INSTRUCTOR'S SOLUTIONS MANUAL

DIFFERENTIAL EQUATIONS AND LINEAR ALGEBRA

FOURTH EDITION

Stephen W. Goode
and
Scott A. Annin
California State University, Fullerton

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Contents

Chapter 1 Solutions

Solutions to Section 1.1

True-False Review:

(a): FALSE. A derivative must involve *some* derivative of the function $y = f(x)$, not necessarily the first derivative.

(b): FALSE. The order of a differential equation is the order of the *highest*, not the lowest, derivative appearing in the differential equation.

(c): FALSE. This differential equation has order two, since the highest order derivative that appears in the equation is the second order expression y''.

(d): FALSE. The carrying capacity refers to the maximum population size that the environment can support in the long run; it is not related to the initial population in any way.

(e): TRUE. The value $y(0)$ is called an initial condition to the differential equation for $y(t)$.

(f): TRUE. According to Newton's Law of Cooling, the rate of cooling is proportional to the *difference* between the object's temperature and the medium's temperature. Since that difference is greater for the object at $100°F$ than the object at $90°F$, the object whose temperature is $100°F$ has a greater rate of cooling.

(g): FALSE. The temperature of the object is given by $T(t) = T_m + ce^{-kt}$, where T_m is the temperature of the medium, and c and k are constants. Since $e^{-kt} \neq 0$, we see that $T(t) \neq T_m$ for all times t. The temperature of the object *approaches* the temperature of the surrounding medium, but never equals it.

(h): TRUE. Since the temperature of the coffee is falling, the temperature *difference* between the coffee and the room is higher initially, during the first hour, than it is later, when the temperature of the coffee has already decreased.

(i): FALSE. The slopes of the two curves are *negative* reciprocals of each other.

(j): TRUE. If the original family of parallel lines have slopes k for $k \neq 0$, then the family of orthogonal trajectories are parallel lines with slope $-\frac{1}{k}$. If the original family of parallel lines are vertical (resp. horizontal), then the family of orthogonal trajectories are horizontal (resp. vertical) parallel lines.

(k): FALSE. The family of orthogonal trajectories for a family of circles centered at the origin is the family of lines passing through the origin.

(l): TRUE. If $v(t)$ denotes the velocity of the object at time t and $a(t)$ denotes the velocity of the object at time t, then we have $a(t) = v'(t)$, which is a differential equation for the unknown function $v(t)$.

(m): FALSE. The restoring force is directed in the direction *opposite* to the displacement from the equilibrium position.

(n): TRUE. The allometric relationship $B = B_0 m^{3/4}$, where B_0 is a constant, relates the metabolic rate and total body mass for any species.

Problems:

1. The order is 2.

2. The order is 1.

3. The order is 3.

4. The order is 2.

5. We compute the first three derivatives of $y(t) = \ln t$:

$$\frac{dy}{dt} = \frac{1}{t}, \qquad \frac{d^2y}{dt^2} = -\frac{1}{t^2}, \qquad \frac{d^3y}{dt^3} = \frac{2}{t^3}.$$

Therefore,

$$2\left(\frac{dy}{dt}\right)^3 = \frac{2}{t^3} = \frac{d^3y}{dt^3},$$

as required.

6. We compute the first two derivatives of $y(x) = x/(x+1)$:

$$\frac{dy}{dx} = \frac{1}{(x+1)^2} \qquad \text{and} \qquad \frac{d^2y}{dx^2} = -\frac{2}{(x+1)^3}.$$

Then

$$y + \frac{d^2y}{dx^2} = \frac{x}{x+1} - \frac{2}{(x+1)^3} = \frac{x^3+2x^2+x-2}{(x+1)^3} = \frac{(x+1)+(x^3+2x^2-3)}{(x+1)^3} = \frac{1}{(x+1)^2} + \frac{x^3+2x^2-3}{(1+x)^3},$$

as required.

7. We compute the first two derivatives of $y(x) = e^x \sin x$:

$$\frac{dy}{dx} = e^x(\sin x + \cos x) \qquad \text{and} \qquad \frac{d^2y}{dx^2} = 2e^x \cos x.$$

Then

$$2y \cot x - \frac{d^2y}{dx^2} = 2(e^x \sin x)\cot x - 2e^x \cos x = 0,$$

as required.

8. $(T - T_m)^{-1}\dfrac{dT}{dt} = -k \Longrightarrow \dfrac{d}{dt}(\ln|T - T_m|) = -k$. The preceding equation can be integrated directly to yield $\ln|T - T_m| = -kt + c_1$. Exponentiating both sides of this equation gives $|T - T_m| = e^{-kt+c_1}$, which can be written as

$$T - T_m = ce^{-kt},$$

where $c = \pm e^{c_1}$. Rearranging yields $T(t) = T_m + ce^{-kt}$.

9. After 4 p.m. In the first two hours after noon, the water temperature increased from 50° F to 55° F, an increase of five degrees. Because the temperature of the water has grown closer to the ambient air temperature, the temperature difference $|T - T_m|$ is smaller, and thus, the rate of change of the temperature of the water grows smaller, according to Newton's Law of Cooling. Thus, it will take longer for the water temperature to increase another five degrees. Therefore, the water temperature will reach 60° F more than two hours later than 2 p.m., or after 4 p.m.

10. The object temperature cools a total of 40° F during the 40 minutes, but according to Newton's Law of Cooling, it cools faster in the beginning (since $|T - T_m|$ is greater at first). Thus, the object cooled half-way from 70° F to 30° F in less than half the total cooling time. Therefore, it took less than 20 minutes for the object to reach 50° F.

11. The given family of curves satisfies: $x^2 + 9y^2 = c \Longrightarrow 2x + 18y\dfrac{dy}{dx} = 0 \Longrightarrow \dfrac{dy}{dx} = -\dfrac{x}{9y}$.
Orthogonal trajectories satisfy:

$$\frac{dy}{dx} = \frac{9y}{x} \Longrightarrow \frac{1}{y}\frac{dy}{dx} = \frac{9}{x} \Longrightarrow \frac{d}{dx}(\ln|y|) = \frac{9}{x} \Longrightarrow \ln|y| = 9\ln|x| + c_1 \Longrightarrow y = kx^9, \text{where } \; k = \pm e^{c_1}$$

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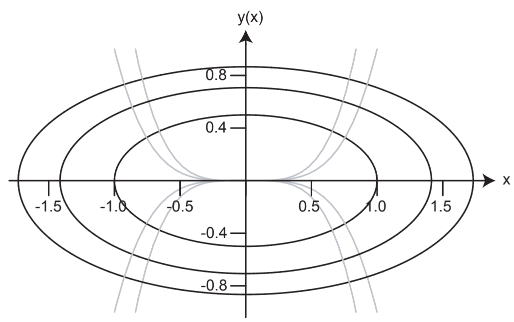

Figure 0.0.1: Figure for Problem 11

12. Given family of curves satisfies: $y = cx^2 \Longrightarrow c = \dfrac{y}{x^2}$. Hence,

$$\frac{dy}{dx} = 2cx = c\left(\frac{y}{x^2}\right)x = \frac{2y}{x}.$$

Orthogonal trajectories satisfy:

$$\frac{dy}{dx} = -\frac{x}{2y} \Longrightarrow 2y\frac{dy}{dx} = -x \Longrightarrow \frac{d}{dx}(y^2) = -x \Longrightarrow y^2 = -\frac{1}{2}x^2 + c_1 \Longrightarrow 2y^2 + x^2 = c_2,$$

where $c_2 = 2c_1$.

13. Given a family of curves satisfies: $y = \dfrac{c}{x} \Longrightarrow x\dfrac{dy}{dx} + y = 0 \Longrightarrow \dfrac{dy}{dx} = -\dfrac{y}{x}$.
Orthogonal trajectories satisfy:

$$\frac{dy}{dx} = \frac{x}{y} \Longrightarrow y\frac{dy}{dx} = x \Longrightarrow \frac{d}{dx}\left(\frac{1}{2}y^2\right) = x \Longrightarrow \frac{1}{2}y^2 = \frac{1}{2}x^2 + c_1 \Longrightarrow y^2 - x^2 = c_2, \text{where } \; c_2 = 2c_1.$$

14. The given family of curves satisfies: $y = cx^5 \Longrightarrow c = \dfrac{y}{x^5}$. Hence,

$$\frac{dy}{dx} = 5cx^4 = 5\left(\frac{y}{x^5}\right)x^4 = \frac{5y}{x}.$$

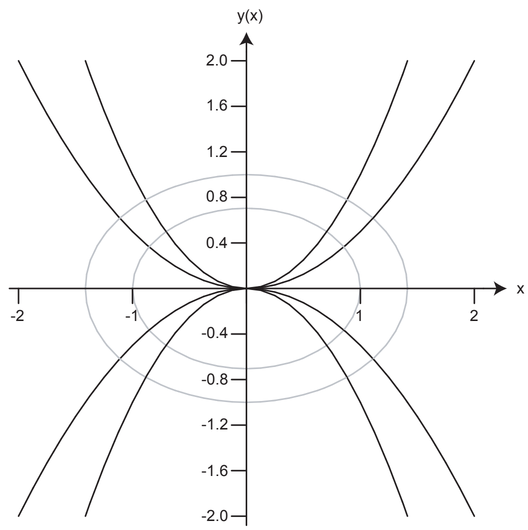

Figure 0.0.2: Figure for Problem 12

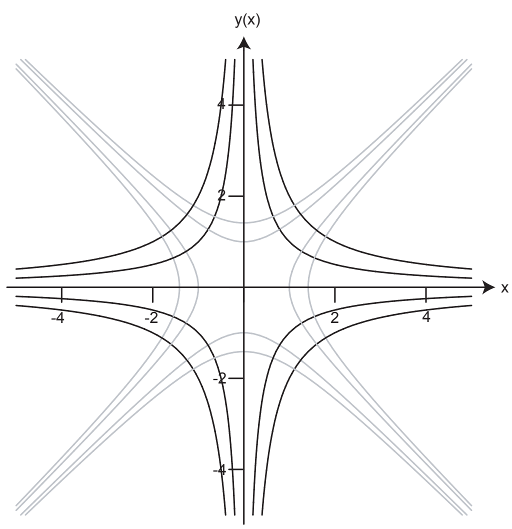

Figure 0.0.3: Figure for Problem 13

Orthogonal trajectories satisfy:

$$\frac{dy}{dx} = -\frac{x}{5y} \Longrightarrow 5y\frac{dy}{dx} = -x \Longrightarrow \frac{d}{dx}\left(\frac{5}{2}y^2\right) = -x \Longrightarrow \frac{5}{2}y^2 = -\frac{1}{2}x^2 + c_1 \Longrightarrow 5y^2 + x^2 = c_2,$$

where $c_2 = 2c_1$.

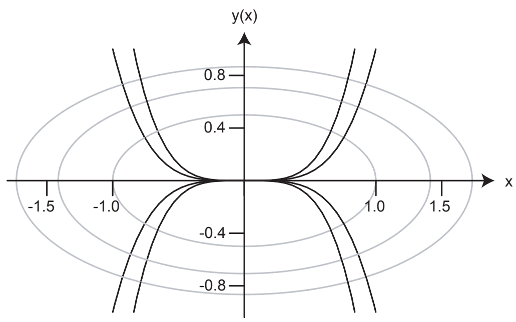

Figure 0.0.4: Figure for Problem 14

15. Given family of curves satisfies: $y = ce^x \Longrightarrow \dfrac{dy}{dx} = ce^x = y$. Orthogonal trajectories satisfy:

$$\frac{dy}{dx} = -\frac{1}{y} \Longrightarrow y\frac{dy}{dx} = -1 \Longrightarrow \frac{d}{dx}\left(\frac{1}{2}y^2\right) = -1 \Longrightarrow \frac{1}{2}y^2 = -x + c_1 \Longrightarrow y^2 = -2x + c_2.$$

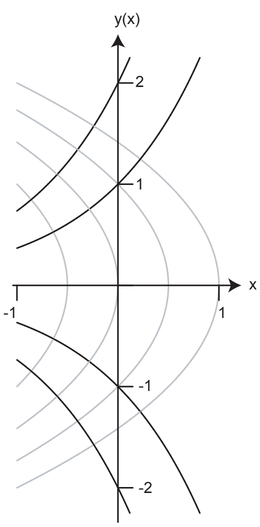

Figure 0.0.5: Figure for Problem 15

16. Given family of curves satisfies: $y^2 = 2x + c \Longrightarrow \dfrac{dy}{dx} = \dfrac{1}{y}$.Orthogonal trajectories satisfy:

$$\frac{dy}{dx} = -y \Longrightarrow y^{-1}\frac{dy}{dx} = -1 \Longrightarrow \frac{d}{dx}(\ln|y|) = -1 \Longrightarrow \ln|y| = -x + c_1 \Longrightarrow y = c_2e^{-x}.$$

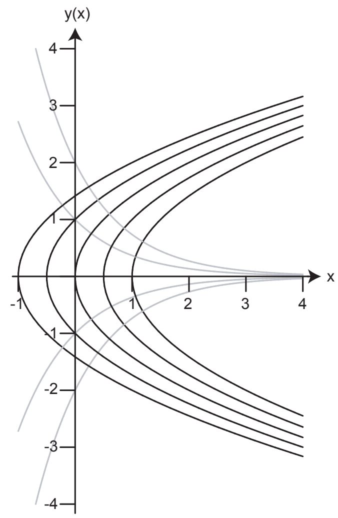

Figure 0.0.6: Figure for Problem 16

17. $y = cx^m \Longrightarrow \dfrac{dy}{dx} = cmx^{m-1}$, but $c = \dfrac{y}{x^m}$ so $\dfrac{dy}{dx} = \dfrac{my}{x}$. Orthogonal trajectories satisfy:

$$\frac{dy}{dx} = -\frac{x}{my} \Longrightarrow y\frac{dy}{dx} = -\frac{x}{m} \Longrightarrow \frac{d}{dx}\left(\frac{1}{2}y^2\right) = -\frac{x}{m} \Longrightarrow \frac{1}{2}y^2 = -\frac{1}{2m}x^2 + c_1 \Longrightarrow y^2 = -\frac{1}{m}x^2 + c_2.$$

18. $y = mx + c \Longrightarrow \dfrac{dy}{dx} = m$.
Orthogonal trajectories satisfy:

$$\frac{dy}{dx} = -\frac{1}{m} \Longrightarrow y = -\frac{1}{m}x + c_1.$$

19. $y^2 = mx + c \Longrightarrow 2y\dfrac{dy}{dx} = m \Longrightarrow \dfrac{dy}{dx} = \dfrac{m}{2y}$.
Orthogonal trajectories satisfy:

$$\frac{dy}{dx} = -\frac{2y}{m} \Longrightarrow y^{-1}\frac{dy}{dx} = -\frac{2}{m} \Longrightarrow \frac{d}{dx}(\ln|y|) = -\frac{2}{m} \Longrightarrow \ln|y| = -\frac{2}{m}x + c_1 \Longrightarrow y = c_2e^{-\frac{2x}{m}}.$$

20. $y^2 + mx^2 = c \Longrightarrow 2y\dfrac{dy}{dx} + 2mx = 0 \Longrightarrow \dfrac{dy}{dx} = -\dfrac{mx}{y}$.
Orthogonal trajectories satisfy:

$$\frac{dy}{dx} = \frac{y}{mx} \Longrightarrow y^{-1}\frac{dy}{dx} = \frac{1}{mx} \Longrightarrow \frac{d}{dx}(\ln|y|) = \frac{1}{mx} \Longrightarrow m\ln|y| = \ln|x| + c_1 \Longrightarrow y^m = c_2x.$$

21. The given family of curves satisfies: $x^2 + y^2 = 2cx \Longrightarrow c = \dfrac{x^2+y^2}{2x}$. Hence,

$$2x + 2y\frac{dy}{dx} = 2c = \frac{x^2+y^2}{x}.$$

Therefore,

$$2y\frac{dy}{dx} = \frac{x^2+y^2}{x} - 2x = \frac{y^2-x^2}{x},$$

so that

$$\frac{dy}{dx} = \frac{y^2-x^2}{2xy}.$$

Orthogonal trajectories satisfy:

$$\frac{dy}{dx} = -\frac{2xy}{y^2-x^2} = \frac{2xy}{x^2-y^2}.$$

22. $u = x^2 + 2y^2 \Longrightarrow 0 = 2x + 4y\frac{dy}{dx} \Longrightarrow \frac{dy}{dx} = -\frac{x}{2y}$.
Orthogonal trajectories satisfy:

$$\frac{dy}{dx} = \frac{2y}{x} \Longrightarrow y^{-1}\frac{dy}{dx} = \frac{2}{x} \Longrightarrow \frac{d}{dx}(\ln|y|) = \frac{2}{x} \Longrightarrow \ln|y| = 2\ln|x| + c_1 \Longrightarrow y = c_2x^2.$$

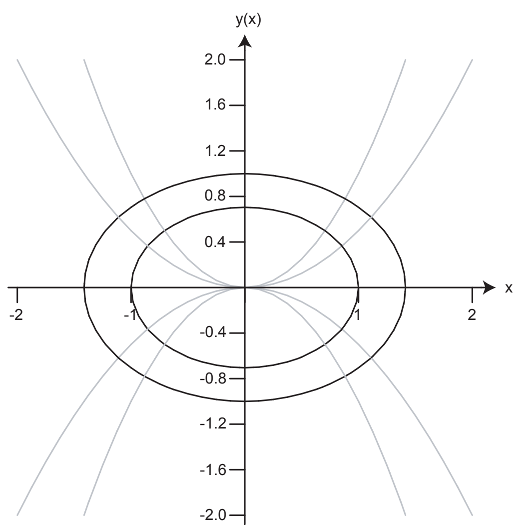

Figure 0.0.7: Figure for Problem 22

23. $m_1 = \tan(a_1) = \tan(a_2 - a) = \dfrac{\tan(a_2) - \tan(a)}{1 + \tan(a_2)\tan(a)} = \dfrac{m_2 - \tan(a)}{1 + m_2\tan(a)}$.

24. $\frac{d^2y}{dt^2} = g \Longrightarrow \frac{dy}{dt} = gt + c_1 \Longrightarrow y(t) = \frac{gt^2}{2} + c_1t + c_2$. Now impose the initial conditions. $y(0) = 0 \Longrightarrow c_2 = 0$. $\frac{dy}{dt}(0) \Longrightarrow c_1 = 0$. Hence, the solution to the initial-value problem is: $y(t) = \frac{gt^2}{2}$. The object hits the ground at time, t_0, when $y(t_0) = 100$. Hence $100 = \frac{gt_0^2}{2}$, so that $t_0 = \sqrt{\frac{200}{g}} \approx 4.52$ s, where we have taken $g = 9.8\ \text{ms}^{-2}$.

25. From $\dfrac{d^2y}{dt^2} = g$, we integrate twice to obtain the general equations for the velocity and the position of the ball, respectively: $\dfrac{dy}{dt} = gt + c$ and $y(t) = \dfrac{1}{2}gt^2 + ct + d$, where c, d are constants of integration. Setting $y = 0$ to be at the top of the boy's head (and positive direction downward), we know that $y(0) = 0$. Since the object hits the ground 8 seconds later, we have that $y(8) = 5$ (since the ground lies at the position $y = 5$). From the values of $y(0)$ and $y(8)$, we find that $d = 0$ and $5 = 32g + 8c$. Therefore, $c = \dfrac{5 - 32g}{8}$.

(a). The ball reaches its maximum height at the moment when $y'(t) = 0$. That is, $gt + c = 0$. Therefore,

$$t = -\frac{c}{g} = \frac{32g - 5}{8g} \approx 3.98 \text{ s}.$$

(b). To find the maximum height of the tennis ball, we compute

$$y(3.98) \approx -253.51 \text{ feet}.$$

So the ball is 253.51 feet *above* the top of the boy's head, which is 258.51 feet above the ground.

26. From $\dfrac{d^2y}{dt^2} = g$, we integrate twice to obtain the general equations for the velocity and the position of the rocket, respectively: $\dfrac{dy}{dt} = gt + c$ and $y(t) = \dfrac{1}{2}gt^2 + ct + d$, where c, d are constants of integration. Setting $y = 0$ to be at ground level, we know that $y(0) = 0$. Thus, $d = 0$.

(a). The rocket reaches maximum height at the moment when $y'(t) = 0$. That is, $gt + c = 0$. Therefore, the time that the rocket achieves its maximum height is $t = -\dfrac{c}{g}$. At this time, $y(t) = -90$ (the negative sign accounts for the fact that the positive direction is chosen to be downward). Hence,

$$-90 = y\left(-\frac{c}{g}\right) = \frac{1}{2}g\left(-\frac{c}{g}\right)^2 + c\left(-\frac{c}{g}\right) = \frac{c^2}{2g} - \frac{c^2}{g} = -\frac{c^2}{2g}.$$

Solving this for c, we find that $c = \pm\sqrt{180g}$. However, since c represents the initial velocity of the rocket, and the initial velocity is negative (relative to the fact that the positive direction is downward), we choose $c = -\sqrt{180g} \approx -42.02 \text{ ms}^{-1}$, and thus the initial speed at which the rocket must be launched for optimal viewing is approximately 42.02 ms^{-1}.

(b). The time that the rocket reaches its maximum height is $t = -\dfrac{c}{g} \approx -\dfrac{-42.02}{9.81} = 4.28$ s.

27. From $\dfrac{d^2y}{dt^2} = g$, we integrate twice to obtain the general equations for the velocity and the position of the rocket, respectively: $\dfrac{dy}{dt} = gt + c$ and $y(t) = \dfrac{1}{2}gt^2 + ct + d$, where c, d are constants of integration. Setting $y = 0$ to be at the level of the platform (with positive direction downward), we know that $y(0) = 0$. Thus, $d = 0$.

(a). The rocket reaches maximum height at the moment when $y'(t) = 0$. That is, $gt + c = 0$. Therefore, the time that the rocket achieves its maximum height is $t = -\dfrac{c}{g}$. At this time, $y(t) = -85$ (this is 85 m above the platform, or 90 m above the ground). Hence,

$$-85 = y\left(-\frac{c}{g}\right) = \frac{1}{2}g\left(-\frac{c}{g}\right)^2 + c\left(-\frac{c}{g}\right) = \frac{c^2}{2g} - \frac{c^2}{g} = -\frac{c^2}{2g}.$$

Solving this for c, we find that $c = \pm\sqrt{170g}$. However, since c represents the initial velocity of the rocket, and the initial velocity is negative (relative to the fact that the positive direction is downward), we choose $c = -\sqrt{170g} \approx -40.84$ ms^{-1}, and thus the initial speed at which the rocket must be launched for optimal viewing is approximately 40.84 ms^{-1}.

(b). The time that the rocket reaches its maximum height is $t = -\frac{c}{g} \approx -\frac{-40.84}{9.81} = 4.16$ s.

28. If $y(t)$ denotes the displacement of the object from its initial position at time t, the motion of the object can be described by the initial-value problem

$$\frac{d^2y}{dt^2} = g, \quad y(0) = 0, \quad \frac{dy}{dt}(0) = -2.$$

We first integrate this differential equation: $\frac{d^2y}{dt^2} = g \Longrightarrow \frac{dy}{dt} = gt + c_1 \Longrightarrow y(t) = \frac{gt^2}{2} + c_1 t + c_2$. Now impose the initial conditions. $y(0) = 0 \Longrightarrow c_2 = 0$. $\frac{dy}{dt}(0) = -2 \Longrightarrow c_1 = -2$. Hence the solution to the initial-value problem is $y(t) = \frac{gt^2}{2} - 2t$. We are given that $y(10) = h$. Consequently, $h = \frac{g(10)^2}{2} - 2 \cdot 10 \Longrightarrow h = 10(5g - 2) \approx 470$ m where we have taken $g = 9.8$ ms^{-2}.

29. If $y(t)$ denotes the displacement of the object from its initial position at time t, the motion of the object can be described by the initial-value problem

$$\frac{d^2y}{dt^2} = g, \quad y(0) = 0, \quad \frac{dy}{dt}(0) = v_0.$$

We first integrate the differential equation: $\frac{d^2y}{dt^2} = g \Longrightarrow \frac{dy}{dt} = gt + c_1 \Longrightarrow y(t) = \frac{gt^2}{2} + c_1 t + c_2$. Now impose the initial conditions. $y(0) = 0 \Longrightarrow c_2 = 0$. $\frac{dy}{dt}(0) = v_0 \Longrightarrow c_1 = v_0$. Hence the solution to the initial-value problem is $y(t) = \frac{gt^2}{2} + v_0 t$. We are given that $y(t_0) = h$. Consequently, $h = gt_0^2 + v_0 t_0$. Solving for v_0 yields $v_0 = \frac{2h - gt_0^2}{2t_0}$.

30. From $y(t) = A\cos(\omega t - \phi)$, we obtain

$$\frac{dy}{dt} = -A\omega \sin(\omega t - \phi) \quad \text{and} \quad \frac{d^2y}{dt^2} = -A\omega^2 \cos(\omega t - \phi).$$

Hence,

$$\frac{d^2y}{dt^2} + \omega^2 y = -A\omega^2 \cos(\omega t - \phi) + A\omega^2 \cos(\omega t - \phi) = 0.$$

Substituting $y(0) = a$, we obtain $a = A\cos(-\phi) = A\cos(\phi)$. Also, from $\frac{dy}{dt}(0) = 0$, we obtain $0 = -A\omega\sin(-\phi) = A\omega\sin(\phi)$. Since $A \neq 0$ and $\omega \neq 0$ and $|\phi| < \pi$, we have $\phi = 0$. It follows that $a = A$.

31. $y(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t) \Longrightarrow \frac{dy}{dt} = -c_1\omega \sin(\omega t) + c_2\omega\cos(\omega t) \Longrightarrow \frac{d^2y}{dt^2} = -c_1\omega^2\cos(\omega t) - c_2\omega^2 \sin(\omega t) = -\omega^2[c_1\cos(\omega t) + c_2\cos(\omega t)] = -\omega^2 y$. Consequently, $\frac{d^2y}{dt^2} + \omega^2 y = 0$. To determine the amplitude of the motion we write the solution to the differential equation in the equivalent form:

$$y(t) = \sqrt{c_1^2 + c_2^2}\left[\frac{c_1}{\sqrt{c_1^2+c_2^2}}\cos(\omega t) + \frac{c_2}{\sqrt{c_1^2+c_2^2}}\sin(\omega t)\right].$$

We can now define an angle ϕ by

$$\cos\phi = \frac{c_1}{\sqrt{c_1^2+c_2^2}} \quad \text{and} \quad \sin\phi = \frac{c_2}{\sqrt{c_1^2+c_2^2}}.$$

Then the expression for the solution to the differential equation is

$$y(t) = \sqrt{c_1^2+c_2^2}[\cos(\omega t)\cos\phi + \sin(\omega t)\sin\phi] = \sqrt{c_1^2+c_2^2}\cos(\omega t + \phi).$$

Consequently the motion corresponds to an oscillation with amplitude $A = \sqrt{c_1^2+c_2^2}$.

32. In this problem we have $m_0 = 3$g, $M = 2700$g, $a = 1.5$. Substituting these values into Equation (1.1.26) yields

$$m(t) = 2700\left\{1 - \left[1 - \left(\frac{1}{900}\right)^{1/4}\right]e^{-1.5t/(4(2700)^{1/4})}\right\}^4.$$

Therefore the mass of the heron after 30 days is

$$m(30) = 2700\left\{1 - \left[1 - \left(\frac{1}{900}\right)^{1/4}\right]e^{-45/(4(2700)^{1/4})}\right\}^4 \approx 1271.18\,\text{g}.$$

33. In this problem we have $m_0 = 8$g, $M = 280$g, $a = 0.25$. Substituting these values into Equation (1.1.26) yields

$$m(t) = 280\left\{1 - \left[1 - \left(\frac{1}{35}\right)^{1/4}\right]e^{-t/(16(280)^{1/4})}\right\}^4.$$

We need to find the time, t when the mass of the rat reaches 75% of its fully grown size. Therefore we need to find t such that

$$\frac{75}{100}\cdot 280 = 280\left\{1 - \left[1 - \left(\frac{1}{35}\right)^{1/4}\right]e^{-t/(16(280)^{1/4})}\right\}^4.$$

Solving algebraically for t yields

$$t = 16\cdot(280)^{1/4}\cdot\ln\left[\frac{1-(1/35)^{1/4}}{1-(75/100))^{1/4}}\right] \approx 140 \text{ days}.$$

Solutions to Section 1.2

True-False Review:

(a): TRUE. This is condition 1 in Definition 1.2.8.

(b): TRUE. This is the content of Theorem 1.2.12.

(c): FALSE. There are solutions to $y'' + y = 0$ that do not have the form $c_1 \cos x + 5c_2 \cos x$, such as $y(x) = \sin x$. Therefore, $c_1 \cos x + 5c_2 \cos x$ does not meet the second requirement set forth in Definition 1.2.8 for the general solution.

(d): FALSE. There are solutions to $y'' + y = 0$ that do not have the form $c_1 \cos x + 5c_1 \sin x$, such as $y(x) = \cos x + \sin x$. Therefore, $c_1 \cos x + 5c_1 \sin x$ does not meet the second requirement set form in Definition 1.2.8 for the general solution.

(e): TRUE. Since the right-hand side of the differential equation is a function of x only, we can integrate both sides n times to obtain the formula for the solution $y(x)$.

Problems:

1. Linear.

2. Non-linear, because of the y^2 expression on the right side of the equation.

3. Non-linear, because of the term yy'' on tthe left side of the equation.

4. Non-linear, because of the expression $\tan y$ appearing on the left side of the equation.

5. Linear.

6. Non-linear, because of the expression $\frac{1}{y'}$ on the left side of the equation.

7. $y(x) = c_1 e^{-5x} + c_2 e^{5x} \implies y' = -5c_1 e^{-5x} + 5c_2 e^{5x} \implies y'' = 25c_1 e^{-5x} + 25c_2 e^{5x} \implies y'' - 25y = (25c_1 e^{-5x} + 25c_2 e^{5x}) - 25(c_1 e^{-5x} + c_2 e^{5x}) = 0$. Thus $y(x) = c_1 e^{-5x} + c_2 e^{5x}$ is a solution of the given differential equation for all $x \in \mathbb{R}$.

8. $y(x) = c_1 \cos 2x + c_2 \sin 2x \implies y' = -2c_1 \sin 2x + 2c_2 \cos 2x \implies y'' = -4c_1 \cos 2x - 4c_2 \sin 2x \implies y'' + 4y = (-4c_1 \cos 2x - 4c_2 \sin 2x) + 4(c_1 \cos 2x + c_2 \sin 2x) = 0$. Thus $y(x) = c_1 \cos 2x + c_2 \sin 2x$ is a solution of the given differential equation for all $x \in \mathbb{R}$.

