1 COUNTING

1.1 BASIC COUNTING

Pages 7 to 8

Problem 1

Solution

The value of i ranges from 2 to n. When $i = k$, the variable j ranges from 2 to k. Thus, there are at most $k - 1$ comparisons (because we stop if $j = 2$). Thus, the total number of comparisons is

$$1 + 2 + \cdots + n - 1 = \frac{n(n-1)}{2}.$$

The algorithm will make this number of comparisons if the original ordering is the reverse of the sorted ordering.

Problem 2

Solution

Number the five teams 1–5. Team 1 must play all four others. Team 2 will be in one of these games but must play in three more games with Teams 3, 4, and 5. Team 3 is in two of the games already mentioned and must still play Teams 4 and 5 for two more games. Team 4 must play Team 5, in addition to playing in three of the games already mentioned. Thus, there are $4 + 3 + 2 + 1 = 10$ games. Alternatively, there are five teams, each of which must play in four games, giving us 20 pairings of two teams each. However, each game involves two of these pairings, so there are $20/2 = 10$ games.

Problem 3

Solution

The set of possible draws is a union of 52 sets (one for each possible first card), each of size 51. So, by the product principle, there are $52 \cdot 51$ ways to draw the two cards.

Problem 4

Solution

The answer is the same as in Problem 3, except we can draw the cards in either order. Therefore, the number of ways is $52 \cdot 51/2 = 1326$.

Problem 5

Solution

$52 \cdot 51 \cdot 50$, by two applications of the product principle.

Problem 6

Solution

$10 \cdot 9 = 90$.

Problem 7

Solution

$\binom{10}{2} = 10 \cdot 9/2 = 45$.

Problem 8

Solution

$10 \cdot \binom{9}{2}$, or $8\binom{10}{2}$.

Problem 9

Solution

This formula counts the number of ways to choose a president and an executive advisory board (not including the president) from a club of n people. The left side chooses the president first, then the committee. The right side chooses the committee first, then the president.

Problem 10

Solution

$m \cdot n$.

Problem 11

Solution

By the product rule, there are $10 \cdot 9 = 90$ ways to choose two-scoop cones with two different flavors. However, according to your mother's rule, the order of scoops doesn't matter. Because each two-scoop cone can be ordered in two different ways (e.g., chocolate over vanilla and vanilla over chocolate), we have $90/2 = 45$ ways of choosing two-scoop cones with different flavors. There are an additional ten cones with the same flavor for both scoops, giving 55 possible cones.

Problem 12 **Solution**

Because order does matter, we have $10 \cdot 9 = 90$ ways to choose ice cream cones with two distinct flavors, plus ten more with the same flavor for both scoops, giving 100 choices.

Problem 13 **Solution**

$1 + 2 + 4 + \cdots + 2^{19} = 2^{20} - 1 = 1{,}048{,}575$. Your justification may be neither principle, only the sum principle (the set of all pennies is the union of the set of pennies on Day 1 with those on Day 2, and so on), or both principles (the set of pennies you receive on Day i is the union of two sets of pennies, each of the size that you received on Day $i - 1$). As long as your explanation makes sense, any of these answers is fine.

Problem 14 **Solution**

$5 \cdot 3 \cdot 3 \cdot 3 = 135$.

Problem 15 **Solution**

Yes; in Line 4, j could start at $i + 1$ rather than i.

1.2 COUNTING LISTS, PERMUTATIONS, AND SUBSETS

Pages 17 to 19

Problem 1 **Solution**

For each piece of fruit, we have n choices of who to give it to. So, by version 2 of the product principle, the number of ways to pass out the fruit is n^k.

Problem 2 **Solution**

$$\begin{array}{lll}
f_1(1) = a & f_1(2) = a & f_1(3) = a \\
f_2(1) = a & f_2(2) = a & f_2(3) = b \\
f_3(1) = a & f_3(2) = b & f_3(3) = a \\
f_4(1) = a & f_4(2) = b & f_4(3) = b \\
f_5(1) = b & f_5(2) = a & f_5(3) = a \\
f_6(1) = b & f_6(2) = a & f_6(3) = b \\
f_7(1) = b & f_7(2) = b & f_7(3) = a \\
f_8(1) = b & f_8(2) = b & f_8(3) = b
\end{array}$$

None are one-to-one; all but f_1 and f_8 are onto.

