CHAPTER 2

P. E. 2.1

(a) At P(1,3,5), x = 1, y = 3, z =5,

$\rho = \sqrt{x^2 + y^2} = \sqrt{10}$, z = 5, $\phi = \tan^{-1} y/x = \tan^{-1} 3 = 71.6^o$

$P(\rho, \phi, z) = P(\sqrt{10}, \tan^{-1} 3, 5) = \underline{\underline{P(3.162, 71.6^o, 5)}}$

Spherical system:

$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{35} = 5.916$

$\theta = \tan^{-1} \sqrt{x^2 + y^2} / z = \tan^{-1} \sqrt{10}/5 = \tan^{-1} 0.6325 = 32.31°$

$P(r, \theta, \varphi) = \underline{\underline{P(5.916, 32.31°, 71.57°)}}$

At T(0,-4,3), x = 0 y =-4, z =3;

$\rho = \sqrt{x^2 + y^2} = 4, z = 3, \varphi = \tan^{-1} y/x = \tan^{-1} -4/0 = 270°$

$\underline{\underline{T(\rho, \varphi, z) = T(4, 270°, 3).}}$

Spherical system:

$r = \sqrt{x^2 + y^2 + z^2} = 5, \theta = \tan^{-1} \rho / z = \tan^{-1} 4/3 = 53.13°$.

$\underline{\underline{T(r, \theta, \varphi) = T(5, 53.13°, 270°).}}$

At S(-3-4-10), x =-3, y =-4, z =-10;

$$\rho = \sqrt{x^2 + y^2} = 5, \phi = \tan^{-1}\left(\frac{-4}{-3}\right) = 233.1°$$

$\underline{\underline{S(\rho, \phi, z) = S(5, 233.1, -10).}}$

Spherical system:

$r = \sqrt{x^2 + y^2 + z^2} = 5\sqrt{5} = 11.18.$

$$\theta = \tan^{-1} \rho/z = \tan^{-1} \frac{5}{-10} = 153.43°;$$

$\underline{\underline{S(r, \theta, \phi) = S(11.18, 153.43°, 233.1°).}}$

(b) In Cylindrical system, $\rho = \sqrt{x^2 + y^2}$; $yz = z\rho \sin\phi$,

$$Q_x = \frac{\rho}{\sqrt{\rho^2 + z^2}}; \quad Q_y = 0; \quad Q_z = -\frac{z\rho\sin\phi}{\sqrt{\rho^2 + z^2}};$$

$$\begin{bmatrix} Q_\rho \\ Q_\phi \\ Q_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Q_x \\ 0 \\ Q_z \end{bmatrix};$$

$$Q_\rho = Q_x \cos\phi = \frac{\rho\cos\phi}{\sqrt{\rho^2+z^2}}, \qquad Q_\phi = -Q_x \sin\phi = \frac{-\rho\sin\phi}{\sqrt{\rho^2+z^2}}$$

Hence,

$$\underline{\underline{\boldsymbol{Q} = \frac{\rho}{\sqrt{\rho^2+z^2}}(\cos\phi\boldsymbol{a}_\rho - \sin\phi\,\boldsymbol{a}_\phi - z\sin\phi\,\boldsymbol{a}_z).}}$$

In Spherical coordinates:

$$Q_x = \frac{r\sin\theta}{r} = \sin\theta;$$

$$Q_z = -r\sin\phi\sin\theta\, r\cos\theta\,\frac{1}{r} = -r\sin\theta\cos\theta\sin\phi.$$

$$\begin{bmatrix} Q_r \\ Q_\theta \\ Q_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} Q_x \\ 0 \\ Q_z \end{bmatrix};$$

$$Q_r = Q_x \sin\theta\cos\phi + Q_z\cos\theta = \sin^2\theta\cos\phi - r\sin\theta\cos^2\theta\sin\phi.$$

$$Q_\theta = Q_x\cos\theta\cos\phi - Q_z\sin\theta = \sin\theta\cos\theta\cos\phi + r\sin^2\theta\cos\theta\sin\phi.$$

$$Q_\phi = -Q_x\sin\phi = -\sin\theta\sin\phi.$$

$$\underline{\underline{\therefore\ \boldsymbol{Q} = \sin\theta\left(\sin\theta\cos\phi - r\cos^2\theta\sin\phi\right)\boldsymbol{a}_r + \sin\theta\cos\theta(\cos\phi + r\sin\theta\sin\phi)\boldsymbol{a}_\theta - \sin\theta\sin\phi\,\boldsymbol{a}_\phi\ .}}$$

At T :

$$\boldsymbol{Q}(x,y,z) = \frac{4}{5}\boldsymbol{a}_x + \frac{12}{5}\boldsymbol{a}_z = 0.8\boldsymbol{a}_x + 2.4\boldsymbol{a}_z;$$

$$\boldsymbol{Q}(\rho,\phi,z) = \frac{4}{5}(\cos 270^\circ\,\boldsymbol{a}_\rho - \sin 270^\circ\,\boldsymbol{a}_\phi - 3\sin 270^\circ\boldsymbol{a}_z$$

$$= 0.8\boldsymbol{a}_\phi + 2.4\boldsymbol{a}_z;$$

$$\boldsymbol{Q}(r,\theta,\phi) = \frac{4}{5}(0 - \frac{45}{25}(-1))\boldsymbol{a}_r + \frac{4}{5}(\frac{3}{5})(0 + \frac{20}{5}(-1))\boldsymbol{a}_\theta - \frac{4}{5}(-1)\boldsymbol{a}_\phi$$

$$= \frac{36}{25}\boldsymbol{a}_r - \frac{48}{25}\boldsymbol{a}_\theta + \frac{4}{5}\boldsymbol{a}_\phi = \underline{\underline{1.44\boldsymbol{a}_r - 1.92\boldsymbol{a}_\theta + 0.8\boldsymbol{a}_\phi}};$$

Note, that the magnitude of vector Q = 2.53 in all 3 cases above.

P.E. 2.2 (a)

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho z\sin\phi \\ 3\rho\cos\phi \\ \rho\cos\phi\sin\phi \end{bmatrix}$$

$$\boldsymbol{A} = (\rho z\cos\phi\sin\phi - 3\rho\cos\phi\sin\phi)\boldsymbol{a}_x + (\rho z\sin^2\phi + 3\rho\cos^2\phi)\boldsymbol{a}_y + \rho\cos\phi\sin\phi\boldsymbol{a}_z.$$

But $\rho = \sqrt{x^2+y^2}$, $\tan\phi = \dfrac{y}{x}$, $\cos\phi = \dfrac{x}{\sqrt{x^2+y^2}}$, $\sin\phi = \dfrac{y}{\sqrt{x^2+y^2}}$;

Substituting all this yields:

$$\boldsymbol{A} = \frac{1}{\sqrt{x^2+y^2}}[(xyz - 3xy)\boldsymbol{a}_x + (zy^2 + 3x^2)\boldsymbol{a}_y + xy\boldsymbol{a}_z].$$

$$\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} r^2 \\ 0 \\ \sin\theta \end{bmatrix}$$

