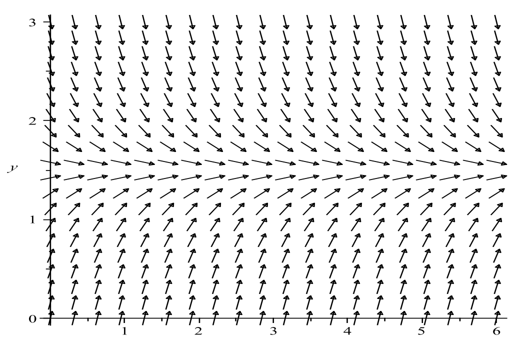
CHAPTER
1

Introduction

1.1

1.



For $y > 3/2$, the slopes are negative, therefore the solutions are decreasing. For $y < 3/2$, the slopes are positive, hence the solutions are increasing. The equilibrium solution appears to be $y(t) = 3/2$, to which all other solutions converge.

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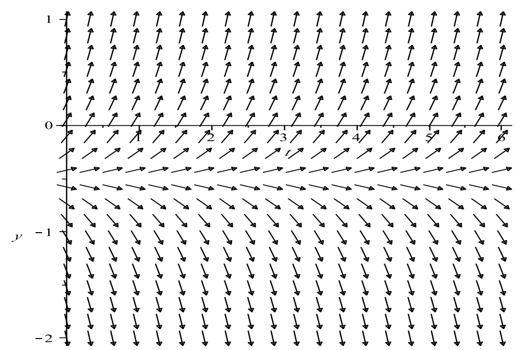
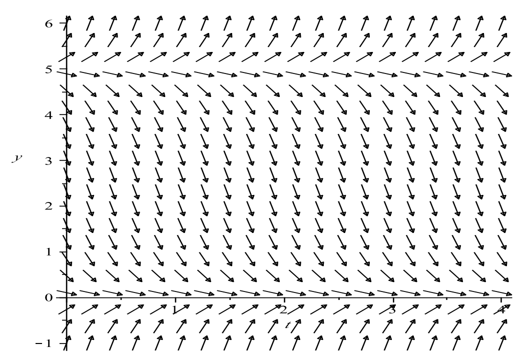


For $y > -1/2$, the slopes are positive, and hence the solutions increase. For $y < -1/2$, the slopes are negative, and hence the solutions decrease. All solutions diverge away from the equilibrium solution $y(t) = -1/2$.

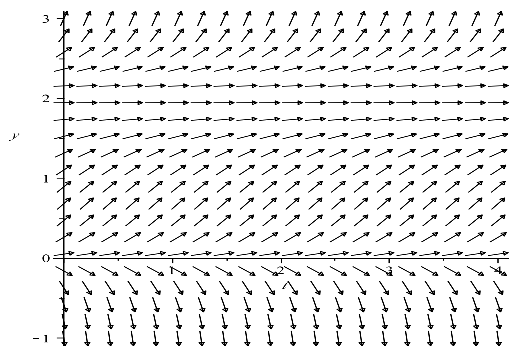
5. For all solutions to approach the equilibrium solution $y(t) = 2/3$, we must have $y' < 0$ for $y > 2/3$, and $y' > 0$ for $y < 2/3$. The required rates are satisfied by the differential equation $y' = 2 - 3y$.

8.



Note that $y' = 0$ for $y = 0$ and $y = 5$. The two equilibrium solutions are $y(t) = 0$ and $y(t) = 5$. Based on the direction field, $y' > 0$ for $y > 5$; thus solutions with initial values greater than 5 diverge from the solution $y(t) = 5$. For $0 < y < 5$, the slopes are negative, and hence solutions with initial values between 0 and 5 all decrease toward the solution $y(t) = 0$. For $y < 0$, the slopes are all positive; thus solutions with initial values less than 0 approach the solution $y(t) = 0$.

10.



Observe that $y' = 0$ for $y = 0$ and $y = 2$. The two equilibrium solutions are $y(t) = 0$ and $y(t) = 2$. Based on the direction field, $y' > 0$ for $y > 2$; thus solutions with initial values greater than 2 diverge from $y(t) = 2$. For $0 < y < 2$, the slopes are also positive, and hence solutions with initial values between 0 and 2 all increase toward the solution $y(t) = 2$. For $y < 0$, the slopes are all negative; thus solutions with initial values less than 0 diverge from the solution $y(t) = 0$.

