordinates r and θ, and the other with coordinates ℓ and ϕ. The coordinate $r = R =$ const., whereas the coordinate ℓ varies with time. Because $h = 2R$, geometry tells us that



$$\text{when } \theta = 30°, \quad \hat u_\phi = \hat u_r, \quad \hat u_\ell = -\hat u_\theta, \quad \phi = \theta, \quad \text{and} \quad \ell = h\cos\theta. \tag{1}$$

The position of B can be described relative to the fixed points A and C: $\vec r_{B/A} = r\,\hat u_r$ and $\vec r_{B/C} = \ell\,\hat u_\ell$. Hence, the velocity of B can be given the following two expressions:

$$\vec v = \dot r\,\hat u_r + r\dot\theta\,\hat u_\theta = -R\omega_{AB}\,\hat u_\theta \quad \text{and} \quad \vec v = \dot\ell\,\hat u_\ell + \ell\dot\phi\,\hat u_\phi, \tag{2}$$

where we accounted for the fact that $r = R =$ const. and $\dot\theta = -\omega_{AB}$. Setting the above two expressions of velocity equal to each other and using Eqs. (1), when $\theta = 30°$, we have

$$-R\omega_{AB}\,\hat u_\theta = -\dot\ell\,\hat u_\theta + (h\cos\theta)\dot\phi\,\hat u_r \quad \Rightarrow \quad \dot\ell = -R\omega_{AB} \quad \text{and} \quad \boxed{\dot\phi = 0.} \tag{3}$$

Similarly to $\vec v$, the acceleration of B has the following two expressions:

$$\vec a = \left(\ddot r - r\dot\theta^2\right)\hat u_r + \left(r\ddot\theta + 2\dot r\dot\theta\right)\hat u_\theta = -R\omega_{AB}^2\,\hat u_r \quad \text{and} \quad \vec a = \left(\ddot\ell - \ell\dot\phi^2\right)\hat u_\ell + \left(\ell\ddot\phi + 2\dot\ell\dot\phi\right)\hat u_\phi, \tag{4}$$

where we recalled that $r = R =$ const. and $\dot\theta = -\omega_{AB} =$ const. Setting the above two expressions of acceleration equal to each other, using Eqs. (1) and the last of Eqs. (3), when $\theta = 30°$, we have

$$-R\omega_{AB}^2\,\hat u_r = -\ddot\ell\,\hat u_\theta + (h\cos\theta)\ddot\phi\,\hat u_r \quad \Rightarrow \quad \ddot\ell = 0 \quad \text{and} \quad \ddot\phi = -\frac{R\omega_{AB}^2}{h\cos\theta}. \tag{5}$$

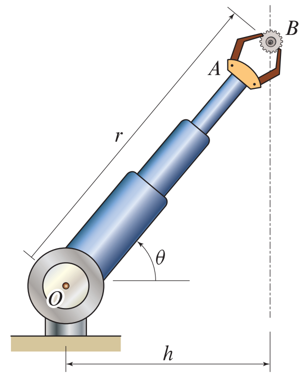
Recalling that $R = 0.3\,\text{ft}$, $\omega_{AB} = 50\,\text{rpm} = 50\left(\frac{2\pi}{60}\right)\text{rad/s}$, $h = 0.6\,\text{ft}$, and $\theta = 30°$, the last of Eqs. (5) can be evaluated to obtain

$$\boxed{\ddot\phi = -15.83\,\text{rad/s}^2.}$$

Problem 2.206

As a part of an assembly process, the end effector at A on the robotic arm needs to move the gear at B along the vertical line shown with some known velocity v_0 and acceleration a_0. Arm OA can vary its length by telescoping via internal actuators, and a motor at O allows it to pivot in the vertical plane.

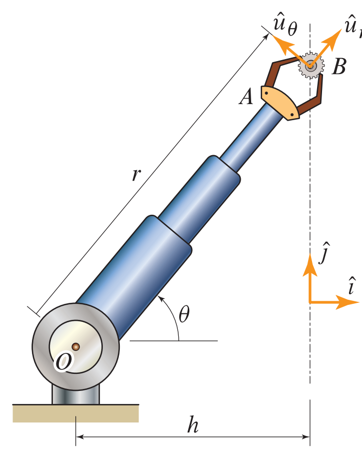
When $\theta = 50°$, it is required that $v_0 = 8\,\text{ft/s}$ (down) and that it be slowing down at $a_0 = 2\,\text{ft/s}^2$. Using $h = 4\,\text{ft}$, determine, at this instant, the values for $\ddot{r}$ (the extensional acceleration) and $\ddot{\theta}$ (the angular acceleration).



Solution

Referring to the figure at the right, the length of the arm as a function of θ is

$$r = h/\cos\theta. \tag{1}$$



The velocity of B can be expressed in both the Cartesian and polar component systems shown. Since B moves downward, this gives

$$\vec{v}_B = -v_0\,\hat{\jmath} = \dot{r}\,\hat{u}_r + r\dot{\theta}\,\hat{u}_\theta. \tag{2}$$

We note that

$$\hat{\jmath} = \sin\theta\,\hat{u}_r + \cos\theta\,\hat{u}_\theta, \tag{3}$$

so that Eq. (2) can be written as $-v_0(\sin\theta\,\hat{u}_r + \cos\theta\,\hat{u}_\theta) = \dot{r}\,\hat{u}_r + r\dot{\theta}\,\hat{u}_\theta$, which implies $\dot{r} = -v_0\sin\theta$ and $r\dot{\theta} = -v_0\cos\theta$, i.e.,

$$\dot{r} = -v_0\sin\theta \quad \text{and} \quad \dot{\theta} = -\frac{v_0\cos^2\theta}{h}, \tag{4}$$

where we have used Eq. (1). Since B is slowing down (in its downward motion), the acceleration of B, using both component systems, is

$$\vec{a}_B = a_0\,\hat{\jmath} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{u}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{u}_\theta$$

$$\Rightarrow \quad a_0(\sin\theta\,\hat{u}_r + \cos\theta\,\hat{u}_\theta) = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{u}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{u}_\theta, \tag{5}$$

where we have used Eq. (3). Equating components, we obtain

$$\ddot{r} - r\dot{\theta}^2 = a_0\sin\theta \quad \text{and} \quad r\ddot{\theta} + 2\dot{r}\dot{\theta} = a_0\cos\theta. \tag{6}$$

Using the results from Eqs. (1) and (4), Eqs. (6) give

$$\ddot{r} = a_0\sin\theta + \frac{h}{\cos\theta}\left(-\frac{v_0\cos^2\theta}{h}\right)^2 \quad \Rightarrow \quad \ddot{r} = a_0\sin\theta + \frac{v_0^2\cos^3\theta}{h} \quad \Rightarrow \quad \boxed{\ddot{r} = 5.781\,\text{ft/s}^2,}$$

and

$$\ddot{\theta} = \frac{a_0\cos^2\theta}{h} - 2(-v_0\sin\theta)\left(-\frac{v_0\cos^2\theta}{h}\right)\left(\frac{\cos\theta}{h}\right) \quad \Rightarrow \quad \ddot{\theta} = \frac{a_0\cos^2\theta}{h} - \frac{2v_0^2\cos^3\theta\sin\theta}{h^2}$$

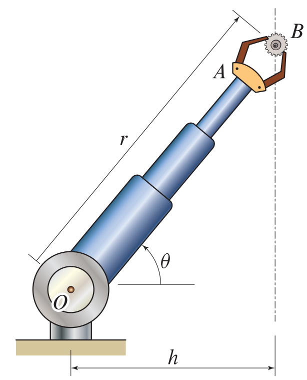
$$\Rightarrow \quad \boxed{\ddot{\theta} = -1.421\,\text{rad/s}^2,}$$

where we have used the following numerical data $\theta = 50°$, $v_0 = 8\,\text{ft/s}$, $a_0 = 2\,\text{ft/s}^2$, and $h = 4\,\text{ft}$.

Problem 2.207

As a part of an assembly process, the end effector at A on the robotic arm needs to move the gear at B along the vertical line shown with some known velocity v_0 and acceleration a_0. Arm OA can vary its length by telescoping via internal actuators, and a motor at O allows it to pivot in the vertical plane.

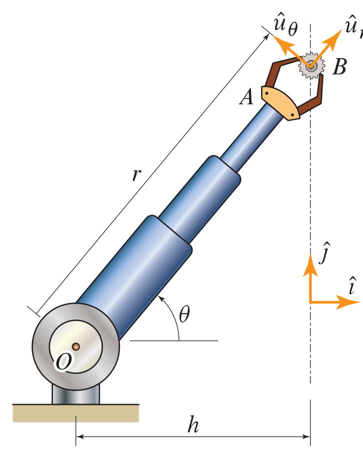
Letting v_0 and a_0 be positive if the gear moves and accelerates upward, determine expressions for r, $\dot{r}$, $\ddot{r}$, $\dot{\theta}$, and $\ddot{\theta}$ that are valid for any value of θ.



Solution

Referring to the figure at the right, the length of the arm as a function of θ is

$$r = h/\cos\theta. \tag{1}$$



The velocity of B can be expressed in both the Cartesian and polar component systems shown. Since $v_0 > 0$ when B moves upward, this gives

$$\vec{v}_B = v_0\,\hat{\jmath} = \dot{r}\,\hat{u}_r + r\dot{\theta}\,\hat{u}_\theta. \tag{2}$$

We note that

$$\hat{\jmath} = \sin\theta\,\hat{u}_r + \cos\theta\,\hat{u}_\theta, \tag{3}$$

so that Eq. (2) can be written as $v_0\,(\sin\theta\,\hat{u}_r + \cos\theta\,\hat{u}_\theta) = \dot{r}\,\hat{u}_r + r\dot{\theta}\,\hat{u}_\theta$, which implies $\dot{r} = v_0\sin\theta$ and $r\dot{\theta} = v_0\cos\theta$, i.e.,

$$\dot{r} = v_0\sin\theta \quad \text{and} \quad \dot{\theta} = \frac{v_0\cos^2\theta}{h}, \tag{4}$$

where we have used Eq. (1). Since $a_0 > 0$ when B accelerates upward, the acceleration of B, in both component systems, is

$$\vec{a}_B = a_0\,\hat{\jmath} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{u}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{u}_\theta$$
$$\Rightarrow \quad a_0\,(\sin\theta\,\hat{u}_r + \cos\theta\,\hat{u}_\theta) = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{u}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{u}_\theta, \tag{5}$$

where used Eq. (3). Equating components, we obtain

$$\ddot{r} - r\dot{\theta}^2 = a_0\sin\theta \quad \text{and} \quad r\ddot{\theta} + 2\dot{r}\dot{\theta} = a_0\cos\theta. \tag{6}$$

Using the results from Eqs. (1) and (4), Eqs. (6) give

$$\ddot{r} = a_0\sin\theta + \frac{h}{\cos\theta}\left(\frac{v_0\cos^2\theta}{h}\right)^2 \quad \Rightarrow \quad \ddot{r} = a_0\sin\theta + \frac{v_0^2\cos^3\theta}{h},$$

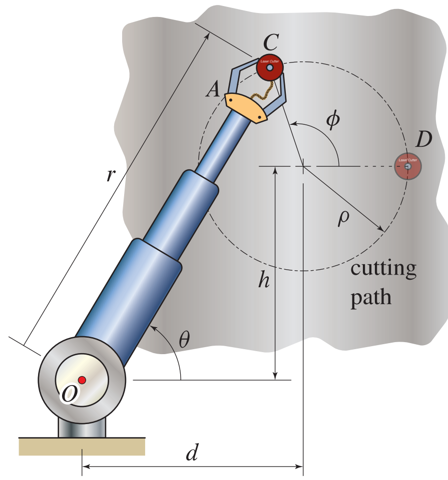
and

$$\ddot{\theta} = \frac{a_0\cos^2\theta}{h} - 2v_0\sin\theta\left(\frac{v_0\cos^2\theta}{h}\right)\left(\frac{\cos\theta}{h}\right) \quad \Rightarrow \quad \ddot{\theta} = \frac{a_0\cos^2\theta}{h} - \frac{2v_0^2\cos^3\theta\sin\theta}{h^2}.$$

Problem 2.208

In the cutting of sheet metal, the robotic arm OA needs to move the cutting tool at C counterclockwise at a constant speed v_0 along a circular path of radius ρ. The center of the circle is located in the position shown relative to the base of the robotic arm at O.

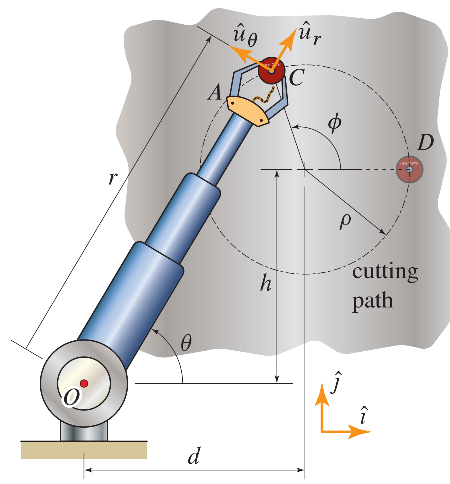
When the cutting tool is at D ($\phi = 0$), determine r, $\dot{r}$, $\dot{\theta}$, $\ddot{r}$, and $\ddot{\theta}$ as functions of the given quantities (i.e., d, h, ρ, v_0).



Solution

Referring to the figure at the right, when C is at D

$$r = \sqrt{h^2 + (d+\rho)^2}. \tag{1}$$



We note that the unit vectors $\hat{\imath}$ and $\hat{\jmath}$ can be expressed as follows:

$$\hat{\imath} = \cos\theta\,\hat{u}_r - \sin\theta\,\hat{u}_\theta \quad \text{and} \quad \hat{\jmath} = \sin\theta\,\hat{u}_r + \cos\theta\,\hat{u}_\theta. \tag{2}$$

The velocity of C must be tangent to the cutting path. Since C moves counterclockwise around the cutting path, and since the speed of C is v_0, when at D we must have

$$\vec{v} = v_0\,\hat{\jmath}. \tag{3}$$

Substituting the second of Eqs. (2) into Eq. (3), we have

$$\vec{v} = v_0 \sin\theta\,\hat{u}_r + v_0\cos\theta\,\hat{u}_\theta. \tag{4}$$

Since in polar coordinates we have $\vec{v} = \dot{r}\,\hat{u}_r + r\dot{\theta}\,\hat{u}_\theta$, from Eq. (4) we deduce that, when C is at D,

$$\dot{r} = v_0\sin\theta \quad \text{and} \quad \dot{\theta} = v_0\cos\theta/r. \tag{5}$$

Now, we observe that when C is at D, we have

$$\sin\theta = h\Big/\sqrt{h^2+(d+\rho)^2} \quad \text{and} \quad \cos\theta = (d+\rho)\Big/\sqrt{h^2+(d+\rho)^2}. \tag{6}$$

Substituting Eqs. (1) and (6) into Eqs. (5), when C is at D, we have

$$\dot{r} = \frac{v_0 h}{\sqrt{h^2+(d+\rho)^2}} \quad \text{and} \quad \dot{\theta} = \frac{v_0(d+\rho)}{h^2+(d+\rho)^2}. \tag{7}$$

Since the speed of C is constant, the acceleration of C must always be directed toward the center of the cutting path. Therefore, when C is at D, we have

$$\vec{a} = -\frac{v_0^2}{\rho}\,\hat{\imath}. \tag{8}$$

Substituting the first of Eqs. (2) into Eq. (8), we have

$$\vec{a} = -\frac{v_0^2\cos\theta}{\rho}\,\hat{u}_r + \frac{v_0^2\sin\theta}{\rho}\,\hat{u}_\theta. \tag{9}$$

Now we recall that, in polar coordinates, we have $\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{u}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{u}_\theta$, which, by comparison with Eq. (9), implies that, when C is at D

$$\ddot{r} = r\dot{\theta}^2 - \frac{v_0^2\cos\theta}{\rho} \quad \text{and} \quad \ddot{\theta} = -2\frac{\dot{r}\dot{\theta}}{r} + \frac{v_0^2\sin\theta}{r\rho}. \tag{10}$$

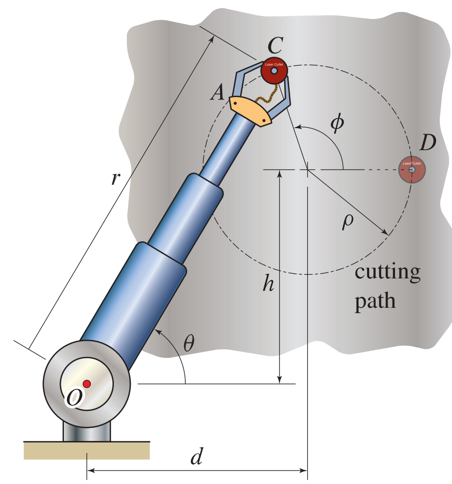
Substituting the expressions for r in Eq. (1), $\sin\theta$ and $\cos\theta$ in Eqs. (6), and for $\dot{r}$ and $\dot{\theta}$ in Eqs. (7) into Eqs. (10), after simplification, when C is at D we have

$$\boxed{\ddot{r} = -\frac{v_0^2}{\rho}\frac{(\rho+d)\left(d^2+h^2+d\rho\right)}{\left(d^2+h^2+\rho^2+2d\rho\right)^{3/2}} \quad \text{and} \quad \ddot{\theta} = \frac{v_0^2}{\rho}\frac{h\left(d^2+h^2-\rho^2\right)}{\left(d^2+h^2+\rho^2+2d\rho\right)^2}.} \tag{11}$$

Problem 2.209

In the cutting of sheet metal, the robotic arm OA needs to move the cutting tool at C counterclockwise at a constant speed v_0 along a circular path of radius ρ. The center of the circle is located in the position shown relative to the base of the robotic arm at O.

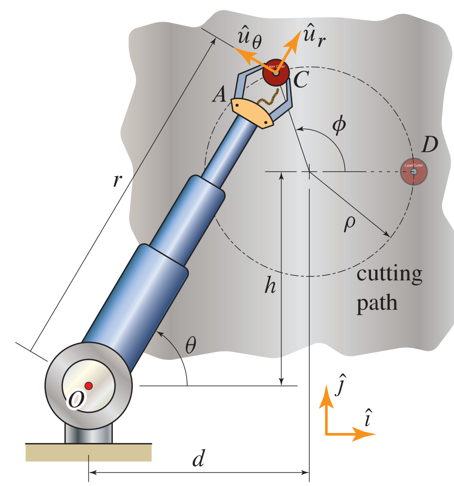
For all positions along the circular cut (i.e., for any value of ϕ), determine r, $\dot{r}$, $\dot{\theta}$, $\ddot{r}$, and $\ddot{\theta}$ as functions of the given quantities (i.e., d, h, ρ, v_0). These quantities can be found "by hand," but it is tedious, so you might consider using symbolic algebra software, such as Mathematica or Maple.



Solution

Referring to the figure at the right, for a generic value of ϕ we have

$$r = \sqrt{(h + \rho\sin\phi)^2 + (d + \rho\cos\phi)^2}. \tag{1}$$



We note that the unit vectors $\hat{\imath}$ and $\hat{\jmath}$ can be expressed as follows:

$$\hat{\imath} = \cos\theta\,\hat{u}_r - \sin\theta\,\hat{u}_\theta \quad \text{and} \quad \hat{\jmath} = \sin\theta\,\hat{u}_r + \cos\theta\,\hat{u}_\theta. \tag{2}$$

The velocity of C must be tangent to the cutting path. Since C moves counterclockwise around the cutting path and since the speed of C is v_0, we must have

$$\vec{v} = v_0(-\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath}). \tag{3}$$

Substituting Eqs. (2) into Eq. (3), we have

$$\vec{v} = v_0(\cos\phi\sin\theta - \cos\theta\sin\phi)\,\hat{u}_r + v_0(\cos\theta\cos\phi + \sin\theta\sin\phi)\,\hat{u}_\theta. \tag{4}$$

Since in polar coordinates we have $\vec{v} = \dot{r}\,\hat{u}_r + r\dot{\theta}\,\hat{u}_\theta$, then from Eq. (4) we deduce that

$$\dot{r} = v_0(\sin\theta\cos\phi - \cos\theta\sin\phi) \quad \text{and} \quad \dot{\theta} = \frac{v_0}{r}(\cos\theta\cos\phi + \sin\theta\sin\phi). \tag{5}$$

Using geometry, we have $\sin\theta = (h + \rho\sin\phi)/r$ and $\cos\theta = (d + \rho\cos\phi)/r$, which, in view of Eq. (1), give

$$\sin\theta = \frac{h + \rho\sin\phi}{\sqrt{(h + \rho\sin\phi)^2 + (d + \rho\cos\phi)^2}} \quad \text{and} \quad \cos\theta = \frac{d + \rho\cos\phi}{\sqrt{(h + \rho\sin\phi)^2 + (d + \rho\cos\phi)^2}}. \tag{6}$$

Substituting Eqs. (1) and (6) into Eqs. (5), and simplifying, we have

$$\dot{r} = \frac{v_0(h\cos\phi - d\sin\phi)}{\sqrt{(d + \rho\cos\phi)^2 + (h + \rho\sin\phi)^2}} \quad \text{and} \quad \dot{\theta} = \frac{v_0(\rho + d\cos\phi + h\sin\phi)}{d^2 + h^2 + \rho^2 + 2d\rho\cos\phi + 2h\rho\sin\phi}. \tag{7}$$

Since the speed of C is constant, the acceleration of C must always be directed toward the center of the cutting path. Therefore, we have

$$\vec{a} = -\frac{v_0^2}{\rho}(\cos\phi\,\hat{\imath} + \sin\phi\,\hat{\jmath}), \tag{8}$$

which, along with Eqs. (2), imply

$$\vec{a} = -\frac{v_0^2}{\rho}(\cos\theta\cos\phi + \sin\theta\sin\phi)\,\hat{u}_r + \frac{v_0^2}{\rho}(\cos\phi\sin\theta - \cos\theta\sin\phi)\,\hat{u}_\theta. \tag{9}$$

Substituting Eqs. (6) into Eq. (9), we have

$$\vec{a} = -\frac{v_0^2}{\rho}\frac{\rho + d\cos\phi + h\sin\phi}{\sqrt{(h+\rho\sin\phi)^2 + (d+\rho\cos\phi)^2}}\,\hat{u}_r + \frac{v_0^2}{\rho}\frac{h\cos\phi - d\sin\phi}{\sqrt{(h+\rho\sin\phi)^2 + (d+\rho\cos\phi)^2}}\,\hat{u}_\theta. \tag{10}$$

Now recall that in polar coordinates, we have $\vec{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{u}_\theta$, and therefore, by comparison with the above expression, we can conclude that

$$\ddot{r} = r\dot{\theta}^2 - \frac{v_0^2}{\rho}\frac{\rho + d\cos\phi + h\sin\phi}{\sqrt{(h+\rho\sin\phi)^2 + (d+\rho\cos\phi)^2}} \tag{11}$$

and

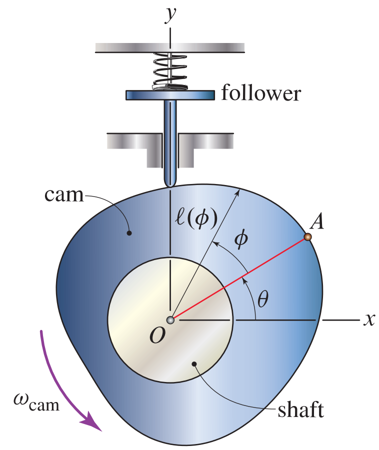
$$\ddot{\theta} = -2\frac{\dot{r}}{r}\dot{\theta} + \frac{v_0^2}{r\rho}\frac{h\cos\phi - d\sin\phi}{\sqrt{(h+\rho\sin\phi)^2 + (d+\rho\cos\phi)^2}}. \tag{12}$$

Substituting Eqs. (1) and (7) into Eqs. (11) and (12), after simplification, we have

$$\ddot{r} = -\frac{v_0^2}{\rho}\frac{(\rho + d\cos\phi + h\sin\phi)\big(d^2 + h^2 + d\rho\cos\phi + h\rho\sin\phi\big)}{\big(d^2 + h^2 + \rho^2 + 2d\rho\cos\phi + 2h\rho\sin\phi\big)^{3/2}},$$
$$\ddot{\theta} = \frac{v_0^2}{\rho}\frac{(h\cos\phi - d\sin\phi)\big(d^2 + h^2 - \rho^2\big)}{\big(d^2 + h^2 + \rho^2 + 2d\rho\cos\phi + 2h\rho\sin\phi\big)^2}.$$

Problem 2.210

The cam is mounted on a shaft that rotates about O with constant angular velocity ω_{cam}. The profile of the cam is described by the function $\ell(\phi) = R_0(1 + 0.25\cos^3\phi)$, where the angle ϕ is measured relative to the segment OA, which rotates with the cam. Letting $\omega_{\text{cam}} = 3000\,\text{rpm}$ and $R_0 = 3\,\text{cm}$, determine the velocity and acceleration of the follower when $\theta = 33°$. Express the acceleration of the follower in terms of g, the acceleration due to gravity.



Solution

The point on the follower in contact with the cam corresponds to $\phi = 90° - \theta$. Therefore, letting r be the distance between the point on the follower in contact with the cam and point O, we had that $r = \ell(\phi = 90° - \theta)$ (θ is measured in degrees), which gives

$$r = R_0\big[1 + 0.25\cos^3(90° - \theta)\big] = R_0\big(1 + 0.25\sin^3\theta\big), \tag{1}$$

where we have used the fact that $\cos 90° - \theta = \sin\theta$. Next, we observe that the velocity and acceleration of the follower are

$$\vec{v} = \dot{r}\,\hat{\jmath} \quad \text{and} \quad \vec{a} = \ddot{r}\,\hat{\jmath}. \tag{2}$$

From Eq. (1), we have

$$\dot{r} = 0.75R_0\dot{\theta}\sin^2\theta\cos\theta \tag{3}$$

and

$$\ddot{r} = 0.75R_0\ddot{\theta}\sin^2\theta\cos\theta + 0.75R_0\dot{\theta}^2\big(2\sin\theta\cos^2\theta - \sin^3\theta\big). \tag{4}$$

Recognizing that $\dot{\theta} = \omega_{\text{cam}} = \text{constant}$, Eqs. (3) and (4) simplify to

$$\dot{r} = 0.75R_0\omega_{\text{cam}}\sin^2\theta\cos\theta \quad \text{and} \quad \ddot{r} = 0.75R_0\omega_{\text{cam}}^2\big(2\sin\theta\cos^2\theta - \sin^3\theta\big). \tag{5}$$

Substituting Eqs. (5) into Eqs. (2), we have

$$\vec{v} = 0.75R_0\omega_{\text{cam}}\sin^2\theta\cos\theta\,\hat{\jmath} \quad \text{and} \quad \vec{a} = 0.75R_0\omega_{\text{cam}}^2\big(2\sin\theta\cos^2\theta - \sin^3\theta\big)\,\hat{\jmath}. \tag{6}$$

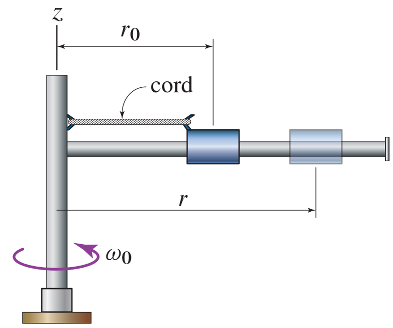
Recalling that $R_0 = 3\,\text{cm} = 0.03000\,\text{m}$, $\omega_{\text{cam}} = 3000\,\text{rpm} = 3000\frac{2\pi}{60}\,\text{rad/s}$, for $\theta = 33°$, we have

$$\boxed{\vec{v} = 1.758\,\hat{\jmath}\,\text{m/s} \quad \text{and} \quad \vec{a} = 136.9g\,\hat{\jmath}.}$$

where we have expressed the acceleration in terms of the acceleration due to gravity $g = 9.81\,\text{m/s}^2$.

Problem 2.211

The collar is mounted on the horizontal arm shown, which is originally rotating with the angular velocity ω_0. Assume that after the cord is cut, the collar slides along the arm in such a way that the collar's total acceleration is equal to zero. Determine an expression of the radial component of the collar's velocity as a function of r, the distance from the spin axis. *Hint:* Using polar coordinates, observe that $d(r^2\dot{\theta})/dt = ra_\theta$.



Solution

We are told that the collar's acceleration is zero, which implies that *each* component of its acceleration must be zero. Looking first at the transverse component of acceleration, we must have

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \quad \Rightarrow \quad ra_\theta = 0.$$

We now use the hint and notice that

$$\frac{d\left(r^2\dot{\theta}\right)}{dt} = 2r\dot{r}\dot{\theta} + r^2\ddot{\theta} = ra_\theta = 0 \quad \Rightarrow \quad r^2\dot{\theta} = \text{constant} = K,$$

where we have used the fact that if the time derivative of a quantity is zero, then that quantity must be constant. Since initially $r = r_0$ and $\dot{\theta} = \omega_0$, the constant K can easily be calculated to be

$$K = r^2\dot{\theta} = r_0^2\omega_0 \quad \Rightarrow \quad \dot{\theta} = \frac{r_0^2\omega_0}{r^2}. \tag{1}$$

Now that we know $\dot{\theta}$ for every value of r, we can use the fact that the radial component of acceleration is also zero to obtain

$$a_r = 0 \quad \Rightarrow \quad \ddot{r} - r\dot{\theta}^2 = \ddot{r} - r\left(\frac{r_0^2\omega_0}{r^2}\right)^2 = \ddot{r} - \frac{r_0^4\omega_0^2}{r^3} = 0, \tag{2}$$

where we have used Eq. (1) for $\dot{\theta}$. Using the chain rule on $\ddot{r}$ and then rearranging Eq. (2), we obtain

$$\ddot{r} = \frac{d\dot{r}}{dr}\frac{dr}{dt} \quad \Rightarrow \quad \dot{r}\,\frac{d\dot{r}}{dr} = \frac{r_0^4\omega_0^2}{r^3} \quad \Rightarrow \quad \dot{r}\,d\dot{r} = \frac{r_0^4\omega_0^2}{r^3}\,dr. \tag{3}$$

Integrating Eq. (3), we obtain

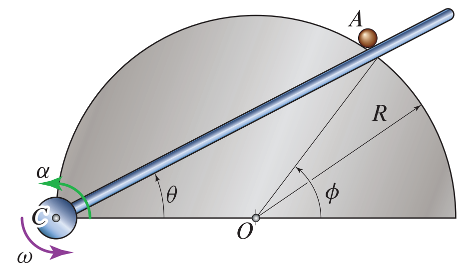
$$\int_0^{\dot{r}} \dot{r}\,d\dot{r} = \int_{r_0}^{r} \frac{r_0^4\omega_0^2}{r^3}\,dr \quad \Rightarrow \quad \tfrac{1}{2}\dot{r}^2 = \left.\frac{-r_0^4\omega_0^2}{2r^2}\right|_{r_0}^{r} \quad \Rightarrow \quad \dot{r} = r_0^2\omega_0\sqrt{\frac{1}{r_0^2} - \frac{1}{r^2}}, \tag{4}$$

where, in choosing the sign of the square root, we have used the fact that $\dot{r}$ is positive since the collar is sliding outward and therefore r is increasing. Simplifying the last of Eqs. (4), we have

$$\boxed{\dot{r} = \frac{r_0\omega_0}{r}\sqrt{r^2 - r_0^2}.}$$

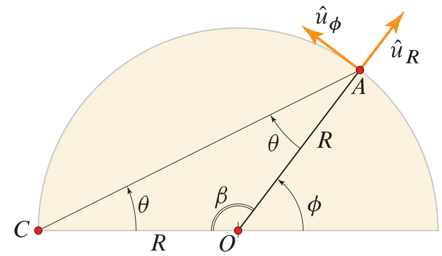
Problem 2.212

Particle A slides over the semicylinder while pushed by the arm pinned at C. The motion of the arm is controlled such that it starts from rest at $\theta = 0$, ω increases uniformly as a function of θ, and $\omega = 0.5\,\text{rad/s}$ for $\theta = 45°$. Letting $R = 4\,\text{in.}$, determine the speed and the magnitude of the acceleration of A when $\phi = 32°$.



Solution

We observe that A moves along a circle of radius R and center O. For this reason we describe the motion of A using a polar coordinate system with center at O so that the radial coordinate of A is $R =$ constant, and the transverse coordinate of A is ϕ. Then, the velocity of A is



$$\vec{v} = \dot{R}\,\hat{u}_R + R\dot{\phi}\,\hat{u}_\phi = R\dot{\phi}\,\hat{u}_\phi, \tag{1}$$

and the acceleration of A is

$$\vec{a} = \big(\ddot{R} - R\dot{\phi}^2\big)\,\hat{u}_R + \big(R\ddot{\phi} + 2\dot{R}\dot{\phi}\big)\,\hat{u}_\phi = R\big(-\dot{\phi}^2\,\hat{u}_R + \ddot{\phi}\,\hat{u}_\phi\big), \tag{2}$$

where, in both Eq. (1) and Eq. (2) we have accounted for the fact that R is constant. Therefore the speed and the magnitude of the acceleration of A are

$$v = R\big|\dot{\phi}\big| \quad \text{and} \quad \big|\vec{a}\big| = R\sqrt{\dot{\phi}^4 + \ddot{\phi}^2}. \tag{3}$$

Equations (3) show that the solution of the problem revolves around the determination of the quantities $\dot{\phi}$ and $\ddot{\phi}$ in terms of ϕ.

To determine the desired expression for $\dot{\phi}$, we begin by observing that the triangle COA is isosceles, with base CA and sides CO and OA. This implies that the angles $O\hat{C}A$ and $C\hat{A}O$ are both equal to θ. Hence, denoting by β the angle $C\hat{O}A$, we have

$$2\theta + \beta = 180° \quad \text{and} \quad \beta + \phi = 180° \quad \Rightarrow \quad \phi = 2\theta. \tag{4}$$

Differentiating the last of Eqs. (4) with respect to time, we have

$$\dot{\phi} = 2\dot{\theta} \quad \text{and} \quad \ddot{\phi} = 2\ddot{\theta}. \tag{5}$$

Therefore, we turn to the determination of $\dot{\theta}$ and $\ddot{\theta}$ and we begin by observing that

$$\dot{\theta} = \omega. \tag{6}$$

Next, we observe that the problem statement indicates that "ω increases uniformly as a function of θ" from $\omega = 0$, for $\theta = 0$, to $\omega = \omega_f = 0.5\,\text{rad/s}$, for $\theta = 45° = (\pi/4)\,\text{rad}$ (where the subscript f stands for final). This implies that $d\omega/d\theta = K$, where K is a constant (with dimensions of time to the power negative one). With this in mind, we have

$$K = \frac{d\omega}{d\theta} \quad \Rightarrow \quad K\,d\theta = d\omega \quad \Rightarrow \quad \int_0^\theta K\,d\theta = \int_0^\omega d\omega. \tag{7}$$

Carrying out the integration, we have

$$K\theta = \omega. \tag{8}$$

Recalling that $\omega = \omega_f = 0.5\,\text{rad/s}$ for $\theta = (\pi/4)\,\text{rad}$, from Eq. (8) we have

$$K(\pi/4) = \omega_f \quad \Rightarrow \quad K = 4\omega_f/\pi. \tag{9}$$

Substituting the last of Eqs. (9) into Eq. (8) and recalling Eq. (6), we have

$$\dot{\theta} = \frac{4\omega_f}{(\pi\ \text{rad})}\theta, \tag{10}$$

where it is understood that θ is expressed in radians. Then, differentiating Eq. (10) with respect to time we have

$$\ddot{\theta} = \frac{4\omega_f}{(\pi\ \text{rad})}\dot{\theta} \quad \Rightarrow \quad \ddot{\theta} = \left[\frac{4\omega_f}{(\pi\ \text{rad})}\right]^2\theta, \tag{11}$$

where we have used Eq. (10) to express $\dot{\theta}$, and where we note again that θ is expressed in radians.

Keeping in mind that the last of Eqs. (4) implies $\theta = \phi/2$, we can now rewrite Eqs. (5) using the results in Eqs. (10) and the last of Eqs. (11). This gives

$$\dot{\phi} = \frac{4\omega_f}{(\pi\ \text{rad})}\phi \quad \text{and} \quad \ddot{\phi} = \left[\frac{4\omega_f}{(\pi\ \text{rad})}\right]^2\phi, \tag{12}$$

where it is understood that ϕ must be expressed in radians. Substituting Eqs. (12) in Eqs. (3), we have

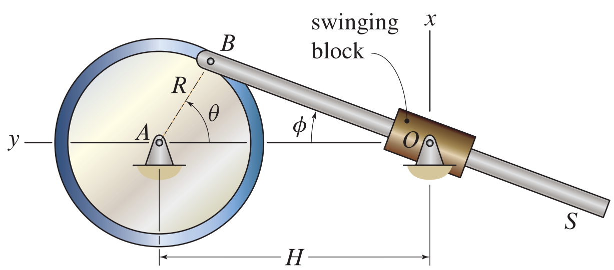
$$v = \frac{4\omega_f R}{(\pi\ \text{rad})}|\phi| \quad \text{and} \quad |\vec{a}| = R\left[\frac{4\omega_f}{(\pi\ \text{rad})}\right]^2\sqrt{\phi^4 + \phi^2}. \tag{13}$$

Recalling that $\omega_f = 0.5\,\text{rad/s}$, $R = 4\,\text{in.} = \frac{4}{12}\,\text{ft}$, $\phi = 32^\circ = 32\frac{\pi}{180}\,\text{rad}$, we can evaluate Eqs. (13) to obtain

$$\boxed{v = 0.1185\,\text{ft/s} \quad \text{and} \quad |\vec{a}| = 0.08642\,\text{ft/s}^2.}$$

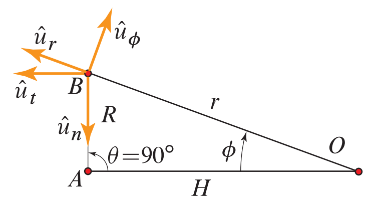
Problem 2.213

The mechanism shown is called a *swinging block* slider crank. First used in various steam locomotive engines in the 1800s, this mechanism is often found in door-closing systems. If the disk is rotating with a constant angular velocity $\dot{\theta} = 60\,\text{rpm}$, $H = 4\,\text{ft}$, $R = 1.5\,\text{ft}$, and r denotes the distance between B and O, compute $\dot{r}$, $\dot{\phi}$, $\ddot{r}$, and $\ddot{\phi}$ when $\theta = 90°$.



Solution

Using the diagram at the right, the velocity of B at $\theta = 90°$ in the $(\hat{u}_n, \hat{u}_t)$ and $(\hat{u}_r, \hat{u}_\phi)$ component systems is, respectively,



$$\vec{v} = R\dot{\theta}\,\hat{u}_t \quad \text{and} \quad \vec{v} = \dot{r}\,\hat{u}_r + r\dot{\phi}\,\hat{u}_\phi, \tag{1}$$

where the first of Eqs. (1) is true because B is in circular motion about A. We note that

$$\hat{u}_t = \cos\phi\,\hat{u}_r - \sin\phi\,\hat{u}_\phi \quad \text{and} \quad \hat{u}_n = -\sin\phi\,\hat{u}_r - \cos\phi\,\hat{u}_\phi, \tag{2}$$

where, for $\theta = 90°$,

$$\sin\phi = R/\sqrt{R^2 + H^2} \quad \text{and} \quad \cos\phi = H/\sqrt{R^2 + H^2}. \tag{3}$$

Substituting the first of Eqs. (2) into the first of Eqs. (1), we have $\vec{v} = R\dot{\theta}(\cos\phi\,\hat{u}_r - \sin\phi\,\hat{u}_\phi)$, which, when compared to the second of Eqs. (1) component by component, yields

$$R\dot{\theta}\cos\phi = \dot{r} \quad \Rightarrow \quad \dot{r} = \frac{RH\dot{\theta}}{\sqrt{R^2 + H^2}} \quad \Rightarrow \quad \boxed{\dot{r} = 8.825\,\text{ft/s}} \tag{4}$$

and

$$-R\dot{\theta}\sin\phi = r\dot{\phi} \quad \Rightarrow \quad \dot{\phi} = -\frac{R^2\dot{\theta}}{R^2 + H^2} \quad \Rightarrow \quad \boxed{\dot{\phi} = -0.7746\,\text{rad/s},} \tag{5}$$

where we have used Eqs. (3), the fact that $r = \sqrt{R^2 + H^2}$, and the following data: $R = 1.5\,\text{ft}$, $H = 4\,\text{ft}$, and $\dot{\theta} = 60\,\text{rpm} = 60\frac{2\pi}{60}\,\text{rad/s}$.

The acceleration of B at $\theta = 90°$ in the two component systems is

$$\vec{a} = R\dot{\theta}^2\,\hat{u}_n \quad \text{and} \quad \vec{a} = \left(\ddot{r} - r\dot{\phi}^2\right)\hat{u}_r + \left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right)\hat{u}_\phi, \tag{6}$$

where, in the first of the above equations, we have used the fact that B is in uniform circular motion about A. Substituting the second of Eqs. (2) into the first of Eqs. (6) gives $\vec{a} = -R\dot{\theta}^2(\sin\phi\,\hat{u}_r + \cos\phi\,\hat{u}_\phi)$, which, when compared to the second of Eqs. (6) component by component, yields

$$-R\dot{\theta}^2\sin\phi = \ddot{r} - r\dot{\phi}^2 \quad \Rightarrow \quad \ddot{r} = \frac{R^4\dot{\theta}^2\sqrt{R^2 + H^2}}{\left(R^2 + H^2\right)^2} - \frac{R^2\dot{\theta}^2}{\sqrt{R^2 + H^2}} \quad \Rightarrow \quad \boxed{\ddot{r} = -18.23\,\text{ft/s}^2}$$

and

$$-R\dot{\theta}^2\cos\phi = r\ddot{\phi} + 2\dot{r}\dot{\phi} \quad \Rightarrow \quad \ddot{\phi} = -\frac{RH\dot{\theta}^2}{R^2 + H^2} + 2\frac{R^3H\dot{\theta}^2}{\left(R^2 + H^2\right)^2} \quad \Rightarrow \quad \boxed{\ddot{\phi} = -9.779\,\text{rad/s}^2,}$$

where we have used Eqs. (3)–(5), the fact that $r = \sqrt{R^2 + H^2}$, and the following data: $R = 1.5\,\text{ft}$, $H = 4\,\text{ft}$, and $\dot{\theta} = 60\,\text{rpm} = 60\frac{2\pi}{60}\,\text{rad/s}$.

Problem 2.214

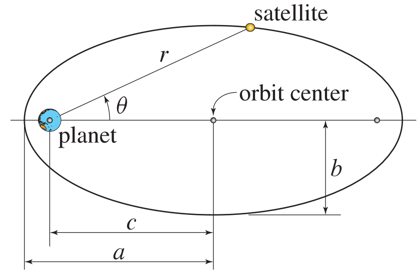
A satellite is moving along the elliptical orbit shown. Using the polar coordinate system in the figure, the satellite's orbit is described by the equation

$$r(\theta) = 2b^2 \frac{a + \sqrt{a^2 - b^2}\cos\theta}{a^2 + b^2 - (a^2 - b^2)\cos(2\theta)},$$

which implies the following identity

$$\frac{rr'' - 2(r')^2 - r^2}{r^3} = -\frac{a}{b^2},$$

where the prime indicates differentiation with respect to θ. Using this identity and knowing that the satellite moves so that $K = r^2\dot{\theta}$ with K constant (i.e., according to Kepler's laws), show that the radial component of acceleration is proportional to $-1/r^2$, which is in agreement with Newton's universal law of gravitation.



Solution

We need to show that $a_r = (\text{constant}) \times \left(-\frac{1}{r^2}\right)$. First we will rewrite Kepler's law as

$$K = r^2\dot{\theta} \quad \Rightarrow \quad \dot{\theta} = \frac{K}{r^2}. \tag{1}$$

We are given $r = r(\theta)$ so we use the chain rule to write its derivative with respect to time as

$$\dot{r} = r'\dot{\theta} \quad \Rightarrow \quad \dot{r} = \frac{Kr'}{r^2}, \tag{2}$$

where the prime denotes differentiation with respect to θ, and where we have substituted Eq. (1) for $\dot{\theta}$. Next we take the second derivative of r with respect to time to obtain

$$\ddot{r} = \frac{-2K}{r^3}(r'\dot{\theta})r' + \frac{K}{r^2}(r''\dot{\theta}) \quad \Rightarrow \quad \ddot{r} = \frac{K^2}{r^2}\left[\frac{r''}{r^2} - \frac{2(r')^2}{r^3}\right], \tag{3}$$

where we have factored $K\dot{\theta}$ out of the first of Eq. (3) and substituted Eq. (1) for $\dot{\theta}$. Now recall that $a_r = \ddot{r} - r\dot{\theta}^2$ and we have expressions for r, $\ddot{r}$, and $\dot{\theta}$ so we can write

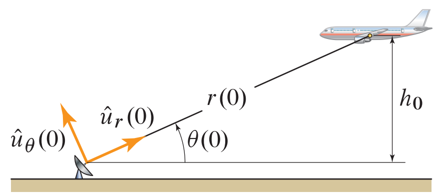
$$a_r = \frac{K^2}{r^2}\left[\frac{r''}{r^2} - \frac{2(r')^2}{r^3}\right] - r\frac{K^2}{r^4} \quad \Rightarrow \quad a_r = \frac{K^2}{r^2}\left[\frac{rr'' - 2(r')^2 - r^2}{r^3}\right]. \tag{4}$$

We see that the expression in brackets is the identity given in the problem statement, so we will replace it with $-a/b^2$. Recalling that K, a, and b are constants, we prove the radial component of acceleration is proportional to $-1/r^2$.

$$a_r = \frac{K^2 a}{b^2}\left(-\frac{1}{r^2}\right).$$

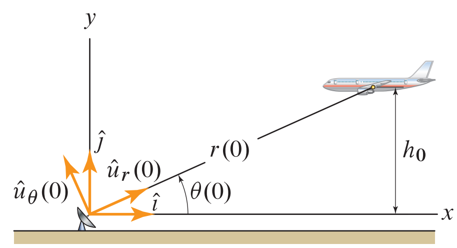
Problem 2.215

At a given instant, an airplane flying at an altitude $h_0 = 10{,}000\,\text{ft}$ begins its descent in preparation for landing when it is $r(0) = 20\,\text{mi}$ from the radar station at the destination's airport. At that instant, the aiplane's speed is $v_0 = 300\,\text{mph}$, the climb rate is constant and equal to $-5\,\text{ft/s}$, and the horizontal component of velocity is decreasing steadily at a rate of $15\,\text{ft/s}^2$. Determine the $\dot{r}$, $\dot{\theta}$, $\ddot{r}$, and $\ddot{\theta}$ that would be observed by the radar station.



