# Solutions Manual for Elementary Particle Physics: An Intuitive Introduction

Andrew Larkoski*

Physics Department

Reed College

April 5, 2019

*larkoski@reed.edu

# Contents

# 1 Introduction

## 1.1 *Energy of a Mosquito*

The mass of a mosquito is approximately $m = 2.5 \times 10^{-6}$ kg and it flies at approximately $v = 0.1$ m/s, or so. Therefore, its kinetic energy is

$$\mathrm{K} = \frac{1}{2}mv^2 = 1.25 \times 10^{-8}\ \mathrm{J}\,. \tag{1}$$

One electron volt is about $1.6 \times 10^{-19}$ J, so the energy in eV of a flying mosquito is

$$\mathrm{K} = \frac{1.25 \times 10^{-8}}{1.6 \times 10^{-19}}\ \mathrm{eV} \simeq 7.8 \times 10^{10}\ \mathrm{eV}\,. \tag{2}$$

The energy per nucleon of the flying mosquito can be found by dividing the total energy found above by the number of protons and neutrons in the mosquito. With a total mass of $2.5 \times 10^{-6}$ kg and the mass of the proton/neutron is approximately $m_p = 1.67 \times 10^{-27}$ kg, the total number of nucleons in the mosquito are

$$N_n \simeq \frac{2.5 \times 10^{-6}}{1.67 \times 10^{-27}} \simeq 1.5 \times 10^{21}\,. \tag{3}$$

Therefore the kinetic energy per nucleon of the mosquito is about

$$\frac{\mathrm{K}}{N_n} \simeq \frac{7.8 \times 10^{10}}{1.5 \times 10^{21}}\ \mathrm{eV} \simeq 5.2 \times 10^{-11}\ \mathrm{eV}\,. \tag{4}$$

This is about 23 orders of magnitude smaller than the energy of protons at the LHC!

## 1.2 *Yukawa's Theory*

The radius of an atomic nucleus is on the order of a femtometer, $10^{-15}$ m. To turn this into a mass or energy, we divide the product $\hbar c$ by this distance. This is

$$E = \frac{\hbar c}{x} = \frac{(1.05 \times 10^{-34}) \cdot (3 \times 10^{8})}{10^{-15}}\ \mathrm{J} \simeq 3 \times 10^{-11}\ \mathrm{J}\,. \tag{5}$$

To convert to eV, we divide by the ratio eV/J$\simeq 1.6 \times 10^{-19}$, so that

$$E = \frac{3 \times 10^{-11}}{1.6 \times 10^{-19}} \simeq 2 \times 10^{8}\ \mathrm{eV} = 200\ \mathrm{MeV}\,. \tag{6}$$

That is, the pion has a mass of about 200 MeV.

## 1.3 *Mass of the Photon*

If Maxwell's equations describe the magnetic field of the Milky Way galaxy, this sets an upper bound on the mass of the photon. The diameter of the Milky Way is about 100,000 light-years, which in meters is approximately

$$100,000 \text{ l-y} = 10^5 \cdot \left(3 \times 10^8\right) \cdot \left(\pi \times 10^7\right) \simeq 10^{21} \text{ m}\,. \tag{7}$$

In this expression, we used the fact that, to better than 1% accuracy, the number of seconds in a year is $\pi \times 10^7$. If electromagnetism as we understand it describes the galactic magnetic field at this distance, the photon must be able to have a wavelength that is at least this size. The corresponding upper bound on the minimum photon energy is

$$E < \frac{\hbar c}{x} \simeq \frac{(1.05 \times 10^{-34}) \cdot (3 \times 10^8)}{10^{21}} \text{ J} \simeq 3 \times 10^{-47} \text{ J}\,. \tag{8}$$

In electron volts, this corresponds to

$$E < \frac{3 \times 10^{-47}}{1.6 \times 10^{-19}} \text{ eV} \simeq 2 \times 10^{-28} \text{ eV}\,. \tag{9}$$

So, the mass of the photon must be less than about $10^{-28}$ eV. Converting this to kg, we divide the energy in Joules by $c^2$:

$$m < \frac{3 \times 10^{-47}}{\left(3 \times 10^8\right)^2} \simeq 3 \times 10^{-64} \text{ kg}\,. \tag{10}$$

The mass of the electron is about 1/2 MeV, so this is about 34 orders of magnitude smaller.