Since $r = \sqrt{x^2+y^2+z^2}$, $\tan\theta = \dfrac{\sqrt{x^2+y^2}}{z}$, $\tan\phi = \dfrac{y}{z}$;

and $\sin\theta = \dfrac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}$, $\cos\theta = \dfrac{z}{\sqrt{x^2+y^2+z^2}}$;

and $\sin\phi = \dfrac{y}{\sqrt{x^2+y^2}}$, $\cos\phi = \dfrac{x}{\sqrt{x^2+y^2}}$;

$$B_x = r^2\sin\theta\cos\phi - \sin\theta\sin\phi = rx - \frac{y}{r} = \frac{1}{r}(r^2x - y).$$

$$B_y = r^2\sin\theta\sin\phi + \sin\theta\cos\phi = ry + \frac{x}{r} = \frac{1}{r}(r^2y + x).$$

$$B_z = r^2\cos\theta = r\,z = \frac{1}{r}(r^2z).$$

Hence,

$$\boldsymbol{B} = \frac{1}{\sqrt{x^2+y^2+z^2}}[\{x(x^2+y^2+z^2) - y\}\boldsymbol{a}_x + \{y(x^2+y^2+z^2) + x\}\boldsymbol{a}_y + z(x^2+y^2+z^2)\boldsymbol{a}_z].$$

P.E.2.3 (a) At:

$(1, \pi/3, 0), \quad \boldsymbol{H} = (0, 0.06767, 1)$

$$\boldsymbol{a}_x = \cos\phi\,\boldsymbol{a}_\rho - \sin\phi\,\boldsymbol{a}_\phi = \frac{1}{2}(\boldsymbol{a}_\rho - \sqrt{3}\,\boldsymbol{a}_\phi)$$

$$\boldsymbol{H} \bullet \boldsymbol{a}_x = \underline{\underline{-0.0586.}}$$

(b) At:

$$(1, \pi/3,\ 0), \quad \boldsymbol{a}_\theta = \cos\theta\,\boldsymbol{a}_\rho - \sin\theta\,\boldsymbol{a}_z = -\boldsymbol{a}_z\,.$$

$$H \times \boldsymbol{a}_z = \begin{vmatrix} \boldsymbol{a}_\rho & a_\phi & a_z \\ 0 & 0.06767 & 1 \\ 0 & 0 & 1 \end{vmatrix} = \underline{\underline{-0.06767\,a_{\rho.}}}$$

(c) $(\boldsymbol{H} \bullet \boldsymbol{a}_\rho)\boldsymbol{a}_\rho = \underline{\underline{0\boldsymbol{a}_\rho}}.$

(d)

$$\boldsymbol{H} \times \boldsymbol{a}_z = \begin{vmatrix} \boldsymbol{a}_\rho & \boldsymbol{a}_\phi & \boldsymbol{a}_z \\ 0 & 0.06767 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 0.06767\,\boldsymbol{a}_\rho.$$

$$|\boldsymbol{H} \times \boldsymbol{a}_z| = \underline{\underline{0.06767}}$$

P.E. 2.4

(a)

$$A \cdot \boldsymbol{B} = (3, 2, -6) \bullet (4, 0, 3) = \underline{\underline{-6.}}$$

(b)

$$|\boldsymbol{A} \times B| = \begin{vmatrix} 3 & 2 & -6 \\ 4 & 0 & 3 \end{vmatrix} = |6\,\boldsymbol{a}_r - 33\boldsymbol{a}_\theta - 8\boldsymbol{a}_\phi|.$$

Thus the magnitude of $A \times B = \underline{\underline{34.48.}}$

(c)

$At\ (1,\ \pi/3,\ 5\pi/4), \quad \theta = \pi/3,$

$$\boldsymbol{a}_z = \cos\theta\,\boldsymbol{a}_r - \sin\theta\,\boldsymbol{a}_\theta = \frac{1}{2}\boldsymbol{a}_r - \frac{\sqrt{3}}{2}\boldsymbol{a}_\theta.$$

$$(A \cdot \boldsymbol{a}_z)\boldsymbol{a}_z = \left(\frac{3}{2} - \sqrt{3}\right)\left(\frac{1}{2}\boldsymbol{a}_r - \frac{\sqrt{3}}{2}\boldsymbol{a}_\theta\right) = \underline{\underline{-0.116\boldsymbol{a}_r + 0.201\boldsymbol{a}_\theta}}$$

Prob. 2.1

(a)

$$\rho = \sqrt{x^2 + y^2} = \sqrt{4+25} = 5.3852, \qquad \phi = \tan^{-1}\frac{y}{x} = \tan^{-1} 2.5 = 68.2^o$$

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{4+25+1} = 5.477, \qquad \theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z} = \tan^{-1}\frac{5.3852}{1} = 79.48^o$$

$$P(\rho,\phi,z) = \underline{\underline{P(5.3852, 68.2^o, 1)}}, \qquad P(r,\theta,\phi) = \underline{\underline{P(5.477, 79.48^o, 68.2^o)}}$$

(b)

$$\rho = \sqrt{x^2 + y^2} = \sqrt{9+16} = 5, \qquad \phi = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{4}{-3} = 360^o - 53.123^o = 306.88^o$$

$$r = \sqrt{x^2 + y^2 + z^2} = 5, \qquad \theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z} = \tan^{-1}\infty = 90^o$$

$$\mathrm{Q}(\rho,\phi,z) = \underline{\underline{\mathrm{Q}(5, 306.88^o, 0)}}, \qquad P(r,\theta,\phi) = \underline{\underline{P(5, 90^o, 306.88^o)}}$$

(c)

$$\rho = \sqrt{x^2 + y^2} = \sqrt{36+4} = 6.325, \qquad \phi = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{2}{6} = 18.43^o$$

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{36+4+16} = 7.483,$$

$$\theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z} = \tan^{-1}\frac{6.325}{-4} = 180^o - 57.69^o = 122.31^o$$

$$\mathrm{R}(\rho,\phi,z) = \underline{\underline{\mathrm{R}(6.325, 18.43^o, -4)}}, \qquad \mathrm{R}(r,\theta,\phi) = \underline{\underline{\mathrm{R}(7.483, 122.31^o, 18.43^o)}}$$

Prob. 2.2

(a)

$$x = \rho\cos\phi = 2\cos 30^\circ = 1.732;$$

$$y = \rho\sin\phi = 2\sin 30^\circ = 1;$$

$$z = 5;$$

$$P_1(x,y,z) = \underline{\underline{P_1(1.732, 1, 5).}}$$

(b)

$$x = 1\cos 90^\circ = 0; \qquad y = 1\sin 90^\circ = 1; \qquad z = -3.$$

$$P_2(x,y,z) = \underline{\underline{P_2(0, 1, -3).}}$$

(c)

$$x = r\sin\theta\cos\phi = 10\sin(\pi/4)\cos(\pi/3) = 3.535;$$
$$y = r\sin\theta\sin\phi = 10\sin(\pi/4)\sin(\pi/3) = 6.124;$$
$$z = r\cos\theta = 10\cos(\pi/4) = 7.0711$$
$$P_3(x,y,z) = \underline{\underline{P_3(3.535,\ 6.124, 7.0711).}}$$

(d)

$$x = 4\sin 30^\circ \cos 60^\circ = 1$$
$$y = 4\sin 30^\circ \sin 60^\circ = 1.7321$$
$$z = r\cos\theta = 4\cos 30^\circ = 3.464$$
$$P_4(x,y,z) = \underline{\underline{P_4(1, 1.7321, 3.464).}}$$

