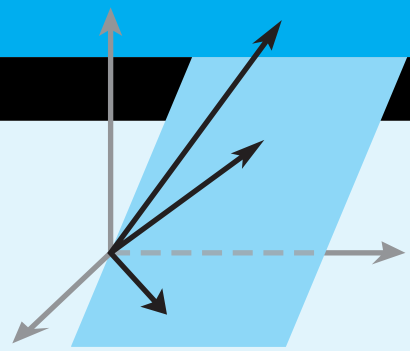


MODULE 1

Linear Programming: A Geometric Approach

In this module a geometric technique is described for maximizing or minimizing a linear expression in two variables subject to a set of linear constraints.

PREREQUISITES: Linear Systems
Linear Inequalities

Linear Programming

The study of linear programming theory has expanded greatly since the pioneer work of George Dantzig in the late 1940s. Today, linear programming is applied to a wide variety of problems in industry and science. In this section we present a geometric approach to the solution of simple linear programming problems. Let us begin with some examples.

▶ EXAMPLE 1 Maximizing Sales Revenue

A candy manufacturer has 130 pounds of chocolate-covered cherries and 170 pounds of chocolate-covered mints in stock. The manufacturer decides to sell them in the form of two different mixtures. One mixture will contain half cherries and half mints by weight and will sell for \$2.00 per pound. The other mixture will contain one-third cherries and two-thirds mints by weight and will sell for \$1.25 per pound. How many pounds of each mixture should the candy manufacturer prepare in order to maximize his sales revenue?

Solution Let us first formulate this problem mathematically. Let the mixture of half cherries and half mints be called mix A, and let x_1 be the number of pounds of this mixture to be prepared. Let the mixture of one-third cherries and two-thirds mints be called mix B, and let x_2 be the number of pounds of this mixture to be prepared. Since mix A sells for \$2.00 per pound and mix B sells for \$1.25 per pound, the total sales z (in dollars) will be

$$z = 2.00x_1 + 1.25x_2$$

Since each pound of mix A contains $\frac{1}{2}$ pound of cherries and each pound of mix B contains $\frac{1}{3}$ pound of cherries, the total number of pounds of cherries used in both mixtures is

$$\tfrac{1}{2}x_1 + \tfrac{1}{3}x_2$$

Similarly, since each pound of mix A contains $\frac{1}{2}$ pound of mints and each pound of mix B contains $\frac{2}{3}$ pound of mints, the total number of pounds of mints used in both mixtures is

$$\tfrac{1}{2}x_1 + \tfrac{2}{3}x_2$$

Because the manufacturer can use at most 130 pounds of cherries and 170 pounds of mints, we must have

$$\tfrac{1}{2}x_1 + \tfrac{1}{3}x_2 \le 130$$
$$\tfrac{1}{2}x_1 + \tfrac{2}{3}x_2 \le 170$$

Furthermore, since x_1 and x_2 cannot be negative numbers, we must have

$$x_1 \geq 0 \quad \text{and} \quad x_2 \geq 0$$

The problem can therefore be formulated mathematically as follows: Find values of x_1 and x_2 that maximize

$$z = 2.00x_1 + 1.25x_2$$

subject to

$$\begin{aligned} \tfrac{1}{2}x_1 + \tfrac{1}{3}x_2 &\leq 130 \\ \tfrac{1}{2}x_1 + \tfrac{2}{3}x_2 &\leq 170 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

Later in this module we will show how to solve this type of mathematical problem geometrically. ◀

▶ EXAMPLE 2 Maximizing Annual Yield

A woman has up to \$10,000 to invest. Her broker suggests investing in two bonds, A and B. Bond A is a rather risky bond with an annual yield of 10%, and bond B is a rather safe bond with an annual yield of 7%. After some consideration, she decides to invest at most \$6000 in bond A, at least \$2000 in bond B, and to invest at least as much in bond A as in bond B. How should she invest her \$10,000 in order to maximize her annual yield?