While this limit is extremely impressive, the assumptions necessary to describe the galactic magnetic field and connect it to Maxwell's equations in particular are a bit tenuous, so this result is not used by the PDG to set a limit on the photon mass.

## 1.4 *Planck Units*

### 1.4 (a)

The Planck time, $t_P$, can be expressed as a product of Newton's constant $G_N$, $\hbar$, and the speed of light $c$ raised to some powers:

$$t_P = G_N^\alpha \hbar^\beta c^\gamma\,, \tag{11}$$

where $\alpha$, $\beta$, and $\gamma$ are some numerical powers. We can find the powers by matching units on both sides of the expression. $c$ is a velocity and so

$$[c] = LT^{-1}\,, \tag{12}$$

where $L$ is a length and $T$ is a time unit. $\hbar$ has units of energy times time or that

$$[\hbar] = ML^2T^{-1}\,, \tag{13}$$

where $M$ is a mass unit. Finally, the units of $G_N$ can be determined by Newton's law of gravitation and Newton's second law:

$$\vec{F}_g = -\frac{G_N m_1 m_2}{r^2}\hat{r} = m_1 \frac{d^2\vec{r}}{dt^2}\,, \tag{14}$$

from which it follows that

$$[G_N] = M^{-1}L^3T^{-2}\,. \tag{15}$$

Plugging these into the expression for the Planck time $t_P$ we have

$$[t_P] = T = [G_N]^\alpha[\hbar]^\beta[c]^\gamma = M^{-\alpha}L^{3\alpha}T^{-2\alpha}M^\beta L^{2\beta}T^{-\beta}L^\gamma T^{-\gamma}\,. \tag{16}$$

Demanding that there is no mass unit requires that

$$-\alpha + \beta = 0\,, \tag{17}$$

or that $\alpha = \beta$. Demanding that there be no length unit requires

$$3\alpha + 2\beta + \gamma = 0 = 5\alpha + \gamma = 0\,, \tag{18}$$

or that $\gamma = -5\alpha$. Finally, demanding that there be one unit of time requires that

$$-2\alpha - \beta - \gamma = 1 = 2\alpha\,, \tag{19}$$

or that $\alpha = 1/2$. It then follows that the Planck time is

$$t_P = \sqrt{\frac{G_N\hbar}{c^5}}\,. \tag{20}$$

The value of Newton's constant in SI units is $G_N = 6.67\times10^{-11}$ kg$^{-1}$m$^3$s$^{-2}$. It then follows that the Planck time is

$$t_P = \sqrt{\frac{(6.67\times10^{-11})\cdot(1.05\times10^{-34})}{(3\times10^8)^5}}\ \text{s} \simeq 5.4\times10^{-44}\ \text{s}\,, \tag{21}$$

which is pretty small!

### 1.4 (b)

Now, we're asked to find the Planck mass, $m_P$. Because we already have the Planck time, $t_P$, we can find the Planck mass pretty easily. Note that the quantity

$$E_P = \frac{\hbar}{t_P}\,, \tag{22}$$

is an energy. Then, the Planck mass is found by dividing by $c^2$:

$$m_P = \frac{\hbar}{t_P c^2} = \sqrt{\frac{\hbar c}{G_N}} \simeq 2\times10^{-8}\ \text{kg}\,. \tag{23}$$

Expressed in eV, the Planck mass is

$$\frac{m_P c^2}{1.6 \times 10^{-19}} \simeq 1.2 \times 10^{28} \text{ eV} . \tag{24}$$

The proton mass is about a GeV, or $10^9$ eV, so the Planck mass is about 19 orders of magnitude larger!

