CHAPTER 2 – 9th Edition

2.1. Three positive point charges of equal magnitude q are located at $x = -2$, $y = 2$, and $y = -\sqrt{2}$. Find the coordinates of a fourth positive charge, also of magnitude q, that will yield a zero net electric field at the origin: The field at the origin that arises from the three charges can be expressed as

$$\mathbf{E}_o = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{2^2}\mathbf{a}_x + \left(\frac{1}{(\sqrt{2})^2} - \frac{1}{2^2}\right)\mathbf{a}_y\right] = \frac{q}{16\pi\epsilon_0}\left[\mathbf{a}_x + \mathbf{a}_y\right]$$

The magnitude of this field is

$$|\mathbf{E}_o| = \left(\mathbf{E}_o \cdot \mathbf{E}_o\right)^{1/2} = \frac{\sqrt{2}q}{16\pi\epsilon_0}$$

To counter this field, the fourth charge must be positioned along a 45° line in the first quadrant. Its distance from the origin along this line will be $d = \sqrt{4/\sqrt{2}} = 2^{1/4}\sqrt{2} = 1.68$. This translates into equal x and y coordinates of $2^{1/4} = 1.19$. Therefore the fourth charge of positive magnitude q is located at $\underline{(1.19, 1.19)}$

2.2. Point charges of 1nC and -2nC are located at (0,0,0) and (1,1,1), respectively, in free space. Determine the vector force acting on each charge.

First, the electric field intensity associated with the 1nC charge, evalutated at the -2nC charge location is:

$$\mathbf{E}_{12} = \frac{1}{4\pi\epsilon_0(3)}\left(\frac{1}{\sqrt{3}}\right)\left(\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z\right) \quad \text{nC/m}$$

in which the distance between charges is $\sqrt{3}$ m. The force on the -2nC charge is then

$$\mathbf{F}_{12} = q_2\mathbf{E}_{12} = \frac{-2}{12\sqrt{3}\,\pi\epsilon_0}\left(\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z\right) = \underline{\frac{-1}{10.4\,\pi\epsilon_0}\left(\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z\right) \quad \text{nN}}$$

The force on the 1nC charge at the origin is just the opposite of this result, or

$$\mathbf{F}_{21} = \underline{\frac{+1}{10.4\,\pi\epsilon_0}\left(\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z\right) \quad \text{nN}}$$

2.3. Point charges of 50nC each are located at $A(1,0,0)$, $B(-1,0,0)$, $C(0,1,0)$, and $D(0,-1,0)$ in free space. Find the total force on the charge at A.

The force will be:

$$\mathbf{F} = \frac{(50\times10^{-9})^2}{4\pi\epsilon_0}\left[\frac{\mathbf{R}_{CA}}{|\mathbf{R}_{CA}|^3} + \frac{\mathbf{R}_{DA}}{|\mathbf{R}_{DA}|^3} + \frac{\mathbf{R}_{BA}}{|\mathbf{R}_{BA}|^3}\right]$$

where $\mathbf{R}_{CA} = \mathbf{a}_x - \mathbf{a}_y$, $\mathbf{R}_{DA} = \mathbf{a}_x + \mathbf{a}_y$, and $\mathbf{R}_{BA} = 2\mathbf{a}_x$. The magnitudes are $|\mathbf{R}_{CA}| = |\mathbf{R}_{DA}| = \sqrt{2}$, and $|\mathbf{R}_{BA}| = 2$. Substituting these leads to

$$\mathbf{F} = \frac{(50\times10^{-9})^2}{4\pi\epsilon_0}\left[\frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} + \frac{2}{8}\right]\mathbf{a}_x = \underline{21.5\mathbf{a}_x\ \mu\text{N}}$$

where distances are in meters.

2.4. Eight identical point charges of Q C each are located at the corners of a cube of side length a, with one charge at the origin, and with the three nearest charges at $(a, 0, 0)$, $(0, a, 0)$, and $(0, 0, a)$. Find an expression for the total vector force on the charge at $P(a, a, a)$, assuming free space:

The total electric field at $P(a, a, a)$ that produces a force on the charge there will be the sum of the fields from the other seven charges. This is written below, where the charge locations associated with each term are indicated:

$$\mathbf{E}_{net}(a, a, a) = \frac{q}{4\pi\epsilon_0 a^2}\left[\underbrace{\frac{\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z}{3\sqrt{3}}}_{(0,0,0)} + \underbrace{\frac{\mathbf{a}_y + \mathbf{a}_z}{2\sqrt{2}}}_{(a,0,0)} + \underbrace{\frac{\mathbf{a}_x + \mathbf{a}_z}{2\sqrt{2}}}_{(0,a,0)} + \underbrace{\frac{\mathbf{a}_x + \mathbf{a}_y}{2\sqrt{2}}}_{(0,0,a)} + \underbrace{\mathbf{a}_x}_{(0,a,a)} + \underbrace{\mathbf{a}_y}_{(a,0,a)} + \underbrace{\mathbf{a}_z}_{(a,a,0)}\right]$$

The force is now the product of this field and the charge at (a, a, a). Simplifying, we obtain

$$\mathbf{F}(a, a, a) = q\mathbf{E}_{net}(a, a, a) = \frac{q^2}{4\pi\epsilon_0 a^2}\left[\frac{1}{3\sqrt{3}} + \frac{1}{\sqrt{2}} + 1\right](\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z) = \underline{\frac{1.90\, q^2}{4\pi\epsilon_0 a^2}(\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z)}$$

in which the magnitude is $|\mathbf{F}| = 3.29\, q^2/(4\pi\epsilon_0 a^2)$.

2.5. A point charge of 3nC is located at (1,1,1) in free space. What charge must be located at (1,3,2) to cause the y component of $\mathbf{E}$ to be zero at the origin?

For two point charges, we may write:

$$\mathbf{E} = \frac{q_1(\mathbf{r} - \mathbf{r}'_1)}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}'_1|^3} + \frac{q_2(\mathbf{r} - \mathbf{r}'_2)}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}'_2|^3}$$

where $q_1 = 3$nC, and where q_2 is to be found. With q_1 located at (1,1,1), $\mathbf{r}'_1 = \mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z$. The position vector for q_2 is then $\mathbf{r}'_2 = \mathbf{a}_x + 3\mathbf{a}_y + 2\mathbf{a}_z$. Because the observation point is at the origin, we have $\mathbf{r} = 0$. The field now becomes:

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\left[\frac{-3(\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z)}{(1^2 + 1^2 + 1^2)^{3/2}} + \frac{-q_2(\mathbf{a}_x + 3\mathbf{a}_y + 2\mathbf{a}_z)}{(1^2 + 3^2 + 2^2)^{3/2}}\right]$$

For a zero y component, we thus find $q_2 = -(14)^{3/2}/3^{3/2} = \underline{-10.1 \text{ nC}}$.