Problem 3 Solution

$$
\begin{array}{ll}
f_1(1) = a & f_1(2) = a \\
f_2(1) = a & f_2(2) = b \\
f_3(1) = a & f_3(2) = c \\
f_4(1) = b & f_4(2) = a \\
f_5(1) = b & f_5(2) = b \\
f_6(1) = b & f_6(2) = c \\
f_7(1) = c & f_7(2) = a \\
f_8(1) = c & f_8(2) = b \\
f_9(1) = c & f_9(2) = c
\end{array}
$$

None of the functions are onto; all except f_1, f_5, and f_9 are one-to-one.

Problem 4 Solution

If we list S as $x_1, x_2, \ldots, x_s$, then there is a bijection between functions from S to T and lists $f(x_1), f(x_2), \ldots, f(x_s)$. For each i, there are t choices for $f(x_i)$. So, by the product principle, there are t^s functions from S to T.

Problem 5 Solution

We are asking for the number of k-element permutations of n children, which is $n^{\underline{k}}$, or zero, if $k > n$.

Problem 6 Solution

What matters is what subset of the n children get fruit, so the answer is $\binom{n}{k}$. If $k > n$, the answer is zero.

Problem 7 Solution

First, note that "a five-digit base 10 number" means a string of five digits, where the first digit is not 0 and each digit is in the set $\{0, 1, \ldots, 9\}$. By the product rule, the number of these is $9 \cdot 10^4$, or 90,000. If no two consecutive digits can be equal, then there are nine choices for the first digit, nine for the second (any digit other than the first), nine for the third (any digit other than the second), and so on. By the product principle, the total number is 9^5.

By the sum principle, the total number of five-digit numbers equals the number that have no two consecutive digits equal plus the number that have at least one pair of consecutive digits equal. Thus, letting x denote the number of the latter, we have $9 \cdot 10^4 = 9^5 + x$; so, $x = 9 \cdot 10^4 - 9^5 = 30{,}951$.

Problem 8

Solution

In both cases, there are two ways to decide whether the leftmost spot is for a student or for an administrator. This decision determines which four places are for students and which are for administrators. Thus, there are 4! ways to assign the students to their places and 4! ways to assign the administrators to their places. In both cases, the product principle leads us to conclude that there are $2 \cdot 4! \cdot 4!$ lists.

Problem 9

Solution

123	124	125	132	134	$\underline{135}$	142	143	145	152	$\underline{153}$	154
213	214	215	231	234	235	241	243	$\boxed{245}$	251	253	$\boxed{254}$
312	314	$\underline{315}$	321	324	325	341	342	345	$\underline{351}$	352	354
412	413	415	421	423	$\boxed{425}$	431	432	435	451	$\boxed{452}$	453
512	$\underline{513}$	514	521	523	$\boxed{524}$	$\underline{531}$	532	534	541	$\boxed{542}$	543

Six permutations correspond to any given three-element set. We have 60 permutations, so there are 10 three-element sets.

Problem 10

Solution

$\binom{20}{3} = 20 \cdot 19 \cdot 18/6 = 1140.$

Problem 11

Solution

$\binom{10}{4}\binom{20}{4}.$

Problem 12

Solution

$2\binom{10}{4}\binom{20}{4}4!4! = 2 \cdot 10^{\underline{4}}\,20^{\underline{4}}.$

Problem 13

Solution

When both scoops have the same flavor, we have 10 possibilities, because there are only 10 flavors. When the scoops have different flavors, we have 45 possibilities, according to your mother's rule, as solved before. So, for ice cream, we have $10 + 45 = 55$ possibilities. For topping, we have 3 possibilities. For whipped cream, nuts, and cherry, because we may either have any, all, or none, we have 2 possibilities for each of them (for example, either have cherry or don't have cherry). By the product rule, we have $2 \cdot 2 \cdot 2 = 8$ possibilities. Thus, we have $55 \cdot 3 \cdot 8 = 1320$ possible sundaes.