### 1.4 (c)

For two particles of mass $m_1$ and $m_2$ and electric charges $q_1$ and $q_2$, the ratio of the electric force $\vec{F}_E$ to the gravitational force $\vec{F}_g$ between them is

$$\frac{|\vec{F}_E|}{|\vec{F}_g|} = \frac{1}{4\pi\epsilon_0 G_N} \frac{|q_1||q_2|}{m_1 m_2} , \tag{25}$$

where $\epsilon_0$ is the permittivity of free space. The proton and electron both have an electric charge magnitude of the fundamental unit of charge $e = 1.6 \times 10^{-19}$, so plugging in numbers, the ratio of forces is

$$\frac{|\vec{F}_E|}{|\vec{F}_g|} \simeq \frac{1}{4\pi\,(8.85 \times 10^{-12})\,(6.67 \times 10^{-11})} \frac{(1.6 \times 10^{-19})^2}{(1.67 \times 10^{-27})\,(9.11 \times 10^{-31})} \simeq 2.2 \times 10^{39} . \tag{26}$$

## 1.5 *Expansion of the Universe*

### 1.5 (a)

The CMB has a temperature of 2.7 K. With Boltzmann's constant $k_B$, we can turn this into a corresponding energy. We find

$$E_{\text{CMB}} = k_B T = 2.7 \times 1.38 \times 10^{-23} \text{ J} \simeq 3.7 \times 10^{-23} \text{ J} . \tag{27}$$

To determine the energy in eV, we divide by $1.6 \times 10^{-19}$:

$$E_{\text{CMB}} \simeq \frac{3.7 \times 10^{-23}}{1.6 \times 10^{-19}} \text{ eV} \simeq 2.3 \times 10^{-4} \text{ eV} . \tag{28}$$

### 1.5 (b)

The ground state energy of hydrogen is $E_g = -13.6$ eV. So, when the temperature of the universe was less than the energy of 13.6 eV, electrons and protons could become bound and form hydrogen. To determine this temperature, we work backward from the steps of the first part of this problem, multiplying by the factor $1.6 \times 10^{-19}$ and then dividing by $k_B$. We find

$$T_{\text{recomb.}} = 1.6 \times 10^{-19} \frac{|E_g|}{k_B} \text{ K} \simeq 1.6 \times 10^5 \text{ K} . \tag{29}$$

**1.5 (c)**

We want to calculate the ratio of the wavelength of CMB photons observed today, $\lambda_{\text{today}}$, by the wavelength at recombination, $\lambda_{\text{recomb.}}$. This ratio is

$$\frac{\lambda_{\text{today}}}{\lambda_{\text{recomb.}}} = \frac{f_{\text{recomb.}}}{f_{\text{today}}} = \frac{E_{\text{recomb.}}}{E_{\text{today}}} = \frac{T_{\text{recomb.}}}{T_{\text{today}}} \,. \tag{30}$$

In this chain of equalities, we used the fact that wavelength $\lambda$ is inversely proportional to frequency $f$ and the frequency of light is proportional to its energy. From what was developed in the previous parts, the energy of the photons is proportional to its temperature. So, the redshift factor is just the ratio of the temperature at recombination to the temperature today:

$$\frac{\lambda_{\text{today}}}{\lambda_{\text{recomb.}}} = \frac{T_{\text{recomb.}}}{T_{\text{today}}} \simeq \frac{1.6 \times 10^5}{2.7} \simeq 5.9 \times 10^4 \,. \tag{31}$$

As mentioned in the problem, this is a factor of about 30 larger than the true result when thermodynamics are properly taken into account.

## 1.6 *Decay Width of the Z boson*

**1.6 (a)**

From the plot of Fig. 1.5, the maximum value of the peak of the distribution is about 32 nb. Therefore, half of this is 16 nb. The lower point of the distribution at which it takes a value of 16 nb is at approximately 90 GeV, while the higher point is at about 92.5 GeV. Therefore, the width, or full-width at half-maximum is the difference of these two values, or 2.5 GeV.