Solution To formulate this problem mathematically, let x_1 be the number of dollars to be invested in bond A and let x_2 be the number of dollars to be invested in bond B. Since each dollar invested in bond A earns \$.10 per year and each dollar invested in bond B earns \$.07 per year, the total dollar amount z earned each year by both bonds is

$$z = .10x_1 + .07x_2$$

The constraints imposed can be formulated mathematically as follows:

Invest no more than \$10,000:	$x_1 + x_2 \leq 10{,}000$
Invest at most \$6000 in bond A:	$x_1 \leq 6000$
Invest at least \$2000 in bond B:	$x_2 \geq 2000$
Invest at least as much in bond A as in bond B:	$x_1 \geq x_2$

We also have the implicit assumption that x_1 and x_2 are nonnegative:

$$x_1 \geq 0 \quad \text{and} \quad x_2 \geq 0$$

Thus, the complete mathematical formulation of the problem is as follows: Find values of x_1 and x_2 that maximize

$$z = .10x_1 + .07x_2$$

subject to

$$\begin{aligned} x_1 + x_2 &\leq 10{,}000 \\ x_1 &\leq 6000 \\ x_2 &\geq 2000 \\ x_1 - x_2 &\geq 0 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$ ◀

EXAMPLE 3 Minimizing Cost

A student desires to design a breakfast of corn flakes and milk that is as economical as possible. On the basis of what he eats during his other meals, he decides that his breakfast should supply him with at least 9 grams of protein, at least $\frac{1}{3}$ the recommended daily allowance (RDA) of vitamin D, and at least $\frac{1}{4}$ the RDA of calcium. He finds the following nutrition information on the milk and corn flakes containers:

	Milk ($\frac{1}{2}$ cup)	**Corn Flakes (1 ounce)**
Cost	7.5 cents	5.0 cents
Protein	4 grams	2 grams
Vitamin D	$\frac{1}{8}$ of RDA	$\frac{1}{10}$ of RDA
Calcium	$\frac{1}{6}$ of RDA	None

In order not to have his mixture too soggy or too dry, the student decides to limit himself to mixtures that contain 1 to 3 ounces of corn flakes per cup of milk, inclusive. What quantities of milk and corn flakes should he use to minimize the cost of his breakfast?

Solution For the mathematical formulation of this problem, let x_1 be the quantity of milk used (measured in $\frac{1}{2}$-cup units), and let x_2 be the quantity of corn flakes used (measured in 1-ounce units). Then if z is the cost of the breakfast in cents, we can write the following.

Cost of breakfast:	$z = 7.5x_1 + 5.0x_2$
At least 9 grams protein:	$4x_1 + 2x_2 \geq 9$
At least $\frac{1}{3}$ RDA vitamin D:	$\frac{1}{8}x_1 + \frac{1}{10}x_2 \geq \frac{1}{3}$
At least $\frac{1}{4}$ RDA calcium:	$\frac{1}{6}x_1 \geq \frac{1}{4}$
At least 1 ounce corn flakes per cup (two $\frac{1}{2}$-cups) of milk:	$\dfrac{x_2}{x_1} \geq \dfrac{1}{2}$ (or $x_1 - 2x_2 \leq 0$)
At most 3 ounces corn flakes per cup (two $\frac{1}{2}$-cups) of milk:	$\dfrac{x_2}{x_1} \leq \dfrac{3}{2}$ (or $3x_1 - 2x_2 \geq 0$)

As before, we also have the implicit assumption that $x_1 \geq 0$ and $x_2 \geq 0$. Thus the complete mathematical formulation of the problem is as follows: Find values of x_1 and x_2 that minimize

$$z = 7.5x_1 + 5.0x_2$$

subject to

$$\begin{aligned}
4x_1 + 2x_2 &\geq 9 \\
\tfrac{1}{8}x_1 + \tfrac{1}{10}x_2 &\geq \tfrac{1}{3} \\
\tfrac{1}{6}x_1 &\geq \tfrac{1}{4} \\
x_1 - 2x_2 &\leq 0 \\
3x_1 - 2x_2 &\geq 0 \\
x_1 &\geq 0 \\
x_2 &\geq 0
\end{aligned}$$

Geometric Solution of Linear Programming Problems

Each of the preceding three examples is a special case of the following problem.

Problem 1 Find values of x_1 and x_2 that either maximize or minimize

$$z = c_1x_1 + c_2x_2 \tag{1}$$

subject to

$$\begin{aligned} a_{11}x_1 &+ a_{12}x_2 \ (\le)(\ge)(=)\ b_1 \\ a_{21}x_1 &+ a_{22}x_2 \ (\le)(\ge)(=)\ b_2 \\ \vdots \quad & \quad \vdots \qquad\qquad\quad \vdots \\ a_{m1}x_1 &+ a_{m2}x_2 \ (\le)(\ge)(=)\ b_m \end{aligned} \tag{2}$$

and

$$x_1 \ge 0, \qquad x_2 \ge 0 \tag{3}$$

In each of the m conditions of (2), any one of the symbols $\le$, $\ge$, or $=$ can be used.