Problem 14

Solution

Although this problem is similar to Problem 13, we may have three different flavors (there are $10 \cdot 9 \cdot 8/6 = 2880$ ways to choose three flavors), two scoops of one flavor and one of a second (there are $10 \cdot 9 = 90$ ways to do this), or three scoops of the same flavor (there are 10 ways to choose these). Thus, we have $220 \cdot 3 \cdot 8 = 5280$ different sundaes.

Problem 15

Solution

Suppose we list the people in the club in some way and keep that list for the remainder of the problem. Take the first person from the list and pair that person with any of the $2n-1$ remaining people. Now take the next *unpaired* person from the list and pair that person with any of the remaining $2n-3$ unpaired people. Continuing in this way, once k pairs have been selected, take the next unpaired person from the list and pair that person with any of the remaining $2n-2k-1$ unpaired people. Every pairing can arise in this way, and no pairing can arise twice in this process. Thus, the number of outcomes is $\prod_{i=0}^{n-1} 2n-2i-1$. For another solution, choose people in pairs. There are $\binom{2n}{2}$ ways to choose one pair, $\binom{2n-2}{2}$ ways to choose a second pair, and once k pairs have been chosen, there are $\binom{2n-2k}{2}$ ways to choose the next pair. The number of *lists* of pairs we get in this way is $\prod_{i=0}^{n-1}\binom{2n-2i}{2} = \frac{(2n)!}{2^i}$. Both ways of pairing people gets listed $n!$ times because we see all possible lists of pairs of length n. Therefore, the number of actual pairings is

$$\frac{(2n)!}{(2^n)n!} = \frac{2n!}{2n \cdot 2n-2 \cdot 2n-4 \cdot \cdots \cdot 2} = \prod_{i=0}^{n-1}(2n-2i-1)\ .$$

For the second question, multiply the answer to the first question by 2^n to give $(2n)!/n!$.

Problem 16

Solution

$\binom{12}{5}$. $\binom{5}{2}\binom{4}{2}\binom{3}{1} = 180$. $\binom{5}{2}\binom{4}{2}\binom{1}{1} + \binom{5}{2}\binom{5}{2}\binom{2}{1}$—that is, either the versatile player is playing center or not; if not, that player is available to play forward.

Problem 17

Solution

If f is one-to-one, it has n distinct values. Therefore, all elements of the range must be values of f, so it must be onto. Thus, a one-to-one function from an n-element set to an n-element set is onto. If we now suppose that f is onto, then f has n distinct values because it maps onto a set of size n. But in this case, we may conclude that because there are only n values of x, all the values of $f(x)$ are different. Therefore, f must be one-to-one.

Problem 18 **Solution**

a. If f is a bijection, we may define $g(y)$ to be the unique x such that $f(x) = y$. This defines a function because different values of y are related to different values of x. But then $f(g(y))$ is the result of applying f to the unique x with $f(x) = y$, so $f(g(y)) = y$. Also, $g(f(x))$ is the unique x that is related to $f(x)$, so $g(f(x)) = x$. This shows that if f is a bijection, then f has an inverse function. On the other hand, if f has an inverse function g and if $f(x) = f(x')$, then $g(f(x)) = g(f(x'))$, which gives us $x = x'$. Therefore, if f has an inverse function, then f is one-to-one. Further, if f has an inverse function g, then because $y = f(g(y))$ and $g(y)$ is in the domain of f, there is an x in the domain of f (namely, $g(y)$) such that $f(x) = y$; so, f is onto.

b. Suppose g and h both satisfy the definition of being inverses to f. Suppose y is in the range of f and $g(y) = x$. Then, $f(g(y)) = f(x)$ and $h(f(g(y))) = h(f(x)) = x$. Because $f(g(y)) = y$, we have $h(y) = x$ as well. Thus, for any y in the range of f, $h(y) = g(y)$, which means that g and h are equal. Thus, f has only one inverse function.