2.6. Two point charges of equal magnitude q are positioned at $z = \pm d/2$.

a) find the electric field everywhere on the z axis: For a point charge at any location, we have

$$\mathbf{E} = \frac{q(\mathbf{r} - \mathbf{r}')}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|^3}$$

In the case of two charges, we would therefore have

$$\mathbf{E}_T = \frac{q_1(\mathbf{r} - \mathbf{r}_1')}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}_1'|^3} + \frac{q_2(\mathbf{r} - \mathbf{r}_2')}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}_2'|^3} \qquad (1)$$

In the present case, we assign $q_1 = q_2 = q$, the observation point position vector as $\mathbf{r} = z\mathbf{a}_z$, and the charge position vectors as $\mathbf{r}_1' = (d/2)\mathbf{a}_z$, and $\mathbf{r}_2' = -(d/2)\mathbf{a}_z$ Therefore

$$\mathbf{r} - \mathbf{r}_1' = [z - (d/2)]\mathbf{a}_z, \quad \mathbf{r} - \mathbf{r}_2' = [z + (d/2)]\mathbf{a}_z,$$

then

$$|\mathbf{r} - \mathbf{r}_1|^3 = [z - (d/2)]^3 \quad \text{and} \quad |\mathbf{r} - \mathbf{r}_2|^3 = [z + (d/2)]^3$$

Substitute these results into (1) to obtain:

$$\mathbf{E}_T(z) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{[z - (d/2)]^2} + \frac{1}{[z + (d/2)]^2}\right]\mathbf{a}_z \quad \text{V/m} \qquad (2)$$

b) find the electric field everywhere on the xy plane: We proceed as in part a, except that now $\mathbf{r}$ lies in the xy plane. For simplicity, we can choose the x axis on which to evaluate the field, so that $\mathbf{r} = x\mathbf{a}_x$. Eq. (1) becomes

$$\mathbf{E}_T(x) = \frac{q}{4\pi\epsilon_0}\left[\frac{x\mathbf{a}_x - (d/2)\mathbf{a}_z}{|x\mathbf{a}_x - (d/2)\mathbf{a}_z|^3} + \frac{x\mathbf{a}_x + (d/2)\mathbf{a}_z}{|x\mathbf{a}_x + (d/2)\mathbf{a}_z|^3}\right] \qquad (3)$$

where

$$|x\mathbf{a}_x - (d/2)\mathbf{a}_z| = |x\mathbf{a}_x + (d/2)\mathbf{a}_z| = \left[x^2 + (d/2)^2\right]^{1/2}$$

Therefore (3) becomes

$$\mathbf{E}_T(x) = \frac{2qx\,\mathbf{a}_x}{4\pi\epsilon_0\left[x^2 + (d/2)^2\right]^{3/2}}$$

This result can be generalized to apply anywhere in the xy plane by noting that the problem exhibits cylindrical symmetry – any rotation of the x axis about the z axis will produce no change. Therefore, we may use cylindrical coordinates, and replace the x variable by the radial variable ρ, and use $\mathbf{a}_\rho$ instead of $\mathbf{a}_x$. The field then becomes:

$$\mathbf{E}_T(\rho) = \frac{2q\rho\,\mathbf{a}_\rho}{4\pi\epsilon_0\left[\rho^2 + (d/2)^2\right]^{3/2}}$$

2.7. Two point charges of equal magnitude but of opposite sign are positioned with charge $+q$ at $z = d/2$, and charge $-q$ at $z = -d/2$. The pair form an *electric dipole*.

a) Find the electric field intensity everywhere on the z axis: For the two charges we would write in general:

$$\mathbf{E} = \frac{q(\mathbf{r} - \mathbf{r}'_+)}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}'_+|^3} - \frac{q(\mathbf{r} - \mathbf{r}'_-)}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}'_-|^3}$$

where $\mathbf{r} = z\mathbf{a}_z$, $\mathbf{r}'_+ = +d/2\,\mathbf{a}_z$, and $\mathbf{r}'_- = -d/2\,\mathbf{a}_z$. Using these substitutions, we find:

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\left[\frac{(z - \frac{d}{2})\mathbf{a}_z}{|z - \frac{d}{2}|^3} - \frac{(z + \frac{d}{2})\mathbf{a}_z}{|z + \frac{d}{2}|^3}\right]$$

b) Evaluate your part a result at the origin: We set $z = 0$ in the above result to obtain

$$\mathbf{E}(z = 0) = \frac{-2q\,\mathbf{a}_z}{4\pi\epsilon_0}\left[\frac{2}{d}\right]^2 = \frac{-2q\,\mathbf{a}_z}{\pi\epsilon_0 d^2} \quad \text{as expected}$$

c) Find the electric field intensity everywhere on the xy plane, expressing your result as a function of radius ρ in cylindrical coordinates: This will begin with the same initial setup as in part a, except now $\mathbf{r} = \rho\mathbf{a}_\rho$ describes the observation point in the xy plane. With this change, we have

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\left[\frac{\rho\mathbf{a}_\rho - (d/2)\mathbf{a}_z}{[\rho^2 + (d/2)^2]^{3/2}} - \frac{\rho\mathbf{a}_\rho + (d/2)\mathbf{a}_z}{[\rho^2 + (d/2)^2]^{3/2}}\right] = \frac{-qd\,\mathbf{a}_z}{4\pi\epsilon_0[\rho^2 + (d/2)^2]^{3/2}}$$

d) Evaluate your part c result at the origin: Setting $\rho = 0$ in the part c expression, we find:

$$\mathbf{E}(\rho = 0) = \frac{-2q\,\mathbf{a}_z}{\pi\epsilon_0 d^2} \quad \text{as in part } b, \text{ -- and as expected}$$

e) Simplify your part c result for the case in which $\rho >> d$: With this requirement, we find

$$\mathbf{E}(\rho >> d) \doteq \frac{-qd\,\mathbf{a}_z}{4\pi\epsilon_0\rho^3}$$

2.8. A crude device for measuring charge consists of two small insulating spheres of radius a, one of which is fixed in position. The other is movable along the x axis, and is subject to a restraining force kx, where k is a spring constant. The uncharged spheres are centered at $x = 0$ and $x = d$, the latter fixed. If the spheres are given equal and opposite charges of Q coulombs:

a) Obtain the expression by which Q may be found as a function of x: The spheres will attract, and so the movable sphere at $x = 0$ will move toward the other until the spring and Coulomb forces balance. This will occur at location x for the movable sphere. With equal and opposite forces, we have

$$\frac{Q^2}{4\pi\epsilon_0(d - x)^2} = kx$$

from which $\underline{Q = 2(d - x)\sqrt{\pi\epsilon_0 kx}}$.

2.8 b) Determine the maximum charge that can be measured in terms of ϵ_0, k, and d, and state the separation of the spheres then:

With increasing charge, the spheres move toward each other until they just touch at $x_{max} = d - 2a$. Using the part a result, we find the maximum measurable charge: $\underline{Q_{max} = 4a\sqrt{\pi\epsilon_0 k(d-2a)}}$. Presumably some form of stop mechanism is placed at $x = x_{max}^-$ to prevent the spheres from actually touching.

c) What happens if a larger charge is applied? No further motion is possible, so nothing happens.