Solution

When expressed in the Cartesian coordinate system shown, the velocity and acceleration of the airplane are



$$\vec{v} = \dot{x}\,\hat{\imath} + \dot{y}\,\hat{\jmath} \quad \text{and} \quad \vec{a} = \ddot{x}\,\hat{\imath} + \ddot{y}\,\hat{\jmath}, \tag{1}$$

where

$$\dot{y} = -5\,\text{ft/s}, \quad \dot{x} = -\sqrt{v_0^2 - \dot{y}^2}, \quad \ddot{x} = 15\,\text{ft/s}^2, \quad \ddot{y} = 0, \tag{2}$$

where it is understood that these values are at $t = 0$. Using the polar component system shown, the velocity and acceleration are

$$\vec{v} = \dot{r}\,\hat{u}_r + r\dot{\theta}\,\hat{u}_\theta \quad \text{and} \quad \vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{u}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{u}_\theta, \tag{3}$$

where the unit vectors $\hat{u}_r$ and $\hat{u}_\theta$ are related to the unit vectors $\hat{\imath}$ and $\hat{\jmath}$ as follows:

$$\hat{\imath} = \cos\theta\,\hat{u}_r - \sin\theta\,\hat{u}_\theta \quad \text{and} \quad \hat{\jmath} = \sin\theta\,\hat{u}_r + \cos\theta\,\hat{u}_\theta. \tag{4}$$

Using Eqs. (4), we can rewrite Eqs. (1) as follows:

$$\vec{v} = (\dot{x}\cos\theta + \dot{y}\sin\theta)\,\hat{u}_r + (-\dot{x}\sin\theta + \dot{y}\cos\theta)\,\hat{u}_\theta \tag{5}$$

and

$$\vec{a} = (\ddot{x}\cos\theta + \ddot{y}\sin\theta)\,\hat{u}_r + (-\ddot{x}\sin\theta + \ddot{y}\cos\theta)\,\hat{u}_\theta. \tag{6}$$

Equating the components of the first of Eqs. (3) and (5), we have

$$\dot{r} = \dot{x}\cos\theta + \dot{y}\sin\theta \quad \text{and} \quad \dot{\theta} = r^{-1}(-\dot{x}\sin\theta + \dot{y}\cos\theta). \tag{7}$$

At the instant considered in the problem, we have that $\theta = \sin^{-1}(h/r)$. With this in mind, using the data in Eqs. (2) along with the fact that $r = 20\,\text{mi} = 20(5280)\,\text{ft}$, $h = 10{,}000\,\text{ft}$, $v_0 = 300\,\text{mph} = 300\frac{5280}{3600}\,\text{ft/s}$, we can evaluate Eqs. (7) to obtain:

$$\boxed{\dot{r} = -438.5\,\text{ft/s} \quad \text{and} \quad \dot{\theta} = 347.4\times 10^{-6}\,\text{rad/s}.}$$

Similarly, equating the components of the second of Eqs. (3) and (6), we have

$$\ddot{r} = \ddot{x}\cos\theta + \ddot{y}\sin\theta + r\dot{\theta}^2 \quad \text{and} \quad \ddot{\theta} = r^{-1}\left(-\ddot{x}\sin\theta + \ddot{y}\cos\theta - 2\dot{r}\dot{\theta}\right), \tag{8}$$

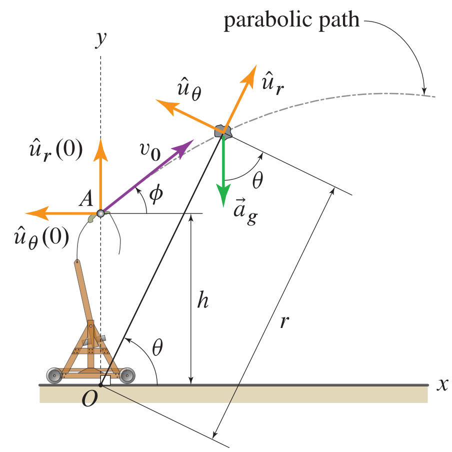
which can be evaluated with the help of Eqs. (7) and the data used in the evaluation of Eq. (7) to obtain

$$\boxed{\ddot{r} = 14.95\,\text{ft/s}^2 \quad \text{and} \quad \ddot{\theta} = -10.59\times 10^{-6}\,\text{rad/s}^2.}$$

Problem 2.216

Considering the system analyzed in Example 2.21, let $h = 15$ ft, $v_0 = 55$ mph, and $\phi = 25°$. Plot the trajectory of the projectile in two different ways: (1) by solving the projectile motion problem using Cartesian coordinates and plotting y versus x and (2) by using a computer to solve Eqs. (3), (4), (9), and (10) in Example 2.21. You should, of course, get the same trajectory regardless of the coordinate system used.

Solution



Using the Cartesian coordinate system shown, the acceleration of the projectile is

$$\ddot{x} = 0 \quad \text{and} \quad \ddot{y} = -g. \tag{1}$$

Since the acceleration is constant, we can use constant acceleration equations to write the position of the projectile as follows:

$$x = x_0 + \dot{x}_0 t \quad \text{and} \quad y = y_0 + \dot{y}_0 t - \tfrac{1}{2} g t^2, \tag{2}$$

where (x_0, y_0) are coordinates of the projectile at time $t = 0$, and $(\dot{x}_0, \dot{y}_0)$ are the components of the velocity of the projectile at $t = 0$. Considering the conditions at release, we have

$$x_0 = 0, \quad y_0 = h, \quad \dot{x}_0 = v_0 \cos\phi, \quad \text{and} \quad \dot{y}_0 = v_0 \sin\phi. \tag{3}$$

From the first of Eqs. (2), we can solve for t to obtain $t = (x - x_0)/\dot{x}_0$. Substituting this expression into the second of Eqs. (2), we obtain the expression for the trajectory of the projectile

$$y = y_0 + \frac{\dot{y}_0}{\dot{x}_0}(x - x_0) - \frac{g}{2\dot{x}_0^2}(x - x_0)^2 \quad \Rightarrow \quad y = h + (\tan\phi)x - \frac{g x^2}{2 v_0^2 \cos^2\phi}, \tag{4}$$

where we have used Eqs. (3). Before plotting the trajectory, we determine the time instant, denoted by t_f, at which the projectile hits the ground, i.e, $y(t_f) = 0$. This condition yields the following result:

$$0 = y_0 + \dot{y}_0 t_f - \tfrac{1}{2} g t_f^2 \quad \Rightarrow \quad t_f = \frac{1}{g}\left(\dot{y}_0 + \sqrt{\dot{y}_0^2 + 2gh}\right) \quad \Rightarrow \quad t_f = 2.491\,\text{s}, \tag{5}$$

where we have used the following numerical data: $h = 15$ ft, $v_0 = 55$ mph $= 55\frac{5280}{3600}$ ft/s, $\phi = 25°$, and $g = 32.2\,\text{ft/s}^2$. The value of x for $t = t_f$ is obtained by substituting t_f into the first of Eqs. (2). This gives

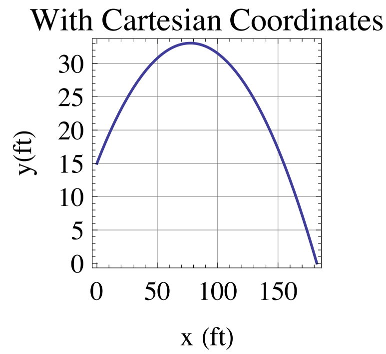
$$x(t_f) = 182.1\,\text{ft}. \tag{6}$$

We can plot the last of Eqs. (4) for $0 \le x \le 182.1$ ft. This can be done using any appropriate mathematical software. The plots shown below were obtained using *Mathematica* with the following code:

```
Parameters = {h → 15., v0 → 55. · 5280./3600., ϕ → 25. Degree, g → 32.2};

Plot[h + Tan[ϕ] x - (g x^2)/(2 v0^2 Cos[ϕ]^2) /. Parameters, {x, 0, 182.1}, Frame → True,
 GridLines → Automatic, FrameLabel → {"x (ft)", "y(ft)"},
 PlotLabel → "With Cartesian Coordinates", AspectRatio → 1]
```

Executing the above code we obtain the following plot:



We now determine the trajectory of the projectile starting from Eqs. (3) and (4) derived in Example 2.21 on p. 110 of the textbook, and subject to the initial conditions in Eqs. (9) and (10) of the example in question. For convenience, we repeat the equations we need here below:

$$\ddot{r} - r\dot{\theta}^2 = -g \sin\theta, \tag{7}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = -g \cos\theta, \tag{8}$$

with the initial conditions

$$r(0) = h, \qquad \theta(0) = \frac{\pi}{2}\,\text{rad}, \tag{9}$$

$$\dot{r}(0) = v_0 \sin\phi, \qquad \dot{\theta}(0) = -\frac{v_0}{h}\cos\phi. \tag{10}$$

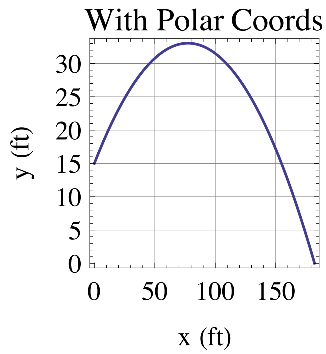
The above system of differential equations and initial conditions can be integrated with any appropriate mathematical software. The solution of these equations will be in terms of r and θ as a function of time. To obtain the plot of the trajectory, we must resort to a parametric plot, i.e., a plot of the coordinates of the projectile for $0 \leq t \leq t_f = 2.491$ s. To produce a plot that can be compared to the one shown above, we must plot values of x and y corresponding to the values of r and θ given by the numerical solution. We do so by observing that

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta. \tag{11}$$

With the above in mind, we have used *Mathematica* with the following code to obtain a solution with time t going from 0 to 2.491 s:

```
Parameters = {h → 15., v0 → 55. · 5280./3600., ϕ → 25. Degree, g → 32.2};
Equations = {r''[t] - r[t] (θ'[t])^2 == -g Sin[θ[t]],
   r[t] θ''[t] + 2 r'[t] θ'[t] == -g Cos[θ[t]], r[0] == h, θ[0] == π/2, r'[0] == v0 Sin[ϕ],
   θ'[0] == -(v0 Cos[ϕ])/h};
Motion = NDSolve[Equations /. Parameters, {r, θ}, {t, 0, 2.491}]
ParametricPlot[{r[t] Cos[θ[t]], r[t] Sin[θ[t]]} /. Motion[[1]], {t, 0, 2.491},
 Frame → True, GridLines → Automatic, FrameLabel → {"x (ft)", "y (ft)"},
 PlotLabel → "With Polar Coords", AspectRatio → 1]
```

The code above yields the following trajectory, which can be seen to be identical to the one obtained earlier (as expected).


With Polar Coords
y (ft)
30
25
20
15
10
5
0
0
50
100
150
x (ft)

Problem 2.217

Reference frame A is translating relative to reference frame B. Both frames track the motion of a particle C. If at one instant the velocity of particle C is the same in the two frames, what can you infer about the motion of frames A and B at that instant?

Solution

For two frames to measure the same velocity these frames must be at rest relative to one another at least at the instant at which the measurement is made. Therefore, we can conclude that frame A has zero velocity relative to frame B at the instant considered.

Problem 2.218

Reference frame A is translating relative to reference frame B with velocity $\vec{v}_{A/B}$ and acceleration $\vec{a}_{A/B}$. A particle C appears to be stationary relative to frame A. What can you say about the velocity and acceleration of particle C relative to frame B?

Solution

If the particle is stationary relative to frame A, then, relative to B, it moves just like frame A. Therefore, the velocity and acceleration of particle C relative to frame B are $\vec{v}_{A/B}$ and $\vec{a}_{A/B}$.

Problem 2.219

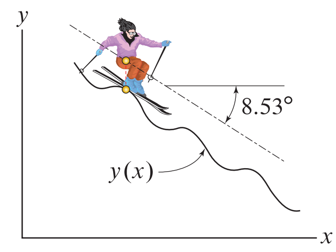
Reference frame A is translating relative to reference frame B with constant velocity $\vec{v}_{A/B}$. A particle C appears to be in uniform rectilinear motion relative to frame A. What can you say about the motion of particle C relative to frame B?

Solution

Because the relative velocity of frame A is constant, then A is in uniform rectilinear motion relative to B. Since the velocity of particle C is constant relative to A, and since the velocity of C relative to B is the sum of the velocity of A relative to B and of the velocity of C relative to A, then the velocity of C relative to be will also be constant. In turn, this implies that particle C will appear to be in uniform rectilinear motion relative to frame B.

Problem 2.220

A skier is going down a slope with moguls. Let the skis be short enough for us to assume that the skier's feet are tracking the moguls' profile. Then, if the skier is skilled enough to maintain her hips on a straight-line trajectory and vertically aligned over her feet, determine the velocity and acceleration of her hips relative to her feet when her speed is equal to 15 km/h. For the profile of the moguls, use the formula $y(x) = h_I - 0.15x + 0.125\sin(\pi x/2)$ m, where h_I is the elevation at which the skier starts the descent.



Solution

The expressions of the vertical position of the skier's hips and feet are denoted by y_H and y_F, respectively.

$$y_H = (h_I - x\tan 8.53°)\text{ m}, \tag{1}$$

$$y_F = \left[h_I - 0.15x + 0.125\sin\left(\frac{\pi x}{2}\right)\right]\text{ m}. \tag{2}$$

Now, calculating the relative position of her hips with respect to her feet,

$$y_{H/F} = y_H - y_F = \left[(0.15 - \tan 8.53°)x - 0.125\sin\left(\frac{\pi x}{2}\right)\right]\text{ m}, \tag{3}$$

and taking the derivative of Eq. (3) with respect to time yields

$$v_{H/F} = \dot{x}\left[0.15 - \tan 8.53° - \frac{0.125\pi}{2}\cos\left(\frac{\pi x}{2}\right)\right]\text{ m/s}. \tag{4}$$

However, $\dot{x} = v_0\cos 8.53°$ and $v_0 = 15\text{ km/h} = 15\frac{1000}{3600}$ m/s. With this substitution Eq. (4) becomes

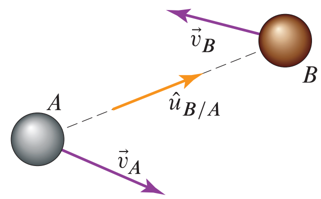
$$\boxed{v_{H/F} = \left[5.630\times10^{-5} - 0.8091\cos\left(\frac{\pi x}{2}\right)\right]\text{ m/s.}} \tag{5}$$

The relative acceleration of the skier's hips with respect to her feet can now be determined by differentiating Eq. (4) and substituting $\dot{x} = v_0\cos 8.53°$. This yields

$$a_{H/F} = 1.271\dot{x}\ \sin\left(\frac{\pi x}{2}\right)\text{ m/s}^2 \quad\Rightarrow\quad \boxed{a_{H/F} = 5.237\sin\left(\frac{\pi x}{2}\right)\text{ m/s}^2.}$$

Problem 2.221

Two particles A and B are moving in a plane with arbitrary velocity vectors $\vec{v}_A$ and $\vec{v}_B$, respectively. Letting the *rate of separation* (ROS) be defined as the component of the relative velocity vector along the line connecting particles A and B, determine a general expression for ROS. Express your result in terms of $\vec{r}_{B/A} = \vec{r}_B - \vec{r}_A$, where $\vec{r}_A$ and $\vec{r}_B$ are the position vectors of A and B, respectively, relative to some chosen fixed point in the plane of motion.



Solution

We begin by writing $\hat{u}_{B/A}$ in terms of $\vec{r}_{B/A}$:

$$\hat{u}_{B/A} = \frac{\vec{r}_{B/A}}{|\vec{r}_{B/A}|}. \qquad (1)$$

The velocity of B relative to A is:

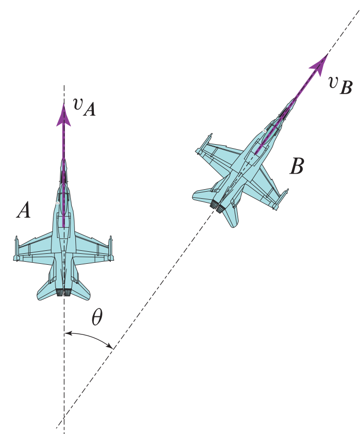
$$\vec{v}_{B/A} = \vec{v}_B - \vec{v}_A. \qquad (2)$$

The component of Eq. (2) in the direction of Eq. (1) is the ROS, which is therefore obtained by dotting $\vec{v}_{B/A}$ with $\hat{u}_{B/A}$. This gives

$$\text{ROS} = \vec{v}_{B/A} \cdot \hat{u}_{B/A} \quad \Rightarrow \quad \boxed{\text{ROS} = (\vec{v}_B - \vec{v}_A) \cdot \frac{\vec{r}_{B/A}}{|\vec{r}_{B/A}|}.}$$

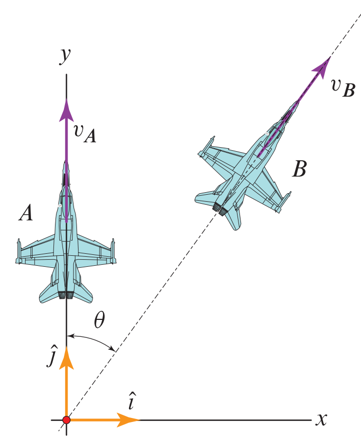
Problem 2.222

Airplanes A and B are flying along straight lines at the same altitude and with speeds $v_A = 660\,\text{km/h}$ and $v_B = 550\,\text{km/h}$, respectively. Determine the speed of A as perceived by B if $\theta = 50°$.



Solution

Referring to the figure at the right, we use the Cartesian coordinate system shown to express the velocities of the airplanes A and B:



$$\vec{v}_A = v_A\,\hat{\jmath} \quad \text{and} \quad \vec{v}_B = v_B(\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}). \tag{1}$$

Using relative kinematics, the velocity of A as perceived by B is

$$\vec{v}_{A/B} = \vec{v}_A - \vec{v}_B = -v_B \sin\theta\,\hat{\imath} + (v_A - v_B\cos\theta)\,\hat{\jmath}. \tag{2}$$

The speed of A as perceived by B is the magnitude of the vector $\vec{v}_{A/B}$. Using the result in Eq. (2), we have

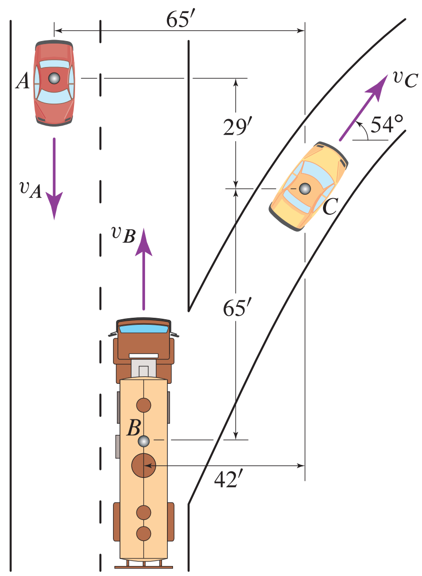
$$v_{A/B} = \left|\vec{v}_{A/B}\right| = \sqrt{v_B^2 \sin^2\theta + (v_A - v_B\cos\theta)^2}. \tag{3}$$

Recalling that $v_A = 660\,\text{km/h} = 660\frac{1000}{3600}\,\text{m/s}$, $v_B = 550\,\text{km/h} = 550\frac{1000}{3600}\,\text{m/s}$ and $\theta = 50°$, we can evaluate the expression in Eq. (3) to obtain

$$v_{A/B} = 144.7\,\text{m/s}.$$

Problem 2.223

Three vehicles A, B, and C are in the positions shown and are moving with the indicated directions. We define the *rate of separation* (ROS) of two particles P_1 and P_2 as the component of the relative velocity of, say, P_2 with respect to P_1 in the direction of the relative position vector of P_2 with respect to P_1, which is along the line that connects the two particles. At the given instant, determine the rates of separation ROS_{AB} and ROS_{CB}, that is, the rate of separation between A and B and between C and B. Let $v_A = 60\,\text{mph}$, $v_B = 55\,\text{mph}$, and $v_C = 35\,\text{mph}$. Furthermore, treat the vehicles as particles and use the dimensions shown in the figure.



Solution

We need to derive a convenient expression for the rate of separation. We begin by writing $\hat{u}_{B/A}$ in terms of $\vec{r}_{B/A}$

$$\hat{u}_{B/A} = \frac{\vec{r}_{B/A}}{|\vec{r}_{B/A}|}. \tag{1}$$

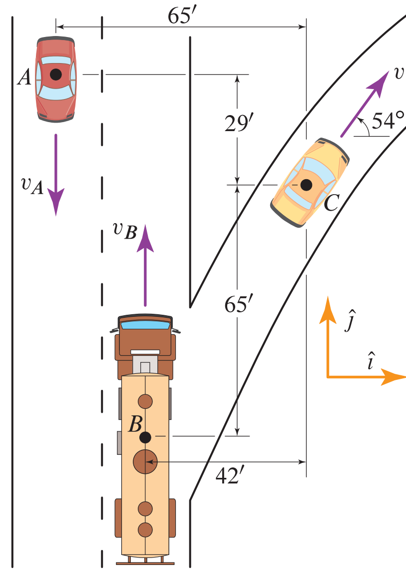
The velocity of B relative to A is:

$$\vec{v}_{B/A} = \vec{v}_B - \vec{v}_A. \tag{2}$$

The component of Eq. (2) in the direction of Eq. (1) is the ROS, i.e.,

$$\text{ROS} = \vec{v}_{B/A} \cdot \hat{u}_{B/A} \quad \Rightarrow \quad \text{ROS} = \left(\vec{v}_B - \vec{v}_A\right) \cdot \frac{\vec{r}_{B/A}}{|\vec{r}_{B/A}|}. \tag{3}$$

Now that we have a formula for the ROS, consider the $(\hat{\imath}, \hat{\jmath})$ component system shown in the figure at the right.



$$\frac{\vec{r}_{B/A}}{|r_{B/A}|} = \frac{\left(23\,\hat{\imath} - 94\,\hat{\jmath}\right)\text{ft}}{\sqrt{(23)^2 + (-94)^2}\,\text{ft}} = (0.2377\,\hat{\imath} - 0.9713\,\hat{\jmath})\,, \tag{4}$$

$$\frac{\vec{r}_{C/B}}{|r_{C/B}|} = \frac{\left(42\,\hat{\imath} + 65\,\hat{\jmath}\right)\text{ft}}{\sqrt{(42)^2 + (65)^2}\,\text{ft}} = (0.5427\,\hat{\imath} + 0.8399\,\hat{\jmath})\,. \tag{5}$$

Recalling that $v_A = 60\,\text{mph} = 60\frac{5280}{3600}\,\text{ft/s}$, $v_B = 55\,\text{mph} = 55\frac{5280}{3600}\,\text{ft/s}$, and $v_C = 35\,\text{mph} = 35\frac{5280}{3600}\,\text{ft/s}$, and observing that $\vec{v}_A = -v_A\,\hat{\jmath}$, $\vec{v}_B = v_B\,\hat{\jmath}$, and $\vec{v}_C = v_C(\cos 54°\,\hat{\imath} + \sin 54°\,\hat{\jmath})$, we have

$$\vec{v}_A = (-88.00\,\hat{\jmath})\ \text{ft/s}, \quad \vec{v}_B = (80.67\,\hat{\jmath})\,\text{ft/s}, \quad \vec{v}_C = (30.17\,\hat{\imath} + 41.53\,\hat{\jmath})\ \text{ft/s}. \tag{6}$$

The rate of separation between vehicles A and B is found as

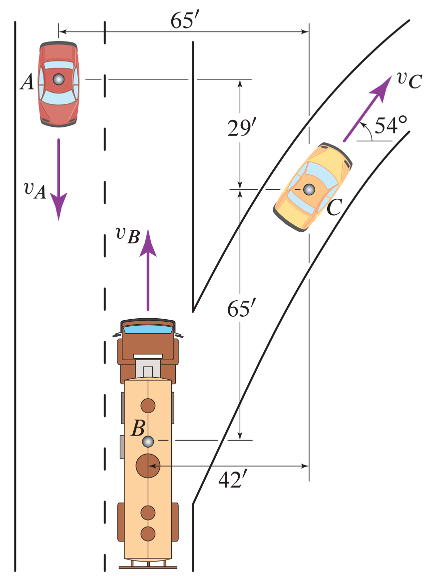
$$\mathrm{ROS}_{AB} = \left(\vec{v}_B - \vec{v}_A\right) \cdot \frac{\vec{r}_{B/A}}{\left|r_{B/A}\right|} \quad \Rightarrow \quad \boxed{\mathrm{ROS}_{AB} = -163.8\,\mathrm{ft/s.}}$$

Similarly, the rate of separation between vehicles C and B is found as

$$\mathrm{ROS}_{CB} = \left(\vec{v}_B - \vec{v}_C\right) \cdot \frac{\vec{r}_{C/B}}{\left|r_{C/B}\right|} \quad \Rightarrow \quad \boxed{\mathrm{ROS}_{CB} = -16.50\,\mathrm{ft/s.}} \tag{7}$$

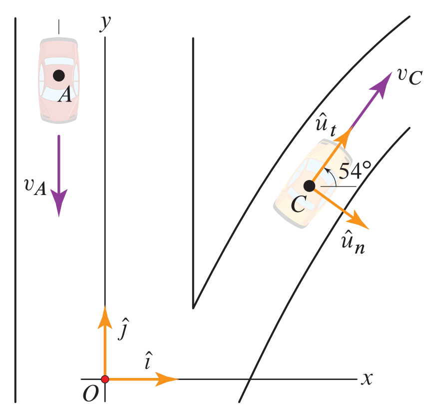
Problem 2.224

Car A is moving at a constant speed $v_A = 75\,\text{km/h}$, while car C is moving at a constant speed $v_C = 42\,\text{km/h}$ on a circular exit ramp with radius $\rho = 80\,\text{m}$. Determine the velocity and acceleration of C relative to A.



Solution

Referring to the figure at the right, the xy frame of reference with origin at O and component system with unit vectors $\hat{\imath}$ and $\hat{\jmath}$ is stationary relative to the ground. Expressing the velocities of cars A and C in this frame, we have



$$\vec{v}_A = -v_A\,\hat{\jmath} \quad \text{and} \quad \vec{v}_C = v_C(\cos 54°\,\hat{\imath} + \sin 54°\,\hat{\jmath}). \tag{1}$$

Recalling that the relative velocity of C with respect to A is $\vec{v}_{C/A} = \vec{v}_C - \vec{v}_A$, using Eqs. (1), we have

$$\vec{v}_{C/A} = v_C \cos 54°\,\hat{\imath} + (v_C \sin 54° + v_A)\,\hat{\jmath}. \tag{2}$$

Recalling that $v_A = 75\,\text{km/h} = 75\frac{1000}{3600}\,\text{m/s}$ and $v_C = 42\,\text{km/h} = 42\frac{1000}{3600}\,\text{m/s}$, we can evaluate Eq. (2) to obtain

$$\vec{v}_{C/A} = (6.857\,\hat{\imath} + 30.27\,\hat{\jmath})\,\text{m/s}.$$

To determine the acceleration of C relative to A, we begin by observing that

$$\vec{a}_A = \vec{0}. \tag{3}$$

This is because A travels at a constant speed along a straight line. By contrast, while the speed of C is also constant, the acceleration of C is not equal to zero because C travels along a curved path. To determine the acceleration of C, we use the normal-tangential component system shown in the figure, in which we have

$$\vec{a}_C = \dot{v}_C\,\hat{u}_t + \left(v_C^2/\rho\right)\hat{u}_n. \tag{4}$$

Since v_C is constant, $\dot{v}_C = 0$ and Eq. (4) simplifies to

$$\vec{a}_C = \left(v_C^2/\rho\right)\hat{u}_n. \tag{5}$$

Observing that $\hat{u}_n = \sin 54° \,\hat{\imath} - \cos 54° \,\hat{\jmath}$, recalling that $\vec{a}_{C/A} = \vec{a}_C - \vec{a}_A$, and using Eqs. (5) and (3), we have

$$\vec{a}_{C/A} = \left(v_C^2/\rho\right)(\sin 54° \,\hat{\imath} - \cos 54° \,\hat{\jmath}). \tag{6}$$

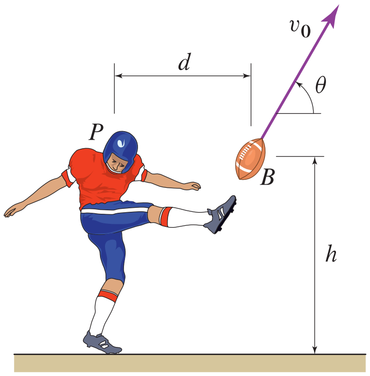
Recalling that $v_C = 42\,\text{km/h} = 42\frac{1000}{3600}\,\text{m/s}$ and $\rho = 80\,\text{m}$, we can evaluate Eq. (6) to obtain

$$\vec{a}_{C/A} = (1.376\,\hat{\imath} - 1.000\,\hat{\jmath})\,\text{m/s}^2.$$

Problem 2.225

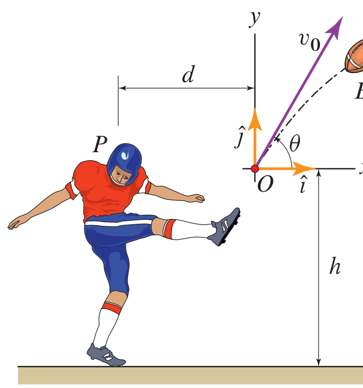
During practice, a player P punts a ball B with a speed $v_0 = 25\,\text{ft/s}$, at an angle $\theta = 60°$, and at a height h from the ground. Then the player sprints along a straight line and catches the ball at the same height from the ground at which the ball was initially kicked. The length d denotes the horizontal distance between the player's position at the start of the sprint and the ball's position when the ball leaves the player's foot. Also, let Δt denote the time interval between the instant at which the ball leaves the player's foot and the instant at which the player starts sprinting.

Assume that $d = 0$ and $\Delta t = 0$, and determine the average speed of the player so that he catches the ball.



Solution

Referring to the figure at the right, we model the motion of the ball as projectile motion and since we are interested in computing the player's average speed, we will model the player's motion as rectilinear motion with constant speed. Under the stated modeling assumptions, we have that the horizontal velocity of the ball is constant and equal to $v_0 \cos\theta$. For the player to catch the ball when it comes back down under the assumption of constant velocity motion, along with $d = 0$ and $\Delta t = 0$, the horizontal component of the relative velocity of the player with respect to the ball must be equal to zero. Therefore, the velocity of the player must match the horizontal component of the velocity of the ball exactly. This implies that the average speed of the player must be



$$(v_P)_{\text{avg}} = v_0 \cos\theta. \tag{1}$$

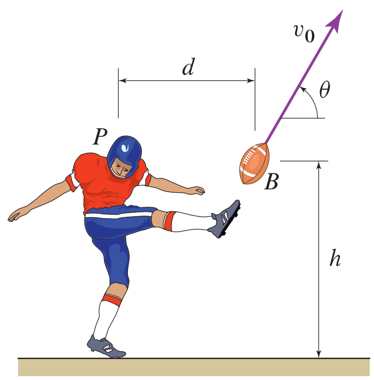
Recalling that $v_0 = 25\,\text{ft/s}$ and $\theta = 60°$, we can evaluate Eq. (1) to obtain

$$\boxed{(v_P)_{\text{avg}} = 12.50\,\text{ft/s}.}$$

Problem 2.226

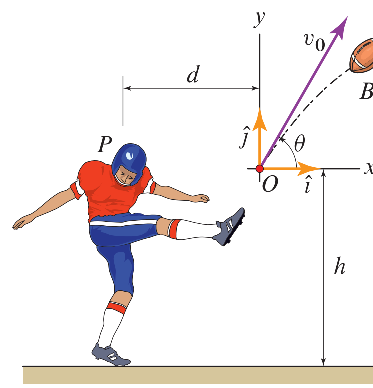
During practice, a player P punts a ball B with a speed $v_0 = 25\,\mathrm{ft/s}$, at an angle $\theta = 60°$, and at a height h from the ground. Then the player sprints along a straight line and catches the ball at the same height from the ground at which the ball was initially kicked. The length d denotes the horizontal distance between the player's position at the start of the sprint and the ball's position when the ball leaves the player's foot. Also, let Δt denote the time interval between the instant at which the ball leaves the player's foot and the instant at which the player starts sprinting.

Assume that $d = 3\,\mathrm{ft}$ and $\Delta t = 0.2\,\mathrm{s}$, and determine the average speed of the player so that he catches the ball.



Solution

Referring to the figure at the right, we model the motion of the ball as projectile motion and since we are interested in computing the player's average speed, we will model the player's motion as rectilinear motion with constant speed. We adopt a Cartesian coordinate system with origin at point O, which we choose as the point at which the ball leaves the player's foot. Let t_f (f stands for final) denote the time the player catches the ball. Then at time t_f, the relative horizontal position of the player with respect to the ball must be equal zero:



$$x_{P/B}(t_f) = 0 \quad\Rightarrow\quad x_P(t_f) - x_B(t_f) = 0 \quad\Rightarrow\quad x_P(t_f) = x_B(t_f). \quad (1)$$

Let $t = 0$ be the time at which the ball leaves the player's foot. Since the horizontal component of the acceleration of the ball is equal to zero, the ball moves in the x direction with a velocity component that is constant and equal to $v_0 \cos\theta$. Therefore, we have that

$$x_B(t_f) = v_0 t_f \cos\theta. \tag{2}$$

Since the player starts sprinting Δt after the ball is kicked, the motion of the player occurs during the time interval that starts at time $t = \Delta t$ and ends at $t = t_f$. Therefore, denoting by τ the time during which the player sprints, we have that

$$\tau = t_f - \Delta t. \tag{3}$$

We also observe that the total distance traveled by the player, which we will denote by L, is given by

$$L = x_P(t_f) + d \quad\Rightarrow\quad L = x_B(t_f) + d, \tag{4}$$

where, in writing the last of Eqs. (4), we have used the last of Eqs. (1). By definition, the average speed of the player is now given by

$$(v_P)_{\mathrm{avg}} = \frac{L}{\tau} \quad\Rightarrow\quad (v_P)_{\mathrm{avg}} = \frac{d + v_0 t_f \cos\theta}{t_f - \Delta t}. \tag{5}$$

This result tells us that the value of $(v_P)_{\mathrm{avg}}$ can be determined if we determine the value of t_f. To this end we observe that since the player catches the ball at the same height from the ground at which the ball leaves the player's foot, and given our choice of coordinate system, we can calculate t_f as the time at which the ball comes back down to $y = 0$. Recalling that the motion of the ball is projectile motion, we have that the y coordinate of the ball as a function of time is given by

$$y_B = v_0 t \sin\theta - \tfrac{1}{2} g t^2, \tag{6}$$

where g is the acceleration due to gravity. Setting $y_B = 0$, we obtain

$$t_f = 0 \quad \text{and} \quad t_f = \frac{2v_0 \sin\theta}{g}. \tag{7}$$

The first root can be neglected since it coincides with the initial time. Hence, substituting the second of Eqs. (7) into the last of Eq. (5) and simplifying, we have

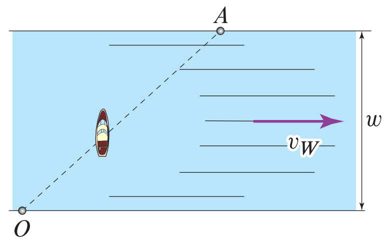
$$(v_P)_{\text{avg}} = \frac{dg + v_0^2 \sin 2\theta}{2v_0 \sin\theta - g\Delta t}, \tag{8}$$

where we have used the trigonometric identity $2\sin\theta\cos\theta = \sin 2\theta$. Recalling that $d = 3\,\text{ft}$, $g = 32.2\,\text{ft/s}^2$, $v_0 = 25\,\text{ft/s}$, $\theta = 60°$, and $\Delta t = 0.2\,\text{s}$, we can evaluate the above expression to obtain

$$\boxed{(v_P)_{\text{avg}} = 17.30\,\text{ft/s}.}$$

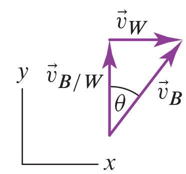
Problem 2.227

A remote controlled boat, capable of a maximum speed of 10 ft/s in still water, is made to cross a stream with a width $w = 35$ ft that is flowing with a speed $v_W = 7$ ft/s. If the boat starts from point O and keeps its orientation parallel to the cross-stream direction, find the location of point A at which the boat reaches the other bank while moving at its maximum speed. Furthermore, determine how much time the crossing requires.



Solution

Using a Cartesian coordinate system with its origin at O, as the boat is crossing the stream, its velocity $\vec{v}_B$ can be written as



$$\vec{v}_B = \vec{v}_W + \vec{v}_{B/W} = (7\,\hat{\imath} + 10\,\hat{\jmath})\,\text{ft/s},$$

where $\vec{v}_W = 7\,\hat{\imath}$ ft/s is the velocity of the water and $\vec{v}_{B/W} = 10\,\hat{\jmath}$ ft/s is the velocity of the boat relative to the water. Using the the y component of velocity, the time of crossing is

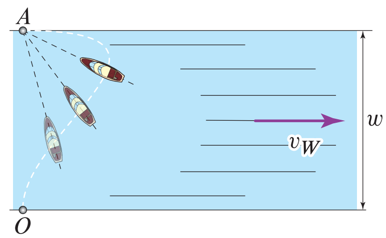
$$t = \frac{w}{v_{By}} \quad \Rightarrow \quad \boxed{t = 3.500\,\text{s},} \tag{1}$$

where $w = 35$ ft. Since the x component of velocity is constant, using the crossing time in Eq. (1), we can calculate the downstream position of A as

$$x = v_{Bx} t \quad \Rightarrow \quad \boxed{x = 24.50\,\text{ft.}}$$

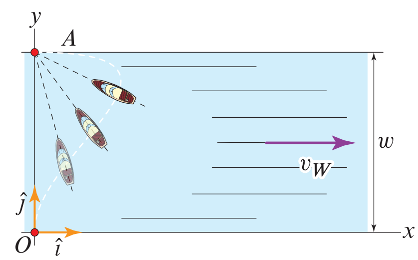
Problem 2.228

A remote controlled boat, capable of a maximum speed of 10 ft/s in still water, is made to cross a stream of width $w = 35$ ft that is flowing with a speed $v_W = 7$ ft/s. The boat is placed in the water at O, and it is *intended* to arrive at A by using a homing device that makes the boat always point toward A. Determine the time the boat takes to get to A and the path it follows. Also, consider a case in which the maximum speed of the boat is equal to the speed of the current. In such a case, does the boat ever make it to point A? *Hint:* To solve the problem, write $\vec{v}_{B/W} = v_{B/W}\,\hat{u}_{A/B}$, where the unit vector $\hat{u}_{A/B}$ always points from the boat to point A and is therefore, a function of time.



Solution

Referring to the figure at the right, we will use the Cartesian coordinate system shown. We denote the boat by B and the water by W. Since the boat is always heading toward point A, we describe the heading of the boat via the unit vector $\hat{u}_{A/B}$ given by



$$\hat{u}_{A/B} = \frac{\vec{r}_{A/B}}{|\vec{r}_{A/B}|}. \tag{1}$$

Denoting the coordinates of the boat by (x, y), the position of points A and B are $\vec{r}_A = w\,\hat{\jmath}$ and $\vec{r}_B = x\,\hat{\imath} + y\,\hat{\jmath}$, so that

$$\vec{r}_{A/B} = \vec{r}_A - \vec{r}_B = -x\,\hat{\imath} + (w - y)\,\hat{\jmath}. \tag{2}$$

Substituting Eq. (2) into Eq. (1), we have

$$\hat{u}_{A/B} = \frac{-x}{\sqrt{x^2 + (w-y)^2}}\,\hat{\imath} + \frac{w-y}{\sqrt{x^2 + (w-y)^2}}\,\hat{\jmath}. \tag{3}$$

Next, letting $\vec{v}_W$ and $\vec{v}_{B/W}$ be the velocity of water and the velocity of the boat relative to the water, respectively, we have

$$\vec{v}_W = v_W\,\hat{\imath} \quad \text{and} \quad \vec{v}_{B/W} = v_{B/W}\,\hat{u}_{A/B}, \tag{4}$$

where $v_W = 7$ ft/s, $v_{B/W} = 10$ ft/s when the boat is moving at its maximum speed (relative to the water), and where, in writing $\vec{v}_{B/W} = v_{B/W}\,\hat{u}_{A/B}$, we have used the hint given in the problem statement.

Using relative kinematics and denoting the velocity of the boat by $\vec{v}_B$, we have $\vec{v}_B = \vec{v}_W + \vec{v}_{B/W}$. Hence, using Eqs. (3) and (4), $\vec{v}_B$ takes on the following form:

$$\vec{v}_B = \left(v_W - \frac{v_{B/W}\,x}{\sqrt{x^2 + (w-y)^2}}\right)\hat{\imath} + \frac{v_{B/W}(w-y)}{\sqrt{x^2 + (w-y)^2}}\,\hat{\jmath}. \tag{5}$$

Now recall that the velocity of the boat can be written as

$$\vec{v}_B = \dot{x}\,\hat{\imath} + \dot{y}\,\hat{\jmath}. \tag{6}$$

Hence, equating Eqs. (5) and (6) component by component we find that the motion of the boat is govered by the following differential equations:

$$\dot{x} = v_W - \frac{v_{B/W}\,x}{\sqrt{x^2 + (w-y)^2}} \quad \text{and} \quad \dot{y} = \frac{v_{B/W}(w-y)}{\sqrt{x^2 + (w-y)^2}}, \tag{7}$$

which, given that the boat starts at O at time $t = 0$, are subject to the following initial conditions:

$$x(0) = 0 \quad \text{and} \quad y(0) = 0. \tag{8}$$

These equations can be integrated using appropriate mathematical software. We have used *Mathematica*. Before presenting the code used to solve the problem, a remark about Eqs. (7) is necessary. Let's suppose that the boat does make it to point A. In this case, the coordinates of the boat would be $(x = 0, y = w)$. When this happens, the argument of the square roots in Eqs. (7) becomes equal to zero. In turn, because the square roots appear at the denominator of fractions, when the boat makes it to A, the equations suffer a division by zero that will cause the numerical software to fail. Now, when using a numerical method to solve Eqs. (7) we must specify the time interval we want the solution to cover. As just discussed, as soon as the boat makes it to A the numerical integration fails. Therefore, we need to use trial and error to find the maximum amount of time for which the equations can be integrated before numerical failure. However, this maximum amount of time will correspond to the time taken by the boat to reach A, which is one of the quantities we need to determine in this problem. With all the above in mind, we have integrated the equations in question for the boat traveling at the maximum speed relative to water, i.e., $w_{B/W} = 10\,\text{ft/s}$, using *Mathematica* with the following code:

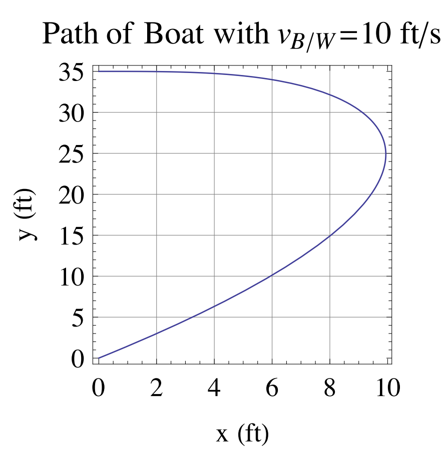
```
Parameters = {vW → 7., vBrelW → 10., w → 35.};
Equations = {x'[t] == vW - vBrelW x[t] / Sqrt[x[t]^2 + (w - y[t])^2], y'[t] == vBrelW (w - y[t]) / Sqrt[x[t]^2 + (w - y[t])^2],
    x[0] == 0, y[0] == 0};
Motion = NDSolve[Equations /. Parameters, {x, y}, {t, 0, 6.8627}]
```

Notice that, using trial and error, we were able to integrate our equations only up to $t = 6.8627\,\text{s}$. Hence, expressing this result to three significant figures, we will say that

For $v_{B/W} = 10\,\text{s}$, the boat reaches A in $6.863\,\text{s}$.

The solution obtained using *Mathematica* with the above code, can be plotted to depict the path followed by the boat. This path was plotted with the following code:

```
ParametricPlot[{x[t], y[t]} /. Motion[[1]], {t, 0, 6.8627}, Frame → True,
 GridLines → Automatic, AspectRatio → 1, FrameLabel → {"x (ft)", "y (ft)"},
 PlotLabel → "Path of Boat with vB/W=10 ft/s"]
```

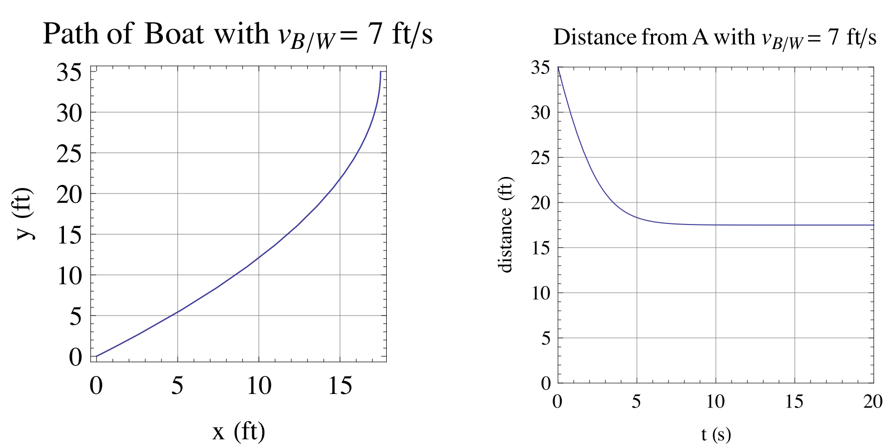


Now we consider the case for which the speed of the boat is equal to the speed of the water current. We repeat the same calculations just described but with $v_{B/W} = 7$ ft/s. This time we find that the mathematical software does not fail for any amount of time, indicating that

When $v_{B/W} = 7$ s, the boat does not reach A.