1.3 BINOMIAL COEFFICIENTS

Pages 26 to 28

Problem 1 **Solution**

220, 220. $\binom{n}{k}$ equals $\binom{n}{n-k}$.

Problem 2 **Solution**

1 8 28 56 70 56 28 8 1

Problem 3 **Solution**

a. $(x + 1)^5 = x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$.

b. $(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$.

c. $(x + 2)^5 = x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32$.

d. $(x - 1)^5 = x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1$.

Problem 4 **Solution**

$(x + y)^4 = x^4 + \binom{4}{1}x^3y^1 + \binom{4}{2}x^2y^2 + \binom{4}{3}x^1y^3 + y^4$. To expand the product, we create all possible products of x's and y's by selecting y from some of the factors and x from the remaining ones. Then we add all these products. There is $1 = \binom{4}{0}$ way to choose a y from none of our four $x + y$ factors. Thus, the coefficient of x^4 is 1. There are $\binom{4}{1}$ ways to choose a y from one of the four factors. Because we choose x from the remaining factors, this choice gives us the term $\binom{4}{1}x^3y$. There are $\binom{4}{2}$ ways to choose y from two of our four factors, so the coefficient of x^2y^2 is $\binom{4}{2}$. There are $\binom{4}{3}$ ways to choose y from three of our factors, so the coefficient of xy^3 is $\binom{4}{3}$. There is $1 = \binom{4}{4}$ way to choose y from all the factors, so the coefficient of y^4 is 1.

Problem 5

Solution

$10!/(3!3!4!) = 4200$. This is the number of ways to label three of the chairs with the label green, three of the chairs with the label blue, and four of the chairs with the label red.

Problem 6

Solution

The multinomial coefficient $\binom{n}{n_1, n_2, \ldots, n_k}$ is the coefficient of $x_1^{n_1}x_2^{n_2}\cdots x_k^{n_k}$ in $(x_1 + x_2 + \cdots + x_k)^n$. The proof is just like the proofs of the binomial theorem and the trinomial theorem.

Problem 7

Solution

If N is a set and K is a subset of N, let $N - K$ stand for the set of all elements of N that are not in K. If $f(K) = N - K$, then f is a bijection that maps the k-element subsets of N onto the $(n - k)$-element subsets of N.

Problem 8

Solution

$\binom{m+n}{n}$ or $\binom{m+n}{m}$, because of the following:

- From any line segment on our path, we have two choices: horizontal and vertical. Clearly, we need n vertical line segments and m horizontal line segments to reach (m, n) in exactly $m + n$ steps.
- By deciding on our choice of n vertical lines, we have also decided the choice of m horizontal lines, because we must choose a horizontal line for each step that is not a vertical line. We could also start by choosing m horizontal lines, which automatically decides the choice of n vertical lines. Thus, the number of ways to choose the path is the number of ways to choose the n places for vertical lines from $m + n$ places or the m places for horizontal lines from $m + n$ places.

Problem 9

Solution

$\sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i} = y^n + \binom{n}{1} x y^{n-1} + \binom{n}{2} x^2 y^{n-2} + \cdots + x^n$.

Problem 10

Solution

Once we choose four disjoint subsets from a 12-element set, we may label them in 4! ways. That is, choosing the sets does not tell us which set to label with which label. Therefore, the number of choices of the four sets is the number of labelings divided by 4!, namely, $\frac{12!}{4!(3!)^4}$. The number of ways to choose three disjoint subsets of size four from a 12-element set is $\frac{12!}{3!(4!)^3}$.

Problem 11

Solution

For the first case, the president may be chosen from 20 persons, and after that, the vice president may chosen from 19 persons, the secretary from 18 persons, and the treasurer from 17 persons. The nominations committee may be chosen from the remaining 16 persons without order mattering. So, there are $20 \cdot 19 \cdot 18 \cdot 17 \cdot \binom{16}{3}$ ways to choose the committee. For the second case, the nominating committee may be chosen from the 20 persons without order mattering. So, there are $20 \cdot 19 \cdot 18 \cdot 17 \cdot \binom{20}{3}$ ways to choose the committee.