2.9. A 100 nC point charge is located at $A(-1, 1, 3)$ in free space.

a) Find the locus of all points $P(x, y, z)$ at which $E_x = 500$ V/m: The total field at P will be:

$$\mathbf{E}_P = \frac{100 \times 10^{-9}}{4\pi\epsilon_0} \frac{\mathbf{R}_{AP}}{|\mathbf{R}_{AP}|^3}$$

where $\mathbf{R}_{AP} = (x+1)\mathbf{a}_x + (y-1)\mathbf{a}_y + (z-3)\mathbf{a}_z$, and where $|\mathbf{R}_{AP}| = [(x+1)^2 + (y-1)^2 + (z-3)^2]^{1/2}$. The x component of the field will be

$$E_x = \frac{100 \times 10^{-9}}{4\pi\epsilon_0} \left[\frac{(x+1)}{[(x+1)^2 + (y-1)^2 + (z-3)^2]^{1.5}} \right] = 500 \text{ V/m}$$

And so our condition becomes:

$$\underline{(x+1) = 0.56\,[(x+1)^2 + (y-1)^2 + (z-3)^2]^{1.5}}$$

b) Find y_1 if $P(0, y_1, 3)$ lies on that locus: At point P, the condition of part a becomes

$$3.19 = \left[1 + (y_1 - 1)^2\right]^3$$

from which $(y_1 - 1)^2 = 0.47$, or $y_1 = \underline{1.69 \text{ or } 0.31}$

2.10. A configuration of point charges consists of a single charge of value $-2q$ at the origin, and two charges of value $+q$ at locations $z = -d$ and $+d$. The charges as positioned form an *electric quadrupole*, equivalent to two dipoles of opposite orientation that are separated by distance d along the z axis.

a) Find the electric field intensity $\mathbf{E}$ everywhere in the xy plane, expressing your result as a function of cylindrical radius ρ: We begin by applying the general formula for the point charge field, where the three terms apply to the three charges:

$$\mathbf{E} = \frac{q(\mathbf{r} - \mathbf{r}'_{lower})}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}'_{lower}|^3} - \frac{2q(\mathbf{r} - \mathbf{r}'_{middle})}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}'_{middle}|^3} + \frac{q(\mathbf{r} - \mathbf{r}'_{upper})}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}'_{upper}|^3}$$

The position vectors will be $\mathbf{r} = \rho\mathbf{a}_\rho$, $\mathbf{r}'_{lower} = -d\mathbf{a}_z$, $\mathbf{r}'_{upper} = +d\mathbf{a}_z$, and $\mathbf{r}'_{middle} = 0$. With these substitutions, the field expression becomes:

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0} \left[\frac{\rho\mathbf{a}_\rho + d\mathbf{a}_z}{(\rho^2 + d^2)^{3/2}} - \frac{2\rho\mathbf{a}_\rho}{\rho^3} + \frac{\rho\mathbf{a}_\rho - d\mathbf{a}_z}{(\rho^2 + d^2)^{3/2}} \right] = \frac{-q\mathbf{a}_\rho}{2\pi\epsilon_0\rho^2} \left[1 - \frac{1}{(1 + d^2/\rho^2)^{3/2}} \right]$$

2.10 b) Specialize your part *a* result for large distances, $\rho >> d$: Under this condition, we may use the expansion:

$$\frac{1}{(1+d^2/\rho^2)^{3/2}} = \left(1 + d^2/\rho^2\right)^{-1}\left(1 + d^2/\rho^2\right)^{-1/2} \doteq \left(1 - d^2/\rho^2\right)\left(1 - d^2/2\rho^2\right)$$

Carrying out the product and neglecting the term involving d^4/ρ^4, we find:

$$\left(1 - d^2/\rho^2\right)\left(1 - d^2/2\rho^2\right) \doteq 1 - \frac{3}{2}\frac{d^2}{\rho^2}$$

from which

$$\mathbf{E}(\rho >> d) \doteq \frac{-q\mathbf{a}_\rho}{2\pi\epsilon_0\rho^2}\left[1 - 1 + \frac{3}{2}\frac{d^2}{\rho^2}\right] = \frac{-3qd^2\mathbf{a}_\rho}{4\pi\epsilon_0\rho^4}$$

2.11. A charge Q_0 located at the origin in free space produces a field for which $E_z = 1$ kV/m at point $P(-2, 1, -1)$.

a) Find Q_0: The field at P will be

$$\mathbf{E}_P = \frac{Q_0}{4\pi\epsilon_0}\left[\frac{-2\mathbf{a}_x + \mathbf{a}_y - \mathbf{a}_z}{6^{1.5}}\right]$$

Since the z component is of value 1 kV/m, we find $Q_0 = -4\pi\epsilon_0 6^{1.5} \times 10^3 = \underline{-1.63\ \mu\text{C}}$.

b) Find $\mathbf{E}$ at $M(1, 6, 5)$ in cartesian coordinates: This field will be:

$$\mathbf{E}_M = \frac{-1.63 \times 10^{-6}}{4\pi\epsilon_0}\left[\frac{\mathbf{a}_x + 6\mathbf{a}_y + 5\mathbf{a}_z}{[1 + 36 + 25]^{1.5}}\right]$$

or $\mathbf{E}_M = \underline{-30.11\mathbf{a}_x - 180.63\mathbf{a}_y - 150.53\mathbf{a}_z}$.

c) Find $\mathbf{E}$ at $M(1, 6, 5)$ in cylindrical coordinates: At M, $\rho = \sqrt{1 + 36} = 6.08$, $\phi = \tan^{-1}(6/1) = 80.54°$, and $z = 5$. Now

$$E_\rho = \mathbf{E}_M \cdot \mathbf{a}_\rho = -30.11\cos\phi - 180.63\sin\phi = -183.12$$

$$E_\phi = \mathbf{E}_M \cdot \mathbf{a}_\phi = -30.11(-\sin\phi) - 180.63\cos\phi = 0 \quad \text{(as expected)}$$

so that $\mathbf{E}_M = \underline{-183.12\mathbf{a}_\rho - 150.53\mathbf{a}_z}$.

d) Find $\mathbf{E}$ at $M(1, 6, 5)$ in spherical coordinates: At M, $r = \sqrt{1 + 36 + 25} = 7.87$, $\phi = 80.54°$ (as before), and $\theta = \cos^{-1}(5/7.87) = 50.58°$. Now, since the charge is at the origin, we expect to obtain only a radial component of $\mathbf{E}_M$. This will be:

$$E_r = \mathbf{E}_M \cdot \mathbf{a}_r = -30.11\sin\theta\cos\phi - 180.63\sin\theta\sin\phi - 150.53\cos\theta = \underline{-237.1}$$

2.12. Electrons are in random motion in a fixed region in space. During any 1μs interval, the probability of finding an electron in a subregion of volume 10^{-15} m^2 is 0.27. What volume charge density, appropriate for such time durations, should be assigned to that subregion?

The finite probabilty effectively reduces the net charge quantity by the probability fraction. With $e = -1.602 \times 10^{-19}$ C, the density becomes

$$\rho_v = -\frac{0.27 \times 1.602 \times 10^{-19}}{10^{-15}} = \underline{-43.3\ \mu\text{C/m}^3}$$

2.13. A uniform volume charge density of 0.2 μC/m^3 is present throughout the spherical shell extending from $r = 3$ cm to $r = 5$ cm. If $\rho_v = 0$ elsewhere:

a) find the total charge present throughout the shell: This will be

$$Q = \int_0^{2\pi} \int_0^{\pi} \int_{.03}^{.05} 0.2\, r^2 \sin\theta\, dr\, d\theta\, d\phi = \left[4\pi(0.2)\frac{r^3}{3}\right]_{.03}^{.05} = 8.21 \times 10^{-5}\ \mu\text{C} = \underline{82.1\ \text{pC}}$$

b) find r_1 if half the total charge is located in the region 3 cm $< r < r_1$: If the integral over r in part a is taken to r_1, we would obtain

$$\left[4\pi(0.2)\frac{r^3}{3}\right]_{.03}^{r_1} = 4.105 \times 10^{-5}$$

Thus

$$r_1 = \left[\frac{3 \times 4.105 \times 10^{-5}}{0.2 \times 4\pi} + (.03)^3\right]^{1/3} = \underline{4.24\ \text{cm}}$$