9. $y(x) = c_1 e^{x} + c_2 e^{-2x} \implies y' = c_1 e^{x} - 2c_2 e^{-2x} \implies y'' = c_1 e^{x} + 4c_2 e^{-2x} \implies y'' + y' - 2y = (c_1 e^{x} + 4c_2 e^{-2x}) + (c_1 e^{x} - 2c_2 e^{-2x}) - 2(c_1 e^{x} + c_2 e^{-2x}) = 0$. Thus $y(x) = c_1 e^{x} + c_2 e^{-2x}$ is a solution of the given differential equation for all $x \in \mathbb{R}$.

10. $y(x) = \dfrac{1}{x+4} \implies y' = -\dfrac{1}{(x+4)^2} = -y^2$. Thus $y(x) = \dfrac{1}{x+4}$ is a solution of the given differential equation for $x \in (-\infty, -4)$ or $x \in (-4, \infty)$.

11. $y(x) = c_1\sqrt{x} \implies y' = \dfrac{c_1}{2\sqrt{x}} = \dfrac{y}{2x}$. Thus $y(x) = c_1\sqrt{x}$ is a solution of the given differential equation for all $x \in \{x : x > 0\}$.

12. $y(x) = c_1 e^{-x} \sin(2x) \implies y' = 2c_1 e^{-x} \cos(2x) - c_1 e^{-x} \sin(2x) \implies y'' = -3c_1 e^{-x} \sin(2x) - 4c_1 e^{-x} \cos(2x) \implies y'' + 2y' + 5y = -3c_1 e^{-x} \sin(2x) - 4c_1 e^{-x} \cos(2x) + 2[2c_1 e^{-x} \cos(2x) - c_1 e^{-x} \sin(2x)] + 5[c_1 e^{-x} \sin(2x)] = 0$. Thus $y(x) = c_1 e^{-x} \sin(2x)$ is a solution to the given differential equation for all $x \in \mathbb{R}$.

13. $y(x) = c_1 \cosh(3x) + c_2 \sinh(3x) \implies y' = 3c_1 \sinh(3x) + 3c_2 \cosh(3x) \implies y'' = 9c_1 \cosh(3x) +$

$9c_2 \sinh(3x) \implies y'' - 9y = [9c_1 \cosh(3x) + 9c_2 \sinh(3x)] - 9[c_1 \cosh(3x) + c_2 \sinh(3x)] = 0$. Thus $y(x) = c_1 \cosh(3x) + c_2 \sinh(3x)$ is a solution to the given differential equation for all $x \in \mathbb{R}$.

14. $y(x) = \dfrac{c_1}{x^3} + \dfrac{c_2}{x} \implies y' = -\dfrac{3c_1}{x^4} - \dfrac{c_2}{x^2} \implies y'' = \dfrac{12c_1}{x^5} + \dfrac{2c_2}{x^3} \implies x^2y'' + 5xy' + 3y = x^2\left(\dfrac{12c_1}{x^5} + \dfrac{2c_2}{x^3}\right) + 5x\left(-\dfrac{3c_1}{x^4} - \dfrac{c_2}{x^2}\right) + 3\left(\dfrac{c_1}{x^3} + \dfrac{c_2}{x}\right) = 0$. Thus $y(x) = \dfrac{c_1}{x^3} + \dfrac{c_2}{x}$ is a solution to the given differential equation for all $x \in (-\infty, 0)$ or $x \in (0, \infty)$.

15. $y(x) = c_1x^2 \ln x \implies y' = c_1(2x \ln x + x) \implies y'' = c_1(2 \ln x + 3) \implies x^2y'' - 3xy' + 4y = x^2 \cdot c_1(2 \ln x + 3) - 3x \cdot c_1(2x \ln x + x) + 4c_1x^2 \ln x = c_1x^2 [(2 \ln x + 3) - 3(2x \ln x + 1) + 4 \ln x] = 0$. Thus $y(x) = c_1x^2 \ln x$ is a solution of the given differential equation for all $x > 0$.

16. $y(x) = c_1x^2 \cos(3 \ln x) \implies y' = c_1[2x \cos(3 \ln x) - 3x \sin(3 \ln x)] \implies y'' = c_1[-7 \cos(3 \ln x) - 6 \sin(3 \ln x)] \implies x^2y'' - 3xy' + 13y = x^2 \cdot c_1[-7 \cos(3 \ln x) - 9 \sin(3 \ln x)] - 3x \cdot c_1[2x \cos(3 \ln x) - 3x \sin(3 \ln x)] + 13c_1x^2 \cos(3 \ln x) = c_1x^2 \{[-7 \cos(3 \ln x) - 9 \sin(3 \ln x)] - 3[2 \cos(3 \ln x) - 3 \sin(3 \ln x)] + 13 \cos(3 \ln x)\} = 0$. Thus $y(x) = c_1x^2 \cos(3 \ln x)$ is a solution of the given differential equation for all $x > 0$.

17. $y(x) = c_1\sqrt{x} + 3x^2 \implies y' = \dfrac{c_1}{2\sqrt{x}} + 6x \implies y'' = -\dfrac{c_1}{4\sqrt{x^3}} + 6 \implies 2x^2y'' - xy' + y = 2x^2\left(-\dfrac{c_1}{4\sqrt{x^3}} + 6\right) - x\left(\dfrac{c_1}{2\sqrt{x}} + 6x\right) + (c_1\sqrt{x} + 3x^2) = 9x^2$. Thus $y(x) = c_1\sqrt{x} + 3x^2$ is a solution to the given differential equation for all $x \in \{x : x > 0\}$.

18. $y(x) = c_1x^2 + c_2x^3 - x^2 \sin x \implies y' = 2c_1x + 3c_2x^2 - x^2 \cos x - 2x \sin x \implies y'' = 2c_1 + 6c_2x + x^2 \sin x - 2x \cos x - 2x \cos -2 \sin x$. Substituting these results into the given differential equation yields

$$\begin{aligned} x^2y'' - 4xy' + 6y &= x^2(2c_1 + 6c_2x + x^2 \sin x - 4x \cos x - 2 \sin x) - 4x(2c_1x + 3c_2x^2 - x^2 \cos x - 2x \sin x) \\ &\qquad + 6(c_1x^2 + c_2x^3 - x^2 \sin x) \\ &= 2c_1x^2 + 6c_2x^3 + x^4 \sin x - 4x^3 \cos x - 2x^2 \sin x - 8c_1x^2 - 12c_2x^3 + 4x^3 \cos x + 8x^2 \sin x \\ &\qquad + 6c_1x^2 + 6c_2x^3 - 6x^2 \sin x \\ &= x^4 \sin x. \end{aligned}$$

Hence, $y(x) = c_1x^2 + c_2x^3 - x^2 \sin x$ is a solution to the differential equation for all $x \in \mathbb{R}$.

19. $y(x) = c_1e^{ax} + c_2e^{bx} \implies y' = ac_1e^{ax} + bc_2e^{bx} \implies y'' = a^2c_1e^{ax} + b^2c_2e^{bx}$. Substituting these results into the differential equation yields

$$\begin{aligned} y'' - (a+b)y' + aby &= a^2c_1e^{ax} + b^2c_2e^{bx} - (a+b)(ac_1e^{ax} + bc_2e^{bx}) + ab(c_1e^{ax} + c_2e^{bx}) \\ &= (a^2c_1 - a^2c_1 - abc_1 + abc_1)e^{ax} + (b^2c_2 - abc_2 - b^2c_2 + abc_2)e^{bx} \\ &= 0. \end{aligned}$$

Hence, $y(x) = c_1e^{ax} + c_2e^{bx}$ is a solution to the given differential equation for all $x \in \mathbb{R}$.

20. $y(x) = e^{ax}(c_1 + c_2x) \implies y' = e^{ax}(c_2) + ae^{ax}(c_1 + c_2x) = e^{ax}(c_2 + ac_1 + ac_2x) \implies y'' = ea^{ax}(ac_2) + ae^{ax}(c_2 + ac_1 + ac_2x) = ae^{ax}(2c_2 + ac_1 + ac_2x)$. Substituting these into the differential equation yields

$$\begin{aligned} y'' - 2ay' + a^2y &= ae^{ax}(2c_2 + ac_1 + ac_2x) - 2ae^{ax}(c_2 + ac_1 + ac_2x) + a^2e^{ax}(c_1 + c_2x) \\ &= ae^{ax}(2c_2 + ac_1 + ac_2x - 2c_2 - 2ac_1 - 2ac_2x + ac_1 + ac_2x) \\ &= 0. \end{aligned}$$

Thus, $y(x) = e^{ax}(c_1 + c_2x)$ is a solution to the given differential eqaution for all $x \in \mathbb{R}$.

21. $y(x) = e^{ax}(c_1 \cos bx + c_2 \sin bx)$ so,

$$\begin{aligned} y' &= e^{ax}(-bc_1 \sin bx + bc_2 \cos bx) + ae^{ax}(c_1 \cos bx + c_2 \sin bx) \\ &= e^{ax}[(bc_2 + ac_1)\cos bx + (ac_2 - bc_1)\sin bx] \text{ so,} \\ y'' &= e^{ax}[-b(bc_2 + ac_1)\sin bx + b(ac_2 + bc_1)\cos bx] + ae^{ax}[(bc_2 + ac_1)\cos bx + (ac_2 + bc_1)\sin bx] \\ &= e^{ax}[(a^2c_1 - b^2c_1 + 2abc_2)\cos bx + (a^2c_2 - b^2c_2 - abc_1)\sin bx]. \end{aligned}$$

Substituting these results into the differential equation yields

$$\begin{aligned} y'' - 2ay' + (a^2+b^2)y = {} & (e^{ax}[(a^2c_1 - b^2c_1 + 2abc_2)\cos bx + (a^2c_2 - b^2c_2 - abc_1)\sin bx]) \\ & - 2a(e^{ax}[(bc_2 + ac_1)\cos bx + (ac_2 - bc_1)\sin bx]) + (a^2+b^2)(e^{ax}(c_1\cos bx + c_2 \sin bx)) \\ & = e^{ax}[(a^2c_1 - b^2c_1 + 2abc_2 - 2abc_2 - 2a^2c_1 + a^2c_1 + b^2c_1)\cos bx \\ & + (a^2c_2 - b^2c_2 - 2abc_1 + 2abc_1 - 2a^2c_2 + a^2c_2 + b^2c_2)\sin bx] \\ & = 0 \end{aligned}.$$

Thus, $y(x) = e^{ax}(c_1 \cos bx + c_2 \sin bx)$ is a solution to the given differential equation for all $x \in \mathbb{R}$.

22. $y(x) = e^{rx} \Longrightarrow y' = re^{rx} \Longrightarrow y'' = r^2e^{rx}$. Substituting these results into the given differential equation yields $e^{rx}(r^2 - r - 6) = 0$, so that r must satisfy $r^2 - r - 6 = 0$, or $(r-3)(r+2) = 0$. Consequently $r = 3$ and $r = -2$ are the only values of r for which $y(x) = e^{rx}$ is a solution to the given differential equation. The corresponding solutions are $y(x) = e^{3x}$ and $y(x) = e^{-2x}$.

23. $y(x) = e^{rx} \Longrightarrow y' = re^{rx} \Longrightarrow y'' = r^2e^{rx}$. Substituting these results into the given differential equation yields $e^{rx}(r^2 + 6r + 9) = 0$, so that r must satisfy $r^2 + 6r + 9 = 0$, or $(r+3)^2 = 0$. Consequently $r = -3$ is the only value of r for which $y(x) = e^{rx}$ is a solution to the given differential equation. The corresponding solution are $y(x) = e^{-3x}$.

24. $y(x) = x^r \Longrightarrow y' = rx^{r-1} \Longrightarrow y'' = r(r-1)x^{r-2}$. Substitution into the given differential equation yields $x^r[r(r-1) + r - 1] = 0$, so that r must satisfy $r^2 - 1 = 0$. Consequently $r = -1$ and $r = 1$ are the only values of r for which $y(x) = x^r$ is a solution to the given differential equation. The corresponding solutions are $y(x) = x^{-1}$ and $y(x) = x$.

25. $y(x) = x^r \Longrightarrow y' = rx^{r-1} \Longrightarrow y'' = r(r-1)x^{r-2}$. Substitution into the given differential equation yields $x^r[r(r-1) + 5r + 4] = 0$, so that r must satisfy $r^2 + 4r + 4 = 0$, or equivalently $(r+2)^2 = 0$. Consequently $r = -2$ is the only value of r for which $y(x) = x^r$ is a solution to the given differential equation. The corresponding solution is $y(x) = x^{-2}$.

26. $y(x) = \frac{1}{2}x(5x^2 - 3) = \frac{1}{2}(5x^3 - 3x) \Longrightarrow y' = \frac{1}{2}(15x^2 - 3) \Longrightarrow y'' = 15x$. Substitution into the Legendre equation with $N = 3$ yields $(1 - x^2)y'' - 2xy' + 12y = (1 - x^2)(15x) + x(15x^2 - 3) + 6x(5x^2 - 3) = 0$. Consequently the given function is a solution to the Legendre equation with $N = 3$.

27. $y(x) = a_0 + a_1x + a_2x^2 \Longrightarrow y' = a_1 + 2a_2x \Longrightarrow y'' = 4a_2$. Substitution into the given differential equation yields $(1-x^2)(2a_2) - x(a_1 + 2a_2x) + 4(a_0 + a_1x + a_2x^2) = 0 \Longrightarrow 3a_1x + 2a_2 + 4a_0 = 0$. For this equation to hold for all x we require $3a_1 = 0$, and $2a_2 + 4a_0 = 0$. Consequently $a_1 = 0$, and $a_2 = -2a_0$. The corresponding solution to the differential equation is $y(x) = a_0(1 - 2x^2)$. Imposing the normalization condition $y(1) = 1$ requires that $a_0 = -1$. Hence, the required solution to the differential equation is $y(x) = 2x^2 - 1$.

28. $x \sin y - e^x = c \Longrightarrow x \cos y \dfrac{dy}{dx} + \sin y - e^x = 0 \Longrightarrow \dfrac{dy}{dx} = \dfrac{e^x - \sin y}{x \cos y}$.

29. $xy^2 + 2y - x = c \Longrightarrow 2xy\dfrac{dy}{dx} + y^2 + 2\dfrac{dy}{dx} - 1 = 0 \Longrightarrow \dfrac{dy}{dx} = \dfrac{1-y^2}{2(xy+1)}$.

30. $e^{xy} + x = c \Longrightarrow e^{xy}[x\dfrac{dy}{dx} + y] - 1 = 0 \Longrightarrow xe^{xy}\dfrac{dy}{dx} + ye^{xy} = 1 \Longrightarrow \dfrac{1-ye^{xy}}{xe^{xy}}$. Given $y(1) = 0 \Longrightarrow$ $e^{0(1)} - 1 = c \Longrightarrow c = 0$. Therefore, $e^{xy} - x = 0$, so that $y = \dfrac{\ln x}{x}$.

31. $e^{y/x} + xy^2 - x = c \Longrightarrow e^{y/x}\dfrac{x\dfrac{dy}{dx} - y}{x^2} + 2xy\dfrac{dy}{dx} + y^2 - 1 = 0 \Longrightarrow \dfrac{dy}{dx} = \dfrac{x^2(1-y^2) + ye^{y/x}}{x(e^{y/x} + 2x^2y)}$.

32. $x^2y^2 - \sin x = c \Longrightarrow 2x^2y\dfrac{dy}{dx} + 2xy^2 - \cos x = 0 \Longrightarrow \dfrac{dy}{dx} = \dfrac{\cos x - 2xy^2}{2x^2y}$. Since $y(\pi) = \dfrac{1}{\pi}$, then $\pi^2\left(\dfrac{1}{\pi}\right)^2 - \sin\pi = c \Longrightarrow c = 1$. Hence, $x^2y^2 - \sin x = 1$ so that $y^2 = \dfrac{1+\sin x}{x^2}$. Since $y(\pi) = \dfrac{1}{\pi}$, take the branch of y where $x < 0$ so $y(x) = \dfrac{\sqrt{1+\sin x}}{x}$.

33. $\dfrac{dy}{dx} = \sin x \Longrightarrow y(x) = -\cos x + c$ for all $x \in \mathbb{R}$.

34. $\dfrac{dy}{dx} = x^{-2/3} \Longrightarrow y(x) = 3x^{1/3} + c$ for all $x \neq 0$.

35. $\dfrac{d^2y}{dx^2} = xe^x \Longrightarrow \dfrac{dy}{dx} = xe^x - e^x + c_1 \Longrightarrow y(x) = xe^x - 2e^x + c_1x + c_2$ for all $x \in \mathbb{R}$.

36. $\dfrac{d^2y}{dx^2} = x^n$, where n is an integer.
If $n = -1$ then $\dfrac{dy}{dx} = \ln|x| + c_1 \Longrightarrow y(x) = x\ln|x| + c_1x + c_2$ for all $x \in (-\infty, 0)$ or $x \in (0, \infty)$.
If $n = -2$ then $\dfrac{dy}{dx} = -x^{-1} + c_1 \Longrightarrow y(x) = c_1x + c_2 - \ln|x|$ for all $x \in (-\infty, 0)$ or $x \in (0, \infty)$.
If $n \neq -1$ and $n \neq -2$ then $\dfrac{dy}{dx} = \dfrac{x^{n+1}}{n+1} + c_1 \Longrightarrow y = \dfrac{x^{n+2}}{(n+1)(n+2)} + c_1x + c_2$ for all $x \in \mathbb{R}$.

37. $\dfrac{dy}{dx} = x^2\ln x \Longrightarrow y(x) = \dfrac{1}{3}x^3\ln x - \dfrac{1}{9}x^3 + c_1 = \dfrac{1}{9}x^3(3\ln x - 1) + c_1$. $y(1) = 2 \Longrightarrow 2 = \dfrac{1}{9}(0-1) + c_1 \Longrightarrow$ $c_1 = \dfrac{19}{9}$. Therefore, $y(x) = \dfrac{1}{9}x^3(3\ln x - 1) + \dfrac{19}{9} = \dfrac{1}{9}\left[x^3(3\ln x - 1) + 19\right]$.

38. $\dfrac{d^2y}{dx^2} = \cos x \Longrightarrow \dfrac{dy}{dx} = \sin x + c_1 \Longrightarrow y(x) = -\cos x + c_1x + c_2$.
Thus, $y'(0) = 1 \Longrightarrow c_1 = 1$, and $y(0) = 2 \Longrightarrow c_2 = 3$. Thus, $y(x) = 3 + x - \cos x$.

39. $\dfrac{d^3y}{dx^3} = 6x \Longrightarrow \dfrac{d^2y}{dx^2} = 3x^2 + c_1 \Longrightarrow \dfrac{dy}{dx} = x^3 + c_1x + c_2 \Longrightarrow y = \frac{1}{4}x^4 + \frac{1}{2}c_1x^2 + c_2x + c_3$.
Thus, $y''(0) = 4 \Longrightarrow c_1 = 4$, and $y'(0) = -1 \Longrightarrow c_2 = -1$, and $y(0) = 1 \Longrightarrow c_3 = 1$. Thus, $y(x) = \frac{1}{4}x^4 + 2x^2 - x + 1$.

40. $y'' = xe^x \Longrightarrow y' = xe^x - e^x + c_1 \Longrightarrow y = xe^x - 2e^x + c_1x + c_2$.

Thus, $y'(0) = 4 \Longrightarrow c_1 = 5$, and $y(0) = 3 \Longrightarrow c_2 = 5$. Thus, $y(x) = xe^x - 2e^x + 5x + 5$.

41. Starting with $y(x) = c_1e^x + c_2e^{-x}$, we find that $y'(x) = c_1e^x - c_2e^{-x}$ and $y''(x) = c_1e^x + c_2e^{-x}$. Thus, $y'' - y = 0$, so $y(x) = c_1e^x + c_2e^{-x}$ is a solution to the differential equation on $(-\infty, \infty)$. Next we establish that every solution to the differential equation has the form $c_1e^x + c_2e^{-x}$. Suppose that $y = f(x)$ is any solution to the differential equation. Then according to Theorem 1.2.12, $y = f(x)$ is the unique solution to the initial-value problem

$$y'' - y = 0, \qquad y(0) = f(0), \qquad y'(0) = f'(0).$$

However, consider the function

$$y(x) = \frac{f(0) + f'(0)}{2}e^x + \frac{f(0) - f'(0)}{2}e^{-x}.$$

This is of the form $y(x) = c_1e^x + c_2e^{-x}$, where $c_1 = \frac{f(0)+f'(0)}{2}$ and $c_2 = \frac{f(0)-f'(0)}{2}$, and therefore solves the differential equation $y'' - y = 0$. Furthermore, evaluation this function at $x = 0$ yields

$$y(0) = f(0) \qquad \text{and} \qquad y'(0) = f'(0).$$

Consequently, this function solves the initial-value problem above. However, by assumption, $y(x) = f(x)$ solves the same initial-value problem. Owing to the uniqueness of the solution to this initial-value problem, it follows that these two solutions are the same:

$$f(x) = c_1e^x + c_2e^{-x}.$$

Consequently, every solution to the differential equation has the form $y(x) = c_1e^x + c_2e^{-x}$, and therefore this is the general solution on any interval I.

42. $\dfrac{d^2y}{dx^2} = e^{-x} \Longrightarrow \dfrac{dy}{dx} = -e^{-x} + c_1 \Longrightarrow y(x) = e^{-x} + c_1x + c_2$. Thus, $y(0) = 1 \Longrightarrow c_2 = 0$, and $y(1) = 0 \Longrightarrow c_1 = -\frac{1}{e}$. Hence, $y(x) = e^{-x} - \frac{1}{e}x$.

43. $\dfrac{d^2y}{dx^2} = -6 - 4\ln x \Longrightarrow \dfrac{dy}{dx} = -2x - 4x\ln x + c_1 \Longrightarrow y(x) = -2x^2\ln x + c_1x + c_2$. Since, $y(1) = 0 \Longrightarrow$ $c_1 + c_2 = 0$, and since, $y(e) = 0 \Longrightarrow ec_1 + c_2 = 2e^2$. Solving this system yields $c_1 = \dfrac{2e^2}{e-1}, c_2 = -\dfrac{2e^2}{e-1}$. Thus, $y(x) = \dfrac{2e^2}{e-1}(x-1) - 2x^2\ln x$.

44. $y(x) = c_1\cos x + c_2\sin x$

(a). $y(0) = 0 \Longrightarrow 0 = c_1(1) + c_2(0) \Longrightarrow c_1 = 0$. $y(\pi) = 1 \Longrightarrow 1 = c_2(0)$, which is impossible. No solutions.

(b). $y(0) = 0 \Longrightarrow 0 = c_1(1) + c_2(0) \Longrightarrow c_1 = 0$. $y(\pi) = 0 \Longrightarrow 0 = c_2(0)$, so c_2 can be anything. Infinitely many solutions.

45-50. Use some kind of technology to define each of the given functions. Then use the technology to simplify the expression given on the left-hand side of each differential equation and verify that the result corresponds to the expression on the right-hand side.

51. (a). Use some form of technology to substitute $y(x) = a + bx + cx^2 + dx^3 + ex^4 + fx^5$ where a, b, c, d, e, f are constants, into the given Legendre equation and set the coefficients of each power of x in the resulting equation to zero. The result is:

$$e = 0, \;\; 20f + 18d = 0, \;\; e + 2c = 0, \;\; 3d + 14b = 0, \;\; c + 15a = 0.$$

Now solve for the constants to find: $a = c = e = 0$, $d = -\frac{14}{3}b$, $f = -\frac{9}{10}d = \frac{21}{5}b$. Consequently the corresponding solution to the Legendre equation is:

$$y(x) = bx\left(1 - \frac{14}{3}x^2 + \frac{21}{5}x^4\right).$$

Imposing the normalization condition $y(1) = 1$ requires $1 = b(1 - \frac{14}{3} + \frac{21}{5}) \implies b = \frac{15}{8}$. Consequently the required solution is $y(x) = \frac{1}{8}x(15 - 70x^2 + 63x^4)$.

52. (a). $J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2}\left(\frac{x}{2}\right)^{2k} = 1 - \frac{1}{4}x^2 + \frac{1}{64}x^4 + ...$

(b). A Maple plot of $J(0, x, 4)$ is given in the accompanying figure.

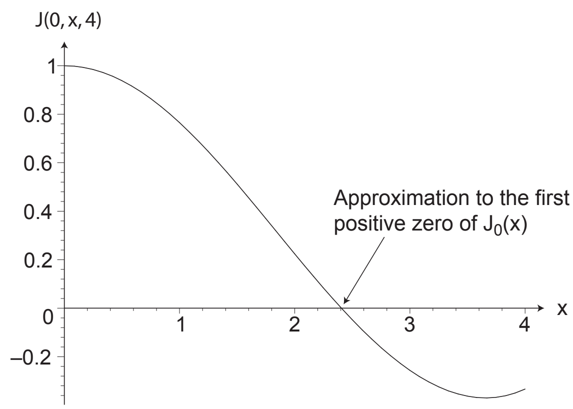

Figure 0.0.8: Figure for Problem 52(b)

(c). From this graph, an approximation to the first positive zero of $J_0(x)$ is 2.4. Using the Maple internal function BesselJZeros gives the approximation 2.404825558.

(c) A Maple plot of the functions $J_0(x)$ and $J(0, x, 4)$ on the interval [0,2] is given in the accompanying figure. We see that to the printer resolution, these graphs are indistinguishable. On a larger interval, for example, [0,3], the two graphs would begin to differ dramatically from one another.

(d). By trial and error, we find the smallest value of m to be $m = 11$. A plot of the functions $J(0, x)$ and $J(0, x, 11)$ is given in the accompanying figure.

Solutions to Section 1.3

True-False Review:

(a): TRUE. This is precisely the remark after Theorem 1.3.2.

(b): FALSE. For instance, the differential equation in Example 1.3.7 has no equilibrium solutions.

(c): FALSE. This differential equation has equilibrium solutions $y(x) = 2$ and $y(x) = -2$.

(d): TRUE. For this differential equation, we have $f(x, y) = x^2 + y^2$. Therefore, any equation of the form $x^2 + y^2 = k$ is an isocline, by definition.

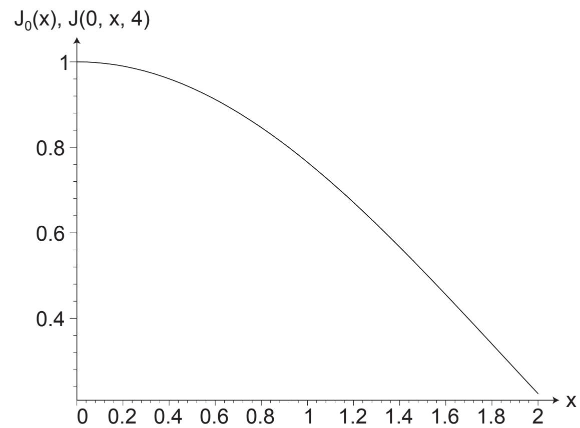

Figure 0.0.9: Figure for Problem 52(c)

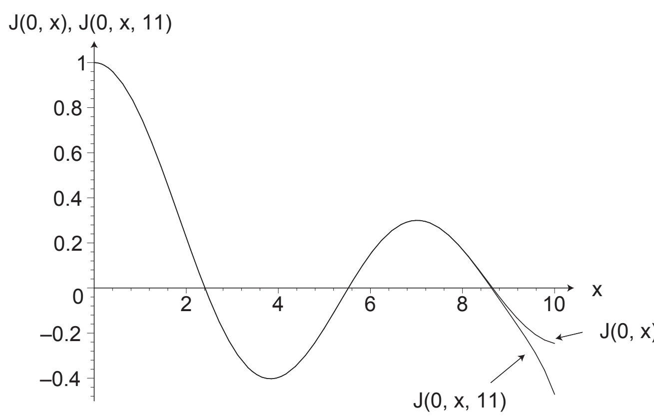

Figure 0.0.10: Figure for Problem 52(d)

(e): TRUE. Equilibrium solutions are always horizontal lines. These are always parallel to each other.

(f): TRUE. The isoclines have the form $\frac{x^2+y^2}{2y} = k$, or $x^2+y^2 = 2ky$, or $x^2+(y-k)^2 = k^2$, so the statement is valid.

(g): TRUE. An equilibrium solution *is* a solution, and two solution curves to the differential equation $\frac{dy}{dx} = f(x,y)$ do not intersect.

Problems:

1. $y = ce^{2x} \Longrightarrow c = ye^{-2x}$. Hence, $\dfrac{dy}{dx} = 2ce^{2x} = 2y$.

2. $y = e^{cx} \Longrightarrow \ln y = cx \Longrightarrow c = \dfrac{\ln y}{x}, x \neq 0$. Hence, $\dfrac{dy}{dx} = ce^{cx} = \dfrac{y}{x}\ln y, x \neq 0$.

3. $y = cx^2 \Longrightarrow c = \dfrac{y}{x^2}$. Hence, $\dfrac{dy}{dx} = 2cx = 2\dfrac{y}{x^2}x = \dfrac{2y}{x}$.

4. $y = cx^{-1} \Longrightarrow c = xy$. Hence, $\dfrac{dy}{dx} = -cx^{-2} = -(xy)x^{-2} = -\dfrac{y}{x}$.

5. $y^2 = cx \Longrightarrow c = \dfrac{y^2}{x}$. Hence, $2y\dfrac{dy}{dx} = c$, so that, $\dfrac{dy}{dx} = \dfrac{c}{2y} = \dfrac{y}{2x}$.

6. $x^2 + y^2 = 2cx \Longrightarrow \dfrac{x^2+y^2}{2x} = c$. Hence, $2x + 2y\dfrac{dy}{dx} = 2c = \dfrac{x^2+y^2}{x}$, so that, $y\dfrac{dy}{dx} = \dfrac{x^2+y^2}{2x} - x$. Consequently, $\dfrac{dy}{dx} = \dfrac{y^2-x^2}{2xy}$.

7. $(x-c)^2 + (y-c)^2 = 2c^2 \Longrightarrow x^2 - 2cx + y^2 - 2cy = 0 \Longrightarrow c = \dfrac{x^2+y^2}{2(x+y)}$. Differentiating the given equation yields $2(x-c) + 2(y-c)\dfrac{dy}{dx} = 0$, so that $2\left[x - \dfrac{x^2+y^2}{2(x+y)}\right] + 2\left[y - \dfrac{x^2+y^2}{2(x+y)}\right]\dfrac{dy}{dx} = 0$, that is $\dfrac{dy}{dx} = -\dfrac{x^2+2xy-y^2}{y^2+2xy-x^2}$.