The problem above is called the ***general linear programming problem*** in two variables. The linear function z in (1) is called the ***objective function***. Equations (2) and (3) are called the ***constraints***; in particular, the equations in (3) are called the ***nonnegativity constraints*** on the variables x_1 and x_2.

We will now show how to solve a linear programming problem in two variables graphically. A pair of values (x_1, x_2) that satisfy all of the constraints is called a ***feasible solution***. The set of all feasible solutions determines a subset of the x_1x_2-plane called the ***feasible set***. Our goal is to find a feasible solution that maximizes the objective function. Such a solution is called an ***optimal solution***.

To examine the feasible set for a linear programming problem, we note that each constraint of the form

$$a_{i1}x_1 + a_{i2}x_2 = b_i$$

defines a line in the x_1x_2-plane, whereas each constraint of the form

$$a_{i1}x_1 + a_{i2}x_2 \le b_i \quad \text{or} \quad a_{i1}x_1 + a_{i2}x_2 \ge b_i$$

defines a half plane that includes its boundary line

$$a_{i1}x_1 + a_{i2}x_2 = b_i$$

Thus, the feasible region is always an intersection of finitely many lines and half planes. For example, the four constraints

$$\begin{aligned} \tfrac{1}{2}x_1 + \tfrac{1}{3}x_2 &\le 130 \\ \tfrac{1}{2}x_1 + \tfrac{2}{3}x_2 &\le 170 \\ x_1 &\ge 0 \\ x_2 &\ge 0 \end{aligned}$$

of Example 1 define the half planes illustrated in parts (*a*), (*b*), (*c*), and (*d*) of Figure 1. The feasible region of this problem is thus the intersection of these four half planes, which is illustrated in Figure 1*e*.

It can be shown that the feasible set for a linear programming problem has a boundary consisting of a finite number of straight line segments. If the feasible set can be enclosed in a sufficiently large circle, it is called ***bounded*** (Figure 1*e*); otherwise it is called ***unbounded*** (Figure 5). If the feasible set is *empty* (contains no points), then the constraints are inconsistent and the linear programming problem has no solution (Figure 6).

Those boundary points of a feasible set that are intersections of two of the straight line boundary segments are called ***extreme points***. (They are also called *corner points* or *vertex points*.) For example, from Figure 1*e* the feasible set of Example 1 has four extreme points:

$$(0, 0), \quad (0, 255), \quad (180, 120), \quad (260, 0) \tag{4}$$

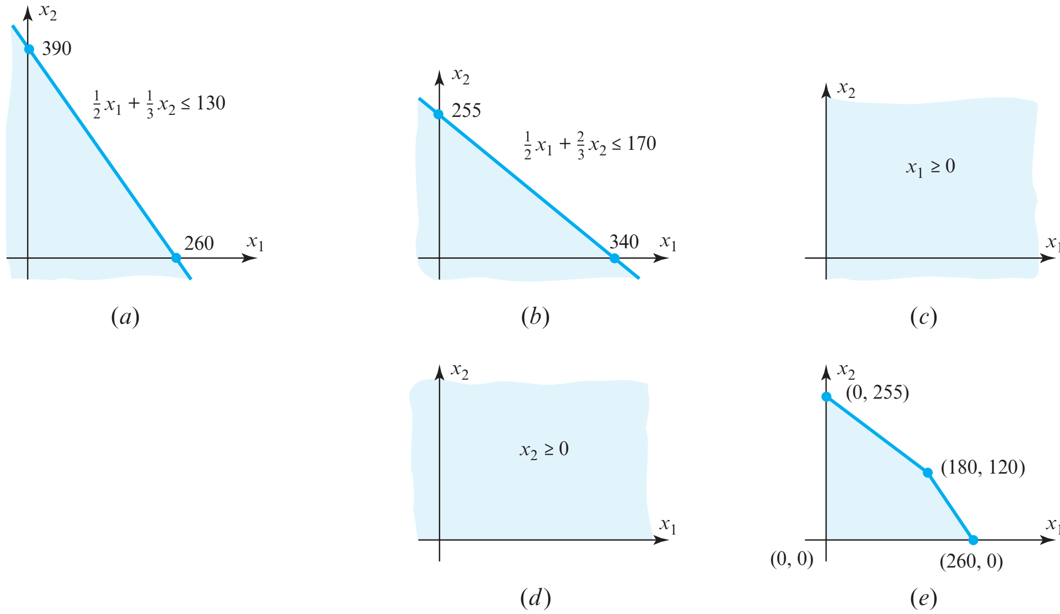


Figure 1

The importance of the extreme points of a feasible set is shown by the following theorem.