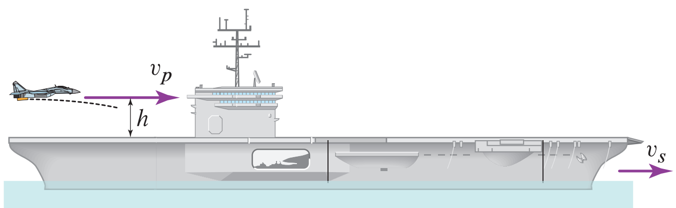
We present our solution for t going from zero to 20 s and we plot both the path of the boat and the distance $d_{AB} = \sqrt{x^2 + (w - y)^2}$ of the boat from A as a function of time. As can be seen from the plot of the path of the boat, the boat does reach the other side of the stream but when it does it points toward A while moving relative to the water with speed equal to the (absolute) speed of the water. As a result, an observer stationary with the banks of the stream sees the boat become stationary. The fact that the boat no longer moves relative to the banks can be seen from the plot of the distance d_{AB}, which, after about 9 s becomes constant (and remains different from zero).

```
Parameters = {vW → 7., vBrelW → 7., w → 35.};
Equations = {x'[t] == vW - vBrelW x[t] / Sqrt[x[t]^2 + (w - y[t])^2], y'[t] == vBrelW (w - y[t]) / Sqrt[x[t]^2 + (w - y[t])^2],
   x[0] == 0, y[0] == 0};
Motion = NDSolve[Equations /. Parameters, {x, y}, {t, 0, 20}];
ParametricPlot[{x[t], y[t]} /. Motion[[1]], {t, 0, 20}, PlotRange → All, Frame → True,
 GridLines → Automatic, AspectRatio → 1, FrameLabel → {"x (ft)", "y (ft)"},
 PlotLabel → "Path of Boat with vB/W= 7 ft/s"]
Plot[Sqrt[x[t]^2 + (w - y[t])^2] /. Parameters /. Motion[[1]], {t, 0, 20},
 PlotRange → {{0, 20}, {0, 35}}, Frame → True, GridLines → Automatic, AspectRatio → 1,
 FrameLabel → {"t (s)", "distance (ft)"}, PlotLabel → "Distance from A with vB/W= 7 ft/s"]
```



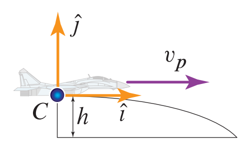
Problem 2.229

An airplane flying horizontally with a speed $v_p = 110\,\text{km/h}$ relative to the water drops a crate onto a carrier when vertically over the back end of the ship, which is traveling at a speed $v_s = 26\,\text{km/h}$ relative to the water. If the airplane drops the crate from a height $h = 20\,\text{m}$, at what distance from the back of the ship will the crate first land on the deck of the ship?



Solution

Referring to the figure at the right, the crate is denoted by C. The motion of C is analyzed using a fixed Cartesian coordinate system with origin at the point where C is released. After the drop, C is in projectile motion so that the acceleration of C is $\vec{a}_C = -g\,\hat{\jmath}$. Using constant acceleration equations and recalling that the crate is traveling with the airplane before being dropped, the velocity and position of C are



$$\vec{v}_C = v_p\,\hat{\imath} - gt\,\hat{\jmath} \quad \text{and} \quad \vec{r}_C = v_p t\,\hat{\imath} - \tfrac{1}{2}gt^2\,\hat{\jmath}. \tag{1}$$

Relative to the same frame of reference used to study the motion of C, the position of the back of the ship is

$$\vec{r}_s = v_s t\,\hat{\imath}, \tag{2}$$

where the subscript s stands for "ship." The vector describing the position of the crate relative to the back of the ship:

$$\vec{r}_{C/s} = \vec{r}_C - \vec{r}_s \quad \Rightarrow \quad \vec{r}_{C/s} = (v_p - v_s)t\,\hat{\imath} - \tfrac{1}{2}gt^2\,\hat{\jmath}. \tag{3}$$

Let t_d be the time at which the crate hits the deck. Given the choice of coordinate system, at $t = t_d$ $r_{Cy} = -h$. Using the second of Eqs. (1), this gives

$$-\tfrac{1}{2}gt_d^2 = -h \quad \Rightarrow \quad t_d = \sqrt{\frac{2h}{g}}. \tag{4}$$

The answer to the problem is the value of the horizontal component of $\vec{r}_{C/s}$ at $t = t_d$:

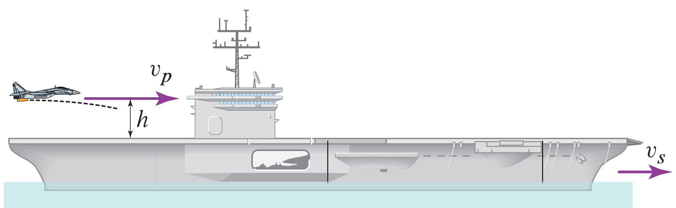
$$(r_{C/s})_x\big|_{t=t_d} = (v_p - v_s)\sqrt{\frac{2h}{g}}. \tag{5}$$

Recalling that $v_p = 110\,\text{km/h} = 110\frac{1000}{3600}\,\text{m/s}$, $v_s = 26\,\text{km/h} = 26\frac{1000}{3600}\,\text{m/s}$, $h = 20\,\text{m}$, and $g = 9.81\,\text{m/s}^2$, we have

$$\boxed{\text{distance} = 47.12\,\text{m}.}$$

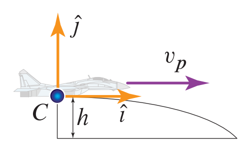
Problem 2.230

An airplane flying horizontally with a speed v_p relative to the water drops a crate onto a carrier when vertically over the back end of the ship, which is traveling at a speed $v_s = 32$ mph relative to the water. The length of the carrier's deck is $\ell = 1000$ ft, and the drop height is $h = 50$ ft. Determine the maximum value of v_p so that the crate will first impact within the rear half of the deck.



Solution

Referring to the figure at the right, the crate is denoted by C. The motion of C is analyzed using a fixed Cartesian coordinate system with origin at the point where C is released. After the drop, C is in projectile motion so that the acceleration of C is $\vec{a}_C = -g\,\hat{\jmath}$. Using constant acceleration equations and recalling that the crate is traveling with the airplane before being dropped, the velocity and position of C are



$$\vec{v}_C = v_p\,\hat{\imath} - gt\,\hat{\jmath} \quad \text{and} \quad \vec{r}_C = v_p t\,\hat{\imath} - \tfrac{1}{2}gt^2\,\hat{\jmath}. \tag{1}$$

Relative to the same frame of reference used to study the motion of C, the position of the back of the ship is

$$\vec{r}_s = v_s t\,\hat{\imath}, \tag{2}$$

where the subscript s stands for "ship." The vector describing the position of the crate relative to the back of the ship:

$$\vec{r}_{C/s} = \vec{r}_C - \vec{r}_s \quad \Rightarrow \quad \vec{r}_{C/s} = (v_p - v_s)t\,\hat{\imath} - \tfrac{1}{2}gt^2\,\hat{\jmath}. \tag{3}$$

Let t_d be the time at which the crate hits the deck. Given the choice of coordinate system, at $t = t_d$ $r_{Cy} = -h$. Using the second of Eqs. (1), this gives

$$-\tfrac{1}{2}gt_d^2 = -h \quad \Rightarrow \quad t_d = \sqrt{\frac{2h}{g}}. \tag{4}$$

$v_{p_{\max}}$ is the value of v_p such that the horizontal component of $\vec{r}_{C/s}$ at $t = t_d$ is equal to $\ell/2$:

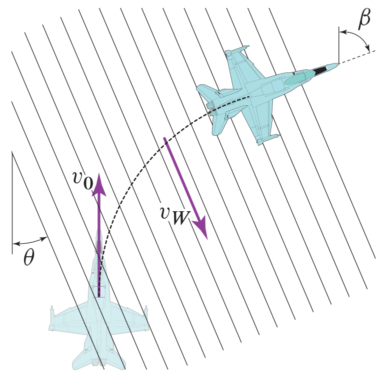
$$(r_{C/s})_x\big|_{t=t_d} = \tfrac{1}{2}\ell \quad \Rightarrow \quad (v_{p_{\max}} - v_s)\sqrt{\frac{2h}{g}} = \tfrac{1}{2}\ell \quad \Rightarrow \quad v_{p_{\max}} = v_s + \sqrt{\frac{g\ell^2}{8h}}. \tag{5}$$

Recalling that $v_s = 32\,\text{mph} = 32\frac{5280}{3600}\,\text{ft/s}$, $\ell = 1000\,\text{ft}$, $h = 50\,\text{ft}$, and $g = 32.2\,\text{ft/s}^2$, we have

$$\boxed{v_{p_{\max}} = 330.7\,\text{ft/s} = 225.4\,\text{mph}.}$$

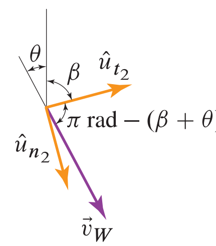
Problem 2.231

An airplane is initially flying north with a speed $v_0 = 430$ mph relative to the ground, while the wind has a constant speed $v_W = 12$ mph, forming an angle $\theta = 23°$ with the north-south direction. The airplane performs a course change of $\beta = 75°$ eastward while maintaining a constant reading of the airspeed indicator. Letting $\vec{v}_{P/A}$ be the velocity of the airplane relative to the air and assuming that the airspeed indicator measures the magnitude of the component of $\vec{v}_{P/A}$ in the direction of motion of the airplane, determine the speed of the airplane relative to the ground after the course correction.



Solution

We will express the velocity of the airplane using a normal tangential component system. In this manner, the tangent direction will always be the direction of motion of the airplane. In addition, we will use subscripts 1 and 2 to denote quantities before and after the change in course, respectively. Before turning, the velocity of the airplane and wind relative to the ground are



$$\vec{v}_{P_1} = v_0\,\hat{u}_{t_1} \quad \text{and} \quad \vec{v}_{W_1} = v_W(-\cos\theta\,\hat{u}_{t_1} + \sin\theta\,\hat{u}_{n_1}). \tag{1}$$

The airspeed of the plane before the turn is

$$v_1 = (\vec{v}_{P_1} - \vec{v}_{W_1}) \cdot \hat{u}_{t_1} \quad \Rightarrow \quad v_1 = v_0 + v_W \cos\theta. \tag{2}$$

After turning, the velocity of the wind is

$$\begin{aligned}\vec{v}_{W_2} &= v_W\{\cos[\pi\text{ rad} - (\beta+\theta)]\,\hat{u}_{t_2} + \sin[\pi\text{ rad} - (\beta+\theta)]\,\hat{u}_{n_2}\}\\ &= v_W[-\cos(\beta+\theta)\,\hat{u}_{t_2} + \sin(\beta+\theta)\,\hat{u}_{n_2}],\end{aligned} \tag{3}$$

where we have used the trigonometric identities $\cos(\pi\text{ rad} - \gamma) = -\cos\gamma$ and $\sin(\pi\text{ rad} - \gamma) = \sin\gamma$. The airspeed of the plane after the turn is

$$v_2 = (\vec{v}_{P_2} - \vec{v}_{W_2}) \cdot \hat{u}_{t_2} \quad \Rightarrow \quad v_2 = v_{P_2} + v_W\cos(\beta+\theta). \tag{4}$$

Enforcing the condition that $v_2 = v_1$, we have

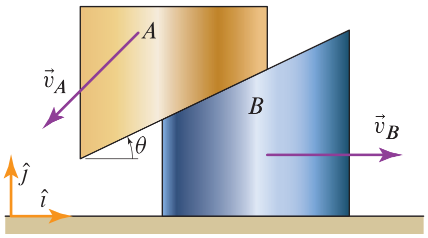
$$v_0 + v_W\cos\theta = v_{P_2} + v_W\cos(\beta+\theta) \quad \Rightarrow \quad v_{P_2} = v_0 + v_W[\cos\theta - \cos(\beta+\theta)]. \tag{5}$$

Recalling that $v_0 = 430$ mph, $v_W = 12$ mph, $\theta = 23°$, and $\beta = 75°$, we can evaluate v_{P_2} to obtain

$$\boxed{v_{P_2} = 442.7\text{ mph.}}$$

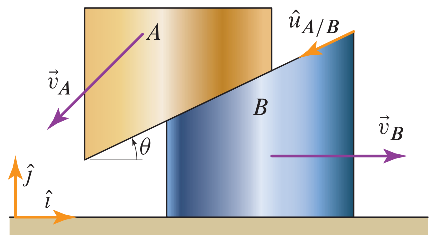
Problem 2.232

At the instant shown, block B is sliding over the ground with a velocity $\vec{v}_B$ while block A is sliding over block B and has an absolute velocity $\vec{v}_A = -(4\,\hat{\imath} + 4\,\hat{\jmath})$ ft/s. Determine $\vec{v}_B$ if $\theta = 30°$.



Solution

Using the Cartesian component system shown at the right, the velocities of A and B are



$$\vec{v}_A = v_{Ax}\,\hat{\imath} + v_{Ay}\,\hat{\jmath} \quad \text{and} \quad \vec{v}_B = v_{Bx}\,\hat{\imath}, \tag{1}$$

where we expressed the fact that B moves only in the x direction. Since A slides over B, the velocity of A relative to B is directed along unit vector $\hat{u}_{A/B}$, which, written in terms of $\hat{\imath}$ and $\hat{\jmath}$, is

$$\hat{u}_{A/B} = -\cos\theta\,\hat{\imath} - \sin\theta\,\hat{\jmath}. \tag{2}$$

Denoting by $v_{A/B}$ the component of the velocity of A relative to B along $\hat{u}_{A/B}$, and using Eq. (2), we have

$$\vec{v}_{A/B} = v_{A/B}\,\hat{u}_{A/B} = -v_{A/B}\cos\theta\,\hat{\imath} - v_{A/B}\sin\theta\,\hat{\jmath}. \tag{3}$$

Using relative kinematics, we have that

$$\vec{v}_{A/B} = \vec{v}_A - \vec{v}_B. \tag{4}$$

Substituting Eqs. (1) and (3) into Eq. (4), and equating component by component, we obtain the following system of two equations in the two unknowns $v_{A/B}$ and v_{Bx}:

$$-v_{A/B}\cos\theta = v_{Ax} - v_{Bx} \quad \text{and} \quad -v_{A/B}\sin\theta = v_{Ay}, \tag{5}$$

which can solved to obtain

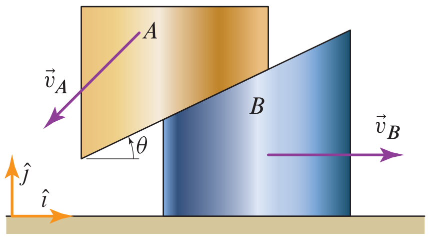
$$v_{Bx} = v_{Ax} - v_{Ay}\frac{\cos\theta}{\sin\theta} \quad \text{and} \quad v_{A/B} = -\frac{v_{Ay}}{\sin\theta}. \tag{6}$$

Recalling that $v_{Ax} = -4$ ft/s, $v_{Ay} = -4$ ft/s, and $\theta = 30°$, and substituting the first of Eqs. (6) into the second of Eqs. (1), we obtain

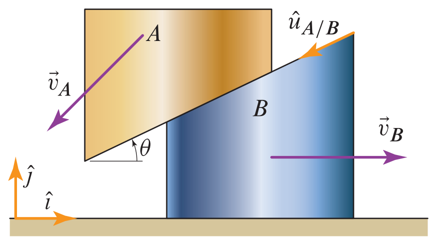
$$\boxed{\vec{v}_B = 2.928\,\hat{\imath}\ \text{ft/s}.}$$

Problem 2.233

At the instant shown, $\vec{v}_B = 5\,\hat{\imath}$ m/s. If $\theta = 25°$, determine the speed of A relative to B in order for A to travel only in the vertical direction while sliding over B.



Solution



Using the Cartesian component system shown at the right, the velocities of A and B are

$$\vec{v}_A = v_{Ay}\,\hat{\jmath} \quad \text{and} \quad \vec{v}_B = v_{Bx}\,\hat{\imath}, \tag{1}$$

where, following the problem statement, we expressed the fact that A and B move only in the y and x directions, respectively. Since A slides over B, the velocity of A relative to B must be directed along the unit vector $\hat{u}_{A/B}$, which, using the unit vectors $\hat{\imath}$ and $\hat{\jmath}$, is

$$\hat{u}_{A/B} = -\cos\theta\,\hat{\imath} - \sin\theta\,\hat{\jmath}. \tag{2}$$

Denoting by $v_{A/B}$ the component of the velocity of A relative to B along $\hat{u}_{A/B}$, we have

$$\vec{v}_{A/B} = v_{A/B}\,\hat{u}_{A/B} = -v_{A/B}\cos\theta\,\hat{\imath} - v_{A/B}\sin\theta\,\hat{\jmath}, \tag{3}$$

where we have used the expression for $\hat{u}_{A/B}$ in Eq. (2). Using relative kinematics, we have

$$\vec{v}_{A/B} = \vec{v}_A - \vec{v}_B. \tag{4}$$

Substituting Eqs. (1) and (3) into Eq. (4), and equating component by component, we obtain the following system of two equations in the two unknowns $v_{A/B}$ and v_{Ay}:

$$-v_{A/B}\cos\theta = -v_{Bx} \quad \text{and} \quad -v_{A/B}\sin\theta = v_{Ay}, \tag{5}$$

which can solved to obtain

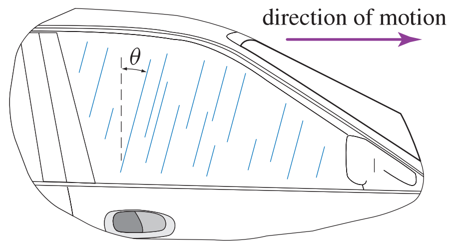
$$v_{Ay} = -v_{Bx}\tan\theta \quad \text{and} \quad v_{A/B} = \frac{v_{Bx}}{\cos\theta}. \tag{6}$$

Recalling that $v_{Bx} = 5$ m/s and $\theta = 25°$ we observe that $v_{A/B}$ in the second of Eqs. (6) is a positive quantity and therefore be taken to represent the speed of A relative to B. Evaluating $v_{A/B}$ we then obtain

$$\boxed{v_{A/B} = 5.517\ \text{m/s}.}$$

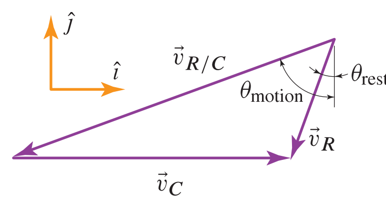
Problem 2.234

An interesting application of the relative motion equations is the experimental determination of the speed at which rain falls. Say you perform an experiment in your car in which you park your car in the rain and measure the angle the falling rain makes on your side window. Let this angle be $\theta_{\text{rest}} = 20°$. Next you drive forward at 25 mph and measure the new angle $\theta_{\text{motion}} = 70°$ that the rain makes with the vertical. Determine the speed of the falling rain.



Solution

In the figure shown, $\hat{\imath}$ and $\hat{\jmath}$ are horizontal and vertical, respectively. Since θ_{rest} is the orientation of the velocity of raindrops relative to the ground, the velocity in question is



$$\vec{v}_R = -v_R(\sin\theta_{\text{rest}}\,\hat{\imath} + \cos\theta_{\text{rest}}\,\hat{\jmath}), \tag{1}$$

where v_R is the speed of the falling rain. The angle θ_{motion} describes the orientation of $\vec{v}_{R/C}$, the velocity of the raindrops relative to the moving car:

$$\vec{v}_{R/C} = -v_{R/C}(\sin\theta_{\text{motion}}\,\hat{\imath} + \cos\theta_{\text{motion}}\,\hat{\jmath}), \tag{2}$$

where $v_{R/C}$ is the speed of the raindrops as perceived by an observer moving with the car. Relative kinematics requires that

$$\vec{v}_R = \vec{v}_C + \vec{v}_{R/C}, \tag{3}$$

where $\vec{v}_C$ is the velocity of the car. Substituting Eqs. (1) and (2) into Eq. (3),

$$-v_R\sin\theta_{\text{rest}}\,\hat{\imath} - v_R\cos\theta_{\text{rest}}\,\hat{\jmath} = v_C\,\hat{\imath} - v_{R/C}\sin\theta_{\text{motion}}\,\hat{\imath} - v_{R/C}\cos\theta_{\text{motion}}\,\hat{\jmath}. \tag{4}$$

Equating components, Eq. (4) yields the following system of two equations in two unknowns v_R and $v_{R/C}$:

$$-v_R\sin\theta_{\text{rest}} = v_C - v_{R/C}\sin\theta_{\text{motion}} \quad \text{and} \quad -v_R\cos\theta_{\text{rest}} = -v_{R/C}\cos\theta_{\text{motion}}, \tag{5}$$

whose solution is

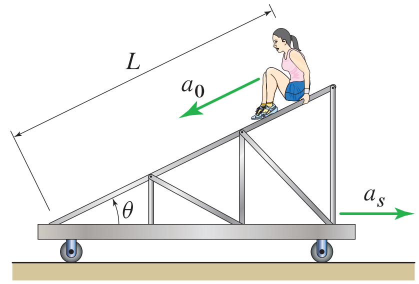
$$v_R = \frac{v_C}{\cos\theta_{\text{rest}}\tan\theta_{\text{motion}} - \sin\theta_{\text{rest}}} \quad \text{and} \quad v_{R/C} = \frac{v_C}{\sin\theta_{\text{motion}} - \cos\theta_{\text{motion}}\tan\theta_{\text{rest}}}. \tag{6}$$

Recalling that we have $\theta_{\text{rest}} = 20°$, $\theta_{\text{motion}} = 70°$, $v_C = 25\,\text{mph} = 25\frac{5280}{3600}\,\text{ft/s}$, we can evaluate the first of Eqs. (6) to obtain

$$\boxed{v_{\text{rain}} = 16.37\,\text{ft/s}.}$$

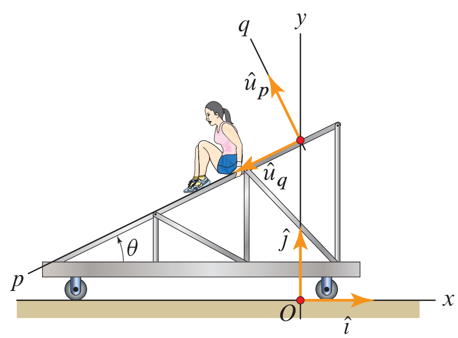
Problem 2.235

A woman is sliding down an incline with a constant acceleration of $a_0 = 2.3\,\mathrm{m/s^2}$ relative to the incline. At the same time the incline is accelerating to the right at $1.2\,\mathrm{m/s^2}$ relative to the ground. Letting $\theta = 34°$ and $L = 4\,\mathrm{m}$ and assuming that both the woman and the incline start from rest, determine the horizontal distance traveled by the woman with respect to the ground when she reaches the bottom of the slide.



Solution

The xy and pq frames shown at the right are attached to the ground and to the incline, respectively. The origins of these frames are chosen so that the x and p coordinates of the woman are both equal to zero at the initial time. We denote by $\vec{a}_W$ and $\vec{a}_{W/I}$ the accelerations of the woman relative to the ground and to the incline, respectively. Denoting by $\vec{a}_I$ the acceleration of the incline relative to the ground, from the problem's statement we have



$$\vec{a}_{W/I} = a_0\,\hat{u}_p \quad \text{and} \quad \vec{a}_I = a_s\,\hat{\imath}. \tag{1}$$

Then, using relative kinematics, we have

$$\vec{a}_W = \vec{a}_I + \vec{a}_{W/I}. \tag{2}$$

Substituting Eqs. (1) into Eq. (2) and observing that $\hat{u}_p = -\cos\theta\,\hat{\imath} - \sin\theta\,\hat{\jmath}$, we obtain

$$\vec{a}_W = (a_s - a_0\cos\theta)\,\hat{\imath} - a_0\sin\theta\,\hat{\jmath}, \tag{3}$$

which implies that the x component of $\vec{a}_W$ is constant. Recalling that $x_W = 0$ for $t = 0$ and that the woman starts from rest, using constant acceleration equations, we have

$$x_W = \tfrac{1}{2}(a_s - a_0\cos\theta)t^2. \tag{4}$$

Denoting by t_f and d the time taken by the woman to slide to the bottom of the incline and the corresponding horizontal distance traveled (with respect to the ground), respectively, we have

$$d = \left|\tfrac{1}{2}(a_s - a_0\cos\theta)t_f^2\right|. \tag{5}$$

We observe that t_f is also the time needed to travel the distance L in the p direction with the constant acceleration a_0. Hence, recalling that $p = 0$ and $\dot{p} = 0$ for $t = 0$, applying constant acceleration equations in the p direction we have

$$p = \tfrac{1}{2}a_0t^2 \quad\Rightarrow\quad L = \tfrac{1}{2}a_0t_f^2 \quad\Rightarrow\quad t_f = \sqrt{2L/a_0}. \tag{6}$$

Substituting the last of Eqs. (6) into Eq. (5) we have

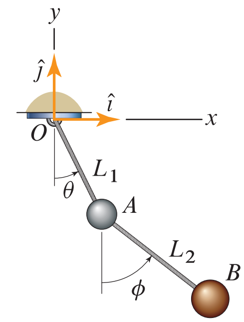
$$d = \left|(a_s - a_0\cos\theta)(L/a_0)\right|. \tag{7}$$

Recalling that $a_s = 1.2\,\mathrm{m/s^2}$, $a_0 = 2.3\,\mathrm{m/s^2}$, $\theta = 34°$, $L = 4\,\mathrm{m}$, we can evaluate Eq. (7) to obtain

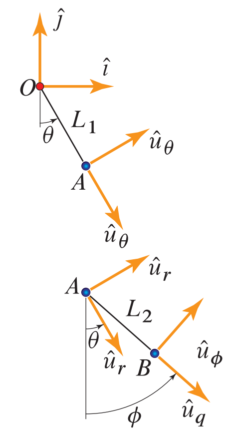
$$\boxed{d = 1.229\,\mathrm{m}.}$$

Problem 2.236

The pendulum bob A swings about O, which is a fixed point, while bob B swings about A. Express the components of the acceleration of B relative to the Cartesian component system shown with origin at the fixed point O in terms of L_1, L_2, θ, ϕ, and the necessary time derivatives of ϕ and θ.



Solution



Referring to the figure at the right, we use two polar component systems, one with unit vectors $\hat{u}_r$ and $\hat{u}_\theta$ attached to A, the other with unit vectors $\hat{u}_q$ and $\hat{u}_\phi$ attached to B. When expressed in terms of the unit vectors $\hat{\imath}$ and $\hat{\jmath}$, the unit vectors of the chosen polar component systems are:

$$\hat{u}_r = \sin\theta\,\hat{\imath} - \cos\theta\,\hat{\jmath}, \quad \hat{u}_\theta = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}, \tag{1}$$

$$\hat{u}_q = \sin\phi\,\hat{\imath} - \cos\phi\,\hat{\jmath}, \quad \hat{u}_\phi = \cos\phi\,\hat{\imath} + \sin\phi\,\hat{\jmath}. \tag{2}$$

Using the $(\hat{u}_r, \hat{u}_\theta)$ component system, and denoting by r and θ the corresponding polar coordinates of A, the acceleration of A is

$$\vec{a}_A = \big(\ddot{r} - r\dot{\theta}^2\big)\,\hat{u}_r + \big(r\ddot{\theta} + 2\dot{r}\dot{\theta}\big)\,\hat{u}_\theta. \tag{3}$$

Recalling that $r = L_1 =$ constant, Eq. (3) can be written

$$\vec{a}_A = -L_1\dot{\theta}^2\,\hat{u}_r + L_1\ddot{\theta}\,\hat{u}_\theta \quad \Rightarrow \quad \vec{a}_A = L_1\big(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\big)\,\hat{\imath} + L_1\big(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\big)\,\hat{\jmath}, \tag{4}$$

where we used Eqs. (1) to express $\vec{a}_A$ in terms of $\hat{\imath}$ and $\hat{\jmath}$. Using the $(\hat{u}_q, \hat{u}_\phi)$ component system, and denoting by q and ϕ the polar coordinates of B relative to A, the acceleration of B *relative* to A is

$$\vec{a}_{B/A} = \big(\ddot{q} - q\dot{\phi}^2\big)\,\hat{u}_q + \big(q\ddot{\phi} + 2\dot{q}\dot{\theta}\big)\,\hat{u}_\phi. \tag{5}$$

Recalling that $q_B = L_2 =$ constant, Eq. (5) can be written

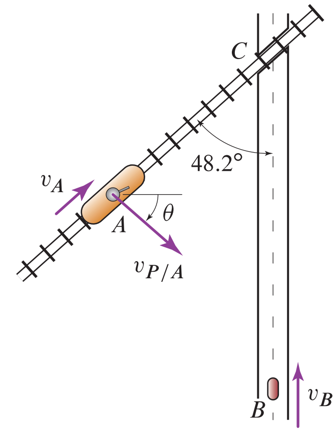
$$\vec{a}_{B/A} = -L_2\dot{\phi}^2\,\hat{u}_q + L_2\ddot{\phi}\,\hat{u}_\phi \quad \Rightarrow \quad \vec{a}_{B/A} = L_2\big(\ddot{\phi}\cos\phi - \dot{\phi}^2\sin\phi\big)\,\hat{\imath} + L_2\big(\ddot{\phi}\sin\phi + \dot{\phi}^2\cos\phi\big)\,\hat{\jmath}, \tag{6}$$

where we used Eqs. (2) to express $\vec{a}_{B/A}$ in terms of $\hat{\imath}$ and $\hat{\jmath}$. Relative kinematics demands that $\vec{a}_B = \vec{a}_A + \vec{a}_{B/A}$. Therefore, using the last of Eqs. (4) and (6), we have

$$\vec{a}_B = \big(L_1\ddot{\theta}\cos\theta - L_1\dot{\theta}^2\sin\theta + L_2\ddot{\phi}\cos\phi - L_2\dot{\phi}^2\sin\phi\big)\,\hat{\imath} + \big(L_1\ddot{\theta}\sin\theta + L_1\dot{\theta}^2\cos\theta + L_2\ddot{\phi}\sin\phi + L_2\dot{\phi}^2\cos\phi\big)\,\hat{\jmath}.$$

Problem 2.237

Revisit Example 2.24 in which the movie's hero is traveling on train car A with constant speed $v_A = 18\,\text{m/s}$ while the target B is moving at a constant speed $v_B = 40\,\text{m/s}$ (so that $a_B = 0$). Recall that 4 s before an otherwise inevitable collision between A and B, a projectile P traveling at a speed of 300 m/s relative to A is shot toward B. Take advantage of the solution in Example 2.24 and determine the time it takes the projectile P to reach B and the projectile's distance traveled.



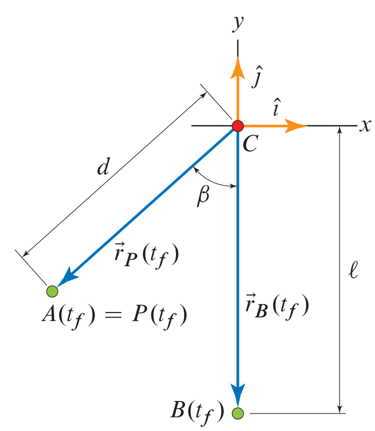
Solution

Referring to Example 2.24 on p. 127 of the textbook and to the figure on the right, (Fig. 3 in Example 2.24), we denote by β the angle $A\hat{C}B$, so that $\beta = 48.2°$. As was done in Example 2.24, we denote by d the distance between A and C at the time of firing, so that $d = (4\,\text{s})v_A = 72.00\,\text{m}$. To find the time taken by the projectile to hit the target, we observe that the distance traveled by the projectile in the x direction, is given by

$$d_x = d\sin\beta. \tag{1}$$

In Example 2.24, we had determined that the (absolute) velocity of the projectile was

$$\vec{v}_P = (v_A \sin\beta + v_{P/A}\cos\theta)\,\hat{\imath} + (v_A\cos\beta - v_{P/A}\sin\theta)\,\hat{\jmath}. \tag{2}$$



Since $\vec{v}_P$ is constant, the time taken by the projectile to hit the target, which we denote by Δt, is simply equal to d_x in Eq. (1) divided by v_{Px}, i.e.,

$$\Delta t = \frac{d\sin\beta}{v_A\sin\beta + v_{P/A}\cos\theta}, \tag{3}$$

which, recalling that we have $d = 72.00\,\text{m}$, $\beta = 48.2°$, $v_A = 18\,\text{m/s}$, $v_{P/A} = 300\,\text{m/s}$, and $\theta = 64.40°$ (final result in Example 2.24), can be evaluated to obtain

$$\boxed{\Delta t = 0.3752\,\text{s}.}$$

To find the distance traveled, we observe again that $\vec{v}_P$ is constant, so that the distance traveled is

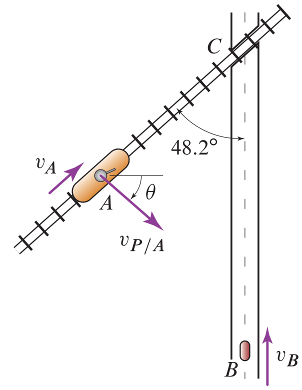
$$|\vec{v}_P\,\Delta t| = \frac{\sqrt{(v_A\sin\beta + v_{P/A}\cos\theta)^2 + (v_A\cos\beta - v_{P/A}\sin\theta)^2}(d\sin\beta)}{v_A\sin\beta + v_{P/A}\cos\theta}, \tag{4}$$

where we have used the expressions for $\vec{v}_P$ in Eq. (2) and Δt in Eq. (3). Since $d = 72.00\,\text{m}$, $\beta = 48.2°$, $v_{P/A} = 300\,\text{m/s}$, and $\theta = 64.40°$, we can evaluate the right-hand side of Eq. (4) to obtain

$$\boxed{\text{Distance traveled by } P = 110.9\,\text{m}.}$$

Problem 2.238

Consider the following variation of the problem in Example 2.24 in which a movie hero needs to destroy a mobile robot B, except this time they are not going to collide at C. Assume that the hero is traveling on the train car A with constant speed $v_A = 18\,\text{m/s}$, while the robot B travels at a constant speed $v_B = 50\,\text{m/s}$. In addition, assume that at time $t = 0\,\text{s}$ the train car A and the robot B are 72 and 160 m away from C, respectively. To prevent B from reaching its intended target, at $t = 0\,\text{s}$ the hero fires a projectile P at B. If P can travel at a constant speed of 300 m/s relative to the gun, determine the orientation θ that must be given to the gun to hit B. *Hint:* An equation of the type $\sin\beta \pm A\cos\beta = C$ has the solution $\beta = \mp\gamma + \sin^{-1}(C\cos\gamma)$, if $|C\cos\gamma| \le 1$, where $\gamma = \tan^{-1} A$.



Solution

We base the solution of this problem on the solution of Example 2.24 on p. 127 of the textbook. All the quantities used in this solution are defined in Example 2.24. We report here Eq. (14) from the Example, which remains valid under the conditions stated in this problem and which determines the value of the angle θ that we want to determine:

$$(\ell - d\cos\beta)\cos\theta - d\sin\beta\sin\theta = \frac{\sin\beta}{v_{P/A}}(v_B d - v_A\ell). \tag{1}$$

Dividing both sides of this equation by $-d\sin\beta$, we have

$$\sin\theta - \frac{\ell - d\cos\beta}{d\sin\beta}\cos\theta = \frac{v_A\ell - v_B d}{v_{P/A}d}. \tag{2}$$

The above equation is a transcendental equation in θ whose solution can be obtained using the following technique. We consider the term multiplying the $\cos\theta$ on the left-hand side of the equation and we define an angle ϕ such that

$$\tan\phi = \frac{\ell - d\cos\beta}{d\sin\beta} \quad\Rightarrow\quad \phi = \tan^{-1}\left(\frac{\ell - d\cos\beta}{d\sin\beta}\right). \tag{3}$$

Then, recalling that $\tan\phi = \sin\phi/\cos\phi$, we can rewrite Eq. (2) as

$$\sin\theta - \frac{\sin\phi}{\cos\phi}\cos\theta = \frac{v_A\ell - v_B d}{v_{P/A}d} \quad\Rightarrow\quad \sin\theta\cos\phi - \sin\phi\cos\theta = \frac{(v_A\ell - v_B d)\cos\phi}{v_{P/A}d}$$

$$\Rightarrow\quad \sin(\theta - \phi) = \frac{(v_A\ell - v_B d)\cos\phi}{v_{P/A}d}, \tag{4}$$

where we have used the trigonometric identity $\sin\theta\cos\phi - \sin\phi\cos\theta = \sin(\theta - \phi)$. The last of Eq. (4) can now be solved for θ to obtain

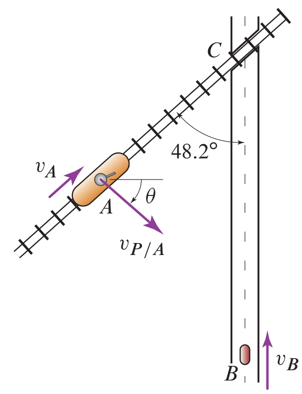
$$\theta = \phi + \sin^{-1}\left[\frac{(v_A\ell - v_B d)\cos\phi}{v_{P/A}d}\right]. \tag{5}$$

Recalling that $v_A = 18\,\text{m/s}$, $v_B = 50\,\text{m/s}$, $v_{P/A} = 300\,\text{m/s}$, and where, using the results in Example 2.24, $\ell = 160.0\,\text{m}$, $d = 72.00\,\text{m}$, we can evaluate ϕ in in the last of Eqs. (3) and then evaluate θ in Eq. (5) to obtain

$$\boxed{\theta = 63.57^\circ.}$$

Problem 2.239

Consider the following variation of the problem in Example 2.24 in which a movie hero needs to destroy a mobile robot B. As was done in that problem, assume that the movie hero is traveling on the train car A with constant speed $v_A = 18\,\text{m/s}$ and that, 4 s before an otherwise inevitable collision at C, the hero fires a projectile P traveling at 300 m/s relative to A. Unlike Example 2.24, assume here that the robot B travels with a constant acceleration $a_B = 10\,\text{m/s}^2$ and that $v_B(0) = 20\,\text{m/s}$, where $t = 0$ is the time of firing. Determine the orientation θ of the gun fired by the hero so that B can be destroyed before the collision at C.



Solution

The general strategy for the solution of moving target problems has been discussed in the Road Map of Example 2.24 on p. 127. According to this strategy, letting $t = 0$ be the time at which the projectile is fired, there is a time $t_I > 0$ such that

$$\vec{r}_{P/B}(t_I) = \vec{r}_P(t_I) - \vec{r}_B(t_I) = \vec{0}, \tag{1}$$

that is, there is a time t_I at which the projectile and the target meet. To solve this problem we need to find the positions of the projectile and of the target as functions of time and then set them equal to each other as required by the above equation.

Because A is moves at a constant speed along a straight line, then $\vec{v}_A$ is constant. Once P is fired, its velocity is also a constant given by

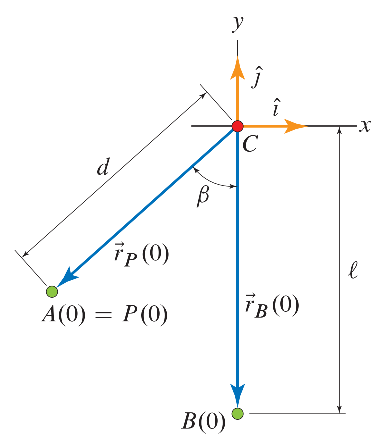
$$\vec{v}_P = \vec{v}_A + v_{P/A}\,\hat{u}_{P/A}(0), \tag{2}$$

where $v_{P/A}$ in known. Hence,

$$\vec{r}_P(t) = \vec{r}_P(0) + \big[\vec{v}_A(0) + v_{P/A}\,\hat{u}_{P/A}(0)\big]\,t. \tag{3}$$

Because B has a constant acceleration, we can use constant acceleration equations to write

$$\vec{r}_B(t) = \vec{r}_B(0) + \vec{v}_B(0)t + \tfrac{1}{2}\vec{a}_B t^2. \tag{4}$$



Substituting Eqs. (3) and (4) into Eq. (1), we have

$$\vec{r}_P(0) + \big(\vec{v}_A(0) + v_{P/A}\,\hat{u}_{P/A}(0)\big)\,t_I - \vec{r}_B(0) - \vec{v}_B(0)t_I - \tfrac{1}{2}\vec{a}_B t_I^2 = \vec{0}, \tag{5}$$

Referring to the figure at the right (similar to Fig. 3 in Example 2.24 and showing the geometry at the time of firing) we have that $\vec{r}_A(0) = \vec{r}_P(0)$, so that Eq. (5) can be rewritten as

$$\vec{r}_{A/B}(0) + [\vec{v}_{A/B}(0) + v_{P/A}\,\hat{u}_{P/A}(0)]t_I - \tfrac{1}{2}\vec{a}_B t_I^2 = \vec{0}. \tag{6}$$

The problem is solved when we are able to express all of the terms in Eq. (6) via known quantities and the only two unknowns of the problem, which are t_I and the firing angle θ. We therefore proceed to determine convenient expressions for each of the vectors in Eq. (6).

Since the position at $t = 0$ in this problem is the same as that in the figure shown, we have

$$\vec{r}_{A/B}(0) = -d \sin\beta\,\hat{\imath} + (\ell - d\cos\beta)\,\hat{\jmath}. \tag{7}$$

Again referring to the geometry at $t = 0$, and recalling that β is known ($\beta = 48.2°$), we have

$$\vec{v}_A(0) = v_A\,\hat{u}_{C/A} = v_A(\sin\beta\,\hat{\imath} + \cos\beta\,\hat{\jmath}) \quad \text{and} \quad \vec{v}_B(0) = v_B(0)\,\hat{\jmath}, \tag{8}$$

so that

$$\vec{v}_{A/B}(0) = v_A \sin\beta\,\hat{\imath} + (v_A\cos\beta - v_B(0))\,\hat{\jmath}. \tag{9}$$

Since $t = 0$ is the time of firing, we must have

$$\hat{u}_{P/A}(0) = \cos\theta\,\hat{\imath} - \sin\theta\,\hat{\jmath}. \tag{10}$$

Finally, the problem statement and the geometry of the problem tell us that

$$\vec{a}_B = a_B\,\hat{\jmath}. \tag{11}$$

Substituting Eqs. (7) and Eqs. (9)–(11) into Eq. (6) and expressing the result on a component by component basis, we have

$$-d\sin\beta + (v_A\sin\beta + v_{P/A}\cos\theta)t_I = 0 \tag{12}$$

and

$$\ell - d\cos\beta + [v_A\cos\beta - v_B(0) - v_{P/A}\sin\theta]t_I - \tfrac{1}{2}a_B t_I^2 = 0, \tag{13}$$

where $v_A = 18\,\text{m/s}$, $v_B(0) = 20\,\text{m/s}$, $v_{P/A} = 300\,\text{m/s}$, $a_B = 10\,\text{m/s}^2$, $\beta = 48.2°$, and where, using the results in Example 2.24, $d = 72.00\,\text{m}$ and $\ell = 160.0\,\text{m}$.

Equations(12) and (13) form a system of two equations in the two unknowns θ and t_I that can be solved numerically with appropriate mathematical software. We have used *Mathematica* with the following code:

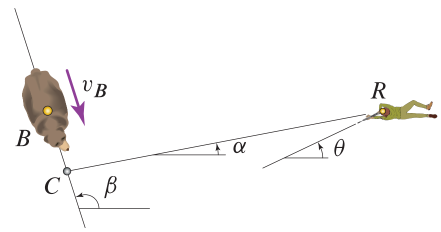
```
Parameters = {vA → 18., vB0 → 20., vPrA → 300., aB → 10., β → 48.2 Degree, d → 72.00, ℓ → 160.0};
FindRoot[
 {-d Sin[β] + (vA Sin[β] + vPrA Cos[θ Degree]) tI == 0,
   ℓ - d Cos[β] + (vA Cos[β] - vB0 - vPrA Sin[θ Degree]) tI - 1/2 aB tI^2 == 0} /. Parameters,
 {{θ, 64.40}, {tI, 2}}]
```

Notice that we have provided the root finding algorithm an initial guess for the solution consisting of the values $\theta = 64.40°$ and $t_I = 2\,\text{s}$, the first of which is the solution to the case discussed in Example 2.24 and the second is between the initial time and the time of collision. The above code yields the following solution:

$$\boxed{\theta = 65.88° \quad \text{and} \quad t_I = 0.3947\,\text{s}.}$$

Problem 2.240

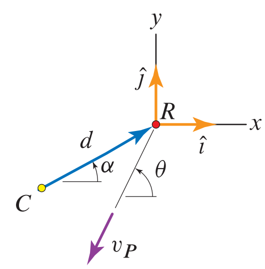
A park ranger R is aiming a rifle armed with a tranquilizer dart at a bear (the figure is not to scale). The bear is moving in the direction shown at a constant speed $v_B = 25$ mph. The ranger fires the rifle when the bear is at C at a distance of 150 ft. Knowing that $\alpha = 10°$, $\beta = 108°$, the dart travels with a constant speed of 425 ft/s, and the dart and the bear are moving in a horizontal plane, determine the orientation θ of the rifle so that the ranger can hit the bear. *Hint:* An equation of the type $\sin\beta \pm A\cos\beta = C$ has the solution $\beta = \mp\gamma + \sin^{-1}(C\cos\gamma)$, if $|C\cos\gamma| \leq 1$, where $\gamma = \tan^{-1} A$.



Solution

We will use the Cartesian component system shown at the right with origin at R. We will denote the dart by P. The velocity of the bear and of the dart are

$$\vec{v}_B = -v_B(\cos\beta\,\hat{\imath} + \sin\beta\,\hat{\jmath}) \quad \text{and} \quad \vec{v}_P = -v_P(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}), \tag{1}$$



where $v_B = 25$ mph and $v_P = 425$ ft/s. Let $t = 0$ be the time of firing, and t_I be the time at which P hits B. Then it must be the case that

$$\vec{r}_{P/B}(t_I) = \vec{0}, \tag{2}$$

where, since both the dart and the bear move at constant velocity (constant speed and fixed direction), we have $\vec{r}_P = \vec{v}_P t$ and $\vec{r}_B = \vec{r}_B(0) + \vec{v}_B t$, which, using Eqs. (1), we can write as

$$\vec{r}_P = -v_P t(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}) \quad \text{and} \quad \vec{r}_B = -(d\cos\alpha + v_B t\cos\beta)\,\hat{\imath} - (d\sin\alpha + v_B t\cos\beta\,\hat{\jmath}), \tag{3}$$

where we have used the fact that $\vec{r}_B(0) = -d(\cos\alpha\,\hat{\imath} + \sin\alpha\,\hat{\jmath})$. Recalling that $\vec{r}_{P/B}(t_I) = \vec{r}_P(t_I) - \vec{r}_B(t_I)$, and using Eqs. (3) to enforce Eq. (2) on a component by component basis, we have

$$-v_P t_I\cos\theta + d\cos\alpha + v_B t_I\cos\beta = 0 \quad \text{and} \quad -v_P t_I\sin\theta + d\sin\alpha + v_B t_I\cos\beta = 0. \tag{4}$$

This is a system of two equations in the two unknowns t_I and θ. Eliminating t_I from Eqs. (4) and rearranging terms so as to take advantage of the hint, we have

$$\tan\alpha(v_P\cos\theta - v_B\cos\beta) = v_P\sin\theta - v_B\sin\beta \quad \Rightarrow \quad \sin\theta - \tan\alpha\cos\theta = \frac{v_B}{v_P}(\sin\beta - \tan\alpha\cos\beta). \tag{5}$$

Letting $A = \tan\alpha$ and $C = (v_B/v_P)(\sin\beta - \tan\alpha\cos\beta)$, and taking advantage of the hint, we can solve for θ to obtain

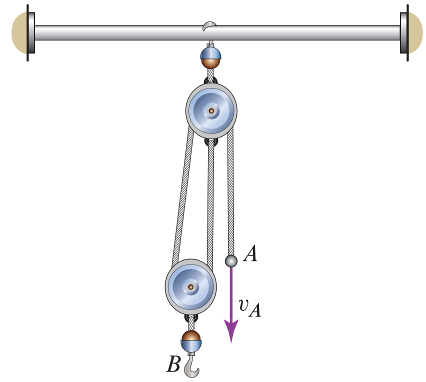
$$\theta = \alpha + \sin^{-1}\left[\frac{v_B}{v_P}(\sin\beta - \tan\alpha\cos\beta)\cos\alpha\right]. \tag{6}$$

Recalling that $\alpha = 10°$, $\beta = 108°$, $v_B = 25\,\text{mph} = 25\frac{5280}{3600}$ ft/s, and $v_P = 425$ ft/s, we can evaluate θ to obtain

$$\boxed{\theta = 14.90°.}$$

Problem 2.241

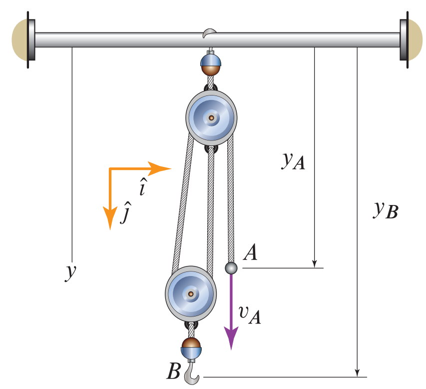
The object in the figure is called a *gun tackle*, and it used to be very common on sailboats to help in the operation of front-loaded guns. If the end at A is pulled down at a speed of 1.5 m/s, determine the velocity of B. Neglect the fact that some portions of the rope are not vertically aligned.