8. $2cy = x^2 - c^2 \Longrightarrow c^2 + 2cy - x^2 = 0 \Longrightarrow c = \dfrac{-2y \pm \sqrt{4y^2+4x^2}}{2} = -y \pm \sqrt{x^2+y^2}$. Hence, $2c\frac{dy}{dx} = 2x$, so that $\frac{dy}{dx} = \frac{x}{c} = \frac{x}{-y\pm\sqrt{x^2+y^2}}$.

9. $x^2 + y^2 = c \Longrightarrow 2x + 2y\dfrac{dy}{dx} = 0 \Longrightarrow \dfrac{dy}{dx} = -\dfrac{x}{y}$.

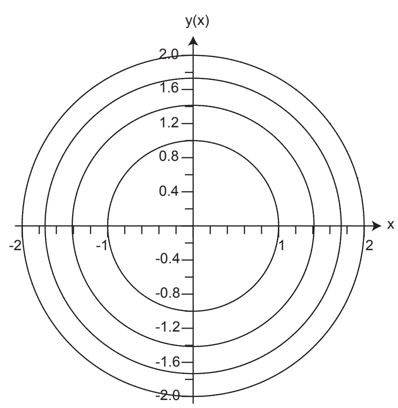

Figure 0.0.11: Figure for Problem 9

10. $y = cx^3 \Longrightarrow \dfrac{dy}{dx} = 3cx^2 = 3\dfrac{y}{x^3}x^2 = \dfrac{3y}{x}$. The initial condition $y(2) = 8 \Longrightarrow 8 = c(2)^3 \Longrightarrow c = 1$. Thus the unique solution to the initial value problem is $y = x^3$.

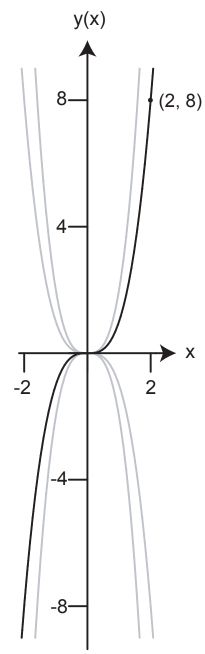

Figure 0.0.12: Figure for Problem 10

11. $y^2 = cx \Longrightarrow 2y\dfrac{dy}{dx} = c \Longrightarrow 2y\dfrac{dy}{dx} = \dfrac{y^2}{x} \Longrightarrow \dfrac{dy}{dx} = y2x \Longrightarrow 2x \cdot dy - y \cdot dx = 0$. The initial condition $y(1) = 2 \Longrightarrow c = 4$, so that the unique solution to the initial value problem is $y^2 = 4x$.

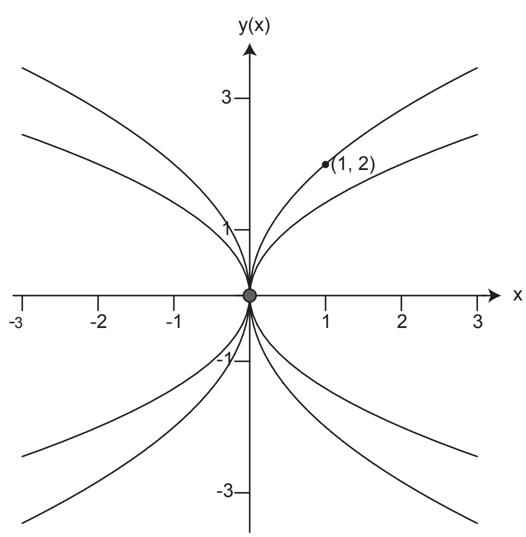

Figure 0.0.13: Figure for Problem 11

12. $(x-c)^2+y^2=c^2 \Longrightarrow x^2-2cx+c^2+y^2=c^2$, so that

$$x^2-2cx+y^2=0. \tag{0.0.1}$$

Differentiating with respect to x yields

$$2x-2c+2y\frac{dy}{dx}=0. \tag{0.0.2}$$

But from (0.0.1), $c=\dfrac{x^2+y^2}{2x}$ which, when substituted into (0.0.2), yields $2x-\left(\dfrac{x^2+y^2}{x}\right)+2y\dfrac{dy}{dx}=0$, that is, $\dfrac{dy}{dx}=\dfrac{y^2-x^2}{2xy}$. Imposing the initial condition $y(2)=2 \Longrightarrow$ from (0.0.1) $c=2$, so that the unique solution to the initial value problem is $y=+\sqrt{x(4-x)}$.

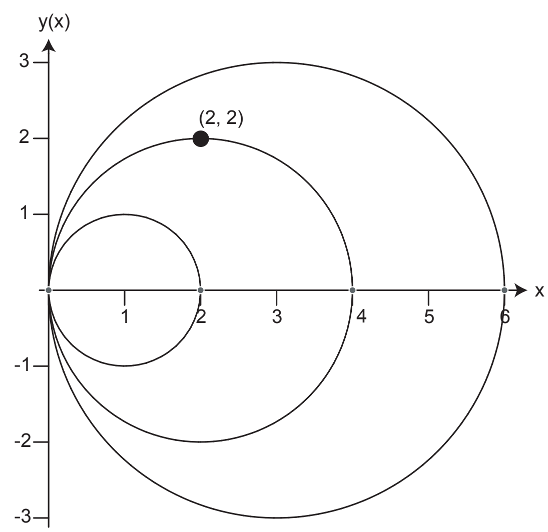

Figure 0.0.14: Figure for Problem 12

13. Let $f(x,y)=x\sin(x+y)$, which is continuous for all $x,y\in\mathbb{R}$.
$\dfrac{\partial f}{\partial y}=x\cos(x+y)$, which is continuous for all $x,y\in\mathbb{R}$.
By Theorem 1.3.2, $\dfrac{dy}{dx}=x\sin(x+y),\;\; y(x_0)=y_0$ has a unique solution for some interval $I\in\mathbb{R}$.

14. $\dfrac{dy}{dx}=\dfrac{x}{x^2+1}(y^2-9),\;\; y(0)=3.$
$f(x,y)=\dfrac{x}{x^2+1}(y^2-9)$, which is continuous for all $x,y\in\mathbb{R}$.
$\dfrac{\partial f}{\partial y}=\dfrac{2xy}{x^2+1}$, which is continuous for all $x,y\in\mathbb{R}$.
So the initial value problem stated above has a unique solution on any interval containing $(0,3)$. By inspection we see that $y(x)=3$ is the unique solution.

15. The initial-value problem does not necessarily have a unique solution since the hypothesis of the existence and uniqueness theorem are not satisfied at (0,0). This follows since $f(x,y) = xy^{1/2}$, so that $\frac{\partial f}{\partial y} = \frac{1}{2}xy^{-1/2}$ which is not continuous at $(0,0)$.

16. (a). $f(x,y) = -2xy^2 \Longrightarrow \frac{\partial f}{\partial y} = -4xy$. Both of these functions are continuous for all (x,y), and therefore the hypothesis of the uniqueness and existence theorem are satisfied for any (x_0, y_0).

(b). $y(x) = \frac{1}{x^2+c} \Longrightarrow y' = -\frac{2x}{(x^2+c)^2} = -2xy^2$.

(c). $y(x) = \frac{1}{x^2+c}$.

(i). $y(0) = 1 \Longrightarrow 1 = \frac{1}{c} \Longrightarrow c = 1$. Hence, $y(x) = \frac{1}{x^2+1}$. The solution is valid on the interval $(-\infty, \infty)$.

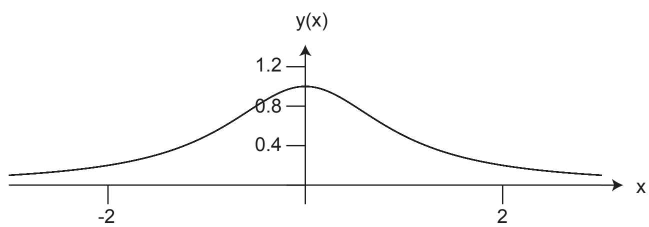

Figure 0.0.15: Figure for Problem 16c(i)

(ii). $y(1) = 1 \Longrightarrow 1 = \frac{1}{1+c} \Longrightarrow c = 0$. Hence, $y(x) = \frac{1}{x^2}$. This solution is valid on the interval $(0, \infty)$.

(iii). $y(0) = -1 \Longrightarrow -1 = \frac{1}{c} \Longrightarrow c = -1$. Hence, $y(x) = \frac{1}{x^2-1}$. This solution is valid on the interval $(-1, 1)$.

(d). Since, by inspection, $y(x) = 0$ satisfies the given initial-value problem, it must be the unique solution to the initial-value problem.

17. (a). Both $f(x,y) = y(y-1)$ and $\frac{\partial f}{\partial y} = 2y - 1$ are continuous at all points (x,y). Consequently, the hypothesis of the existence and uniqueness theorem are satisfied by the given initial-value problem for any x_0, y_0.

(b). Equilibrium solutions: $y(x) = 0, y(x) = 1$.

(c). Differentiating the given differential equation yields $\frac{d^2y}{dx^2} = (2y-1)\frac{dy}{dx} = (2y-1)y(y-1)$. Hence the solution curves are concave up for $0 < y < \frac{1}{2}$, and $y > 1$, and concave down for $y < 0$, and $\frac{1}{2} < y < 1$.

(d). The solutions will be bounded provided $0 \leq y_0 \leq 1$.

18. (a). Equilibrium solutions: $y(x) = -2$, $y(x) = 1$.

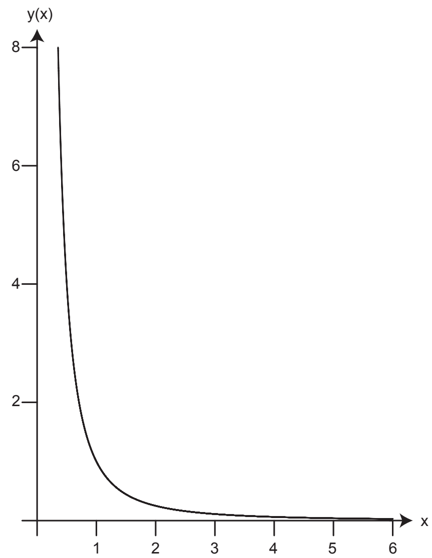

Figure 0.0.16: Figure for Problem 16c(ii)

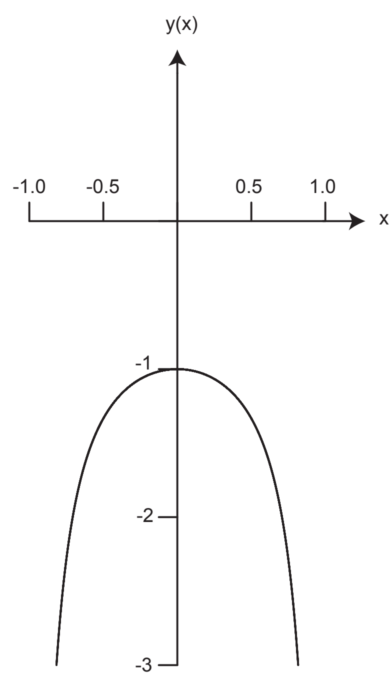

Figure 0.0.17: Figure for Problem 16c(iii)

(b). $\dfrac{dy}{dx} = (y+2)(y-1) \Longrightarrow$ the solutions are increasing when $y < -2$ and $y > 1$, and the solutions are

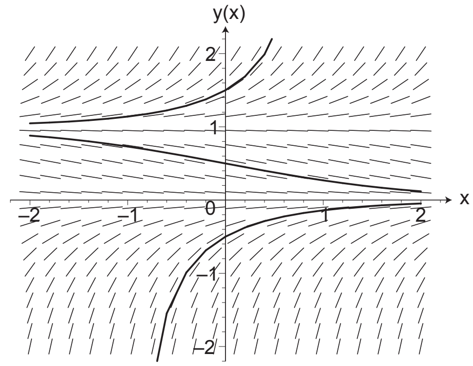

Figure 0.0.18: Figure for Problem 17(d)

decreasing when $-2 < y < 1$.

(c). Differentiating the given differential equation yields $\dfrac{d^2y}{dx^2} = (2y+1)\dfrac{dy}{dx} = (2y+1)(y+2)(y-1)$. Hence the solution curves are concave up for $-2 < y < -\frac{1}{2}$, and $y > 1$, and concave down for $y < -2$, and $-\frac{1}{2} < y < 1$.

19. (a). Equilibrium solution: $y(x) = 2$.

(b). $\dfrac{dy}{dx} = (y-2)^2 \Longrightarrow$ the solutions are increasing when $y < 2$ and $y > 2$.

(c). Differentiating the given differential equation yields $\dfrac{d^2y}{dx^2} = 2(y-2)\dfrac{dy}{dx} = 2(y-2)^3$. Hence the solution curves are concave up for $y > 2$, and concave down for $y < 2$.

20. (a). Equilibrium solutions: $y(x) = 0$, $y(x) = 1$.

(b). $\dfrac{dy}{dx} = y^2(y-1) \Longrightarrow$ the solutions are increasing when $y < 1$, and the solutions are decreasing when $y < 1$.

(c). Differentiating the given differential equation yields $\dfrac{d^2y}{dx^2} = (3y^2-2y)\dfrac{dy}{dx} = y^3(3y-2)(y-1)$. Hence the solution curves are concave up for $0 < y < -\frac{2}{3}$, and $y > 1$, and concave down for $y < 0$, and $2/3 < y < 1$.

21. (a). Equilibrium solutions: $y(x) = 0$, $y(x) = 1$, $y(x) = -1$.

(b). $\dfrac{dy}{dx} = (y+2)(y-1) \Longrightarrow$ the solutions are increasing when $-1 < y < 0$ and $y > 1$, and the solutions are decreasing when $y < -1$, and $0 < y < 1$.

(c). Differentiating the given differential equation yields $\dfrac{d^2y}{dx^2} = (3y^2-1)\dfrac{dy}{dx} = (3y^2-1)y(y-1)(y+1)$. Hence the solution curves are concave up for $-1 < y < -\frac{1}{\sqrt{3}}$, and $0 < y < \frac{1}{\sqrt{3}}$, and $y > 1$, and concave down for $y < -1$, and $-\frac{1}{\sqrt{3}} < y < 0$, and $\frac{1}{\sqrt{3}} < y < 1$.

22. $y' = 4x$. There are no equilibrium solutions. The slope of the solution curves is positive for $x > 0$ and is negative for $x < 0$. The isoclines are the lines $x = \frac{k}{4}$.

Slope of Solution Curve	Equation of Isocline
-4	$x = -1$
-2	$x = -1/2$
0	$x = 0$
2	$x = 1/2$
4	$x = 1$

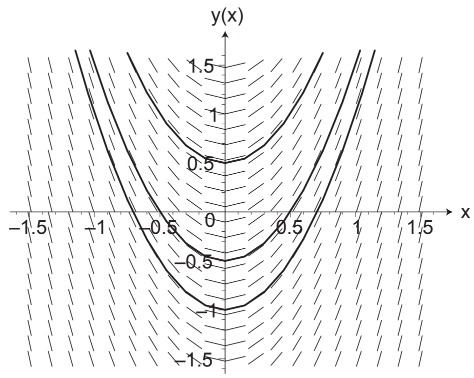

Figure 0.0.19: Figure for Problem 22

23. $y' = \frac{1}{x}$. There are no equilibrium solutions. The slope of the solution curves is positive for $x > 0$ and increases without bound as $x \to 0^+$. The slope of the curve is negative for $x < 0$ and decreases without bound as $x \to 0^-$. The isoclines are the lines $\frac{1}{x} = k$.

Slope of Solution Curve	Equation of Isocline
± 4	$x = \pm 1/4$
± 2	$x = \pm 1/2$
$\pm 1/2$	$x = \pm 2$
$\pm 1/4$	$x = \pm 4$
$\pm 1/10$	$x = \pm 10$

24. $y' = x + y$. There are no equilibrium solutions. The slope of the solution curves is positive for $y > -x$, and negative for $y < -x$. The isoclines are the lines $y + x = k$.

Slope of Solution Curve	Equation of Isocline
-2	$y = -x - 2$
-1	$y = -x - 1$
0	$y = -x$
1	$y = -x + 1$
2	$y = -x + 2$

Since the slope of the solution curve along the isocline $y = -x - 1$ coincides with the slope of the isocline, it follows that $y = -x - 1$ is a solution to the differential equation. Differentiating the given differential

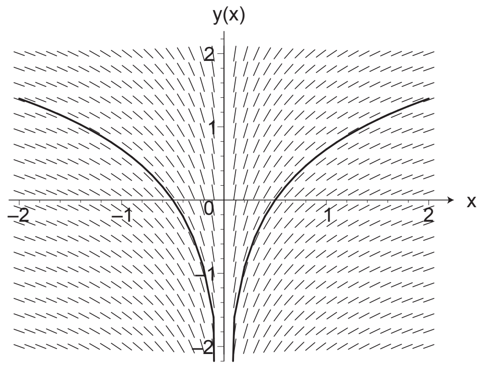

Figure 0.0.20: Figure for Problem 23

equation yields: $y'' = 1 + y' = 1 + x + y$. Hence the solution curves are concave up for $y > -x - 1$, and concave down for $y < -x - 1$. Putting this information together leads to the slope field in the accompanying figure.

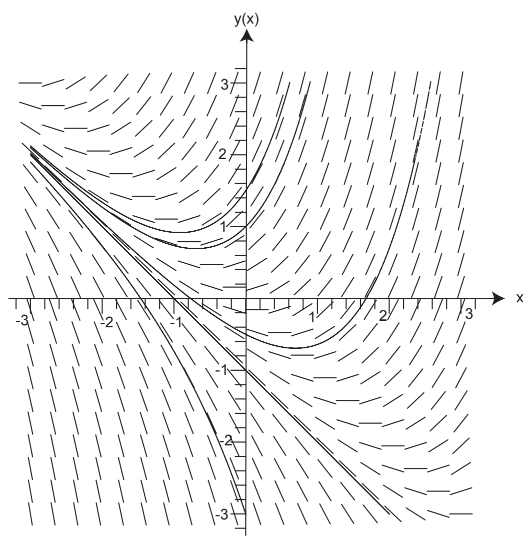

Figure 0.0.21: Figure for Problem 24

25. $y' = \frac{x}{y}$. There are no equilibrium solutions. The slope of the solution curves is zero when $x = 0$. The solution has a vertical tangent line at all points along the x-axis (except the origin). Differentiating the differential equation yields: $y' = \dfrac{1}{y} - \dfrac{x}{y^2}y' = \dfrac{1}{y} - \dfrac{x^2}{y^3} = \dfrac{1}{y^3}(y^2 - x^2)$. Hence the solution curves are concave up for $y > 0$ and $y^2 > x^2$; $y < 0$ and $y^2 < x^2$ and concave down for $y > 0$ and $y^2 < x^2$; $y < 0$ and $y^2 > x^2$. The isoclines are the lines $\frac{x}{y} = k$.

Slope of Solution Curve	Equation of Isocline
± 2	$y = \pm x/2$
± 1	$y = \pm x$
$\pm 1/2$	$y = \pm 2x$
$\pm 1/4$	$y = \pm 4x$
$\pm 1/10$	$y = \pm 10x$

Note that $y = \pm x$ are solutions to the differential equation.

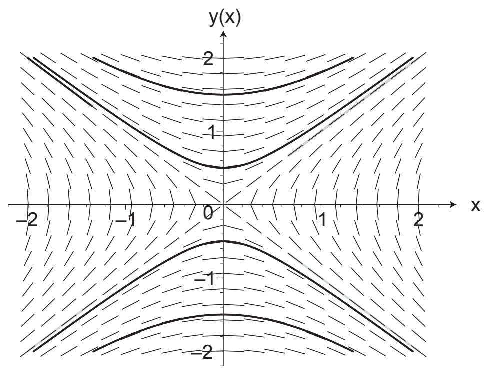

Figure 0.0.22: Figure for Problem 25

26. $y' = -\frac{4x}{y}$. Slope is zero when $x = 0\,(y \neq 0)$. The solutions have a vertical tangent line at all points along the x-axis(except the origin). The isoclines are the lines $-\frac{4x}{y} = k$. Some values are given in the table below.

Slope of Solution Curve	Equation of Isocline
± 1	$y = \pm 4x$
± 2	$y = \pm 2x$
± 3	$y = \pm 4x/3$

Differentiating the given differential equation yields: $y' = -\dfrac{4}{y} + \dfrac{4xy'}{y^2} = -\dfrac{4}{y} - \dfrac{16x^2}{y^3} = -\dfrac{4(y^2+4x^2)}{y}$. Consequently the solution curves are concave up for $y < 0$, and concave down for $y > 0$. Putting this information together leads to the slope field in the accompanying figure.

27. $y' = x^2 y$. Equilibrium solution: $y(x) = 0 \implies$ no solution curve can cross the x-axis. Slope: zero when $x = 0$ or $y = 0$. Positive when $y > 0\,(x \neq 0)$, negative when $y < 0\,(x \neq 0)$. Differentiating the given differential equation yields: $\dfrac{d^2y}{dx^2} = 2xy + x^2\dfrac{dy}{dx} = 2xy + x^4 y = xy(2+x^3)$. So, when $y > 0$, the solution curves are concave up for $x \in (-\infty, (-2)^{1/3})$, and for $x > 0$, and are concave down for $x \in ((-2)^{1/3}, 0)$. When $y < 0$, the solution curves are concave up for $x \in ((-2)^{1/3}, 0)$, and concave down for $x \in (-\infty, (-2)^{1/3})$ and for $x > 0$. The isoclines are the hyperbolas $x^2 y = k$.

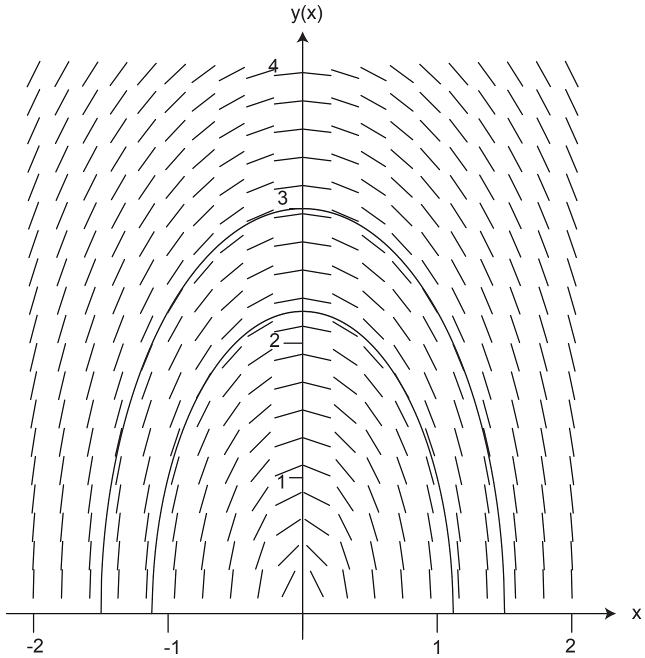

Figure 0.0.23: Figure for Problem 26

Slope of Solution Curve	Equation of Isocline
± 2	$y = \pm 2/x^2$
± 1	$y = \pm 1/x^2$
$\pm 1/2$	$y = \pm 1/(2x)^2$
$\pm 1/4$	$y = \pm 1/(4x)^2$
$\pm 1/10$	$y = \pm 1/(10x)^2$
0	$y = 0$

28. $y' = x^2 \cos y$. The slope is zero when $x = 0$. There are equilibrium solutions when $y = (2k+1)\frac{\pi}{2}$. The slope field is best sketched using technology. The accompanying figure gives the slope field for $-\frac{\pi}{2} < y < \frac{3\pi}{2}$.

29. $y' = x^2 + y^2$. The slope of the solution curves is zero at the origin, and positive at all the other points. There are no equilibrium solutions. The isoclines are the circles $x^2 + y^2 = k$.

Slope of Solution Curve	Equation of Isocline
1	$x = \pm 1/4$
2	$x = \pm 1/2$
3	$x = \pm 2$
4	$x = \pm 4$
5	$x = \pm 10$

30. $\frac{dT}{dt} = -\frac{1}{80}(T - 70)$. Equilibrium solution: $T(t) = 70$. The slope of the solution curves is positive for

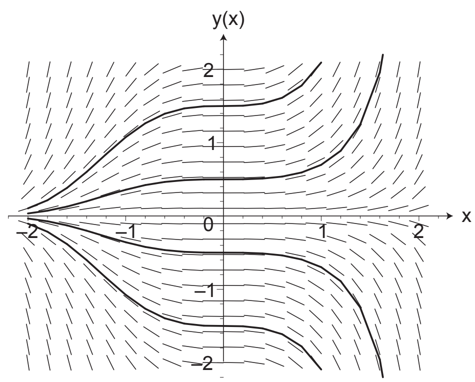

Figure 0.0.24: Figure for Problem 27

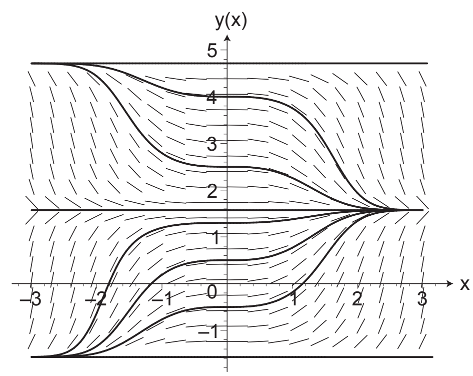

Figure 0.0.25: Figure for Problem 28

$T > 70$, and negative for $T < 70$. $\dfrac{d^2T}{dt^2} = -\dfrac{1}{80}\dfrac{dT}{dt} = \dfrac{1}{6400}(T-70)$. Hence the solution curves are concave up for $T > 70$, and concave down for $T < 70$. The isoclines are the horizontal lines $-\frac{1}{80}(T-70) = k$.

Slope of Solution Curve	Equation of Isocline
$-1/4$	$T = 90$
$1/5$	$T = 86$
0	$T = 70$
$1/5$	$T = 54$
$1/4$	$T = 50$

31. $y' = -2xy$.

32. $y' = \dfrac{x\sin x}{1+y^2}$.

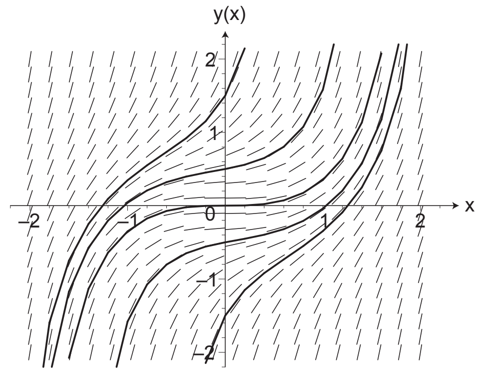

Figure 0.0.26: Figure for Problem 29

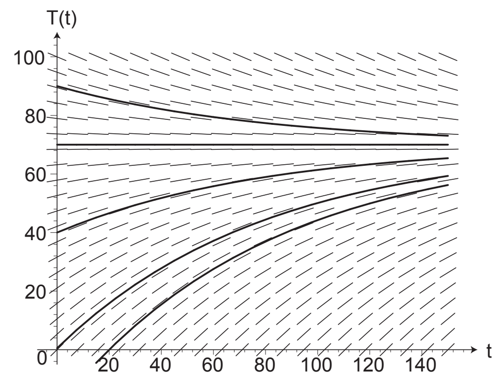

Figure 0.0.27: Figure for Problem 30

33. $y' = 3x - y$.

34. $y' = 2x^2 \sin y$.

35. $y' = \dfrac{2 + y^2}{3 + 0.5x^2}$.

36. $y' = \dfrac{1 - y^2}{2 + 0.5x^2}$.

37. (a). Slope field for the differential equation $y' = x^{-1}(3 \sin x - y)$.

(b). Slope field with solution curves included.

The figure suggests that the solution to the differential equation are unbounded as $x \to 0^+$.

(c). Slope field with solution curve corresponding to the initial condition $y(\frac{\pi}{2}) = \frac{6}{\pi}$.

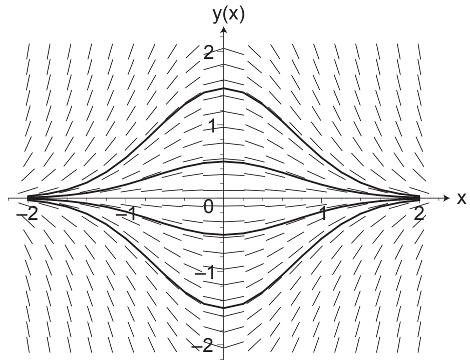

Figure 0.0.28: Figure for Problem 31

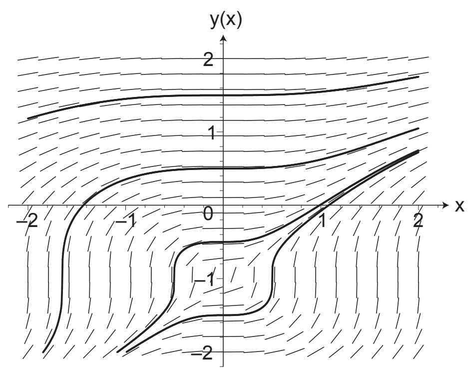

Figure 0.0.29: Figure for Problem 32

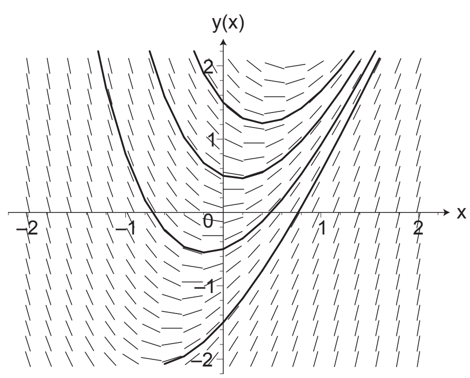

Figure 0.0.30: Figure for Problem 33

This solution curve is bounded as $x \to 0^+$.

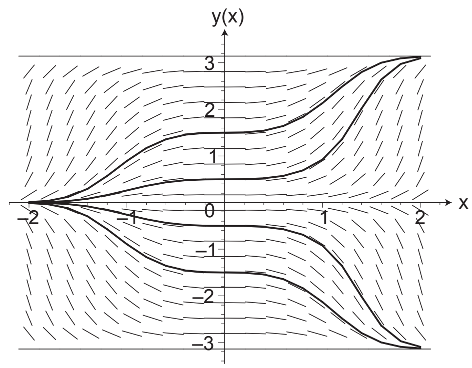

Figure 0.0.31: Figure for Problem 34

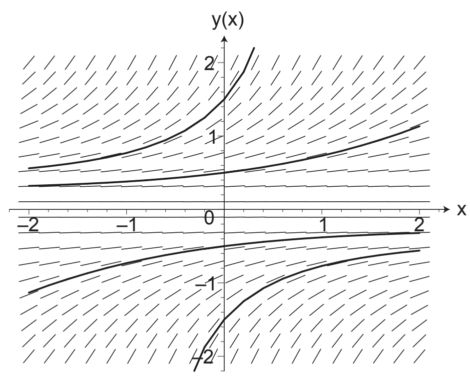

Figure 0.0.32: Figure for Problem 35

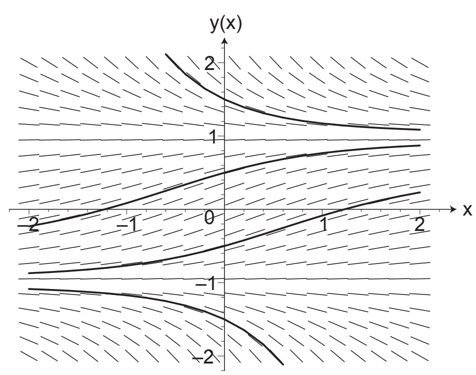

Figure 0.0.33: Figure for Problem 36

(d). In the accompanying figure we have sketched several solution curves on the interval (0,15].