**1.6 (b)**

To determine the lifetime in seconds of the $Z$ boson from its width, we need to relate the width to a time through the energy-time uncertainty relation:

$$\Delta t \simeq \frac{\hbar}{\Delta E} \,. \tag{32}$$

To convert the width from natural units to SI, we need to multiply by the factor of $1.6 \times 10^{-19}$ so that the lifetime in seconds is

$$\Delta t \simeq \frac{\hbar}{\Delta E} \simeq \frac{1.05 \times 10^{-34}}{(2.5 \times 10^9) \cdot (1.6 \times 10^{-19})} \simeq 2.6 \times 10^{-25} \text{ s} \,. \tag{33}$$

We also needed to include a factor of $10^9$ to account for the fact that the width is 2.5 GeV = $2.5 \times 10^9$ eV.

### 1.6 (c)

Through the energy-time uncertainty principle, lifetime and decay width are inversely proportional. Therefore, if the width approaches 0, then the lifetime diverges: the particle lives forever. Correspondingly, if the width gets very large, then the lifetime approaches 0: the particle decays instantly.

## 1.7 *Decay of Strange Hadrons*

The $\Omega^-$ hadron travels about 3 cm = 0.03 m before decaying which corresponds to a lifetime by multiplication by its velocity. While its velocity is not known, it will be an appreciable fraction of the speed of light $c$, so we can just assume that it is $c$. Dividing by $c$, the lifetime $\tau$ is

$$\tau \simeq \frac{0.03}{3 \times 10^8} \text{ s} \simeq 10^{-10} \text{ s}\,. \tag{34}$$

To convert to the decay width $\Delta E$, we use the energy-time uncertainty relationship, and the factor $1.6 \times 10^{-19}$ to convert to eV. We then find

$$\Delta E = \frac{\hbar}{\tau} \simeq \frac{1.05 \times 10^{-34}}{10^{-10} \cdot (1.6 \times 10^{-19})} \text{ eV} \simeq 6.6 \times 10^{-5} \text{ eV}\,. \tag{35}$$

## 1.8 *PDG Review*

### 1.8 (a)

From the PDG, the lower bound on the lifetime of the proton is $2.1 \times 10^{29}$ years. To have a reasonable probability to observe one proton decay in a year, you would need at least $2.1 \times 10^{29}$ protons, if the lifetime were exactly at the lower bound. Water, $H_2O$, consists of 10 protons (and 8 neutrons), so we would need to observe $2.1 \times 10^{28}$ water molecules for a year. The volume of this amount of water can be found by first identifying the number of moles of water:

$$\frac{2.1 \times 10^{28}}{6.02 \times 10^{23}} \text{ mol} \simeq 3.5 \times 10^4 \text{ mol}\,. \tag{36}$$

Because water consists of 18 total protons and neutrons, this amount of water has a mass of

$$\left(3.5 \times 10^4\right) \cdot 18 \text{ g} \simeq 630 \text{ kg}\,. \tag{37}$$

Water has a density of 1000 kg/m$^3$, so the total volume of water needed is about

$$\frac{630}{1000} \text{ m}^3 = 0.63 \text{ m}^3\,. \tag{38}$$

This is roughly the volume of a large bathtub.

**1.8 (b)**

From the "Particle Listings" section of the PDG, the masses of the $W$, $Z$, and Higgs bosons and the top quark are:

$$m_W = 80.379 \text{ GeV}, \quad (39)$$
$$m_Z = 91.1876 \text{ GeV}, \quad (40)$$
$$m_H = 125.18 \text{ GeV}, \quad (41)$$
$$m_t = 173.0 \text{ GeV}. \quad (42)$$

The mass of the proton or neutron is approximately 1 GeV (actually slightly less), and so the value of the mass in GeV can be approximately used to identify the atomic mass of elements with about the same mass. For example, Krypton has an atomic mass of about 83, which is close to the mass of the $W$ boson. Zirconium has an atomic mass of about 91, close to the mass of the $Z$ boson. Tellurium has an atomic mass of about 127, close to the mass of the Higgs boson. Finally, Ytterbium has an atomic mass of about 174, close to the mass of the top quark.