Prob. 2.3

$$\rho = \sqrt{x^2 + y^2} = \sqrt{4+36} = 6.324$$

(a) $\phi = \tan^{-1}\dfrac{y}{x} = \tan^{-1}\dfrac{6}{2} = 71.56^o$

P is $\underline{\underline{(6.324, 71.56^o, -4)}}$

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{4+36+16} = 7.485$$

(b) $\theta = \tan^{-1}\dfrac{\sqrt{x^2+y^2}}{z} = \tan^{-1}\dfrac{6.324}{-4} = 90^o + \tan^{-1}\dfrac{4}{6.324} = 122.3^o$

P is $\underline{\underline{(7.483, 122.3^o, 71.56^o)}}$

Prob. 2.4

(a)

$$x = \rho\cos\phi = 5\cos 120^o = -2.5$$
$$y = \rho\sin\phi = 5\sin 120^o = 4.33$$
$$z = 1$$

Hence Q $= \underline{\underline{(-2.5, 4.33, 1)}}$

(b)

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{\rho^2 + z^2} = \sqrt{25+1} = 5.099$$

$$\theta = \tan^{-1}\frac{\sqrt{x^2 + y^2}}{z} = \tan^{-1}\frac{\rho}{z} = \tan^{-1}\frac{5}{1} = 78.69^o$$

$$\phi = 120^o$$

Hence Q $= \underline{\underline{(5.099, 78.69^o, 120^o)}}$

Prob. 2.5

$$T(r,\theta,\phi) \longrightarrow r = 10, \theta = 60^o, \phi = 30^o$$

$$x = r\sin\theta\cos\phi = 10\sin 60^o \cos 30^o = 7.5$$

$$y = r\sin\theta\sin\phi = 10\sin 60^o \sin 30^o = 4.33$$

$$z = r\cos\theta = 10\cos 60^o = 5$$

$$T(x,y,z) = \underline{\underline{(7.5, 4.33, 5)}}$$

$$\rho = r\sin\theta = 10\sin 60^o = 8.66$$

$$T(\rho,\phi,z) = \underline{\underline{(8.66, 30^o, 5)}}$$

Prob. 2.6

(a)

$$x = \rho\cos\phi, \qquad y = \rho\sin\phi,$$

$$V = \underline{\underline{\rho z\cos\phi - \rho^2\sin\phi\cos\phi + \rho z\sin\phi}}$$

(b)

$$U = x^2 + y^2 + z^2 + y^2 + 2z^2$$

$$= r^2 + r^2\sin^2\theta\sin^2\phi + 2r^2\cos^2\theta$$

$$= \underline{\underline{r^2[1 + \sin^2\theta\sin^2\phi + 2\cos^2\theta]}}$$

Prob. 2.7

(a)

$$\begin{bmatrix} F_\rho \\ F_\phi \\ F_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dfrac{x}{\sqrt{\rho^2+z^2}} \\ \dfrac{y}{\sqrt{\rho^2+z^2}} \\ \dfrac{4}{\sqrt{\rho^2+z^2}} \end{bmatrix}$$

$$F_\rho = \frac{1}{\sqrt{\rho^2+z^2}}[\rho\cos^2\phi + \rho\sin^2\phi] = \frac{\rho}{\sqrt{\rho^2+z^2}};$$

$$F_\phi = \frac{1}{\sqrt{\rho^2+z^2}}[-\rho\cos\phi\sin\phi + \rho\cos\phi\sin\phi] = 0;$$

$$F_z = \frac{4}{\sqrt{\rho^2+z^2}};$$

$$\bar{F} = \underline{\underline{\frac{1}{\sqrt{\rho^2+z^2}}(\rho \boldsymbol{a}_\rho + 4\boldsymbol{a}_z)}}$$

In Spherical:

$$\begin{bmatrix} F_r \\ F_\theta \\ F_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \dfrac{x}{r} \\ \dfrac{y}{r} \\ \dfrac{4}{r} \end{bmatrix}$$

$$F_r = \frac{r}{r}\sin^2\theta\cos^2\phi + \frac{r}{r}\sin^2\theta\sin^2\phi + \frac{4}{r}\cos\theta \quad = \sin^2\theta + \frac{4}{r}\cos\theta;$$

$$F_\theta = \sin\theta\cos\theta\cos^2\phi + \sin\theta\cos\theta\sin^2\phi - \frac{4}{r}\sin\theta = \sin\theta\cos\theta - \frac{4}{r}\sin\theta;$$

$$F_\phi = -\sin\theta\cos\phi\sin\phi + \sin\theta\sin\phi\cos\phi = 0;$$

$$\therefore \bar{F} = \underline{\underline{(\sin^2\theta + \frac{4}{r}\cos\theta)\boldsymbol{a}_r + \sin\theta(\cos\theta - \frac{4}{r})\boldsymbol{a}_\theta}}$$

(b)

$$\begin{bmatrix} G_\rho \\ G_\phi \\ G_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dfrac{x\rho^2}{\sqrt{\rho^2+z^2}} \\ \dfrac{y\rho^2}{\sqrt{\rho^2+z^2}} \\ \dfrac{z\rho^2}{\sqrt{\rho^2+z^2}} \end{bmatrix}$$

$$G_\rho = \frac{\rho^2}{\sqrt{\rho^2+z^2}}[\rho\cos^2\phi + \rho\sin^2\phi] = \frac{\rho^3}{\sqrt{\rho^2+z^2}};$$

$$G_\phi = 0;$$

$$G_z = \frac{z\rho^2}{\sqrt{\rho^2+z^2}};$$

$$\boldsymbol{G} = \frac{\rho^2}{\sqrt{\rho^2+z^2}}(\rho\,\boldsymbol{a}_\rho + z\,\boldsymbol{a}_z)$$

Spherical :

$$\boldsymbol{G} = \frac{\rho^2}{r}(x\boldsymbol{a}_x + y\boldsymbol{a}_y + z\boldsymbol{a}_z) = \frac{r^2\sin^2\theta}{r} r\boldsymbol{a}_r = r^2\sin^2\theta\boldsymbol{a}_r$$

Prob. 2.8

$$\boldsymbol{B} = \rho\boldsymbol{a}_x + \frac{y}{\rho}\boldsymbol{a}_y + z\boldsymbol{a}_z$$

$$\begin{bmatrix} B_\rho \\ B_\phi \\ B_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho \\ y/\rho \\ z \end{bmatrix}$$

$$B_\rho = \rho\cos\phi + \frac{y}{\rho}\sin\phi$$

$$B_\phi = -\rho\sin\phi + \frac{y}{\rho}\cos\phi$$

$$B_z = z$$

But $y = \rho\sin\phi$

$$B_\rho = \rho\cos\phi + \sin^2\phi, B_\phi = -\rho\sin\phi + \sin\phi\cos\phi$$

Hence,

$$\boldsymbol{B} = (\rho\cos\phi + \sin^2\phi)\boldsymbol{a}_\rho + \sin\phi(\cos\phi - \rho)\boldsymbol{a}_\phi + z\boldsymbol{a}_z$$

Prob. 2.9

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

At P, $\rho = 2, \quad \phi = \pi/2, \quad z = -1$

$$A_x = 2\cos\phi - 3\sin\phi = 2\cos 90^o - 3\sin 90^o = -3$$

$$A_y = 2\sin\phi + 3\cos\phi = 2\sin 90^o + 3\cos 90^o = 2$$

$$A_z = 4$$

Hence, $\quad \underline{\underline{\boldsymbol{A} = -3\boldsymbol{a}_x + 2\boldsymbol{a}_y + 4\boldsymbol{a}_z}}$