Problem 12

Solution

$$\begin{aligned}\binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!} \\ &= \frac{(n-1)!k}{(k)!(n-k)!} + \frac{(n-1)!(n-k)}{k!(n-k)!} \\ &= \frac{(n-1)!(k+n-k)}{k!(n-k)!} \\ &= \frac{n!}{k!(n-k)!} = \binom{n}{k}.\end{aligned}$$

Problem 13

Solution

Proof 1:

$$\frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!}.$$

Proof 2: The number of ways to choose the k elements that are in a k-element subset is the same as the number of ways to choose the elements that are *not* in the subset, and there are $n-k$ such elements.

Problem 14

Solution

Proof 1:

$$\binom{n}{k}\binom{k}{j} = \frac{n!}{k!(n-k)!}\,\frac{k!}{j!(k-j)!}$$
$$= \frac{n!}{(n-k)!\,j!(k-j)!}$$
$$= \frac{n!}{j!(n-j)!}\,\frac{(n-j)!}{(k-j)!(n-k)!}$$
$$= \binom{n}{j}\binom{n-j}{k-j}.$$

Proof 2: The number of ways to choose k items from n items and then choose j items from the chosen k items is $\binom{n}{k}\binom{k}{j}$. We can also carry out this process in the following way: First choose j items from n items, and then choose $k-j$ more items from the remaining $n-j$ items. The number of ways to do this is $\binom{n}{j}\binom{n-j}{k-j}$.

Problem 15

Solution

Proof 1:

$$\binom{n}{k}\binom{n-k}{j} = \frac{n!(n-k)!}{k!(n-k)!\,j!(n-k-j!)} = \frac{n!}{k!\,j!(n-k-j)!}.$$

Similarly, $\binom{n}{k}\binom{n-k}{j}$ equals the same expression.

Proof 2: There are two ways of choosing two disjoint sets, one with k elements and one with j elements. We can pick the k-element set first, then choose j elements from what is left, or we can pick the j-element set first, then choose k elements from what is left.

Problem 16

Solution

$n \backslash k$	3	4	5	6
6	20	15	6	1
7		35	21	7
8			56	28
9				84

Problem 17

Solution

The formula is simply the expansion of $(1-1)^n$.

Problem 18 **Solution**

$(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i$. Taking the derivative of both sides, we get

$$n(1+x)^{n-1} = \sum_{i=1}^{n} i \binom{n}{i} x^{i-1} .$$

Thus, if we let $x = 1$, we have $n2^{n-1} = \sum_{i=1}^{n} i \binom{n}{i}$.

Problem 19 **Solution**

False. $\binom{4}{2}$ is 6, but $\binom{2}{0} + \binom{2}{1} + \binom{2}{2}$ is 4. The correct statement is

$$\binom{n}{k} = \binom{n-2}{k-2} + 2\binom{n-2}{k-1} + \binom{n-2}{k} .$$

The proof consists of applying the Pascal relationship to both $\binom{n-1}{k-1}$ and $\binom{n-1}{k}$ and adding the results.

1.4 EQUIVALENCE RELATIONS AND COUNTING

Pages 38 to 41

Problem 1 **Solution**

$(n-1)!$ ways.

Problem 2 **Solution**

If we rotate a scarf through 180 degrees, we do not change the scarf, but we do get the reversed row of circles. Thus, two arrangements of n circles are equivalent if one is the reverse of the other. The equivalence classes have two members, so the number of ways to embroider the circles on the scarf is $n!/2$.

Problem 3 **Solution**

There are $\binom{5}{2} = 10$ ways to choose the two places out of five in which the golden apples are placed. For equivalence class counting, there are 5! ways to line up the apples. Two ways are equivalent if we interchange the golden apples or mix up the red apples in any way. Thus, there are $2 \cdot 3!$ arrangements per equivalence class. Therefore, there are $5!/(2 \cdot 3!)$ equivalence classes. This means that there are $5!/(2 \cdot 3!)$ ways to line up the apples. Note that the second answer is simply the usual formula for the first answer.