2.14. The electron beam in a certain cathode ray tube possesses cylindrical symmetry, and the charge density is represented by $\rho_v = -0.1/(\rho^2 + 10^{-8})$ pC/m^3 for $0 < \rho < 3 \times 10^{-4}$ m, and $\rho_v = 0$ for $\rho > 3 \times 10^{-4}$ m.

a) Find the total charge per meter along the length of the beam: We integrate the charge density over the cylindrical volume having radius 3×10^{-4} m, and length 1m.

$$q = \int_0^1 \int_0^{2\pi} \int_0^{3\times10^{-4}} \frac{-0.1}{(\rho^2 + 10^{-8})}\, \rho\, d\rho\, d\phi\, dz$$

From integral tables, this evaluates as

$$q = -0.2\pi \left(\frac{1}{2}\right) \ln\left(\rho^2 + 10^{-8}\right)\Big|_0^{3\times10^{-4}} = 0.1\pi \ln(10) = \underline{-0.23\pi\ \text{pC/m}}$$

b) if the electron velocity is 5×10^7 m/s, and with one ampere defined as 1C/s, find the beam current:

$$\text{Current} = \text{charge/m}\times v = -0.23\pi\ [\text{pC/m}]\times5\times10^7\ [\text{m/s}] = -11.5\pi\times10^6\ [\text{pC/s}] = \underline{-11.5\pi\ \mu\text{A}}$$

2.15. A spherical volume having a 2 μm radius contains a uniform volume charge density of 10^5 C/m^3.

a) What total charge is enclosed in the spherical volume?
This will be $Q = (4/3)\pi(2\times10^{-6})^3 \times 10^5 = \underline{3.35\times10^{-12}\text{ C}}$.

b) Now assume that a large region contains one of these little spheres at every corner of a cubical grid 3mm on a side, and that there is no charge between spheres. What is the average volume charge density throughout this large region? Each cube will contain the equivalent of one little sphere. Neglecting the little sphere volume, the average density becomes

$$\rho_{v,avg} = \frac{3.35\times10^{-12}}{(0.003)^3} = \underline{1.24\times10^{-4}\text{ C/m}^3}$$

2.16. Within a region of free space, charge density is given as $\rho_v = \left(\rho_0 r/a\right)\cos\theta$ C/m^3, where ρ_0 and a are constants. Find the total charge lying within:

a) the sphere, $r \le a$: This will be

$$Q_a = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \frac{\rho_0 r}{a}\cos\theta\, r^2\sin\theta\, dr\, d\theta\, d\phi = \underline{0}$$

It is the integral over θ, performed first, that gives the zero result.

b) the cone, $r \le a$, $0 \le \theta \le 0.1\pi$:

$$Q_b = \int_0^{2\pi}\int_0^{0.1\pi}\int_0^{a} \frac{\rho_0 r}{a}\cos\theta\, r^2\sin\theta\, dr\, d\theta\, d\phi = \pi\frac{\rho_0 a^3}{4}\left[1-\cos^2(0.1\pi)\right] = \underline{0.024\pi\rho_0 a^3}$$

c) the region, $r \le a$, $0 \le \theta \le 0.1\pi$, $0 \le \phi \le 0.2\pi$.

$$Q_c = \int_0^{0.2\pi}\int_0^{0.1\pi}\int_0^{a} \frac{\rho_0 r}{a}\cos\theta\, r^2\sin\theta\, dr\, d\theta\, d\phi = 0.024\pi\rho_0 a^3\left(\frac{0.2\pi}{2\pi}\right) = \underline{0.0024\pi\rho_0 a^3}$$

2.17. A length d of line charge lies on the z axis in free space. The charge density on the line is $\rho_L = +\rho_0$ C/m $(0 < z < d/2)$ and $\rho_L = -\rho_0$ C/m $(-d/2 < z < 0)$, where ρ_0 is a positive constant.

a) Find the electric field intensity $\mathbf{E}$ everywhere in the xy plane, expressing your result as a function of cylindrical radius ρ: Begin by constructing the differential field at radius ρ in the xy plane that arises from a point charge $dq = \rho_L dz$ on the z axis. To do this, use the general expression:

$$d\mathbf{E} = \frac{dq(\mathbf{r}-\mathbf{r}')}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|^3}$$

where in this case, $\mathbf{r} = \rho\mathbf{a}_\rho$ and $\mathbf{r}' = z\mathbf{a}_z$. With these substitutions, we find

$$d\mathbf{E} = \frac{\pm\rho_0 dz(\rho\mathbf{a}_\rho - z\mathbf{a}_z)}{4\pi\epsilon_0(\rho^2 - z^2)^{3/2}}$$

where the positive sign applies to the region $z > 0$; the negative sign to $z < 0$. The total field at radius ρ is then found by integrating $d\mathbf{E}$ over the total charge length:

$$\mathbf{E} = \int_{-d/2}^{0} \frac{-\rho_0 dz(\rho\mathbf{a}_\rho - z\mathbf{a}_z)}{4\pi\epsilon_0(\rho^2 - z^2)^{3/2}} + \int_0^{+d/2} \frac{+\rho_0 dz(\rho\mathbf{a}_\rho - z\mathbf{a}_z)}{4\pi\epsilon_0(\rho^2 - z^2)^{3/2}}$$

2.17. a) (continued) Note that the radial component will integrate to zero through odd parity, leaving only the z component, as would be expected. The integral simplifies to:

$$\mathbf{E} = 2\int_0^{d/2} \frac{-\rho_0 z \mathbf{a}_z dz}{4\pi\epsilon_0(\rho^2+z^2)^{3/2}} = \frac{\rho_0 \mathbf{a}_z}{2\pi\epsilon_0(\rho^2+z^2)^{1/2}}\Bigg|_0^{d/2} = \frac{-\rho_0 \mathbf{a}_z}{2\pi\epsilon_0 \rho}\left[1 - \frac{1}{\sqrt{1+(d/2\rho)^2}}\right]$$

b) Simplify your part *a* result for the case in which radius $\rho >> d$, and express this result in terms of charge $q = \rho_0 d/2$: At large radii, we can use the binomial expansion to the first two terms:

$$\frac{1}{\sqrt{1+(d/2\rho)^2}} \doteq 1 - \frac{1}{2}\left(\frac{d}{2\rho}\right)^2$$

with which

$$\mathbf{E} \doteq \frac{-\rho_0 d^2}{16\pi\epsilon_0\rho^3}\,\mathbf{a}_z = \frac{-qd}{4\pi\epsilon_0\rho^3}\,\mathbf{a}_z$$

where $q = \rho_0 d/2$.