Prob. 2.10

(a)

$$\begin{bmatrix} A_x \\ A_y \\ A_y \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho\sin\phi \\ \rho\cos\phi \\ -2z \end{bmatrix}$$

$$A_x = \rho\sin\phi\cos\phi - \rho\cos\phi\sin\phi = 0$$

$$A_y = \rho\sin^2\phi + \rho\cos^2\phi = \rho = \sqrt{x^2 + y^2}$$

$$A_z = -2z$$

Hence,

$$\underline{\underline{\boldsymbol{A} = \sqrt{x^2 + y^2}\,\boldsymbol{a}_y - 2z\boldsymbol{a}_z}}$$

(b)

$$\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} 4r\cos\phi \\ r \\ 0 \end{bmatrix}$$

$$B_x = 4r\sin\theta\cos^2\phi + r\cos\theta\cos\phi$$

$$B_y = 4r\sin\theta\sin\phi\cos\phi + r\cos\theta\sin\phi$$

$$B_z = 4r\cos\theta\cos\phi - r\sin\theta$$

But $\quad r = \sqrt{x^2 + y^2 + z^2}, \quad \sin\theta = \dfrac{\sqrt{x^2 + y^2}}{r}, \quad \cos\theta = \dfrac{z}{r}$

$$\sin\phi = \frac{y}{\sqrt{x^2 + y^2}}, \quad \cos\phi = \frac{x}{\sqrt{x^2 + y^2}}$$

$$B_x = 4\sqrt{x^2+y^2}\,\frac{x^2}{x^2+y^2} + \frac{zx}{\sqrt{x^2+y^2}}$$

$$B_y = 4\sqrt{x^2+y^2}\,\frac{xy}{x^2+y^2} + \frac{zy}{\sqrt{x^2+y^2}}$$

$$B_z = 4z\frac{x}{\sqrt{x^2+y^2}} - \sqrt{x^2+y^2}$$

$$\boldsymbol{B} = \frac{1}{\sqrt{x^2+y^2}}\left[x(4x+z)\boldsymbol{a}_x + y(4x+z)\boldsymbol{a}_y + (4xz - x^2 - y^2)\boldsymbol{a}_z\right]$$

Prob. 2.11

Method 1:

$$\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} 4/r^2 \\ 0 \\ 0 \end{bmatrix}$$

$$F_x = \frac{4}{r^2}\sin\theta\cos\phi, \qquad F_y = \frac{4}{r^2}\sin\theta\sin\phi, \qquad F_z = \frac{4}{r^2}\cos\theta$$

$$r^2 = x^2+y^2+z^2, \qquad \sin\theta = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}, \qquad \cos\theta = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$\sin\phi = \frac{y}{\sqrt{x^2+y^2}}, \qquad \cos\phi = \frac{x}{\sqrt{x^2+y^2}}$$

$$F_x = \frac{4}{x^2+y^2+z^2}\,\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}\,\frac{x}{\sqrt{x^2+y^2}} = \frac{4x}{(x^2+y^2+z^2)^{3/2}}$$

$$F_y = \frac{4}{x^2+y^2+z^2}\,\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}\,\frac{y}{\sqrt{x^2+y^2}} = \frac{4y}{(x^2+y^2+z^2)^{3/2}}$$

$$F_x = \frac{4}{x^2+y^2+z^2}\,\frac{z}{(x^2+y^2+z^2)} = \frac{4z}{(x^2+y^2+z^2)^{3/2}}$$

Thus,

$$\boldsymbol{F} = \frac{4}{(x^2+y^2+z^2)^{3/2}}\left[x\boldsymbol{a}_x + y\boldsymbol{a}_y + z\boldsymbol{a}_z\right]$$

Method 2:

$$\boldsymbol{F} = \frac{4\boldsymbol{a}_r}{r^2} \cdot \frac{\boldsymbol{r}}{r} = \frac{4r\boldsymbol{a}_r}{r^3}$$

$$\boldsymbol{F} = \frac{4}{(x^2+y^2+z^2)^{3/2}}\left[x\boldsymbol{a}_x + y\boldsymbol{a}_y + z\boldsymbol{a}_z \right]$$

Prob. 2.12

(a)
$$r = 2, \quad \theta = \pi/2, \quad \phi = 3\pi/2$$
$$\boldsymbol{B} = 2\sin(\pi/2)\boldsymbol{a}_r - 4\cos(3\pi/2)\boldsymbol{a}_\phi = \underline{\underline{2\boldsymbol{a}_r}}$$

(b)

$$\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \cos\theta\cos\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} r\sin\theta \\ 0 \\ -r^2\cos\phi \end{bmatrix}$$

$$B_x = r\sin^2\theta\cos\phi - r^2\sin\phi\cos\phi, \qquad B_y = r\sin\theta\cos\theta\cos\phi - r^2\cos^2\phi$$

$$B_z = r\sin\theta\cos\theta$$

But $\quad r = \sqrt{x^2+y^2+z^2}, \cos\theta = \dfrac{z}{r}, \sin\theta = \dfrac{\rho}{r} = \dfrac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}$

$$\cos\phi = \frac{x}{\rho} = \frac{x}{\sqrt{x^2+y^2}}, \qquad \sin\phi = \frac{y}{\rho} = \frac{y}{\sqrt{x^2+y^2}}$$

$$B_x = \sqrt{x^2+y^2+z^2}\,\frac{x^2+y^2}{x^2+y^2+z^2}\frac{x}{\sqrt{x^2+y^2}} - (x^2+y^2+z^2)\frac{xy}{x^2+y^2}$$

$$= \frac{x\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}} - \frac{xy(x^2+y^2+z^2)}{x^2+y^2}$$

$$B_y = \sqrt{x^2+y^2+z^2}\,\frac{z\sqrt{x^2+y^2}}{x^2+y^2+z^2}\frac{x}{\sqrt{x^2+y^2}} - (x^2+y^2+z^2)\frac{x^2}{x^2+y^2}$$

$$= \frac{xz}{\sqrt{x^2+y^2+z^2}} - \frac{x^2(x^2+y^2+z^2)}{x^2+y^2}$$

$$B_z = \sqrt{x^2+y^2+z^2}\,\frac{z\sqrt{x^2+y^2}}{x^2+y^2+z^2} = \frac{z\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}$$

$$\boldsymbol{B} = B_x\boldsymbol{a}_x + B_y\boldsymbol{a}_y + B_z\boldsymbol{a}_z$$

Prob. 2.13

(a) $x = \rho\cos\phi$

$$\underline{\underline{\boldsymbol{B} = \rho\cos\phi\boldsymbol{a}_z}}$$

$x = r\sin\theta\cos\phi$

(b) $\boldsymbol{B} = r\sin\theta\cos\phi\boldsymbol{a}_z, \qquad B_x = 0 = B_y, B_z = r\sin\theta\cos\phi$

$$\begin{bmatrix} B_r \\ B_\theta \\ B_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ r\sin\theta\cos\phi \end{bmatrix}$$

$$B_r = r\sin\theta\cos\theta\cos\phi = 0.5r\sin(2\theta)\cos\phi$$

$$B_\theta = -r\sin^2\theta\cos\phi, \qquad B_\phi = 0$$

$$\underline{\underline{\boldsymbol{B} = 0.5r\sin(2\theta)\cos\phi\boldsymbol{a}_r - r\sin^2\theta\cos\phi\boldsymbol{a}_\theta}}$$