11. -(j) $y' = 2 - y$.

13. -(g) $y' = -2 - y$.

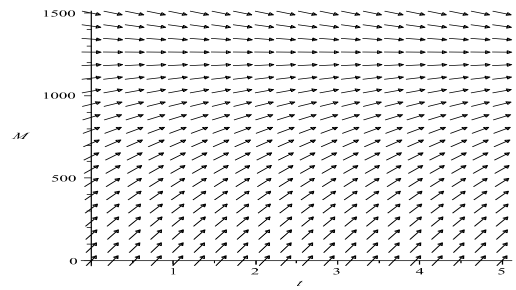
14. -(b) $y' = 2 + y$.

16. -(e) $y' = y\,(y - 3)$.

19. The difference between the temperature of the object and the ambient temperature is $u - 70$ (u in °F). Since the object is cooling when $u > 70$, and the rate constant is $k = 0.05\ \text{min}^{-1}$, the governing differential equation for the temperature of the object is $du/dt = -.05\,(u - 70)$.

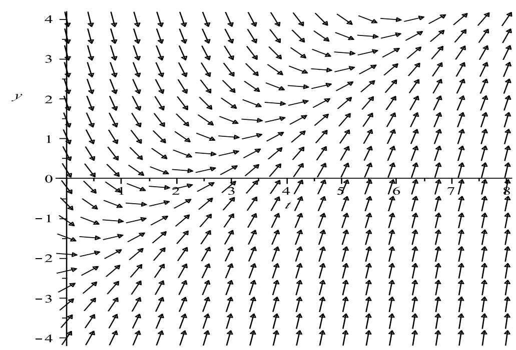
20.(a) Let $M(t)$ be the total amount of the drug (in milligrams) in the patient's body at any given time t (hr). The drug enters the body at a constant rate of 500 mg/hr. The rate at which the drug leaves the bloodstream is given by $0.4M(t)$. Hence the accumulation rate of the drug is described by the differential equation $dM/dt = 500 - 0.4\,M$ (mg/hr).

(b)



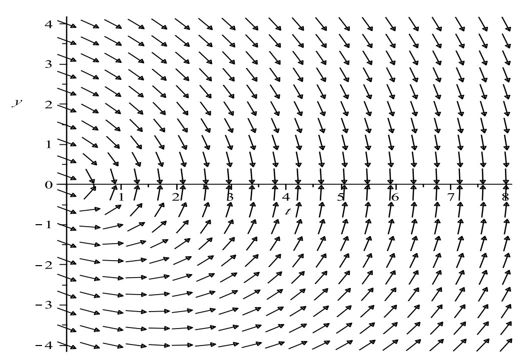
Based on the direction field, the amount of drug in the bloodstream approaches the equilibrium level of 1250 mg (within a few hours).

22.



All solutions become asymptotic to the line $y = t - 3$ as $t \to \infty$.

25.



For all $y(0)$, there is a number t_f that depends on the value of $y(0)$ such that $y(t_f) = 0$ and the solution does not exist for $t > t_f$.

1.2

4.(a) The equilibrium solution satisfies the differential equation $dy_e/dt = 0$. Setting $ay_e - b = 0$, we obtain $y_e(t) = b/a$.

(b) Since $dY/dt = dy/dt$, it follows that $dY/dt = a(Y + y_e) - b = a\,Y$.

6.(a) Consider the simpler equation $dy_1/dt = -ay_1$. As in the previous solutions, rewrite the equation as $(1/y_1)dy_1 = -a\,dt$. Integrating both sides results in $y_1(t) = c\,e^{-at}$.

(b) Now set $y(t) = y_1(t) + k$, and substitute into the original differential equation. We find that $-ay_1 + 0 = -a(y_1 + k) + b$. That is, $-ak + b = 0$, and hence $k = b/a$.

(c) The general solution of the differential equation is $y(t) = c\,e^{-at} + b/a$. This is exactly the form given by Eq.(17) in the text. Invoking an initial condition $y(0) = y_0$, the solution may also be expressed as $y(t) = b/a + (y_0 - b/a)e^{-at}$.