Students may find a slightly different selection of elements from a more precise accounting of the proton and neutron masses or identification of different isotopes.

**1.8 (c)**

We want to identify the masses of the particles involved in the bubble chamber trace of Fig. 1.6. Again, we use the "Particle Listings" section of the PDG, and we have to do a bit of sleuthing to identify all the particles by their symbols. Their masses are:

$$m_{K^-} = 493.677 \text{ MeV}, \quad (43)$$
$$m_{\Omega^-} = 1672.43 \text{ MeV}, \quad (44)$$
$$m_{K^0} = 497.611 \text{ MeV}, \quad (45)$$
$$m_{\pi^-} = 139.57061 \text{ MeV}, \quad (46)$$
$$m_{\Xi^0} = 1314.82 \text{ MeV}, \quad (47)$$
$$m_{K^+} = 493.677 \text{ MeV}, \quad (48)$$
$$m_{\Lambda^0} = 1115.683 \text{ MeV}, \quad (49)$$
$$m_p = 938.2720813 \text{ MeV}. \quad (50)$$

The width, or lifetime, from the PDG of the $\Omega^-$ baryon is $0.821 \times 10^{-10}$ s, which is very close to our very simple estimate!

## 1.9 *InSpire and arXiv*

**1.9 (a)**

At InSpire, we can search for Noether's papers and find her most highly-cited paper. In fact, InSpire only has one of her papers listed (all others are pure mathematics), which is

"Invariant Variation Problems."

**1.9 (b)**

We can also search by date. To find all papers from 1967, we use the command "find date = 1967". Then, we can sort by decreasing order in citation count. The two most highly-cited papers are:

S. Weinberg, "A Model of Leptons," Phys. Rev. Lett. **19**, 1264 (1967).

A. D. Sakharov, "Violation of CP Invariance, C asymmetry, and baryon asymmetry of the universe," Pisma Zh. Eksp. Teor. Fiz. **5**, 32 (1967) [JETP Lett. **5**, 24 (1967)] [Sov. Phys. Usp. **34**, no. 5, 392 (1991)] [Usp. Fiz. Nauk **161**, no. 5, 61 (1991)].

# 2 Special Relativity

## 2.1 *Properties of Lorentz Transformations*

We're asked to verify that the matrix

$$\Lambda_\mu{}^\nu = \begin{pmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{pmatrix} \tag{51}$$

leaves the metric invariant:

$$\Lambda^\intercal \eta \Lambda = \eta\,. \tag{52}$$

First, note that the matrix $\Lambda$ is symmetric: $\Lambda^\intercal = \Lambda$. So, multiplying from the left, we have

$$\Lambda^\intercal \eta = \begin{pmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \gamma\beta & 0 & 0 & -\gamma \end{pmatrix}. \tag{53}$$

Continuing, multiplying by $\Lambda$ on the right produces

$$\begin{aligned} \Lambda^\intercal \eta \Lambda &= \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \gamma\beta & 0 & 0 & -\gamma \end{pmatrix} \begin{pmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{pmatrix} \\ &= \begin{pmatrix} \gamma^2(1-\beta^2) & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -\gamma^2(1-\beta^2) \end{pmatrix} = \eta\,. \end{aligned} \tag{54}$$

Recall that $1-\beta^2 = \gamma^{-2}$, from which the result follows.

## 2.2 *Rapidity*

The rapidity $y$ is defined to be

$$y = \frac{1}{2}\log\frac{E+p_z}{E-p_z}\,. \tag{55}$$

We can perform a Lorentz boost along the $\hat{z}$ axis by a velocity $\beta$ by multiplication of the momentum four-vector by a Lorentz matrix:

$$\left(\begin{array}{cccc} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{array}\right)\left(\begin{array}{c} E \\ 0 \\ 0 \\ p_z \end{array}\right) = \left(\begin{array}{c} \gamma(E+\beta p_z) \\ 0 \\ 0 \\ \gamma(p_z+\beta E) \end{array}\right). \tag{56}$$

Therefore, under a Lorentz transformation, the rapidity transforms as

$$y \to \frac{1}{2}\log\frac{\gamma(E+\beta p_z)+\gamma(p_z+\beta E)}{\gamma(E+\beta p_z)-\gamma(p_z+\beta E)} = \frac{1}{2}\log\frac{(1+\beta)(E+p_z)}{(1-\beta)(E-p_z)} = y + \frac{1}{2}\log\frac{1+\beta}{1-\beta}\,. \tag{57}$$

That is, under a Lorentz boost along the $\hat{z}$ axis, the rapidity transforms additively.