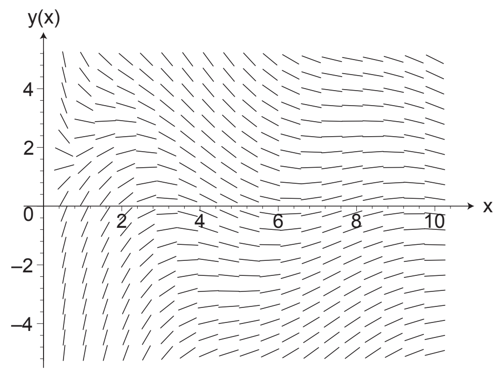

Figure 0.0.34: Figure for Problem 37(a)

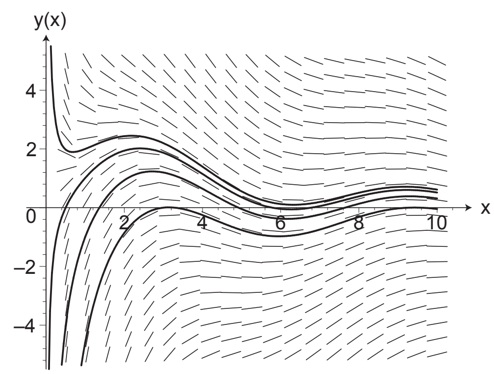

Figure 0.0.35: Figure for Problem 37(b)

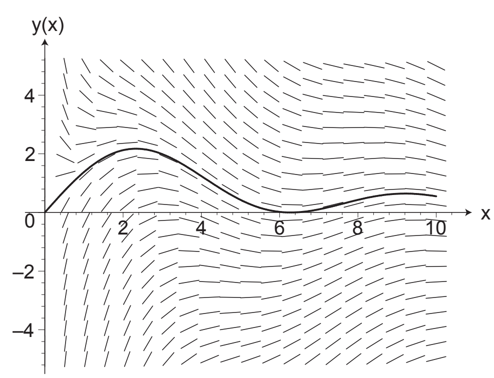

Figure 0.0.36: Figure for Problem 37(c)

The figure suggests that the solution curves approach the x-axis as $x \to \infty$.

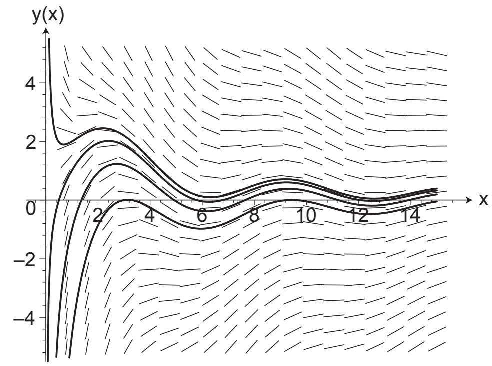

Figure 0.0.37: Figure for Problem 37(d)

38. (a). Differentiating the given equation gives $\frac{dy}{dx} = 2kx = 2\frac{y}{x}$. Hence the differential equation of the orthogonal trajectories is $\frac{dy}{dx} = -\frac{x}{2y}$.

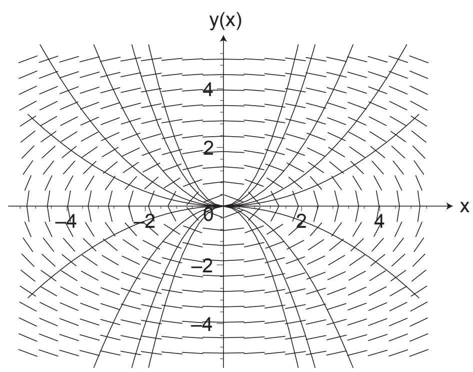

Figure 0.0.38: Figure for Problem 38(a)

(b). The orthogonal trajectories appear to be ellipses. This can be verified by integrating the differential equation derived in (a).

39. If $a > 0$, then as illustrated in the following slope field ($a = 0.5, b = 1$), it appears that $\lim_{t\to\infty} i(t) = \frac{b}{a}$.

If $a < 0$, then as illustrated in the following slope field ($a = -0.5, b = 1$) it appears that $i(t)$ diverges as $t \to \infty$.

If $a = 0$ and $b \neq 0$, then once more $i(t)$ diverges as $t \to \infty$. The accompanying figure shows a representative case when $b > 0$. Here we see that $\lim_{t\to\infty} i(t) = +\infty$. If $b < 0$, then $\lim_{t\to\infty} i(t) = -\infty$.

If $a = b = 0$, then the general solution to the differential equation is $i(t) = i_0$ where i_0 is a constant.

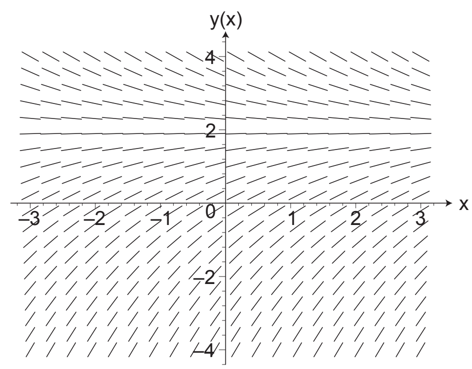

Figure 0.0.39: Figure for Problem 39 when $a > 0$

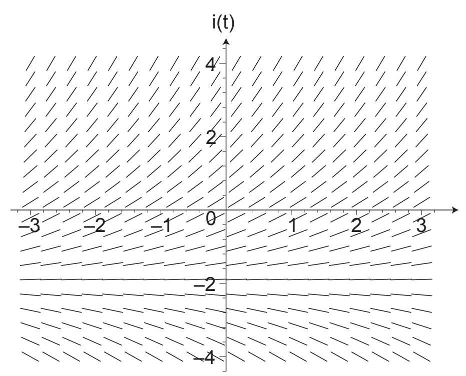

Figure 0.0.40: Figure for Problem 39 when $a < 0$

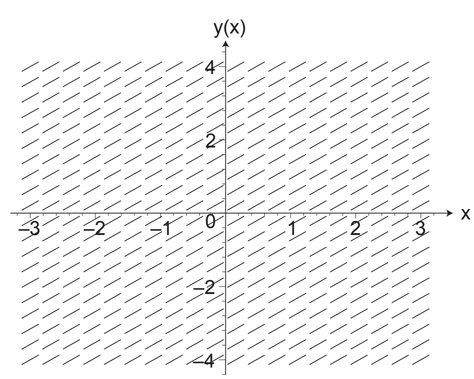

Figure 0.0.41: Figure for Problem 39 when $a = 0$

Solutions to Section 1.4

True-False Review:

(a): TRUE. The differential equation $\frac{dy}{dx} = f(x)g(y)$ can be written $\frac{1}{g(y)}\frac{dy}{dx} = f(x)$, which is the proper form, according to Definition 1.4.1, for a separable differential equation.

(b): TRUE. A separable differential equation is a first-order differential equation, so the general solution contains one constant. The value of that constant can be determined from an initial condition, as usual.

(c): TRUE. Newton's Law of Cooling is usually expressed as $\frac{dT}{dt} = -k(T - T_m)$, and this can be rewritten as

$$\frac{1}{T - T_m}\frac{dT}{dt} = -k,$$

and this form shows that the equation is separable.

(d): FALSE. The expression $x^2 + y^2$ cannot be separated in the form $f(x)g(y)$, so the equation is not separable.

(e): FALSE. The expression $x\sin(xy)$ cannot be separated in the form $f(x)g(y)$, so the equation is not separable.

(f): TRUE. We can write the given equation as $e^{-y}\frac{dy}{dx} = e^x$, which is the proper form for a separable equation.

(g): TRUE. We can write the given equation as $(1 + y^2)\frac{dy}{dx} = \frac{1}{x^2}$, which is the proper form for a separable equation.

(h): FALSE. The expression $\frac{x+4y}{4x+y}$ cannot be separated in the form $f(x)g(y)$, so the equation is not separable.

(i): TRUE. We can write $\frac{x^3y+x^2y^2}{x^2+xy} = xy$, so we can write the given differential equation as $\frac{1}{y}\frac{dy}{dx} = x$, which is the proper form for a separable equation.

Problems:

1. Separating the variables and integrating yields

$$\int \frac{dy}{y} = 2\int xdx \Longrightarrow \ln|y| = x^2 + c_1 \Longrightarrow y(x) = ce^{x^2}.$$

2. Separating the variables and integrating yields

$$\int y^{-2}dy = \int \frac{dx}{x^2+1} \Longrightarrow y(x) = -\frac{1}{\tan^{-1}x + c}.$$

3. Separating the variables and integrating yields

$$\int e^y dy = \int e^{-x}dx = 0 \Longrightarrow e^y + e^{-x} = c \Longrightarrow y(x) = \ln(c - e^{-x}).$$

4. Separating the variables and integrating yields

$$\int \frac{dy}{y} = \int \frac{(\ln x)^{-1}}{x}dx \Longrightarrow y(x) = c\ln x.$$

5. Separating the variables and integrating yields

$$\int \frac{dx}{x-2} = \int \frac{dy}{y} \Longrightarrow \ln|x-2| - \ln|y| = c_1 \Longrightarrow y(x) = c(x-2).$$

6. Separating the variables and integrating yields

$$\int \frac{dy}{y-1} = \int \frac{2x}{x^2+3}dx \Longrightarrow \ln|y-1| = \ln|x^2+3| + c_1 \Longrightarrow y(x) = c(x^2+3)+1.$$

7. $y - x\dfrac{dy}{dx} = 3 - 2x^2\dfrac{dy}{dx} \Longrightarrow x(2x-1)\dfrac{dy}{dx} = (3-y)$. Separating the variables and integrating yields

$$\begin{aligned}-\int \frac{dy}{y-3} = \int \frac{dx}{x(2x-1)} &\Longrightarrow -\ln|y-3| = -\int \frac{dx}{x} + \int \frac{2}{2x-1}dx \\ &\Longrightarrow -\ln|y-3| = -\ln|x| + \ln|2x-1| + c_1 \\ &\Longrightarrow \frac{x}{(y-3)(2x-1)} = c_2 \Longrightarrow y(x) = \frac{cx-3}{2x-1}.\end{aligned}$$

8. $\dfrac{dy}{dx} = \dfrac{\cos(x-y)}{\sin x \sin y} - 1 \Longrightarrow \dfrac{dy}{dx} = \dfrac{\cos x \cos y}{\sin x \sin y} \Longrightarrow \int \dfrac{\sin y}{\cos y}dy = \int \dfrac{\cos x}{\cos y}dx \Longrightarrow -\ln|\cos y| = \ln|\sin x| + c_1 \Longrightarrow$ $\cos y = c\csc x$.

9. $\dfrac{dy}{dx} = \dfrac{x(y^2-1)}{2(x-2)(x-1)} \Longrightarrow \int \dfrac{dy}{(y+1)(y-1)} = \dfrac{1}{2}\int \dfrac{xdx}{(x-2)(x-1)}, y \neq \pm 1$. Thus,

$-\dfrac{1}{2}\displaystyle\int \frac{dy}{y+1} + \frac{1}{2}\int \frac{dy}{y-1} = \frac{1}{2}\left(2\int \frac{dx}{x-2} - \int \frac{dx}{x-1}\right) \Longrightarrow -\ln|y+1| + \ln|y-1| = 2\ln|x-2| - \ln|x-1| + c_1$

$\Longrightarrow \dfrac{y-1}{y+1} = c\dfrac{(x-2)^2}{x-1} \Longrightarrow y(x) = \dfrac{(x-1)+c(x-2)^2}{(x-1)-c(x-2)^2}$. By inspection we see that $y(x) = 1$, and $y(x) = -1$ are solutions of the given differential equation. The former is included in the above solution when $c = 0$.

10. $\dfrac{dy}{dx} = \dfrac{x^2y-32}{16-x^2} + 2 \Longrightarrow \int \dfrac{dy}{y-2} = \int \dfrac{x^2}{16-x^2}dx \Longrightarrow \ln|y-2| = -\int\left(1 + \dfrac{16}{x^2-16}\right)dx \Longrightarrow \ln|y-2| =$ $-x - 16\int \dfrac{dx}{x^2-16} \Longrightarrow \ln|y-2| = -x - 16\left(-\frac{1}{8}\int \dfrac{dx}{x+4} + \frac{1}{8}\int \dfrac{dx}{x-4}\right) \Longrightarrow \ln|y-2| = -x + 2\ln|x+4| -$ $2\ln|x-4| + c_1 \Longrightarrow y(x) = 2 + c\left(\dfrac{x+4}{x-4}\right)^2 e^{-x}$.

11. $(x-a)(x-b)\dfrac{dy}{dx} - (y-c) = 0 \Longrightarrow \int \dfrac{dy}{y-c} = \int \dfrac{dx}{(x-a)(x-b)} \Longrightarrow \int \dfrac{dy}{y-c} = \dfrac{1}{a-b}\int\left(\dfrac{1}{x-a} - \dfrac{1}{x-b}\right)dx \Longrightarrow$ $\ln|y-c| = \ln\left[c_1\left|\dfrac{x-a}{x-b}\right|^{1/(a-b)}\right] \Longrightarrow \left|(y-c)\left(\dfrac{x-b}{x-a}\right)^{1/(a-b)}\right| = c_1 \Longrightarrow y - c = c_2\left(\dfrac{x-a}{x-b}\right)^{1/(a-b)} \Longrightarrow$ $y(x) = c + c_2\left(\dfrac{x-a}{x-b}\right)^{1/(a-b)}$.

12. $(x^2+1)\dfrac{dy}{dx}+y^2=-1 \Longrightarrow \int \dfrac{dy}{1+y^2}=-\int \dfrac{dx}{1+x^2} \Longrightarrow \tan^{-1} y=\tan^{-1} x+c$, but $y(0)=1$ so $c=\dfrac{\pi}{4}$. Thus, $\tan^{-1} y=\tan^{-1} x+\dfrac{\pi}{4}$ or $y(x)=\dfrac{1-x}{1+x}$.

13. $(1-x^2)\dfrac{dy}{dx}+xy=ax \Longrightarrow \int \dfrac{dy}{a-y}=-\frac{1}{2}\int -\dfrac{2x}{1-x^2}dx \Longrightarrow -\ln|a-y|=-\frac{1}{2}\ln|1-x^2|+c_1 \Longrightarrow y(x)=a+c\sqrt{1-x^2}$, but $y(0)=2a$ so $c=a$ and therefore, $y(x)=a(1+\sqrt{1-x^2})$.

14. $\dfrac{dy}{dx}=1-\dfrac{\sin(x+y)}{\sin x\sin y} \Longrightarrow \dfrac{dy}{dx}=-\tan x\cot y \Longrightarrow -\int \dfrac{\sin y}{\cos y}dy=\int \dfrac{\sin x}{\cos x}dx \Longrightarrow -\ln|\cos x\cos y|=c$, but $y(\dfrac{\pi}{4})=\dfrac{\pi}{4}$ so $c=\ln(2)$. Hence, $-\ln|\cos x\cos y|=\ln(2) \Longrightarrow y(x)=\cos^{-1}\left(\frac{1}{2}\sec x\right)$.

15. $\dfrac{dy}{dx}=y^3\sin x \Longrightarrow \int \dfrac{dy}{y^3}=\int \sin x dx$ for $y\neq 0$. Thus $-\dfrac{1}{2y^2}=-\cos x+c$. However, we cannot impose the initial condition $y(0)=0$ on the last equation since it is not defined at $y=0$. But, by inspection, $y(x)=0$ is a solution to the given differential equation and further, $y(0)=0$; thus, the unique solution to the initial value problem is $y(x)=0$.

16. $\dfrac{dy}{dx}=\frac{2}{3}(y-1)^{1/2} \Longrightarrow \int \dfrac{dy}{(y-1)^{1/2}}=\frac{2}{3}\int dx$ if $y\neq 1 \Longrightarrow 2(y-1)^{1/2}=\frac{2}{3}x+c$ but $y(1)=1$ so $c=-\frac{2}{3} \Longrightarrow 2\sqrt{y-1}=\frac{2}{3}x-\frac{2}{3} \Longrightarrow \sqrt{y-1}=\frac{1}{3}(x-1)$. This does not contradict the Existence-Uniqueness theorem because the hypothesis of the theorem is not satisfied when $x=1$.

17. (a). $m\dfrac{dv}{dt}=mg-kv^2 \Longrightarrow \dfrac{m}{k\left[(mg/k)-v^2\right]}dv=dt$. If we let $a=\sqrt{\frac{mg}{k}}$ then the preceding equation can be written as $\dfrac{m}{k}\int \dfrac{1}{a^2-v^2}dv=\int dt$ which can be integrated directly to obtain

$$\frac{m}{2ak}\ln\left(\frac{a+v}{a-v}\right)=t+c,$$

that is, upon exponentiating both sides,

$$\frac{a+v}{a-v}=c_1e^{\frac{2ak}{m}t}.$$

Imposing the initial condition $v(0)=0$, yields $c=0$ so that

$$\frac{a+v}{a-v}=e^{\frac{2ak}{m}t}.$$

Therefore,

$$v(t)=a\left(\frac{e^{\frac{2akt}{m}}-1}{e^{\frac{2akt}{m}}+1}\right)$$

which can be written in the equivalent form

$$v(t)=a\tanh\left(\frac{gt}{a}\right).$$

(b). No. As $t\to\infty, v\to a$ and as $t\to 0^+, v\to 0$.

(c). $v(t) = a\tanh\left(\frac{gt}{a}\right) \Longrightarrow \dfrac{dy}{dt} = a\tanh\left(\frac{gt}{a}\right) \Longrightarrow a\int\tanh\left(\frac{gt}{a}\right)dt \Longrightarrow y(t) = \dfrac{a^2}{g}\ln(\cosh\left(\frac{gt}{a}\right)) + c_1$ and if $y(0) = 0$ then $y(t) = \dfrac{a^2}{g}\ln\left[\cosh\left(\frac{gt}{a}\right)\right]$.

18. The required curve is the solution curve to the initial-value problem $\dfrac{dy}{dx} = -\dfrac{x}{4y}, y(0) = \frac{1}{2}$. Separating the variables in the differential equation yields $4y^{-1}dy = -1dx$, which can be integrated directly to obtain $2y^2 = -\dfrac{x^2}{2} + c$. Imposing the initial condition we obtain $c = \frac{1}{2}$, so that the solution curve has the equation $2y^2 = -x^2 + \frac{1}{2}$, or equivalently, $4y^2 + 2x^2 = 1$.

19. The required curve is the solution curve to the initial-value problem $\dfrac{dy}{dx} = e^{x-y}, y(3) = 1$. Separating the variables in the differential equation yields $e^y dy = e^x dx$, which can be integrated directly to obtain $e^y = e^x + c$. Imposing the initial condition we obtain $c = e - e^3$, so that the solution curve has the equation $e^y = e^x + e - e^3$, or equivalently, $y = \ln(e^x + e - e^3)$.

20. The required curve is the solution curve to the initial-value problem $\dfrac{dy}{dx} = x^2y^2, y(-1) = 1$. Separating the variables in the differential equation yields $\frac{1}{y^2}dy = x^2 dx$, which can be integrated directly to obtain $-\frac{1}{y} = \frac{1}{3}x^3 + c$. Imposing the initial condition we obtain $c = -\frac{2}{3}$, so that the solution curve has the equation $y = -\frac{1}{\frac{1}{3}x^3 - \frac{2}{3}}$, or equivalently, $y = \frac{3}{2-x^3}$.

21. (a). Separating the variables in the given differential equation yields $\dfrac{1}{1+v^2}dv = -dt$. Integrating we obtain $\tan^{-1}(v) = -t + c$. The initial condition $v(0) = v_0$ implies that $c = \tan^{-1}(v_0)$, so that $\tan^{-1}(v) = -t + \tan^{-1}(v_0)$. The object will come to rest if there is time t, at which the velocity is zero. To determine t_r, we set $v = 0$ in the previous equation which yields $\tan^{-1}(0) = t_r + \tan^{-1}(v_0)$. Consequently, $t_r = \tan^{-1}(v_0)$. The object does not remain at rest since we see from the given differential equation that $\dfrac{dv}{dt} < 0$ at $t = t_r$, and so v is decreasing with time. Consequently v passes through zero and becomes negative for $t < t_r$.

(b). From the chain rule we have $\dfrac{dv}{dt} = \dfrac{dx}{dt}$. Then $\dfrac{dv}{dx} = v\dfrac{dv}{dx}$. Substituting this result into the differential equation (1.4.22) yields $v\dfrac{dv}{dx} = -(1+v^2)$. We now separate the variables: $\dfrac{v}{1+v^2}dv = -dx$. Integrating we obtain $\ln(1+v^2) = -2x + c$. Imposing the initial condition $v(0) = v_0, x(0) = 0$ implies that $c = \ln(1+v_0^2)$, so that $\ln(1+v^2) = -2x + \ln(1+v_0^2)$. When the object comes to rest the distance travelled by the object is $x = \frac{1}{2}\ln(1+v_0^2)$.

22. (a). $\dfrac{dv}{dt} = -kv^n \Longrightarrow v^{-n}dv = -kdt$.
$n \neq 1 :\Longrightarrow \dfrac{1}{1-n}v^{1-n} = -kt + c$. Imposing the initial condition $v(0) + v_0$ yields $c = \dfrac{1}{1-n}v_0^{1-n}$, so that $v = [v_0^{1-n} + (n-1)kt]^{1/(1-n)}$. The object comes to rest in a finite time if there is a positive value of t for which $v = 0$.
$n = 1 :\Longrightarrow$ Integrating$v^{-n}dv = -kdt$ and imposing the initial conditions yields $v = v_0e^{-kt}$, and the object does not come to rest in a finite amount of time.

(b). If $n \neq 1, 2$, then $\dfrac{dx}{dt} = [v_0^{1-n} + (n-1)kt]^{1/(1-n)}$, where $x(t)$ denotes the distanced travelled by the

object. Consequently, $x(t) = -\frac{1}{k(2-n)}[v_0^{1-n} + (n-1)kt]^{(2-n)/(1-n)} + c$. Imposing the initial condition $x(0) = 0$ yields $c = \frac{1}{k(2-n)}v_0^{2-n}$, so that $x(t) = -\frac{1}{k(2-n)}[v_o^{1-n} + n(n-1)kt]^{(2-n)/(1-n)} + \frac{1}{k(2-n)}v_0^{2-n}$. For $1 < n < 2$, we have $\frac{2-n}{1-n} < 0$, so that $\lim_{t\to\infty} x(t) = \frac{1}{k(2-n)}$. Hence the maximum distance that the object can travel in a finite time is less than $\frac{1}{k(2-n)}$.
If $n = 1$, then we can integrate to obtain $x(t) = \frac{v_0}{k}(1 - e^{-kt})$, where we have imposed the initial condition $x(0) = 0$. Consequently, $\lim_{t\to\infty} x(t) = \frac{v_0}{k}$. Thus in this case the maximum distance that the object can travel in a finite time is less than $\frac{v_0}{k}$.

(c). If $n > 2$, then $x(t) = -\frac{1}{k(2-n)}[v_o^{1-n} + n(n-1)kt]^{(2-n)/(1-n)} + \frac{1}{k(2-n)}v_0^{2-n}$ is still valid. However, in this case $\frac{2-n}{1-n} > 0$, and so $\lim_{t\to\infty} x(t) = +\infty$. Consequently, there is no limit to the distance that the object can travel.
If $n = 2$, then we return to $v = [v_0^{1-n} + (n-1)kt]^{1/(1-n)}$. In this case $\frac{dx}{dt} = (v_0^{-1} + kt)^{-1}$, which can be integrated directly to obtain $x(t) = \frac{1}{k}\ln(1 + v_0kt)$, where we have imposed the initial condition that $x(0) = 0$. Once more we see that $\lim_{t\to\infty} x(t) = +\infty$, so that there is no limit to the distance that the object can travel.

23. Solving $p = p_0(\frac{\rho}{\rho_0})^{1/\gamma}$. Consequently the given differential equation can be written as $dp = -g\rho_0(\frac{p}{p_0})^{1/\gamma}dy$, or equivalently, $p^{-1/\gamma}dp = -\frac{g\rho_0}{p_0^{1/\gamma}}dy$. This can be integrated directly to obtain $\frac{\gamma p^{(\gamma-1)/\gamma}}{\gamma - 1} = -\frac{g\rho_0 y}{p_0^{1/\gamma}} + c$. At the center of the Earth we have $p = p_0$. Imposing this initial condition on the preceding solution gives $c = \frac{\gamma p_0^{(\gamma-1)/\gamma}}{\gamma - 1}$. Substituting this value of c into the general solution to the differential equation we find, after some simplification, $p^{(\gamma-1)/\gamma} = p_0^{(\gamma-1)/\gamma}\left[1 - \frac{(\gamma-1)\rho_0 g y}{\gamma p_0}\right]$, so that $p = p_0\left[1 - \frac{(\gamma-1)\rho_0 g y}{\gamma p_0}\right]^{(\gamma-1)/\gamma}$.

24. $\frac{dT}{dt} = -k(T - T_m) \Longrightarrow \frac{dT}{dt} = -k(T - 75) \Longrightarrow \frac{dT}{T - 75} = -kdt \Longrightarrow \ln|T - 75| = -kt + c_1 \Longrightarrow T(t) = 75 + ce^{-kt}$. $T(0) = 135 \Longrightarrow c = 60$ so $T = 75 + 60e^{-kt}$. $T(1) = 95 \Longrightarrow 95 = 75 + 60e^{-k} \Longrightarrow k = \ln 3 \Longrightarrow T(t) = 75 + 60e^{-t\ln 3}$. Now if $T(t) = 615$ then $615 = 75 + 60^{-t\ln 3} \Longrightarrow t = -2$h. Thus the object was placed in the room at 2p.m.

25. $\frac{dT}{dt} = -k(T - 450) \Longrightarrow T(t) = 450 + Ce^{-kt}.T(0) = 50 \Longrightarrow C = -400$ so $T(t) = 450 - 400e^{-kt}$ and $T(20) = 150 \Longrightarrow k = \frac{1}{20}\ln\frac{4}{3}$; hence, $T(t) = 450 - 400(\frac{3}{4})^{t/20}$.
(i) $T(40) = 450 - 400(\frac{3}{4})^2 = 225°$F.
(ii) $T(t) = 350 = 450 - 400(\frac{3}{4})^{t/20} \Longrightarrow (\frac{3}{4})^{t/20} = \frac{1}{4} \Longrightarrow t = \frac{20\ln 4}{\ln(4/3)} \approx 96.4$ minutes.

26. $\frac{dT}{dt} = -k(T-34) \Longrightarrow \frac{dT}{T-34} = -kdt \Longrightarrow T(t) = 34 + ce^{-kt}$. $T(0) = 38 \Longrightarrow c = 4$ so that $T(t) = 34 + 4e^{-kt}$.$T(1) = 36 \Longrightarrow k = \ln 2$; hence, $T(t) = 34 + 4e^{-t\ln 2}$. Now $T(t) = 98 \Longrightarrow T(t) = 34 + 4e^{-kt} = 98 \Longrightarrow 2^{-t} = 16 \Longrightarrow t = -4$h. Thus $T(-4) = 98$ and Holmes was right, the time of death was 10 a.m.

27. $T(t) = 75 + ce^{-kt}$. $T(10) = 415 \Longrightarrow 75 + ce^{-10k} = 415 \Longrightarrow 340 = ce^{-10k}$ and $T(20) = 347 \Longrightarrow 75 + ce^{-20k} = 347 \Longrightarrow 272 = ce^{-20k}$. Solving these two equations yields $k = \frac{1}{10}\ln\frac{5}{4}$ and $c = 425$; hence, $T = 75 + 425(\frac{4}{5})^{t/10}$

(a) Furnace temperature: $T(0) = 500°$F.

(b) If $T(t) = 100$ then $100 = 75 + 425(\frac{4}{5})^{t/10} \Longrightarrow t = \frac{10\ln 17}{\ln\frac{5}{4}} \approx 126.96$ minutes. Thus the temperature of the coal was 100°F at 6:07 p.m.

28. $\frac{dT}{dt} = -k(T-72) \Longrightarrow \frac{dT}{T-72} = -kdt \Longrightarrow T(t) = 72 + ce^{-kt}$. Since $\frac{dT}{dt} = -20, -k(T-72) = -20$ or $k = \frac{10}{39}$. Since $T(1) = 150 \Longrightarrow 150 = 72 + ce^{-10/39} \Longrightarrow c = 78e^{10/39}$; consequently, $T(t) = 72 + 78e^{10(1-t)/39}$.

(i). Initial temperature of the object: $t = 0 \Longrightarrow T(t) = 72 + 78e^{10/30} \approx 173°$F

(ii). Rate of change of the temperature after 10 minutes: $T(10) = 72 + 78e^{-30/13}$ so after 10 minutes, $\frac{dT}{dt} = -\frac{10}{39}(72 + 78e^{-30/13} - 72) \Longrightarrow \frac{dT}{dt} = -\frac{260}{13}e^{-30/13} \approx 2°$F per minute.