2.18. a) Find $\mathbf{E}$ in the plane $z = 0$ that is produced by a uniform line charge, ρ_L, extending along the z axis over the range $-L < z < L$ in a cylindrical coordinate system: We find $\mathbf{E}$ through

$$\mathbf{E} = \int_{-L}^{L} \frac{\rho_L dz(\mathbf{r}-\mathbf{r}')}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|^3}$$

where the observation point position vector is $\mathbf{r} = \rho\mathbf{a}_\rho$ (anywhere in the x-y plane), and where the position vector that locates any differential charge element on the z axis is $\mathbf{r}' = z\mathbf{a}_z$. So $\mathbf{r}-\mathbf{r}' = \rho\mathbf{a}_\rho - z\mathbf{a}_z$, and $|\mathbf{r}-\mathbf{r}'| = (\rho^2+z^2)^{1/2}$. These relations are substituted into the integral to yield:

$$\mathbf{E} = \int_{-L}^{L} \frac{\rho_L dz(\rho\mathbf{a}_\rho - z\mathbf{a}_z)}{4\pi\epsilon_0(\rho^2+z^2)^{3/2}} = \frac{\rho_L\,\rho\,\mathbf{a}_\rho}{4\pi\epsilon_0}\int_{-L}^{L}\frac{dz}{(\rho^2+z^2)^{3/2}} = E_\rho\,\mathbf{a}_\rho$$

Note that the second term in the left-hand integral (involving $z\mathbf{a}_z$) has effectively vanished because it produces equal and opposite sign contributions when the integral is taken over symmetric limits (odd parity). Evaluating the integral results in

$$E_\rho = \frac{\rho_L\,\rho}{4\pi\epsilon_0}\,\frac{z}{\rho^2\sqrt{\rho^2+z^2}}\Bigg|_{-L}^{L} = \frac{\rho_L}{2\pi\epsilon_0\rho}\,\frac{L}{\sqrt{\rho^2+L^2}} = \frac{\rho_L}{2\pi\epsilon_0\rho}\,\frac{1}{\sqrt{1+(\rho/L)^2}}$$

Note that as $L \to \infty$, the expression reduces to the expected field of the infinite line charge in free space, $\rho_L/(2\pi\epsilon_0\rho)$.

b) if the finite line charge is approximated by an infinite line charge ($L \to \infty$), by what percentage is E_ρ in error if $\rho = 0.5L$? The percent error in this situation will be

$$\%\,\text{error} = \left[1 - \frac{1}{\sqrt{1+(\rho/L)^2}}\right] \times 100$$

For $\rho = 0.5L$, this becomes $\%\,\text{error} = \underline{10.6\,\%}$

c) repeat *b* with $\rho = 0.1L$. For this value, obtain $\%\,\text{error} = \underline{0.496\,\%}$.

2.19. A line having charge density $\rho_0|z|$ C/m and of length ℓ is oriented along the z axis at $-\ell/2 < z < \ell/2$.

a) Find the electric field intensity **E** everywhere in the xy plane, expressing your result in cylindrical coordinates: As the problem exhibits cylindrical symmetry, we may write the position vector for the observation point in the xy plane as $\mathbf{r} = \rho\mathbf{a}_\rho$. Then, with $\mathbf{r}' = z\mathbf{a}_z$, we may write

$$\mathbf{E} = \int_{-\ell/2}^{\ell/2} \frac{\rho_L(z)dz(\mathbf{r}-\mathbf{r}')}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|^3} = \int_{-\ell/2}^{\ell/2} \frac{\rho_0|z|dz(\rho\mathbf{a}_\rho - z\mathbf{a}_z)}{4\pi\epsilon_0(\rho^2+z^2)^{3/2}}$$

Note that the second term in the integrand (the z component) is zero, because of odd parity. We are left with

$$\mathbf{E} = \frac{\rho_0\rho\,\mathbf{a}_\rho}{4\pi\epsilon_0}\int_{-\ell/2}^{\ell/2}\frac{|z|dz}{(\rho^2+z^2)^{3/2}} = \frac{2\rho_0\rho\,\mathbf{a}_\rho}{4\pi\epsilon_0}\int_0^{\ell/2}\frac{zdz}{(\rho^2+z^2)^{3/2}} = \frac{-\rho_0\rho\,\mathbf{a}_\rho}{2\pi\epsilon_0}\frac{1}{(\rho^2+z^2)^{1/2}}\Bigg|_0^{\ell/2}$$

Evaluating the limits the final result can be written as

$$\mathbf{E} = \frac{\rho_0\mathbf{a}_\rho}{2\pi\epsilon_0}\left[1 - \frac{1}{\sqrt{1-(\ell/2\rho)^2}}\right] \text{ V/m}$$

b) Evaluate your result of part *a* in the limit as ℓ (not z) approaches infinity: In this limit, the second term in the bracket tends to zero, and we have

$$\mathbf{E}(\ell\to\infty) = \frac{\rho_0\mathbf{a}_\rho}{2\pi\epsilon_0} \text{ V/m}$$

thus exhibiting no radial variation!

2.20. A line charge of uniform charge density ρ_0 C/m and of length ℓ, is oriented along the z axis at $-\ell/2 < z < \ell/2$.

a) Find the electric field strength, **E**, in magnitude and direction at any position along the x axis: This follows the method in Problem 2.18. We find **E** through

$$\mathbf{E} = \int_{-\ell/2}^{\ell/2}\frac{\rho_0\,dz(\mathbf{r}-\mathbf{r}')}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|^3}$$

where the observation point position vector is $\mathbf{r} = x\mathbf{a}_x$ (anywhere on the x axis), and where the position vector that locates any differential charge element on the z axis is $\mathbf{r}' = z\mathbf{a}_z$. So $\mathbf{r}-\mathbf{r}' = x\mathbf{a}_x - z\mathbf{a}_z$, and $|\mathbf{r}-\mathbf{r}'| = (x^2+z^2)^{1/2}$. These relations are substituted into the integral to yield:

$$\mathbf{E} = \int_{-\ell/2}^{\ell/2}\frac{\rho_0 dz(x\mathbf{a}_x - z\mathbf{a}_z)}{4\pi\epsilon_0(x^2+z^2)^{3/2}} = \frac{\rho_0\,x\,\mathbf{a}_x}{4\pi\epsilon_0}\int_{-\ell/2}^{\ell/2}\frac{dz}{(x^2+z^2)^{3/2}} = E_x\,\mathbf{a}_x$$

Note that the second term in the left-hand integral (involving $z\mathbf{a}_z$) has effectively vanished because it produces equal and opposite sign contributions when the integral is taken over symmetric limits (odd parity). Evaluating the integral results in

$$E_x = \frac{\rho_0\,x}{4\pi\epsilon_0}\frac{z}{x^2\sqrt{x^2+z^2}}\Bigg|_{-\ell/2}^{\ell/2} = \frac{\rho_0}{2\pi\epsilon_0 x}\frac{\ell/2}{\sqrt{x^2+(\ell/2)^2}} = \frac{\rho_0}{2\pi\epsilon_0 x}\frac{1}{\sqrt{1+(2x/\ell)^2}}$$

2.20. b) with the given line charge in position, find the force acting on an identical line charge that is oriented along the x axis at $\ell/2 < x < 3\ell/2$: The differential force on an element of the x-directed line charge will be $d\mathbf{F} = dq\mathbf{E} = (\rho_0\, dx)\mathbf{E}$, where $\mathbf{E}$ is the field as determined in part a. The net force is then the integral of the differential force over the length of the horizontal line charge, or

$$\mathbf{F} = \int_{\ell/2}^{3\ell/2} \frac{\rho_0^2}{2\pi\epsilon_0 x} \frac{1}{\sqrt{1+(2x/\ell)^2}}\, dx\, \mathbf{a}_x$$

This can be re-written and then evaluated using integral tables as

$$\mathbf{F} = \frac{\rho_0^2\,\ell\,\mathbf{a}_x}{4\pi\epsilon_0} \int_{\ell/2}^{3\ell/2} \frac{dx}{x\sqrt{x^2+(\ell/2)^2}} = \frac{-\rho_0^2\,\ell\,\mathbf{a}_x}{4\pi\epsilon_0} \left(\frac{1}{(\ell/2)} \ln \left[\frac{\ell/2 + \sqrt{x^2+(\ell/2)^2}}{x} \right]_{\ell/2}^{3\ell/2} \right)$$

$$= \frac{-\rho_0^2\,\mathbf{a}_x}{2\pi\epsilon_0} \ln \left[\frac{(\ell/2)\left(1+\sqrt{10}\right)}{3(\ell/2)\left(1+\sqrt{2}\right)} \right] = \frac{\rho_0^2\,\mathbf{a}_x}{2\pi\epsilon_0} \ln \left[\frac{3(1+\sqrt{2})}{1+\sqrt{10}} \right] = \frac{0.55\rho_0^2}{2\pi\epsilon_0}\,\mathbf{a}_x \text{ N}$$