## 2.3 *Lorentz-Invariant Measure*

Under a Lorentz transformation, the coordinate four-vector $x^\mu$ transforms to

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu\,. \tag{58}$$

Then, under a Lorentz transformation, the coordinate measure $d^4x$ transforms as

$$d^4x' = |J|\, d^4x\,, \tag{59}$$

where $J$ is the Jacobian formed from the determinant of the derivative matrix:

$$J = \det\frac{\partial x'^\mu}{\partial x^\nu}\,. \tag{60}$$

From the Lorentz transformation above in terms of the matrix $\Lambda$, this partial derivative is

$$\frac{\partial x'^\mu}{\partial x^\nu} = \Lambda^\mu{}_\nu\,. \tag{61}$$

Therefore, the Jacobian is just the determinant of the Lorentz-transformation matrix:

$$J = \det\Lambda\,. \tag{62}$$

By the definition of $\Lambda$, it satisfies

$$\Lambda^\intercal\eta\Lambda = \eta\,, \tag{63}$$

which enables us to calculate the determinant of $\Lambda$. First, note that transposition doesn't change the determinant:

$$\det \Lambda = \det \Lambda^{\intercal} \,. \tag{64}$$

Also, the determinant of the metric $\eta$ is just the product of its non-zero elements:

$$\det \eta = -1 \,. \tag{65}$$

Then, taking the determinant of the Lorentz transformation equation, we have

$$\det(\Lambda^{\intercal}\eta\Lambda) = -\left(\det \Lambda\right)^2 = -1 \,. \tag{66}$$

Therefore, $|\det \Lambda| = 1 = J$, and so after Lorentz transformation the measure $d^4x$ is unchanged.

## 2.4 *Properties of Klein-Gordon Equation*

The solution of the Klein-Gordon equation is an exponential phase function,

$$\phi(x) = e^{-ip\cdot x} = e^{-i(Et-\vec{p}\cdot\vec{x})} \,. \tag{67}$$

We can find the frequency by adding the period $T$ to the time $t$ and demanding that the field $\phi(x)$ is unchanged:

$$e^{-i(E(t+T)-\vec{p}\cdot\vec{x})} = e^{-i(Et-\vec{p}\cdot\vec{x})} \,, \tag{68}$$

or that $\exp[-iET] = 1$. Then, the period $T = 2\pi/E$ and so the frequency $f = E/(2\pi)$. A similar procedure can be used to determine the wavelength of the solution. The wavelength $\lambda$ is then

$$\lambda = \frac{2\pi}{|\vec{p}|} \,. \tag{69}$$

It then follows that the phase velocity is the just the product of the frequency and wavelength:

$$v = \lambda f = \frac{E}{|\vec{p}|} \,. \tag{70}$$

## 2.5 *Maxwell's Equations*

Gauss's law has already been identified as the $0^{\text{th}}$ component of the equations of motion of

$$\partial_\mu F^{\mu\nu} = J^\nu \,. \tag{71}$$

Now, let's take $\nu = i$, a spatial coordinate. Then, $J_i$ is the $i^{\text{th}}$ component of the current vector and the left side of this equation is

$$\partial_\mu F^{\mu i} = -\partial_0 F_{0i} + \partial_i F_{ii} + \partial_j F_{ji} + \partial_k F_{ki} = -\frac{\partial E_i}{\partial t} + \left(\frac{\partial B_k}{\partial j} - \frac{\partial B_j}{\partial k}\right) \,. \tag{72}$$