Problem 4

Solution

Passing out the apples corresponds to choosing a k-element multiset from the n children. Choosing a k-element multiset from the children tells us how many apples to give each child. Thus, there are $\binom{n+k-1}{k}$ ways to pass out the apples.

Problem 5

Solution

Number the places around the table consecutively from 1 to $2n$. A seating gives a list of the $2n$ people. Once we decide the gender of the person in Seat 1, we have $n!$ ways to seat that gender and $n!$ ways to seat the other gender. By the product principle, we have $2 \cdot n! \cdot n!$ lists of people corresponding to seating arrangements. But there are $2n$ lists that correspond to the same circular arrangement. Therefore, the number of ways to seat the people is $2n!n!/2n = n!(n-1)!$.

Another way is to seat one gender in $(n-1)!$ ways and then seat the other gender in the n places between members of the first group in $n!$ ways. Then, by the product principle, we have $(n-1)!n!$ seating arrangements.

Problem 6

Solution

Give one apple to each child. Then pass out $k-n$ apples to the children in $\binom{n+k-n-1}{k-n} = \binom{k-1}{k-n}$ ways.

Problem 7

Solution

Select k books in $\binom{n}{k}$ ways, and place one on the far left of each shelf in $n!$ ways. Then shelve the remaining $k-n$ books in $n^{\overline{k-n}}$ ways. This gives

$$\binom{k}{n} n! n^{\overline{k-n}} = \frac{k!(k-1)!}{(k-n)!(n-1)!}$$

ways to shelve all the books.

Problem 8

Solution

a. $\binom{k+n-1}{k}$, or the number of ways to choose the places where the red checkers go. This is also the way to put k identical books and $n-1$ identical blocks of wood in a line, because it is the number of ways to choose where the books (or the blocks of wood) go.

b. The number of red checkers between black checker number $i-1$ and black checker number i (or before black checker number 1) is the multiplicity of i (or 1). The number of red checkers after the last black checker is the multiplicity of black checker n.

c. Parts a and b count the same thing, so the number of k-element multisets chosen from an n-element set is $\binom{n+k-1}{k}$.

Problem 9

Solution

If you think of the value of x_i as the multiplicity of i in a multiset, then the number of solutions is the number of multisets, $\binom{n+k-1}{k}$.

Problem 10

Solution

If you think of x_i as the multiplicity of i in a multiset, then you are counting the number of k-element multisets chosen from $\{1, 2, \ldots, n\}$ in which every element has multiplicity at least 1. This is the same as the number of $k - n$ element multisets of $\{1, 2, \ldots, n\}$, which is $\binom{k-1}{k-n}$. To see why this is the answer, think of giving k identical apples to n children. Give each child one apple and then distribute the remaining $k-n$ apples however you want.

Problem 11

Solution

Imagine this circular arrangement as a linear arrangement. If both the n red checkers and the $n + 1$ black checkers were distinguishable, there would be $(n + n + 1)! = (2n + 1)!$ possible linear arrangements. When we think of the red checkers (and the black ones) as indistinguishable, we have $n!(n + 1)!$ lists per equivalence class, because these lists would be equivalent if we mixed n red checkers among themselves and $n + 1$ black checkers among themselves. Each linear arrangement of n red and $n + 1$ black checkers corresponds to $2n + 1$ other arrangements, which become identical when we put them into a circle. (This would not be the case if we had n red and n black checkers; think about alternating red and black.) So, the number of arrangements is

$$\frac{(2n+1)!}{(2n+1)n!(n+1)!} = \frac{(2n)!}{n!(n+1)!} = \frac{1}{n+1}\binom{2n}{n}.$$

Problem 12

Solution

a. For $S(n, n)$, each element must be in a part by itself. There is one such partition, so $S(n, n) = 1$. For $S(n, 1)$, every element must be in the same part, so $S(n, 1) = 1$.