11. The general solution of the differential equation $dQ/dt = -r\,Q$ is $Q(t) = Q_0 e^{-rt}$, in which $Q_0 = Q(0)$ is the initial amount of the substance. Let τ be the time that it takes the substance to decay to one-half of its original amount, Q_0. Setting $t = \tau$ in the solution, we have $0.5\,Q_0 = Q_0 e^{-r\tau}$. Taking the natural logarithm of both sides, it follows that $-r\tau = \ln(0.5)$ or $r\tau = \ln 2$.

14.(a) The accumulation rate of the chemical is (0.01)(300) grams per hour. At any given time t, the concentration of the chemical in the pond is $Q(t)/10^6$ grams per gallon. Consequently, the chemical leaves the pond at a rate of $(3 \times 10^{-4})Q(t)$ grams per hour. Hence, the rate of change of the chemical is given by

$$\frac{dQ}{dt} = 3 - 0.0003\,Q(t) \quad \text{g/hr.}$$

Since the pond is initially free of the chemical, $Q(0) = 0$.

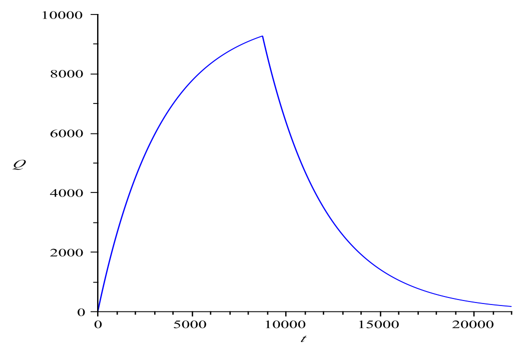
(b) The differential equation can be rewritten as $dQ/(10000 - Q) = 0.0003\,dt$. Integrating both sides of the equation results in $-\ln|10000 - Q| = 0.0003t + C$. Taking the exponential of both sides gives $10000 - Q = c\,e^{-0.0003t}$. Since $Q(0) = 0$, the value of the constant is $c = 10000$. Hence the amount of chemical in the pond at any time is $Q(t) = 10000(1 - e^{-0.0003t})$ grams. Note that 1 year= 8760 hours. Setting $t = 8760$, the amount of chemical present after one year is $Q(8760) \approx 9277.77$ grams, that is, 9.27777 kilograms.

(c) With the accumulation rate now equal to zero, the governing equation becomes $dQ/dt = -0.0003\,Q(t)$ g/hr. Resetting the time variable, we now assign the new initial value as $Q(0) = 9277.77$ grams.

(d) The solution of the differential equation in part (c) is $Q(t) = 9277.77\,e^{-0.0003t}$. Hence, one year after the source is removed, the amount of chemical in the pond is $Q(8760) \approx 670.1$ grams.

(e) Letting t be the amount of time after the source is removed, we obtain the equation $10 = 9277.77\, e^{-0.0003t}$. Taking the natural logarithm of both sides, $-0.0003\, t = \ln(10/9277.77)$ or $t \approx 22,776$ hours≈ 2.6 years.

(f)



1.3

1. The differential equation is second order, since the highest derivative in the equation is of order two. The equation is linear, since the left hand side is a linear function of y and its derivatives.

3. The differential equation is fourth order, since the highest derivative of the function y is of order four. The equation is also linear, since the terms containing the dependent variable is linear in y and its derivatives.

4. The differential equation is second order. Furthermore, the equation is nonlinear, since the dependent variable y is an argument of the sine function, which is not a linear function.

5. $y_1(t) = e^t$, so $y_1'(t) = y_1''(t) = e^t$. Hence $y_1'' - y_1 = 0$. Also, $y_2(t) = \cosh t$, so $y_1'(t) = \sinh t$ and $y_2''(t) = \cosh t$. Thus $y_2'' - y_2 = 0$.

7. $y(t) = 3t + t^2$, so $y\,'(t) = 3 + 2t$. Substituting into the differential equation, we have $t(3 + 2t) - (3t + t^2) = 3t + 2t^2 - 3t - t^2 = t^2$. Hence the given function is a solution.