THEOREM 1 Maximum and Minimum Values

If the feasible set of a linear programming problem in two variables is nonempty and bounded, then the objective function attains both a maximum and minimum value and these occur at extreme points of the feasible set. If the feasible set is unbounded, then the objective function may or may not attain a maximum or minimum value; however, if it attains a maximum or minimum value, it does so at an extreme point.

Figure 2 suggests the idea behind the proof of this theorem. Since the objective function

$$z = c_1x_1 + c_2x_2$$

of a linear programming problem is a linear function of x_1 and x_2, its level curves (the curves along which z has constant values) are straight lines. As we move in a direction perpendicular to these level curves, the objective function either increases or decreases

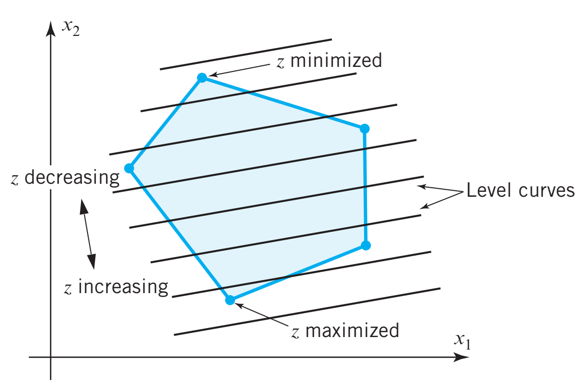


Figure 2

monotonically. Within a bounded feasible region, the maximum and minimum values of z must therefore occur at extreme points, as Figure 2 indicates.

In the next few examples we use Theorem 1 to solve several linear programming problems and illustrate the variations in the nature of the solutions that may occur.

EXAMPLE 4 Example 1 Revisited

From Figure 1*e* we see that the feasible set of Example 1 is bounded. Consequently, from Theorem 1 the objective function

$$z = 2.00x_1 + 1.25x_2$$

attains both its minimum and maximum values at extreme points. The four extreme points and the corresponding values of z are given in the following table.

Extreme Point (x_1, x_2)	**Value of** $z = 2.00x_1 + 1.25x_2$
(0, 0)	0
(0, 255)	318.75
(180, 120)	510.00
(260, 0)	520.00

We see that the largest value of z is 520.00 and the corresponding optimal solution is (260, 0). Thus the candy manufacturer attains maximum sales of \$520 by producing 260 pounds of mixture A and none of mixture B.

EXAMPLE 5 Using Theorem 1

Find values of x_1 and x_2 that maximize

$$z = x_1 + 3x_2$$

subject to

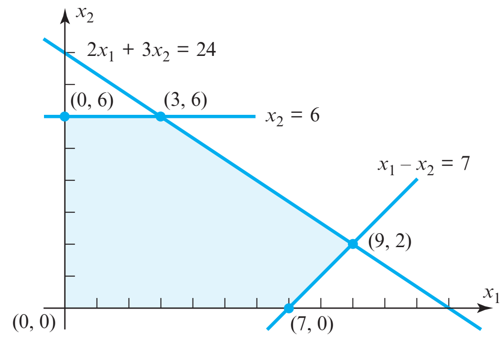
$$\begin{aligned} 2x_1 + 3x_2 &\le 24 \\ x_1 - x_2 &\le 7 \\ x_2 &\le 6 \\ x_1 &\ge 0 \\ x_2 &\ge 0 \end{aligned}$$

Solution In Figure 3 we have drawn the feasible set for this problem. Since it is bounded, the maximum value of z is attained at one of the five extreme points. The values of the objective function at the five extreme points are given in the table that accompanies Figure 3. From this table the maximum value of z is 21, which is attained at $x_1 = 3$ and $x_2 = 6$.