Prob. 2.14

(a)

$$\boldsymbol{a}_x \times \boldsymbol{a}_\rho = (\cos\phi\,\boldsymbol{a}_\rho - \sin\phi\boldsymbol{a}_\phi)\times\boldsymbol{a}_\rho = \cos\phi$$

$$\boldsymbol{a}_x \times \boldsymbol{a}_\phi = (\cos\phi\boldsymbol{a}_\rho - \sin\phi\,\boldsymbol{a}_\phi)\times\boldsymbol{a}_\phi = -\sin\phi$$

$$\boldsymbol{a}_y \times \boldsymbol{a}_\rho = (\sin\phi\,\boldsymbol{a}_\rho + \cos\phi\boldsymbol{a}_\phi)\times\boldsymbol{a}_\rho = \sin\phi$$

$$\bar{a}_y \times \bar{a}_\phi = (\sin\phi\,\boldsymbol{a}_\rho + \sin\phi\boldsymbol{a}_\phi)\times\boldsymbol{a}_\phi = \cos\phi$$

(b) and (c)

In spherical system:

$$\boldsymbol{a}_x = \sin\theta\cos\phi\boldsymbol{a}_r + \cos\theta\cos\phi\,\boldsymbol{a}_\theta - \sin\phi\boldsymbol{a}_\phi.$$

$$\boldsymbol{a}_y = \sin\theta\sin\phi\,\boldsymbol{a}_r + \cos\theta\sin\phi\boldsymbol{a}_\theta - \cos\phi\boldsymbol{a}_\phi.$$

$$\boldsymbol{a}_z = \cos\theta\,\boldsymbol{a}_x - \sin\theta\boldsymbol{a}_\theta.$$

Hence,

$$\mathbf{a}_x \times \mathbf{a}_r = \sin\theta\cos\phi;$$

$$\mathbf{a}_x \times \mathbf{a}_\theta = \cos\theta\cos\phi;$$

$$\mathbf{a}_y \times \mathbf{a}_r = \sin\theta\sin\phi;$$

$$\mathbf{a}_y \times \mathbf{a}_\theta = \cos\theta\sin\phi;$$

$$\bar{a}_z \times \bar{a}_r = \cos\theta;$$

$$\bar{a}_z \times \bar{a}_\theta = -\sin\theta;$$

Prob. 2.15

(a)

$$\mathbf{a}_\rho = \cos\phi\mathbf{a}_x + \sin\phi\mathbf{a}_y, \qquad \mathbf{a}_\phi = -\sin\phi\mathbf{a}_x + \cos\phi\mathbf{a}_y$$

$$\mathbf{a}_\rho \times \mathbf{a}_\phi = \begin{vmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \end{vmatrix} = (\cos^2\phi + \sin^2\phi)\mathbf{a}_z = \mathbf{a}_z$$

$$\mathbf{a}_z \times \mathbf{a}_\rho = \begin{vmatrix} 0 & 0 & 1 \\ \cos\phi & \sin\phi & 0 \end{vmatrix} = -\sin\phi\mathbf{a}_x + \cos\phi\mathbf{a}_y = \mathbf{a}_\phi$$

$$\mathbf{a}_\phi \times \mathbf{a}_z = \begin{vmatrix} -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = \cos\phi\mathbf{a}_x + \sin\phi\mathbf{a}_y = \mathbf{a}_\rho$$

(b)

$$\mathbf{a}_r = \sin\theta\cos\phi\mathbf{a}_x + \sin\theta\sin\phi\mathbf{a}_y + \cos\theta\mathbf{a}_z$$

$$\mathbf{a}_\theta = \cos\theta\cos\phi\mathbf{a}_x + \cos\theta\sin\phi\mathbf{a}_y - \sin\theta\mathbf{a}_z$$

$$\mathbf{a}_\phi = -\sin\phi\mathbf{a}_x + \cos\phi\mathbf{a}_y$$

$$\mathbf{a}_r \times \mathbf{a}_\theta = \begin{vmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \end{vmatrix}$$

$$= (-\sin^2\theta\sin\phi - \cos^2\theta\sin\phi)\mathbf{a}_x + (\cos^2\theta\cos\phi + \sin^2\theta\cos\phi)\mathbf{a}_y$$

$$+(\sin\theta\cos\theta\sin\phi\cos\phi - \sin\theta\cos\theta\sin\phi\cos\phi)\mathbf{a}_z$$

$$= -\sin\phi\mathbf{a}_x + \cos\phi\mathbf{a}_y = \mathbf{a}_\phi$$

$$\boldsymbol{a}_\phi \times \boldsymbol{a}_r = \begin{vmatrix} -\sin\phi & \cos\phi & 0 \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{vmatrix}$$

$$= \cos\theta\cos\phi\ \boldsymbol{a}_x + \cos\theta\sin\phi\boldsymbol{a}_y + (-\sin\theta\sin^2\phi - \sin\theta\cos^2\phi)\boldsymbol{a}_z$$

$$= \cos\theta\cos\phi\boldsymbol{a}_x + \cos\theta\sin\phi\boldsymbol{a}_y - \sin\theta\boldsymbol{a}_z = \boldsymbol{a}_\theta$$

$$\boldsymbol{a}_\theta \times \boldsymbol{a}_\phi = \begin{vmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{vmatrix}$$

$$= \sin\theta\cos\phi\ \boldsymbol{a}_x + \sin\theta\sin\phi\boldsymbol{a}_y + (\cos\theta\cos^2\phi + \cos\theta\sin^2\phi)\boldsymbol{a}_z$$

$$= \sin\theta\cos\phi\boldsymbol{a}_x + \sin\theta\sin\phi\boldsymbol{a}_y + \cos\theta\boldsymbol{a}_z = \boldsymbol{a}_r$$

Prob. 2.16

(a)

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{\rho^2 + z^2}.$$

$$\theta = \tan^{-1}\frac{\rho}{z}; \qquad \phi = \phi.$$

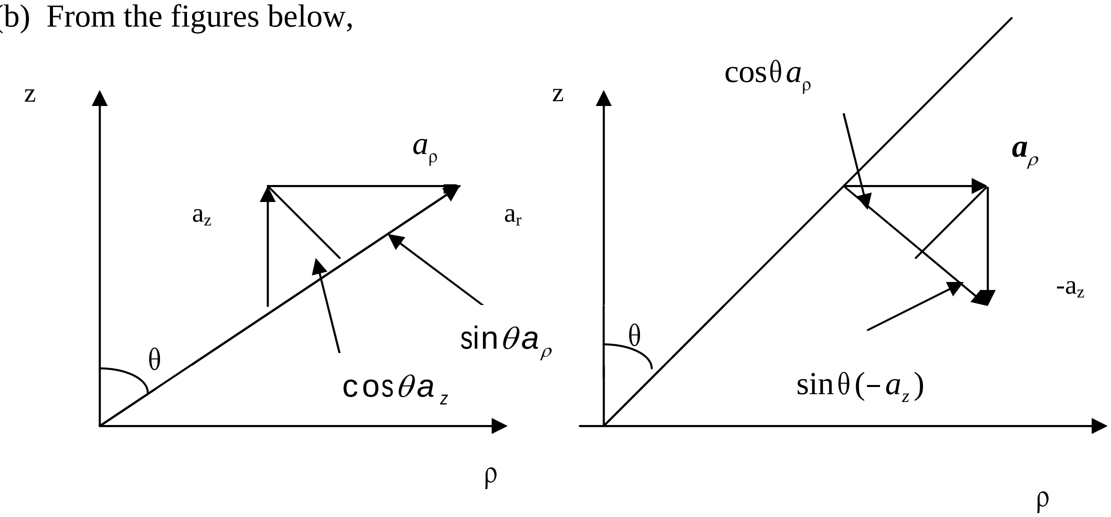
or

$$\rho = \sqrt{x^2 + y^2} = \sqrt{r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi}.$$