8. $y_1(t) = t/3$, so $y_1'(t) = 1/3$ and $y_1''(t) = y_1'''(t) = y_1''''(t) = 0$. Clearly, $y_1(t)$ is a solution. Likewise, $y_2(t) = e^{-t} + t/3$, so $y_2'(t) = -e^{-t} + 1/3$, $y_2''(t) = e^{-t}$, $y_2'''(t) = -e^{-t}$, $y_2''''(t) = e^{-t}$. Substituting into the left hand side of the equation, we find that $e^{-t} + 4(-e^{-t}) + 3(e^{-t} + t/3) = e^{-t} - 4e^{-t} + 3e^{-t} + t = t$. Hence both functions are solutions of the differential equation.

9. $y_1(t) = t^{-2}$, so $y_1'(t) = -2t^{-3}$ and $y_1''(t) = 6\,t^{-4}$. Substituting into the left hand side of the differential equation, we have $t^2(6\,t^{-4}) + 5t(-2t^{-3}) + 4\,t^{-2} = 6\,t^{-2} - 10\,t^{-2} + 4\,t^{-2} = 0$. Likewise, $y_2(t) = t^{-2}\ln\, t$, so $y_2'(t) = t^{-3} - 2t^{-3}\ln\, t$ and $y_2''(t) = -5\,t^{-4} + 6\,t^{-4}\ln\, t$. Substituting into the left hand side of the equation, we have

$$t^2(-5\,t^{-4} + 6\,t^{-4}\ln\, t) + 5t(t^{-3} - 2t^{-3}\ln\, t) + 4(t^{-2}\ln\, t) =$$

$$= -5\,t^{-2} + 6\,t^{-2}\ln\, t + 5\,t^{-2} - 10\,t^{-2}\ln\, t + 4\,t^{-2}\ln\, t = 0.$$

Hence both functions are solutions of the differential equation.

11. Let $y(t) = e^{rt}$. Then $y'(t) = re^{rt}$, and substitution into the differential equation results in $re^{rt} + 2e^{rt} = 0$. Since $e^{rt} \neq 0$, we obtain the algebraic equation $r + 2 = 0$. The root of this equation is $r = -2$.

12. $y(t) = e^{rt}$, so $y'(t) = r\,e^{rt}$ and $y''(t) = r^2e^{rt}$. Substituting into the differential equation, we have $r^2e^{rt} + re^{rt} - 6\,e^{rt} = 0$. Since $e^{rt} \neq 0$, we obtain the algebraic equation $r^2 + r - 6 = 0$, that is, $(r-2)(r+3) = 0$. The roots are $r_1 = 2$, $r_2 = -3$.

13. Let $y(t) = e^{rt}$. Then $y'(t) = re^{rt}$, $y''(t) = r^2e^{rt}$ and $y'''(t) = r^3e^{rt}$. Substituting the derivatives into the differential equation, we have $r^3e^{rt} - 3r^2e^{rt} + 2re^{rt} = 0$. Since $e^{rt} \neq 0$, we obtain the algebraic equation $r^3 - 3r^2 + 2r = 0$. By inspection, it follows that $r(r-1)(r-2) = 0$. Clearly, the roots are $r_1 = 0$, $r_2 = 1$ and $r_3 = 2$.

15. $y(t) = t^r$, so $y'(t) = r\,t^{r-1}$ and $y''(t) = r(r-1)t^{r-2}$. Substituting the derivatives into the differential equation, we have $t^2\left[r(r-1)t^{r-2}\right] - 4t(r\,t^{r-1}) + 4\,t^r = 0$. After some algebra, it follows that $r(r-1)t^r - 4r\,t^r + 4\,t^r = 0$. For $t \neq 0$, we obtain the algebraic equation $r^2 - 5r + 4 = 0$. The roots of this equation are $r_1 = 1$ and $r_2 = 4$.

16. The order of the partial differential equation is two, since the highest derivative, in fact each one of the derivatives, is of second order. The equation is linear, since the left hand side is a linear function of the partial derivatives.