EXAMPLE 6 Using Theorem 1

Find values of x_1 and x_2 that maximize

$$z = 4x_1 + 6x_2$$



▲ Figure 3

Extreme Point (x_1, x_2)	Value of $z = x_1 + 3x_2$
(0, 6)	18
(3, 6)	21
(9, 2)	15
(7, 0)	7
(0, 0)	0

subject to

$$\begin{aligned} 2x_1 + 3x_2 &\leq 24 \\ x_1 - x_2 &\leq 7 \\ x_2 &\leq 6 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

Solution The constraints in this problem are identical to the constraints in Example 5, so the feasible set for this problem is also given by Figure 3. The values of the objective function at the extreme points are given in the following table.

Extreme Point (x_1, x_2)	Value of $z = 4x_1 + 6x_2$
(0, 6)	36
(3, 6)	48
(9, 2)	48
(7, 0)	28
(0, 0)	0

We see that the objective function attains a maximum value of 48 at two adjacent extreme points, (3, 6) and (9, 2). This shows that an optimal solution to a linear programming problem need not be unique. As we ask you to show in Exercise 9, if the objective function has the same value at two adjacent extreme points, it has the same value at all points on the straight line boundary segment connecting the two extreme points. Thus, in this example the maximum value of z is attained at all points on the straight line segment connecting the extreme points (3, 6) and (9, 2). ◀

▶ EXAMPLE 7 A Feasible Set That Is a Line Segment

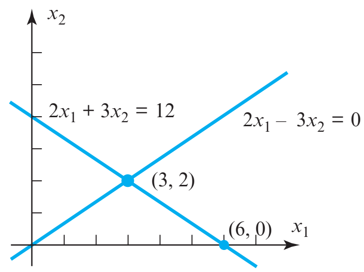
Find values of x_1 and x_2 that minimize

$$z = 2x_1 - x_2$$

subject to

$$\begin{aligned} 2x_1 + 3x_2 &= 12 \\ 2x_1 - 3x_2 &\geq 0 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

Solution In Figure 4 we have drawn the feasible set for this problem. Because one of the constraints is an equality constraint, the feasible region is a straight line segment with two extreme points. The values of z at the two extreme points are given in the following table.



▲ Figure 4

Extreme Point (x_1, x_2)	Value of $z = 2x_1 - x_2$
(3, 2)	4
(6, 0)	12

The minimum value of z is thus 4 and is attained at $x_1 = 3$ and $x_2 = 2$. ◀

▶ EXAMPLE 8 Using Theorem 1

Find values of x_1 and x_2 that maximize

$$z = 2x_1 + 5x_2$$

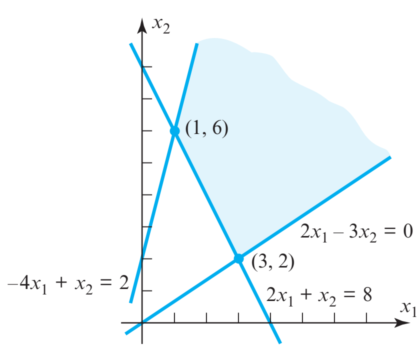
subject to

$$\begin{aligned} 2x_1 + x_2 &\geq 8 \\ -4x_1 + x_2 &\leq 2 \\ 2x_1 - 3x_2 &\leq 0 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

Solution The feasible set for this linear programming problem is illustrated in Figure 5. Since it is unbounded, we are not assured by Theorem 1 that the objective function attains a maximum value. In fact, it is easily seen that since the feasible set contains points for which both x_1 and x_2 are arbitrarily large and positive, the objective function

$$z = 2x_1 + 5x_2$$

can be made arbitrarily large and positive. This problem has no optimal solution, so we say the problem has an ***unbounded solution***. ◀



▶ Figure 5

▶ EXAMPLE 9 Using Theorem 1

Find values of x_1 and x_2 that maximize

$$z = -5x_1 + x_2$$

subject to

$$\begin{aligned} 2x_1 + x_2 &\geq 8 \\ -4x_1 + x_2 &\leq 2 \\ 2x_1 - 3x_2 &\leq 0 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

Solution The above constraints are the same as those in Example 8, so the feasible set for this problem is also given by Figure 5. In Exercise 10 we ask you to show that the objective function in this problem attains a maximum within the feasible set. By Theorem 1, this maximum must be attained at an extreme point. The values of z at the two extreme points of the feasible set are given in the following table.