2.21. A charged filament forms a circle of radius a in the xy plane with center at the origin. The filament carries uniform line charge density $+\rho_0$ C/m for $-\pi/2 < \phi < \pi/2$, and $-\rho_0$ C/m for $\pi/2 < \phi < 3\pi/2$. Find the electric field intensity at the origin:

The field at the origin arising from a differential length $ad\phi$ of charge on the ring will be

$$d\mathbf{E}_o = \frac{\pm\rho_0 a d\phi}{4\pi\epsilon_0 a^2}\mathbf{a}_\rho$$

where the positive sign applies to the negative charge contribution, the negative sign to the positive charge contribution. Now the total field will be the piecewise integral of the differential field over the two semi-circles. Using $\mathbf{a}_\rho = \cos\phi\,\mathbf{a}_x + \sin\phi\,\mathbf{a}_y$ (as is required to include all ϕ dependence), we have

$$\mathbf{E}_o = \int_{-\pi/2}^{\pi/2} \frac{-\rho_0 d\phi}{4\pi\epsilon_0 a}(\cos\phi\,\mathbf{a}_x + \sin\phi\,\mathbf{a}_y) + \int_{\pi/2}^{3\pi/2} \frac{+\rho_0 d\phi}{4\pi\epsilon_0 a}(\cos\phi\,\mathbf{a}_x + \sin\phi\,\mathbf{a}_y)$$

Noting that the two terms in the above expression are equal to each other, we may evaluate the first integral and introduce a factor of 2:

$$\mathbf{E}_o = \frac{-2\rho_0}{4\pi\epsilon_0 a}\left[\underbrace{\sin\phi\Big|_{-\pi/2}^{\pi/2}}_{2} \mathbf{a}_x - \underbrace{\cos\phi\Big|_{-\pi/2}^{\pi/2}}_{0} \mathbf{a}_y \right] = \underline{\frac{-\rho_0\mathbf{a}_x}{\pi\epsilon_0 a}\ \text{V/m}}$$

2.22. Two identical uniform sheet charges with $\rho_s = 100$ nC/m^2 are located in free space at $z = \pm 2.0$ cm. What force per unit area does each sheet exert on the other?

The field from the top sheet is $\mathbf{E} = -\rho_s/(2\epsilon_0)\,\mathbf{a}_z$ V/m. The differential force produced by this field on the bottom sheet is the charge density on the bottom sheet times the differential area there, multiplied by the electric field from the top sheet: $d\mathbf{F} = \rho_s da\mathbf{E}$. The force per unit area is then just $\mathbf{F} = \rho_s\mathbf{E} = (100\times10^{-9})(-100\times10^{-9})/(2\epsilon_0)\,\mathbf{a}_z = \underline{-5.6\times10^{-4}\,\mathbf{a}_z \text{ N/m}^2}$.

2.23. A disk of radius a in the xy plane carries surface charge of density $\rho_s = \rho_0/\rho$, where ρ_0 is a constant. Find the electric field strength, $\mathbf{E}$, everywhere on the z axis.

We find the field through

$$\mathbf{E} = \int\int \frac{\rho_s da(\mathbf{r} - \mathbf{r}')}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}'|^3}$$

where the integral is taken over the surface of the disk, and where $\mathbf{r} = z\mathbf{a}_z$ and $\mathbf{r}' = \rho\mathbf{a}_\rho$. The integral then becomes

$$\mathbf{E} = \int_0^{2\pi}\int_0^a \frac{(\rho_0/\rho)\,\rho\,d\rho\,d\phi\left(z\mathbf{a}_z - \rho\mathbf{a}_\rho\right)}{4\pi\epsilon_0(z^2+\rho^2)^{3/2}}$$

In evaluating this integral, we need to introduce the ϕ dependence in $\mathbf{a}_\rho$ by writing it as $\mathbf{a}_\rho = \cos\phi\,\mathbf{a}_x + \sin\phi\,\mathbf{a}_y$, where $\mathbf{a}_x$ and $\mathbf{a}_y$ are invariant in their orientation as ϕ varies. So the integral now simplifies to

$$\mathbf{E} = \frac{2\pi\rho_0\,z\,\mathbf{a}_z}{4\pi\epsilon_0}\int_0^\infty \frac{d\rho}{(z^2+\rho^2)^{3/2}} = \frac{2\pi\rho_0\,z\,\mathbf{a}_z}{4\pi\epsilon_0}\left[\frac{\rho}{z^2\sqrt{z^2+\rho^2}}\right]_{\rho=0}^{a} = \frac{2\pi a\rho_0\mathbf{a}_z}{4\pi\epsilon_0 z^2[1+(a/z)^2]^{1/2}}$$

2.24. a) Find the electric field on the z axis produced by an annular ring of uniform surface charge density ρ_s in free space. The ring occupies the region $z = 0$, $a \le \rho \le b$, $0 \le \phi \le 2\pi$ in cylindrical coordinates: We find the field through

$$\mathbf{E} = \int\int \frac{\rho_s da(\mathbf{r} - \mathbf{r}')}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}'|^3}$$

where the integral is taken over the surface of the annular ring, and where $\mathbf{r} = z\mathbf{a}_z$ and $\mathbf{r}' = \rho\mathbf{a}_\rho$. The integral then becomes

$$\mathbf{E} = \int_0^{2\pi}\int_a^b \frac{\rho_s\,\rho\,d\rho\,d\phi\left(z\mathbf{a}_z - \rho\mathbf{a}_\rho\right)}{4\pi\epsilon_0(z^2+\rho^2)^{3/2}}$$

In evaluating this integral, we first note that the term involving $\rho\mathbf{a}_\rho$ integrates to zero over the ϕ integration range of 0 to 2π. This is because we need to introduce the ϕ dependence in $\mathbf{a}_\rho$ by writing it as $\mathbf{a}_\rho = \cos\phi\,\mathbf{a}_x + \sin\phi\,\mathbf{a}_y$, where $\mathbf{a}_x$ and $\mathbf{a}_y$ are invariant in their orientation as ϕ varies. So the integral now simplifies to

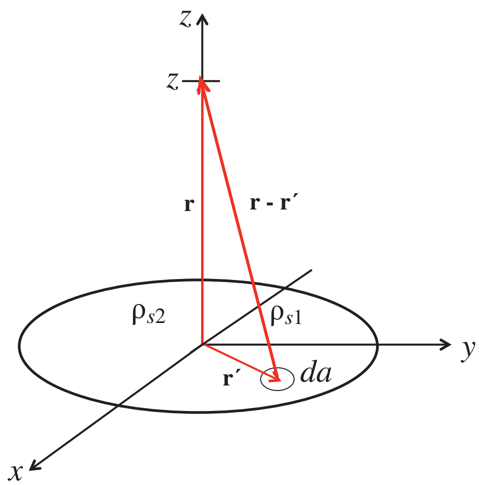
$$\mathbf{E} = \frac{2\pi\rho_s\,z\,\mathbf{a}_z}{4\pi\epsilon_0}\int_a^b \frac{\rho\,d\rho}{(z^2+\rho^2)^{3/2}} = \frac{\rho_s\,z\,\mathbf{a}_z}{2\epsilon_0}\left[\frac{-1}{\sqrt{z^2+\rho^2}}\right]_a^b$$

$$= \frac{\rho_s}{2\epsilon_0}\left[\frac{1}{\sqrt{1+(a/z)^2}} - \frac{1}{\sqrt{1+(b/z)^2}}\right]\mathbf{a}_z$$

b) from your part a result, obtain the field of an infinite uniform sheet charge by taking appropriate limits. The infinite sheet is obtained by letting $a \to 0$ and $b \to \infty$, in which case $\mathbf{E} \to \rho_s/(2\epsilon_0)\,\mathbf{a}_z$ as expected.