17. The partial differential equation is fourth order, since the highest derivative, and in fact each of the derivatives, is of order four. The equation is linear, since the left hand side is a linear function of the partial derivatives.

18. The partial differential equation is second order, since the highest derivative of the function $u(x,y)$ is of order two. The equation is nonlinear, due to the product $u \cdot u_x$ on the left hand side of the equation.

19. If $u_1(x,y) = \cos\, x \cosh\, y$, then $\partial^2 u_1/\partial x^2 = -\cos\, x \cosh\, y$ and $\partial^2 u_1/\partial y^2 = \cos\, x \cosh\, y$. It is evident that $\partial^2 u_1/\partial x^2 + \partial^2 u_1/\partial y^2 = 0$. Also, when $u_2(x,y) = \ln(x^2+y^2)$, the second derivatives are

$$\frac{\partial^2 u_2}{\partial x^2} = \frac{2}{x^2+y^2} - \frac{4x^2}{(x^2+y^2)^2} \quad \text{and} \quad \frac{\partial^2 u_2}{\partial y^2} = \frac{2}{x^2+y^2} - \frac{4y^2}{(x^2+y^2)^2}.$$

Adding the partial derivatives,

$$\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} = \frac{2}{x^2+y^2} - \frac{4x^2}{(x^2+y^2)^2} + \frac{2}{x^2+y^2} - \frac{4y^2}{(x^2+y^2)^2} =$$

$$= \frac{4}{x^2+y^2} - \frac{4(x^2+y^2)}{(x^2+y^2)^2} = 0.$$

Hence $u_2(x, y)$ is also a solution of the differential equation.

21. Let $u_1(x,t) = \sin(\lambda x)\,\sin(\lambda a t)$. Then the second derivatives are

$$\frac{\partial^2 u_1}{\partial x^2} = -\lambda^2 \sin\lambda x\,\sin\lambda a t \quad \text{and} \quad \frac{\partial^2 u_1}{\partial t^2} = -\lambda^2 a^2 \sin\lambda x\,\sin\lambda a t.$$

It is easy to see that $a^2\partial^2 u_1/\partial x^2 = \partial^2 u_1/\partial t^2$. Likewise, given $u_2(x,t) = \sin(x - at)$, we have

$$\frac{\partial^2 u_2}{\partial x^2} = -\sin(x - at) \quad \text{and} \quad \frac{\partial^2 u_2}{\partial t^2} = -a^2 \sin(x - at).$$

Clearly, $u_2(x, t)$ is also a solution of the partial differential equation.

23.(a) The kinetic energy of a particle of mass m is given by $T = m\,v^2/2$, in which v is its speed. A particle in motion on a circle of radius L has speed $L\,(d\theta/dt)$, where θ is its angular position and $d\theta/dt$ is its angular speed.

(b) Gravitational potential energy is given by $V = mgh$, where h is the height above a certain datum. Choosing the lowest point of the swing as the datum, it follows from trigonometry that $h = L(1 - \cos\theta)$.

(c) From parts (a) and (b),

$$E = \frac{1}{2} m L^2 (\frac{d\theta}{dt})^2 + mgL(1 - \cos\theta)\,.$$

Applying the chain rule for differentiation,

$$\frac{dE}{dt} = mL^2\,\frac{d\theta}{dt}\,\frac{d^2\theta}{dt^2} + mgL\,\sin\theta\,\frac{d\theta}{dt}\,.$$

Setting $dE/dt = 0$ and dividing both sides of the equation by $d\theta/dt$ results in

$$mL^2\,\frac{d^2\theta}{dt^2} + mgL\,\sin\theta = 0,$$

which leads to Equation (12).

24.(a) The angular momentum is the moment of the linear momentum about a given point. The linear momentum is given by $mv = mLd\theta/dt$. Taking the moment about the point of support, the angular momentum is

$$M = mvL = mL^2\frac{d\theta}{dt}\,.$$

(b) The moment of the gravitational force is $-mgL\sin\theta$. The negative sign is included since positive moments are counterclockwise. Setting dM/dt equal to the moment of the gravitational force gives

$$\frac{dM}{dt} = mL^2\frac{d^2\theta}{dt^2} = -mgL\sin\theta,$$

which leads to Equation (12).