Extreme Point (x_1, x_2)	Value of $z = -5x_1 + x_2$
(1, 6)	1
(3, 2)	–13

The maximum value of z is thus 1 and is attained at the extreme point $x_1 = 1$, $x_2 = 6$. ◀

▶ EXAMPLE 10 Inconsistent Constraints

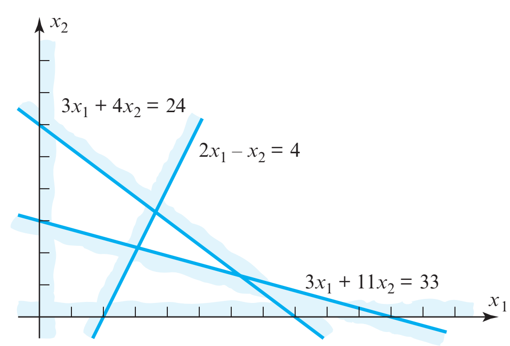
Find values of x_1 and x_2 that minimize

$$z = 3x_1 - 8x_2$$

subject to

$$\begin{aligned} 2x_1 - x_2 &\leq 4 \\ 3x_1 + 11x_2 &\leq 33 \\ 3x_1 + 4x_2 &\geq 24 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

Solution As can be seen from Figure 6, the intersection of the five half planes defined by the five constraints is empty. This linear programming problem has no feasible solutions since the constraints are inconsistent. ◀



▶ Figure 6 There are no points common to all five shaded half planes.

Exercise Set

1. Find values of x_1 and x_2 that maximize

$$z = 3x_1 + 2x_2$$

subject to

$$\begin{aligned} 2x_1 + 3x_2 &\leq 6 \\ 2x_1 - x_2 &\geq 0 \\ x_1 &\leq 2 \\ x_2 &\leq 1 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

2. Find values of x_1 and x_2 that minimize

$$z = 3x_1 - 5x_2$$

subject to

$$\begin{aligned} 2x_1 - x_2 &\leq -2 \\ 4x_1 - x_2 &\geq 0 \\ x_2 &\leq 3 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

3. Find values of x_1 and x_2 that minimize

$$z = -3x_1 + 2x_2$$

subject to

$$\begin{aligned} 3x_1 - x_2 &\geq -5 \\ -x_1 + x_2 &\geq 1 \\ 2x_1 + 4x_2 &\geq 12 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

4. Solve the linear programming problem posed in Example 2.

5. Solve the linear programming problem posed in Example 3.

6. A trucking firm ships the containers of two companies, A and B. Each container from company A weighs 40 pounds and is 2 cubic feet in volume. Each container from company B weighs 50 pounds and is 3 cubic feet in volume. The trucking firm charges company A \$2.20 for each container shipped and charges company B \$3.00 for each container shipped. If one of the firm's trucks cannot carry more than 37,000 pounds and cannot hold more than 2000 cubic feet, how many containers from companies A and B should a truck carry to maximize the shipping charges?

7. Repeat Exercise 6 if the trucking firm raises its price for shipping a container from company A to \$2.50.

8. A manufacturer produces sacks of chicken feed from two ingredients, A and B. Each sack is to contain at least 10 ounces of nutrient N_1, at least 8 ounces of nutrient N_2, and at least 12 ounces of nutrient N_3. Each pound of ingredient A contains 2 ounces of nutrient N_1, 2 ounces of nutrient N_2, and 6 ounces of nutrient N_3. Each pound of ingredient B contains 5 ounces of nutrient N_1, 3 ounces of nutrient N_2, and 4 ounces of nutrient N_3. If ingredient A costs 8 cents per pound and ingredient B costs 9 cents per pound, how much of each ingredient should the manufacturer use in each sack of feed to minimize the total cost?

9. If the objective function of a linear programming problem has the same value at two adjacent extreme points, show that it has the same value at all points on the straight line segment connecting the two extreme points. [*Hint:* If (x_1', x_2') and (x_1'', x_2'') are any two points in the plane, a point (x_1, x_2) lies on the straight line segment connecting them if

$$x_1 = tx_1' + (1-t)x_1''$$

and

$$x_2 = tx_2' + (1-t)x_2''$$

where t is a number in the interval $[0, 1]$.]

10. Show that the objective function in Example 9 attains a maximum value in the feasible set. [*Hint:* Examine the level curves of the objective function.]