29. Substituting $a = 0.5$, $M = 2000$ g, and $m_0 = 4$ g into the initial-value problem (1.4.17) yields

$$\frac{dm}{dt} = 0.5m^{3/4}\left[1 - \left(\frac{m}{2000}\right)^{1/4}\right], \quad m(0) = 4.$$

Separating the variables in the preceding differential equation gives

$$\frac{1}{m^{3/4}\left[1 - \left(\frac{m}{2000}\right)^{1/4}\right]}dm = 0.5\,dt$$

so that

$$\int \frac{1}{m^{3/4}\left[1 - \left(\frac{m}{2000}\right)^{1/4}\right]}dm = 0.5t + c.$$

To evaluate the integral on the left-hand-side of the preceding equation, we make the change of variable

$$w = \left(\frac{m}{2000}\right)^{1/4}, \quad dw = \frac{1}{4}\cdot\frac{1}{2000}\left(\frac{m}{2000}\right)^{-3/4}dm$$

and simplify to obtain

$$4\cdot(2000)^{1/4}\int\frac{1}{1-w}dw = 0.5t + c$$

which can be integrated directly to obtain

$$-4\cdot(2000)^{1/4}\ln(1-w) = 0.5t + c.$$

Exponentiating both sides of the preceding equation, and solving for w yields

$$w = 1 - c_1e^{-0.125t/(2000)^{1/4}}$$

or equivalently,

$$\left(\frac{m}{2000}\right)^{1/4} = 1 - c_1 e^{-0.125t/(2000)^{1/4}}.$$

Consequently,

$$m(t) = 2000\left[1 - c_1 e^{-0.125t/(2000)^{1/4}}\right]^4. \tag{0.0.3}$$

Imposing the initial condition $m(0) = 4$ yields

$$4 = 2000\,(1 - c_1)^4$$

so that

$$c_1 = 1 - \left(\frac{1}{500}\right)^{1/4} \approx 0.7885.$$

Inserting this expression for c_1 into Equation (0.0.3) gives

$$m(t) = 2000\left[1 - 0.7885e^{-0.125t/(2000)^{1/4}}\right]^4.$$

Consequently,

$$m(100) = 2000\left[1 - 0.7885e^{-12.5/(2000)^{1/4}}\right]^4 \approx 1190.5 \text{ g}.$$

30. Substituting $a = 0.10$, $M = 0.15$ g, and $m_0 = 0.008$ g into the initial-value problem (1.4.17) yields

$$\frac{dm}{dt} = 0.1m^{3/4}\left[1 - \left(\frac{m}{0.15}\right)^{1/4}\right], \quad m(0) = 0.008.$$

Separating the variables in the preceding differential equation gives

$$\frac{1}{m^{3/4}\left[1 - \left(\frac{m}{0.15}\right)^{1/4}\right]}dm = 0.1\,dt$$

so that

$$\int \frac{1}{m^{3/4}\left[1 - \left(\frac{m}{0.15}\right)^{1/4}\right]}dm = 0.1t + c.$$

To evaluate the integral on the left-hand-side of the preceding equation, we make the change of variable

$$w = \left(\frac{m}{0.15}\right)^{1/4}, \quad dw = \frac{1}{4}\cdot\frac{1}{0.15}\left(\frac{m}{0.15}\right)^{-3/4} dm$$

and simplify to obtain

$$4\cdot(0.15)^{1/4}\int \frac{1}{1-w}dw = 0.1t + c$$

which can be integrated directly to obtain

$$-4\cdot(0.15)^{1/4}\ln(1 - w) = 0.1t + c.$$

Exponentiating both sides of the preceding equation, and solving for w yields

$$w = 1 - c_1 e^{-0.025t/(0.15)^{1/4}}$$

or equivalently,

$$\left(\frac{m}{0.15}\right)^{1/4} = 1 - c_1 e^{-0.025t/(0.15)^{1/4}}.$$

Consequently,

$$m(t) = 0.15\left[1 - c_1 e^{-0.025t/(0.15)^{1/4}}\right]^4. \tag{0.0.4}$$

Imposing the initial condition $m(0) = 0.008$ yields

$$0.008 = 0.15\,(1 - c_1)^4$$

so that

$$c_1 = 1 - \left(\frac{4}{75}\right)^{1/4} \approx 0.5194.$$

Inserting this expression for c_1 into Equation (0.0.4) gives

$$m(t) = 0.15\left[1 - 0.5194 e^{-0.025t/(0.15)^{1/4}}\right]^4.$$

Consequently,

$$m(30) = 0.15\left[1 - 0.5194 e^{-0.75/(0.15)^{1/4}}\right]^4 \approx 0.076 \text{ g}.$$

The guppy will have reached 90% of its fully grown mass at time t where

$$0.9 \cdot 0.15 = 0.15\left[1 - 0.5194 e^{-0.025t/(0.15)^{1/4}}\right]^4.$$

Solving algebraically for t yields

$$t = -\frac{(0.15)^{1/4}}{0.025} \ln\left[\frac{1 - (0.9)^{1/4}}{0.5194}\right] \approx 74.5 \text{ days}.$$

31. Since the chemicals A and B combine in the ratio 2:1, the amounts of A and B that are unconverted at time t are $(20 - \frac{2}{3}Q)$ grams and $(20 - \frac{1}{3}Q)$ grams, respectively. Thus, according to the law of mass action, the differential equation governing the behavior of $Q(t)$ is

$$\frac{dQ}{dt} = k_1(20 - \frac{2}{3}Q)(20 - \frac{1}{3}Q) \Longrightarrow \frac{dQ}{dt} = k(30 - Q)(60 - Q) \Longrightarrow \int \frac{1}{(30 - Q)(60 - Q)}\,dQ = \int k\,dt$$

$$\Longrightarrow \int \frac{1}{30}\left(\frac{1}{(30 - Q)} - \frac{1}{60 - Q)}\right)\frac{dQ}{dt} = kt + c \Longrightarrow \ln\left(\frac{60 - Q}{30 - Q}\right) = kt + c \Longrightarrow \frac{60 - Q}{30 - Q} = c_1 e^{30kt}.$$

Imposing the initial condition $Q(0) = 0$ yields $c_1 = 2$. Further, $Q(10) = 15 \Longrightarrow 45/15 = 2e^{300k}$, so that $k = \dfrac{1}{300}\ln(3/2)$. Therefore,

$$\frac{60 - Q}{30 - Q} = 2e^{\frac{t}{10}\ln(3/2)} = 2\left(\frac{3}{2}\right)^{t/10} \Longrightarrow Q(t) = \frac{60\left[(3/2)^{t/10} - 1\right]}{2(3/2)^{t/10} - 1}.$$

Therefore, $Q(20) = \dfrac{60\left[(3/2)^2 - 1\right]}{2(3/2)^2 - 1} = \dfrac{150}{7}$. Hence, $\dfrac{150}{7}$ grams of C are produced in 20 minutes.

32. Since the chemicals A and B combine in the ratio 2:3, the amounts of A and B that are unconverted at time t are $(10 - \frac{2}{5}Q)$ grams and $(15 - \frac{3}{5}Q)$ grams, respectively. Thus, according to the law of mass action, the differential equation governing the behavior of $Q(t)$ is

$$\frac{dQ}{dt} = k_1(10 - \frac{2}{5}Q)(15 - \frac{3}{5}Q) \Longrightarrow \frac{dQ}{dt} = k(25 - Q)^2 \Longrightarrow \int \frac{1}{(25-Q)^2}\,dQ = \int k\,dt \Longrightarrow \frac{1}{25-Q} = kt + c.$$

Hence, $Q(t) = 25 - \dfrac{1}{kt+c}$. Imposing the initial condition $Q(0) = 0$ yields $c = 1/25$, so that

$$Q(t) = 25\left(1 - \frac{1}{25kt+1}\right) = \frac{625kt}{25kt+1}.$$

$Q(5) = 10 \Longrightarrow k = 2/375$, so that $Q(t) = \dfrac{50t}{2t+15}$. Therefore, $Q(30) = 1500/75 = 20$ grams. Hence, 20 grams of C are produced in 30 minutes. The reaction will be 50% complete when $Q(t) = 12.5$ this will occur after t minutes where $12.5 = \dfrac{50t}{2t+15} \Longrightarrow t \approx 7.5$ minutes.

33. Since A and B combine in the ratio 3:5 to produce C. Therefore, producing 30 g of C will require $\dfrac{5}{8} \cdot 30 = \dfrac{150}{8}$ g of A.

34. (a). Since the chemicals A and B combine in the ratio $a : b$ to produce chemical C, when Q grams of C are produced , the consist of $\dfrac{a}{a+b}Q$ grams of A and $\dfrac{b}{a+b}$ grams of B. Consequently, the amounts of A and B that are unconverted at time t are $A_0 - \dfrac{a}{a+b}Q$ grams and $B_0 - \dfrac{b}{a+b}Q$ grams, respectively. Therefore, according to the law of mass action, the chemical reaction is governed by the differential equation

$$\frac{dQ}{dt} = k\left(A_0 - \frac{a}{a+b}Q\right)\left(B_0 - \frac{b}{a+b}Q\right).$$

(b). $\dfrac{dQ}{dt} = k\left(A_0 - \dfrac{a}{a+b}Q\right)\left(B_0 - \dfrac{b}{a+b}Q\right) = \dfrac{abk}{(a+b)^2}\left(\dfrac{a+b}{a}A_0 - Q\right)\left(\dfrac{a+b}{b}B_0 - Q\right)$, which can be written as $\dfrac{dQ}{dt} = r(\alpha - Q)(\beta - Q)$, where $r = \dfrac{abk}{(a+b)^2}, \alpha = \dfrac{a+b}{a}A_0, \beta = \dfrac{a+b}{b}B_0$.

35. $\dfrac{dQ}{dt} = r(\alpha - Q)(\beta - Q) \Longrightarrow \displaystyle\int \frac{1}{(\alpha-Q)(\beta-Q)}\,dQ = \int r\,dt + c \Longrightarrow \frac{1}{\alpha-\beta}\int\left(\frac{1}{\beta-Q} - \frac{1}{\alpha-Q}\right)dQ =$ $rt + c$. Hence, $\dfrac{1}{\alpha-\beta}\ln\left(\dfrac{\alpha-Q}{\beta-Q}\right) = rt + c \Longrightarrow \dfrac{\alpha-Q}{\beta-Q} = ce^{r(\alpha-\beta)t} \Longrightarrow Q(t) = \dfrac{\alpha - \beta ce^{r(\alpha-\beta)t}}{1 - ce^{r(\alpha-\beta)t}}$. Imposing the initial condition $Q(0) = 0$ yields $c = \alpha/\beta$, so that $Q(t) = \dfrac{\alpha\beta\left[1 - e^{r(\alpha-\beta)t}\right]}{\beta - \alpha e^{r(\alpha-\beta)t}}$. When $\alpha > \beta$, $\lim_{t\to\infty} Q(t) =$ $\dfrac{\alpha\beta}{\alpha} = \beta$.

36. When $\alpha = \beta$, Equation (1.4.24) reduces to $\dfrac{dQ}{dt} = r(Q - \alpha)^2 \Longrightarrow (\alpha - Q)^{-2}\dfrac{dQ}{dt} = r \Longrightarrow -(\alpha - Q)^{-1} =$ $rt + c$.

Hence, $Q(t) = \alpha - \dfrac{1}{rt+c}$. Imposing the initial condition $Q(0) = 0$ yields $c = 1/\alpha$ so that $Q(t) = \dfrac{\alpha^2 rt}{\alpha rt + 1}$. Therefore, $\lim_{t\to\infty} Q(t) = \alpha$.

37. Separating the variables in the given differential equation yields

$$\int \frac{1}{(\alpha - Q)(\beta - Q)(\gamma - Q)}\, dQ = \int k dt$$

so that

$$\int \left[\frac{1}{(\beta-\alpha)(\gamma-\alpha)(\alpha-Q)} + \frac{1}{(\alpha-\beta)(\gamma-\beta)(\beta-Q)} + \frac{1}{(\alpha-\gamma)(\beta-\gamma)(\gamma-Q)}\right] dQ = kt + c,$$

so that

$$\frac{1}{(\beta-\alpha)(\gamma-\alpha)}\ln(\alpha-Q) + \frac{1}{(\alpha-\beta)(\gamma-\beta)}\ln(\beta-Q) + \frac{1}{(\alpha-\gamma)(\beta-\gamma)}\ln(\gamma-Q) = -kt - c,$$

which can be written as

$$(\beta-\gamma)\ln(\alpha-Q) + (\gamma-\alpha)\ln(\alpha-Q) + (\alpha-\beta)\ln(\gamma-Q) = (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)kt + c_1$$

or equivalently,

$$\ln(\alpha-Q)^{\beta-\gamma} + \ln(\alpha-Q)^{\gamma-\alpha} + \ln(\gamma-Q)^{\alpha-\beta} = (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)kt + c_1.$$

Exponentiating both sides and simplifying yields

$$(\alpha-Q)^{\beta-\gamma}(\beta-Q)^{\gamma-\alpha}(\gamma-Q)^{\alpha-\beta} = c_2 e^{(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)kt}.$$

$Q(0) = 0 \Longrightarrow c_2 = \alpha^{\beta-\gamma}\beta^{\gamma-\alpha}\gamma^{\alpha-\beta}$, so that

$$(1-Q/\alpha)^{\beta-\gamma}(1-Q/\beta)^{\gamma-\alpha}(1-Q/\gamma)^{\alpha-\beta} = e^{(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)kt}.$$

Solutions to Section 1.5

True-False Review:

(a): TRUE. The differential equation for such a population growth is $\frac{dP}{dt} = kP$, where $P(t)$ is the population as a function of time, and this is the Malthusian growth model described at the beginning of this section.

(b): FALSE. The initial population could be greater than the carrying capacity, although in this case the population will asymptotically decrease towards the value of the carrying capacity.

(c): TRUE. The differential equation governing the logistic model is (1.5.2), which is certainly separable as

$$\frac{D}{P(C-P)}\frac{dP}{dt} = r.$$

Likewise, the differential equation governing the Malthusian growth model is $\frac{dP}{dt} = kP$, and this is separable as $\frac{1}{P}\frac{dP}{dt} = k$.

(d): TRUE. As (1.5.3) shows, as $t \to \infty$, the population does indeed tend to the carrying capacity C independently of the initial population P_0. As it does so, its rate of change $\frac{dP}{dt}$ slows to zero (this is best seen from (1.5.2) with $P \approx C$).

(e): TRUE. Every five minutes, the population doubles (increase 2-fold). Over 30 minutes, this population will double a total of 6 times, for an overall $2^6 = 64$-fold increase.

(f): TRUE. An 8-fold increase would take 30 years, and a 16-fold increase would take 40 years. Therefore, a 10-fold increase would take between 30 and 40 years.

(g): FALSE. The growth rate is $\frac{dP}{dt} = kP$, and so as P changes, $\frac{dP}{dt}$ changes. Therefore, it is not always constant.

(h): TRUE. From (1.5.2), the equilibrium solutions are $P(t) = 0$ and $P(t) = C$, where C is the carrying capacity of the population.

(i): FALSE. If the initial population is in the interval $(\frac{C}{2}, C)$, then although it is less than the carrying capacity, its concavity does not change. To get a true statement, it should be stated instead that the initial population is less than *half* of the carrying capacity.

(j): TRUE. Since $P'(t) = kP$, then $P''(t) = kP'(t) = k^2P > 0$ for all t. Therefore, the concavity is always positive, and does not change, regardless of the initial population.

Problems:

1. $\dfrac{dP}{dt} = kP \Longrightarrow P(t) = P_0e^{kt}$. Since $P(0) = 10$, then $P = 10e^{kt}$. Since $P(3) = 20$, then $2 = e^{3k} \Longrightarrow k = \dfrac{\ln 2}{3}$. Thus $P(t) = 10e^{(t/3)\ln 3}$. Therefore, $P(24) = 10e^{(24/3)\ln 3} = 10 \cdot 2^8 = 2560$ bacteria.

2. Using $P(t) = P_0e^{kt}$ we obtain $P(10) = 5000 \Longrightarrow 5000 = p_0e^{10k}$ and $P(12) = 6000 \Longrightarrow 6000 = P_0e^{12k}$ which implies that $e^{2k} = \frac{6}{5} \Longrightarrow k = \frac{1}{2}\ln\frac{6}{5}$. Hence, $P(0) = 5000(\frac{5}{6})^5 = 2009.4$. Also, $P = 2P_0$ when $t = \frac{1}{2}\ln 2 = \dfrac{2\ln 2}{\ln\frac{6}{5}} \approx 7.6$h.

3. From $P(t) = P_0e^{kt}$ and $P(0) = 2000$ it follows that $P(t) = 2000e^{kt}$. Since $t_d = 4, k = \frac{1}{4}\ln 2$ so $P = 2000e^{t\ln 2/4}$. Therefore, $P(t) = 10^6 \Longrightarrow 10^6 = 2000e^{t\ln 2/4} \Longrightarrow t \approx 35.86$ hours.

4. $\dfrac{dP}{dt} = kP \Longrightarrow P(t) = P_0e^{kt}$. Since, $P(0) = 10000$ then $P(t) = 10000e^{kt}$. Since $P(5) = 20000$ then $20000 = 10000e^{kt} \Longrightarrow k = \frac{1}{5}\ln 2$. Hence $P(t) = 10000e^{(t\ln 2)/5}$.

(a). $P(20) = 10000e^{4\ln 2} = 160000$.

(b). $1000000 = 10000e^{(t\ln 2)/5} \Longrightarrow 100 = e^{(t\ln 2)/5} \Longrightarrow t = \dfrac{5\ln 100}{\ln 2} \approx 33.22$ years.

5. $P(t) = \dfrac{50C}{50 + (C - 50)e^{-rt}}$. In formulas (1.5.5) and (1.5.6) we have $P_0 = 500, P_1 = 800, P_2 = 1000, t_1 = 5$, and $t_2 = 10$. Hence, $r = \dfrac{1}{5}\ln\left[\dfrac{(1000)(300)}{(500)(200)}\right] = \dfrac{1}{5}\ln 3, C = \dfrac{800[(800)(1500) - 2(500)(1000)]}{800^2 - (500)(1000)} \approx 1142.86$, so that $P(t) = \dfrac{1142.86)(500)}{500 + 642.86e^{-0.2t\ln 3}} \approx \dfrac{571430}{500 + 642.86e^{-0.2t\ln 3}}$. Inserting $t = 15$ into the preceding formula yields $P(15) = 1091$.

6. $P(t) = \dfrac{50C}{50 + (C-50)e^{-rt}}$ In formulas (1.5.5) and (1.5.6) we have $P_0 = 50, P_1 = 62, P_2 = 76, t_1 = 2$, and $t_2 = 2t_1 = 4$. Hence, $r = \dfrac{1}{2}\ln\left[\dfrac{(76)(12)}{(50)(14)}\right] \approx 0.132, C = \dfrac{62[(62)(126) - 2(50)(76)]}{62^2 - (50)(76)} \approx 298.727$, so that $P(t) = \dfrac{14936.35}{50 + 248.727e^{-0.132t}}$. Inserting $t = 20$ into the preceding formula yields $P(20) \approx 221$.

7. From equation (1.5.5) $r > 0$ requires $\dfrac{P_2(P_1 - P_0)}{P_0(P_2 - P_1)} > 1$. Rearranging the terms in this inequality and using the fact that $P_2 > P_1$ yields $P_1 > \dfrac{2P_0P_2}{P_0 + P_2}$. Further, $C > 0$ requires that $\dfrac{P_1(P_0 + P_2) - 2P_0P_2}{P_1^2 - P_0P_2} > 0$. From $P_1 > \dfrac{2P_0P_2}{P_0 + P_2}$ we see that the numerator in the preceding inequality is positive, and therefore the denominator must also be positive. Hence in addition to $P_1 > \dfrac{2P_0P_2}{P_0 + P_2}$, we must also have $P_1^2 > P_0P_2$.

8. Let $y(t)$ denote the number of passengers who have the flu at time t. Then we must solve $\dfrac{dy}{dt} = ky(1500 - y), y(0) = 5, y(1) = 10$, where k is a positive constant. Separating the differential equation and integrating yields $\int \dfrac{1}{y(1500-y)}dy = k\int dt$. Using a partial fraction decomposition on the left-hand side gives $\int \left[\dfrac{1}{1500y} + \dfrac{1}{1500(1500-y)}\right] dy = kt + c$, so that $\dfrac{1}{1500}\ln\left(\dfrac{y}{1500-y}\right) = kt + c$, which upon exponentiation yields $\dfrac{y}{1500-y} = c_1e^{1500kt}$. Imposing the initial condition $y(0) = 5$, we find that $c_1 = \dfrac{1}{299}$. Hence, $\dfrac{y}{1500-y} = \dfrac{1}{299}e^{1500kt}$. The further condition $y(1) = 10$ requires $\dfrac{10}{1490} = \dfrac{1}{299}e^{1500k}$. solving for k gives $k = \dfrac{1}{1500}\ln\dfrac{299}{149}$. Therefore, $\dfrac{y}{1500-y} = \dfrac{1}{299}e^{t\ln(299/149)}$. Solving algebraically for y we find $y(t) = \dfrac{1500e^{t\ln(299/149)}}{299 + e^{t\ln(299/149)}} = \dfrac{1500}{1 + 299e^{-t\ln(299/149)}}$. Hence, $y(14) = \dfrac{1500}{1 + 299e^{-14\ln(299/149)}} = 1474$.

9.(a). Equilibrium solutions: $P(t) = 0, P(t) = T$.
Slope: $P > T \Longrightarrow \dfrac{dP}{dt} > 0, 0 < P < T \Longrightarrow \dfrac{dP}{dt} < 0$.
Isoclines: $r(P - T) = k \Longrightarrow P^2 - TP - \dfrac{k}{r} = 0 \Longrightarrow P = \dfrac{1}{2}\left(T \pm \sqrt{\dfrac{rT^2 + 4k}{r}}\right)$. We see that slope of the solution curves satisfies $k \geq \dfrac{-rT^2}{4}$.
Concavity: $\dfrac{d^2P}{dt^2} = r(2P - T)\dfrac{dP}{dt} = r^2(2P - T)(P - T)P$. Hence, the solution curves are concave up for $P > \dfrac{t}{2}$, and are concave down for $0 < P < \dfrac{T}{2}$.

(b). See accompanying figure.

(c). For $0 < P_0 < T$, the population dies out with time. For $P_0 > T$, there is a population growth. The term threshold level is appropriate since T gives the minimum value of P_0 above which there is a population growth.

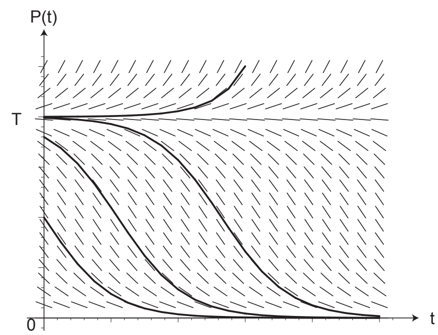

Figure 0.0.42: Figure for Problem 9(b)

10. (a). Separating the variables in differential equation (1.5.7) gives $\frac{1}{P(P-T)}\frac{dP}{dt} = r$, which can be written in the equivalent form $\left[\frac{1}{T(P-T)} - \frac{1}{TP}\right]\frac{dP}{dt} = r$. Integrating yields $\frac{1}{T}\ln\left(\frac{P-T}{P}\right) = rt+c$, so that $\frac{P-T}{P} = c_1 e^{Trt}$. The initial condition $P(0) = P_0$ requires $\frac{P_0 - T}{P_0} = c_1$, so that $\frac{P-T}{P} = \left(\frac{P_0-T}{P_0}\right)e^{rTt}$. Solving algebraically for P yields $P(t) = \frac{TP_0}{P_0 - (P_0 - T)e^{rTt}}$.

(b). If $P_0 < T$, then the denominator in $\frac{TP_0}{P_0 - (P_0 - T)e^{rTt}}$ is positive, and increases without bound as $t \to \infty$. Consequently $\lim_{t\to\infty} P(t) = 0$. In this case the population dies out as t increases.

(c). If $P_0 > T$, then the denominator of $\frac{TP_0}{P_0 - (P_0 - T)e^{rTt}}$ vanishes when $(P_0 - T)e^{rTt} = P_0$, that is when $t = \frac{1}{rT}\ln\left(\frac{P_0}{P_0 - T}\right)$. This means that within a finite time the population grows without bound. We can interpret this as a mathematical model of a population explosion.

11. $\frac{dP}{dt} = r(C-P)(P-T)P, P(0) = P_0, r > 0, 0 < T < C.$
Equilibrium solutions: $P(t) = 0, P(t) = T, P(t) = C$. The slope of the solution curves is negative for $0 < P < T$, and for $P > C$. It is positive for $T < P < C$.
Concavity: $\frac{d^2P}{dt^2} = r^2[(C-P)(P-T) - (P-T)P + (C-P)P](C-P)(P-T)P$, which simplifies to $\frac{d^2P}{dt^2} = r^2(-3P^2 + 2PT + 2CP - CT)(C-P)(P-T)$. Hence changes in concavity occur when $P = \frac{1}{3}(C + T \pm \sqrt{C^2 - CT + T^2})$. A representative slope field with some solution curves is shown in the accompanying figure. We see that for $0 < P_0 < T$ the population dies out, whereas for $T < P_0 < C$ the population grows and asymptotes to the equilibrium solution $P(t) = C$. If $P_0 > C$, then the solution decays towards the equilibrium solution $P(t) = C$.

12. $\frac{dP}{dt} = rP(\ln C - \ln P), P(0) = P_0$, and r, C, and P_0 are positive constants.
Equilibrium solutions: $P(t) = C$. The slope of the solution curves is positive for $0 < P < C$, and negative

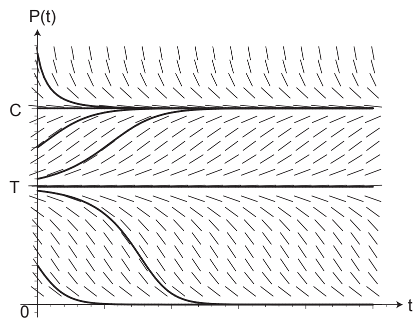

Figure 0.0.43: Figure for Problem 11

for $P > C$.
Concavity: $\frac{d^2P}{dt^2} = r\left[\ln\left(\frac{C}{P}\right) - 1\right]\frac{dP}{dt} = r^2\left[\ln\left(\frac{C}{P}\right) - 1\right]P\ln\frac{C}{P}$. Hence, the solution curves are concave up for $0 < P < \frac{C}{e}$ and $P > C$. They are concave down for $\frac{C}{e} < P < C$. A representative slope field with some solution curves are shown in the accompanying figure.

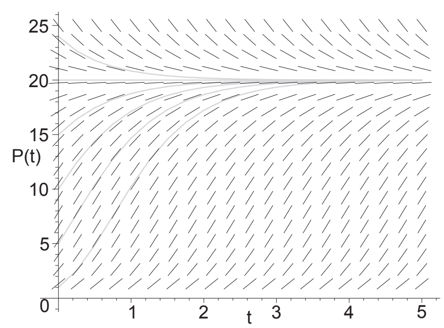

Figure 0.0.44: Figure for Problem 12

13. Separating the variables in (1.5.8) yields $\frac{1}{P(\ln C - \ln P)}\frac{dP}{dt} = r$ which can be integrated directly to obtain $-\ln(\ln C - \ln P) = rt + c$ so that $\ln\left(\frac{C}{P}\right) = c_1e^{-rt}$. The initial condition $P(0) = P_0$ requires that $\ln\left(\frac{C}{P_0}\right) = c_1$. Hence, $\ln\left(\frac{C}{P}\right) = e^{-rt}\ln\left(\frac{C}{P_0}\right)$ so that $P(t) = Ce^{\ln(P_0/k)e^{-rt}}$. Since $\lim_{t\to\infty} e^{-rt} = 0$, it follows that $\lim_{t\to\infty} P(t) = C$.

14. Using the exponential decay model we have $\frac{dP}{dt} = kP$, which is easily integrated to obtain $P(t) = P_0e^{kt}$. The initial condition $P(0) = 400$ requires that $P_0 = 400$, so that $P(t) = 400e^{kt}$. We also know that $P(30) = 340$. This requires that $340 = 400e^{30k}$ so that $k = \frac{1}{30}\ln\left(\frac{17}{20}\right)$. Consequently,

$$P(t) = 400e^{\frac{t}{30}\ln\left(\frac{17}{20}\right)}. \quad (0.0.5)$$

(a). From (0.0.5), $P(60)400e^{2\ln\left(\frac{17}{20}\right)} = 289$.

(b). From (0.0.5), $P(100) = 400e^{\frac{10}{3}\ln\left(\frac{17}{20}\right)} \approx 233$

(c). From (0.0.5), the half-life, t_H, is determine from

$$200 = 400e^{\frac{t_H}{30}\ln\left(\frac{17}{20}\right)} \implies t_H = 30\frac{\ln 2}{\ln\left(\frac{20}{17}\right)} \approx 128 \text{ days.}$$

15. (a). More.

(b). Using the exponential decay model we have $\frac{dP}{dt} = kP$, which is easily integrated to obtain $P(t) = P_0e^{kt}$. The initial condition $P(0) = 100,000$ requires that $P_0 = 100,000$, so that $P(t) = 100,000e^{kt}$. We also know that $P(10) = 80,000$. This requires that $100,000 = 80,000e^{10k}$ so that $k = \frac{1}{10}\ln\left(\frac{4}{5}\right)$. Consequently,

$$P(t) = 100,000e^{\frac{t}{10}\ln\left(\frac{4}{5}\right)}. \tag{0.0.6}$$

Using (0.0.6), the half-life is determined from

$$50,000 = 100,000e^{\frac{t_H}{10}\ln\left(\frac{4}{5}\right)} \implies t_H = 10\frac{\ln 2}{\ln\left(\frac{5}{4}\right)} \approx 31.06 \text{ min.}$$

(c). Using (0.0.6) there will be 15,000 fans left in the stadium at time t_0, where

$$15,000 = 100,000e^{\frac{t_0}{10}\ln\left(\frac{4}{5}\right)} \implies t_0 = 10\frac{\ln\left(\frac{3}{20}\right)}{\ln\left(\frac{4}{15}\right)} \approx 85.02 \text{ min.}$$

16. Using the exponential decay model we have $\frac{dP}{dt} = kP$, which is easily integrated to obtain $P(t) = P_0e^{kt}$. Since the half-life is 5.2 years, we have

$$\frac{1}{2}P_0 = P_0e^{5.2k} \implies k = -\frac{\ln 2}{5.2}.$$

Therefore,

$$P(t) = P_0e^{-t\frac{\ln 2}{5.2}}.$$

Consequently, only 4% of the original amount will remain at time t_0 where

$$\frac{4}{100}P_0 = P_0e^{-t_0\frac{\ln 2}{5.2}} \implies t_0 = 5.2\,\frac{\ln 25}{\ln 2} \approx 24.15 \text{ years.}$$

17. Maple, or even a TI 92 plus, has no problem in solving these equations.

18. (a). Malthusian model is $P(t) = 151.3e^{kt}$. Since $P(1) = 179.4$, then $179.4 = 151.3e^{10k} \implies k = \frac{1}{10}\ln\frac{179.4}{151.3}$. Hence, $P(t) = 151.3e^{\frac{t\ln(179.4/151.1)}{10}}$.