$$= r\sin\theta;$$

$$z = r\cos\theta; \qquad \phi = \phi.$$

(b) From the figures below,



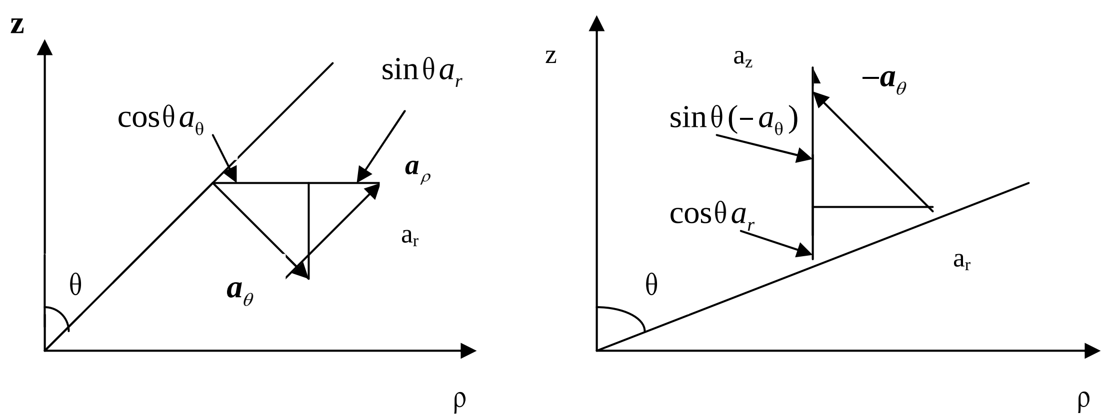
$\boldsymbol{a}_r = \sin\theta\,\boldsymbol{a}_\rho + \cos\theta\,\boldsymbol{a}_z$; $\boldsymbol{a}_\theta = \cos\theta\,\boldsymbol{a}_\rho - \sin\theta\,\boldsymbol{a}_z$; $\boldsymbol{a}_\phi = \boldsymbol{a}_\phi$.

Hence,

$$\begin{bmatrix} \boldsymbol{a}_r \\ \boldsymbol{a}_\theta \\ \boldsymbol{a}_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{a}_\rho \\ \boldsymbol{a}_\phi \\ \boldsymbol{a}_z \end{bmatrix}$$

From the figures below,

$\boldsymbol{a}_\rho = \cos\theta\boldsymbol{a}_\theta + \sin\theta\,\boldsymbol{a}_r; \boldsymbol{a}_z = \cos\theta\boldsymbol{a}_r - \sin\theta\boldsymbol{a}_\theta; \boldsymbol{a}_\phi = \boldsymbol{a}_\phi$.



$$\begin{bmatrix} \boldsymbol{a}_\rho \\ \boldsymbol{a}_\phi \\ \boldsymbol{a}_z \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{a}_r \\ \boldsymbol{a}_\theta \\ \boldsymbol{a}_z \end{bmatrix}$$

Prob. 2.17

At $P(2,0,-1)$, $\phi = 0$, $\theta = \cos^{-1}\left(\dfrac{z}{\sqrt{x^2+y^2+z^2}}\right) = \cos^{-1}\left(\dfrac{-1}{\sqrt{5}}\right) = 116.56^o$

(a) $\boldsymbol{a}_\rho \bullet \boldsymbol{a}_x = \cos\phi = \underline{\underline{1}}$

(b) $\boldsymbol{a}_\phi \bullet \boldsymbol{a}_y = \cos\phi = \underline{\underline{1}}$

(c) $\boldsymbol{a}_r \bullet \boldsymbol{a}_z = \cos\theta = \underline{\underline{-0.4472}}$

Prob. 2.18

If **A** and **B** are perpendicular to each other, **A**□**B** = 0

$\boldsymbol{A}\cdot\boldsymbol{B} = \rho^2 \sin^2\phi + \rho^2\cos^2\phi - \rho^2$

$= \rho^2(\sin^2\phi + \cos^2\phi) - \rho^2$

$= \rho^2 - \rho^2$

$= 0$

As expected.

Prob. 2.19

$(a)\ \boldsymbol{A}+\boldsymbol{B} = \underline{\underline{8\boldsymbol{a}_\rho + 2\boldsymbol{a}_\phi - 7\boldsymbol{a}_z}}$

$(b)\ \boldsymbol{A}\cdot\boldsymbol{B} = 15 + 0 - 8 = \underline{\underline{7}}$

$$(c)\ \boldsymbol{A}\times\boldsymbol{B} = \begin{vmatrix} 3 & 2 & 1 \\ 5 & 0 & -8 \end{vmatrix}$$

$= -16\mathbf{a}_\rho + (5+24)\boldsymbol{a}_\phi - 10\boldsymbol{a}_z$

$= \underline{\underline{-16\mathbf{a}_\rho + 29\boldsymbol{a}_\phi - 10\boldsymbol{a}_z}}$

$$(d)\ \cos\theta_{AB} = \frac{\boldsymbol{A}\cdot\boldsymbol{B}}{AB} = \frac{7}{\sqrt{9+4+1}\sqrt{25+64}} = \frac{7}{\sqrt{14}\sqrt{89}}$$

$= 0.19831$

$\underline{\underline{\theta_{AB} = 78.56^o}}$

Prob. 2.20

$$\begin{bmatrix} G_x \\ G_y \\ G_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} G_\rho \\ G_\phi \\ G_z \end{bmatrix}$$

$G_x = G_\rho\cos\phi - G_\phi\sin\phi = 3\rho\cos\phi - \rho\cos\phi\sin\phi$

$= 3x - x\sin\phi = 3(3) - (3)\sin(306.87^o) = 11.4$

$\boldsymbol{G}_x = G_x\boldsymbol{a}_x = \underline{\underline{11.4\boldsymbol{a}_x}}$

Prob. 2.21

$$\begin{bmatrix} G_\rho \\ G_\phi \\ G_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} yz \\ xz \\ xy \end{bmatrix}$$

$$G_\rho = yz\cos\phi + xz\sin\phi$$

$$x = \rho\cos\phi, y = \rho\sin\phi, \qquad yz = \rho z\sin\phi, xz = \rho z\cos\phi$$

$$G_\rho = \rho z\sin\phi\cos\phi + \rho z\cos\phi\sin\phi = 2\rho z\sin\phi\cos\phi = \rho z\sin 2\phi$$

$$G_\phi = -yz\sin\phi + xz\cos\phi = \rho z(\cos^2\phi - \sin^2\phi) = \rho z\cos 2\phi$$

$$G_z = xy = \rho^2\cos\phi\sin\phi = 0.5\rho^2\sin 2\phi$$

$$\mathbf{G} = \rho z\sin 2\phi\mathbf{a}_y + \rho z\cos 2\phi\mathbf{a}_\phi + 0.5\rho^2\sin 2\phi\mathbf{a}_z$$