Solution

By neglecting vertical misalignments, and using the y axis shown at the right, we can expresss the length of the cord as follows:

$$L = y_A + 2y_B. \tag{1}$$



Since L is constant, taking its time derivative, we have

$$0 = \dot{y}_A + 2\dot{y}_B. \tag{2}$$

Solving for $\dot{y}_B$, and recalling that $\vec{v}_B = \dot{y}_B\,\hat{\jmath}$, we have

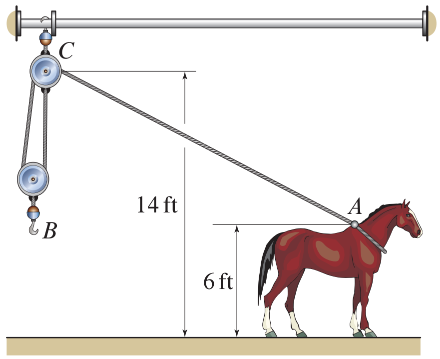
$$\vec{v}_B = -\tfrac{1}{2}\dot{y}_A\,\hat{\jmath}. \tag{3}$$

Recalling that $\dot{y}_A = 1.5\text{m/s}$, we can evaluate the above expression to obtain

$$\vec{v}_B = -0.7500\,\hat{\jmath}\ \text{m/s}.$$

Problem 2.242

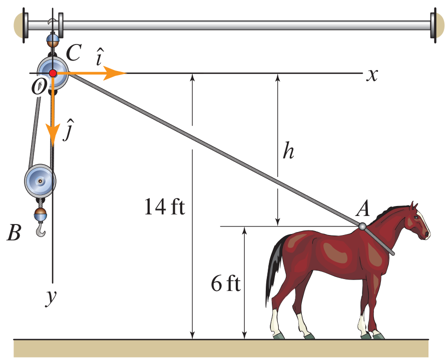
The gun tackle shown is operated with the help of a horse. If the horse moves to the right at a constant speed of 7 ft/s, determine the velocity and acceleration of B when the horizontal distance from B to A is 15 ft. Except for the part of the rope between C and A, neglect the fact that some portions are not vertically aligned. Also neglect the change in the amount of rope wrapped around pulley C as the horse moves to the right.



Solution

Using the coordinate system shown in the figure at the right, and denoting the length of the rope by L, we have

$$L = 2y_B + \sqrt{x_A^2 + h^2}. \tag{1}$$



Since L is constant, differentiating Eq. (1) with respect to time gives

$$0 = 2\dot{y}_B + \frac{x_A \dot{x}_A}{\sqrt{x_A^2 + h^2}}. \tag{2}$$

Solving Eq. (2) for $\dot{y}_B$, we have

$$\dot{y}_B = -\frac{x_A \dot{x}_A}{2\sqrt{x_A^2 + h^2}}. \tag{3}$$

Since $\vec{v}_B = \dot{y}_B\,\hat{\jmath}$, recalling that $h = 8$ ft, $\dot{x}_A = 7$ ft/s, and $x_A = 15$ ft, we can evaluate Eq. (3) to obtain

$$\boxed{\vec{v}_B = -3.088\,\hat{\jmath}\ \text{ft/s}.} \tag{4}$$

To obtain the acceleration of B we differentiate Eq. (3) with respect to time and, observing that $\ddot{x}_A = 0$, we obtain

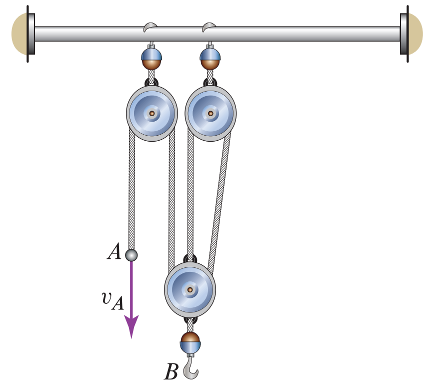
$$\ddot{y}_B = -\frac{h^2 \dot{x}_A^2}{2\left(x_A^2 + h^2\right)^{3/2}}. \tag{5}$$

Since $\vec{a}_B = \ddot{y}_B\,\hat{\jmath}$, again recalling that we have $h = 8$ ft, $\dot{x}_A = 7$ ft/s, and $x_A = 15$ ft, we can evaluate Eq. (5) to obtain

$$\boxed{\vec{a}_B = -0.3192\,\hat{\jmath}\ \text{ft/s}^2.}$$

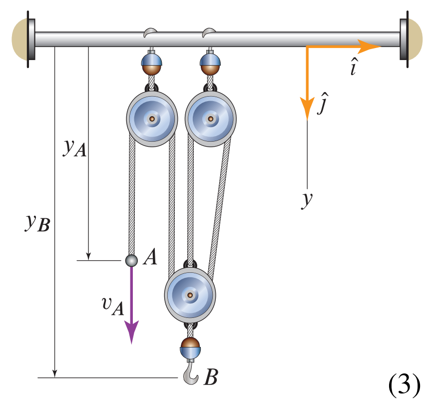
Problem 2.243

The figure shows an inverted gun tackle with snatch block, which used to be common on sailboats. If the end at A is pulled at a speed of $1.5\,\mathrm{m/s}$, determine the velocity of B. Neglect the fact that some portions of the rope are not vertically aligned.



Solution

Neglecting vertical misalignments, and using the Cartesian coordinate system shown at the right, we can express the length of the cord as follows:



$$L = y_A + 3y_B. \quad (1)$$

Since L is constant, taking its derivative with respect to time, we have

$$0 = \dot{y}_A + 3\dot{y}_B. \quad (2)$$

Recalling that $\vec{v}_B = \dot{y}_B\,\hat{\jmath}$, solving Eq. (2) for $\dot{y}_B$, we have

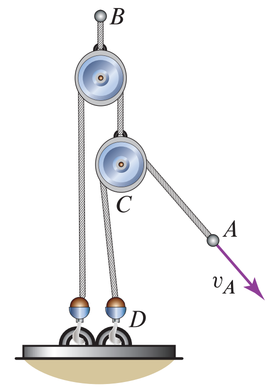
$$\vec{v}_B = -\tfrac{1}{3}\dot{y}_A\,\hat{\jmath}. \quad (3)$$

Recalling that $\dot{y}_A = 1.5\,\mathrm{m/s}$, we can evaluate $\vec{v}_B$ to obtain

$$\boxed{\vec{v}_B = -0.5000\,\hat{\jmath}\ \mathrm{m/s}.}$$

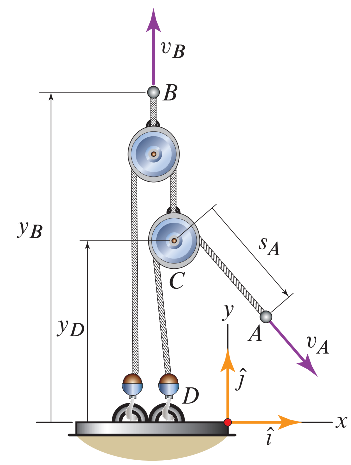
Problem 2.244

In maritime speak, the system in the figure is often called a *whip-upon-whip purchase* and is used for controlling certain types of sails on small cutters (by attaching point B to the sail to be unfurled). If the end of the rope at A is pulled with a speed of $4\,\text{m/s}$, determine the velocity of B. Neglect the fact that the segment of the rope between C and D is not vertically aligned, and assume that the slope of segment AC is constant.



Solution

Neglecting the vertical misalignment of the segment CD, assuming that the slope of the segment AC is constant, and using the coordinate system shown at the right, we can write the lengths of the two rope in the system as follows:



$$L_1 = 2y_B - y_C \quad \text{and} \quad L_2 = y_C + s_A. \tag{1}$$

Since L_1 and L_2 are constant, taking their time derivative we have

$$0 = 2\dot{y}_B - \dot{y}_C \quad \text{and} \quad 0 = \dot{y}_C + \dot{s}_A. \tag{2}$$

Recalling that $\dot{s}_A = v_A$ is known and eliminating $\dot{y}_C$ from Eqs. (2), we can solve for $\dot{y}_B$ to obtain

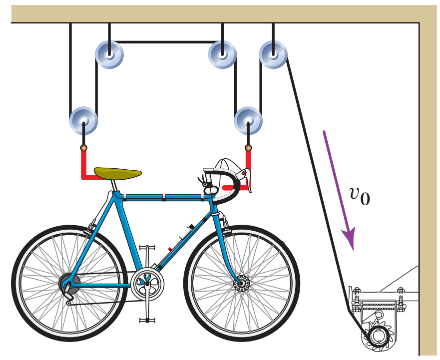
$$\dot{y}_B = -\tfrac{1}{2} v_A \quad \Rightarrow \quad \vec{v}_B = -\tfrac{1}{2} v_A\,\hat{\jmath}. \tag{3}$$

Recalling that $v_A = 4\,\text{m/s}$, we have

$$\vec{v}_B = -2.000\,\hat{\jmath}\ \text{m/s}.$$

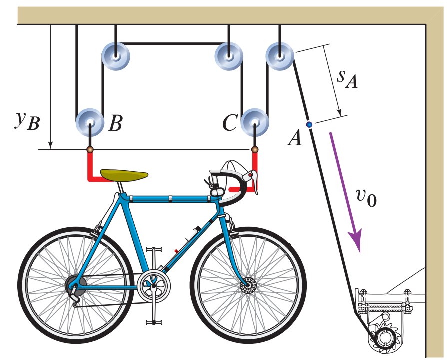
Problem 2.245

The pulley system shown is used to store a bicycle in a garage. If the bicycle is hoisted by a winch that winds the rope at a rate $v_0 = 5$ in./s, determine the vertical speed of the bicycle.



Solution

Referring to the figure at the right, we have that pulleys B and C are at the same height. Also, A is a point on the branch of the rope that is being pulled in by the winch. The length of the rope between the left end of the system and point A can be written as follows:



$$L = 4y_B + s_A. \tag{1}$$

Since L is constant, taking its time derivative with respect to time, and recalling that $\dot{s}_A - v_0$, we have

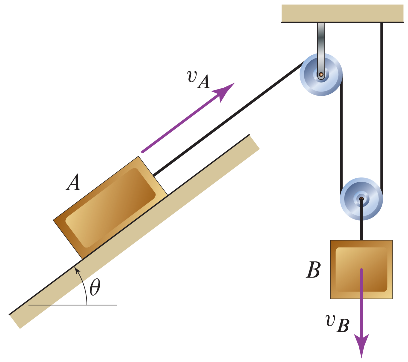
$$0 = 4\dot{y}_B + \dot{s}_A \quad \Rightarrow \quad \dot{y}_B = -\tfrac{1}{4}v_0. \tag{2}$$

Recalling that we have $v_0 = 5\,\text{in.} = \frac{5}{12}\,\text{ft/s}$, and observing that the speed of the bicycle is equal to $|\dot{y}_B|$, we have

$$\boxed{v_{\text{bicycle}} = 0.1042\,\text{ft/s}.}$$

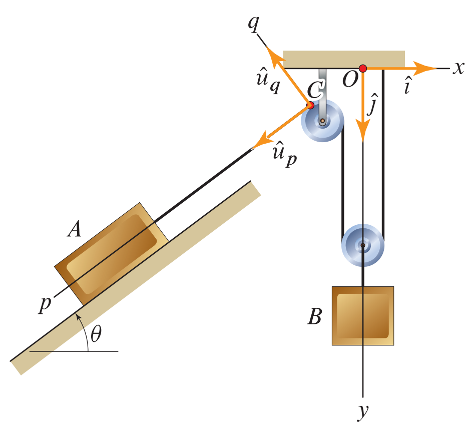
Problem 2.246

Letting $\theta = 50°$, determine the vertical component of the velocity of A if B is moving downward with a speed $v_B = 3\,\text{ft/s}$.



Solution

Referring to the figure at the right, we select a Cartesian coordinate system with axes p and q and origin C such that p and q are parallel and perpendicular to the incline, respectively. We will also use a Cartesian coordinate system with origin at O and axes x and y. Since A slides over the incline, the velocity of A is



$$\vec{v}_A = \dot{p}_A\,\hat{u}_p. \tag{1}$$

The vertical component of the velocity of A is then given by

$$v_{Ay} = \vec{v}_A \cdot \hat{\jmath} = \dot{p}_A \sin\theta. \tag{2}$$

In order to determine $\dot{p}_A$, letting L denote the length of the cord, we write

$$L = p_A + 2y_B. \tag{3}$$

Next, observing that L is constant, we can differentiate the above expression with respect to time to obtain

$$0 = \dot{p}_A + 2\dot{y}_B \quad\Rightarrow\quad \dot{p}_A = -2\dot{y}_B \quad\Rightarrow\quad \dot{p}_A = -2v_B, \tag{4}$$

where, in writing the last of Eqs. (4), we have taken into account that B moves only in the positive y direction, so that $\dot{y}_B$ can be taken to represent the speed of B. Substituting the last of Eqs. (4) into Eq. (2) we then obtain

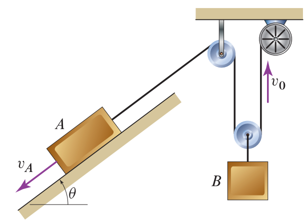
$$v_{Ay} = -2v_B \sin\theta. \tag{5}$$

Recalling that $v_B = 3\,\text{ft/s}$ and that $\theta = 50°$, we can evaluate Eq. (5) to obtain

$$v_{Ay} = -4.596\,\text{ft/s}.$$

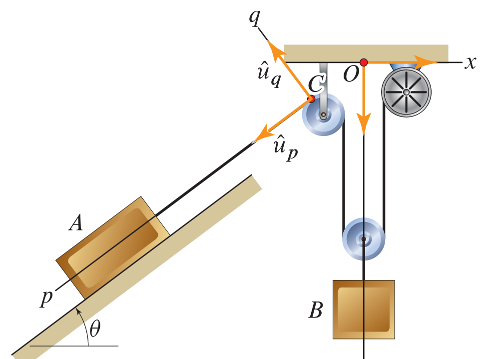
Problem 2.247

Determine the speed of block B if block A is sliding down the incline with a speed $v_A = 1.5\,\text{m/s}$ while the cord is retracted by a winch at a constant rate $v_0 = 2.5\,\text{m/s}$.



Solution

Referring to the figure at the right, we select a Cartesian coordinate system with axes p and q and origin C such that p and q are parallel and perpendicular to the incline, respectively. We will also use a Cartesian coordinate system with origin at O and axes x and y. The velocity of block B is given by



$$\vec{v}_B = \dot{y}_B\,\hat{\jmath}. \tag{1}$$

In order to determine $\dot{y}_B$, letting L denote the length of the cord, we write

$$L = p_A + 2y_B, \tag{2}$$

where p_A identifies the position of A along the p direction. Next, observing that L is decreasing at the rate v_0, we can differentiate the above expression with respect to time to obtain

$$-v_0 = \dot{p}_A + 2\dot{y}_B \quad \Rightarrow \quad \dot{y}_B = -\frac{v_0 + \dot{p}_A}{2}. \tag{3}$$

We now observe that $\dot{p}_A > 0$ because block A is sliding downward. This implies that $\dot{p}_A = v_A$, namely, the speed of A. With this in mind, and recalling that $v_0 > 0$, Eq. (3) implies that $\dot{y}_B < 0$ so that the speed of B is given by

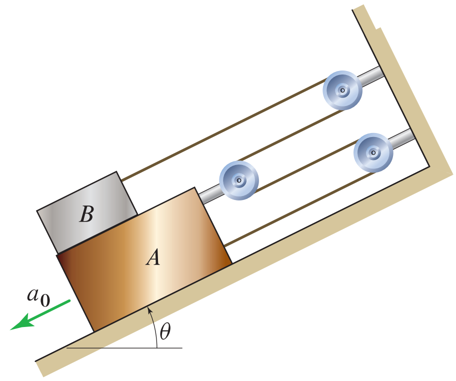
$$v_B = \frac{v_0 + v_A}{2}. \tag{4}$$

Therefore, recalling that $v_0 = 2.5\,\text{m/s}$ and $v_A = 1.5\,\text{m/s}$, we can evaluate the above expression to obtain

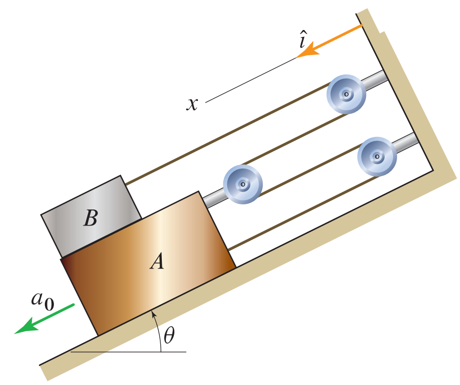
$$\boxed{v_B = 2.000\,\text{m/s}.}$$

Problem 2.248

Block A is released from rest and starts sliding down the incline with an acceleration $a_0 = 3.7\,\mathrm{m/s^2}$. Determine the acceleration of block B relative to the incline. Also, determine the time needed for B to move a distance $d = 0.2\,\mathrm{m}$ relative to A.



Solution



Expressing the length of the rope L in terms of the coordinates of A and B, we have

$$L = 3x_A + x_B. \tag{1}$$

Since L is constant, differentiating Eq. (1) twice with respect to time gives

$$0 = 3\ddot{x}_A + \ddot{x}_B. \tag{2}$$

Solving Eq. (2) with respect to $\ddot{x}_B$, gives

$$\ddot{x}_B = -3\ddot{x}_A = -3a_0, \tag{3}$$

where we have used the fact that $\ddot{x}_A = a_0$. Recalling that $a_0 = 3.7\,\mathrm{m/s^2}$ and that $\vec{a}_B = \ddot{x}_B\,\hat{\imath}$, evaluating Eq. (3), we have

$$\boxed{\vec{a}_B = -11.10\,\hat{\imath}\ \mathrm{m/s^2}. \quad \hat{\imath}\,\hat{\jmath}\ @\theta}$$

Using the result in Eq. (3), we observe that $\ddot{x}_{B/A} = \ddot{x}_B - \ddot{x}_A = -4a_0 =$ constant. Therefore, $x_{B/A}$, namely, the relative position of B with respect to A, can be expressed as a function of time as follows:

$$x_{B/A}(t) = x_{B/A}(0) + \dot{x}_{B/A}(0)\,t - 2a_0\,t^2, \tag{4}$$

where $x_{B/A}(0)$ and $\dot{x}_{B/A}(0)$ denote the initial values of $x_{B/A}$ and $\dot{x}_{B/A}$, respectively. Recalling that $\dot{x}_{B/A}(0) = 0$ since the blocks are released from rest, Eq. (4) simplifies to

$$x_{B/A}(t) = x_{B/A}(0) - 2a_0\,t^2. \tag{5}$$

Denoting by t_d the time needed to travel the distance d, observing that $d = \left|x_{B/A}(t_d) - x_{B/A}(0)\right|$, and letting $t = t_d$ in Eq. (5), we have

$$d = 2a_0\,t_d^2 \quad \Rightarrow \quad t_d = \sqrt{d/(2a_0)}. \tag{6}$$

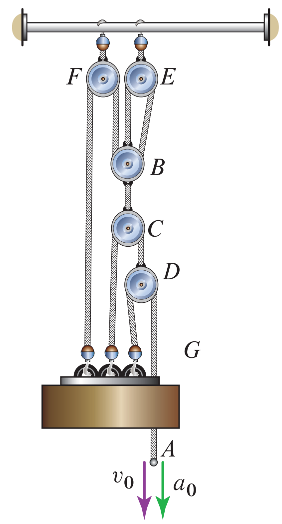
Recalling that $d = 0.2\,\mathrm{m}$ and $a_0 = 3.7\,\mathrm{m/s^2}$, the last of Eqs. (6) can be evaluated to obtain

$$\boxed{t_d = 0.1644\,\mathrm{s}.}$$

Problem 2.249

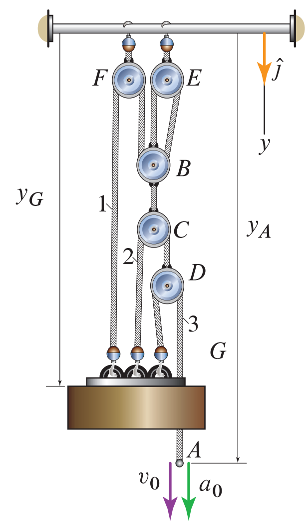
In the pulley system shown, the segment AD and the motion of A are not impeded by the load G. Assume all ropes are vertically aligned.

Determine the velocity and acceleration of the load G if $v_0 = 3\,\text{ft/s}$ and $a_0 = 1\,\text{ft/s}^2$.



Solution

Referring to the figure at the right, and using the y coordinate shown, the length of the ropes labeled 1, 2, and 3 are



$$L_1 = y_G + 3y_B, \quad L_2 = y_G + y_D - 2y_B, \quad L_3 = y_G + y_A - 2y_D. \quad (1)$$

Since L_1, L_2, and L_3 are constant, taking the time derivative of Eqs. (1) gives

$$0 = \dot{y}_G + 3\dot{y}_B, \quad 0 = \dot{y}_G + \dot{y}_D - 2\dot{y}_B, \quad 0 = \dot{y}_G + \dot{y}_A - 2\dot{y}_D. \quad (2)$$

Equations (2) forms a system of three equations in the three unknowns $\dot{y}_B$, $\dot{y}_D$, and $\dot{y}_G$ whose solution is

$$\dot{y}_B = \tfrac{1}{13}v_0, \quad \dot{y}_D = \tfrac{5}{13}v_0, \quad \dot{y}_G = -\tfrac{3}{13}v_0, \quad (3)$$

where we have used the fact that $\dot{y}_A = v_0$. Differentiating Eqs. (3) with respect to time, we have

$$\ddot{y}_B = \tfrac{1}{13}a_0, \quad \ddot{y}_D = \tfrac{5}{13}a_0, \quad \ddot{y}_G = -\tfrac{3}{13}a_0, \quad (4)$$

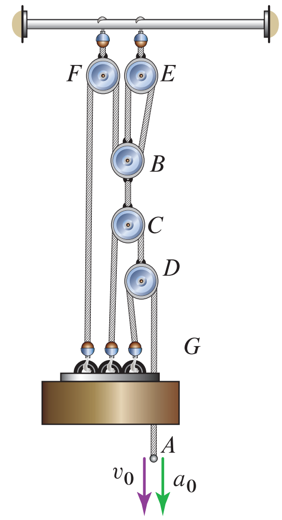
where we have used the fact that $\dot{v}_0 = a_0$. Recalling that $\vec{v}_G = \dot{y}_G\,\hat{\jmath}$ and $\vec{a}_G = \ddot{y}_G\,\hat{\jmath}$, and recalling that $v_0 = 3\,\text{ft/s}$ and $a_0 = 1\,\text{ft/s}^2$, we can evaluate the last of Eqs. (3) and (4) to obtain

$$\vec{v}_G = -0.6923\,\hat{\jmath}\,\text{ft/s} \quad \text{and} \quad \vec{a}_G = -0.2308\,\hat{\jmath}\,\text{ft/s}^2.$$

Problem 2.250

In the pulley system shown, the segment AD and the motion of A are not impeded by the load G. Assume all ropes are vertically aligned.

The load G is initially at rest when the end A of the rope is pulled with the constant acceleration a_0. Determine a_0 so that G is lifted 2 ft in 4.3 s.



Solution

Referring to the figure at the right, and using the y coordinate shown, the lengths of the ropes labeled 1, 2, and 3 are

$$L_1 = y_G + 3y_B, \quad L_2 = y_G + y_D - 2y_B, \quad L_3 = y_G + y_A - 2y_D. \tag{1}$$

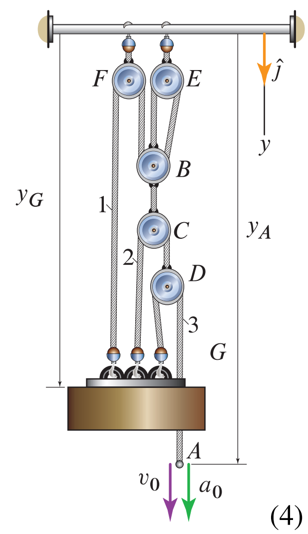
Since L_1, L_2, and L_3 are constant, taking the time derivative of Eqs. (1) gives

$$0 = \dot{y}_G + 3\dot{y}_B, \quad 0 = \dot{y}_G + \dot{y}_D - 2\dot{y}_B, \quad 0 = \dot{y}_G + \dot{y}_A - 2\dot{y}_D. \tag{2}$$

Equations (2) form a system of three equations in the three unknowns $\dot{y}_B$, $\dot{y}_D$, and $\dot{y}_G$ whose solution is

$$\dot{y}_B = \tfrac{1}{13}v_0, \quad \dot{y}_D = \tfrac{5}{13}v_0, \quad \dot{y}_G = -\tfrac{3}{13}v_0, \tag{3}$$

where we have used the fact that $\dot{y}_A = v_0$. Differentiating Eqs. (3) with respect to time, we have



$$\ddot{y}_B = \tfrac{1}{13}a_0, \quad \ddot{y}_D = \tfrac{5}{13}a_0, \quad \ddot{y}_G = -\tfrac{3}{13}a_0, \tag{4}$$

where we have used the fact that $\dot{v}_0 = a_0$. The last of Eqs. (4) implies that the acceleration of G is constant. Therefore, using constant acceleration equations, we can write the following expression of y_G as a function of time:

$$y_G(t) = y_G(0) + \dot{y}_G(0)t - \tfrac{3}{26}a_0 t^2. \tag{5}$$

Let $t_f = 4.3$ s (f stands for final). Hence, recalling that G starts from rest and that therefore $\dot{y}_G(0) = 0$, from Eq. (5) we have

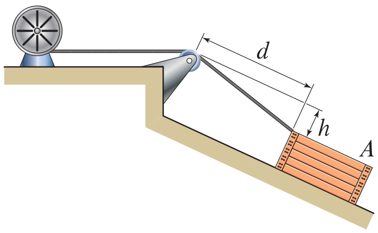
$$y_G(t_f) = y_G(0) - \tfrac{3}{26}a_0 t_f^2 \quad \Rightarrow \quad a_0 = \frac{26[y_G(0) - y_G(t_f)]}{3t_f^2}. \tag{6}$$

Recalling that $y_G(0) - y_G(t_f) = 2$ ft and that $t_f = 4.3$ s, we can evaluate the last of Eqs. (6) to obtain

$$\boxed{a_0 = 0.9374\,\text{ft/s}^2.}$$

Problem 2.251

A crate A is being pulled up an inclined ramp by a winch retracting the cord at a constant rate $v_0 = 2\,\text{ft/s}$. Letting $h = 1.5\,\text{ft}$, determine the speed of the crate when $d = 4\,\text{ft}$.



Solution

Referring to the problem's figure, we begin by observing that the motion of the crate is rectilinear and, denoting the speed of the crate by v_A, we have

$$v_A = \left|\dot{d}\right|. \tag{1}$$

To determine $\dot{d}$, we can write

$$L = w + \sqrt{h^2 + d^2}, \tag{2}$$

where L denotes the length of the cord and w denotes the (constant) horizontal distance between the winch and the pulley. Recalling that L decreases at the rate v_0, we have that

$$-v_0 = \frac{d\,\dot{d}}{\sqrt{h^2 + d^2}}. \tag{3}$$

Solving Eq. (3) for $\dot{d}$ and substituting the result in Eq. (1) gives

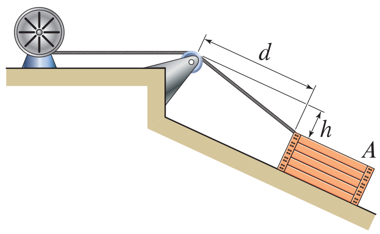
$$v_A = \frac{v_0}{d}\sqrt{h^2 + d^2}. \tag{4}$$

Therefore, recalling that $h = 1.5\,\text{ft}$ and $v_0 = 2\,\text{ft/s}$, we can evaluate Eq. (4) for $d = 4\,\text{ft}$ to obtain

$$\boxed{v_A = 2.136\,\text{ft/s}.}$$

Problem 2.252

A crate A is being pulled up an inclined ramp by a winch. The rate of winding of the cord is controlled so as to hoist the crate up the incline with a constant speed v_0. Letting $\dot{\ell}$ denote the length of cord retracted by the winch per unit time, determine an expression for $\dot{\ell}$ in terms of v_0, h, and d.



Solution

Letting L denote the length of the cord and w denote the (constant) horizontal distance between the winch and the pulley, we can write

$$L = w + \sqrt{h^2 + d^2}. \tag{1}$$

Recalling that L decreases at the rate $\dot{\ell}$, i.e., $\dot{L} = -\dot{\ell}$, differentiating Eq. (1) with respect to time, we have

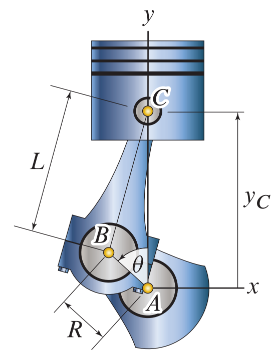
$$-\dot{\ell} = \frac{d\dot{d}}{\sqrt{h^2 + d^2}}. \tag{2}$$

where $\dot{d}$ is the time rate of change of the distance d. When the crate moves up the incline, $\dot{d} < 0$ and is equal in magnitude to the speed of the crate, i.e., $\dot{d} = -v_0$. Substituting this relation into Eq. (2), we obtain the desired relation:

$$\dot{\ell} = \frac{dv_0}{\sqrt{h^2 + d^2}}.$$

Problem 2.253

The piston head at C is constrained to move along the y axis. Let the crank AB be rotating counterclockwise at a constant angular speed $\dot{\theta} = 2000\,\text{rpm}$, $R = 3.5\,\text{in.}$, and $L = 5.3\,\text{in.}$ Determine the velocity of C when $\theta = 35°$.



Solution

Using the figure in the problem statement and the law of cosines, we have

$$L^2 = R^2 + y_C^2 - 2Ry_C\cos\theta \quad\Rightarrow\quad y_C = R\cos\theta \pm \sqrt{L^2 - R^2 + R^2\cos^2\theta}$$

$$\Rightarrow\quad y_C = R\cos\theta \pm \sqrt{L^2 - R^2\sin^2\theta}, \quad (1)$$

where we have used the identity $\sin^2\theta = 1 - \cos^2\theta$. To determine the appropriate root, observe that for $\theta = 0$ we expect $y_C = R + L$. Therefore, we have

$$y_C = R\cos\theta + \sqrt{L^2 - R^2\sin^2\theta}. \quad (2)$$

Differentiating Eq. (2) with respect to time and simplifying yields

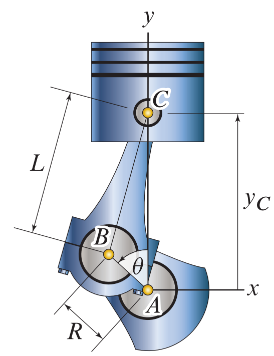
$$\dot{y}_C = -R\dot{\theta}\sin\theta - \frac{R^2\dot{\theta}\sin\theta\cos\theta}{\sqrt{L^2 - R^2\sin^2\theta}}. \quad (3)$$

Recalling that $\vec{v}_C = \dot{y}_C\,\hat{\jmath}$, and recalling that $R = 3.5\,\text{in.} = \frac{3.5}{12}\,\text{ft}$, $L = 5.3\,\text{in.} = \frac{5.3}{12}\,\text{ft}$, and $\dot{\theta} = 2000\,\text{rpm} = 2000\frac{2\pi}{60}\,\text{rad/s}$, we can evaluate Eq. (3) for $\theta = 35°$ to obtain

$$\boxed{\vec{v}_C = -55.52\,\hat{\jmath}\,\text{ft/s}.}$$

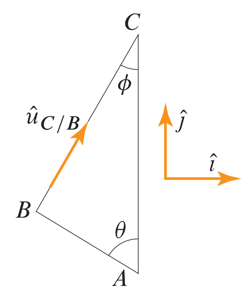
Problem 2.254

Let $\vec{\omega}_{BC}$ denote the angular velocity of the relative position vector $\vec{r}_{C/B}$. As such, $\vec{\omega}_{BC}$ is also the angular velocity of the connecting rod BC. Using the concept of time derivative of a vector given in Section 2.4 on p. 80, determine the component of the relative velocity of C with respect to B along the direction of the connecting rod BC.



Solution

We being by observing that $\vec{v}_{C/B} = \dot{\vec{r}}_{C/B}$, where $\vec{r}_{C/B} = L\,\hat{u}_{C/B}$. Since L is constant, we must have $\dot{\vec{r}}_{C/B} = L\,\dot{\hat{u}}_{C/B}$. Using the formula for the time derivative of a unit vector, we have $\dot{\hat{u}}_{C/B} = \vec{\omega}_{BC} \times \hat{u}_{C/B}$, so that



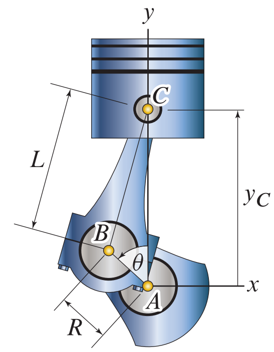
$$\vec{v}_{C/B} = L\vec{\omega}_{BC} \times \hat{u}_{C/B}.$$

We now observe that $\vec{\omega}_{BC} \times \hat{u}_{C/B}$ is perpendicular to $\hat{u}_{C/B}$. Therefore the component of $\vec{v}_{C/B}$ along the direction $\hat{u}_{C/B}$ is

Component of $\vec{v}_{C/B}$ along $\overline{BC} = 0$.

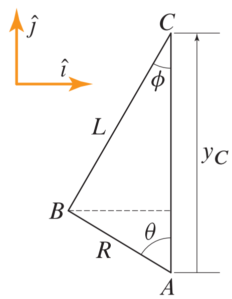
Problem 2.255

The piston head at C is constrained to move along the y axis. Let the crank AB be rotating counterclockwise at a constant angular speed $\dot{\theta} = 2000\,\text{rpm}$, $R = 3.5\,\text{in.}$, and $L = 5.3\,\text{in}$. Determine expressions for the velocity and acceleration of C as a function of θ and the given parameters.



Solution

Using the diagram at the right and the law of cosines, we have



$$L^2 = R^2 + y_C^2 - 2Ry_C\cos\theta \quad\Rightarrow\quad y_C = R\cos\theta \pm \sqrt{L^2 - R^2 + R^2\cos^2\theta}$$

$$\Rightarrow\quad y_C = R\cos\theta \pm \sqrt{L^2 - R^2\sin^2\theta}, \quad (1)$$

where we have used the identity $\sin^2\theta = 1 - \cos^2\theta$. To determine the appropriate root, observe that for $\theta = 0$ we expect $y_C = R + L$. Therefore, we have

$$y_C = R\cos\theta + \sqrt{L^2 - R^2\sin^2\theta}. \quad (2)$$

Differentiating Eq. (2) with respect to time and simplifying yields

$$\dot{y}_C = -\sin\theta\left(R\dot{\theta} + \frac{R^2\dot{\theta}\cos\theta}{\sqrt{L^2 - R^2\sin^2\theta}}\right). \quad (3)$$

Recalling that $\vec{v}_C = \dot{y}_C\,\hat{\jmath}$, and recalling that $R = 3.5\,\text{in.} = \frac{3.5}{12}\,\text{ft}$, $L = 5.3\,\text{in.} = \frac{5.3}{12}\,\text{ft}$, and $\dot{\theta} = 2000\,\text{rpm} = 2000\frac{2\pi}{60}\,\text{rad/s}$, we can evaluate Eq. (3) to obtain

$$\vec{v}_C = -\sin\theta\left(61.09 + \frac{17.82\cos\theta}{\sqrt{0.1951 - 0.08507\sin^2\theta}}\right)\hat{\jmath}\ \text{ft/s}.$$

Recalling that $\dot{\theta} = $ constant, differentiating Eq. (3), we have

$$\ddot{y}_C = -R\dot{\theta}^2\left\{\cos\theta\left(1 + \frac{R\cos\theta}{\sqrt{L^2 - R^2\sin^2\theta}}\right) + \sin\theta\left[\frac{R^3\cos^2\theta\sin\theta}{\left(L^2 - R^2\sin^2\theta\right)^{3/2}} - \frac{R\sin\theta}{\sqrt{L^2 - R^2\sin^2\theta}}\right]\right\}. \quad (4)$$

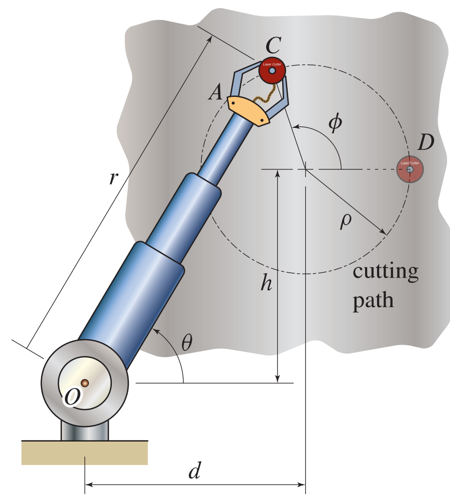
Recalling that $\vec{a}_C = \ddot{y}_C\,\hat{\jmath}$, and recalling again that $R = 3.5\,\text{in.} = \frac{3.5}{12}\,\text{ft}$, $L = 5.3\,\text{in.} = \frac{5.3}{12}\,\text{ft}$, and $\dot{\theta} = 2000\,\text{rpm} = 2000\frac{2\pi}{60}\,\text{rad/s}$, we can evaluate Eq. (4) to obtain

$$\vec{a}_C = -1.279\times10^4\left\{\cos\theta\left(1 + \frac{0.2917\cos\theta}{\sqrt{0.1951 - 0.08507\sin^2\theta}}\right) + \sin\theta\left[\frac{0.02481\cos^2\theta\sin\theta}{\left(0.1951 - 0.08507\sin^2\theta\right)^{3/2}} - \frac{0.2917\sin\theta}{\sqrt{0.1951 - 0.08507\sin^2\theta}}\right]\right\}\hat{\jmath}\ \text{ft/s}^2.$$

Problem 2.256

In the cutting of sheet metal, the robotic arm OA needs to move the cutting tool at C counterclockwise at a constant speed v_0 along a circular path of radius ρ. The center of the circle is located in the position shown relative to the base of the robotic arm at O.

For all positions along the circular cut (i.e., for any value of ϕ), determine r, $\dot{r}$, and $\dot{\theta}$ as functions of the given quantities (i.e., d, h, ρ, v_0). Use one or more geometric constraints and their derivatives to do this. These quantities can be found "by hand," but it is tedious, so you might consider using symbolic algebra software, such as Mathematica or Maple.



Solution

Referring to the figure at the right, we define a Cartesian coordinate system with origin at O. Then the coordinates of point C are such that

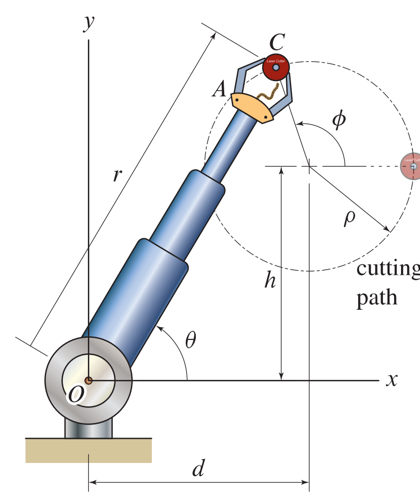
$$r^2 = x_C^2 + y_C^2, \tag{1}$$

where

$$x_C = d + \rho\cos\phi = r\cos\theta \quad \text{and} \quad y_C = h + \rho\sin\phi = r\sin\theta, \tag{2}$$

so that r can be expressed as

$$\boxed{r = \sqrt{(d + \rho\cos\phi)^2 + (h + \rho\sin\phi)^2}.} \tag{3}$$



Differentiating Eqs. (2) with respect to time, we have

$$-\rho\dot{\phi}\sin\phi = \dot{r}\cos\theta - r\dot{\theta}\sin\theta \quad \text{and} \quad \rho\dot{\phi}\cos\phi = \dot{r}\sin\theta + r\dot{\theta}\cos\theta. \tag{4}$$

Equations (4) can be viewed as a system of two equations in the two unknowns $\dot{\theta}$ and $\dot{r}$ whose solution is

$$\dot{\theta} = (\rho\dot{\phi}/r)(\cos\theta\cos\phi - \sin\theta\sin\phi) \quad \text{and} \quad \dot{r} = \rho\dot{\phi}(\sin\theta\cos\phi - \cos\theta\sin\phi). \tag{5}$$

Next, observe that we have

$$\sin\theta = \frac{h + \rho\sin\phi}{r}, \quad \cos\theta = \frac{d + \rho\cos\phi}{r}, \quad \text{and} \quad \dot{\phi} = v_0/\rho. \tag{6}$$

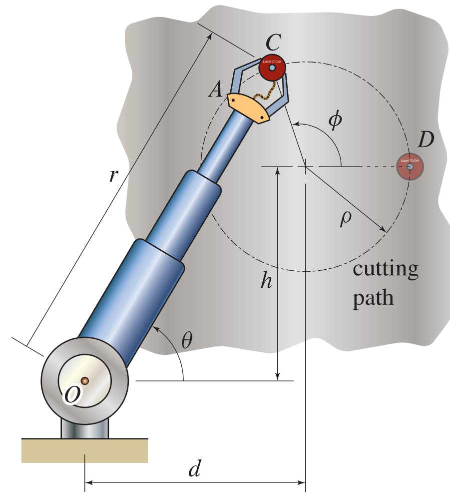
Substituting Eqs. (3) and (6) into Eqs. (5) and simplifying, we have

$$\boxed{\dot{\theta} = \frac{v_0(\rho + d\cos\phi + h\sin\phi)}{(d + \rho\cos\phi)^2 + (h + \rho\sin\phi)^2} \quad \text{and} \quad \dot{r} = \frac{v_0(h\cos\phi - d\sin\phi)}{\sqrt{(d + \rho\cos\phi)^2 + (h + \rho\sin\phi)^2}}.}$$

Problem 2.257

In the cutting of sheet metal, the robotic arm OA needs to move the cutting tool at C counterclockwise at a constant speed v_0 along a circular path of radius ρ. The center of the circle is located in the position shown relative to the base of the robotic arm at O.

For all positions along the circular cut (i.e., for any value of ϕ), determine $\ddot{r}$ and $\ddot{\theta}$ as functions of the given quantities (i.e., d, h, ρ, v_0). These quantities can be found by hand, but it is very tedious, so you might consider using symbolic algebra software, such as Mathematica or Maple.



Solution

Referring to the figure at the right, the coordinates C are

$$x_C = d + \rho\cos\phi = r\cos\theta \quad \text{and} \quad y_C = h + \rho\sin\phi = r\sin\theta. \quad (1)$$

Observing that $r^2 = x_C^2 + y_C^2$, we have

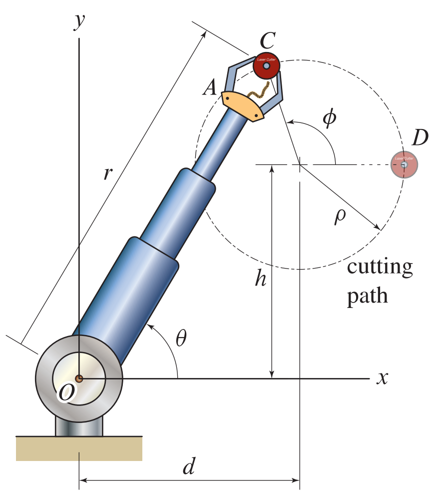
$$r = \sqrt{(d+\rho\cos\phi)^2 + (h+\rho\sin\phi)^2}. \quad (2)$$

Differentiating Eqs. (1) with respect to time, we have

$$-\rho\dot{\phi}\sin\phi = \dot{r}\cos\theta - r\dot{\theta}\sin\theta \quad \text{and} \quad \rho\dot{\phi}\cos\phi = \dot{r}\sin\theta + r\dot{\theta}\cos\theta. \quad (3)$$

Equations (3) can be viewed as a system of two equations in the two unknowns $\dot{\theta}$ and $\dot{r}$ whose solution is

$$\dot{\theta} = (\rho\dot{\phi}/r)(\cos\theta\cos\phi - \sin\theta\sin\phi),$$
$$\dot{r} = \rho\dot{\phi}(\sin\theta\cos\phi - \cos\theta\sin\phi). \quad (4)$$



Next, observe that we have

$$\sin\theta = \frac{h+\rho\sin\phi}{r}, \quad \cos\theta = \frac{d+\rho\cos\phi}{r}, \quad \text{and} \quad \dot{\phi} = v_0/\rho. \quad (5)$$

Hence, substituting Eqs. (2) and (5) into Eqs. (4) and simplifying, we have

$$\dot{\theta} = \frac{v_0(\rho + d\cos\phi + h\sin\phi)}{(d+\rho\cos\phi)^2 + (h+\rho\sin\phi)^2} \quad \text{and} \quad \dot{r} = \frac{v_0(h\cos\phi - d\sin\phi)}{\sqrt{(d+\rho\cos\phi)^2 + (h+\rho\sin\phi)^2}}. \quad (6)$$

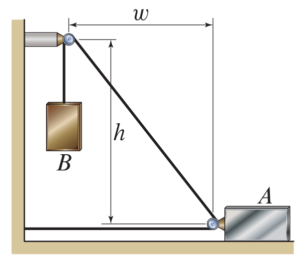
To find expressions for $\ddot{r}$ and $\ddot{\theta}$ we must take the time derivative of Eqs. (6) and then replace $\dot{\phi}$ with its expressions in the last of Eqs. (5). Doing so, after simplification, yields the following expressions:

$$\ddot{r} = -\frac{v_0^2(\rho + d\cos\phi + h\sin\phi)(d^2 + h^2 + \rho d\cos\phi + h\rho\sin\phi)}{\rho\big[(d+\rho\cos\phi)^2 + (h+\rho\sin\phi)^2\big]^{3/2}},$$

$$\ddot{\theta} = -\frac{v_0^2(d^2+h^2-\rho^2)(d\sin\phi - h\cos\phi)}{\rho\big[(d+\rho\cos\phi)^2 + (h+\rho\sin\phi)^2\big]^2}.$$

Problem 2.258

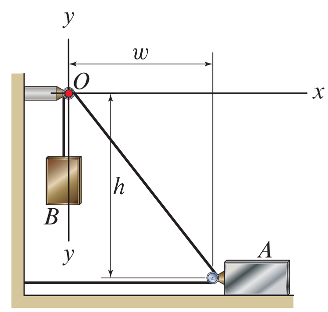
At the instant shown, block A is moving at a constant speed $v_0 = 3\,\mathrm{m/s}$ to the left and $w = 2.3\,\mathrm{m}$. Using $h = 2.7\,\mathrm{m}$, determine how much time is needed to lower B 0.75 m from this position.