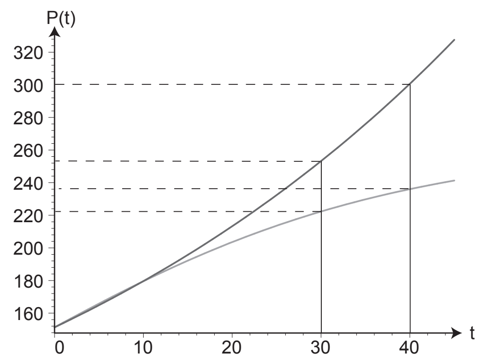

Figure 0.0.45: Figure for Problem 18(c)

(b). $P(t) = \dfrac{151.3C}{151.3 + (C - 151.3)e^{-rt}}$. Imposing the initial conditions $P(10) = 179.4$ and $P(20) = 203.3$ gives the pair of equations $179.4 = 151.3e^{\frac{10\ln(179.4/151.1)}{10}}$ and $203.3 = 151.3e^{\frac{20\ln(179.4/151.1)}{10}}$ whose solution is $C \approx 263.95, r \approx 0.046$. Using these values for C and r gives $P(t) = \dfrac{39935.6}{151.3 + 112.65e^{-0.046t}}$.

(c). Malthusian model: $P(30) \approx 253$ million; $P(40) \approx 300$ million.
Logistics model: $P(30) \approx 222$ million; $P(40) \approx 236$ million.
The logistics model fits the data better than the Malthusian model, but still gives a significant underestimate of the 1990 population.

19. $P(t) = \dfrac{50C}{50 + (C - 50)e^{-rt}}$. Imposing the conditions $P(5) = 100, P(15) = 250$ gives the pair of equations $100 = \dfrac{50C}{50 + (C - 50)e^{-5r}}$ and $250 = \dfrac{50C}{50 + (C - 50)e^{-15r}}$ whose positive solutions are $C \approx 370.32, r \approx 0.17$. Using these values for C and r gives $P(t) = \dfrac{18500}{50 + 18450e^{-0.17t}}$. From the figure we see that it will take approximately 52 years to reach 95% of the carrying capacity.

Solutions to Section 1.6

True-False Review:

(a): FALSE. Any solution to the differential equation (1.6.7) serves as an integrating factor for the differential equation. There are infinitely many solutions to (1.6.7), taking the form $I(x) = c_1 e^{\int p(x)dx}$, where c_1 is an arbitrary constant.

(b): TRUE. Any solution to the differential equation (1.6.7) serves as an integrating factor for the differential equation. There are infinitely many solutions to (1.6.7), taking the form $I(x) = c_1 e^{\int p(x)dx}$, where c_1 is an arbitrary constant. The most natural choice is $c_1 = 1$, giving the integrating factor $I(x) = e^{\int p(x)dx}$.

(c): TRUE. Multiplying $y' + p(x)y = q(x)$ by $I(x)$ yields $y'I + pIy = qI$. Assuming that $I' = pI$, the requirement on the integrating factor, we have $y'I + I'y = qI$, or by the product rule, $(I \cdot y)' = qI$, as requested.

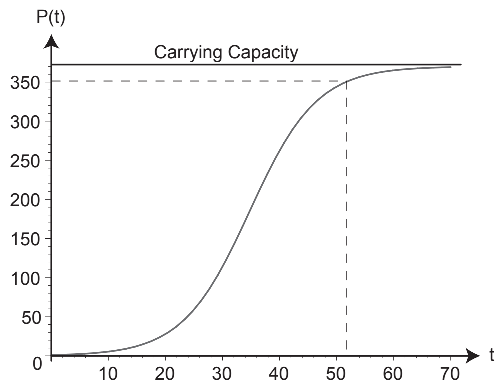

Figure 0.0.46: Figure for Problem 19

(d): FALSE. Rewriting the differential equation as

$$\frac{dy}{dx} - x^2 y = \sin x,$$

we have $p(x) = -x^2$, and so an integrating factor must have the form $I(x) = e^{\int p(x)dx} = e^{\int (-x^2)dx} = e^{-x^3/3}$, or any constant multiple of $e^{-x^3/3}$. Since e^{x^2} is not of this form, then it is not an integrating factor.

(e): FALSE. Rewriting the differential equation as

$$\frac{dy}{dx} + \frac{1}{x} y = x,$$

we have $p(x) = \frac{1}{x}$, and so an integrating factor must have the form $I(x) = e^{\int p(x)dx} = e^{\int (1/x)dx} = x$, or any constant multiple of x. Since $x + 5$ is not of this form, then it is not an integrating factor.

Problems:

In this section the function $I(x) = e^{\int p(x)dx}$ will represent the integrating factor for a differential equation of the form $y' + p(x)y = q(x)$.

1. $y' + y = 4e^x$. $I(x) = e^{\int dx} = e^x \Longrightarrow \frac{d(e^x y)}{dx} = 4e^{2x} \Longrightarrow e^x y = 2e^{2x} + c \Longrightarrow y(x) = e^{-x}(2e^{2x} + c)$.

2. $y' + \frac{2}{x} y = 5x^2$. $I(x) = e^{\int (2/x)\,dx} = x^2 \Longrightarrow \frac{d(x^2 y)}{dx} = 5x^4 \Longrightarrow x^2 y = x^5 + c \Longrightarrow y(x) = x^{-2}(x^5 + c)$.

3. $x^2 y' - 4xy = x^7 \sin x, x > 0 \Longrightarrow y' - \frac{4}{x} y = x^5 \sin x$. $I(x) = x^{-4} \Longrightarrow \frac{d(x^{-4} y)}{dx} = x \sin x \Longrightarrow x^{-4} y = \sin x - x \cos x + c \Longrightarrow y(x) = x^4(\sin x - x\cos x + c)$.

4. $y' + 2xy = 2x^3$. $I(x) = e^{2\int x dx} = e^{x^2} \Longrightarrow \frac{d}{dx}(e^{x^2} y) = 2e^{x^2} x^3 \Longrightarrow e^{x^2} y = 2\int e^{x^2} x^3 dx \Longrightarrow e^{x^2} y = e^{x^2}(x^2 - 1) + c \Longrightarrow y(x) = x^2 - 1 + ce^{-x^2}$.

5. $y' + \dfrac{2x}{1-x^2}y = 4x, -1 < x < 1$. $I(x) = \dfrac{1}{1-x^2} \Longrightarrow \dfrac{d}{dx}\left(\dfrac{y}{1-x^2}\right) = \dfrac{4x}{1-x^2} \Longrightarrow \dfrac{y}{1-x^2} = -\ln(1-x^2)^2 + c \Longrightarrow y(x) = (1-x^2)[-\ln(1-x^2)^2 + c]$.

6. $y' + \dfrac{2x}{1+x^2}y = \dfrac{4}{(1+x^2)^2}$. $I(x) = e^{\int \frac{2x}{1+x^2}dx} = 1 + x^2 \Longrightarrow \dfrac{d}{dx}[(1+x^2)y] = \dfrac{4}{(1+x^2)^2} \Longrightarrow (1+x^2)y = 4\int \dfrac{dx}{1+x^2} \Longrightarrow (1+x^2)y = 4\tan^{-1}x + c \Longrightarrow y(x) = \dfrac{1}{1+x^2}(4\tan^{-1}x + c)$.

7. $2\cos^2 x\dfrac{dy}{dx} + y\sin 2x = 4\cos^4 x, 0 \le x \le \frac{\pi}{2} \Longrightarrow y' + \dfrac{\sin 2x}{2\cos^2 x}y = 2\cos^2 x$. $I(x) = \dfrac{1}{\cos x} \Longrightarrow \dfrac{d}{dx}(y\sec x) = \cos x \Longrightarrow y(x) = \cos x(2\sin x + c) \Longrightarrow y(x) = \sin 2x + c\cos x$.

8. $y' + \dfrac{1}{x\ln x}y = 9x^2$. $I(x) = e^{\int \frac{dx}{x\ln x}} = \ln x \Longrightarrow \dfrac{d}{dx}(y\ln x) = 9\int x^2\ln x dx \Longrightarrow y\ln x = 3x^3\ln x - x^3 + c \Longrightarrow y(x) = \dfrac{3x^3\ln x - x^3 + c}{\ln x}$.

9. $y' - y\tan x = 8\sin^3 x$. $I(x) = \cos x \Longrightarrow \dfrac{d}{dx}(y\cos x) = 8\cos x\sin^3 x \Longrightarrow y\cos x = 8\int \cos x\sin^3 x dx + c \Longrightarrow y\cos x = 2\sin^4 x + c \Longrightarrow y(x) = \dfrac{1}{\cos x}(2\sin^4 x + c)$.

10. $t\dfrac{dx}{dt} + 2x = 4e^t \Longrightarrow x' + \dfrac{2}{t}x = \dfrac{4e^t}{t}$. $I(x) = e^{2\int \frac{dt}{t}} = t^2 \Longrightarrow \dfrac{d}{dt}(t^2x) = 4te^t \Longrightarrow t^2x = 4\int te^t dt + c \Longrightarrow t^2x = 4e^t(t-1) + c \Longrightarrow x(t) = \dfrac{4e^t(t-1)+c}{t^2}$.

11. $y' = (\sin x\sec x)y - 2\sin x \Longrightarrow y' - (\sin x\sec x)y = -2\sin x$. $I(x) = \cos x \Longrightarrow \dfrac{d}{dx}(y\cos x) = -2\sin x\cos x \Longrightarrow y\cos x = -2\int \sin x\cos x dx + c = \dfrac{1}{2}\cos 2x + c \Longrightarrow y(x) = \dfrac{1}{\cos x}\left(\dfrac{1}{2}\cos 2x + c\right)$.

12. $(1 - y\sin x)dx - \cos x dy = 0 \Longrightarrow y' + (\sin x\sec x)y = \sec x$. $I(x) = e^{\int \sin x\sec x dx} = \sec x \Longrightarrow \dfrac{d}{dx}(y\sec x) = \sec^2 x \Longrightarrow y\sec x = \int \sec^2 x dx + c \Longrightarrow y\sec x = \tan x + c \Longrightarrow y(x) = \cos x(\tan x + c) \Longrightarrow y(x) = \sin x + c\cos x$.

13. $y' - x^{-1}y = 2x^2\ln x$. $I(x) = e^{-\int \frac{1}{x}dx} = x^{-1} \Longrightarrow \dfrac{d}{dx}(x^{-1}y) = 2x\ln x \Longrightarrow x^{-1}y = 2\int x\ln x dx + c \Longrightarrow x^{-1}y = \dfrac{1}{2}x^2(2\ln x - 1) + c$. Hence, $y(x) = \dfrac{1}{2}x^3(2\ln x - 1) + cx$.

14. $y' + \alpha y = e^{\beta x}$. $I(x) = e^{\alpha\int dx} = e^{\alpha x} \Longrightarrow \dfrac{d}{dx}(e^{\alpha x}y) = e^{(\alpha+\beta)x} \Longrightarrow e^{\alpha x}y = \int e^{(\alpha+\beta)x}dx + c$. If $\alpha + \beta = 0$, then $e^{\alpha x}y = x + c \Longrightarrow y(x) = e^{-\alpha x}(x + c)$. If $\alpha + \beta \ne 0$, then $e^{\alpha x}y = \dfrac{e^{(\alpha+\beta)x}}{\alpha+\beta} + c \Longrightarrow y(x) = \dfrac{e^{\beta x}}{\alpha+\beta} + ce^{-\alpha x}$.

15. $y' + \dfrac{m}{x}y = \ln x$. $I(x) = x^m \Longrightarrow \dfrac{d}{dx}(x^m y) = x^m\ln x \Longrightarrow x^m y = \int x^m\ln x dx + c$. If $m = -1$, then $x^m y = \dfrac{(\ln x)^2}{2} + c \Longrightarrow y(x) = x\left[\dfrac{(\ln x)^2}{2} + c\right]$. If $m \ne -1$, then $x^m y = \dfrac{x^{m+1}}{m+1}\ln x - \dfrac{x^{m+1}}{(m+1)^2} + c \Longrightarrow y(x) =$

$\frac{x}{m+1}\ln x - \frac{x}{(m+1)^2} + \frac{c}{x^m}$.

16. $y' + \frac{2}{x}y = 4x$. $I(x) = e^{2\int \frac{dx}{x}} = e^{2\ln x} = x^2 \Longrightarrow \frac{d}{dx}(x^2y) = 4x^3 \Longrightarrow x^2y = 4\int x^3dx + c \Longrightarrow x^2y = x^4 + c$, but $y(1) = 2$ so $c = 1$; thus, $y(x) = \frac{x^4+1}{x^2}$.

17. $y'\sin x - y\cos x = \sin 2x \Longrightarrow y' - y\cot x = 2\cos x$. $I(x) = \csc x \Longrightarrow \frac{d}{dx}(y\csc x) = 2\csc x\cos x \Longrightarrow$ $y\csc x = 2\ln(\sin x) + c$, but $y(\frac{\pi}{2}) = 2$ so $c = 2$; thus, $y(x) = 2\sin x[\ln(\sin x) + 1]$.

18. $x' + \frac{2}{4-t}x = 5$. $I(t) = e^{2\int \frac{dt}{4-t}} = e^{-2\ln(4-t)} = (4-t)t^{-2} \Longrightarrow \frac{d}{dt}((4-t)^{-2}x) = 5(4-t)^{-2} \Longrightarrow (4-t)^{-2}x =$ $5\int(4-t)^{-2}dt + c \Longrightarrow (4-t)^{-2}x = 5(4-t)^{-1} + c$, but $x(0) = 4$ so $c = -1$; thus, $x(t) = (4-t)^2[5(4-t)^{-1} - 1]$ or $x(t) = (4-t)(1+t)$.

19. $(y - e^{-x})dx + dy = 0 \Longrightarrow y' + y = e^x$. $I(x) = e^x \Longrightarrow \frac{d}{dx}(e^xy) = e^{2x} \Longrightarrow e^xy = \frac{e^{2x}}{2} + c$, but $y(0) = 1$ so $c = \frac{1}{2}$; thus, $y(x) = \frac{1}{2}(e^x + e^{-x}) = \cosh x$.

20. $y' + y = f(x), y(0) = 3$,

$$f(x) = \begin{cases} 1, & \text{if } x \le 1, \\ 0, & \text{if } x > 1. \end{cases}$$

$I(x) = e^{\int dx} = e^x \Longrightarrow \frac{d}{dx}(e^xy) = e^xf(x) \Longrightarrow [e^xy]_0^x = \int_0^x e^xf(x)dx \Longrightarrow e^xy - y(0) = \int_0^x e^xf(x)dx \Longrightarrow$ $e^xy - 3 = \int_0^x e^xdx \Longrightarrow y(x) = e^{-x}\left[3 + \int_0^x e^xf(x)dx\right]$.
If $x \le 1, \int_0^x e^xf(x)dx = \int_0^x e^xdx = e^x - 1 \Longrightarrow y(x) = e^{-x}(2 + e^x)$
If $x > 1, \int_0^x e^xf(x)dx = \int_0^x e^xdx = e - 1 \Longrightarrow y(x) = e^{-x}(2 + e)$.

21. $y' - 2y = f(x), y(0) = 1$,

$$f(x) = \begin{cases} 1 - x, & \text{if } x < 1, \\ 0, & \text{if } x \ge 1. \end{cases}$$

$I(x) = e^{-\int 2dx} = e^{-2x} \Longrightarrow \frac{d}{dx}(e^{-2x}y) = e^{-2x}f(x) \Longrightarrow [e^{-2x}y]_0^x = \int_0^x e^{-2x}f(x)dx \Longrightarrow e^{-2x}y - y(0) =$ $\int_0^x e^{-2x}f(x)dx \Longrightarrow e^{-2x}y - 1 = \int_0^x e^{-2x}f(x) \Longrightarrow y(x) = e^{2x}\left[1 + \int_0^x e^{-2x}f(x)dx\right]$.
If $x < 1, \int_0^x e^{-2x}f(x)dx = \int_0^x e^{-2x}(1-x)dx = \frac{1}{4}e^{-2x}(2x-1+e^{2x}) \Longrightarrow y(x) = e^{2x}\left[1 + \frac{1}{4}e^{-2x}(2x - 1 + e^{2x})\right] =$ $\frac{1}{4}(5e^{2x} + 2x - 1)$.
If $x \ge 1, \int_0^x e^{-2x}f(x)dx = \int_0^x e^{-2x}(1-x)dx = \frac{1}{4}(1+e^{-2}) \Longrightarrow y(x) = e^{2x}\left[1 + \frac{1}{4}(1 + e^{-2})\right] = \frac{1}{4}e^{2x}(5 + e^{-2})$.

22. On $(-\infty, 1), y' - y = 1 \Longrightarrow I(x) = e^{-x} \Longrightarrow y(x) = c_1e^x - 1$. Imposing the initial condition $y(0) = 0$ requires $c_1 = 1$, so that $y(x) = e^x - 1$, for $x < 1$.
On $[1, \infty), y' - y = 2 - x \Longrightarrow I(x) = e^{-x} \Longrightarrow \frac{d}{dx}(e^{-x}y) = (2-x)e^{-x} \Longrightarrow y(x) = x - 1 + c_2e^{-x}$.
Continuity at $x = 1$ requires that $\lim_{x\to 1} y(x) = y(1)$. Consequently we must choose c_2 to satisfy $c_2e = e - 1$, so that $c_2 = 1 - e^{-1}$. Hence, for $x \ge 1, y(x) = x - 1 + (1 - e^{-1})e^x$.

23. $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} = 9x, x > 0$. Let $u = \frac{dy}{dx}$ so $\frac{du}{dx} = \frac{d^2y}{dx^2}$. The first equation becomes $\frac{du}{dx} + \frac{1}{x}u = 9x$ which is first-order linear. An integrating factor for this is $I(x) = x$ so $\frac{d}{dx}(xu) = 9x^2 \Longrightarrow xu = \int x^2 dx + c \Longrightarrow xu = 3x^3 + c_1 \Longrightarrow u = 3x^2 + c_1 x^{-1}$, but $u = \frac{dy}{dx}$ so $\frac{dy}{dx} = 3x^2 + c_1 x^{-1} \Longrightarrow y = \int (3x^2 + c_1 x^{-1})dx + c_2 \Longrightarrow y(x) = x^3 + c_1 \ln x + c_2$.

24. The differential equation for Newton's Law of Cooling is $\frac{dT}{dt} = -k(T - T_m)$. We can re-write this equation in the form of a first-order linear differential equation: $\frac{dT}{dt} + kT = kT_m$. An integrating factor for this differential equation is $I = e^{\int k\,dt} = e^{kt}$. Thus, $\frac{d}{dt}(Te^{kt}) = kT_m e^{kt}$. Integrating both sides, we get $Te^{kt} = T_m e^{kt} + c$, and hence, $T = T_m + ce^{-kt}$, which is the solution to Newton's Law of Cooling.

25. $\frac{dT_m}{dt} = \alpha \Longrightarrow T_m = \alpha t + c_1$ so $\frac{dT}{dt} = -k(T - \alpha t - c_1) \Longrightarrow \frac{dT}{dt} + kT = k(\alpha t + c_1)$. An integrating factor for this differential equation is $I = e^{k\int dt} = e^{kt}$. Thus, $\frac{d}{dt}(e^{kt}T) = ke^{kt}(\alpha t + c_1) \Longrightarrow e^{kt}T = e^{kt}(\alpha t - \frac{\alpha}{k} + c_1) + c_2 \Longrightarrow T = \alpha t - \frac{\alpha}{k} + c_1 + c_2 e^{-kt} \Longrightarrow T(t) = \alpha(t - \frac{1}{k}) + \beta + T_0 e^{-kt}$ where $\beta = c_1$ and $T_0 = c_2$.

26. $\frac{dT_m}{dt} = 10 \Longrightarrow T_m = 10t + c_1$ but $T_m = 65$ when $t = 0$ so $c_1 = 65$ and $T_m = 10t + 65$. $\frac{dT}{dt} = -k(T - T_m) \Longrightarrow \frac{dT}{dt} = -k(T - 10t - 65)$, but $\frac{dT}{dt}(1) = 5$, so $k = \frac{1}{8}$. The last differential equation can be written $\frac{dT}{dt} + kT = k(10t + 65) \Longrightarrow \frac{d}{dt}(e^{kt}T) = 5ke^{kt}(2t + 13) \Longrightarrow e^{kt}T = 5ke^{kt}\left(\frac{2}{k}t - \frac{2}{k^2} + \frac{13}{k}\right) + c \Longrightarrow T = 5(2t - \frac{2}{k} + 13) + ce^{-kt}$, but $k = \frac{1}{8}$ so $T(t) = 5(2t - 3) + ce^{-\frac{t}{8}}$. Since $T(1) = 35, c = 40e^{\frac{1}{8}}$. Thus, $T(t) = 10t - 15 + 40e^{\frac{1}{8}(1-t)}$.

27. (a). In this case, Newton's law of cooling is $\frac{dT}{dt} = -\frac{1}{40}(t - 80e^{-t/20})$. This linear differential equation has standard form $\frac{dT}{dt} + \frac{1}{40}T = 2e^{-t/20}$, with integrating factor $I(t) = e^{t/40}$. Consequently the differential equation can be written in the integrable form $\frac{d}{dt}(e^{t/40}T) = 2e^{-t/40}$, so that $T(t) = -80e^{-t/20} + ce^{-t/40}$. Then $T(0) = 0 \Longrightarrow c = 80$, so that $T(t) = 80(e^{-t/40} - e^{-t/20})$.

(b). We see that $\lim_{t\to\infty} = 0$. This is a reasonable result since the temperature of the surrounding medium also approaches zero as $t \to \infty$. We would expect the temperature of the object to approach to the temperature of the surrounding medium at late times.

(c). $T(t) = 80(e^{-t/40} - e^{-t/20}) \Longrightarrow \frac{dT}{dt} = 80\left(-\frac{1}{40}e^{-t/40} + \frac{1}{20}e^{-t/20}\right)$. So $T(t)$ has only one critical point when $80\left(-\frac{1}{40}e^{-t/40} + \frac{1}{20}e^{-t/20}\right) = 0 \Longrightarrow t = 40\ln 2$. Since $T(0) = 0$, and $\lim_{t\to\infty} T(t) = 0$ the function assumes a maximum value at $t_{max} = 40\ln 2$. $T(t_{max}) = 80(e^{-\ln 2} - e^{-2\ln 2}) = 20°$F, $T_m(t_{max}) = 80e^{-2\ln 2} = 20°$F.

(d). The behavior of $T(t)$ and $T_m(t)$ is given in the accompanying figure.

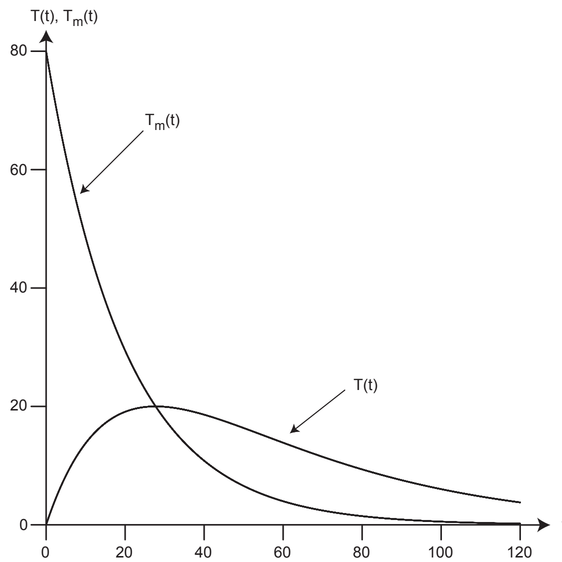

Figure 0.0.47: Figure for Problem 27(d)

28. (a). The temperature varies from a minimum of $A - B$ at $t = 0$ to a maximum of $A + B$ when $t = 12$.

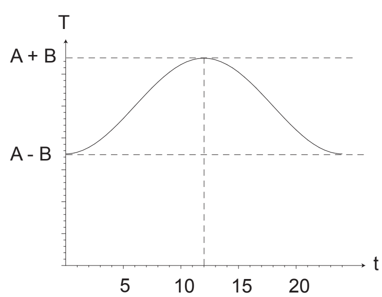

Figure 0.0.48: Figure for Problem 28(a)

(b). First write the differential equation in the linear form $\frac{dT}{dt} + k_1 T = k_1(A - B\cos\omega t) + T_0$. Multiplying by the integrating factor $I = e^{k_1 t}$ reduces this differential equation to the integrable form

$$\frac{d}{dt}(e^{k_1 t}T) = k_1 e^{k_1 t}(A - B\cos\omega t) + T_0 e^{k_1 t}.$$

Consequently,

$$e^{k_1 t}T(t) = \left(Ae^{k_1 t} - Bk_1 \int e_1^k t \cos\omega t dt + \frac{T_0}{k_1}e^{k_1 t} + c\right)$$

so that

$$T(t) = A + \frac{T_0}{k_1} - \frac{Bk_1}{k_1^2 + \omega^2}(k_1 \cos \omega t + \omega \sin \omega t) + ce^{-k_1 t}.$$

This can be written in the equivalent form

$$T(t) = A + \frac{T_0}{k_1} - \frac{Bk_1}{\sqrt{k_1^2 + \omega^2}} \cos(\omega t - \alpha) + ce^{-k_1 t}$$

for an approximate phase constant α.

29. (a). $\frac{dy}{dx} + p(x)y = 0 \Longrightarrow \frac{dy}{y} = -p(x)dx \Longrightarrow \int \frac{dy}{y} = -\int p(x)dx \Longrightarrow \ln|y| = -\int p(x)dx + c \Longrightarrow y_H = c_1 e^{-\int p(x)dx}$.

(b). Replace c_1 in part (a) by $u(x)$ and let $v = e^{-\int p(x)dx}$. $y = uv \Longrightarrow \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$. Substituting this last result into the original differential equation, $\frac{dy}{dx} + p(x) = q(x)$, we obtain $u\frac{dv}{dx} + v\frac{du}{dx} + p(x)y = q(x)$, but since $\frac{dv}{dx} = -vp$, the last equation reduces to $v\frac{du}{dx} = q(x) \Longrightarrow du = v^{-1}(x)q(x)dx \Longrightarrow u = \int v^{-1}(x)q(x)dx + c$. Substituting the values for u and v into $y = uv$, we obtain $y = e^{-\int p(x)dx}\left[\int e^{\int p(x)dx} q(x)dx + c\right]$.

30. The associated homogeneous equation is $\frac{dy}{dx} + x^{-1}y = 0$, with solution $y_H = cx^{-1}$. According to Problem 29, we determine the function $u(x)$ such that $y(x) = x^{-1}u(x)$ is a solution to the given differential equation. We have $\frac{dy}{dx} = x^{-1}\frac{du}{dx} - x^{-2}u$. Substituting into $\frac{dy}{dx} + x^{-1}y = \cos x$ yields $x^{-1}\frac{du}{dx} - \frac{1}{x^2}u + x^{-1}(x^{-1}u) = \cos x$, so that $\frac{du}{dx} = x\cos x$. Integrating we obtain $u = x\sin x + \cos x + c$, so that $y(x) = x^{-1}(x\sin x + \cos x + c)$.

31. The associated homogeneous equation is $\frac{dy}{dx} + y = 0$, with solution $y_H = ce^{-x}$. According to Problem 29, we determine the function $u(x)$ such that $y(x) = e^{-x}u(x)$ is a solution to the given differential equation. We have $\frac{dy}{dx} = \frac{du}{dx}e^{-x} - e^{-x}u$. Substituting into $\frac{dy}{dx} + y = e^{-2x}$ yields $\frac{du}{dx}e^{-x} - e^{-x}u + e^{-x}u(x) = e^{-2x}$, so that $\frac{du}{dx} = e^{-x}$. Integrating we obtain $u = -e^{-x} + c$, so that $y(x) = e^{-x}(-e^{-x} + c)$.

32. The associated homogeneous equation is $\frac{dy}{dx} + \cot x \cdot y = 0$, with solution $y_H = c \cdot \csc x$. According to Problem 29, we determine the function $u(x)$ such that $y(x) = \csc x \cdot u(x)$ is a solution to the given differential equation. We have $\frac{dy}{dx} = \csc x \cdot \frac{du}{dx} - \csc x \cdot \cot x \cdot u$. Substituting into $\frac{dy}{dx} + \cot x \cdot y = 2\cos x$ yields $\csc x \cdot \frac{du}{dx} - \csc x \cdot \cot x \cdot u + \csc x \cdot \cot x \cdot u = \cos x$, so that $\frac{du}{dx} = 2\cos x \sin x$. Integrating we obtain $u = \sin^2 x + c$, so that $y(x) = \csc x(\sin^2 x + c)$.

33. The associated homogeneous equation is $\frac{dy}{dx} - \frac{1}{x}y = 0$, with solution $y_H = cx$. We determine the function $u(x)$ such that $y(x) = xu(x)$ is a solution of the given differential equation. We have $\frac{dy}{dx} = x\frac{du}{dx} + u$. Substituting into $\frac{dy}{dx} - \frac{1}{x}y = x\ln x$ and simplifying yields $\frac{du}{dx} = \ln x$, so that $u = x\ln x - x + c$. Consequently,

$y(x) = x(x \ln x - x + c)$.

Problems **34 - 39** are easily solved using a differential equation solver such as the *dsolve* package in Maple.

Solutions to Section 1.7

True-False Review:

(a): TRUE. Concentration of chemical is defined as the ratio of mass to volume; that is, $c(t) = \frac{A(t)}{V(t)}$. Therefore, $A(t) = c(t)V(t)$.

(b): FALSE. The rate of change of volume is "rate in" − "rate out", which is $r_1 - r_2$, not $r_2 - r_1$.

(c): TRUE. This is reflected in the fact that c_1 is always assumed to be a constant.

(d): FALSE. The concentration of chemical leaving the tank is $c_2(t) = \frac{A(t)}{V(t)}$, and since both $A(t)$ and $V(t)$ can be nonconstant, $c_2(t)$ can also be nonconstant.

(e): FALSE. Kirchhoff's second law states that the sum of the voltage drops around a closed circuit is *zero*, not that it is independent of time.

(f): TRUE. This is essentially Ohm's law, (1.7.10).

(g): TRUE. Due to the negative exponential in the formula for the transient current, $i_T(t)$, it decays to zero as $t \to \infty$. Meanwhile, the steady-state current, $i_S(t)$, oscillates with the same frequency ω as the alternating current, albeit with a phase shift.

(h): TRUE. The amplitude is given in (1.7.19) as $A = \frac{E_0}{\sqrt{R^2+\omega^2L^2}}$, and so as ω gets larger, the amplitude A gets smaller.