Prob. 2.22

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

$$= \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} & -\frac{y}{\sqrt{x^2+y^2}} & 0 \\ \frac{y}{\sqrt{x^2+y^2}} & \frac{x}{\sqrt{x^2+y^2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

$$= \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2+z^2}} & \frac{xz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}} & \frac{-y}{\sqrt{x^2+y^2}} \\ \frac{y}{\sqrt{x^2+y^2+z^2}} & \frac{yz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}} & \frac{x}{\sqrt{x^2+y^2}} \\ \frac{z}{\sqrt{x^2+y^2+z^2}} & -\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}} & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\phi \\ A_\phi \end{bmatrix}$$

Prob. 2.23 (a) Using the results in Prob.2.14,

$$A_\rho = \rho z \sin\phi = r^2 \sin\theta \cos\theta \sin\phi$$
$$A_\phi = 3\rho\cos\phi = 3r\sin\theta\cos\phi$$
$$A_z = \rho\cos\phi\sin\phi = r\sin\theta\cos\phi\sin\phi$$

Hence,

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} r^2 \sin\theta\cos\theta\sin\phi \\ 3r\sin\theta\cos\phi \\ r\sin\theta\cos\phi\sin\phi \end{bmatrix}$$

$$\boldsymbol{A}(r,\theta,\phi) = \underline{\underline{r\sin\theta\left[\sin\phi\cos\theta\left(r\sin\theta+\cos\phi\right)\boldsymbol{a}_r + \sin\phi\left(r\cos^2\theta - \sin\theta\cos\phi\right)\boldsymbol{a}_\theta + 3\cos\phi\boldsymbol{a}_\phi\right]}}$$

At $(10, \pi/2, 3\pi/4)$, $r = 10, \theta = \pi/2, \phi = 3\pi/4$

$$\boldsymbol{A} = 10(0\boldsymbol{a}_r + 0.5\boldsymbol{a}_\theta - \frac{3}{\sqrt{2}}\boldsymbol{a}_\phi) = \underline{\underline{5\boldsymbol{a}_\theta - 21.21\boldsymbol{a}_\phi}}$$

(b) $B_r = r^2 = (\rho^2 + z^2)$, $B_\theta = 0$, $B_\phi = \sin\theta = \dfrac{\rho}{\sqrt{\rho^2 + z^2}}$

$$\begin{bmatrix} B_\rho \\ B_\phi \\ B_z \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} B_r \\ B_\theta \\ B_\phi \end{bmatrix}$$

$$\boldsymbol{B}(\rho,\phi,z) = \underline{\underline{\sqrt{\rho^2+z^2}\left(\rho\boldsymbol{a}_\rho + \frac{\rho}{\rho^2+z^2}\boldsymbol{a}_\phi + z\boldsymbol{a}_z\right)}}$$

At $(2, \pi/6, 1)$, $\rho = 2, \phi = \pi/6, z = 1$

$$\boldsymbol{B} = \sqrt{5}(2\boldsymbol{a}_\rho + 0.4\boldsymbol{a}_\phi + \boldsymbol{a}_z) = \underline{\underline{4.472\boldsymbol{a}_\rho + 0.8944\boldsymbol{a}_\phi + 2.236\boldsymbol{a}_z}}$$

Prob. 2.24

(a) $d = \sqrt{(6-2)^2 + (-1-1)^2 + (2-5)^2} = \sqrt{29} = \underline{\underline{5.385}}$

(b)
$$d^2 = 3^2 + 5^2 - 2(3)(5)\cos\pi + (-1-5)^2 = 100$$
$$d = \sqrt{100} = \underline{\underline{10}}$$

(c)

$$d^2 = 10^2 + 5^2 - 2(10)(5)\cos\frac{\pi}{4}\cos\frac{\pi}{6} - 2(10)(5)\sin\frac{\pi}{4}\sin\frac{\pi}{6}\cos(7\frac{\pi}{4} - \frac{3\pi}{4})$$
$$= 125 - 100(0.7071)(0.866) - 100(0.7071)(0.5)(-0.2334)$$
$$= 125 - 61.23 + 35.33 = 99.118$$
$$d = \sqrt{99.118} = \underline{\underline{9.956.}}$$

Prob. 2.25

Using eq. (2.33),

$$d^2 = r_1^2 + r_2^2 - 2r_1r_2\cos\theta_1\cos\theta_2 - 2r_1r_2\sin\theta_1\sin\theta_2\cos(\phi_2 - \phi_1)$$
$$= 16 + 36 - 2(4)(6)\cos 30^o\cos 90^o - 2(4)(6)\sin 30^o\sin 90^o\cos(180^o)$$
$$= 16 + 36 - 0 - 48(0.5)(1)(-1) = 52 + 24 = 76$$
$$d = 8.718$$

Prob. 2.26

$$\begin{bmatrix} \boldsymbol{a}_\rho \\ \boldsymbol{a}_\phi \\ \boldsymbol{a}_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \boldsymbol{a}_x \\ \boldsymbol{a}_y \\ \boldsymbol{a}_z \end{bmatrix}$$

$$\boldsymbol{a}_\rho = \cos\phi\boldsymbol{a}_x + \sin\phi\boldsymbol{a}_y, \qquad \boldsymbol{a}_\phi = -\sin\phi\boldsymbol{a}_x + \cos\phi\boldsymbol{a}_y$$

At (0, 4, -1), $\phi = 90^o$

$$\boldsymbol{a}_\rho = \sin 90^o\boldsymbol{a}_y = \underline{\underline{\boldsymbol{a}_y}}$$
$$\boldsymbol{a}_\phi = -\sin 90^o\boldsymbol{a}_x = \underline{\underline{-\boldsymbol{a}_x}}$$

Prob. 2.27

At $(1, 60^o, -1)$, $\rho = 1, \phi = 60^o, z = -1,$

(a) $\boldsymbol{A} = (-2 - \sin 60^o)\boldsymbol{a}_\rho + (4 + 2\cos 60^o)\boldsymbol{a}_\phi - 3(1)(-1)\boldsymbol{a}_z$

$$= -2.866\boldsymbol{a}_\rho + 5\boldsymbol{a}_\phi + 3\boldsymbol{a}_z$$

$$\boldsymbol{B} = 1\cos 60^o\boldsymbol{a}_\rho + \sin 60^o\boldsymbol{a}_\phi + \boldsymbol{a}_z = 0.5\boldsymbol{a}_\phi + 0.866\boldsymbol{a}_\phi + \boldsymbol{a}_z$$
$$\boldsymbol{A}\cdot\boldsymbol{B} = -1.433 + 4.33 + 3 = 5.897$$
$$AB = \sqrt{2.866^2 + 26 + 9}\sqrt{0.25 + 1 + 0.866^2} = 9.1885$$
$$\cos\theta_{AB} = \frac{\boldsymbol{A}\cdot\boldsymbol{B}}{AB} = \frac{5.897}{9.1885} = 0.6419 \longrightarrow \theta_{AB} = \underline{\underline{50.07^o}}$$

Let $\mathbf{D} = \mathbf{A} \times \mathbf{B}$. At $(1, 90^o, 0)$, $\rho = 1, \phi = 90^o, z = 0$

(b) $\boldsymbol{A} = -\sin 90^o \boldsymbol{a}_\rho + 4\boldsymbol{a}_\phi = -\boldsymbol{a}_\rho + 4\boldsymbol{a}_\phi$