b. An element a may be in a part by itself or not. The number of partitions in which a is in a part by itself is $S(n - 1, k - 1)$, because then we have to choose remaining $k - 1$ parts from remaining $n - 1$ elements. Otherwise a is in a part with other elements. Because a can be in any of the k parts and the other $n - 1$ elements may be partitioned into k parts in $S(n - 1, k)$ ways, we have

$kS(n-1, k)$ partitions of this type. By the sum principle, we get $S(n, k) = S(n-1, k-1) + kS(n-1, k)$.

c.

n \ k	1	2	3	4	5	6
1	1					
2	1	1				
3	1	3	1			
4	1	7	6	1		
5	1	15	25	10	1	
6	1	31	90	65	15	1

Problem 13

Solution

There are six lists of four letters: you can make a list by choosing two of the four places for the red beads in $\binom{4}{2} = 6$ ways. The equivalence classes are {*RRBB*, *RBBR*, *BBRR*, *BRRB*} and {*RBRB*, *BRBR*}. Because the sizes of the equivalence classes are not the same, you cannot apply Theorem 1.5.

Problem 14

Solution

a. "Greater than" is transitive. "Is a brother of" is not transitive (Paul can be a brother to Bob and Bob to Paul, but Paul is not a brother to Paul). "Is a sibling of" is not transitive but is symmetric. "Is a sibling of or is" is transitive, symmetric, and reflexive.

b. x is in the same equivalence class as x, so it is reflexive. If x is in the same equivalence class as y, then y is in the same equivalence class as x. Therefore, it is symmetric. If x is in the same equivalence class as y and y is in the same equivalence class as z, then x is in the same equivalence class as z. Therefore, it is transitive.

c. If S_x and S_y have a common element, such as a, then we have that a is related to both x and y. We also have that y is related to a by symmetry. For any element b in S_y, we have b is related to y. Then b is related to a by transitivity and related to x again by transitivity. Thus, we have $b \in S_x$, which shows that $S_y \subseteq S_x$. Similarly, we have $S_x \subseteq S_y$. Therefore, we have $S_x = S_y$. Thus, if S_x and S_y have a common element, they are identical. If S_x and S_y have no common element, they are disjoint. Therefore, we can use the relationship to divide S into disjoint sets: the sets S_x. Note that by reflexivity, any element x is in the set S_x. Thus, a reflexive symmetric transitive relation is an equivalence relation.

d. Part b says that an equivalence relation, as defined in the text, is reflexive, symmetric, and transitive. Part c says that a reflexive, symmetric, and transitive relation is an equivalence relation, as was defined in the text.

Problem 15

Solution

The original version computes the entire Pascal triangle up to the element in row n and column k each time it is called. Your new version should store that information globally so that it only has to do one addition each time it computes a new value in Pascal's triangle.

Problem 16

Solution

a. This problem asks you to count functions from the candy to the people, so the answer is n^k.

b. This asks for one-to-one functions from the candy to the people, so the answer is $n^{\underline{k}}$. Note how the notation for counting one-to-one functions is similar to the notation for counting functions.

c. A distribution is equivalent to a k-element multiset of the n-element set of people. Thus, there are $\binom{n+k-1}{k}$ distributions.

d. This asks for k-element subsets of the n people, so the answer is $\binom{n}{k}$.

e. This is the number of k-element permutations of the n objects, so the answer is $n^{\underline{k}}$.

f. Because you are counting functions, the answer is n^k.

g. By definition, $\binom{n}{k}$.

h. By the theorem for counting multisets, $\binom{n+k-1}{k}$.

i. This is asking for the number of lists of k distinct people chosen out of n. Such a list is a k-element permutation of the n elements, so the answer is $n^{\underline{k}}$.

j. This is asking for the number of k-element multisets of an n-element set, and this number is $\binom{n+k-1}{k}$.

k. Because it matters who gets which type, this is the same as asking how many one-to-one functions there are from a k-element set to an n-element set. Thus, the answer is $n^{\underline{k}}$.