Problems:

1. Given $V(0) = 600, A(0) = 1500, c_1 = 5, r_1 = 6$, and $r_2 = 3$. We need to find $\dfrac{A(60)}{V(60)}$. $\Delta V = r_1\Delta t - r_2\Delta t \Longrightarrow \dfrac{dV}{dt} = 3 \Longrightarrow V(t) = 3(t+200)$ since $V(0) = 600$. $\Delta A \approx c_1r_1\Delta t - c_2r_2\Delta t \Longrightarrow \dfrac{dA}{dt} = 30 - 3c_2 = 30 - 3\dfrac{A}{V} = 30 - \dfrac{A}{t+200} \Longrightarrow (t+200)A = 15(t+200)^2 + c$. Since $A(0) = 1500, c = -300000$ and therefore $A(t) = \dfrac{15}{t+200}[(t+200)^2 - 20000]$. Thus $\dfrac{A(60)}{V(60)} = \dfrac{596}{169}$ g/L.

2. Given $V(0) = 10, A(0) = 20, c_1 = 4, r_1 = 2$, and $r_2 = 1$. Then $\Delta V = r_1\Delta t - r_2\Delta t \Longrightarrow \dfrac{dV}{dt} = 1 \Longrightarrow V(t) = t+10$ since $V(0) = 10$. $\Delta A \approx c_1r_1\Delta t - c_2r_2\Delta t \Longrightarrow \dfrac{dA}{dt} = 8 - c_2 = 8 - \dfrac{A}{V} = 8 - \dfrac{A}{t+10} \Longrightarrow \dfrac{dA}{dt} + \dfrac{1}{t+10}A = 8 \Longrightarrow (t+10)A = 4(t+10)^2 + c_1$. Since $A(0) = 20 \Longrightarrow c_1 = -200$ so $A(t) = \dfrac{4}{t+10}[(t+10)^2 - 50]$. Therefore, $A(40) = 196$ g.

3. Given $V(0) = 100, A(0) = 100, c_1 = 0.5, r_1 = 6$, and $r_2 = 4$. Then $\Delta V = r_1\Delta t - r_2\Delta t \Longrightarrow \dfrac{dV}{dt} = 2 \Longrightarrow V(t) = 2(t+50)$ since $V(0) = 100$. Then $\dfrac{dA}{dt} + \dfrac{4A}{2(t+50)} = 3 \Longrightarrow \dfrac{dA}{dt} + \dfrac{2A}{t+50} = 3 \Longrightarrow \dfrac{d}{dt}[(t+50)^2A] =$

$3(t+50)^2 \Longrightarrow (t+50)^2A = (t+50)^3+c$ but $A(0) = 100$ so $c = 125000$ and therefore $A(t) = t+50+\dfrac{125000}{(t+50)^2}$. The tank is full when $V(t) = 200$, that is when $2(t+50) = 200$ so that $t = 50$ min. Therefore the concentration just before the tank overflows is: $\dfrac{A(50)}{V(50)} = \dfrac{9}{16}$ g/L.

4. Given $V(0) = 20, A(0) = 0, c_1 = 10, r_1 = 4$, and $r_2 = 2$. Then $\Delta V = r_1\Delta t - r_2\Delta t \Longrightarrow \dfrac{dV}{dt} = 2 \Longrightarrow V = 2(t+10)$ since $V(0) = 20$. Thus $V(t) = 40$ for $t = 10$, so we must find $A(10)$.$\Delta A \approx c_1r_1\Delta t - c_2r_2\Delta t \Longrightarrow \dfrac{dA}{dt} = 40 - 2c_2 = 40 - \dfrac{2A}{V} = 40 - \dfrac{A}{t+10} \Longrightarrow \dfrac{dA}{dt} + \dfrac{1}{t+10}A = 40 \Longrightarrow \dfrac{d}{dt}[(t+10)A] = 40(t+10)dt \Longrightarrow (t+10)A = 20(t+10)^2+c$. Since $A(0) = 0 \Longrightarrow c = -2000$ so $A(t) = \dfrac{20}{t+10}[(t+10)^2 - 100]$ and $A(10) = 300$ g.

5. (a). We are given that $V(0) = 20, c_1 = 1, r_1 = 3$, and $r_2 = 2$. Then $\Delta V = r_1\Delta t - r_2\Delta t \Longrightarrow \dfrac{dV}{dt} = 1 \Longrightarrow V = t+20$ since $V(0) = 20$. Then $\dfrac{dA}{dt}+\dfrac{2}{t+20}A = 3 \Longrightarrow \dfrac{d}{dt}[(t+20)^2A] = 3(t+20)^2 \Longrightarrow A(t) = \dfrac{(20+t)^3+c}{(t+20)^2}$ and since $A(0) = 0, c = -20^3$ which means that $A(t) = \dfrac{(t+20)^3 - 20^3}{(t+20)^2}$.

(b). The concentration of chemical in the tank, c_2, is given by $c_2 = \dfrac{A(t)}{V(t)}$ or $c_2 = \dfrac{A(t)}{t+20}$ so from part (a), $c_2 = \dfrac{(t+20)^3 - 20^3}{(t+20)^3}$. Therefore $c_2 = \dfrac{1}{2}$ g/l when $\dfrac{1}{2} = \dfrac{(t+20)^3 - 20^3}{(t+20)^3} \Longrightarrow t = 20(\sqrt[3]{2} - 1)$minutes.

6. We are given that $V(0) = 10, A(0) = 0, c_1 = 0.5, r_1 = 3, r_2 =$, and $\dfrac{A(5)}{V(5)} = 0.2$.

(a). $\Delta V = r_1\Delta t - r_2\Delta t \Longrightarrow \dfrac{dV}{dt} = 1 \Longrightarrow V(t) = t+10$ since $V(0) = 10$. Then $\Delta A \approx c_1r_1\Delta t - c_2r_2\Delta t \Longrightarrow \dfrac{dA}{dt} = -2c_2 = -2\dfrac{A}{V} = -\dfrac{2A}{t+10} \Longrightarrow \dfrac{dA}{A} = -\dfrac{2dt}{t+10} \Longrightarrow \ln|A| = -2\ln|t+10|+c \Longrightarrow A = k(t+10)^{-2}$. Then $A(5) = 3$ since $V(5) = 15$ and $\dfrac{A(5)}{V(5)} = 0.2$. Thus, $k = 675$ and $A(t) = \dfrac{675}{(t+10)^2}$. In particular, $A(0) = 6.75$ g.

(b). Find $V(t)$ when $\dfrac{A(t)}{V(t)} = 0.1$. From part (a) $A(t) = \dfrac{675}{(t+10)^2}$ and $V(t) = t+10 \Longrightarrow \dfrac{A(t)}{V(t)} = \dfrac{675}{(t+10)^3}$. Since $\dfrac{A(t)}{V(t)} = 0.1 \Longrightarrow (t+10)^3 = 6750 \Longrightarrow t+10 = 15\sqrt[3]{2}$ so $V(t) = t+10 = 15\sqrt[3]{2}$ L.

7. (a). We are given that $V(0) = w, c_1 = k, r_1 = r, r_2 = r$, and $A(0) = A_0$. Then $\Delta V = r_1\Delta r - r_2\Delta t \Longrightarrow \dfrac{dV}{dt} = 0 \Longrightarrow V(t) = V(0) = w$ for all t. Then $\Delta A = c_1r_1\Delta t - c_2r_2\Delta t \Longrightarrow \dfrac{dA}{dt} = kr - r\dfrac{A}{V} = kr - r\dfrac{A}{V} = kr - \dfrac{r}{w}A \Longrightarrow \dfrac{dA}{dt} + \dfrac{r}{w}A = kr \Longrightarrow \dfrac{d}{dt}(e^{-rt/w}A) = kre^{-rt/w} \Longrightarrow A(t) = kw + ce^{-rt/w}$. Since $A(0) = A_0$ so $c = A_0 - kw \Longrightarrow A(t) = e^{-rt/w}[kw(e^{rt/w} - 1) + A_0]$.

(b). $\lim_{t\to\infty} \dfrac{A(t)}{V(t)} = \lim_{t\to\infty} \dfrac{e^{-rt/w}}{w}[kw(e^{rt/w} - 1) + A_0] = \lim_{t\to\infty}[k + \left(\dfrac{A_0}{w} - k\right)e^{-rt/w}] = k$. This is reasonable since the volume remains constant, and the solution in the tank is gradually mixed with and replaced by the solution of concentration k flowing in.

8. (a). For the top tank we have: $\dfrac{dA_1}{dt} = c_1r_1 - c_2r_2 \Longrightarrow \dfrac{dA_1}{dt} = c_1r_1 - r_2\dfrac{A_1(t)}{V_1(t)} \Longrightarrow \dfrac{dA_1}{dt} = c_1r_1 - \dfrac{r_2}{(r_1-r_2)t+V_1}A_1(t) \Longrightarrow \dfrac{dA_1}{dt} + \dfrac{r_2}{(r_1-r_2)t+v_1}A_1 = c_1r_1$.
For the bottom tank we have: $\dfrac{dA_2}{dt} = c_2r_2 - c_3r_3 \Longrightarrow \dfrac{dA_2}{dt} = r_2\dfrac{A_1}{(r_1-r_2)t+V_1} - r_3\dfrac{A_2(t)}{V_2(t)} \Longrightarrow \dfrac{dA_2}{dt} = r_2\dfrac{A_1}{(r_1-r_2)t+V_1} - r_3\dfrac{A_2(t)}{(r_2-r_3)t+V_2} \Longrightarrow \dfrac{dA_2}{dt} + \dfrac{r_3}{(r_1-r_2)t+V_2}A_2 = \dfrac{r_2A_1}{(r_1-r_2)t+V_1}$.

(b). From part (a) $\dfrac{dA_1}{dt} + \dfrac{r_2}{(r_1-r_2)t+v_1}A_1 = c_1r_1 \Longrightarrow \dfrac{dA_1}{dt} + \dfrac{4}{2t+40}A_1 = 3 \Longrightarrow \dfrac{dA_1}{dt} + \dfrac{2}{t+20}A_1 = 3 \Longrightarrow \dfrac{d}{dt}[(t+20)^2A] = 3(t+20)^2 \Longrightarrow A_1 = t+20+\dfrac{c}{(t+20)^2}$ but $A_1(0)=4$ so $c=-6400$. Consequently $A_1(t) = t+20-\dfrac{6400}{(t+20)^2}$. Then $\dfrac{dA_2}{dt} + \dfrac{3}{t+20}A_2 = \dfrac{2}{t+20}\left[t+20-\dfrac{6400}{(t+20)^2}\right] \Longrightarrow \dfrac{dA_2}{dt} + \dfrac{3}{t+20}A_2 = \dfrac{2[(t+20)^3-6400]}{(t+20)^3} \Longrightarrow \dfrac{d}{dt}[(t+20)^3A_2] = (t+20)^3\left\{\dfrac{2[(t+20)^3-6400]}{(t+20)^3}\right\} \Longrightarrow A_2(t) = \dfrac{t+20}{2} - \dfrac{12800t}{(t+20)^3} + \dfrac{k}{(t+20)^3}$ but $A_2(0)=20$ so $k=80000$. Thus $A_2(t) = \dfrac{t+20}{2} - \dfrac{12800t}{(t+20)^3} + \dfrac{80000}{(t+20)^3}$ and in particular $A_2(10) = \dfrac{119}{9} \approx 13.2$ g.

9. Let $E(t)=20, R=4$ and $L=\frac{1}{10}$. Then $\dfrac{di}{dt} + \dfrac{R}{L}i = \dfrac{1}{L}E(t) \Longrightarrow \dfrac{di}{dt} + 40i = 200 \Longrightarrow \dfrac{d}{dt}(e^{40t}i) = 200e^{40t} \Longrightarrow i(t) = 5+ce^{-40t}$. But $i(0)=0 \Longrightarrow c=-5$. Consequently $i(t) = 5(1-e^{-40t})$.

10. Let $R=5, C=\frac{1}{50}$ and $E(t)=100$. Then $\dfrac{dq}{dt} + \dfrac{1}{RC}q = \dfrac{E}{R} \Longrightarrow \dfrac{dq}{dt} + 10q = 20 \Longrightarrow \dfrac{d}{dt}(qe^{10t}) = 20e^{10t} \Longrightarrow q(t) = 2+ce^{-10t}$. But $q(0)=0 \Longrightarrow c=-2$ so $q(t)=2(1-e^{-40t})$.

11. Let $R=2, L=\frac{2}{3}$ and $E(t)=10\sin 4t$. Then $\dfrac{di}{dt} + \dfrac{R}{L}i = \dfrac{1}{L}E(t) \Longrightarrow \dfrac{di}{dt} + 3i = 15\sin 4t \Longrightarrow \dfrac{d}{dt}(e^{3t}i) = 15e3t\sin 4t \Longrightarrow e^{3t}i = \dfrac{3e^{3t}}{5}(3\sin 4t - 4\cos 4t) + c \Longrightarrow i = 3\left(\dfrac{3}{5}\sin 4t - \dfrac{4}{5}\cos 4t\right) + ce^{-3t}$, but $i(0)=0 \Longrightarrow c = \dfrac{12}{5}$ so $i(t) = \dfrac{3}{5}(3\sin 4t - 4\cos 4t + 4e^{-3t})$.

12. Let $R=2, C=\frac{1}{8}$ and $E(t)=10\cos 3t$. Then $\dfrac{dq}{dt} + \dfrac{1}{RC}q = \dfrac{E}{R} \Longrightarrow \dfrac{dq}{dt} + 4q = 5\cos 3t \Longrightarrow \dfrac{d}{dt}(e^{4t}q) = 5e^{4t}\cos 3t \Longrightarrow e^{4t}q = \dfrac{e^{4t}}{5}(4\cos 3t + 3\sin 3t) + c \Longrightarrow q(t) = \frac{1}{5}(4\cos 3t + 3\sin 3t) + ce^{4t}$, but $q(0)=1 \Longrightarrow c = \dfrac{1}{5}(4\cos 3t + 3\sin 3t) + \dfrac{1}{5}e^{-4t}$ and $i(t) = \dfrac{dq}{dt} = \dfrac{1}{5}(9\cos 3t - 12\sin 3t - 4e^{-4t})$.

13. In an RC circuit for $t>0$ the differential equation is given by $\dfrac{dq}{dt} + \dfrac{1}{RC}q = \dfrac{E}{R}$. If $E(t)=0$ then $\dfrac{dq}{dt} + \dfrac{1}{RC}q = 0 \Longrightarrow \dfrac{d}{dt}(e^{t/RC}q) = 0 \Longrightarrow q = ce^{t/RC}$ and if $q(0)=5$ then $q(t) = 5e^{-t/RC}$. Then $\lim_{t\to\infty} q(t) = 0$. Yes, this is reasonable. As the time increases and $E(t)=0$, the charge will dissipate to zero.

14. In an RC circuit the differential equation is given by $i + \frac{1}{RC}q = \frac{E_0}{R}$. Differentiating this equation with respect to t we obtain $\frac{di}{dt} + \frac{1}{RC}\frac{dq}{dt} = 0$, but $\frac{dq}{dt} = i$ so $\frac{di}{dt} + \frac{1}{RC}i = 0 \Longrightarrow \frac{d}{dt}(e^{t/RC}i) = 0 \Longrightarrow i(t) = ce^{-t/RC}$. Since $q(0) = 0, i(0) = \frac{E_0}{R}$ and so $i(t) = \frac{E_0}{R}e^{-t/RC} \Longrightarrow d = E_0 k$ so $q(t) = E_0 k(1 - e^{-t/RC})$. Then $\lim_{t\to\infty} q(t) = E_0 k$, and $\lim t \to \infty i(t) = 0$.

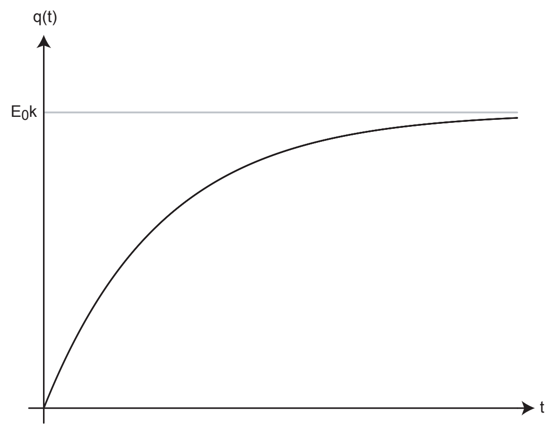

Figure 0.0.49: Figure for Problem 14

15. In an RL circuit, $\frac{di}{dt} + \frac{R}{L}i = \frac{E(t)}{L}$ and since $E(t) = E_0 \sin \omega t$, then $\frac{di}{dt} + \frac{R}{L}i = \frac{E_0}{L}\sin \omega t \Longrightarrow \frac{d}{dt}(e^{Rt/L}i) = \frac{E_0}{L}e^{Rt/L}\sin \omega t \Longrightarrow i(t) = \frac{E_0}{R^2 + L^2\omega^2}[R\sin \omega t - \omega L \cos \omega t] + Ae^{-Rt/L}$. We can write this as $i(t) = \frac{E_0}{\sqrt{R^2 + L^2\omega^2}}\left[\frac{R}{\sqrt{R^2 + L^2\omega^2}}\sin \omega t - \frac{\omega L}{\sqrt{R^2 + L^2\omega^2}}\cos \omega t\right] + Ae^{-Rt/L}$. Defining the phase ϕ by $\cos \phi = \frac{R}{\sqrt{R^2 + L^2\omega^2}}, \sin \phi = \frac{\omega L}{\sqrt{R^2 + L^2\omega^2}}$, we have $i(t) = \frac{E_0}{\sqrt{R^2 + L^2\omega^2}}[\cos \phi \sin \omega t - \sin \phi \cos \omega t] + Ae^{-Rt/L}$.
That is, $i(t) = \frac{E_0}{\sqrt{R^2 + L^2\omega^2}}\sin(\omega t - \phi) + Ae^{-Rt/L}$.
Transient part of the solution: $i_T(t) = Ae^{-Rt/L}$.
Steady state part of the solution: $i_S(t) = \frac{E_0}{\sqrt{R^2 + L^2\omega^2}}\sin(\omega t - \phi)$.

16. We must solve the initial value problem $\frac{di}{dt} + ai = \frac{E_0}{L}, i(0) = 0$, where $a = \frac{R}{L}$, and E_0 denotes the constant EMF. An integrating factor for the differential equation is $I = e^{at}$, so that the differential equation can be written in the form $\frac{d}{dt}(e^{at}i) = \frac{E_0}{L}e^{at}$. Integrating yields $i(t) = \frac{E_0}{aL} + c_1 e^{-at}$. The given initial condition requires $c_1 + \frac{E_0}{aL} = 0$, so that $c_1 = -\frac{E_0}{aL}$. Hence $i(t) = \frac{E_0}{aL}(1 - e^{-at}) = \frac{E_0}{R}(1 - e^{-at})$.

17. $\frac{dq}{dt} + \frac{1}{RC}q = \frac{E(t)}{R} \Longrightarrow \frac{dq}{dt} + \frac{1}{RC}q = \frac{E_0}{R}e^{-at} \Longrightarrow \frac{d}{dt}(e^{t/RC}q) = \frac{E_0}{R}e^{(1/RC-a)t} \Longrightarrow$

$q(t) = e^{-t/RC}\left[\frac{E_0C}{1-aRC}e^{(1/RC-a)t} + k\right] \Longrightarrow q(t) = \frac{E_0C}{1-aRC}e^{-at} + ke^{-t/RC}$. Imposing the initial condition $q(0) = 0$ (capacitor initially uncharged) requires $k = -\frac{E_0C}{1-aRC}$, so that $q(t) = \frac{E_0C}{1-aRC}(e^{-at} - e^{-t/RC})$. Thus $i(t) = \frac{dq}{dt} = \frac{E_0C}{1-aRC}\left(\frac{1}{RC}e^{-t/RC} - ae^{-at}\right)$.

18. $\frac{d^2q}{dt^2} + \frac{1}{LC}q = 0 \Longrightarrow i\frac{di}{dq} + \frac{1}{LC}q = 0$, since $i = \frac{dq}{dt}$. Then $idi = -\frac{1}{LC}qdq \Longrightarrow i^2 = -\frac{1}{LC}q^2 + k$ but $q(0) = q_0$ and $i(0) = 0$ so $k = \frac{q_0^2}{LC} \Longrightarrow i^2 = -\frac{1}{LC}q^2 + \frac{q_0^2}{LC} \Longrightarrow i = \pm\frac{\sqrt{q_0^2 - q^2}}{\sqrt{LC}} \Longrightarrow \frac{dq}{dt} = \pm\frac{\sqrt{q_0^2 - q^2}}{\sqrt{LC}} \Longrightarrow$ $\frac{dq}{\sqrt{q_o^2 - q^2}} = \pm\frac{dt}{\sqrt{LC}} \Longrightarrow \sin^{-1}\left(\frac{q}{q_0}\right) = \pm\frac{t}{\sqrt{LC}} + k_1 \Longrightarrow q = q_0\sin\left(\pm\frac{t}{\sqrt{LC}} + k_1\right)$ but $q(0) = q_0$ so $q_0 = q_0\sin k_1 \Longrightarrow k_1 = \frac{\pi}{2} + 2n\pi$ where n is an integer $\Longrightarrow q = q_0\sin\left(\pm\frac{1}{\sqrt{LC}} + \frac{\pi}{2}\right) \Longrightarrow q(t) = q_0\cos\left(\frac{t}{\sqrt{LC}}\right)$ and $i(t) = \frac{dq}{dt} = -\frac{q_0}{\sqrt{LC}}\sin\left(\frac{t}{\sqrt{LC}}\right)$.

19. $\frac{d^2q}{dt^2} + \frac{1}{LC}q = \frac{E_0}{L}$. Since $i = \frac{dq}{dt}$ then $\frac{d^2}{dt^2} = \frac{di}{dq}\frac{dq}{dt} = i\frac{di}{dq}$. Hence the original equation can be written as $i\frac{di}{dq} + \frac{1}{LC}q = \frac{E_0}{L} \Longrightarrow idi + \frac{1}{LC}qdq = \frac{E_0}{L}dq$ or $\frac{i^2}{2} + \frac{q^2}{2LC} = \frac{E_0q}{L} + A$. Since $i(0) = 0$ and $q(0) = q_0$ then $A = \frac{q_0^2}{2LC} - \frac{E_0q_0}{L}$. From $\frac{i^2}{2} + \frac{q^2}{2LC} = \frac{E_0q}{L} + A$ we get that $i = \left[2A + \frac{2E_0q}{L} - \frac{q^2}{LC}\right]^{1/2} \Longrightarrow i =$ $\left[2A + \frac{(2E_0C)^2}{LC} - \frac{(q-E_0C)^2}{LC}\right]^{1/2}$ and we let $D^2 = 2A + \frac{(E_0C)^2}{LC}$ then $i\frac{dq}{dt} = D\left[1 - \left(\frac{q-E_0C}{D\sqrt{LC}}\right)^2\right]^{1/2} \Longrightarrow$ $\sqrt{LC}\sin^{-1}\left(\frac{q-E_0C}{D\sqrt{LC}}\right) = t + B$. Then since $q(0) = 0$ so $B = \sqrt{LC}\sin^{-1}\left(\frac{q-E_0C}{D\sqrt{LC}}\right)$ and therefore $\frac{q-E_0C}{D\sqrt{LC}} = \sin\left(\frac{t+B}{\sqrt{LC}}\right) \Longrightarrow q(t) = D\sqrt{LC}\sin\left(\frac{t+B}{\sqrt{LC}}\right) + E_0c \Longrightarrow i = \frac{dq}{dt} = D\cos\left(\frac{t+B}{\sqrt{LC}}\right)$. Since $D^2 = \frac{2A + (E_0C)^2}{LC}$ and $A = \frac{q_0^2}{2LC} - \frac{E_0q_0}{L}$ we can substitute to eliminate A and obtain $D = \pm\frac{|q_0 - E_0C|}{\sqrt{LC}}$. Thus $q(t) = \pm|q_0 - E_0C|\sin\left(\frac{t+B}{\sqrt{LC}}\right) + E_0c$.

Solutions to Section 1.8

True-False Review:

(a): TRUE. We have

$$f(tx, ty) = \frac{3(yt)^2 - 5(xt)(yt)}{2(xt)(yt) + (yt)^2} = \frac{3y^2t^2 - 5xyt^2}{2xyt^2 + y^2t^2} = \frac{3y^2 - 5xy}{2xy + y^2} = f(x, y),$$

so f is homogeneous of degree zero.

(b): FALSE. We have

$$f(tx,ty) = \frac{(yt)^2 + xt}{(xt)^2 + 2(yt)^2} = \frac{y^2t^2 + xt}{x^2t^2 + 2y^2t^2} = \frac{y^2t + x}{x^2t + 2y^2t} \neq \frac{y^2 + x}{x^2 + 2y^2},$$

so f is not homogeneous of degree zero.

(c): FALSE. Setting $f(x,y) = \frac{x^3+xy^2}{y^3+1}$, we have

$$f(tx,ty) = \frac{(xt)^3 + (xt)(yt)^2}{(yt)^3 + 1} = \frac{x^3t^3 + xy^2t^3}{y^3t^3 + 1} \neq f(x,y),$$

so f is not homogeneous of degree zero. Therefore, the differential equation is not homogeneous.

(d): TRUE. Setting $f(x,y) = \frac{x^4y^{-2}}{x^2+y^2}$, we have

$$f(tx,ty) = \frac{(xt)^4(yt)^{-2}}{(xt)^2 + (yt)^2} = \frac{x^4y^{-2}t^2}{x^2t^2 + y^2t^2} = \frac{x^4y^{-2}}{x^2 + y^2} = f(x,y).$$

Therefore, f is homogeneous of degree zero, and therefore, the differential equation is homogeneous.

(e): TRUE. This is verified in the calculation leading to Theorem 1.8.5.

(f): TRUE. This is verified in the calculation leading to (1.8.12).

(g): TRUE. We can rewrite the equation as

$$y' - \sqrt{x}y = \sqrt{x}y^{1/2},$$

which is the proper form for a Bernoulli equation, with $p(x) = -\sqrt{x}$, $q(x) = \sqrt{x}$, and $n = 1/2$.

(h): FALSE. The presence of an exponential e^{xy} involving y prohibits this equation from having the proper form for a Bernoulli equation.

(i): TRUE. After dividing the differential equation through by y, it becomes $\dfrac{dy}{dx} + xy = x^2y^{2/3}$, which is a Bernoulli equation with $p(x) = x$, $q(x) = x^2$, and $n = 2/3$.

Unless otherwise indicated in this section $v = \dfrac{y}{x}$, $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ and $t > 0$.

Problems:

1. $f(tx,ty) = \dfrac{5(xt) + 2(yt)}{9(xt) - 4(yt)} = \dfrac{t(5x + 2y)}{t(9x - 4y)} = \dfrac{5x + 2y}{9x - 4y} = f(x,y)$. Thus, f is homogeneous of degree zero. $f(x,y) = \dfrac{5 + 2\frac{y}{x}}{9 - 4\frac{y}{x}} = \dfrac{5 + 2v}{9 - 4v} = F(v)$.

2. $f(tx,ty) = 2(xt) - 5(yt) = t(2x - 5y) \neq f(x,y)$. Thus, f is not homogeneous of degree zero.

3. $f(tx,ty) = \dfrac{(tx)\sin\left(\frac{tx}{ty}\right) - (ty)\cos\left(\frac{ty}{tx}\right)}{\frac{y}{x}} = \dfrac{x\sin\frac{x}{y} - y\cos\frac{y}{x}}{y} = f(x,y)$. Thus f is homogeneous of degree zero. $f(x,y) = \dfrac{\sin\frac{x}{y} - \frac{y}{x}\cos\frac{y}{x}}{\frac{y}{x}} = \dfrac{\sin\frac{1}{v} - v\cos v}{v} = F(v)$.

4. $f(tx,ty) = \dfrac{\sqrt{3(tx)^2+5(ty)^2}}{2(tx)+5(ty)} = \dfrac{\sqrt{3x^2+5y^2}}{2x+5y} = f(x,y)$. Thus f is homogeneous of degree zero. $f(x,y) = \dfrac{\sqrt{3x^2+5y^2}}{2x+5y} = \dfrac{\sqrt{3+5(\frac{y}{x})^2}}{2+5(\frac{y}{x})} = \dfrac{\sqrt{3+5V^2}}{2+5V} = F(v)$.

5. $f(tx,ty) = \dfrac{tx+7}{2ty} \neq \dfrac{x+7}{2y}$. Thus f is not homogeneous of degree zero.

6. $f(tx,ty) = \dfrac{tx-2}{2(ty)} + \dfrac{5(ty)+3}{3(ty)} = \dfrac{(3tx-6)+(10ty+6)}{6ty} = \dfrac{t(3x+10y)}{6ty} = \dfrac{3x+10y}{6y} = \dfrac{x-2}{2y} + \dfrac{5y+3}{3y}$. Thus, f is homogeneous of degree zero. We have $f(x,y) = \dfrac{3x+10y}{6y} = \dfrac{x}{2y} + \dfrac{5}{3} = \dfrac{1}{2v} + \dfrac{5}{3} = F(v)$.

7. $f(tx,ty) = \dfrac{\sqrt{(tx)^2+(ty)^2}}{tx} = \dfrac{\sqrt{x^2+y^2}}{x} = f(x,y)$. Thus f is homogeneous of degree zero. $f(x,y) = \dfrac{\sqrt{x^2+y^2}}{x} = \dfrac{|x|\sqrt{1+(\frac{y}{x})^2}}{x} = -\sqrt{1+\left(\dfrac{y}{x}\right)^2} = -\sqrt{1+v^2} = F(v)$.

8. $f(tx,ty) = \dfrac{\sqrt{(tx)^2+4(ty)^2}-(tx)+(ty)}{(tx)+3(ty)} = \dfrac{\sqrt{x^2+4y^2}-x+y}{x+3y} = f(x,y)$. Thus f is homogeneous of degree zero. $f(x,y) = \dfrac{\sqrt{x^2+4y^2}-x+y}{x+3y} = \dfrac{\sqrt{1+4(\frac{y}{x})^2}-1+\frac{y}{x}}{1+3\frac{y}{x}} = \dfrac{\sqrt{1+4v^2}-1+v}{1+3v} = F(v)$.