Solution

Referring to the figure at the right, we will use the Cartesian coordinate system shown with origin at the fixed pulley O. The length of the rope can be described as

$$L = x_A + y_B + \sqrt{x_A^2 + h^2}. \tag{1}$$



We will denote quantities evaluated at the initial and final positions of the system by the subscripts 1 and 2, respectively. Because the length of the rope is constant, we set the expressions of the length of the rope corresponding to the initial and final positions of the system equal to each other. Initially, we have $x_{A1} = w$. In addition, we have $y_{B2} - y_{B1} = d = 0.75\,\mathrm{m}$. Hence, referring to Eq. (1), we have

$$w + y_{B1} + \sqrt{w^2 + h^2} = x_{A2} + y_{B2} + \sqrt{x_{A2}^2 + h^2}$$

$$\Rightarrow \quad x_{A2} + \left(d - w - \sqrt{w^2 + h^2}\right) = -\sqrt{x_{A2}^2 + h^2}$$

$$\Rightarrow \quad x_{A2}^2 + \left(d - w - \sqrt{w^2 + h^2}\right)^2 + 2x_{A2}\left(d - w - \sqrt{w^2 + h^2}\right) = x_{A2}^2 + h^2. \tag{2}$$

This last equation can be solved for x_{A2} to obtain

$$x_{A2} = \frac{h^2 - \left(d - w - \sqrt{w^2 + h^2}\right)^2}{2\left(d - w - \sqrt{w^2 + h^2}\right)}. \tag{3}$$

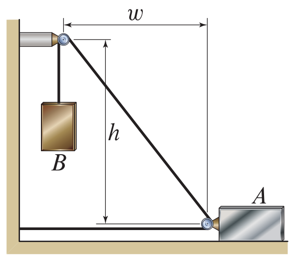
The expression above gives the position of A corresponding to the final position achieved by B as given by the problem statement. Because A and B are connected by an inextensible rope that is assumed not to go slack, the time taken by B to achieve its final position will be equal to the time taken by A to achieve its final position. With this in mind, letting t denote the time to be determined, and recalling that the velocity of A is constant, we then have

$$x_{A2} = x_{A1} - v_0 t \quad \Rightarrow \quad t = \frac{w}{v_0} - \frac{h^2 - \left(d - w - \sqrt{w^2 + h^2}\right)^2}{2v_0\left(d - w - \sqrt{w^2 + h^2}\right)} \quad \Rightarrow \quad \boxed{t = 0.1556\,\mathrm{s},}$$

where we have used the fact that $x_{A1} = w$, and where we have used the following numerical data: $v_0 = 3\,\mathrm{m/s}$, $h = 2.7\,\mathrm{m}$, $d = 0.75\,\mathrm{m}$, and $w = 2.3\,\mathrm{m}$.

Problem 2.259

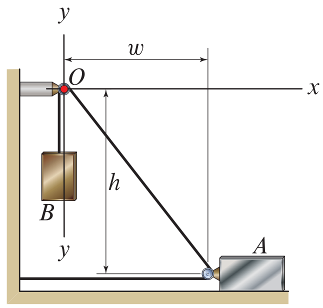
At the instant shown, $h = 10\,\text{ft}$, $w = 8\,\text{ft}$, and block B is moving with a speed $v_0 = 5\,\text{ft/s}$ and an acceleration $a_0 = 1\,\text{ft/s}^2$, both downward. Determine the velocity and acceleration of block A.



Solution

We will use the Cartesian coordinate system shown at the right with origin at the fixed pulley O. The length of the rope can be described as

$$L = x_A + y_B + \sqrt{x_A^2 + h^2}. \tag{1}$$



Since the length of the rope is constant, differentiating the above equation with respect to time, we have

$$0 = \dot{x}_A + \dot{y}_B + \frac{x_A \dot{x}_A}{\sqrt{x_A^2 + h^2}} \quad \Rightarrow \quad \dot{x}_A = -\frac{\dot{y}_B \sqrt{x_A^2 + h^2}}{x_A + \sqrt{x_A^2 + h^2}}. \tag{2}$$

Since $\vec{v}_A = \dot{x}_A\,\hat{\imath}$, recalling that $\dot{y}_B = v_0 = 5\,\text{ft/s}$ and $h = 10\,\text{ft}$, we can evaluate the last of Eqs. (2) for $x = w = 8\,\text{ft}$ to obtain

$$\boxed{\vec{v}_A = -3.078\,\hat{\imath}\ \text{ft/s}.}$$

Taking the derivative with respect to time of the last of Eqs. (2), after simplification, we have

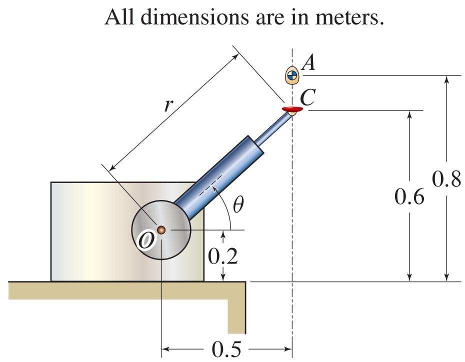
$$\ddot{x}_A = -\frac{\ddot{y}_B \sqrt{x_A^2 + h^2}}{x_A + \sqrt{x_A^2 + h^2}} + \frac{\dot{y}_B^2 \left(x_A - \sqrt{x_A^2 + h^2}\right)}{\left(x_A + \sqrt{x_A^2 + h^2}\right)^2}. \tag{3}$$

Since $\vec{a}_A = \ddot{x}_A\,\hat{\imath}$, recalling that $\dot{y}_B = v_0 = 5\,\text{ft/s}$, $\ddot{y}_B = a_0 = 1\,\text{ft/s}^2$, and $h = 10\,\text{ft}$, we can evaluate the last of Eq. (3) for $x = w = 8\,\text{ft}$ to obtain

$$\boxed{\vec{a}_A = -0.8931\,\hat{\imath}\ \text{ft/s}^2.}$$

Problem 2.260

As a part of a robotics competition, a robotic arm with a rigid open hand at C is to be designed so that the hand catches an egg without breaking it. The egg is released from rest at $t = 0$ from point A. The arm, initially at rest in the position shown, starts moving when the egg is released. The hand must catch the egg without any impact with the egg. This can be done by specifying that the hand and the egg must be at the same position at the same time with identical velocities. A student proposes to do this using a constant value of $\ddot{\theta}$ for which (after a fair bit of work) it is found that the arm catches the egg at $t = 0.4391$ s for $\ddot{\theta} = -13.27\,\text{rad/s}^2$. Using these values of t and $\ddot{\theta}$, determine the acceleration of both the hand and the egg at the time of catch. Then, explain whether or not using a constant value of $\ddot{\theta}$, as has been proposed, is an acceptable strategy.

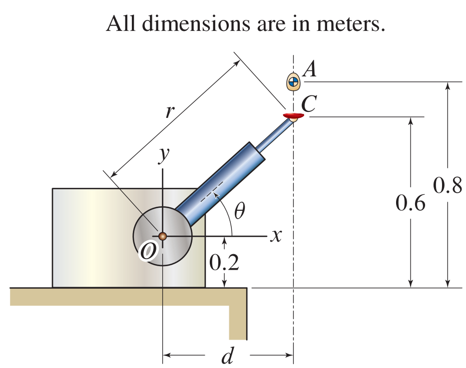


Solution

Referring to the diagram at the right, we will use a Cartesian coordinate system with its origin at O. As far as the acceleration of the egg is concerned, up to the time of catch the acceleration of the egg is

$$\ddot{y}_e = -9.81\,\text{m/s}^2, \tag{1}$$

where the subscript e stands for 'egg'. As far as the determination of the acceleration of point C is concerned, we begin by observing that since $\ddot{\theta}_C$ is constant, and since $\dot{\theta}_C(0) = 0$ because the arm starts from rest, we can use the constant acceleration relations to provide the expression of the angular coordinate θ_C as a function of time:



$$\theta_C(t) = \theta_C(0) + \tfrac{1}{2}\ddot{\theta}_C t^2. \tag{2}$$

Since the trajectory of C is a straight vertical line, $\theta_C(t)$ is related to $y_C(t)$ as follows:

$$y_C(t) = d\tan[\theta_C(t)]. \tag{3}$$

Differentiating Eq. (3) twice with respect to time, we have

$$\ddot{y}_C(t) = d\,\ddot{\theta}_C \sec^2\theta_C(t) + 2d\,\dot{\theta}_C^2(t)\sec^2\theta_C(t)\tan\theta_C(t)$$
$$\Rightarrow\quad \ddot{y}_C(t) = d\sec^2\theta_C(t)\big[\ddot{\theta}_C + 2\dot{\theta}_C^2(t)\tan\theta_C(t)\big]. \tag{4}$$

To evaluate the expression above, in addition to $\ddot{\theta}_C$ (which is given), we need the values of θ_C and $\dot{\theta}_C$ at the time of catch. To determine these values, we begin by denoting by t_f the time of catch, where the subscript f stands for 'final.' Since the egg is falling under the action of gravity, using constant acceleration equations, for $t = t_f$, the y coordinate of the egg is given by

$$y_e(t_f) = y_e(0) - \tfrac{1}{2}g t_f^2, \tag{5}$$

where $y_e(0) = 0.6\,\text{m}$ is the initial vertical position of the egg. At the time of catch we must have $y_e(t_f) = y_C(t_f)$. Therefore, setting $t = t_f$ in Eq. (3), replacing $y_C(t_f)$ with the right-hand side of Eq. (5), and solving for $\theta_C(t_f)$, we have

$$\theta_C(t_f) = \tan^{-1}\left[\frac{2y_e(0) - g t_f^2}{2d}\right]. \tag{6}$$

Next, we determine an expression for $\dot{\theta}_C(t_f)$. To do so, we differentiate Eq. (2) and evaluate it for $t = t_f$:

$$\dot{\theta}_C(t_f) = \ddot{\theta}_C t_f. \tag{7}$$

Substituting Eq. (7) into the last of Eqs. (4), at the time of catch, we have

$$\ddot{y}_C(t_f) = d\sec^2\theta_C(t_f)\big[\ddot{\theta}_C + 2\ddot{\theta}_C^2 t_f^2 \tan\theta_C(t_f)\big]. \tag{8}$$

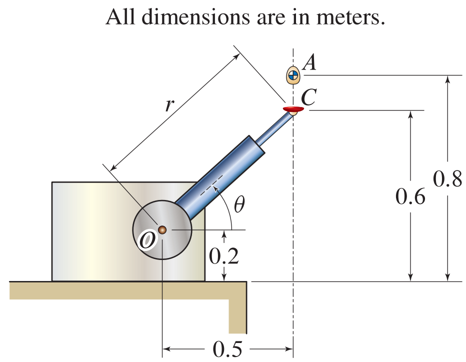
Recalling that $t_f = 0.4391\,\text{s}$, $d = 0.5\,\text{m}$, $g = 9.81\,\text{m/s}^2$, and $y_e(0) = 0.6\,\text{m}$, we can evaluate $\theta_C(t_f)$ in Eq. (6) (this gives $\theta_C(t_f) = -34.66^\circ$) and then use it, along with $\ddot{\theta}_C = -13.27\,\text{rad/s}^2$, in Eq. (8) to obtain

$$\boxed{\ddot{y}_C\big|_{t=0.4391\,\text{s}} = -44.51\,\text{m/s}^2.}$$

Since, at the time of catch, the acceleration $|\ddot{y}_C| > |\ddot{y}_e|$, the arm and egg will only be in contact for an instant and will then separate again. Consequently, the proposed strategy is not acceptable for catching the egg.

Problem 2.261

Referring to the problem of a robot arm catching an egg (Prob. 2.260), the strategy is that the arm and the egg must have the same velocity and the same position at the same time for the arm to gently catch the egg. In addition, what should be true about the accelerations of the arm and the egg for the catch to be successful *after* they rendezvous with the same velocity at the same position and time? Describe what happens if the accelerations of the arm and egg do not match.

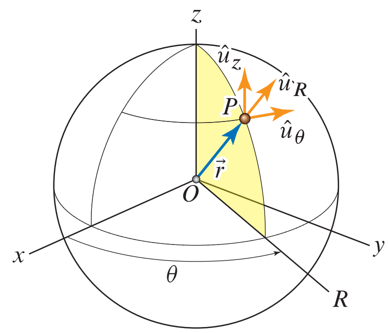


Solution

After they rendezvous, the relative acceleration of the arm with respect to the egg must be zero. If $\ddot{y}_e < \ddot{y}_C$ then the arm and egg will separate right after the catch. If $\ddot{y}_e > \ddot{y}_C$ the egg will experience a jerk.

Problem 2.262

Although point P is moving on a sphere, its motion is being studied with the *cylindrical* coordinate system shown. Discuss in detail whether or not there are incorrect elements in the sketch of the cylindrical component system at P.

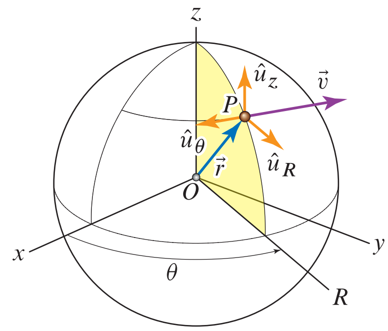


Solution

The unit vector $\hat{u}_R$ points in the direction of $\vec{r}$. This is incorrect. For a cylindrical coordinate system, the unit vector $\hat{u}_R$ must be parallel to the θR plane and point in the R direction.

Problem 2.263

Although point P is moving on a sphere, its motion is being studied with the *cylindrical* coordinate system shown. Discuss in detail whether or not there are incorrect elements in the sketch of the cylindrical component system at P.

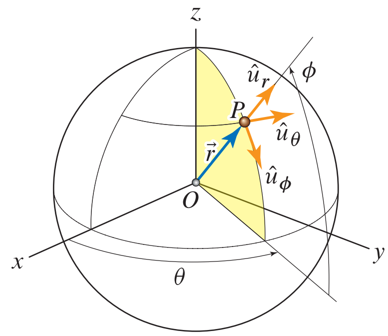


Solution

The unit vector $\hat{u}_\theta$ points in the direction of decreasing θ. This is incorrect, as $\hat{u}_\theta$ must point in the direction of increasing θ.

Problem 2.264

Discuss in detail whether or not (a) there are incorrect elements in the sketch of the spherical component system at P and (b) the formulas for the velocity and acceleration components derived in the section can be used with the coordinate system shown.



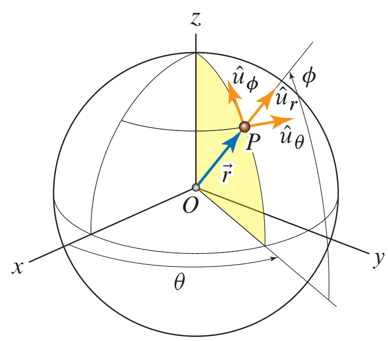
Solution

(a) The unit vector $\hat{u}_\phi$ is pointing in the direction of decreasing ϕ. This is incorrect. It must point in the direction of increasing ϕ.

(b) No, the formulas derived in the section cannot be used since the angle ϕ in this figure is defined from the xy plane to the line OP. The formulas of this section require that ϕ be defined from the z axis to the line OP.

Problem 2.265

Discuss in detail whether or not (a) there are incorrect elements in the sketch of the spherical component system at P and (b) the formulas for the velocity and acceleration components derived in the section can be used with the coordinate system shown.



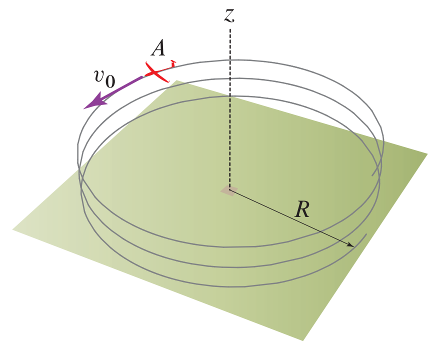
Solution

(a) The orientations of the unit vectors in relation to the positive directions of r, ϕ, and θ are correct.

(b) No, the formulas derived in the section cannot be used since the angle ϕ in this figure is defined from the xy plane to the line OP. The formulas of this section require that ϕ be defined from the z axis to the line OP. Also the $(\hat{u}_r, \hat{u}_\phi, \hat{u}_\theta)$, triad is *not* right-handed.

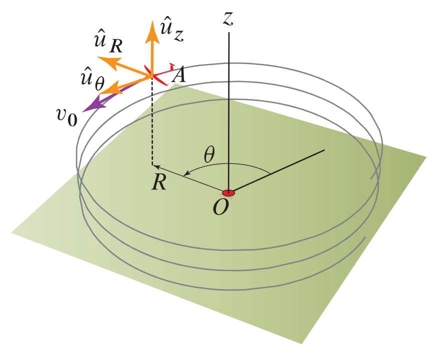
Problem 2.266

A glider is descending with a constant speed $v_0 = 30\,\mathrm{m/s}$ and a constant descent rate of $1\,\mathrm{m/s}$ along a helical path with a constant radius $R = 400\,\mathrm{m}$. Determine the time the glider takes to complete a full 360° turn about the axis of the helix (the z axis).



Solution

Referring to the figure at the right, we adopt a cylindrical coordinate system with origin on the ground and the z axis coinciding with the axis of the helical path. Using the polar component system corresponding to the chosen coordinate system, we can write the velocity of the airplane as follows:



$$\vec{v} = \dot{R}\,\hat{u}_R + R\dot{\theta}\,\hat{u}_\theta + \dot{z}\,\hat{u}_z. \tag{1}$$

Since the radius of the helix is constant, we have that $\dot{R} = 0$ and Eq. (1) simplifies to

$$\vec{v} = R\dot{\theta}\,\hat{u}_\theta + \dot{z}\,\hat{u}_z. \tag{2}$$

Since the speed of the airplane is constant and equal v_0, using Eq. (2) we can write

$$v_0^2 = R^2\dot{\theta}^2 + \dot{z}^2, \tag{3}$$

which we can solve for $\dot{\theta}$ since $\dot{z}$ is known and equal to $-1\,\mathrm{m/s}$:

$$\dot{\theta} = \pm\frac{\sqrt{v_0^2 - \dot{z}^2}}{R}. \tag{4}$$

Of the two roots in Eq. (4) we will select the positive one to match the direction of motion of the glider indicated in the problem's figure. We now observe that since v_0, $\dot{z}$, and R are constant, $\dot{\theta}$ is also a constant. Therefore, letting Δt denote the time needed to complete a full 360° arc around the helix, Δt can be computed by simply dividing the measure of the angle 360° in radians, i.e., 2π, by the angular velocity $\dot{\theta}$. This gives

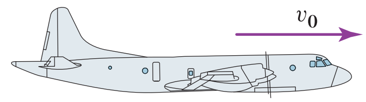
$$\Delta t = \frac{2\pi R}{\sqrt{v_0^2 - \dot{z}^2}}. \tag{5}$$

Recalling that $R = 400\,\mathrm{m}$, $v_0 = 30\,\mathrm{m/s}$, and $\dot{z} = -1\,\mathrm{m/s}$, we can evaluate Eq. (5) to obtain

$$\boxed{\Delta t = 83.82\,\mathrm{s}.}$$

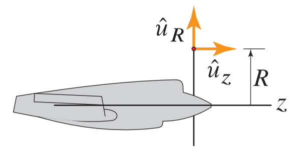
Problem 2.267

An airplane is flying horizontally at a constant speed $v_0 = 320$ mph while its propellers rotate at a constant angular speed $\omega = 1500$ rpm. If the propellers have a diameter $d = 14$ ft, determine the magnitude of the acceleration of a point on the periphery of the propeller blades.



Solution

Referring to the figure at the right, we define a *fixed* cylindrical coordinate system with the z axis coinciding with the propeller's shaft, R direction perpendicular to the shaft and going from the z axis to the point whose acceleration we want to measure, and such that the triad $(\hat{u}_R, \hat{u}_\theta, \hat{u}_z)$ is right-handed. Next, we recall that in cylindrical coordinates the acceleration is given by the following formula:



$$\vec{a} = \left(\ddot{R} - R\dot{\theta}^2\right)\hat{u}_R + \left(R\ddot{\theta} + 2\dot{R}\dot{\theta}\right)\hat{u}_\theta + \ddot{z}\,\hat{u}_z. \tag{1}$$

Using the problem's given information, we have

$$R = \tfrac{1}{2}d = 7.000\,\text{ft}, \quad \dot{R} = 0, \quad \ddot{R} = 0, \quad \dot{\theta} = \omega = 1500\,\text{rpm} = 157.1\,\text{rad/s}, \quad \ddot{z} = 0. \tag{2}$$

Substituting the above information in the formula for the acceleration we have

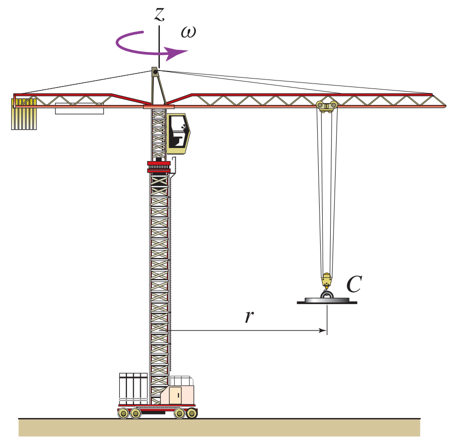
$$\vec{a} = -R\dot{\theta}^2\,\hat{u}_R = (-1.728\times10^5\ \text{ft/s}^2)\,\hat{u}_R. \tag{3}$$

The magnitude of the above vector is

$$\boxed{|\vec{a}| = 17{,}280\,\text{ft/s}^2.}$$

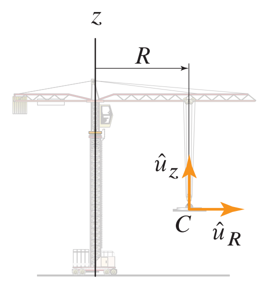
Problem 2.268

A top-slewing crane is lifting an object C at a constant rate of $\dot{z} = 5.3\,\text{ft/s}$ while rotating at a constant rate $\omega = 0.12\,\text{rad/s}$ about the vertical axis. If the distance between the object and the axis of rotation of the crane's boom is $r = 46\,\text{ft}$ and it is being reduced at a constant rate of 6.5 ft/s, find the velocity and acceleration of C, assuming that the swinging motion of C can be neglected.



Solution

Referring to the figure at the right, we define a cylindrical coordinate system with the R direction parallel to the crane's boom going from the vertical axis of the crane toward point C, with the z axis coinciding with the vertical axis of the crane and pointing in the direction opposite to gravity, and with the θ direction defined in such a way the triad $(\hat{u}_R, \hat{u}_\theta, \hat{u}_z)$ is right-handed. Interpreting the data given in the problem statement, in addition to having $R = 46\,\text{ft}$, we can write:



$$\dot{R} = -6.5\,\text{ft/s}, \qquad \dot{\theta} = 0.12\,\text{rad/s}, \qquad \dot{z} = 5.3\,\text{ft/s}, \tag{1}$$

$$\ddot{R} = 0, \qquad \ddot{\theta} = 0, \qquad \ddot{z} = 0. \tag{2}$$

Substituting the values into the equation for the velocity, namely, $\vec{v} = \dot{R}\,\hat{u}_R + R\dot{\theta}\,\hat{u}_\theta + \dot{z}\,\hat{u}_z$, we have

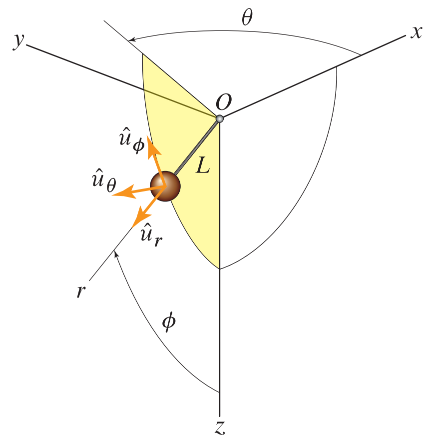
$$\boxed{\vec{v}_C = (-6.500\,\hat{u}_R + 5.520\,\hat{u}_\theta + 5.300\,\hat{u}_z)\,\text{ft/s}.}$$

For the acceleration, substituting the given values into the equation $\vec{a} = \left(\ddot{R} - R\dot{\theta}^2\right)\hat{u}_R + \left(R\ddot{\theta} + 2\dot{R}\dot{\theta}\right)\hat{u}_\theta + \ddot{z}\,\hat{u}_z$, gives

$$\boxed{\vec{a}_C = (-0.6624\,\hat{u}_R - 1.560\,\hat{u}_\theta)\,\text{ft/s}^2.}$$

Problem 2.269

The system depicted in the figure is called a *spherical pendulum*. The fixed end of the pendulum is at O. Point O behaves as a spherical joint; i.e., the location of O is fixed while the pendulum's cord can swing in any direction in the three-dimensional space. Assume that the pendulum's cord has a constant length L, and use the coordinate system depicted in the figure to derive the expression for the acceleration of the pendulum.



Solution

Keeping in mind that the length of the pendulum is constant, we have that the position vector of the pendulum bob is described as $\vec{r} = L\,\hat{u}_r$, where the radial coordinate r is such that

$$r = L = \text{constant}. \tag{1}$$

Therefore, the time derivatives of the radial coordinate must be equal to zero, i.e.,

$$\dot{r} = 0 \quad \text{and} \quad \ddot{r} = 0. \tag{2}$$

Now recall that the acceleration in spherical coordinates is given by

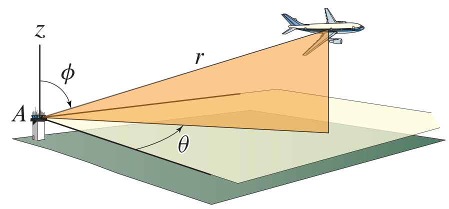
$$\vec{a} = \left(\ddot{r} - r\dot{\phi}^2 - r\dot{\theta}^2 \sin^2\phi\right)\hat{u}_r + \left(r\ddot{\phi} + 2\dot{r}\dot{\phi} - r\dot{\theta}^2 \sin\phi\cos\phi\right)\hat{u}_\phi + \left(r\ddot{\theta}\sin\phi + 2\dot{r}\dot{\theta}\sin\phi + 2r\dot{\phi}\dot{\theta}\cos\phi\right)\hat{u}_\theta.$$

Therefore, using Eqs. (1) and (2) the acceleration becomes

$$\vec{a} = -L\left(\dot{\phi}^2 + \dot{\theta}^2\sin^2\phi\right)\hat{u}_r + L\left(\ddot{\phi} - \dot{\theta}^2\sin\phi\cos\phi\right)\hat{u}_\phi + L\left(\ddot{\theta}\sin\phi + 2\dot{\phi}\dot{\theta}\cos\phi\right)\hat{u}_\theta.$$

Problem 2.270

Revisit Example 2.29, and assuming that the plane is accelerating, determine the relation(s) that the radar readings obtained by the station at A need to satisfy for you to conclude that the jet is flying along a straight line whether at constant altitude or not.



Solution

For the plane to fly along a straight line, the airplane's velocity and acceleration vectors must be parallel. This condition is expressed by the following vector equation:

$$\vec{v} \times \vec{a} = \vec{0}. \tag{1}$$

Recalling that we can write

$$\vec{v} = v_r\,\hat{u}_r + v_\phi\,\hat{u}_\phi + v_\theta\,\hat{u}_\theta \quad \text{and} \quad \vec{a} = a_r\,\hat{u}_r + a_\phi\,\hat{u}_\phi + a_\theta\,\hat{u}_\theta, \tag{2}$$

the condition in Eq. (1) takes on the form

$$\vec{v} \times \vec{a} = (v_\phi a_\theta - v_\theta a_\phi)\,\hat{u}_r + (v_\theta a_r - v_r a_\theta)\,\hat{u}_\phi + (v_r a_\phi - v_\phi a_r)\,\hat{u}_\theta = \vec{0}. \tag{3}$$

The above equation is satisfied if and only if

$$v_\phi a_\theta - v_\theta a_\phi = 0, \quad v_\theta a_r - v_r a_\theta = 0, \quad \text{and} \quad v_r a_\phi - v_\phi a_r = 0. \tag{4}$$

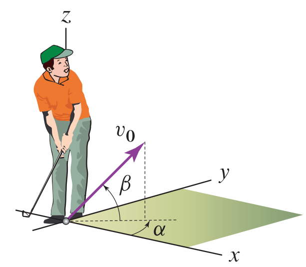
Next, using Eq. (2.105) on p. 143 of the textbook and Eq. (2.107) on p. 144 of the textbook, we can rewrite Eqs. (4) as follows:

$$\begin{aligned}
r\dot{\phi}(r\ddot{\theta}\sin\phi + 2\dot{r}\dot{\theta}\sin\phi + 2r\dot{\phi}\dot{\theta}\cos\phi) - r\dot{\theta}\sin\phi(r\ddot{\phi} + 2\dot{r}\dot{\phi} - r\dot{\theta}^2\sin\phi\cos\phi) &= 0,\\
r\dot{\theta}\sin\phi(\ddot{r} - r\dot{\phi}^2 - r\dot{\theta}^2\sin^2\phi) - \dot{r}(r\ddot{\theta}\sin\phi + 2\dot{r}\dot{\theta}\sin\phi + 2r\dot{\phi}\dot{\theta}\cos\phi) &= 0,\\
\dot{r}(r\ddot{\phi} + 2\dot{r}\dot{\phi} - r\dot{\theta}^2\sin\phi\cos\phi) - r\dot{\phi}(\ddot{r} - r\dot{\phi}^2 - r\dot{\theta}^2\sin^2\phi) &= 0.
\end{aligned}$$

The above equations are those that need to be satisfied by the radar reading to conclude that the plane is flying along a straight line.

Problem 2.271

A golfer chips the ball on a flat, level part of a golf course as shown. Letting $\alpha = 23°$, $\beta = 41°$, and the initial speed be $v_0 = 6\,\mathrm{m/s}$, determine the x and y coordinates of the place where the ball will land.



Solution

For projectile motion $a_x = 0$, $a_y = 0$, and $a_z = -g$. Hence

$$x = x_0 + \dot{x}_0 t, \quad y = y_0 + \dot{y}_0 t, \quad \text{and} \quad z = z_0 + \dot{z}_0 t - \tfrac{1}{2} g t^2.$$

At time $t = 0$ the ball is at the origin of our coordinate system so $x_0 = y_0 = z_0 = 0$. The initial components of the velocity are

$$\dot{x}_0 = v_0 \cos\beta \cos\alpha, \quad \dot{y}_0 = v_0 \cos\beta \sin\alpha, \quad \text{and} \quad \dot{z}_0 = v_0 \sin\beta.$$

Using the equations written so far, we have that the motion of the ball is described by the following equations:

$$x = (v_0 \cos\beta \cos\alpha)t, \quad y = (v_0 \cos\beta \sin\alpha)t, \quad \text{and} \quad z = (v_0 \sin\beta)t - \tfrac{1}{2} g t^2.$$

To determine the location of landing, we observe that the z coordinate of the landing spot must be $z = 0$. Next we find the time corresponding to $z = 0$, i.e.,

$$z = (v_0 \sin\beta)t - \tfrac{1}{2} g t^2 = 0 \quad \Rightarrow \quad t = \frac{2 v_0 \sin\beta}{g}.$$

The x and y components of the position corresponding to this time are

$$x_{\text{land}} = (v_0 \cos\beta \cos\alpha)\left(\frac{2 v_0 \sin\beta}{g}\right) \quad \Rightarrow \quad x_{\text{land}} = \frac{v_0^2 \sin 2\beta \cos\alpha}{g} = 3.345\,\mathrm{m},$$

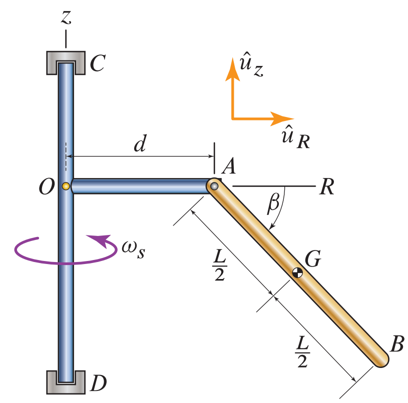
$$y_{\text{land}} = (v_0 \cos\beta \sin\alpha)\left(\frac{2 v_0 \sin\beta}{g}\right) \quad \Rightarrow \quad y_{\text{land}} = \frac{v_0^2 \sin 2\beta \sin\alpha}{g} = 1.420\,\mathrm{m},$$

where we have used the following numerical data: $v_0 = 6\,\mathrm{m/s}$, $\alpha = 23°$, $\beta = 41°$, and $g = 9.81\,\mathrm{m/s^2}$. In summary, the location of the ball's landing spot is identified by the following coordinates

$$\boxed{x_{\text{land}} = 3.345\,\mathrm{m}, \quad y_{\text{land}} = 1.420\,\mathrm{m}, \quad \text{and} \quad z_{\text{land}} = 0.}$$

Problem 2.272

Relative to the cylindrical coordinate system shown, with origin at O, the radial and z coordinates of point G are $R = d + (L/2)\cos\beta$ and $z = -(L/2)\sin\beta$, respectively, where $d = 0.5\,\text{m}$ and $L = 0.6\,\text{m}$. The shaft CD rotates as shown with a constant angular velocity $\omega_s = 10\,\text{rad/s}$, and the angle β varies with time as follows: $\beta = \beta_0 \sin(2\omega t)$, where $\beta_0 = 0.3\,\text{rad}$, $\omega = 2\,\text{rad/s}$, and t is time in seconds. Determine the velocity and the acceleration of G for $t = 3\,\text{s}$ (express the result in the cylindrical component system $(\hat{u}_R, \hat{u}_\theta, \hat{u}_z)$, with $\hat{u}_\theta = \hat{u}_z \times \hat{u}_R$).



Solution

The problem gives the radial and vertical coordinates of point G, which are

$$R = d + \tfrac{1}{2}L\cos\beta \quad \text{and} \quad z = -\tfrac{1}{2}L\sin\beta. \tag{1}$$

In addition, the angle β is given as a function of time: $\beta = \beta_0 \sin(2\omega t)$. Substituting the expression for β into Eqs. (1) we have the R and z coordinates of G directly as a function of time:

$$R = d + \tfrac{1}{2}L\cos[\beta_0 \sin(2\omega t)] \quad \text{and} \quad z = -\tfrac{1}{2}L\sin[\beta_0\sin(2\omega t)]. \tag{2}$$

Now we recall that, in cylindrical coordinates, the velocity is given by

$$\vec{v} = \dot{R}\,\hat{u}_R + R\dot{\theta}\,\hat{u}_\theta + \dot{z}\,\hat{u}_z. \tag{3}$$

The quantities $\dot{R}$ and $\dot{z}$ can be computed by differentiating Eqs. (2) with respect to time. This process, while possibly tedious, is straightforward and gives

$$\dot{R} = -L\beta_0\omega\sin[\beta_0\sin(2\omega t)]\cos(2\omega t), \tag{4}$$
$$\dot{z} = -L\beta_0\omega\cos[\beta_0\sin(2\omega t)]\cos(2\omega t). \tag{5}$$

As far as $\dot{\theta}$ is concerned, we observe that the whole system rotates in the θ direction with a constant angular velocity ω_s:

$$\dot{\theta} = \omega_s = \text{constant}. \tag{6}$$

Recalling that $d = 0.5\,\text{m}$, $L = 0.6\,\text{m}$, $\omega_s = 10\,\text{rad/s}$, $\beta_0 = 0.3\,\text{rad}$, and $\omega = 2\,\text{rad/s}$, for $t = 3\,\text{s}$, we can evaluate the first of Eqs. (2) as well as Eqs. (4)–(6) to obtain

$$R = 0.7961\,\text{m}, \quad \dot{R} = 0.04869\,\text{m/s}, \quad \dot{z} = -0.2999\,\text{m/s}, \quad \text{and} \quad \dot{\theta} = 10.00\,\text{rad/s}. \tag{7}$$

Using the results in Eqs. (7) we can evaluate Eq. (3) to obtain the velocity at $t = 3\,\text{s}$:

$$\boxed{\vec{v} = (0.04869\,\hat{u}_R + 7.961\,\hat{u}_\theta - 0.2999\,\hat{u}_z)\,\text{m/s}.}$$

In cylindrical coordinates, the acceleration is given by

$$\vec{a} = \left(\ddot{R} - R\dot{\theta}^2\right)\hat{u}_r + \left(R\ddot{\theta} + 2\dot{R}\dot{\theta}\right)\hat{u}_\theta + \ddot{z}\,\hat{u}_z. \tag{8}$$

We have already computed the terms R, $\dot{R}$, $\dot{\theta}$. Hence, we only need to determine the terms $\ddot{R}$, $\ddot{\theta}$, and $\ddot{z}$ to compute the acceleration of point G. Differentiating Eqs. (4)–(6) with respect to time we have

$$\dot{R} = 2L\beta_0\omega^2\big\{\sin(2\omega t)\sin[\beta_0\sin(2\omega t)] - \beta_0\cos[\beta_0\sin(2\omega t)]\cos^2(2\omega t)\big\}, \tag{9}$$

$$\ddot{\theta} = 0 \tag{10}$$

$$\dot{z} = 2L\beta_0\omega^2\big\{\sin(2\omega t)\cos[\beta_0\sin(2\omega t)] + \beta_0\sin[\beta_0\sin(2\omega t)]\cos^2(2\omega t)\big\}, \tag{11}$$

Again recalling that $d = 0.5\,\text{m}$, $L = 0.6\,\text{m}$, $\omega_s = 10\,\text{rad/s}$, $\beta_0 = 0.3\,\text{rad}$, and $\omega = 2\,\text{rad/s}$, for $t = 3\,\text{s}$, we can evaluate Eqs. (9)–(11) for $t = 3\,\text{s}$ wo obtain

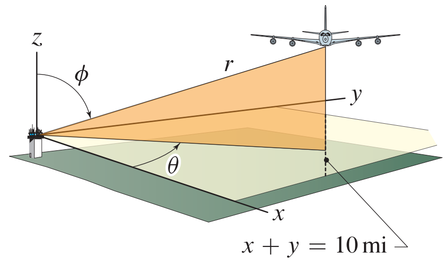
$$\ddot{R} = -0.1798\,\text{m/s}^2, \quad \ddot{\theta} = 0, \quad \text{and} \quad \ddot{z} = -0.8120\,\text{m/s}^2. \tag{12}$$

Hence, using Eqs. (7) and (12), we can evaluate the expression in Eq. (8) for $t = 3\,\text{s}$ to obtain

$$\boxed{\vec{a} = (-79.79\,\hat{u}_R + 0.9738\,\hat{u}_\theta - 0.8120\,\hat{u}_z)\,\text{m/s}^2.}$$

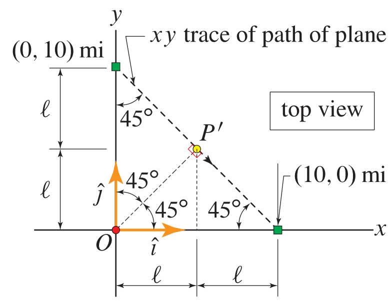
Problem 2.273

An airplane is traveling at a constant altitude of 10,000 ft, with a constant speed of 450 mph, within the plane whose equation is given by $x + y = 10$ mi and in the direction of increasing x. Find the expressions for $\dot{r}$, $\dot{\theta}$, $\dot{\phi}$, $\ddot{r}$, $\ddot{\theta}$, and $\ddot{\phi}$ that would be measured when the airplane is closest to the radar station.



Solution

The figure at the right shows the trace of the path of the airplane on the xy plane. Since the airplane is moving along a straight line with constant speed, then the airplane's velocity is constant. Letting v_0 be the speed of the airplane, the velocity and acceleration of the airplane are



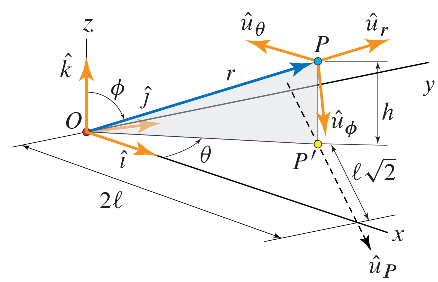
$$\vec{v} = \frac{v_0}{\sqrt{2}}(\hat{\imath} - \hat{\jmath}) \quad \text{and} \quad \vec{a} = \vec{0}. \tag{1}$$

We denote by P the point on the airplane's path that is closest to the origin O. The point P' in the figure is the projection of P on the xy plane. Letting $\ell = 5$ mi, the coordinates of P' are $(\ell, \ell, 0)$, so that the coordinates of P are

$$P\colon (\ell, \ell, h), \tag{2}$$

where we have denoted by h the altitude of the airplane (see bottom right figure).

Using Eq. (2) and the Pythagorean theorem, the distance between P and O is



$$r = \sqrt{2\ell^2 + h^2}. \tag{3}$$

Observing that P' lies on the line bisecting the first quadrant of the xy plane at a distance $\ell\sqrt{2}$ from O, we have that the angles θ and ϕ identifying P are

$$\theta = 45^\circ \quad \text{and} \quad \phi = 90^\circ - \tan^{-1}\left(\frac{h}{\ell\sqrt{2}}\right) = 75.01^\circ, \tag{4}$$

where we have used the data $h = 10{,}000$ ft and $\ell = 5\,\text{mi} = 5(5280)$ ft. We now recall that the unit vectors $\hat{u}_r$, $\hat{u}_\phi$, and $\hat{u}_\theta$ of a spherical coordinate system with origin at O and coordinates (r, θ, ϕ) can be expressed in terms of the Cartesian base vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ as follows

$$\hat{u}_r = \sin\phi\cos\theta\,\hat{\imath} + \sin\phi\sin\theta\,\hat{\jmath} + \cos\phi\,\hat{k}, \tag{5}$$

$$\hat{u}_\phi = \cos\phi\cos\theta\,\hat{\imath} + \cos\phi\sin\theta\,\hat{\jmath} - \sin\phi\,\hat{k}, \tag{6}$$

$$\hat{u}_\theta = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}. \tag{7}$$

Using the first of Eqs. (1) and Eqs. (3)–(6), the components of the velocity in the $\hat{u}_r$, $\hat{u}_\phi$, and $\hat{u}_\theta$ directions are

$$v_r = \dot{r} = \vec{v}\cdot\hat{u}_r \qquad \Rightarrow \qquad \dot{r} = \frac{v_0}{\sqrt{2}}\sin\phi(\cos\theta - \sin\theta), \tag{8}$$

$$v_\phi = r\dot{\phi} = \vec{v}\cdot\hat{u}_\phi \quad\Rightarrow\quad \dot{\phi} = \frac{v_0}{\sqrt{2}\sqrt{2\ell^2+h^2}}\cos\phi(\cos\theta - \sin\theta) \tag{9}$$

$$v_\theta = r\dot{\theta}\sin\phi = \vec{v}\cdot\hat{u}_\theta \quad\Rightarrow\quad \dot{\theta} = -\frac{v_0}{\sin\phi\sqrt{2}\sqrt{2\ell^2+h^2}}(\sin\theta + \cos\theta), \tag{10}$$

where we have used the expression for r in Eq. (3). Recalling that $v_0 = 450\,\text{mph} = 450\frac{5280}{3600}\,\text{ft/s}$, $\ell = 5\,\text{mi} = 5(5280)\,\text{ft}$, and $h = 10{,}000\,\text{ft}$, and using Eqs. (4), we can evaluate Eqs. (8)–(10) to obtain

$$\boxed{\dot{r} = 0, \quad \dot{\phi} = 0, \quad \text{and} \quad \dot{\theta} = -0.01768\,\text{rad/s}.}$$

The second of Eqs. (1) states that all of the components of the acceleration are equal to zero. Therefore, we have

$$a_r = \ddot{r} - r\dot{\phi}^2 - r\dot{\theta}^2\sin^2\phi = 0 \quad\Rightarrow\quad \ddot{r} = r\dot{\phi}^2 + r\dot{\theta}^2\sin^2\phi, \tag{11}$$

$$a_\phi = r\ddot{\phi} + 2\dot{r}\dot{\phi} - r\dot{\theta}^2\sin\phi\cos\phi = 0 \quad\Rightarrow\quad \ddot{\phi} = \dot{\theta}^2\sin\phi\cos\phi - 2\frac{\dot{r}}{r}\dot{\phi} \tag{12}$$

$$a_\theta = r\ddot{\theta}\sin\phi + 2\dot{r}\dot{\theta}\sin\phi + 2r\dot{\phi}\dot{\theta}\cos\phi = 0 \quad\Rightarrow\quad \ddot{\theta} = -2\frac{\dot{r}}{r}\dot{\theta} - 2\dot{\phi}\dot{\theta}(\tan\phi)^{-1}. \tag{13}$$

Recalling again that $v_0 = 450\,\text{mph} = 450\frac{5280}{3600}\,\text{ft/s}$, $\ell = 5\,\text{mi} = 5(5280)\,\text{ft}$, and $h = 10{,}000\,\text{ft}$, and using Eqs. (4), as well as Eqs. (8)–(10), we can evaluate Eqs. (11)–(13) to obtain

$$\boxed{\ddot{r} = 11.27\,\text{ft/s}^2, \quad \ddot{\phi} = 78.10\times 10^{-6}\,\text{rad/s}^2, \quad \text{and} \quad \ddot{\theta} = 0.}$$

Problem 2.274

A carnival ride called the *octopus* consists of eight arms that rotate about the z axis at the constant angular velocity $\dot{\theta} = 6$ rpm. The arms have a length $L = 22$ ft and form an angle ϕ with the z axis. Assuming that ϕ varies with time as $\phi(t) = \phi_0 + \phi_1 \sin \omega t$ with $\phi_0 = 70.5°$, $\phi_1 = 25.5°$, and $\omega = 1$ rad/s, determine the magnitude of the acceleration of the outer end of an arm when ϕ achieves its maximum value.