2.25. A disk of radius a in the xy plane carries surface charge of density $\rho_{s1} = +\rho_{s0}/\rho$ C/m^2 for $0 < \phi < \pi$, and $\rho_{s2} = -\rho_{s0}/\rho$ C/m^2 for $\pi < \phi < 2\pi$, where ρ_{s0} is a constant.

a) Find the electric field intensity **E** everywhere on the z axis:



With the setup shown at right, the field can be found through the general relation

$$\mathbf{E} = \int\int_{disk\ area} \frac{\rho_s da(\mathbf{r}-\mathbf{r}')}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|^3}$$

where $\mathbf{r} = z\mathbf{a}_z$ and $\mathbf{r}' = \rho\mathbf{a}_\rho$. Because the charge is ϕ-dependent, we need to include the ϕ dependence in all terms. This means that we must use $\mathbf{a}_\rho = \mathbf{a}_x\cos\phi + \mathbf{a}_y\sin\phi$. Then, with $|\mathbf{r}-\mathbf{r}'|^3 = (z^2+\rho^2)^{3/2}$ and with the positive and negative charges accounted for, the integral is performed piecewise over the two halves of the disk:

$$\mathbf{E} = \int_0^\pi\int_0^a \frac{+(\rho_{s0}/\rho)[z\mathbf{a}_z - \rho\cos\phi\mathbf{a}_x - \rho\sin\phi\mathbf{a}_y]\,\rho d\rho d\phi}{4\pi\epsilon_0(z^2+\rho^2)^{3/2}}$$
$$+ \int_\pi^{2\pi}\int_0^a \frac{-(\rho_{s0}/\rho)[z\mathbf{a}_z - \rho\cos\phi\mathbf{a}_x - \rho\sin\phi\mathbf{a}_y]\,\rho d\rho d\phi}{4\pi\epsilon_0(z^2+\rho^2)^{3/2}}$$

Noting that the z components will cancel, and performing the ϕ integration first, we find:

$$\mathbf{E} = \frac{\rho_{s0}}{4\pi\epsilon_0}\int_0^a \frac{\left[-\sin\phi|_0^\pi\mathbf{a}_x + \cos\phi|_0^\pi\mathbf{a}_y + \sin\phi|_\pi^{2\pi}\mathbf{a}_x - \cos\phi|_\pi^{2\pi}\mathbf{a}_y\right]\rho d\rho}{(z^2+\rho^2)^{3/2}}$$
$$= \frac{-4\rho_{s0}\mathbf{a}_y}{4\pi\epsilon_0}\int_0^a \frac{\rho d\rho}{(z^2+\rho^2)^{3/2}} = \frac{-\rho_{s0}\mathbf{a}_y}{\pi\epsilon_0}\left[\frac{-1}{(z^2+\rho^2)^{3/2}}\right]_0^a = \underline{\frac{-\rho_{s0}\mathbf{a}_y}{\pi\epsilon_0 z}\left[1 - \frac{1}{\sqrt{1+a^2/z^2}}\right]}$$

b) Specialize your part a result for distances $z >> a$: With this condition we have $(1+a^2/z^2)^{-1/2} \doteq 1 - a^2/2z^2$ and thus

$$\mathbf{E}(z >> a) \doteq \frac{-\rho_{s0}\mathbf{a}_y}{\pi\epsilon_0 z}\left[1 - 1 + \frac{a^2}{2z^2}\right] = \underline{\frac{-\rho_{s0}a^2\mathbf{a}_y}{2\pi\epsilon_0 z^3}}$$

Note the inverse distance cubed dependence that is characteristic of a dipole field.

2.26 a) Find the electric field intensity on the z axis produced by a cone surface that carries charge density $\rho_s(r) = \rho_0/r$ C/m^2 in free space. The cone has its vertex at the origin and occupies the region $\theta = \alpha$, $0 < r < a$, and $0 < \phi < 2\pi$ in spherical coordinates. Differential area in spherical coordinates is given as $da = r \sin\alpha\, dr d\phi$: **E** is found using the general surface integral

$$\mathbf{E} = \int\int_{cone} \frac{\rho_s da(\mathbf{r} - \mathbf{r}')}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|^3}$$

where in this case $\mathbf{r} = z\mathbf{a}_z$ and $\mathbf{r}' = r\mathbf{a}_r$.
Then $|\mathbf{r} - \mathbf{r}'| = \left[(\mathbf{r} - \mathbf{r}') \cdot (\mathbf{r} - \mathbf{r}')\right]^{1/2} = \left[(z\mathbf{a}_z - r\mathbf{a}_r) \cdot (z\mathbf{a}_z - r\mathbf{a}_r)\right]^{1/2} = \sqrt{z^2 - 2rz\cos\alpha + r^2}$, where $\mathbf{a}_z \cdot \mathbf{a}_r = \cos\alpha$ has been used. The integral now becomes:

$$\mathbf{E} = \int_0^{2\pi} \int_0^a \frac{(\rho_0/r) r \sin\alpha\, dr d\phi (z\mathbf{a}_z - r\mathbf{a}_r)}{4\pi\epsilon_0 (z^2 - 2rz\cos\alpha + r^2)^{3/2}}$$

It is necessary to include the ϕ dependence in $\mathbf{a}_r$.
We substitute $\mathbf{a}_r = \sin\alpha\cos\phi\mathbf{a}_x + \sin\alpha\sin\phi\mathbf{a}_y + \cos\alpha\mathbf{a}_z$. The first two terms of this, involving $\sin\phi$ and $\cos\phi$, will integrate to zero when taking ϕ from 0 to 2π. Only the z component survives, and the integral becomes, after performing the ϕ integration:

$$\mathbf{E} = 2\pi \int_0^a \frac{\rho_0 \sin\alpha (z - r\cos\alpha) dr\, \mathbf{a}_z}{4\pi\epsilon_0 (z^2 - 2rz\cos\alpha + r^2)^{3/2}}$$
$$= \frac{2\pi\rho_0 \sin\alpha \mathbf{a}_z}{4\pi\epsilon_0} \int_0^a \left[\frac{z dr}{(z^2 - 2rz\cos\alpha + r^2)^{3/2}} - \frac{r\cos\alpha dr}{(z^2 - 2rz\cos\alpha + r^2)^{3/2}} \right]$$

The two integrals evaluate as

$$\int_0^a \frac{z dr}{(z^2 - 2rz\cos\alpha + r^2)^{3/2}} = \frac{(r - z\cos\alpha) z}{z^2 \sin^2\alpha (z^2 - 2rz\cos\alpha + r^2)^{1/2}}\Big|_0^a$$

$$\int_0^a \frac{r\cos\alpha dr}{(z^2 - 2rz\cos\alpha + r^2)^{3/2}} = \frac{-(z - r\cos\alpha) z\cos\alpha}{z^2 \sin^2\alpha (z^2 - 2rz\cos\alpha + r^2)^{1/2}}\Big|_0^a$$

Combining these, cancelling terms, and rearranging, finally leads to

$$\mathbf{E} = \frac{2\pi a\rho_0 \sin\alpha\, \mathbf{a}_z}{4\pi\epsilon_0 z (z^2 - 2az\cos\alpha + a^2)^{1/2}}$$

b) Find the total charge on the cone: This will be

$$Q_c = \int\int_{cone} \rho_s da = \int_0^{2\pi} \int_0^a \frac{\rho_0}{r} r \sin\alpha\, dr d\phi = \underline{2\pi a\rho_0 \sin\alpha}$$

c) Specialize your result of part a to the case in which $\alpha = 90°$, at which the cone flattens to a disk in the xy plane. Compare this result to the answer to Problem 2.23.