$\boldsymbol{B} = 1\cos 90^o \boldsymbol{a}_\rho + \sin 90^o \boldsymbol{a}_\phi + \boldsymbol{a}_z = \boldsymbol{a}_\phi + \boldsymbol{a}_z$

$$\boldsymbol{D} = \boldsymbol{A} \times \boldsymbol{B} = \begin{vmatrix} \boldsymbol{a}_\rho & \boldsymbol{a}_\phi & \boldsymbol{a}_z \\ -1 & 4 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 4\boldsymbol{a}_\rho + \boldsymbol{a}_\phi - \boldsymbol{a}_z$$

$$\boldsymbol{a}_D = \frac{\boldsymbol{D}}{D} = \frac{(4,1,-1)}{\sqrt{16+1+1}} = \underline{\underline{0.9428\boldsymbol{a}_\rho + 0.2357\boldsymbol{a}_\phi - 0.2357\boldsymbol{a}_z}}$$

Prob.2.28

$At\ P(0, 2, -5)$, $\phi = 90°$;

$$\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} B_\rho \\ B_\phi \\ B_z \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -5 \\ 1 \\ -3 \end{bmatrix}$$

$\boldsymbol{B} = -\boldsymbol{a}_x - 5\boldsymbol{a}_y - 3\boldsymbol{a}_z$

(a) $\boldsymbol{A} + \boldsymbol{B} = (2, 4, 10) + (-1, -5, -3)$

$= \underline{\underline{\boldsymbol{a}_x - \boldsymbol{a}_y + 7\boldsymbol{a}_z.}}$

(b) $\cos\theta_{AB} = \dfrac{\boldsymbol{A} \bullet \boldsymbol{B}}{AB} = \dfrac{-52}{\sqrt{4200}}$

$\theta_{AB} = \cos^{-1}\left(\dfrac{-52}{\sqrt{4200}}\right) = \underline{\underline{143.36°.}}$

(c) $A_B = \boldsymbol{A} \bullet \boldsymbol{a}_B = \dfrac{\boldsymbol{A} \bullet \boldsymbol{B}}{B} = -\dfrac{52}{\sqrt{35}} = \underline{\underline{-8.789.}}$

Prob. 2.29

$\mathbf{B} \bullet \mathbf{a}_x = B_x$

$$\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} B_\rho \\ B_\phi \\ B_z \end{bmatrix}$$

$$B_x = B_\rho \cos\phi - B_\phi \sin\phi = \rho^2 \sin\phi\cos\phi - (z-1)\cos\phi\sin\phi$$

$$= 16(0.5) - (-2)(0.5) = 8 + 1 = \underline{\underline{9}}$$

Prob. 2.30

$$\bar{G} = \cos^2\phi\,\mathbf{a}_x + \frac{2r\cos\theta\sin\phi}{r\sin\theta}\bar{a}_y + (1 - \cos^2\phi)\bar{a}_z$$

$$= \cos^2\phi\,\bar{a}_x + 2\cot\theta\sin\phi\,\bar{a}_y + \sin^2\phi\,\bar{a}_z$$

$$\begin{pmatrix} G_r \\ G_\theta \\ G_\phi \end{pmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \cos^2\phi \\ 2\cot\theta\sin\phi \\ \sin^2\phi \end{bmatrix}$$

$$G_r = \sin\theta\cos^3\phi + 2\cos\theta\sin^2\phi + \cos\theta\sin^2\phi$$

$$= \sin\theta\cos^3\phi + 3\cos\theta\sin^2\phi$$

$$G_\theta = \cos\theta\cos^3\phi + 2\cot\theta\cos\theta\sin^2\phi - \sin\theta\sin^2\phi$$

$$G_\phi = -\sin\phi\cos^2\phi + 2\cot\theta\sin\phi\cos\phi$$

$$\bar{G} = [\sin\theta\cos^3\phi + 3\cos\theta\sin^2\phi]\mathbf{a}_r$$
$$+ [\cos\theta\cos^3\phi + 2\cot\theta\cos\theta\sin^2\phi - \sin\theta\sin^2\phi]\mathbf{a}_\theta$$
$$\underline{\underline{+ \sin\phi\cos\phi(2\cot\theta - \cos\phi)\mathbf{a}_\phi}}$$

Prob. 2.31

(a) An infinite line parallel to the z-axis.

(b) Point (2,-1,10).

(c) A circle of radius $r\sin\theta = 5$, i.e. the intersection of a cone and a sphere.

(d) An infinite line parallel to the z-axis.

(e) A semi-infinite line parallel to the x-y plane.

(f) A semi-circle of radius 5 in the y-z plane.

Prob. 2.32

(a) $\boldsymbol{J}_z = (\boldsymbol{J} \bullet \boldsymbol{a}_z)\boldsymbol{a}_z.$

$At\,(2,\ \pi/2,\ 3\pi/2),\ \boldsymbol{a}_z = \cos\theta\boldsymbol{a}_r - \sin\theta\boldsymbol{a}_\theta = -\boldsymbol{a}_\theta.$

$\boldsymbol{J}_z = -\cos 2\theta \sin\phi\boldsymbol{a}_\theta = -\cos\pi\sin(3\pi/2)\boldsymbol{a}_\theta = -\boldsymbol{a}_\theta.$

$(b)\boldsymbol{J}_\phi = \tan\frac{\theta}{2}\ln r\,\boldsymbol{a}_\phi = \tan\frac{\pi}{4}\ln 2\,\boldsymbol{a}_\phi = \ln 2\boldsymbol{a}_\phi = 0.6931\boldsymbol{a}_\phi.$

$(c)\boldsymbol{J}_t = \boldsymbol{J} - \boldsymbol{J}_n = \boldsymbol{J} - \boldsymbol{J}_r = -\boldsymbol{a}_\theta + \ln 2\,\boldsymbol{a}_\phi = \underline{\underline{-\boldsymbol{a}_\theta + 0.6931\boldsymbol{a}_\phi}}$

$(d)\quad \boldsymbol{J}_P = (\boldsymbol{J} \bullet \boldsymbol{a}_x)\boldsymbol{a}_x$

$\boldsymbol{a}_x = \sin\theta\cos\phi\boldsymbol{a}_r + \cos\theta\cos\phi\boldsymbol{a}_\theta - \sin\phi\boldsymbol{a}_\phi = \boldsymbol{a}_\phi.$

At $(2,\ \pi/2,\ 3\pi/2),$

$\boldsymbol{J}_P = \underline{\underline{\ln 2\boldsymbol{a}_\phi.}}$

Prob. 2.33

$$\boldsymbol{H}\cdot\boldsymbol{a}_x = H_x$$

$$\begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \rho^2\cos\phi \\ -\rho\sin\phi \\ 0 \end{bmatrix}$$

$H_x = \rho^2\cos^2\phi + \rho\sin^2\phi$

At P, $\rho = 2, \phi = 60^o, z = -1$

$H_x = 4(1/4) + 2(3/4) = 1 + 1.5 = \underline{\underline{2.5}}$

Prob. 2.34

(a) $5 = \boldsymbol{r}\cdot\boldsymbol{a}_x + \boldsymbol{r}\cdot\boldsymbol{a}_y = x + y$ $\qquad$ <u>a plane</u>

(b) $10 = |\boldsymbol{r}x\boldsymbol{a}_z| = \begin{vmatrix} x & y & z \\ 0 & 0 & 1 \end{vmatrix} = |y\boldsymbol{a}_x - x\boldsymbol{a}_y| = \sqrt{x^2 + y^2} = \rho$

<u>a cylinder of infinite length</u>