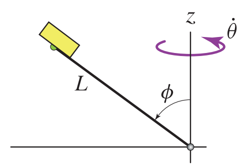


Photo credit: © Gary L. Gray

Solution

Since ϕ_0 and ϕ_1 are positive, the function $\phi(t) = \phi_0 + \phi_1 \sin \omega t$ is maximum when $\sin \omega t = 1$:

$$\phi = \phi_{\max} = \phi_0 + \phi_1 \quad \text{for} \quad \omega t = (\pi/2)\,\text{rad}. \tag{1}$$

Next, we recall that the components of acceleration in spherical coordinates are

$$\begin{aligned} a_r &= \ddot{r} - r\dot{\phi}^2 - r\dot{\theta}^2 \sin^2 \phi, \\ a_\phi &= r\ddot{\phi} + 2\dot{r}\dot{\phi} - r\dot{\theta}^2 \sin \phi \cos \phi, \\ a_\theta &= r\ddot{\theta} \sin \phi + 2\dot{r}\dot{\theta} \sin \phi + 2r\dot{\phi}\dot{\theta} \cos \phi. \end{aligned} \tag{2}$$

To use these equations we need the values of r, $\dot{r}$, $\ddot{r}$, $\dot{\phi}$, $\ddot{\phi}$, $\dot{\theta}$, and $\ddot{\theta}$ for $\phi = \phi_{\max}$. First we observe that

$$r = L = \text{costant} \quad \Rightarrow \quad \dot{r} = 0 \quad \text{and} \quad \ddot{r} = 0. \tag{3}$$

Using the given $\phi(t)$, we have

$$\dot{\phi} = \phi_1 \omega \cos \omega t \quad \text{and} \quad \ddot{\phi} = -\phi_1 \omega^2 \sin \omega t. \tag{4}$$

Finally, for $\dot{\theta}$ and $\ddot{\theta}$ we have

$$\dot{\theta} = 6\,\text{rpm} = \text{constant} \quad \Rightarrow \quad \ddot{\theta} = 0. \tag{5}$$

Using the results in Eqs. (3)–(5), for $\phi = \phi_{\max} = \phi_0 + \phi_1$ the components of acceleration become

$$a_r = -L\dot{\theta}^2 \sin^2(\phi_0 + \phi_1), \quad a_\phi = -L\phi_1\omega^2 - L\dot{\theta}^2 \sin(\phi_0 + \phi_1) \cos(\phi_0 + \phi_1), \quad a_\theta = 0, \tag{6}$$

where we have used the fact that, for $\phi = \phi_{\max}$, $\sin \omega t = 1$ and $\cos \omega t = 0$. Recalling that the magnitude of the acceleration is $|\vec{a}| = \sqrt{a_r^2 + a_\phi^2 + a_\theta^2}$, using Eqs. (6), for $\phi = \phi_{\max}$, we have

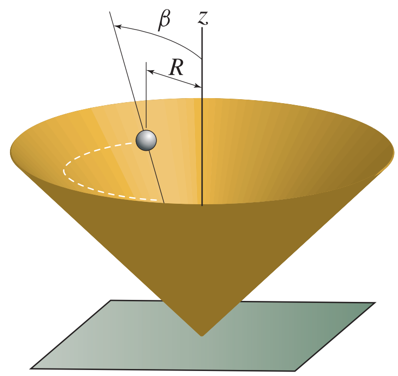
$$\left|\vec{a}_{\phi_{\max}}\right| = \sqrt{\left[-L\dot{\theta}^2 \sin^2(\phi_0 + \phi_1)\right]^2 + \left[-L\phi_1\omega^2 - L\dot{\theta}^2 \sin(\phi_0 + \phi_1) \cos(\phi_0 + \phi_1)\right]^2},$$

which, recalling that $\phi_0 = 70.5° = 70.5\frac{\pi}{180}$ rad, $\phi_1 = 25.5° = 25.5\frac{\pi}{180}$ rad, $\omega = 1$ rad/s, $L = 22$ ft, and $\dot{\theta} = 6\,\text{rpm} = 6\frac{2\pi}{60}$ rad/s, can be evaluated to obtain

$$\boxed{\left|\vec{a}_{\phi_{\max}}\right| = 12.36\,\text{ft/s}^2.}$$

Problem 2.275

A particle is moving over the surface of a right cone with angle β and under the constraint that $R^2\dot{\theta} = K$, where K is a constant. The equation describing the cone is $R = z\tan\beta$. Determine the expressions for the velocity and the acceleration of the particle in terms of K, β, z, and the time derivatives of z.



Solution

We use the the cylindrical coordinate system implied by the problem's figure. Next we recall that the general expressions for the velocity and acceleration in cylindrical coordinates are as follows:

$$\vec{v} = \dot{R}\,\hat{u}_R + R\dot{\theta}\,\hat{u}_\theta + \dot{z}\,\hat{u}_z \quad \text{and} \quad \vec{a} = \big(\ddot{R} - R\dot{\theta}^2\big)\,\hat{u}_R + \big(R\ddot{\theta} + 2\dot{R}\dot{\theta}\big)\,\hat{u}_\theta + \ddot{z}\,\hat{u}_z. \tag{1}$$

The problem is solved by determining all of the terms in the expressions for $\vec{v}$ and $\vec{a}$ and then substituting the terms in question into the above equations.

We begin by determining the terms related to the coordinate R. We are told that $R = z\tan\beta$. Hence, recalling that β is constant, we have

$$R = z\tan\beta \quad \Rightarrow \quad \dot{R} = \dot{z}\tan\beta \quad \Rightarrow \quad \ddot{R} = \ddot{z}\tan\beta. \tag{2}$$

Next we consider the terms related to θ. Specifically, we start with the constraint equation $R^2\dot{\theta} = K$, and obtain

$$K = R^2\dot{\theta} \quad \Rightarrow \quad \dot{\theta} = \frac{K}{R^2} = \frac{K}{z^2\tan^2\beta} \quad \Rightarrow \quad \ddot{\theta} = -\frac{2K\dot{z}}{z^3\tan^2\beta}. \tag{3}$$

Substituting the first two of Eqs. (2) and the second of Eqs. (3) into the first of Eqs. (1), we have

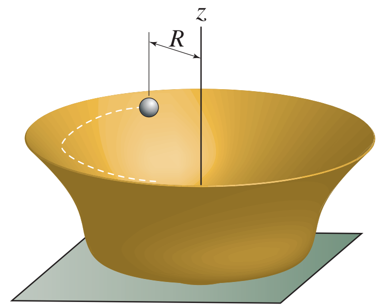
$$\boxed{\vec{v} = \dot{z}\tan\beta\,\hat{u}_R + \frac{K}{z\tan\beta}\,\hat{u}_\theta + \dot{z}\,\hat{u}_z.}$$

Substituting Eqs. (2) and the last two of Eqs. (3) into the second of Eqs. (1), we have

$$\boxed{\vec{a} = \left(\ddot{z}\tan\beta - \frac{K^2}{z^3\tan^3\beta}\right)\hat{u}_R + \ddot{z}\,\hat{u}_z.}$$

Problem 2.276

Solve Prob. 2.275 for general surfaces of revolution; that is, R is no longer equal to $z\tan\beta$, but is now an arbitrary function of z, that is, $R = f(z)$. The expressions you need to find will contain K, $f(z)$, derivatives of $f(z)$ with respect to z, and derivatives of z with respect to time.



Solution

We use the cylindrical coordinate system implied by the problem's figure. Next we recall that the general expressions for the velocity and acceleration in cylindrical coordinates are as follows:

$$\vec{v} = \dot{R}\,\hat{u}_R + R\dot{\theta}\,\hat{u}_\theta + \dot{z}\,\hat{u}_z \quad \text{and} \quad \vec{a} = \left(\ddot{R} - R\dot{\theta}^2\right)\hat{u}_R + \left(R\ddot{\theta} + 2\dot{R}\dot{\theta}\right)\hat{u}_\theta + \ddot{z}\,\hat{u}_z. \tag{1}$$

The problem is solved by determining all of the terms in the expressions for $\vec{v}$ and $\vec{a}$ and then substituting the terms in question into the above equations.

We begin by determining the terms related to the coordinate R. Since we are told that $R = f(z)$, and keeping in mind that $z = z(t)$, using the chain rule, we have

$$\dot{R} = \frac{dR}{dz}\frac{dz}{dt} \quad \Rightarrow \quad \dot{R} = \dot{z}\frac{df}{dz} \quad \text{and} \quad \ddot{R} = \frac{d}{dt}\left(\dot{z}\frac{df}{dz}\right) \quad \Rightarrow \quad \ddot{R} = \ddot{z}\frac{df}{dz} + \dot{z}^2\frac{d^2 f}{dz^2}. \tag{2}$$

Next we consider the terms concerning the coordinate θ. Recalling that we have the constraint equation $K = R^2\dot{\theta}$. Hence, we can write

$$K = R^2\dot{\theta} \quad \Rightarrow \quad \dot{\theta} = \frac{K}{f^2(z)} \quad \Rightarrow \quad \ddot{\theta} = \frac{d\dot{\theta}}{dz}\frac{dz}{dt} = \frac{-2K}{f^3(z)}\frac{df}{dz}\dot{z}. \tag{3}$$

Recalling that $R = f(z)$, substituting the expression for $\dot{R}$ from Eqs.(2) along with the expression for $\dot{\theta}$ from Eqs. (3) into the first of Eqs. (1) we have

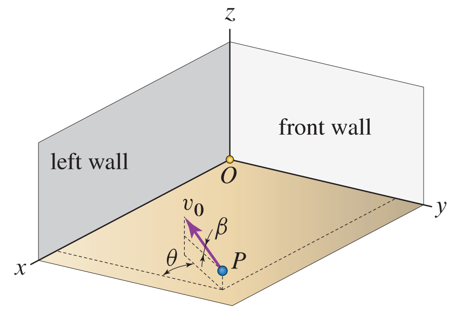
$$\vec{v} = \frac{df}{dz}\dot{z}\,\hat{u}_r + \frac{K}{f(z)}\,\hat{u}_\theta + \dot{z}\,\hat{u}_z.$$

Again recalling that $R = f(z)$, substituting the expressions for $\dot{R}$ and $\ddot{R}$ from Eqs.(2) along with the expressions for $\dot{\theta}$ and $\ddot{\theta}$ from Eqs. (3) into the second of Eqs. (1) we have

$$\vec{a} = \left(\ddot{z} + \dot{z}^2\frac{d^2 f}{dz^2} - \frac{K^2}{f^3(z)}\right)\hat{u}_R + \ddot{z}\,\hat{u}_z.$$

Problem 2.277

In a racquetball court, at point P with coordinates $x_P = 35$ ft, $y_P = 16$ ft, and $z_P = 1$ ft, a ball is imparted a speed $v_0 = 90$ mph and a direction defined by the angles $\theta = 63°$ and $\beta = 8°$ (β is the angle formed by the initial velocity vector and the xy plane). The ball bounces off the left vertical wall to then hit the front wall of the court. Assume that the rebound off the left vertical wall occurs such that (1) the component of the ball's velocity tangent to the wall before and after rebound is the same and (2) the component of velocity normal to the wall right after impact is equal in magnitude and opposite in direction to the same component of velocity right before impact. Accounting for the effect of gravity, determine the coordinates of the point on the front wall that will be hit by the ball after rebounding off the left wall.



Solution

From P to the left wall the racquetball undergoes projectile motion and therefore the components of the ball's acceleration in the given coordinate system are $a_x = 0$, $a_y = 0$, and $a_z = -g$. In turn, the coordinates of the ball as a function of time are

$$x = x_P + \dot{x}_0 t, \quad y = y_P + \dot{y}_0 t, \quad \text{and} \quad z = z_P + \dot{z}_0 t - \tfrac{1}{2} g t^2, \tag{1}$$

where $x_P = 35$ ft, $y_P = 16$ ft, and $z_P = 1$ ft. Next, the initial components of velocity are

$$\dot{x}_0 = -v_0 \cos\beta \sin\theta, \quad \dot{y}_0 = -v_0 \cos\beta \cos\theta, \quad \text{and} \quad \dot{z}_0 = v_0 \sin\beta. \tag{2}$$

The first part of the motion of the ball is described by

$$x = x_P - (v_0 \cos\beta \sin\theta) t, \quad y = y_P - (v_0 \cos\beta \cos\theta) t, \quad \text{and} \quad z = z_P + (v_0 \sin\beta) t - \tfrac{1}{2} g t^2. \tag{3}$$

Letting the subscript lw stand for 'left wall', we must have $y(t_{\text{lw}}) = 0$ so that

$$y_P - (v_0 \cos\beta \cos\theta) t_{\text{lw}} = 0 \quad \Rightarrow \quad t_{\text{lw}} = \frac{y_P}{v_0 \cos\beta \cos\theta} = 0.2696\,\text{s}. \tag{4}$$

The corresponding x and z coordinates at $t = t_{\text{lw}}$ are

$$x_{\text{lw}} = x_P - (v_0 \cos\beta \sin\theta) t_{\text{lw}} \quad \Rightarrow \quad x_{\text{lw}} = 3.598\,\text{ft}, \tag{5}$$

$$z_{\text{lw}} = z_P + (v_0 \sin\beta) t_{\text{lw}} - \tfrac{1}{2} g t_{\text{lw}} \quad \Rightarrow \quad z_{\text{lw}} = 4.783\,\text{ft}, \tag{6}$$

where, in Eqs. (4)–(6), we have used the fact that $x_P = 35$ ft, $y_P = 16$ ft, $z_P = 1$ ft, $v_0 = 90\,\text{mph} = 90\frac{5280}{3600}$ ft/s, $\beta = 8°$, and $\theta = 63°$.

After impact the x and y components of velocity are

$$\dot{x}_{\text{lw}} = -v_0 \cos\beta \sin\theta \quad \text{and} \quad \dot{y}_{\text{lw}} = v_0 \cos\beta \cos\theta. \tag{7}$$

The z component of velocity after impact is calculated with the constant acceleration equation $v = v_0 + at$.

$$\dot{z}_{\text{lw}} = v_0 \sin\beta - g t_{\text{lw}}. \tag{8}$$

After the impact with the left wall, the ball is again in projectile motion. Hence, the equations describing the motion of the ball after the impact are

$$x = x_{\text{lw}} - (v_0 \cos\beta \sin\theta)t, \quad y = (v_0 \cos\beta \cos\theta)t, \quad \text{and} \quad z = z_{\text{lw}} + (v_0 \sin\beta - g t_{\text{lw}})t - \tfrac{1}{2} g t^2, \quad (9)$$

where we have "reset" the time variable so that $t = 0$ now corresponds to when the ball bounces off the the left wall. We are now ready to consider the impact of the ball with the front wall. Letting the subscript fw stand for 'front wall', we must have $x(t_{\text{fw}}) = 0$, so that

$$x_{\text{fw}} = x_{\text{lw}} - (v_0 \cos\beta \sin\theta)t_{\text{fw}} = 0 \quad \Rightarrow \quad t_{\text{fw}} = \frac{x_{\text{lw}}}{v_0 \cos\beta \sin\theta} = 0.03089\,\text{s}. \quad (10)$$

The corresponding values of the y and z coordinates are

$$y_{\text{fw}} = (v_0 \cos\beta \cos\theta)t_{\text{fw}} \quad \Rightarrow \quad y_{\text{fw}} = 1.833\,\text{ft}, \quad (11)$$

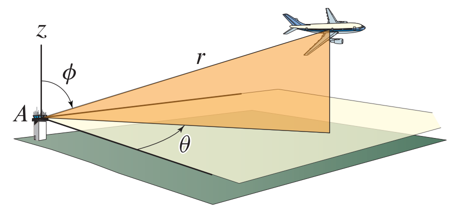
$$z_{\text{fw}} = z_{\text{lw}} + (v_0 \sin\beta - g t_{\text{lw}})t_{\text{fw}} - \tfrac{1}{2} g t_{\text{fw}}^2 \quad \Rightarrow \quad z_{\text{fw}} = 5.067\,\text{ft}. \quad (12)$$

In summary, the coordinates of the point on the front wall that is impacted by the ball, we have

$$\boxed{x_{\text{fw}} = 0, \quad y_{\text{fw}} = 1.833\,\text{ft}, \quad \text{and} \quad z_{\text{fw}} = 5.067\,\text{ft}.}$$

Problem 2.278

An airplane is being tracked by a radar station at A. At the instant $t = 0$, the following data is recorded: $r = 15\,\text{km}$, $\phi = 80°$, $\theta = 15°$, $\dot{r} = 350\,\text{km/h}$, $\dot{\phi} = -0.002\,\text{rad/s}$, $\dot{\theta} = 0.003\,\text{rad/s}$. If the airplane is flying to keep each of the spherical velocity components constant for a few minutes, determine the spherical components of the airplane's acceleration when $t = 30\,\text{s}$.



Solution

In spherical coordinates, the components of acceleration are

$$a_r = \ddot{r} - r\dot{\phi}^2 - r\dot{\theta}^2 \sin^2\phi,$$
$$a_\phi = r\ddot{\phi} + 2\dot{r}\dot{\phi} - r\dot{\theta}^2 \sin\phi\cos\phi, \tag{1}$$
$$a_\theta = r\ddot{\theta}\sin\phi + 2\dot{r}\dot{\theta}\sin\phi + 2r\dot{\phi}\dot{\theta}\cos\phi.$$

To solve this problem, we must provide the values of r, $\dot{r}$, $\ddot{r}$, ϕ, $\dot{\phi}$, $\ddot{\phi}$, $\dot{\theta}$, and $\ddot{\theta}$ for $t = 30\,\text{s}$.

We will now use the assumption that the velocity components are constant to determine the value of the quantities just listed. The expression of the velocity in spherical coordinates is

$$\vec{v} = \dot{r}\,\hat{u}_r + r\dot{\phi}\,\hat{u}_\phi + r\dot{\theta}\sin\phi\,\hat{u}_\theta = v_r\,\hat{u}_r + v_\phi\,\hat{u}_\phi + v_\theta\,\hat{u}_\theta. \tag{2}$$

Then, under the assumption that v_r is constant, for the radial coordinate r we have

$$v_r(0) = \dot{r}(0) = \text{constant} \quad\Rightarrow\quad \ddot{r}(t) = 0 \quad\text{and}\quad r(t) = r(0) + \dot{r}(0)t. \tag{3}$$

Recalling that $r(0) = 15\,\text{km} = 15{,}000\,\text{m}$ and that $\dot{r}(0) = 350\,\text{km/h} = 350\frac{1000}{3600}\,\text{m/s}$, then, using Eqs. (3), at $t = 30\,\text{s}$ we have

$$r(30\,\text{s}) = 17920, \quad \dot{r}(30\,\text{s}) = 97.22\,\text{m/s}, \quad\text{and}\quad \ddot{r}(30\,\text{s}) = 0. \tag{4}$$

Next we proceed to determine the values of ϕ, $\dot{\phi}$, and $\ddot{\phi}$ at $t = 30\,\text{s}$. To do so, referring to Eq. (2), we start from the consideration that $v_\phi = r\dot{\phi}$. Hence, using the expression for r in the last of Eqs. (3) we have

$$v_\phi = r\dot{\phi} = \text{constant} = r(0)\dot{\phi}(0), \quad\Rightarrow\quad \dot{\phi} = \frac{r(0)\dot{\phi}(0)}{r(0) + \dot{r}(0)t} \quad\Rightarrow\quad \ddot{\phi} = -\frac{r(0)\dot{r}(0)\dot{\phi}(0)}{[r(0) + \dot{r}(0)t]^2}. \tag{5}$$

The last two of Eqs. (5) will allow us to compute $\dot{\phi}$ and $\ddot{\phi}$ at $t = 30\,\text{s}$. However, we also need the value of ϕ at $t = 30\,\text{s}$. To compute such a value we now proceed to integrate the second of Eqs. (5) with respect to time. This gives

$$\dot{\phi} = \frac{d\phi}{dt} = \frac{r(0)\dot{\phi}(0)}{r(t)} \quad\Rightarrow\quad \phi(t) - \phi(0) = \int_0^t \frac{r(0)\dot{\phi}(0)}{r(0) + \dot{r}(0)t}\,dt$$
$$\Rightarrow\quad \phi(t) = \phi(0) + \frac{r(0)\dot{\phi}(0)}{\dot{r}(0)} \ln\left(1 + \frac{\dot{r}(0)}{r(0)}t\right). \tag{6}$$

Using the last of Eqs. (6) and the last two of Eqs. (5) we then have

$$\phi(30\,\text{s}) = 76.86°, \quad \dot{\phi}(30\,\text{s}) = -0.001674\,\text{rad/s}, \quad\text{and}\quad \ddot{\phi}(30\,\text{s}) = 9.086\times10^{-6}\,\text{rad/s}^2, \tag{7}$$

where we have used the following numerical data: $r(0) = 15\,\text{km} = 15{,}000\,\text{m}$, $\dot{r}(0) = 350\,\text{km/h} = 350\frac{1000}{3600}\,\text{m/s}$, $\phi(0) = 80^\circ$, and $\dot{\phi}(0) = -0.002\,\text{rad/s}$.

Next we proceed to determine the values of $\dot{\theta}$, and $\ddot{\theta}$ at $t = 30\,\text{s}$. We start from the fact that $v_\theta = r\dot{\theta}\sin\phi$ is assumed to be constant. Hence, we have

$$r(t)\dot{\theta}(t)\sin\phi(t) = r(0)\dot{\theta}(0)\sin\phi(0) \quad\Rightarrow\quad \dot{\theta}(t) = \frac{r(0)\dot{\theta}(0)\sin\phi(0)}{r(t)\sin\phi(t)}$$

$$\Rightarrow\quad \ddot{\theta} = -\frac{r(0)\dot{\theta}(0)\sin\phi(0)\{\dot{r}(t)\sin\phi(t) + r(t)[\cos\phi(t)]\dot{\phi}(t)\}}{r^2(t)\sin^2\phi(t)}. \tag{8}$$

Hence, recalling that $r(0) = 15\,\text{km} = 15{,}000\,\text{m}$, $\dot{r}(0) = 350\,\text{km/h} = 350\frac{1000}{3600}\,\text{m/s}$, $\phi(0) = 80^\circ$, $\dot{\theta}(0) = 0.003\,\text{rad/s}$, and using Eqs. (4) and (7), for $t = 30\,\text{s}$, we have

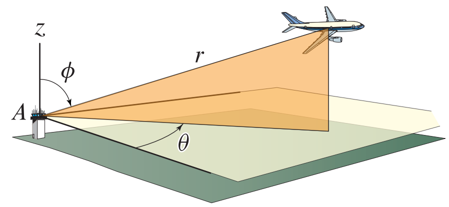
$$\dot{\theta}(30\,\text{s}) = 0.002540\,\text{rad/s} \quad\text{and}\quad \ddot{\theta}(30\,\text{s}) = -0.00001279\,\text{rad/s}^2. \tag{9}$$

In conclusion, substituting Eqs. (4), (7), and (9) into Eqs. (1), for $t = 30\,\text{s}$ we have

$$a_r = -0.1598\,\text{m/s}^2, \quad a_\phi = -0.1884\,\text{m/s}^2, \quad\text{and}\quad a_\theta = 0.2232\,\text{m/s}^2.$$

Problem 2.279

An airplane is being tracked by a radar station at A. At the instant $t = 0$, the following data is recorded: $r = 15\,\text{km}$, $\phi = 80°$, $\theta = 15°$, $\dot{r} = 350\,\text{km/h}$, $\dot{\phi} = -0.002\,\text{rad/s}$, $\dot{\theta} = 0.003\,\text{rad/s}$. If the airplane is flying to keep each of the spherical velocity components constant, plot the trajectory of the airplane for $0 < t < 150\,\text{s}$.



Solution

To plot the trajectory of the airplane we need to find expressions for the coordinates of the airplane as a function of time. Clearly, this must be done while enforcing the condition that the velocity components remain constant during the time interval of interest. For this purpose, we consider the general expression for the velocity in spherical coordinates, which is

$$\vec{v} = \dot{r}\,\hat{u}_r + r\dot{\phi}\,\hat{u}_\phi + r\dot{\theta}\sin\phi\,\hat{u}_\theta = v_r\,\hat{u}_r + v_\phi\,\hat{u}_\phi + v_\theta\,\hat{u}_\theta. \tag{1}$$

Then, under the assumption that v_r is constant, for the radial coordinate r we have

$$v_r(0) = \dot{r}(0) = \text{constant} \quad \Rightarrow \quad r(t) = r(0) + \dot{r}(0)t, \tag{2}$$

where $r(0) = 15\,\text{km}$ and that $\dot{r}(0) = 350\,\text{km/h}$.

Next we proceed to determine an expression for $\phi(t)$. To do so, referring to Eq. (1), we start from the consideration that $v_\phi = r\dot{\phi}$. Hence, using the expression for r in the last of Eqs. (2) we have

$$v_\phi = r\dot{\phi} = \text{constant} = r(0)\dot{\phi}(0), \quad \Rightarrow \quad \dot{\phi} = \frac{r(0)\dot{\phi}(0)}{r(0) + \dot{r}(0)t}. \tag{3}$$

Then, to determine $\phi(t)$ we now proceed to integrate the last of Eqs. (3) with respect to time. This gives

$$\dot{\phi} = \frac{d\phi}{dt} = \frac{r(0)\dot{\phi}(0)}{r(t)} \quad \Rightarrow \quad \phi(t) - \phi(0) = \int_0^t \frac{r(0)\dot{\phi}(0)}{r(0) + \dot{r}(0)t}\,dt$$

$$\Rightarrow \quad \phi(t) = \phi(0) + \frac{r(0)\dot{\phi}(0)}{\dot{r}(0)}\ln\left(1 + \frac{\dot{r}(0)}{r(0)}t\right), \tag{4}$$

where $r(0) = 15\,\text{km} = 15{,}000\,\text{m}$, $\dot{r}(0) = 350\,\text{km/h}$, $\phi(0) = 80°$, and $\dot{\phi}(0) = -0.002\,\text{rad/s}$.

Next we try and provide an expression for $\theta(t)$. We start from the fact that $v_\theta = r\dot{\theta}\sin\phi$ is assumed to be constant. Hence, we have

$$r(t)\dot{\theta}(t)\sin\phi(t) = r(0)\dot{\theta}(0)\sin\phi(0) \quad \Rightarrow \quad \dot{\theta}(t) = \frac{r(0)\dot{\theta}(0)\sin\phi(0)}{r(t)\sin\phi(t)}. \tag{5}$$

Although the expressions for $r(t)$ and $\phi(t)$ are currently known, even if one were to substitute these expressions into the above equation, we would obtain an expression for $\dot{\theta}$ that cannot be integrated with respect to time in closed-form. Hence, we must proceed to integrate with respect to time numerically. This can be done with appropriate mathematical software. We have used *Mathematica* as described below. Once $r(t)$, $\phi(t)$, and $\theta(t)$ are known, in order to plot the trajectory of the airplane, we need to transform the spherical coordinates into corresponding Cartesian coordinates. We do so using the following relations:

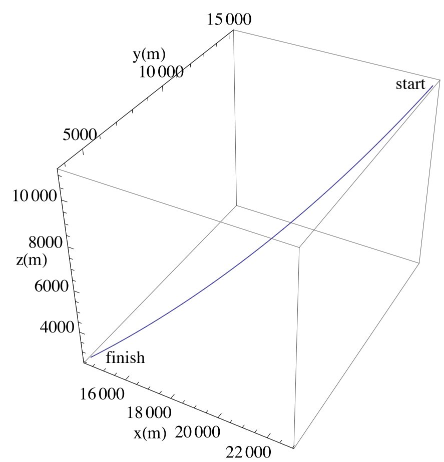
$$x = r\sin\phi\cos\theta, \tag{6}$$

$$y = r \sin\phi \sin\theta, \tag{7}$$
$$z = r \cos\phi. \tag{8}$$

The code provided below first defines the known parameters of the problem. Then the functions giving $r(t)$ and $\phi(t)$ are stated, followed by the instructions necessary to determine $\theta(t)$. Once the solution for $\theta(t)$ is obtained, the spherical coordinates of the airplane are transformed into corresponding Cartesian coordinates and the resulting trajectory is plotted for $0 < t < 150$ s. Referring to the plot of the trajectory, the airplane is moving from the upper right to the lower left.

```
Parameters = {r0 → 15 000., ϕ0 → 80. Degree, θ0 → 15. Degree, rDot0 → 350. · 1000./3600.,
   ϕDot0 → -0.002, θDot0 → 0.003};
r[t_] = r0 + rDot0 t;
ϕ[t_] := ϕ0 + (r0 ϕDot0)/rDot0 Log[1 + rDot0/r0 t];
θSolution = NDSolve[{θ'[t] == (r0 θDot0 Sin[ϕ0])/(r[t] Sin[ϕ[t]]), θ[0] == θ0} /. Parameters, θ[t],
   {t, 0, 150.}];
x[t_] := r[t] Sin[ϕ[t]] Cos[θ[t]] /. θSolution[[1]];
y[t_] := r[t] Sin[ϕ[t]] Sin[θ[t]] /. θSolution[[1]];
z[t_] := r[t] Cos[ϕ[t]] /. θSolution[[1]];
ParametricPlot3D[{x[t], y[t], z[t]} /. Parameters, {t, 0, 150.},
 AxesLabel → {"x(m)", "y(m)", "z(m)"}]
```

Executing the above code yields the following plot (except for the labels “start” and “finish”):

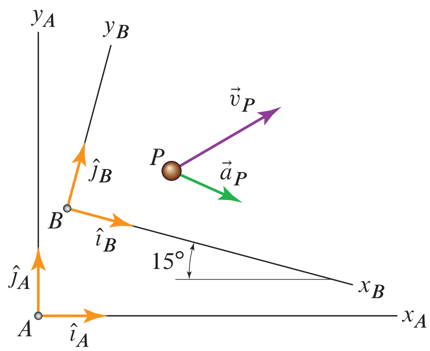


Problem 2.280

The velocity and acceleration of point P expressed relative to frame A at some time t are

$$\vec{v}_{P/A} = (12.5\,\hat{\imath}_A + 7.34\,\hat{\jmath}_A)\text{ m/s} \quad \text{and} \quad \vec{a}_{P/A} = (7.23\,\hat{\imath}_A - 3.24\,\hat{\jmath}_A)\text{ m/s}^2.$$

Knowing that frame B does not move relative to frame A, determine the expressions for the velocity and acceleration of P with respect to frame B. Verify that the speed of P and the magnitude of P's acceleration are the same in the two frames.



Solution

To express the velocity and acceleration vectors relative to the reference frame B, the base vectors of frame A need to be expressed in terms of the base vectors of frame B. Because of the orientation of frame B relative to A, we have

$$\hat{\imath}_A = \cos(15^\circ)\,\hat{\imath}_B + \sin(15^\circ)\,\hat{\jmath}_B \quad \text{and} \quad \hat{\jmath}_A = -\sin(15^\circ)\,\hat{\imath}_B + \cos(15^\circ)\,\hat{\jmath}_B, \tag{1}$$

Substituting the expressions in Eqs. (1) into the given expression for the velocity of P, we have

$$\vec{v}_{P/B} = \big\{[12.5\cos(15^\circ) - 7.34\sin(15^\circ)]\,\hat{\imath}_B + [12.5\sin(15^\circ) + 7.34\cos(15^\circ)]\,\hat{\jmath}_B\big\}\text{ m/s}. \tag{2}$$

The magnitude of $\vec{v}_{P/B}$ is

$$\begin{aligned} v_{B/P} &= \sqrt{[12.5\cos(15^\circ) - 7.34\sin(15^\circ)]^2 + [12.5\sin(15^\circ) + 7.34\cos(15^\circ)]}\text{ m/s} \\ &= \sqrt{12.5^2[\cos^2(15^\circ) + \sin^2(15^\circ)] + 7.34^2[\cos^2(15^\circ) + \sin^2(15^\circ)]}\text{ m/s} \\ &= \sqrt{12.5^2 + 7.34^2}\text{ m/s} = v_{P/A}, \end{aligned} \tag{3}$$

where we have used the trigonometric identity $\sin^2\alpha + \cos^2\alpha = 1$. Equation (3) states that the speed of P as seen by the two frames is the same. Evaluating Eqs. (2) and (3), we have

$$\boxed{\vec{v}_{P/B} = (10.17\,\hat{\imath}_B + 10.33\,\hat{\jmath}_B)\text{ m/s} \quad \text{and} \quad v_{P/B} = v_{P/A} = 14.50\text{ m/s}.}$$

Substituting Eqs. (1) into the given expression for the acceleration of P, we have

$$\vec{a}_{P/B} = \big\{[7.23\cos(15^\circ) + 3.24\sin(15^\circ)]\,\hat{\imath}_B + [7.23\sin(15^\circ) - 3.24\cos(15^\circ)]\,\hat{\jmath}_B\big\}\text{ m/s}^2. \tag{4}$$

The magnitude of $\vec{a}_{P/B}$ is

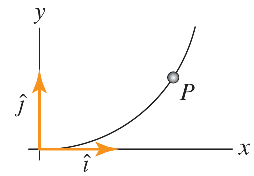
$$\left|\vec{a}_{P/B}\right| = \sqrt{[7.23\cos(15°) + 3.24\sin(15°)]^2 + [7.23\sin(15°) - 3.24\cos(15°)]}\,\text{m/s}^2$$
$$= \sqrt{7.23^2[\cos^2(15°) + \sin^2(15°)] + 3.24^2[\cos^2(15°) + \sin^2(15°)]}\,\text{m/s}^2$$
$$= \sqrt{7.23^2 + 3.24^2}\,\text{m/s}^2 = \left|\vec{a}_{P/A}\right|, \quad (5)$$

where we have used the trigonometric identity $\sin^2\alpha + \cos^2\alpha = 1$. Equation (5) states that the magnitude of the acceleration of P as seen by the two frames is the same. Evaluating Eqs. (4) and (5), we have

$$\vec{a}_{P/B} = (7.822\,\hat{\imath}_B - 1.258\,\hat{\jmath}_B)\,\text{m/s}^2 \quad \text{and} \quad \left|\vec{a}_{P/B}\right| = \left|\vec{a}_{P/A}\right| = 7.923\,\text{m/s}^2.$$

Problem 2.281

The motion of a point P with respect to a Cartesian coordinate system is described by $\vec{r} = \{2\sqrt{t}\,\hat{\imath} + [4\ln(t+1) + 2t^2]\,\hat{\jmath}\}$ ft, where t is time expressed in seconds. Determine the average velocity between $t_1 = 4$ s and $t_2 = 6$ s. Then find the time $\bar{t}$ for which the x component of P's velocity is *exactly* equal to the x component of P's average velocity between times t_1 and t_2. Is it possible to find a time at which P's velocity and P's average velocity are exactly equal? Explain why. *Hint:* Velocity is a vector.



Solution

The average velocity over the time interval $[t_1, t_2]$ is

$$\vec{v}_{\text{avg}} = \frac{\vec{r}(t_2) - \vec{r}(t_1)}{t_2 - t_1}. \tag{1}$$

Using the given function for $\vec{r}(t)$, for $t_1 = 4$ s and $t_2 = 6$ s, we have

$$\boxed{\vec{v}_{\text{avg}} = (0.4495\,\hat{\imath} + 20.67\,\hat{\jmath})\ \text{ft/s},} \tag{2}$$

Next, we compute the velocity of P by differentiating the position vector $\vec{r}(t)$ with respect to time. This gives

$$\vec{v} = \left[\frac{1}{\sqrt{t}}\,\hat{\imath} + \left(4t + \frac{4}{1+t}\right)\hat{\jmath}\right]\ \text{ft/s}. \tag{3}$$

To determine $\bar{t}$ we then need to solve the equation

$$v_x(\bar{t}) = \left(\frac{1}{\sqrt{\bar{t}}}\right)\ \text{ft/s} = 0.4495\ \text{ft/s}, \tag{4}$$

which, can be solved for $\bar{t}$ to obtain

$$\boxed{\bar{t} = 4.949\ \text{s}.} \tag{5}$$

Substituting the result in Eq. (5) to compute the y component of the velocity vector in Eq. (3), we have

$$v_y(\bar{t}) = 20.47\ \text{ft/s} \neq (v_{\text{avg}})_y. \tag{6}$$

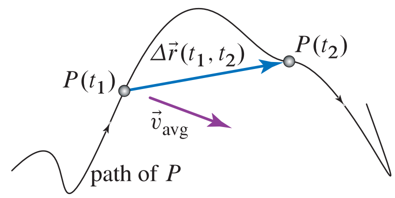
This result implies that

> In general, it is not possible to find a time instant in an interval $[t_1, t_2]$ for which the velocity and the average velocity are equal.

While it is always possible for a scalar function to find a value of time $\bar{t}$ at which the function is equal to its average over a time interval containing $\bar{t}$, when it comes to a vector function, finding a common time that works for every scalar component is, in general, not possible.

Problem 2.282

The figure shows the displacement vector of a point P between two time instants t_1 and t_2. Is it possible for the vector $\vec{v}_{avg}$ shown to be the average velocity of P over the time interval $[t_1, t_2]$?



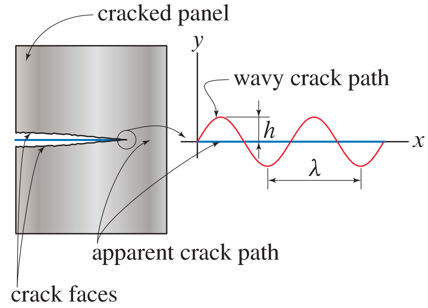
Solution

No. $\vec{v}_{avg}$ must have the same direction as $\Delta\vec{r}\,(t_1, t_2)$.

Problem 2.283

A dynamic fracture model proposed to explain the behavior of cracks propagating at high velocity views the crack path as a *wavy path*. In this model, a crack tip appearing to travel along a straight path actually travels at roughly the speed of sound along a wavy path. Let the wavy path of the crack tip be described by the function $y = h\,\sin(2\pi x/\lambda)$, where h is the amplitude of the crack tip fluctuations in the direction perpendicular to the crack plane and λ is the corresponding period. Assume that the crack tip travels along the wavy path at a constant speed v_s (e.g., the speed of sound).

Find the expression for the x component of the crack tip velocity as a function of v_s, λ, h, and x.



Solution

Using the coordinate system shown, the velocity of the crack tip has the form $\vec{v} = \dot{x}\,\hat{\imath} + \dot{y}\,\hat{\jmath}$. Hence, letting v_s be the speed of the crack, we must have

$$v_s = \sqrt{\dot{x}^2 + \dot{y}^2}. \tag{1}$$

Now recalling that $y = h\sin(2\pi x/\lambda)$, we have

$$\dot{y} = \frac{dy}{dx}\frac{dx}{dt} \quad \Rightarrow \quad \dot{y} = \frac{2h\pi\dot{x}}{\lambda}\cos\left(\frac{2\pi x}{\lambda}\right). \tag{2}$$

Substituting Eq. (2) into Eq. (1) we have

$$v_s = \sqrt{\dot{x}^2 + \left[\frac{2h\pi\dot{x}}{\lambda}\cos\left(\frac{2\pi x}{\lambda}\right)\right]^2} = |\dot{x}|\sqrt{1 + \left[\frac{2h\pi}{\lambda}\cos\left(\frac{2\pi x}{\lambda}\right)\right]^2}, \tag{3}$$

which can be solved to find that the x component of the velocity is

$$\dot{x} = \frac{v_s\lambda}{\sqrt{\lambda^2 + 4h^2\pi^2\cos^2\left(\frac{2\pi x}{\lambda}\right)}},$$

where we have chosen $\dot{x}$ to be positive since the crack is assumed to propagate in the positive x direction.

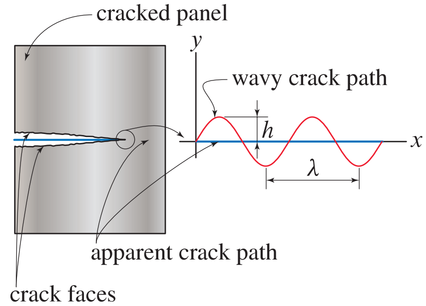
Problem 2.284

A dynamic fracture model proposed to explain the behavior of cracks propagating at high velocity views the crack path as a *wavy path*. In this model, a crack tip appearing to travel along a straight path actually travels at roughly the speed of sound along a wavy path. Let the wavy path of the crack tip be described by the function $y = h\sin(2\pi x/\lambda)$, where h is the amplitude of the crack tip fluctuations in the direction perpendicular to the crack plane and λ is the corresponding period. Assume that the crack tip travels along the wavy path at a constant speed v_s (e.g., the speed of sound).

Denote the *apparent* crack tip velocity by v_a, and define it as the average value of the x component of the crack velocity, that is,

$$v_a = \frac{1}{\lambda}\int_0^\lambda v_x\,dx.$$

In dynamic fracture experiments on polymeric materials, $v_a = 2v_s/3$, v_s is found to be close to 800 m/s, and λ is of the order of 100 μm. What value of h would you expect to find in the experiments if the wavy crack theory were confirmed to be accurate?



Solution

To solve this problem we need to first determine the x component of the velocity of the crack tip. Once, we determine an expression for v_x, then we will solve the equation stated in the problem numerically for h.

We begin by observing that, using the coordinate system shown, the velocity of the crack tip has the form $\vec{v} = \dot{x}\,\hat{\imath} + \dot{y}\,\hat{\jmath}$. Hence, letting v_s be the speed of the crack, we must have

$$v_s = \sqrt{\dot{x}^2 + \dot{y}^2}. \tag{1}$$

Now recalling that $y = h\sin(2\pi x/\lambda)$, we have

$$\dot{y} = \frac{dy}{dx}\frac{dx}{dt} \quad\Rightarrow\quad \dot{y} = \frac{2h\pi\dot{x}}{\lambda}\cos\left(\frac{2\pi x}{\lambda}\right). \tag{2}$$

Substituting Eq. (2) into Eq. (1) we have

$$v_s = \sqrt{\dot{x}^2 + \left[\frac{2h\pi\dot{x}}{\lambda}\cos\left(\frac{2\pi x}{\lambda}\right)\right]^2} = |\dot{x}|\sqrt{1 + \left[\frac{2h\pi}{\lambda}\cos\left(\frac{2\pi x}{\lambda}\right)\right]^2}, \tag{3}$$

which can be solved to find that the x component of the velocity is

$$v_x = \dot{x} = \frac{v_s\lambda}{\sqrt{\lambda^2 + 4h^2\pi^2\cos^2\left(\frac{2\pi x}{\lambda}\right)}}, \tag{4}$$

where we have chosen $\dot{x}$ to be positive since the crack is assumed to propagate in the positive x direction.

Now, that v_x is known, using the formula for v_a given in the problem statement, we have

$$v_a = \frac{1}{\lambda}\int_0^\lambda \frac{v_s\lambda}{\sqrt{\lambda^2 + 4h^2\pi^2\cos^2\left(\frac{2\pi x}{\lambda}\right)}}\,dx = \int_0^\lambda \frac{v_s}{\sqrt{\lambda^2 + 4h^2\pi^2\cos^2\left(\frac{2\pi x}{\lambda}\right)}}\,dx. \tag{5}$$

Recall that we have $v_s = 800\,\mathrm{m/s}$, $v_a = \frac{2}{3}v_s$, and $\lambda = 100\,\mu\mathrm{m}$. Hence, since the variable x in Eq. (5) has the role of dummy variable of integration, Eq. (5) is an equation in the single unknown h. The equation in question is an integral equation and can be solved numerically using appropriate mathematical software. We have used *Mathematica* with the code given below. As is often the case with the numerical solution of equations, we had to supply *Mathematica* with an initial guess for the solution. Since the quantity h is expected to be of the same order of a few μm, our guess for h was set to 1 μm.

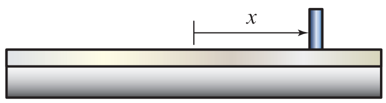
```
Parameters = {vs → 800., va → 2/3 800., λ → 10^-4};
IntegralEq[h_] := NIntegrate[vs / Sqrt[λ^2 + 4 h^2 π^2 Cos[2 π x / λ]^2] /. Parameters,
  {x, 0, λ /. Parameters}]
FindRoot[(va /. Parameters) == IntegralEq[h], {h, 10.^-6}]
```

The execution of the code above yields the following result:

$$h = 29.43\times 10^{-6}\ \mathrm{m.}$$

Problem 2.285

The motion of a peg sliding within a rectilinear guide is controlled by an actuator in such a way that the peg's acceleration takes on the form $\ddot{x} = a_0(2\cos 2\omega t - \beta \sin \omega t)$, where t is time, $a_0 = 3.5\,\text{m/s}^2$, $\omega = 0.5\,\text{rad/s}$, and $\beta = 1.5$. Determine the total distance traveled by the peg during the time interval $0\,\text{s} \le t \le 5\,\text{s}$ if $\dot{x}(0) = a_0\beta/\omega + 0.3\,\text{m/s}$. When compared with Prob. 2.57, why does the addition of 0.3 m/s in the initial velocity turn this into a problem that requires a computer to solve?