When $\alpha = 90°$, $\cos\alpha = 0$, and the field expression in part a can be expressed as:

$$\mathbf{E}(\alpha = 0) = \frac{2\pi a\rho_0 \mathbf{a}_z}{4\pi\epsilon_0 z^2 [1 + (a/z)^2]^{1/2}}$$

which is the same result as found in Problem 2.23.

2.26 d) Show that your part *a* result becomes a point charge field when $z >> a$.

Using the expression for the total charge as found in part *b*, and rearranging terms, the field of part *a* takes the form:

$$\mathbf{E} = \frac{Q_c\,\mathbf{a}_z}{4\pi\epsilon_0 z^2[1-(2a/z)\cos\alpha + (a/z)^2]^{1/2}} \doteq \frac{Q_c\,\mathbf{a}_z}{4\pi\epsilon_0 z^2} \quad (z >> a)$$

where we recognize the second equality as the point charge field.

e) Show that your part *a* result becomes an inverse-z-dependent $\mathbf{E}$ field when $z << a$.

If $z << a$, we have $[1-(2a/z)\cos\alpha + (a/z)^2]^{1/2} \doteq a/z$ and thus

$$\mathbf{E} = \frac{Q_c\,\mathbf{a}_z}{4\pi\epsilon_0 z^2[1-(2a/z)\cos\alpha + (a/z)^2]^{1/2}} \doteq \frac{Q_c a\,\mathbf{a}_z}{4\pi\epsilon_0 z}$$

2.27. Given the electric field $\mathbf{E} = (4x-2y)\mathbf{a}_x - (2x+4y)\mathbf{a}_y$, find:

a) the equation of the streamline that passes through the point $P(2,3,-4)$: We write

$$\frac{dy}{dx} = \frac{E_y}{E_x} = \frac{-(2x+4y)}{(4x-2y)}$$

Thus

$$2(x\,dy + y\,dx) = y\,dy - x\,dx$$

or

$$2\,d(xy) = \frac{1}{2}\,d(y^2) - \frac{1}{2}\,d(x^2)$$

So

$$C_1 + 2xy = \frac{1}{2}y^2 - \frac{1}{2}x^2$$

or

$$y^2 - x^2 = 4xy + C_2$$

Evaluating at $P(2,3,-4)$, obtain:

$$9 - 4 = 24 + C_2, \text{ or } C_2 = -19$$

Finally, at P, the requested equation is

$$\underline{y^2 - x^2 = 4xy - 19}$$

b) a unit vector specifying the direction of $\mathbf{E}$ at $Q(3,-2,5)$: Have $\mathbf{E}_Q = [4(3)+2(2)]\mathbf{a}_x - [2(3) - 4(2)]\mathbf{a}_y = 16\mathbf{a}_x + 2\mathbf{a}_y$. Then $|\mathbf{E}| = \sqrt{16^2+4} = 16.12$ So

$$\mathbf{a}_Q = \frac{16\mathbf{a}_x + 2\mathbf{a}_y}{16.12} = \underline{0.99\mathbf{a}_x + 0.12\mathbf{a}_y}$$

2.28 An electric dipole (discussed in detail in Sec. 4.7) consists of two point charges of equal and opposite magnitude $\pm Q$ spaced by distance d. With the charges along the z axis at positions $z = \pm d/2$ (with the positive charge at the positive z location), the electric field in spherical coordinates is given by $\mathbf{E}(r,\theta) = \left[Qd/(4\pi\epsilon_0 r^3)\right]\left[2\cos\theta\mathbf{a}_r + \sin\theta\mathbf{a}_\theta\right]$, where $r >> d$. Using rectangular coordinates, determine expressions for the vector force on a point charge of magnitude q:

a) at $(0,0,z)$: Here, $\theta = 0$, $\mathbf{a}_r = \mathbf{a}_z$, and $r = z$. Therefore

$$\mathbf{F}(0,0,z) = \frac{qQd\,\mathbf{a}_z}{4\pi\epsilon_0 z^3}\ \text{N}$$

b) at $(0,y,0)$: Here, $\theta = 90°$, $\mathbf{a}_\theta = -\mathbf{a}_z$, and $r = y$. The force is

$$\mathbf{F}(0,y,0) = \frac{-qQd\,\mathbf{a}_z}{4\pi\epsilon_0 y^3}\ \text{N}$$

2.29. If $\mathbf{E} = 20e^{-5y}\left(\cos 5x\mathbf{a}_x - \sin 5x\mathbf{a}_y\right)$, find:

a) $|\mathbf{E}|$ at $P(\pi/6, 0.1, 2)$: Substituting this point, we obtain $\mathbf{E}_P = -10.6\mathbf{a}_x - 6.1\mathbf{a}_y$, and so $|\mathbf{E}_P| = \underline{12.2}$.

b) a unit vector in the direction of $\mathbf{E}_P$: The unit vector associated with $\mathbf{E}$ is $\left(\cos 5x\mathbf{a}_x - \sin 5x\mathbf{a}_y\right)$, which evaluated at P becomes $\mathbf{a}_E = \underline{-0.87\mathbf{a}_x - 0.50\mathbf{a}_y}$.

c) the equation of the direction line passing through P: Use

$$\frac{dy}{dx} = \frac{-\sin 5x}{\cos 5x} = -\tan 5x \quad\Rightarrow\quad dy = -\tan 5x\,dx$$

Thus $y = \frac{1}{5}\ln\cos 5x + C$. Evaluating at P, we find $C = 0.13$, and so

$$\underline{y = \frac{1}{5}\ln\cos 5x + 0.13}$$

2.30. For fields that do not vary with z in cylindrical coordinates, the equations of the streamlines are obtained by solving the differential equation $E_\rho/E_\phi = d\rho(\rho d\phi)$. Find the equation of the line passing through the point $(2, 30°, 0)$ for the field $\mathbf{E} = \rho\cos 2\phi\,\mathbf{a}_\rho - \rho\sin 2\phi\,\mathbf{a}_\phi$:

$$\frac{E_\rho}{E_\phi} = \frac{d\rho}{\rho d\phi} = \frac{-\rho\cos 2\phi}{\rho\sin 2\phi} = -\cot 2\phi \;\Rightarrow\; \frac{d\rho}{\rho} = -\cot 2\phi\,d\phi$$

Integrate to obtain

$$2\ln\rho = \ln\sin 2\phi + \ln C = \ln\left[\frac{C}{\sin 2\phi}\right] \;\Rightarrow\; \rho^2 = \frac{C}{\sin 2\phi}$$

At the given point, we have $4 = C/\sin(60°) \;\Rightarrow\; C = 4\sin 60° = 2\sqrt{3}$. Finally, the equation for the streamline is $\underline{\rho^2 = 2\sqrt{3}/\sin 2\phi}$.