Solution

We begin by obtaining an expression for the velocity of the peg as a function of time. Since the acceleration is given as a function of time, we can integrate it as follows:

$$v = \dot{x}(0) + \int_0^t a_0(2\cos 2\omega t - \beta \sin \omega t)\,dt$$

$$\Rightarrow \quad v = a_0\frac{\beta}{\omega} + 0.3\,\text{m/s} + a_0\left(\frac{2}{2\omega}\sin 2\omega t + \frac{\beta}{\omega}\cos \omega t\right) - a_0\frac{\beta}{\omega}, \quad (1)$$

which we can simplify to obtain

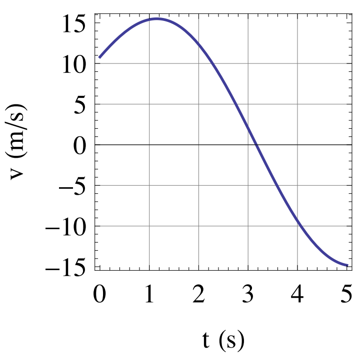
$$v = \frac{a_0}{\omega}\left[\sin 2\omega t + \beta \cos \omega t\right] + 0.3\,\text{m/s}. \quad (2)$$

To find the distance traveled, we need to establish if and when the peg switches direction during the time interval considered. This can be easily done by using any appropriate mathematical software that can plot the function v over the time interval considered. Recalling that $a_0 = 3.5\,\text{m/s}^2$, $\omega = 0.5\,\text{rad/s}$, and $\beta = 1.5$, we have used *Mathematica* with the following code:

```
Parameters = {a0 → 3.5, ω → 0.5, β → 1.5};

Plot[a0/ω (Sin[2 ω t] + β Cos[ω t]) + 0.3 /. Parameters, {t, 0, 5}, Frame → True,
 FrameLabel → {"t (s)", "v (m/s)"}, GridLines → Automatic, AspectRatio → 1]
```

The above code yields the following plot:



As can be seen from the above plot, the velocity changes sign near $t = 3\,\text{s}$. Hence, we need to solve the equation $v = 0$ to determine where exactly the sign switch occurs. Because the expression we have for v

includes the constant term 0.3 m, the equation $v = 0$ cannot be solved analytically. Hence, we will solve it numerically. This can be done with any appropriate mathematical software. As is common with root finding algorithms, we need to provide a guess for the solution. Based on the graph presented above, we set our guess to $t = 3$ s and then use the following *Mathematica* code:

```
Parameters = {a0 → 3.5, ω → 0.5, β → 1.5};
FindRoot[a0/ω (Sin[2 ω t] + β Cos[ω t]) + 0.3 == 0 /. Parameters, {t, 3}]
```

which yields the following solution:

$$t = 3.166\,\text{s}. \tag{3}$$

Using this result, and letting d denote the distance traveled, we have

$$d = \int_0^{3.166\,\text{s}} \left[\frac{a_0}{\omega}\cos\omega t(2\sin\omega t+\beta)+0.3\,\text{m/s}\right]dt - \int_{3.166\,\text{s}}^{5\,\text{s}} \left[\frac{a_0}{\omega}\cos\omega t(2\sin\omega t+\beta)+0.3\,\text{m/s}\right]dt. \tag{4}$$

The above integrals can be computed either numerically or analytically. Since this is a computer problem, we have chosen to carry out the integration numerically. We have used *Mathematica* with the following code:

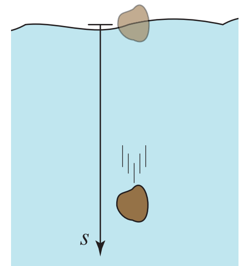
```
Parameters = {a0 → 3.5, ω → 0.5, β → 1.5};
d = NIntegrate[a0/ω (Sin[2 ω t] + β Cos[ω t]) + 0.3 /. Parameters, {t, 0, 3.166}] -
  NIntegrate[a0/ω (Sin[2 ω t] + β Cos[ω t]) + 0.3 /. Parameters, {t, 3.166, 5}]
```

The execution of the above code, yields the following result:

$$\boxed{d = 52.81\,\text{m}.}$$

Problem 2.286

The acceleration of an object in rectilinear free fall while immersed in a linear viscous fluid is $a = g - C_d v/m$, where g is the acceleration of gravity, C_d is a constant drag coefficient, v is the object's velocity, and m is the object's mass. Letting $v = 0$ and $s = 0$ for $t = 0$, where s is position and t is time, determine the position as a function of time.



Solution

The acceleration is given as a function of velocity. Recalling that $a = dv/dt$, we can separate the time and velocity variables as follows:

$$dt = \frac{dv}{a} \quad \Rightarrow \quad \int_0^t dt = \int_0^v \frac{dv}{g - C_d v/m} \quad \Rightarrow \quad t = \frac{-m}{C_d} \ln\left(1 - \frac{C_d}{mg} v\right), \tag{1}$$

where we have enforced the condition that $v = 0$ for $t = 0$. We now solve the last of Eqs. (1) for v. This gives

$$e^{-\frac{C_d}{m}t} = 1 - \frac{C_d}{mg} v \quad \Rightarrow \quad v = \frac{mg}{C_d}\left(1 - e^{-\frac{C_d}{m}t}\right). \tag{2}$$

Now that we have velocity as a function of time, we recall that $v = ds/dt$, so that we can separate the variables s and t by writing

$$ds = v\,dt \quad \Rightarrow \quad \int_0^s ds = \frac{mg}{C_d} \int_0^t \left(1 - e^{-\frac{C_d}{m}t}\right) dt, \tag{3}$$

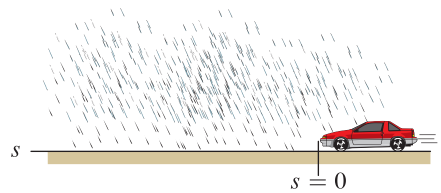
where we have enforced the condition that $s = 0$ for $t = 0$. Carrying out the above integration and simplifying, we have

$$\boxed{s = \frac{mg}{C_d^2}\left[C_d t + m\left(e^{-\frac{C_d}{m}t} - 1\right)\right].}$$

Problem 2.287

Heavy rains cause a stretch of road to have a coefficient of friction that changes as a function of location. Under these conditions, the acceleration of a car skidding while trying to stop can be approximated by $\ddot{s} = -(\mu_k - cs)g$, where μ_k is the friction coefficient under dry conditions, g is the acceleration of gravity, and c, with units of m^{-1}, describes the rate of friction decrement. Let $\mu_k = 0.5$, $c = 0.015\,\mathrm{m}^{-1}$, and $v_0 = 45\,\mathrm{km/h}$, where v_0 is the initial velocity of the car. Determine the time it will take the car to stop and the percent increase in stopping time with respect to dry conditions, i.e., when $c = 0$. *Hint:*

$$\int \frac{1}{\sqrt{1+x^2}}\,dx = \log\Bigl(x + \sqrt{1+x^2}\Bigr).$$



Solution

We are given the acceleration as a function of position. We will first relate the acceleration to the velocity using the chain rule, and then relate the velocity to position:

$$a = \frac{d\dot{s}}{ds}\frac{ds}{dt} \quad \Rightarrow \quad a = \frac{d\dot{s}}{ds}\dot{s} \quad \Rightarrow \quad a\,ds = \dot{s}\,d\dot{s} \quad \Rightarrow \quad \int_{v_0}^{v} \dot{s}\,d\dot{s} = \int_0^s -(\mu_k - cs)g\,ds, \tag{1}$$

where $v = \dot{s}$, $v = v_0$ for $s = 0$, and where we have used the given expression of the acceleration in that last of Eqs. (1). Carrying out the integration, we have

$$\tfrac{1}{2}v^2 - \tfrac{1}{2}v_0^2 = \frac{g}{2c}(cs - \mu_k)^2\Big|_0^s \quad \Rightarrow \quad v = \sqrt{\frac{g}{c}}\sqrt{\frac{c}{g}v_0^2 + (cs-\mu_k)^2 - \mu_k^2}. \tag{2}$$

This result will be needed later. We now determine the expression for the stopping position of the car, which we will denote by s_f (f stands for final). Setting $v = 0$ for $s = s_f$ in Eq. (2), and solving for s_f, we have

$$s_f = \frac{\mu_k}{c} \pm \frac{1}{c}\sqrt{\mu_k^2 - \frac{cv_0^2}{g}}. \tag{3}$$

Only the smaller of the two roots in Eq. (3) makes physical sense (once the car stops, it will not stop at an increased position). Hence, we select the following root:

$$s_f = \frac{\mu_k}{c} - \frac{1}{c}\sqrt{\mu_k^2 - \frac{cv_0^2}{g}}. \tag{4}$$

Note that we could have identified the correct the solution to choose by evaluating the two roots numerically. Recalling that $\mu_k = 0.5$, $c = 0.015\,\mathrm{m}^{-1}$, $g = 9.81\,\mathrm{m/s^2}$, and $v_0 = 45\,\mathrm{km/h} = 45\frac{1000}{3600}$, the roots corresponding to the minus and plus signs are $s_f = 26.31\,\mathrm{m}$ and $s_f = 40.35\,\mathrm{m}$, respectively.

We now go back to Eq. (2) and proceed to determine the relation between position and time. To do so, we recall that $v = ds/dt$. Observing that we have the velocity as a function of position, we can separate the variables s and t as follows: $dt = ds/v$. Hence, using Eq. (2), we have

$$\int_0^{t_f} dt = \sqrt{\frac{c}{g}} \int_0^{s_f} \frac{ds}{\sqrt{\frac{c}{g}v_0^2 + (cs-\mu_k)^2 - \mu_k^2}}, \tag{5}$$

where we have enforced the condition that $s = 0$ for $t = 0$ and we have denoted by t_f the time at which the car comes to a stop. To carry out the integration, we let

$$A = \frac{c}{g} v_0^2 - \mu_k^2, \tag{6}$$

and we perform the following change of variables of integration:

$$x = cs - \mu_k \quad \Rightarrow \quad ds = \frac{1}{c}\, dx. \tag{7}$$

For $s = 0$ we have $x = -\mu_k$ and for $s = s_f$ we have $x = cs_f - \mu_k$. Hence, Eq. (5) becomes

$$t_f = \frac{1}{\sqrt{cg}} \int_{-\mu_k}^{cs_f - \mu_k} \frac{dx}{\sqrt{A + x^2}}. \tag{8}$$

Using the hint, we obtain

$$t_f = \frac{1}{\sqrt{cg}} \ln\Big(x + \sqrt{A + x^2}\Big)\Big|_{-\mu_k}^{cs_f - \mu_k} \quad \Rightarrow \quad t_f = \frac{1}{\sqrt{cg}} \ln \frac{cs_f - \mu_k + \sqrt{A + (cs_f - \mu_k)^2}}{-\mu_k + \sqrt{A + \mu_k^2}}, \tag{9}$$

which, recalling the definition of A in Eq. (6), after simplification yields

$$t_f = \frac{1}{\sqrt{cg}} \ln \frac{cs_f - \mu_k}{v_0 \sqrt{c/g} - \mu_k}. \tag{10}$$

Realling that $c = 0.015\,\mathrm{m}^{-1}$, $g = 9.81\,\mathrm{m/s^2}$, $\mu_k = 0.5$, and $v_0 = 45\,\mathrm{km/s} = 45\frac{1000}{3600}\,\mathrm{m/s}$, we can evaluate s_f in Eq. (4) and then evaluate t_f in Eq. (10) to obtain

$$\boxed{t_f = 5.839\,\mathrm{s}.} \tag{11}$$

With dry conditions, i.e., for $c = 0$, the acceleration of the car would be $\ddot{s} = -\mu_k g$. Hence, we can use the constant acceleration equation $\dot{s} = \dot{s}_0 + a_c t$ to find the stopping time. This gives

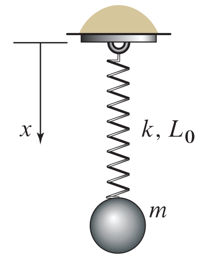
$$0 = v_0 - \mu_k g t_f \quad \Rightarrow \quad t_f = 2.548\,\mathrm{s}. \tag{12}$$

Comparing this result to that in Eq. (11), we find the percent increase to be

$$\text{Stopping time percent increase} = \frac{t_{f_\text{wet}} - t_{f_\text{dry}}}{t_{f_\text{dry}}} \times 100 \quad \Rightarrow \quad \boxed{\text{Stopping time percent increase} = 129.1\%.}$$

Problem 2.288

The acceleration of a particle of mass m suspended by a linear spring with spring constant k and unstretched length L_0 (when the spring length is equal to L_0, the spring exerts no force on the particle) is given by $\ddot{x} = g - (k/m)(x - L_0)$. Assuming that at $t = 0$ the particle is at rest and its position is $x = 0$ m, derive the expression of the particle's position x as a function of time. *Hint:* A good table of integrals will come in handy.



Solution

We have acceleration as a function of position and we integrate it as follows:

$$a = \frac{d\dot{x}}{dx}\frac{dx}{dt} \quad\Rightarrow\quad \int_0^{\dot{x}} \dot{x}\,d\dot{x} = \int_0^x \left[g - \frac{k}{m}(x - L_0)\right]dx.$$

$$\tfrac{1}{2}\dot{x}^2 = gx - \frac{k}{2m}x^2 + \frac{kL_0}{m}x \quad\Rightarrow\quad \dot{x} = \sqrt{2gx - \frac{k}{m}x^2 + \frac{2kL_0}{m}x}.$$

Now that we have $\dot{x}$ as a function of x, we relate it to time as follows:

$$\dot{x} = \frac{dx}{dt} \quad\Rightarrow\quad \int_0^t dt = \int_0^x \frac{dx}{\dot{x}} \quad\Rightarrow\quad t = \int_0^x \frac{dx}{\sqrt{x\left(\frac{2kL_0}{m} + 2g\right) - \frac{k}{m}x^2}}.$$

Now we let $A = \frac{m}{k}\left(\frac{2kL_0}{m} + 2g\right) = 2\left(L_0 + \frac{gm}{k}\right)$ and we rewrite the expression for t as follows:

$$t = \sqrt{\frac{m}{k}}\int_0^x \frac{dx}{\sqrt{Ax - x^2}} \quad\Rightarrow\quad t = 2\sqrt{\frac{m}{k}}\tan^{-1}\left(\frac{\sqrt{x}}{\sqrt{A - x}}\right)\Bigg|_0^x.$$

Making use of the trigonometric identity $1 + \tan^2\theta = \sec^2\theta$, we have that

$$\tan^2\left(\frac{t}{2}\sqrt{\frac{k}{m}}\right) = \frac{x}{A - x} \quad\Rightarrow\quad x\left[\sec^2\left(\frac{t}{2}\sqrt{\frac{k}{m}}\right)\right] = A\tan^2\left(\frac{t}{2}\sqrt{\frac{k}{m}}\right).$$

Now making use of the trigonometric identity $\sin^2(\theta/2) = \frac{1}{2}(1 - \cos\theta)$, we can express x as

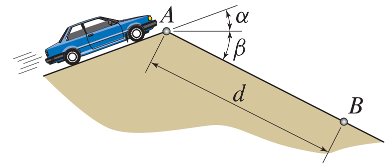
$$x = A\sin^2\left(\frac{t}{2}\sqrt{\frac{k}{m}}\right) \quad\Rightarrow\quad x = \frac{A}{2}\left[1 - \cos\left(\sqrt{\frac{k}{m}}\,t\right)\right].$$

Finally, recalling that A is the quantity $2\left(L_0 + \frac{gm}{k}\right)$, the expression for x can be written as

$$\boxed{x = \left(L_0 + \frac{gm}{k}\right)\left[1 - \cos\left(\sqrt{\frac{k}{m}}\,t\right)\right].}$$

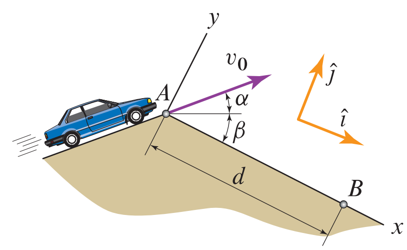
Problem 2.289

In a movie scene involving a car chase, a car goes over the top of a ramp at A and lands at B below. Letting $\alpha = 18°$ and $\beta = 25°$, determine the speed of the car at A if the car is to be airborne for a full 3 s. Furthermore, determine the distance d covered by the car during the stunt, as well as the impact speed and angle at B. Neglect aerodynamic effects. Express your answer using the U.S. Customary system of units.



Solution

This is a projectile problem and we begin by relating the jump speed v_0 to the velocity of the car and then to the car's trajectory. Referring to the figure at the right, the initial velocity and acceleration in the (x, y) coordinate system are



$$\dot{x}_0 = v_0 \cos(\alpha + \beta), \quad \dot{y}_0 = v_0 \sin(\alpha + \beta), \tag{1}$$
$$\ddot{x} = g \sin \beta, \quad \ddot{y} = -g \cos \beta. \tag{2}$$

Using constant acceleration equations, we have that the coordinates of the (airborne) car as a function of time are

$$x = v_0 \cos(\alpha + \beta) t + \tfrac{1}{2} g \sin \beta \, t^2, \tag{3}$$
$$y = v_0 \sin(\alpha + \beta) t - \tfrac{1}{2} g \cos \beta \, t^2. \tag{4}$$

We want to satisfy the condition that $y = 0$ at $t = t_f = 3$ s, where the subscript f stands for flight.

$$0 = v_0 \sin(\alpha + \beta) t_f - \tfrac{1}{2} g \cos \beta \, t_f^2 \quad \Rightarrow \quad v_0 = \frac{g t_f \cos \beta}{2 \sin(\alpha + \beta)}. \tag{5}$$

Recalling that $g = 32.2\,\text{ft/s}^2$, $t_f = 3$ s, $\beta = 25°$, and $\alpha = 18°$, we can evaluate the result in the last of Eqs. (5) to obtain

$$\boxed{v_0 = 64.19\,\text{ft/s}.}$$

We now observe that the distance d is equal to the value of x at time $t = t_f$. Using Eq. (3) and the expression for v_0 in the last of Eqs. (5), we have that the distance d is given by

$$d = \frac{g t_f \cos \beta}{2 \sin(\alpha + \beta)} \cos(\alpha + \beta) t_f + \tfrac{1}{2} g \sin \beta \, t_f^2 \quad \Rightarrow \quad \boxed{d = 202.1\,\text{ft}.} \tag{6}$$

To find the impact speed and the impact angle, we need to determine the velocity at impact. To do so, we use the constant acceleration equation $v = v_0 + a_c t$ to obtain

$$\vec{v}_i = [v_0 \cos(\alpha + \beta) + g \sin \beta \, t_f]\,\hat{\imath} + [v_0 \sin(\alpha + \beta) - g \cos \beta \, t_f]\,\hat{\jmath}$$
$$\Rightarrow \quad \vec{v}_i = g t_f \left[\frac{\cos \beta}{2 \tan(\alpha + \beta)} + \sin \beta \right] \hat{\imath} - \tfrac{1}{2} g t_f \cos \beta \, \hat{\jmath}, \tag{7}$$

where we have used the expression for v_0 in the last of Eqs. (5). Therefore, the impact speed is

$$v_i = g t_f \sqrt{\left[\frac{\cos \beta}{2 \tan(\alpha + \beta)} + \sin \beta \right]^2 + \tfrac{1}{4} \cos^2 \beta}. \tag{8}$$

Recalling that $g = 32.2\,\text{ft/s}^2$, $t_f = 3\,\text{s}$, $\beta = 25°$, and $\alpha = 18°$, we can evaluate the result in Eq. (8) to obtain

$$\boxed{v_i = 98.08\,\text{ft/s.}}$$

We now compute the impact angle of the car with the ground at B measuring it with respect to the slope. This angle is given by

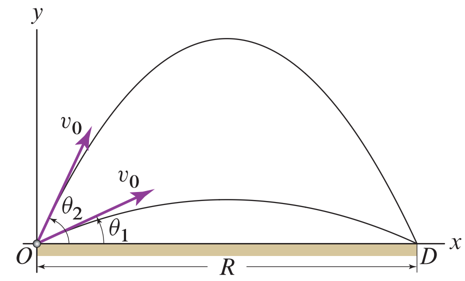
$$\text{Impact angle} = \tan^{-1}\left(-\frac{v_{i_y}}{v_{i_x}}\right) = \tan^{-1}\left\{\tfrac{1}{2}\cos\beta\left[\frac{\cos\beta}{2\tan(\alpha+\beta)} + \sin\beta\right]^{-1}\right\}. \tag{9}$$

Recalling that $\beta = 25°$ and $\alpha = 18°$, we can evaluate the above result to obtain

$$\boxed{\text{Impact angle} = 26.51° \text{ measured clockwise from the slope.}}$$

Problem 2.290

Consider the problem of launching a projectile a distance R from O to D with a known launch speed v_0. It is probably clear to you that you also need to know the launch angle θ if you want the projectile to land exactly at R. But it turns out that the condition determining whether or not v_0 is large enough to get to R does not depend on θ. Determine this condition on v_0. *Hint:* Find v_0 as a function of R and θ, and then remember that the sine function is bounded by 1.



Solution

For projectile motion we have the initial velocity and acceleration in the (x, y) coordinate system, i.e.,

$$\dot{x}_0 = v_0 \cos\theta, \quad \dot{y}_0 = v_0 \sin\theta, \tag{1}$$

$$\ddot{x} = 0, \quad \ddot{y} = -g. \tag{2}$$

Using the constant acceleration equations, we obtain

$$x = v_0 \cos\theta\, t \quad \text{and} \quad y = v_0 \sin\theta\, t - \tfrac{1}{2} g t^2. \tag{3}$$

For $t = t_D$ we have $y(t_D) = 0$. Using this fact in the second of Eqs. (3), yields

$$t_D = \frac{2 v_0 \sin\theta}{g}. \tag{4}$$

Then substituting Eq. (4) into the first of Eqs. (3) and setting $x = R$, we obtain

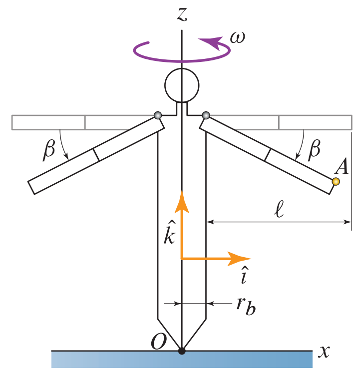
$$R = \frac{v_0^2}{g} 2 \sin\theta \cos\theta \quad \Rightarrow \quad v_0^2 = \frac{gR}{\sin 2\theta}, \tag{5}$$

where we have used the trigonometric identity $2\sin\theta\cos\theta = \sin 2\theta$. Since we must have $|\sin 2\theta| \leq 1$, we have a corresponding condition on v_0 of the following form:

$$\boxed{v_0 \geq \sqrt{gR}.}$$

Problem 2.291

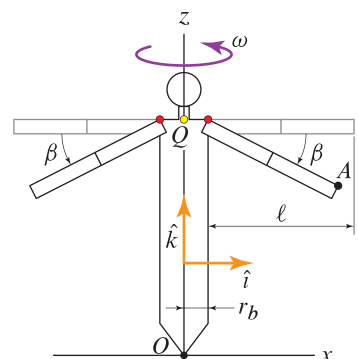
A skater is spinning with her arms completely stretched out and with an angular velocity $\omega = 60\,\text{rpm}$. Letting $r_b = 0.55\,\text{ft}$, and $\ell = 2.2\,\text{ft}$ and neglecting the change in ω as the skater lowers her arms, determine the velocity and acceleration of the hand A right when $\beta = 0°$ if the skater lowers her arms at the constant rate $\dot{\beta} = 0.2\,\text{rad/s}$. Express the answers using the component system shown, which rotates with the skater and for which the unit vector $\hat{\jmath}$ (not shown) is such that $\hat{\jmath} = \hat{k} \times \hat{\imath}$.



Solution

Referring to the figure at the right, let Q be the *fixed* point on the spin axis that is at the same height as the shoulders. Then the position of A relative to Q is

$$\vec{r}_{A/Q} = (r_b + \ell\cos\beta)\,\hat{\imath} - \ell\sin\beta\,\hat{k}. \tag{1}$$



Since Q is a fixed point, the time derivative of $\vec{r}_{A/Q}$ is the velocity A:

$$\vec{v}_A = -\ell\dot{\beta}\sin\beta\,\hat{\imath} + (r_b + \ell\cos\beta)\,\dot{\hat{\imath}} - \ell\dot{\beta}\cos\beta\,\hat{k} - \ell\sin\beta\,\dot{\hat{k}}. \tag{2}$$

Since $\hat{\imath}$ always points towards A, $\hat{\imath}$ changes its orientation with the angular velocity $\vec{\omega} = \omega\,\hat{k}$. In addition, since $\hat{k}$ does not change its orientation, then $\hat{k}$ is constant. Hence, using the concept of time derivative of a unit vector, we have

$$\dot{\hat{\imath}} = \omega\,\hat{k} \times \hat{\imath} = \omega\,\hat{\jmath} \quad \text{and} \quad \dot{\hat{k}} = \vec{0}. \tag{3}$$

Substituting Eqs. (3) into Eq. (2) gives

$$\vec{v}_A = -\ell\dot{\beta}\sin\beta\,\hat{\imath} + (r_b + \ell\cos\beta)\omega\,\hat{\jmath} - \ell\dot{\beta}\cos\beta\,\hat{k}. \tag{4}$$

For $\beta = 0$, we then have

$$\vec{v}_A\big|_{\beta=0} = (r_b + \ell)\omega\,\hat{\jmath} - \ell\dot{\beta}\,\hat{k} \quad \Rightarrow \quad \boxed{\vec{v}_A\big|_{\beta=0} = \big(17.28\,\hat{\jmath} - 0.4400\,\hat{k}\big)\,\text{ft/s},} \tag{5}$$

where we have used the data: $r_b = 0.55\,\text{ft}$, $\ell = 2.2\,\text{ft}$, $\omega = 60\,\text{rpm} = 60\frac{2\pi}{60}\,\text{rad/s}$, and $\dot{\beta} = 0.2\,\text{rad/s}$. To determine $\vec{a}_A$, we differentiate $\vec{v}_A$ in Eq. (4) with respect to time and, recalling that $\dot{\beta}$ and $\hat{k}$ are constant, we obtain

$$\vec{a}_A = -\ell\dot{\beta}^2\cos\beta\,\hat{\imath} - \ell\dot{\beta}\sin\beta\,\dot{\hat{\imath}} - \ell\dot{\beta}\omega\sin\beta\,\hat{\jmath} + (r_b + \ell\cos\beta)\omega\,\dot{\hat{\jmath}} + \ell\dot{\beta}^2\sin\beta\,\hat{k}. \tag{6}$$

Using the first of Eqs. (3), and observing that the same idea that allowed us to derive the first of Eqs. (3) tells us that $\dot{\hat{\jmath}} = \omega\,\hat{k} \times \hat{\jmath} = -\omega\,\hat{\imath}$, Eq. (6) can be simplified to

$$\vec{a}_A = -\big[\ell\dot{\beta}^2\cos\beta + \omega^2(r_b + \ell\cos\beta)\big]\,\hat{\imath} - 2\omega\ell\dot{\beta}\sin\beta\,\hat{\jmath} + \ell\dot{\beta}^2\sin\beta\,\hat{k}. \tag{7}$$

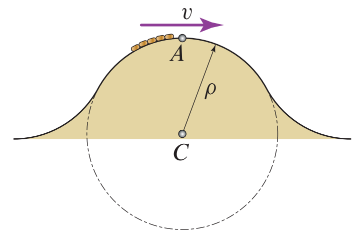
Hence, for $\beta = 0$ the acceleration of A is

$$\vec{a}_A\big|_{\beta=0} = \big[-\ell\dot{\beta}^2 - \omega^2(r_b + \ell)\big]\hat{\imath} \quad \Rightarrow \quad \boxed{\vec{a}_A\big|_{\beta=0} = -108.7\,\hat{\jmath}\,\text{ft/s}^2,} \tag{8}$$

where we have used the data: $r_b = 0.55\,\text{ft}$, $\ell = 2.2\,\text{ft}$, $\omega = 60\,\text{rpm} = 60\frac{2\pi}{60}\,\text{rad/s}$, and $\dot{\beta} = 0.2\,\text{rad/s}$.

Problem 2.292

A roller coaster travels over the top A of the track section shown with a speed $v = 60$ mph. Compute the largest radius of curvature ρ at A such that the passengers on the roller coaster will experience weightlessness at A.



Solution

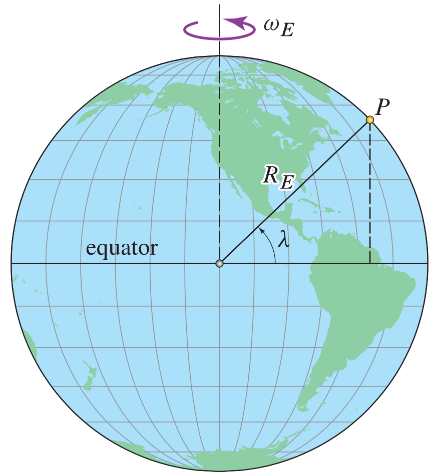
To experience weightlessness, the passengers must be in free fall, i.e., their acceleration must be equal to the acceleration of gravity. Since the tangent to the trajectory at A is horizontal, at A the direction of acceleration must be toward C, i.e., it must be completely in the normal direction. With this in mind, using normal-tangential components and recalling that the normal acceleration is given by v^2/ρ, we must have

$$\frac{v^2}{\rho} = g \quad \Rightarrow \quad \rho = \frac{v^2}{g} \quad \Rightarrow \quad \boxed{\rho = 240.5\,\text{ft},}$$

where we have used the fact that $v = 60\,\text{mph} = 60\frac{5280}{3600}\,\text{ft/s}$ and $g = 32.2\,\text{ft/s}^2$.

Problem 2.293

Determine, as a function of the latitude λ, the normal acceleration of the point P on the surface of the Earth due to the spin ω_E of Earth about its axis. In addition, determine the normal acceleration of the Earth due to its rotation about the Sun. Using these results, determine the latitude above which the acceleration due to the orbital motion of the Earth is more significant than the acceleration due to the spin of the Earth about its axis. Use $R_E = 6371$ km for the mean radius of the Earth, and assume the Earth's orbit about the Sun is circular with radius $R_O = 1.497\times 10^8$ km.



Solution

Assuming that the center of the Earth is fixed and the Earth rotates as shown, the point P moves around a circle centered at the spin axis of the Earth with radius $\rho_P = R_E \cos\lambda$. The speed of $v_P = \rho_P \omega_E \cos\lambda$. Therefore, the normal acceleration of P due to rotation is

$$a_{Pn} = \frac{v_P^2}{\rho_P} = \frac{\omega_E^2 R_E^2 \cos^2\lambda}{R_E \cos\lambda} \quad \Rightarrow \quad \boxed{a_{Pn} = \omega_E^2 R_E \cos\lambda = 0.03369 \cos\lambda \text{ m/s}^2,} \tag{1}$$

where we have used the data $R_E = 6371\text{ km} = 6371\times 10^3$ m and $\omega_E = 1\text{ rev/day} = \frac{2\pi}{(24)(3600)}$ rad/s. We now model the motion of the Earth around the Sun as a circular motion along a circle centered at the Sun with radius R_O and angular speed ω_O. The speed of P due to this motion is the same as that of the Earth, namely, $v_E = R_O\omega_O$. The corresponding normal acceleration is

$$a_{En} = \frac{v_E^2}{\rho_O} = \frac{\omega_O^2 R_O^2}{R_O} \quad \Rightarrow \quad \boxed{a_{En} = \omega_O^2 R_O = 0.005942 \text{ m/s}^2,} \tag{2}$$

where we have used the data $\omega_O = 1\text{ rev/year} = \frac{2\pi}{(365)(24)(3600)}$ rad/s and $R_O = 1.497\times 10^8\text{ km} = 1497\times 10^8$ m.

The latitude λ at which the acceleration due to the motion of the Earth around the Sun is equal to that due to the Earth's spin about its own axis is

$$\omega_E^2 R_E \cos\lambda = \omega_O^2 R_O \quad \Rightarrow \quad \lambda = \cos^{-1}\left(\frac{\omega_O^2 R_O}{\omega_E^2 R_E}\right). \tag{3}$$

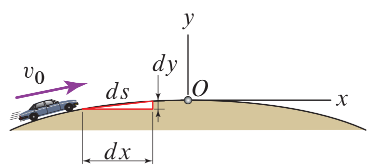
Recalling that $R_E = 6371\text{ km} = 6371\times 10^3$ m, $\omega_E = 1\text{ rev/day} = \frac{2\pi}{(24)(3600)}$ rad/s, $\omega_O = 1\text{ rev/year} = \frac{2\pi}{(365)(24)(3600)}$ rad/s, and $R_O = 1.497\times 10^8\text{ km} = 1497\times 10^8$ m, we can evaluate the last of Eqs. (3) to obtain

$$\boxed{\lambda = 79.84^\circ.}$$

Problem 2.294

A car is traveling over a hill with a speed $v_0 = 160\,\text{km/h}$. Using the Cartesian coordinate system shown, the hill's profile is described by the function $y = -(0.003\,\text{m}^{-1})x^2$, where x and y are measured in meters. At $x = -100\,\text{m}$, the driver realizes that her speed will cause her to lose contact with the ground once she reaches the top of the hill at O. Verify that the driver's intuition is correct, and determine the minimum constant time rate of change of the speed such that the car will not lose contact with the ground at O. *Hint:* To compute the distance traveled by the car along the car's path, observe that $ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + (dy/dx)^2}\,dx$ and that

$$\int \sqrt{1 + C^2x^2}\,dx = \frac{x}{2}\sqrt{1 + C^2x^2} + \frac{1}{2C}\ln\Big(Cx + \sqrt{1 + C^2x^2}\Big).$$



Solution

The minimum speed to lose contact with the ground is such that

$$\frac{v_{\text{min}}^2}{\rho} = g \quad \Rightarrow \quad v_{\text{min}}^2 = g\rho(0). \tag{1}$$

To calculate the radius of curvature at the origin of the coordinate system indicated in the problem's figure we use the following equation:

$$\rho(x) = \frac{\left[1 + (dy/dx)^2\right]^{3/2}}{\left|d^2y/dx^2\right|} \quad \Rightarrow \quad \rho(0) = \frac{\left[1 + (0.006x)^2\right]^{3/2}}{0.006}\Bigg|_{x=0} = 166.7\,\text{m}. \tag{2}$$

Therefore

$$v_{\text{min}} = \sqrt{\rho g} = 40.44\,\text{m/s}. \tag{3}$$

Since $v_0 = 160\,\text{km/h} = 44.44\,\text{m/s}$, we conclude that

The car will lose contact with the ground.

Now we have to find the minimum constant value of $\dot{v} = a_c$ such that the car does not lose contact. Since we need to relate a change in speed to a change in position, we can use the constant acceleration equation $v^2 - v_0^2 = 2a_c(s - s_0)$ with $s_0 = 0$ and $v_f = v_{\text{min}}$, where s is the path coordinate along the profile of the hill and the subscript f stands for 'final.' This gives

$$v_{\text{min}}^2 - v_0^2 = 2a_c s \quad \Rightarrow \quad a_c = \frac{v_{\text{min}}^2 - v_0^2}{2s_f}. \tag{4}$$

To evaluate this equation we need to express the path coordinate s in terms of the Cartesian coordinates x and y. Taking advantage of the hint given in the problem, we can write

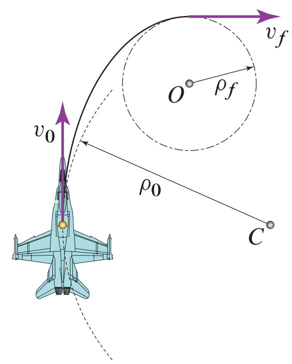
$$s_f = \int_{-100\,\text{m}}^{0} \sqrt{1 + (-0.006x)^2}\,dx = \left[\frac{x}{2}\sqrt{1 + C^2x^2} + \frac{1}{2C}\ln\Big(Cx + \sqrt{1 + C^2x^2}\Big)\right]_{-100\,\text{m}}^{0} = 105.7\,\text{m}, \tag{5}$$

where $C = -0.006\,\mathrm{m}^{-1}$. Using this result in Eq. (4), along with the fact that $v_0 = 160\,\mathrm{km/h} = 44.44\,\mathrm{m/s}$ and $v_{\min} = 40.44\,\mathrm{m/s}$ (see Eq. (3)), we have

$$\boxed{a_c = -1.606\,\mathrm{m/s^2}.}$$

Problem 2.295

A jet is flying straight and level at a speed $v_0 = 1100\,\mathrm{km/h}$ when it turns to change its course by 90° as shown. The turn is performed by decreasing the path's radius of curvature uniformly as a function of the position s along the path while keeping the normal acceleration constant and equal to $8g$, where g is the acceleration due to gravity. At the end of the turn, the speed of the plane is $v_f = 800\,\mathrm{km/h}$. Determine the radius of curvature ρ_f at the end of the turn and the time t_f that the plane takes to complete its change in course.



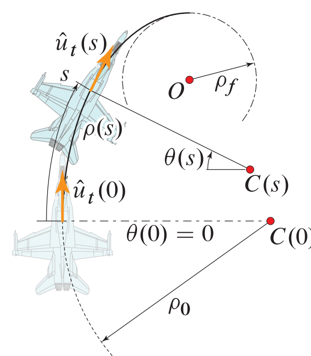
Solution

The radii of curvature at the beginning and end of the maneuver are

$$\rho_0 = \frac{v_0^2}{a_n} \quad \text{and} \quad \rho_f = \frac{v_f^2}{a_n}. \tag{1}$$

Recalling that $a_n = 8g = 8(9.81\,\mathrm{m/s^2})$ and $v_f = 800\,\mathrm{km/s} = 800\frac{1000}{3600}\,\mathrm{m/s}$, the radius of curvature at the end of the turn is

$$\boxed{\rho_f = 629.2\,\mathrm{m}.}$$



Since ρ decreases uniformly with the position s along the airplane's path, we have $d\rho/ds =$ constant. Denoting the (nondimensional) constant in question by γ, we have that the radius of curvature, expressed as a function of s has the following form:

$$\rho(s) = \rho_0 + \gamma s, \tag{2}$$

where γ will be determined later, and where, referring to the figure at the right, $s = 0$ corresponds to the beginning of the turn. At every point on the trajectory, we can find the osculating circle (the circle tangent to the path and center on the concave part of the trajectory). Let $C(s)$ denote the center of the osculating circle at s and consider the radial segment of length $\rho(s)$ going from $C(s)$ to the airplane. Let the orientation of this segment be the angle $\theta(s)$ that the segment forms with the line connecting $C(0)$ and the airplane when $s = 0$ so that $\theta(0) = 0$, $\theta(s_f) = \frac{1}{2}\pi$ rad. The angular velocity $\omega = \dot{\theta}$ is the time rate of change of the orientation of the unit vector $\hat{u}_t$ so that (see Eq. (2.62) on p. 93 of the textbook)

$$\omega(s) = \frac{v(s)}{\rho(s)}, \tag{3}$$

where $v(s)$ is the speed of the plane. Then manipulating the above equation, we have

$$\omega(s) = \frac{v(s)}{\rho(s)} = \frac{d\theta(s)}{dt} = \frac{d\theta(s)}{ds}\frac{ds}{dt} = v(s)\frac{d\theta(s)}{ds} \quad \Rightarrow \quad \frac{d\theta(s)}{ds} = \frac{1}{\rho(s)} \quad \Rightarrow \quad d\theta = \frac{ds}{\rho(s)}, \tag{4}$$

where we have separated the variables θ and s. Using Eq. (2), and recalling that $\theta = 0$ for $s = 0$, and that $\theta = \frac{1}{2}\pi$ rad for $s = s_f$, we can integrate the last of Eqs. (4) as follows:

$$\int_0^{(\pi/2)\,\mathrm{rad}} d\theta = \int_0^{s_f} \frac{ds}{\rho_0 + \gamma s} \quad \Rightarrow \quad \tfrac{1}{2}\pi\,\mathrm{rad} = \frac{1}{\gamma}\ln\left(\frac{\rho_0 + \gamma s_f}{\rho_0}\right) \quad \Rightarrow \quad \tfrac{1}{2}\pi\,\mathrm{rad} = \frac{1}{\gamma}\ln\left(\frac{\rho_f}{\rho_0}\right), \tag{5}$$

which can be solved for γ to obtain

$$\gamma = \frac{2}{\pi\text{ rad}}\ln\left(\frac{\rho_f}{\rho_0}\right) \quad\Rightarrow\quad \gamma = \frac{4}{\pi\text{ rad}}\ln\left(\frac{v_f}{v_0}\right), \tag{6}$$

where we have used Eq. (1) and the following property of logarithms: $\ln x^n = n\ln x$. From Eq. (2) we see that $\gamma = d\rho/ds$. Next recalling that $a_n = 8g$ and $a_n = v^2/\rho$, we can write

$$v^2 = 8g\rho \quad\Rightarrow\quad 2v\frac{dv}{ds} = 8g\frac{d\rho}{ds} \quad\Rightarrow\quad v\frac{dv}{ds} = 4g\gamma. \tag{7}$$

We now observe that $v\frac{dv}{ds}$ is also the quantity $\dot{v} = dv/dt$. Therefore, we can write

$$\frac{dv}{dt} = 4g\gamma \quad\Rightarrow\quad dt = \frac{dv}{4g\gamma}. \tag{8}$$

Letting $t = 0$ correspond to the beginning of the maneuver, we have $v = v_0$ for $t = 0$ and $v = v_f$ for $t = t_f$ so that we can integrate the last of Eqs. (8) as follows:

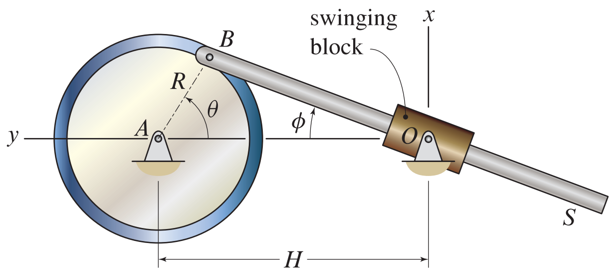
$$\int_0^{t_f} dt = \int_{v_0}^{v_f}\frac{dv}{4g\gamma} \quad\Rightarrow\quad t_f = \frac{v_f - v_0}{4g\gamma} \quad\Rightarrow\quad t_f = \frac{(v_f - v_0)\pi\text{ rad}}{16g\ln(v_f/v_0)}, \tag{9}$$

where we have used the last of Eqs. (6). Recalling that we have $g = 9.81\,\text{m/s}^2$, $v_0 = 1100\,\text{km/h} = 1100\frac{1000}{3600}\,\text{m/s}$, and $v_f = 800\,\text{km/h} = 800\frac{1000}{3600}\,\text{m/s}$, we can evaluate the last of Eqs. (9) to obtain

$$\boxed{t_f = 5.238\,\text{s}.}$$

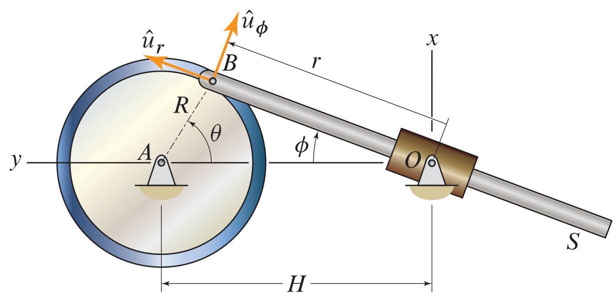
Problem 2.296

The mechanism shown is called a *swinging block* slider crank. First used in various steam locomotive engines in the 1800s, this mechanism is often found in door-closing systems. Let $H = 1.25\,\text{m}$, $R = 0.45\,\text{m}$, and r denote the distance between B and O. Assuming that the speed of B is constant and equal to $5\,\text{m/s}$, determine $\dot{r}$, $\dot{\phi}$, $\ddot{r}$, and $\ddot{\phi}$ when $\theta = 180°$.



Solution

We start by defining a polar coordinate system with origin at O and radial direction along the segment OB so that the coordinate r measures the distance from O to B. We take as transverse coordinate the angle ϕ. Observe that for $0 < \theta < 180°$, r is growing. For $\theta = 180°$, r achieves its maximum value $r_{\text{max}} = R + H$, and for $180° < \theta < 360°$, r decreases. Therefore, the rate of change of r for $\theta = 180°$ must be equal to zero, i.e.,



$$\dot{r}\big|_{\theta=180°} = 0. \tag{1}$$

In polar coordinates, the velocity of B is expressed as

$$\vec{v}_B = \dot{r}\,\hat{u}_r + r\dot{\phi}\,\hat{u}_\phi. \tag{2}$$

For $\theta = 180°$, recalling that B is moving counterclockwise with the constant speed v_0, we have

$$\vec{v}_B\big|_{\theta=180°} = r_{\text{max}}\big(\dot{\phi}\big|_{\theta=180°}\big)\,\hat{u}_\phi = -v_0\,\hat{u}_\phi \quad\Rightarrow\quad \dot{\phi}\big|_{\theta=180°} = -\frac{v_0}{R+H}. \tag{3}$$

Recalling that $v_0 = 5\,\text{m/s}$, $R = 0.45\,\text{m}$, and $H = 1.25\,\text{m}$, we can evaluate the last of Eqs. (3) to obtain

$$\dot{\phi}\big|_{\theta=180°} = -2.941\,\text{rad/s}.$$

Next we recall that the general expression of the acceleration in polar coordinates is

$$\vec{a}_B = (\ddot{r} - r\dot{\phi}^2)\,\hat{u}_r + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\,\hat{u}_\phi. \tag{4}$$

Recalling that, for $\theta = 180°$, $r = R + H$, $\dot{r} = 0$, and $\dot{\phi}$ takes on the expression in the last of Eqs. (3), Eq. (4) reduces to

$$\vec{a}_B\big|_{\theta=180°} = \left(\ddot{r}\big|_{\theta=180°} - \frac{v_0^2}{R+H}\right)\hat{u}_r + \left[(R+H)\ddot{\phi}\big|_{\theta=180°}\right]\hat{u}_\phi. \tag{5}$$

Since B is in uniform circular motion along a circle with center at A, the acceleration of B is always directed toward A and, for $\theta = 180°$, we must have

$$\vec{a}_B\big|_{\theta=180°} = -\frac{v_0^2}{R}\,\hat{u}_r. \tag{6}$$

Setting Eqs. (5) and (6) equal to each other component by component, we have

$$\ddot{r}\big|_{\theta=180°} - \frac{v_0^2}{R+H} = -\frac{v_0^2}{R} \quad \Rightarrow \quad \ddot{r}\big|_{\theta=180°} = v_0^2\left(\frac{1}{R+H} - \frac{1}{R}\right) \tag{7}$$

and

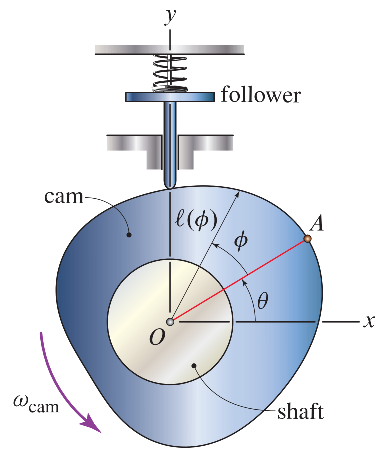
$$(R+H)\ddot{\phi}\big|_{\theta=180°} = 0. \tag{8}$$

Recalling that $v_0 = 5\,\mathrm{m/s}$, $R = 0.45\,\mathrm{m}$, and $H = 1.25\,\mathrm{m}$, we can evaluate the last of Eqs. (7) and Eq. (8) to obtain

$$\boxed{\ddot{r}\big|_{\theta=180°} = -40.85\,\mathrm{m/s^2} \quad \text{and} \quad \ddot{\phi}\big|_{\theta=180°} = 0.}$$

Problem 2.297

The cam is mounted on a shaft that rotates about O with constant angular velocity ω_{cam}. The profile of the cam is described by the function $\ell(\phi) = R_0(1 + 0.25\cos^3\phi)$, where the angle ϕ is measured relative to the segment OA, which rotates with the cam. Letting $R_0 = 3$ cm, determine the maximum value of angular velocity $\omega_{\max}$ such that the maximum speed of the follower is limited to 2 m/s. In addition, compute the smallest angle $\theta_{\min}$ for which the follower achieves it maximum speed.



Solution

Let y denote the position of the follower when in contact with the cam. In addition, let ϕ_f (where the subscript f stands for follower) denote the value of ϕ that identifies the radial line on the cam that goes from point O to the follower. Hence, we have that the relation between ϕ_f and θ is $\phi_f = 90° - \theta$. Keeping in mind that $\dot{y}$ describes the velocity of the follower, we have

$$y = \ell(\phi_f) = R_0(1 + 0.25\cos^3\phi_f) \quad \Rightarrow \quad \dot{y} = \frac{d\ell}{d\phi_f}\frac{d\phi_f}{dt}. \tag{1}$$

Since $\dot{\phi_f} = -\dot{\theta}$ and $\dot{\theta} = \omega_{\text{cam}}$, we have

$$\dot{y} = -\dot{\theta}\frac{d\ell}{d\phi_f} \quad \Rightarrow \quad \dot{y} = -\omega_{\text{cam}}\frac{d\ell}{d\phi_f}. \tag{2}$$

Since ω_{cam} is constant, $\dot{y}$ is maximum when $\frac{d\ell}{d\phi_f}$ is maximum. Hence, taking the derivative of $\ell(\phi_f)$ with respect to ϕ_f, we have

$$\frac{d\ell}{d\phi_f} = -0.75R_0\cos^2\phi_f\sin\phi_f. \tag{3}$$

To maximize $\frac{d\ell}{d\phi_f}$, we differentiate the above quantity with respect to ϕ_f and set the result equal to 0. This gives

$$\frac{d^2\ell}{d\phi_f^2} = 1.5R_0\cos\phi_f\sin^2\phi_f - 0.75R_0\cos^3\phi_f = 0 \quad \Rightarrow \quad \cos^2\phi_f - 2\sin^2\phi_f = 0$$

$$\Rightarrow \quad \sin\phi_f\big|_{\dot{y}_{\max}} = \sqrt{1/3} \quad \text{and} \quad \cos^2\phi_f\big|_{\dot{y}_{\max}} = 2/3. \tag{4}$$

Thus, the maximum magnitude of Eq. (3) and the maximum magnitude of Eq. (2) are, respectively,

$$\left|\frac{d\ell}{d\phi_f}\right|_{\max} = \frac{R_0}{2\sqrt{3}} \quad \text{and} \quad |\dot{y}_{\max}| = \frac{\omega_{\text{cam}}R_0}{2\sqrt{3}}. \tag{5}$$

Setting $|\dot{y}_{\max}| = v_{\max}$ and solving for ω_{cam}, we have

$$\omega_{\max} = \frac{v_{\max}2\sqrt{3}}{R_0} \quad \Rightarrow \quad \boxed{\omega_{\max} = 230.9\,\text{rad/s},} \tag{6}$$

where we have used the data $v_{\max} = 2\,\text{m/s}$ and $R_0 = 3\,\text{cm} = 0.03000\,\text{m}$. The minimum angle θ for which the follower achieves its maximum speed is obtained by recalling that $\phi_f = 90° - \theta$ and that, by the first of Eqs. (4), we have

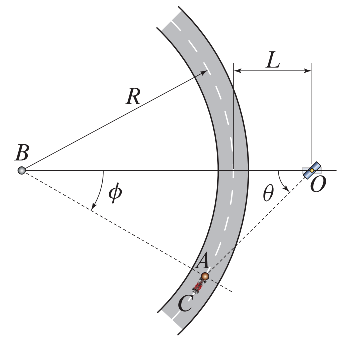
$$\sin\phi_f = \sqrt{1/3} \quad \Rightarrow \quad \phi = \sin^{-1}\sqrt{1/3}\Big|_{\dot{y}_{\max}} = 90° - \theta \quad \Rightarrow \quad \theta_{\min} = 90° - \sin^{-1}\sqrt{1/3}, \tag{7}$$

which can be evaluated to obtain

$$\boxed{\theta_{\min} = 54.74°.}$$

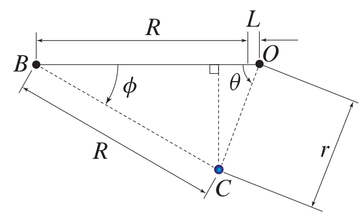
Problem 2.298

A car is traveling at a constant speed $v_0 = 210\,\mathrm{km/h}$ along a circular turn with radius $R = 137\,\mathrm{m}$ (the figure is not to scale). The camera at O is tracking the motion of the car. Letting $L = 15\,\mathrm{m}$, determine the camera's rotation rate, as well as the corresponding time rate of change of the rotation rate when $\phi = 30°$.



Solution

The orientation of the camera is described by the angle θ shown in the figure at the right. Let r represent the distance from O to the car. From the geometry of the figure we have



$$\tan\theta = \frac{R\sin\phi}{R+L-R\cos\phi} \quad\Rightarrow\quad \theta = \tan^{-1}\left(\frac{R\sin\phi}{R+L-R\cos\phi}\right). \tag{1}$$

Differentiating the first of Eqs. (1) with respect to time, we have

$$\frac{\dot\theta}{\cos^2\theta} = \frac{R\cos\phi(R+L-R\cos\phi)-R^2\sin^2\phi}{(R+L-R\cos\phi)^2}\dot\phi \quad\Rightarrow\quad \dot\theta = \frac{[(RL+R^2)\cos\phi - R^2]\cos^2\theta}{(R+L-R\cos\phi)^2}\dot\phi. \tag{2}$$

Since the car is in uniform circular motion with constant speed v_0, we have

$$\dot\phi = -v_0/R, \tag{3}$$

where the minus sign is due to the fact that the car is moving counterclockwise. Substituting Eq. (3) in Eq. (1) gives

$$\dot\theta = \frac{v_0}{R}\frac{[R^2-(RL+R^2)\cos\phi]\cos^2\theta}{(R+L-R\cos\phi)^2}. \tag{4}$$

Recalling that $R = 137\,\mathrm{m}$, $L = 15\,\mathrm{m}$, and $v_0 = 210\,\mathrm{km/h} = 210\frac{1000}{3600}\,\mathrm{m/s}$,for $\phi = 30°$ we can first evaluate θ from the second of Eqs. (1) (this gives $\theta = 64.04°$) and then evaluate $\dot\theta$ in Eq. (4) to obtain

$$\boxed{\dot\theta = 0.05391\,\mathrm{rad/s}.}$$

Differentiating Eq. (4) with respect to time we have

$$\ddot\theta = -\frac{v_0}{R}2\dot\theta\cos\theta\sin\theta\frac{[R^2-(RL+R^2)\cos\phi]}{(R+L-R\cos\phi)^2} + \frac{v_0}{R}\cos^2\theta\frac{d}{d\phi}\left\{\frac{[R^2-(RL+R^2)\cos\phi]}{(R+L-R\cos\phi)^2}\right\}\dot\phi. \tag{5}$$

Carrying out the differentiation with respect to ϕ of the term in braces, we have

$$\frac{d}{d\phi}\left\{\frac{[R^2-(RL+R^2)\cos\phi]}{(R+L-R\cos\phi)^2}\right\} = \frac{(RL+R^2)\sin\phi}{(R+L-R\cos\phi)^2} - 2\frac{R\sin\phi[R^2-(RL+R^2)\cos\phi]}{(R+L-R\cos\phi)^3}. \tag{6}$$

Recalling that $\dot{\phi}$ is given in Eq. (3), and substituting Eqs. (4) and (6) into Eq. (5), we obtain

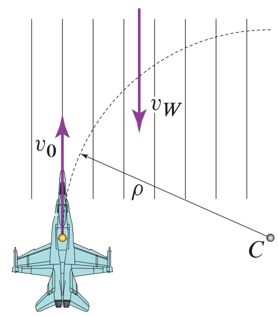
$$\ddot{\theta} = -\frac{v_0^2}{R^2}\cos^3\theta\sin\theta\frac{[R^2 - (RL + R^2)\cos\phi]^2}{(R + L - R\cos\phi)^4} - \frac{v_0^2}{R^2}\cos^2\theta\left\{\frac{(RL + R^2)\sin\phi}{(R + L - R\cos\phi)^2} - 2\frac{R\sin\phi[R^2 - (RL + R^2)\cos\phi]}{(R + L - R\cos\phi)^3}\right\}. \quad (7)$$

Recalling again that $R = 137\,\text{m}$, $L = 15\,\text{m}$, and $v_0 = 210\,\text{km/h} = 210\frac{1000}{3600}\,\text{m/s}$, for $\phi = 30°$, again we can first evaluate θ from the second of Eqs. (1) (this gives $\theta = 64.04°$), and then evaluate $\ddot{\theta}$ in Eq. (7) to obtain

$$\boxed{\ddot{\theta} = -0.2380\,\text{rad/s}^2.}$$

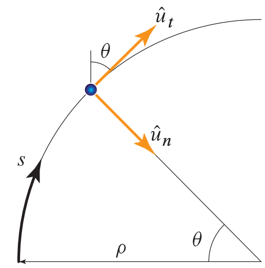
Problem 2.299

A plane is initially flying north with a speed $v_0 = 430$ mph relative to the ground while the wind has a constant speed $v_W = 12$ mph in the north-south direction. The plane performs a circular turn with radius of $\rho = 0.45$ mi. Assume that the airspeed indicator on the plane measures the absolute value of the component of the relative velocity of the plane with respect to the air in the direction of motion. Then determine the value of the tangential component of the airplane's acceleration when the airplane is halfway through the turn, assuming that the airplane maintains constant the reading of the airspeed indicator.



Solution

Let the subscripts P and W denote quantities pertaining to the airplane and wind, respectively. Then, referring to the figure on the right and using a normal tangential component system, the velocity of the airplane and wind are



$$\vec{v}_P = v_P\,\hat{u}_t \quad \text{and} \quad \vec{v}_W = v_W(-\cos\theta\,\hat{u}_t + \sin\theta\,\hat{u}_n). \tag{1}$$

Therefore the velocity of the airplane relative to the wind is

$$\vec{v}_{P/W} = (v_P + v_W\cos\theta)\,\hat{u}_t - v_W\sin\theta\,\hat{u}_n. \tag{2}$$

Now we recall that the quantity measured by the airspeed indicator is the component of $\vec{v}_{P/W}$ that is in the direction of motion, i.e.,

$$v_{\text{ai}} = |\vec{v}_{P/A}\cdot\hat{u}_t| = v_P + v_W\cos\theta,$$

where the subscript 'ai' stands for 'airspeed indicator.' Now, recalling that the measure of θ in radians is given by $\theta = s/\rho$, we can rewrite v_{ai} as follows:

$$v_{\text{ai}} = v_P + v_W\cos\frac{s}{\rho}. \tag{3}$$

Observing that for $\theta = 0$ we have $v_{\text{ai}} = v_0 + v_W$ and recalling that v_{ai} is maintained constant along the turn, we can solve Eq. (3) for v_P as a function of s to obtain

$$v_P(s) = v_0 + v_W\left(1 - \cos\frac{s}{\rho}\right). \tag{4}$$

The tangential component of acceleration is the time derivative of Eq. (4). Hence, we can write

$$a_t = \frac{dv_P}{ds}\frac{ds}{dt} = v_P\frac{dv_P}{ds} \quad \Rightarrow \quad a_t = \left[v_0 + v_W\left(1 - \cos\frac{s}{\rho}\right)\right]\frac{v_W}{\rho}\sin\frac{s}{\rho}. \tag{5}$$

For $s/\rho = (\pi/4)$ rad, we have $\sin(s/\rho) = \cos(s/\rho) = \sqrt{2}/2$. Therefore, midway through the turn, we have

$$a_t = \left[v_0 + v_W\left(1 - \frac{\sqrt{2}}{2}\right)\right]\frac{v_W\sqrt{2}}{2\rho}. \tag{6}$$

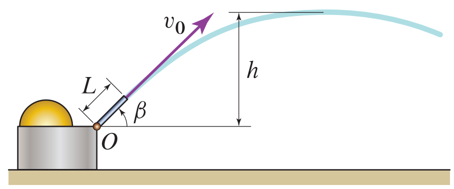
Recalling that $v_0 = 430\,\text{mph} = 430\frac{5280}{3600}\,\text{ft/s}$, $v_W = 12\,\text{mph} = 12\frac{5280}{3600}\,\text{ft/s}$, and $\rho = 0.45\,\text{mi} = 0.45(5280)\,\text{ft}$, we can evaluate the above expression to obtain

$$\boxed{a_t = 3.330\,\text{ft/s}^2.}$$

Problem 2.300

A fountain has a spout that can rotate about O and whose angle β is controlled so as to vary with time according to $\beta = \beta_0[1 + \sin^2(\omega t)]$, with $\beta_0 = 15°$ and $\omega = 0.4\pi$ rad/s. The length of the spout is $L = 1.5$ ft, the water flow through the spout is constant, and the water is ejected at a speed $v_0 = 6$ ft/s, measured relative to the spout.

Determine the largest speed with which the water particles are released from the spout.



Solution

We define a polar coordinate system with origin at O and transverse coordinate coinciding with β. The expression of the velocity of the water particles as they leave the spout is given by

$$\vec{v} = \dot{r}\,\hat{u}_r + L\dot{\beta}\,\hat{u}_\beta \quad \Rightarrow \quad v = \sqrt{\dot{r}^2 + (L\dot{\beta})^2}. \tag{1}$$

where, based on the problem statement,

$$\dot{r} = v_0 = 6\,\text{ft/s} \quad \text{and} \quad \dot{\beta} = 2\beta_0\omega\sin\omega t\cos\omega t = \beta_0\omega\sin(2\omega t), \tag{2}$$

where we have used the trigonometric identity $2\sin x\cos x = \sin(2x)$. Substituting Eqs. (2) into the last of Eqs. (1) we obtain the following expression for the speed:

$$v = \sqrt{v_0^2 + [L\beta_0\omega\sin(2\omega t)]^2}. \tag{3}$$

From the above expression, recalling that v_0, β_0, and ω are constant, we see that v is maximum when $\sin(2\omega t)$ is maximum. Since the sine function has maximum value equal to 1, we have

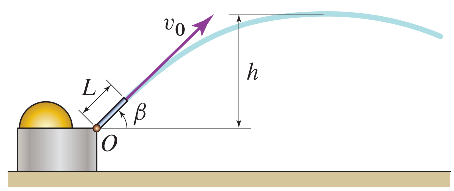
$$v_{max} = \sqrt{v_0^2 + L^2\beta_0^2\omega^2} \quad \Rightarrow \quad \boxed{v_{max} = 6.020\,\text{ft/s},} \tag{4}$$

where we have used the following numerical data: $v_0 = 6$ ft/s, $L = 1.5$ ft, $\beta_0 = 15°$, and $\omega = 0.4\pi$ rad/s.

Problem 2.301

A fountain has a spout that can rotate about O and whose angle β is controlled so as to vary with time according to $\beta = \beta_0[1 + \sin^2(\omega t)]$, with $\beta_0 = 15°$ and $\omega = 0.4\pi$ rad/s. The length of the spout is $L = 1.5$ ft, the water flow through the spout is constant, and the water is ejected at a speed $v_0 = 6$ ft/s, measured relative to the spout.

Determine the magnitude of the acceleration immediately before release when $\beta = 15°$.



Solution

We define a polar coordinate system with origin at O and transverse coordinate coinciding with β. The expression of the acceleration in polar coordinates is

$$\vec{a} = (\ddot{r} - r\dot{\beta}^2)\,\hat{u}_r + (r\ddot{\beta} + 2\dot{r}\dot{\beta})\,\hat{u}_\beta. \tag{1}$$

Now, recalling that v_0 is constant, when the water particles leave the spout we have

$$r = L,\ \dot{r} = v_0,\ \ddot{r} = 0,\ \dot{\beta} = 2\beta_0\omega\sin\omega t\cos\omega t = \beta_0\omega\sin(2\omega t),\ \text{and}\ \ddot{\beta} = 2\omega^2\beta_0\cos(2\omega t), \tag{2}$$

where we have used the trigonometric identity $2\sin x\cos x = \sin(2x)$. Now, we observe that $\beta_0 = 15°$, and therefore for $\beta = 15°$, we must have $\beta = \beta_0$ and $\sin(\omega t) = 0$, which implies $\sin(2\omega t) = 0$ and $\cos(2\omega t) = 1$. Hence, for $\beta = 15°$, we have

$$r = L, \quad \dot{r} = v_0, \quad \ddot{r} = 0, \quad \dot{\beta} = 0, \quad \text{and} \quad \ddot{\beta} = 2\omega^2\beta_0, \tag{3}$$

so that, for $\beta = 15°$, the acceleration takes the form

$$\vec{a} = 2L\omega^2\beta_0\,\hat{u}_\beta. \tag{4}$$

Consequently, for $\beta = 15°$, we have

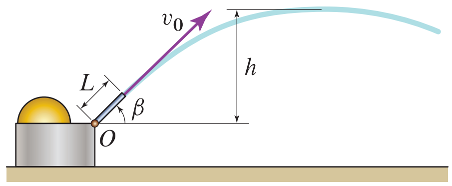
$$|\vec{a}| = 2L\omega^2\beta_0 \quad \Rightarrow \quad \boxed{|\vec{a}| = 1.240\ \text{ft/s}^2,} \tag{5}$$

where we have used the following numerical data: $L = 1.5$ ft, $\beta_0 = 15°$, and $\omega = 0.4\pi$ rad/s.

Problem 2.302

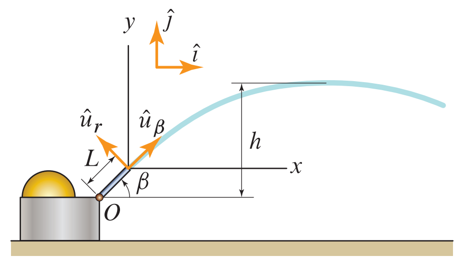
A fountain has a spout that can rotate about O and whose angle β is controlled so as to vary with time according to $\beta = \beta_0[1 + \sin^2(\omega t)]$, with $\beta_0 = 15°$ and $\omega = 0.4\pi$ rad/s. The length of the spout is $L = 1.5$ ft, the water flow through the spout is constant, and the water is ejected at a speed $v_0 = 6$ ft/s, measured relative to the spout.

Determine the highest position reached by the resulting water arc.



Solution

Referring to the figure at the right, in the solution of this problem we will use two coordinate systems. The first is a polar coordinate system with origin at O and transverse coordinate β. Let $\bar{\beta}$ be the value of β corresponding to the trajectory of the water jet achieving the maximum height. Then the second is a fixed Cartesian coordinate system with origin coinciding with the position of the mouth of the spout corresponding to $\bar{\beta}$. Letting v denote the speed of the water particles at the spout's mouth, then the velocity of the water particles at the spout's mouth is



$$\vec{v} = v(\cos\beta\,\hat{\imath} + \sin\beta\,\hat{\jmath}). \tag{1}$$

Once the water particles leave the spout, they are in projectile motion and the trajectory with the maximum height is determined both by the vertical component of the speed, i.e,

$$v_y = v\sin\beta, \tag{2}$$

and the vertical position of mouth of the water spout.

To derive an expression for the speed v of the water particles as they leave the spout, we use the chosen polar coordinate system, for which the velocity at the mouth of the spout is expressed as

$$\vec{v} = \dot{r}\,\hat{u}_r + L\dot{\beta}\,\hat{u}_\beta \quad \Rightarrow \quad v = \sqrt{\dot{r}^2 + (L\dot{\beta})^2}. \tag{3}$$

where, based on the problem statement,

$$\dot{r} = v_0 = 6\,\text{ft/s} \quad \text{and} \quad \dot{\beta} = 2\beta_0\omega\sin\omega t\cos\omega t = \beta_0\omega\sin(2\omega t), \tag{4}$$

where we have used the trigonometric identity $2\sin x\cos x = \sin(2x)$. Substituting Eqs. (2) into the last of Eqs. (3) we obtain the following expression for the speed:

$$v = \sqrt{v_0^2 + [L\beta_0\omega\sin(2\omega t)]^2}. \tag{5}$$

Consequently, the vertical component of velocity at the spout is

$$v_y = \sqrt{v_0^2 + [L\beta_0\omega\sin(2\omega t)]^2}\,\sin\left[\beta_0(1 + \sin^2\omega t)\right]. \tag{6}$$

Now let $h = h_1 + h_2$ where h_1 is the height of the mouth of the spout with respect to the base of the spout, and where h_2 is the maximum elevation of the water jet measured from the horizontal line going through the mouth of the spout. Hence, for h_1 we have

$$h_1 = L \sin\beta = L \sin\big[\beta_0(1 + \sin^2\omega t)\big]. \tag{7}$$

By contrast, h_2 is found using the constant acceleration equation $v^2 = v_0^2 + 2a(s - s_0)$ in the vertical direction. Recalling that $v_y = 0$ for $y = h_2$, we have

$$0 = \left\{\sqrt{v_0^2 + [L\beta_0\omega\sin(2\omega t)]^2}\,\sin\big[\beta_0(1 + \sin^2\omega t)\big]\right\}^2 - 2gh_2,$$

which can be solved for h_2 to obtain

$$h_2 = \frac{1}{2g}\left\{v_0^2 + [L\beta_0\omega\sin(2\omega t)]^2\right\}\sin^2\big[\beta_0(1 + \sin^2\omega t)\big]. \tag{8}$$

Consequently, the expression for the height of the water jet as a function of time is

$$h = L\sin\big[\beta_0(1 + \sin^2\omega t)\big] + \frac{1}{2g}\left\{v_0^2 + [L\beta_0\omega\sin(2\omega t)]^2\right\}\sin^2\big[\beta_0(1 + \sin^2\omega t)\big]. \tag{9}$$

This function needs to be maximized. This can be done by differentiating h with respect to time and setting the result equal to zero. Doing so, after simplification, yields the following equation:

$$\frac{\beta_0\omega}{2g}\sin(2\omega t)\Big\{2Lg\cos\big[\beta_0(1 + \sin^2\omega t)\big] + 2L\beta_0\cos(2\omega t)\sin^2\big[\beta_0(1 + \sin^2\omega t)\big] + [v_0^2 + L^2\beta_0^2 + \omega^2\sin^2(2\omega t)]\sin\big[2\beta_0(1 + \sin^2\omega t)\big]\Big\} = 0 \tag{10}$$

Recalling that we have $L = 1.5\,\text{ft}$, $\beta_0 = 15^\circ$, $\omega = 0.4\pi\,\text{rad/s}$, and $g = 32.2\,\text{ft/s}^2$, and although this may require plotting the terms within braces as a function of time, it turns out that the term within braces can never be equal to zero. Hence, the solution of the above equation reduces to the solution of the equation

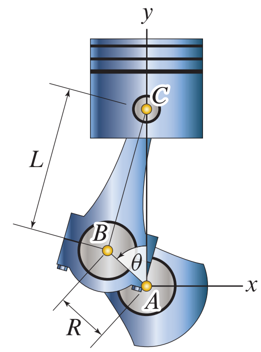
$$\sin(2\omega t) = 0 \quad\Rightarrow\quad t = 0 \pm \frac{n\pi}{\omega} \quad\text{and}\quad t = \frac{\pi}{2\omega} \pm \frac{n\pi}{\omega}, \quad\text{with}\quad n = 0, 1, 2, \ldots \tag{11}$$

Since the function $\beta = \beta_0(1 + \sin^2\omega t)$ is at a maximum for $t = \frac{\pi}{2\omega} \pm \frac{n\pi}{\omega}$ the function h will also achieve its maxima for $t = \frac{\pi}{2\omega} \pm \frac{n\pi}{\omega}$. In addition, since the function h is a periodic function with period $\frac{\pi}{\omega}$, the values of h for $t = \frac{\pi}{2\omega} \pm \frac{n\pi}{\omega}$ are all identical to one another and we can therefore evaluate $h_{\max}$ by simply letting $t = \frac{\pi}{2\omega}$, i.e., for $n = 0$. Hence, recalling that $L = 1.5\,\text{ft}$, $\beta_0 = 15^\circ$, $\omega = 0.4\pi\,\text{rad/s}$, and $g = 32.2\,\text{ft/s}^2$, for $t = \frac{\pi}{2\omega} = 1.250\,\text{s}$, we can evaluate h in Eq. (9) to obtain

$$\boxed{h_{\max} = 0.8898\,\text{ft}.}$$

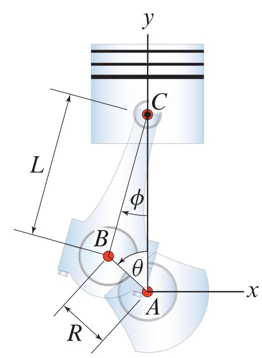
Problem 2.303

The piston head at C is constrained to move along the y axis. Let the crank AB be rotating counterclockwise at a constant angular speed $\dot{\theta} = 2000\,\text{rpm}$, $R = 3.5\,\text{in.}$, and $L = 5.3\,\text{in.}$ Obtain the angular velocity of the connecting rod BC by differentiating the relative position vector of C with respect to B when $\theta = 35°$. *Hint:* You will also need to determine the velocity of B and enforce the constraint that demands that C move only along the y axis.



Solution

Referring to the figure at the right, we can describe the position of C relative to B as follows:



$$\vec{r}_{C/B} = L\,\hat{u}_{C/B}. \tag{1}$$

Then, using the concept of time derivative of a vector, we have that

$$\vec{v}_{C/B} = L\,\dot{\hat{u}}_{C/B} = L\vec{\omega}_{BC} \times \hat{u}_{C/B}, \tag{2}$$

where $\vec{\omega}_{BC}$ is the angular velocity of the vector $\vec{r}_{C/B}$ and therefore the angular velocity of the connecting rod. Next, we note that

$$\vec{\omega}_{BC} = -\dot{\phi}\,\hat{k} \quad \text{and} \quad \hat{u}_{C/B} = \sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath}. \tag{3}$$

Substituting Eqs. (3) into Eq. (2) and carrying out the cross-product, we have

$$\vec{v}_{C/B} = L\,\dot{\hat{u}}_{C/B} = L\dot{\phi}(\cos\phi\,\hat{\imath} - \sin\phi\,\hat{\jmath}). \tag{4}$$

To enforce the condition that C can only move in the y direction, we must compute the velocity of B and then apply the relative kinematics equation $\vec{v}_C = \vec{v}_B + \vec{v}_{C/B}$. Since B is in uniform circular motion along a circle of radius R and center A, we have

$$\vec{v}_B = -\dot{\theta}R(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}). \tag{5}$$

Therefore, combining Eqs. (4) and (5), we have

$$\vec{v}_C = \vec{v}_B + \vec{v}_{C/B} = (L\dot{\phi}\cos\phi - \dot{\theta}R\cos\theta)\,\hat{\imath} - (L\dot{\phi}\sin\phi + \dot{\theta}R\sin\theta)\,\hat{\jmath}. \tag{6}$$

Since $v_{Cx} = 0$, we have

$$L\dot{\phi}\cos\phi - \dot{\theta}R\cos\theta = 0 \quad \Rightarrow \quad \dot{\phi} = \frac{R\cos\theta}{L\cos\phi}\dot{\theta} \quad \Rightarrow \quad \vec{\omega}_{BC} = -\frac{R\cos\theta}{L\cos\phi}\dot{\theta}\,\hat{k}, \tag{7}$$

where we have used the definition in the first of Eqs. (3). In order to complete our calculation, we need to determine the angle ϕ. Using trigonometry, we see that

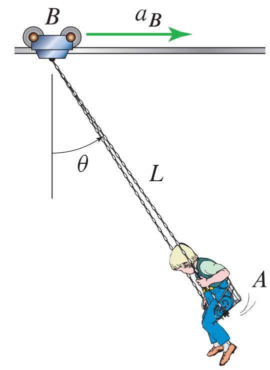
$$L\sin\phi = R\sin\theta \quad \Rightarrow \quad \phi = \sin^{-1}[(R/L)\sin\theta]. \tag{8}$$

For $\theta = 35°$ and recalling that $R = 3.5\,\text{in.}$ and $L = 5.3\,\text{in.}$, we can first compute the angle ϕ from the last of Eqs. (8) (this gives $\phi = 22.26°$), and then, using $\dot{\theta} = 2000\,\text{rpm}$, we can evaluate the last of Eqs. (7) to obtain

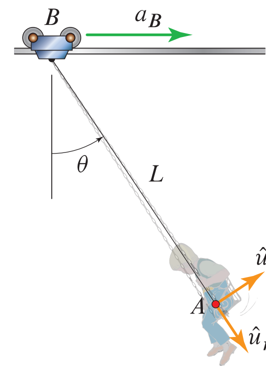
$$\boxed{\vec{\omega}_{BC} = (-1169\,\text{rpm})\,\hat{k}.}$$

Problem 2.304

A child A is swinging from a swing that is attached to a trolley that is free to move along a fixed rail. Letting $L = 3$ m, if at a given instant $a_B = 47.98$ m/s^2, $\theta = 23°$, $\dot{\theta} = -3.512$ rad/s, and $\ddot{\theta} = -16$ rad/s^2, determine the magnitude of the acceleration of the child relative to the rail at that instant.



Solution



The acceleration of A relative to B is most easily described by setting up a polar coordinate system with origin at B, radial coordinate r going from B to A, and transverse coordinate θ. Recalling that the expression of the angular acceleration in polar coordinates is $\vec{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{u}_r = (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{u}_\theta$ and observing that the length of the rope is constant so that $r = L$, we have

$$\vec{a}_{A/B} = -L\dot{\theta}^2\,\hat{u}_r + L\ddot{\theta}\,\hat{u}_\theta, \tag{1}$$

The acceleration of the trolley can be described using the chosen component system as follows:

$$\vec{a}_B = a_B(\sin\theta\,\hat{u}_r + \cos\theta\,\hat{u}_\theta). \tag{2}$$

Hence, using relative kinematics, the acceleration of A relative to the fixed rail is

$$\vec{a}_A = \left(a_B\sin\theta - L\dot{\theta}^2\right)\hat{u}_r + \left(a_B\cos\theta + L\ddot{\theta}\right)\hat{u}_\theta. \tag{3}$$

Consequently, we have

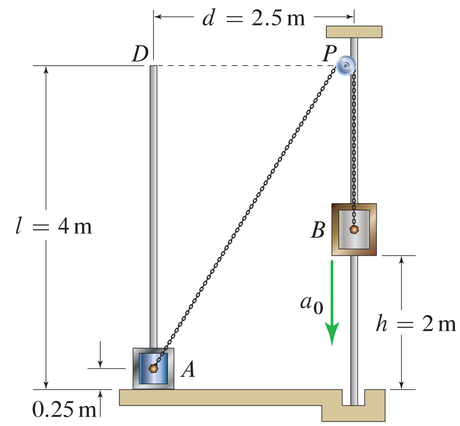
$$|\vec{a}_A| = \sqrt{\left(a_B\sin\theta - L\dot{\theta}^2\right)^2 + \left(a_B\cos\theta + L\ddot{\theta}\right)^2}. \tag{4}$$

Recalling that $L = 3$ m, $a_B = 47.98$ m/s^2, $\theta = 23°$, $\dot{\theta} = -3.512$ rad/s, and $\ddot{\theta} = -16$ rad/s^2, we can evaluate the above expression to obtain

$$|\vec{a}_A| = 18.65\,\text{m/s}^2.$$

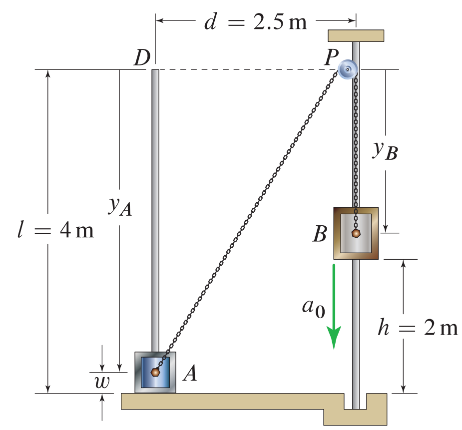
Problem 2.305

Block B is released from rest at the position shown, and it has a constant acceleration downward $a_0 = 5.7\,\mathrm{m/s^2}$. Determine the velocity and acceleration of block A at the instant that B touches the floor.



Solution

The length of the rope is



$$L = \sqrt{d^2 + y_A^2} + y_B. \tag{1}$$

For $t = 0$, we have that $y_A(0) = l - w$. Letting t_f represent the time at the final position, the length of the rope at the initial and final positions is

$$L = \sqrt{d^2 + y_A^2(0)} + y_B(0) \quad \text{and} \quad L = \sqrt{d^2 + y_A^2(t_f)} + y_B(t_f). \tag{2}$$

Equating these two expressions for L and rearranging terms, we have

$$\sqrt{d^2 + y_A^2(t_f)} = \sqrt{d^2 + y_A^2(0)} - [y_B(t_f) - y_B(0)] = 0. \tag{3}$$

Ssquaring both sides of the above equation, with $y_B(t_f) - y_B(0) = h$, we obtain

$$d^2 + y_A^2(t_f) = d^2 + y_A^2(0) - 2h\sqrt{d^2 + y_A^2(0)} + h^2. \tag{4}$$

Solving the above equation for $y_A(t_f)$, we have

$$y_A(t_f) = \sqrt{y_A^2(0) + h^2 - 2h\sqrt{d^2 + y_A^2(0)}} = 0.1864\,\mathrm{m}, \tag{5}$$

where we have used the following numerical data: $y_A(0) = l - w = 3.75\,\mathrm{m}$, $h = 2\,\mathrm{m}$, and $d = 2.5\,\mathrm{m}$. Next, differentiating Eq. (1) with respect to time and solving for $\dot{y}_A$, we have

$$0 = \frac{y_A \dot{y}_A}{\sqrt{d^2 + y_A^2}} + \dot{y}_B \quad \Rightarrow \quad \dot{y}_A = \frac{-\dot{y}_B}{y_A}\sqrt{d^2 + y_A^2}. \tag{6}$$

Now, we use the constant acceleration equation $\dot{s}^2 = \dot{s}_0^2 + 2a_c(s - s_0)$ to find $\dot{y}_B$ after B has traveled a distance h. This gives

$$\dot{y}_B^2 = 2a_0 h \quad \Rightarrow \quad \dot{y}_B = \sqrt{2a_0 h} = 4.775\,\mathrm{m/s}, \tag{7}$$

where we have used the fact that $\ddot{y}_B = a_0 = 5.7\,\mathrm{m/s^2}$ and $h = 2\,\mathrm{m}$. Then, using Eq. (5) and the last of Eqs. (7), and recalling that $d = 2.5\,\mathrm{m}$, we can evaluate Eq. (6) to obtain for $t = t_f$ to obtain

$$\boxed{\dot{y}_A(t_f) = -64.22\,\mathrm{m/s}.} \tag{8}$$

Now we differentiate Eq. (6) with respect to time to obtain

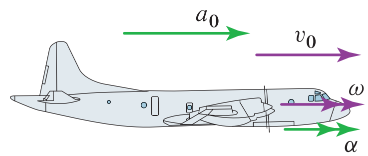
$$\ddot{y}_A = \frac{-\ddot{y}_B}{y_A}\sqrt{d^2 + y_A^2} + \frac{\dot{y}_B\,\dot{y}_A}{y_A^2}\sqrt{d^2 + y_A^2} - \frac{\dot{y}_B\,\dot{y}_A}{\sqrt{d^2 + y_A^2}}.$$

Recalling that $d = 2.5\,\mathrm{m}$ and $\ddot{y}_B = a_0 = 5.7\,\mathrm{m/s^2}$, and using the (full precision) value of $y_A(t_f)$ in Eq. (5), the (full precision) value of $\dot{y}_A(t_f)$ in Eq. (8), and the (full precision) value of $\dot{y}_B(t_f)$ in the last of Eqs. (7), we can evaluate the above expression for $t = t_f$ to obtain

$$\boxed{\ddot{y}_A(t_f) = -22{,}080\,\mathrm{m/s^2}.}$$

Problem 2.306

At a given instant, an airplane is flying horizontally with speed $v_0 = 290$ mph and acceleration $a_0 = 12\,\text{ft/s}^2$. At the same time, the airplane's propellers rotate at an angular speed $\omega = 1500$ rpm while accelerating at a rate $\alpha = 0.3\,\text{rad/s}^2$. Knowing that the propeller diameter is $d = 14$ ft, determine the magnitude of the acceleration of a point on the periphery of the propellers at the given instant.



Solution

Using a cylindrical coordinate system with origin at the propeller's axis of rotation and z axis in the direction of motion, the general expression for acceleration is

$$\vec{a} = \left(\ddot{R} - R\dot{\theta}^2\right)\hat{u}_R + \left(R\ddot{\theta} + 2\dot{R}\dot{\theta}\right)\hat{u}_\theta + \ddot{z}\,\hat{u}_z.$$

For the propeller tip we have $R = d/2$, $\dot{R} = 0$, $\ddot{R} = 0$, $\ddot{z} = a_0$, $\dot{\theta} = \omega$, and $\ddot{\theta} = \alpha$. Thus, the acceleration is

$$\vec{a} = -\frac{d\omega^2}{2}\,\hat{u}_R + \frac{d\alpha}{2}\,\hat{u}_\theta + a_0\,\hat{u}_z,$$

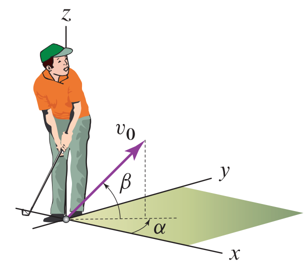
and the magnitude of the acceleration is

$$|\vec{a}| = \sqrt{\left(-\frac{d\omega^2}{2}\right)^2 + \left(\frac{d\alpha}{2}\right)^2 + a_0^2} \quad\Rightarrow\quad \boxed{|\vec{a}| = 172{,}700\,\text{ft/s}^2,}$$

where we have used the data $a_0 = 12\,\text{ft/s}^2$, $\omega = 1500\,\text{rpm} = 1500\frac{2\pi}{60}\,\text{rad/s}$, $\alpha = 0.3\,\text{rad/s}^2$, and $d = 14$ ft.

Problem 2.307

A golfer chips the ball as shown. Treating α, β, and the initial speed v_0 as given, find an expression for the radius of curvature of the ball's trajectory as a function of time and the given parameters. *Hint:* Use the Cartesian coordinate system shown to determine the acceleration and the velocity of the ball. Then reexpress these quantities, using normal-tangential components.



Solution

This is 3-D projectile motion. We will follow the hint and develop equations both in Cartersian components as well as in normal-tangential components. In Cartesian coordinates the components of acceleration are

$$\ddot{x} = 0, \quad \ddot{y} = 0, \quad \text{and} \quad \ddot{z} = -g.$$

Using constant acceleration equations, the velocity vector expressed in the $(\hat{\imath}, \hat{\jmath}, \hat{k})$ and $(\hat{u}_t, \hat{u}_n, \hat{u}_b)$ component systems are

$$\vec{v} = v_0 \cos\beta \cos\alpha\, \hat{\imath} + v_0 \cos\beta \sin\alpha\, \hat{\jmath} + (v_0 \sin\beta - gt)\, \hat{k},$$
$$\vec{v} = v\, \hat{u}_t,$$

where, v is the speed, which can be given the form

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2} \quad \Rightarrow \quad v = \sqrt{v_0^2 + g^2t^2 - 2v_0 gt \sin\beta}, \tag{1}$$

and where the tangent unit vector $\hat{u}_t$ can be related to the base vectors of the Cartesian component system as follows:

$$\hat{u}_t = \frac{v_0 \cos\beta \cos\alpha}{v}\, \hat{\imath} + \frac{v_0 \cos\beta \sin\alpha}{v}\, \hat{\jmath} + \frac{v_0 \sin\beta - gt}{v}\, \hat{k}. \tag{2}$$

To express the acceleration in normal-tangential components, we will need an expression for the time derivative of the speed. Hence, we proceed to differentiate v with respect to time, to obtain

$$\dot{v} = \frac{g^2 t - v_0 g \sin\beta}{\sqrt{v_0^2 + g^2t^2 - 2v_0 gt \sin\beta}} = \frac{g}{v}\big(gt - v_0 \sin\beta\big). \tag{3}$$

The acceleration vector expressed in the $(\hat{\imath}, \hat{\jmath}, \hat{k})$ and $(\hat{u}_t, \hat{u}_n, \hat{u}_b)$ component systems are

$$\vec{a} = \dot{v}\, \hat{u}_t + \frac{v^2}{\rho}\, \hat{u}_n \quad \text{and} \quad \vec{a} = -g\, \hat{k}.$$

Equating the above expressions for the acceleration, we obtain an expression for the normal unit vector

$$\hat{u}_n = -\frac{\rho}{v^2}\big(\dot{v}\, \hat{u}_t + g\, \hat{k}\big). \tag{4}$$

Using Eq. (3), we can rewrite Eq. (4) as follows:

$$\hat{u}_n = -\frac{\rho g}{v^2}\left[\frac{1}{v}\,(gt - v_0 \sin\beta)\, \hat{u}_t + \hat{k}\right].$$

Now, recalling that we must have $\hat{u}_n \cdot \hat{u}_n = 1$, we obtain the following equation:

$$1 = \frac{\rho^2 g^2}{v^4}\left[1 + \frac{1}{v^2}\,(gt - v_0 \sin\beta)^2 + \frac{2}{v}\,(gt - v_0 \sin\beta)\,\hat{k}\cdot\hat{u}_t\right]. \tag{5}$$

Observe that the last term on the right-hand side of Eq. (5) requires the computation of the product $\hat{k}\cdot\hat{u}_t$. This computation can be done by using Eq. (2), which shows that $\hat{k}\cdot\hat{u}_t = \frac{v_0 \sin\beta - gt}{v}$. Consequently, Eq. (5) becomes

$$1 = \frac{\rho^2 g^2}{v^4}\left[1 + \frac{1}{v^2}(gt - v_0 \sin\beta)^2 - \frac{2}{v^2}(gt - v_0 \sin\beta)^2\right] = \frac{\rho^2 g^2}{v^6}\big[v^2 - (gt - v_0 \sin\beta)^2\big]. \tag{6}$$

Keeping in mind that the above expression is to be solved for ρ, the term $v^2 - (gt - v_0 \sin\beta)^2$ can be simplified using Eq. (1) and the trigonometric identity $\cos^2\theta = 1 - \sin^2\theta$ as follows:

$$\begin{aligned} v^2 - (gt - v_0 \sin\beta)^2 = v^2 - g^2t^2 + 2gtv_0 \sin\beta - v_0^2 \sin^2\beta &= v_0^2 - v_0^2 \sin^2\beta \\ &= v_0^2(1 - \sin^2\beta) = v_0^2 \cos^2\beta. \end{aligned} \tag{7}$$

This result allows us to rewrite Eq. (6) as follows:

$$1 = \frac{\rho^2 g^2}{v^6} v_0^2 \cos^2\beta. \tag{8}$$

Solving Eq. (6) for ρ, we have

$$\rho = \frac{v^3}{g v_0 \cos\beta}. \tag{9}$$

Finally, substituting the expression for v in Eq. (1), we have

$$\boxed{\rho = \frac{(v_0^2 + g^2t^2 - 2v_0 gt \sin\beta)^{3/2}}{g v_0 \cos\beta}.}$$

Problem 2.308

A carnival ride called the octopus consists of eight arms that rotate about the z axis with a constant angular velocity $\dot{\theta} = 6\,\text{rpm}$. The arms have a length $L = 8\,\text{m}$ and form an angle ϕ with the z axis. Assuming that ϕ varies with time as $\phi(t) = \phi_0 + \phi_1 \sin\omega t$ with $\phi_0 = 70.5°$, $\phi_1 = 25.5°$, and $\omega = 1\,\text{rad/s}$, determine the magnitude of the acceleration of the outer end of an arm when ϕ achieves its minimum value.

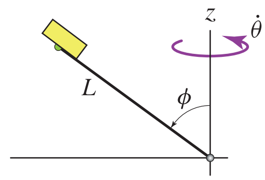
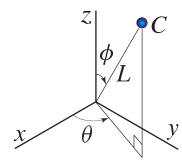


Photo credit: © Gary L. Gray

Solution



Using a spherical coordinate system the components of acceleration of a point C at the end of an arm with constant length L are

$$a_r = -L\phi_1^2\omega^2\cos^2\omega t - L\dot{\theta}^2\sin^2(\phi_0 + \phi_1\sin\omega t), \tag{1}$$

$$a_\phi = -L\omega^2\phi_1\sin\omega t - L\dot{\theta}^2\sin(\phi_0 + \phi_1\sin\omega t)\cos(\phi_0 + \phi_1\sin\omega t), \tag{2}$$

$$a_\theta = 2L\phi_1\omega\dot{\theta}\cos\omega t\cos(\phi_0 + \phi_1\sin\omega t). \tag{3}$$

To determine the minimum value of ϕ, recall that $\phi = \phi_0 + \phi_1\sin\omega t$. Because the mininum of the sine function is equal to -1, then the minimum value of ϕ is

$$\phi_{\text{min}} = \phi_0 - \phi_1 = 45° = \tfrac{1}{4}\pi\,\text{rad}, \tag{4}$$

where we have used the data $\phi_0 = 70.5°$ and $\phi = 25.5°$. We observe that ϕ_{min} occurs when $\sin(\omega t) = -1$, i.e., when

$$\omega t\big|_{\phi_{\text{min}}} = 270° = \tfrac{3}{2}\pi\,\text{rad}. \tag{5}$$

Evaluating the acceleration components for $\phi = \phi_{\text{min}} = \frac{1}{4}\pi$ rad and $\omega t = \frac{3}{2}\pi$ rad, we have

$$a_r = -\tfrac{1}{2}L\dot{\theta}^2, \quad a_\phi = L\big(\omega^2\phi_1 - \tfrac{1}{2}\dot{\theta}^2\big), \quad \text{and} \quad a_\theta = 0. \tag{6}$$

Consequently, the magnitude of the acceleration for $\phi = \phi_{\text{min}}$ is $|\vec{a}|_{\phi_{\text{min}}} = \sqrt{a_r^2 + a_\phi^2 + a_\theta^2}$, which gives

$$|\vec{a}|_{\phi_{\text{min}}} = L\sqrt{\tfrac{1}{2}\dot{\theta}^4 + \omega^4\phi_1^2 - \omega^2\dot{\theta}^2\phi_1} \quad \Rightarrow \quad \boxed{|\vec{a}|_{\phi_{\text{min}}} = 2.534\,\text{m/s}^2,}$$

where we have used the following numerical data: $L = 8\,\text{m}$, $\dot{\theta} = 6\,\text{rpm} = 6\frac{2\pi}{60}\,\text{rad/s}$, $\omega = 1\,\text{rad/s}$, and $\phi_1 = 25.5° = 25.5\frac{\pi}{180}\,\text{rad}$.