9. By inspection the differential equation is first-order homogeneous. We therefore let $y = xV$ in which case $y' = xV' + V$. Substituting these results into the given differential equation yields $xV' + V = V^2 + V + 1$, or equivalently, $xV' = V^2 + 1$. Separating the variables and integrating yields

$$\int \frac{1}{V^2+1}\,dV = \int \frac{1}{x}\,dx \Longrightarrow \arctan\left(\frac{y}{x}\right) = \ln|x| + c_1 \Longrightarrow y(x) = \tan(x\ln cx).$$

10. $(3x-2y)\dfrac{dy}{dx} = 3y \Longrightarrow (3-2\dfrac{y}{x})\dfrac{dy}{dx} = 3\dfrac{y}{x} \Longrightarrow (3-2v)\left(v + x\dfrac{dv}{dx}\right) = 3v \Longrightarrow x\dfrac{dv}{dx} = \dfrac{3v}{3-2v} - v \Longrightarrow$ $\int \dfrac{3-2v}{2v^2}dv = \int \dfrac{dx}{x} \Longrightarrow -\dfrac{3}{2v} - \ln|v| = \ln|x| + c_1 \Longrightarrow -\dfrac{3x}{2y} - \ln|\dfrac{y}{x}| = \ln|x| + c_1 \Longrightarrow \ln y = -\dfrac{3x}{2y} + c_2 \Longrightarrow$ $y^2 = ce^{-3x/y}$.

11. $\dfrac{dy}{dx} = \dfrac{(x+y)^2}{2x^2} \Longrightarrow \dfrac{dy}{dx} = \dfrac{1}{2}\left(1+\dfrac{y}{x}\right)^2 \Longrightarrow v + x\dfrac{dv}{dx} = \dfrac{1}{2}(1+v)^2 \Longrightarrow \int \dfrac{dv}{v^2+1} = \int \dfrac{dx}{x} \Longrightarrow \tan^{-1} v = \dfrac{1}{2}\ln|x| + c \Longrightarrow \tan^{-1}\left(\dfrac{y}{x}\right) = \dfrac{1}{2}\ln|x| + c$.

12. $\sin\left(\dfrac{y}{x}\right)\left(x\dfrac{dy}{dx} - y\right) = x\cos\left(\dfrac{y}{x}\right) \Longrightarrow \sin\left(\dfrac{x}{y}\right)\left(\dfrac{dy}{dx} - \dfrac{y}{x}\right) = \cos\left(\dfrac{y}{x}\right) \Longrightarrow \sin v\left(v + x\dfrac{dv}{dx} - v\right) = \cos v \Longrightarrow \sin v\left(x\dfrac{dv}{dx}\right) = \cos v \Longrightarrow \int \dfrac{\sin v}{\cos v}dv = \int \dfrac{dx}{x} \Longrightarrow -\ln|\cos v| = \ln|x| + c_1 \Longrightarrow \left|x\cos\left(\dfrac{y}{x}\right)\right| = c_2 \Longrightarrow$ $y(x) = x\cos^{-1}\left(\dfrac{c}{x}\right)$.

13. $\frac{dy}{dx} = \frac{\sqrt{16x^2-y^2}+y}{x} \Longrightarrow \frac{dy}{dx} = \sqrt{16-(\frac{y}{x})^2}+\frac{y}{x} \Longrightarrow v + x\frac{dv}{dx} = \sqrt{16-v^2}+v \Longrightarrow \int \frac{dv}{\sqrt{16-v^2}} = \int \frac{dx}{x} \Longrightarrow sin^{-1}(\frac{v}{4}) = \ln|x| + c \Longrightarrow \sin^{-1}(\frac{y}{4x}) = \ln|x|+c.$

14. We first rewrite the given differential equation in the equivalent form $y' = \frac{\sqrt{(9x^2+y^2)}+y}{x}$. Factoring out an x^2 from the square root yields $y' = \frac{|x|\sqrt{9+(\frac{y}{x})^2}+y}{x}$. Since we are told to solve the differential equation on the interval $x > 0$ we have $|x| = x$, so that $y' = 9 + (\frac{y}{x})^2 + \frac{y}{x}$, which we recognize as being homogeneous. We therefore let $y = xV$, so that $y' = xV' + V$. Substitution into the preceding differential equation yields $xV' + V = \sqrt{9+V^2}+V$, that is $xV' = \sqrt{9+V^2}$. Separating the variables in this equation we obtain $\frac{1}{\sqrt{9+V^2}}dV = \frac{1}{x}dx$. Integrating we obtain $\ln(V+\sqrt{9+V^2}) = \ln c_1 x$. Exponentiating both sides yields $V + \sqrt{9+V^2} = c_1 x$. Substituting $\frac{y}{x} = V$ and multiplying through by x yields the general solution $y + \sqrt{9x^2+y^2} = c_1 x^2$.

15. The given differential equation can be written in the equivalent form

$$\frac{dy}{dx} = \frac{y(x^2-y^2)}{x(x^2+y^2)},$$

which we recognize as being first order homogeneous. The substitution $y = xv$ yields

$$v + x\frac{dv}{dx} = \frac{v(1-v^2)}{1+v^2} \Longrightarrow x\frac{dv}{dx} = -\frac{2v^3}{1+v^2},$$

so that

$$\int \frac{1+v^2}{v^3}dv = -2\int \frac{dx}{x} \Longrightarrow -\frac{v^{-2}}{2} + \ln|v| = -2\ln|x| + c_1.$$

Consequently,

$$-\frac{x^2}{2y^2} + \ln|xy| = c_1.$$

16. $x\frac{dy}{dx} + y\ln x = y\ln y \Longrightarrow \frac{dy}{dx} = \frac{y}{x}\ln\frac{y}{x} \Longrightarrow v + x\frac{dv}{dx} = v\ln v \Longrightarrow \int \frac{dv}{v(\ln v - 1)} \int \frac{dx}{x} \Longrightarrow \ln|\ln v - 1| = \ln|x| + c_1 \Longrightarrow \frac{\ln\frac{y}{x}-1}{x} = c \Longrightarrow y(x) = xe^{1+cx}.$

17. $\frac{dy}{dx} = \frac{y^2+2xy-2x^2}{x^2-xy+y^2} \Longrightarrow v + x\frac{dv}{dx} = \frac{v^2+2v-2}{1-v+v^2} \Longrightarrow x\frac{dv}{dx} = \frac{-v^3+2v^2+v-2}{v^2-v+1} \Longrightarrow \int \frac{v^2-v+1}{v^3-2v^2-v+2}dv = -\int \frac{dx}{x} \Longrightarrow \int \frac{v^2-v+1}{(v-1)(v+2)(v+1)}dv = -\int \frac{dx}{x} \Longrightarrow \int \left[\frac{1}{v-2} - \frac{1}{2(v-1)} + \frac{1}{2(v+1)}\right] dv = -\int \frac{dx}{x} \Longrightarrow \ln|v-2| - \frac{1}{2}\ln|v-1| + \frac{1}{2}\ln|v+1| = -\ln|x| + c_1 \Longrightarrow \ln\left|\frac{(v-2)^2(v+1)}{v-1}\right| = -2\ln x + c_2 \Longrightarrow \frac{(y-2x)^2(y+x)}{y-x} = c.$

18. $2xydy - (x^2e^{-y^2/x^2} + 2y^2)dx = 0 \Longrightarrow 2\frac{y}{x}\frac{dy}{dx} - \left(e^{-y^2/x^2} + 2\left(\frac{y}{x}\right)^2\right) = 0 \Longrightarrow 2v\left(v + x\frac{dv}{dx}\right) - (e^{-v^2} +$

$2v^2) = 0 \Longrightarrow 2vx\dfrac{dv}{dx} = e^{-v^2} \Longrightarrow \int e^{v^2}(2vdv) = \int \dfrac{dx}{x} \Longrightarrow e^{v^2} = \ln|x| + c_1 \Longrightarrow e^{y^2/x^2} = \ln(cx) \Longrightarrow y^2 = x^2\ln(\ln(cx))$.

19. $x^2\dfrac{dy}{dx} = y^2 + 3xy + x^2 \Longrightarrow \dfrac{dy}{dx} = (\frac{y}{x})^2 + 3\frac{y}{x} + 1 \Longrightarrow v + x\dfrac{dv}{dx} = v^2 + 3v + 1 \Longrightarrow x\dfrac{dv}{dx} = (v+1)^2 \Longrightarrow \int \dfrac{dv}{(v+1)^2} = \int \dfrac{dx}{x} \Longrightarrow -\dfrac{1}{v+1} = \ln|x| + c_1 \Longrightarrow -\dfrac{1}{\frac{y}{x}+1} = \ln|x| + c_1 \Longrightarrow \dfrac{y}{x} = -\dfrac{1}{\ln(cx)} \Longrightarrow y(x) = -x\left[1 + \dfrac{1}{\ln(cx)}\right]$.

20. $\dfrac{dy}{dx} = \dfrac{\sqrt{x^2+y^2} - x}{y} \Longrightarrow \dfrac{dy}{dx} = \dfrac{\sqrt{1+(\frac{y}{x})^2} - 1}{\frac{y}{x}} \Longrightarrow v + x\dfrac{dv}{dx} = \dfrac{\sqrt{1+v^2}-}{v} \Longrightarrow x\dfrac{dv}{dx} = \dfrac{\sqrt{1+v^2} - x}{v} \Longrightarrow \int \dfrac{v}{\sqrt{1+v^2} - 1 - v^2}dv = \int \dfrac{dx}{x} \Longrightarrow \ln|1-u| = \ln|x| + c_1 \Longrightarrow |x(1-u)| = c_2 \Longrightarrow 1 - u = \dfrac{c}{x} \Longrightarrow u^2 = \dfrac{c^2}{x^2} - 2\dfrac{c}{x} + 1 \Longrightarrow v^2 = \dfrac{c^2}{x^2} - 2\dfrac{c}{x} \Longrightarrow y^2 = c^2 - 2cx$.

21. $2x(y+2x)\dfrac{dy}{dx} = y(4x-y) \Longrightarrow 2\left(\dfrac{y}{x} + 2\right)\dfrac{dy}{dx} = \frac{y}{x}(4 - \frac{y}{x}) \Longrightarrow 2(v+2)\left(v + x\dfrac{dv}{dx}\right) = v(4-v) \Longrightarrow 2x\dfrac{dv}{dx} = -\dfrac{3v^2}{v+2} \Longrightarrow 2\int \dfrac{v+2}{v^2}dv = -3\int \dfrac{dx}{x} \Longrightarrow 2\ln|v| - \frac{4}{v} = -3\ln|x| + c_1 \Longrightarrow y^2 = cxe^{4x/y}$.

22. $x\dfrac{dy}{dx} = x\tan(\frac{y}{x}) + y \Longrightarrow v + x\dfrac{dv}{dx} = \tan v + v \Longrightarrow x\dfrac{dv}{dx} = \tan v \Longrightarrow \int \cot v dv = \int \dfrac{dx}{x} \Longrightarrow \ln|\sin v| = \ln|x| + c_1 \Longrightarrow \sin v = cx \Longrightarrow v = \sin^{-1}(cx) \Longrightarrow y(x) = x\sin^{-1}(cx)$.

23. $\dfrac{dy}{dx} = \dfrac{x\sqrt{x^2+y^2} + y^2}{xy} \Longrightarrow \dfrac{dy}{dx} = \sqrt{\left(\dfrac{x}{y}\right)^2 + 1} + \dfrac{y}{x} \Longrightarrow v + x\dfrac{dv}{dx} = \sqrt{(\frac{1}{v})^2 + 1} + v \Longrightarrow \dfrac{dv}{dx} = \sqrt{(\frac{1}{v})^2 + 1} \Longrightarrow \int \dfrac{dv}{\sqrt{(\frac{1}{v})^2}} = \int \dfrac{dx}{x} \Longrightarrow \ln|v + \sqrt{1+v^2}| = \ln|x| + c \Longrightarrow v + \sqrt{1+v^2} = cx \Longrightarrow \frac{y}{x} + \sqrt{1+(\frac{y}{x})^2} = xc \Longrightarrow 2(\frac{y}{x}) + 1 = (cx)^2 \Longrightarrow y^2 = x^2\dfrac{[(cx)^2 - 1]}{2}$.

24. The given differential equation can be written as $(x-4y)dy = (4x+y)dx$. Converting to polar coordinates we have $x = r\cos\theta \Longrightarrow dx = \cos\theta dr - r\sin\theta d\theta$, and $y = r\sin\theta dr + r\cos\theta d\theta$. Substituting these results into the preceding differential equation and simplifying yields the separable equation $4r^{-1}dr = d\theta$ which can be integrated directly to yield $4\ln r = \theta + c$, so that $r = c_1e^{\theta/4}$.

25. $\dfrac{dy}{dx} = \dfrac{2(2y-x)}{x+y}$. Since $x = 0$, divide the numerator and denominator by y yields $\dfrac{dy}{dx} = \dfrac{\frac{x}{y} + 1}{2(2 - \frac{x}{y})}$. Now let $v = \frac{x}{y}$ so that $\dfrac{dx}{dy} = v + y\dfrac{dv}{dy} \Longrightarrow v + y\dfrac{dv}{dy} = \dfrac{v+1}{2(2-v)} \Longrightarrow \int \dfrac{2(2-v)}{2v^2 - 3v + 1}dv = \int \dfrac{dy}{y} \Longrightarrow -6\int \dfrac{dv}{2v-1} + 2\int \dfrac{dv}{v-1} = \ln|y| + c_1 \Longrightarrow \ln\dfrac{(v-1)^2}{|2v-1|^3} = \ln(c_2|y|) \Longrightarrow (x-y)^2 = c(y-2x)^3$. Since $y(0) = 2$ then $c = \dfrac{1}{2}$. Thus, $(x-y)^2 = \dfrac{1}{2}(y-2x)^3$.

26. $\dfrac{dy}{dx} = \dfrac{2x-y}{x+4y} \Longrightarrow \dfrac{dy}{dx} = \dfrac{2-\frac{y}{x}}{1+4\frac{y}{x}} \Longrightarrow v+x\dfrac{dv}{dx} = \dfrac{2-v}{1+4v} \Longrightarrow x\dfrac{dv}{dx} = \dfrac{2-2v-4v^2}{1+4v} \Longrightarrow \dfrac{1}{2}\int \dfrac{1+4v}{2v^2+v-1}dv = -\int \dfrac{dx}{x} \Longrightarrow \dfrac{1}{2}\ln|2v^2+v-1| = -\ln|x| + c \Longrightarrow \dfrac{1}{2}\ln|x^2(2v^2+v-1)| = c \Longrightarrow \dfrac{1}{2}\ln|2y^2+yx-x^2| = c$, but $y(1)=1$ so $c = \dfrac{1}{2}\ln 2$. Thus $\dfrac{1}{2}\ln|2y^2+yx-x^2| = \dfrac{1}{2}\ln 2$ and since $y(1)=1$ it must be the case that $2y^2+yx-x^2=2$.

27. $\dfrac{dy}{dx} = \dfrac{y-\sqrt{x^2+y^2}}{x} \Longrightarrow \dfrac{dy}{dx} = \dfrac{y}{x} - \sqrt{1+(\dfrac{y}{x})^2} \Longrightarrow x\dfrac{dv}{dx} = -\sqrt{1+v^2} \Longrightarrow \int \dfrac{dv}{\sqrt{1+v^2}} = -\int \dfrac{dx}{x} \Longrightarrow \ln(v+\sqrt{1+v^2}) = -\ln|x| + c_1 \Longrightarrow y\dfrac{|x|}{x} + \sqrt{x^2+y^2} = c_2$. Since $y(3)=4$ then $c_2=9$. Then take $\dfrac{|x|}{x}=1$ since we must have $y(3)=4$; thus $y+\sqrt{x^2+y^2}=9$.

28. $\dfrac{dy}{dx} - \dfrac{y}{x} = \sqrt{4-\left(\dfrac{y}{x}\right)^2} \Longrightarrow v + x\dfrac{dv}{dx} = v + \sqrt{4-v^2} \Longrightarrow \int \dfrac{dv}{\sqrt{4-v^2}} = \int \dfrac{dx}{x} \Longrightarrow \sin^{-1}\dfrac{v}{2} = \ln|x| + c \Longrightarrow \sin^{-1}\dfrac{y}{2x} = \ln x + c$ since $x>0$.

29. (a). $\dfrac{dy}{dx} = \dfrac{x+ay}{ax-y}$. Substituting $y = xv$ and simplifying yields $x\dfrac{dv}{dx} = \dfrac{1+v^2}{a-v}$. Separating the variables and integrating we obtain $a\tan^{-1}v - \dfrac{1}{2}\ln(1+v^2) = \ln x + \ln c$ or equivalently, $a\tan^{-1}\frac{y}{x} - \frac{1}{2}\ln(x^2+y^2) = \ln c$. Substituting for $x = r\cos\theta, y = r\sin\theta$ yields $a\theta - \ln r = \ln c$. Exponentiating then gives $r = ke^{a\theta}$.

(b). The initial condition $y(1)=1$ corresponds to $r(\frac{\pi}{4}) = \sqrt{2}$. Imposing this condition on the polar form of the solution obtained in (a) yields $k = \sqrt{2}e^{-\pi/8}$. Hence, the solution to the initial value problem is $r = \sqrt{2}e^{(\theta-\pi/4)/2}$. When $a = \dfrac{1}{2}$, the differential equation is $\dfrac{dy}{dx} = \dfrac{2x+y}{x-2y}$. Consequently every solution curve has a vertical tangent line at points of intersection with the line $y = \dfrac{x}{2}$. The maximum interval of existence for the solution of the initial value problem can be obtained by determining where $y = \dfrac{x}{2}$ intersects the curve $r = \sqrt{2}e^{(\theta-\pi/4)/2}$. The line $y = \dfrac{x}{2}$ has a polar equation $\tan\theta = \dfrac{1}{2}$. The corresponding values of θ are $\theta = \theta_1 = \tan^{-1}\dfrac{1}{2} \approx 0.464, \theta = \theta_2 = \theta_1 + \pi \approx 3.61$. Consequently, the x-coordinates of the intersection points are $x_1 = r\cos\theta_1 = \sqrt{2}e^{(\theta_1-\pi/4)/2}\cos\theta_1 \approx 1.08, x_2 = r\cos\theta_2 = \sqrt{2}e^{(\theta_2-\pi/4)/2}\cos\theta_2 \approx -5.18$. Hence the maximum interval of existence for the solution is approximately $(-5.18, 1.08)$.

(c). See the accompanying figure.

30. Given family of curves satisfies: $x^2+y^2 = 2cy \Longrightarrow c\dfrac{x^2+y^2}{2y}$. Hence $2x + 2y\dfrac{dy}{dx} = 2c\dfrac{dy}{dx} \Longrightarrow \dfrac{dy}{dx} = \dfrac{x}{c-y} = \dfrac{2xy}{x^2-y^2}$. Orthogonal trajectories satisfies: $\dfrac{dy}{dx} = \dfrac{y^2-x^2}{2xy}$. Let $y = vx$ so that $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$. Substituting these results into the last equation yields $x\dfrac{dv}{dx} = -\dfrac{v^2+1}{2v} \Longrightarrow \ln|v^1+1| = -\ln|x| + c_1 \Longrightarrow \dfrac{y^2}{x^2} + 1 = \dfrac{c_2}{x} \Longrightarrow x^2+y^2 = 2kx$.

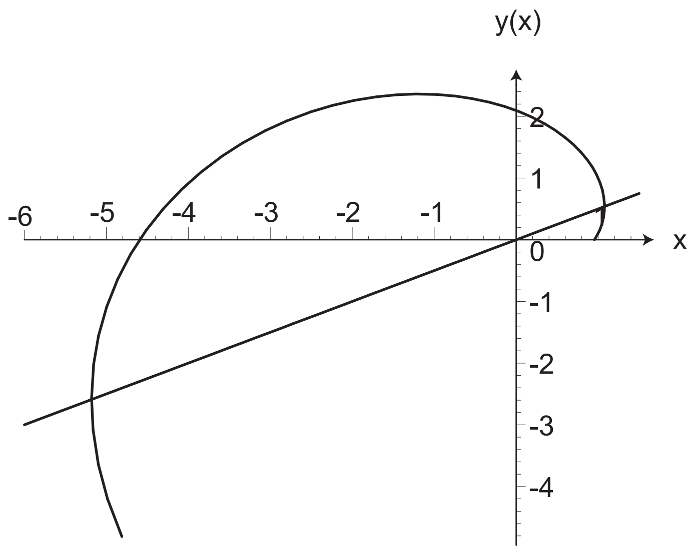

Figure 0.0.50: Figure for Problem 29(c)

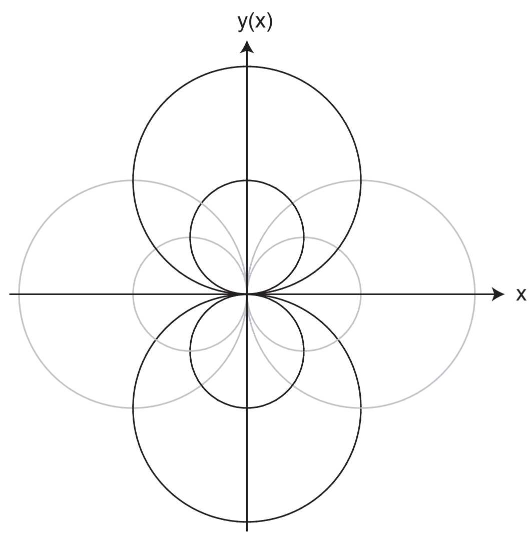

Figure 0.0.51: Figure for Problem 30

31. Given family of curves satisfies: $(x-c)^2+(y-c)^2 = 2c^2 \Longrightarrow c = \dfrac{x^2+y^2}{2(x+y)}$. Hence $2(x-c)+2(y-c)\dfrac{dy}{dx} = 0 \Longrightarrow \dfrac{c-x}{y-c} = \dfrac{y^2-2xy-x^2}{y^2+2xy-x^2}$. Orthogonal trajectories satisfies: $\dfrac{dy}{dx} = \dfrac{y^2+2xy-x^2}{x^2+2xy-y^2}$. Let $y = vx$ so that $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$. Substituting these results into the last equation yields $v + x\dfrac{dv}{dx} = \dfrac{v^2+2v-1}{1+2v-v^2} \Longrightarrow$

$\dfrac{1+2v-v^2}{v^3-v^2+v-1}dv = \dfrac{1}{x}dx \Longrightarrow \left(\dfrac{1}{v-1} - \dfrac{2v}{v^2+1}\right) dv = \dfrac{1}{x}dx \Longrightarrow x^2+y^2 = 2k(x-y) \Longrightarrow (x-k)^2+(y+k)^2 = 2k^2$.

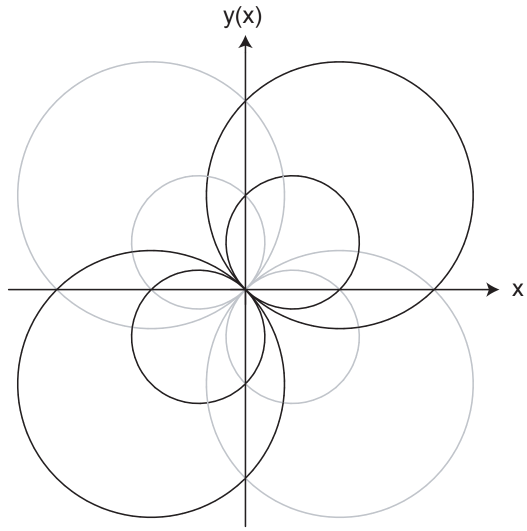

Figure 0.0.52: Figure for Problem 31

32. (a). Let r represent the radius of one of the circles with center at $(a.ma)$ and passing through $(0,0)$. $r = \sqrt{(a-0)^2+(ma-0)^2} = |a|\sqrt{1+m^2}$. Thus, the circle's equation can be written as $(x-a)^2+(y-ma)^2 = (|a|\sqrt{1+m^2})^2$ or $(x-a)^2+(y-ma)^2 = a^2(1+m^2)$.

(b). $(x-a)^2+(y-ma)^2 = a^2(1+m^2) \Longrightarrow a = \dfrac{x^2+y^2}{2(x+my)}$. Differentiating the first equation with respect x and solving we obtain $\dfrac{dy}{dx} = \dfrac{a-x}{y-ma}$. Substituting for a and simplifying yields $\dfrac{dy}{dx} = \dfrac{y^2-x^2-2mxy}{my^2-mx^2+2xy}$. Orthogonal trajectories satisfies: $\dfrac{dy}{dx} = \dfrac{mx^2-my^2-2xy}{y^2-x^2-2mxy} \Longrightarrow \dfrac{dy}{dx} = \dfrac{m-m(\frac{y}{x})^2-2\frac{y}{x}}{(\frac{y}{x})^2-1-2m\frac{y}{x}}$. Let $y = vx$ so that $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$. Substituting these results into the last equation yields $v + x\dfrac{dv}{dx} = \dfrac{m-mv^2-2v}{v^2-1-2mv} \Longrightarrow \dfrac{xdv}{dx} = \dfrac{(m-v)(1+v^2)}{v^2-2mv-1} \Longrightarrow \int \dfrac{v^2-2mv-1}{(m-v)(1+v^2)}dv = \int \dfrac{dx}{x} \Longrightarrow \int \dfrac{dv}{v-m} - \int \dfrac{2v}{1+v^2}dv = \int \dfrac{dx}{x} \Longrightarrow \ln|v-m| - \ln(1+v^2) = \ln|x| + c_1 \Longrightarrow v-m = c_2x(1+v^2) \Longrightarrow y-mx = c_2x^2+c_2y^2 \Longrightarrow x^2+y^2+cmx-cy = 0$. Completing the square we obtain $(x+cm/2)^2+(y-c/2)^2 = c^2/4(m^2+1)$. Now letting $b = c/2$, the last equation becomes $(x+bm)^2+(y-b)^2 = b^2(m^2+1)$ which is a family or circles lying on the line $y = -my$ and passing through the origin.

(c). See the accompanying figure.

33. $x^2+y^2 = c \Longrightarrow \dfrac{dy}{dx} = -\dfrac{x}{y} = m_2$. $m_1 = \dfrac{m_2 - \tan(\frac{\pi}{4})}{1+m_2\tan(\frac{\pi}{4})} = \dfrac{-x/y-1}{1-x/y} = \dfrac{x+y}{x-y}$. Let $y = vx$ so that

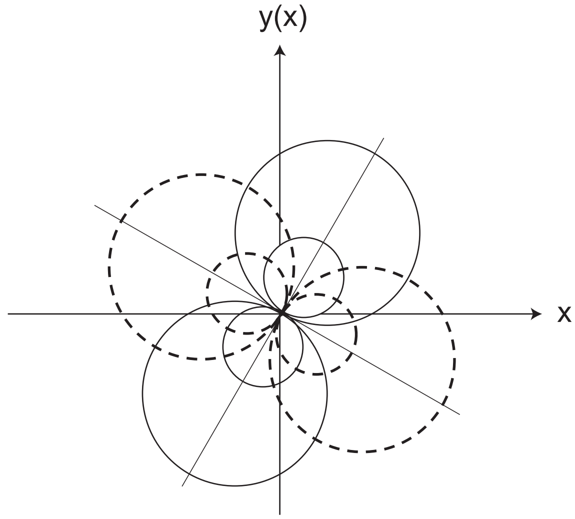

Figure 0.0.53: Figure for Problem 32(c)

$\frac{dy}{dx} = v + x\frac{dv}{dx}$. Substituting these results into the last equation yields $v + x\frac{dv}{dx} = \frac{1+v}{1-v} \Longrightarrow \frac{1-v}{1+v^2}dv = \frac{dx}{x} \Longrightarrow \int\left(-\frac{v}{1+v^2} + \frac{1}{v^2+1}\right)dv = \int\frac{dx}{x} \Longrightarrow -\frac{1}{2}\ln(1+v^2) + \tan^{-1}v = \ln|x| + c_1 \Longrightarrow$ Oblique trajectories: $\ln(x^2+y^2) - 2\tan^{-1}(y/x) = c_2$.

34. $y = cx^6 \Longrightarrow \frac{dy}{dx} = 6y/x = m_2$. $m_1 = \frac{m_2 - \tan\left(\frac{\pi}{4}\right)}{1 + m_2\tan\left(\frac{\pi}{4}\right)} = \frac{6y/x - 1}{1 + 6y/x} = \frac{6y-x}{6y+x}$. Let $y = vx$ so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$. Substitute these results into the last equation yields $v + x\frac{dv}{dx} = \frac{6v-1}{6v+1} \Longrightarrow x\frac{dv}{dx} = \frac{(3v-1)(1-2v)}{6v+1} \Longrightarrow \int\left(\frac{9}{3v-1} - \frac{8}{2v-1}\right)dv = \int\frac{dx}{x} \Longrightarrow 3\ln|3v-1| - 4\ln|2v-1| = \ln|x| + c_1 \Longrightarrow$ Oblique trajectories $(3y-x)^3 = k(2y-x)^4$.

35. $x^2 + y^2 = 2cx \Longrightarrow c = \frac{x^2+y^2}{2x}$ and $\frac{dy}{dx} = \frac{y^2-x^2}{2xy} = m_2$. $m_1 = \frac{m_2 - \tan\left(\frac{\pi}{4}\right)}{1 + m_2\tan\left(\frac{\pi}{4}\right)} = \frac{\frac{y^2-x^2}{2xy} - 1}{1 + \frac{y^2-x^2}{2xy}} = \frac{y^2-x^2-2xy}{y^2-x^2+2xy}$. Let $y = vx$ so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$. Substituting these results into the last equation yields $v + x\frac{dv}{dx} = \frac{v^2-2v-1}{v^2+2v-1} \Longrightarrow x\frac{dv}{dx} = \frac{-v^3-v^2-v-1}{v^2+2v-1} \Longrightarrow \frac{-v^2-2v+1}{v^3+v^2+v+1}dv = \frac{dx}{x} \Longrightarrow \int\left(\frac{1}{v+1} - \frac{2v}{v^2+1}\right)dv = \int\frac{dx}{x} \Longrightarrow \ln|v+1| - \ln(v^2+1) = \ln|x| + c_1 \Longrightarrow \ln|y+x| = \ln|y^2+x^2| + c_1 \Longrightarrow$ Oblique trajectories: $x^2 + y^2 = 2k(x+y)$ or, equivalently, $(x-k)^2 + (y-k)^2 = 2k^2$.

36. **(a).** $y = cx^{-1} \Longrightarrow \dfrac{dy}{dx} = -cx^{-2} = -\dfrac{y}{x}$. $m_1 = \dfrac{m_2 - \tan\alpha_0}{1 + m_2 \tan\alpha_0} = \dfrac{-y/x - \tan\alpha_0}{1 - y/x \tan\alpha_0}$. Let $y = vx$ so that $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$. Substituting these results into the last equation yields $v + x\dfrac{dv}{dx} = \dfrac{\tan\alpha_0 + v}{v\tan\alpha_0 - 1} \Longrightarrow \dfrac{2v\tan\alpha_0 - 2}{v^2\tan\alpha_0 - 2v - \tan\alpha_0}dv = -\dfrac{2dx}{x} \Longrightarrow \ln|v^2\tan\alpha_0 - 2v - \tan\alpha_0| = -2\ln|x| + c_1 \Longrightarrow (y^2 - x^2)\tan\alpha_0 - 2xy = k$.

(b). See the accompanying figure.

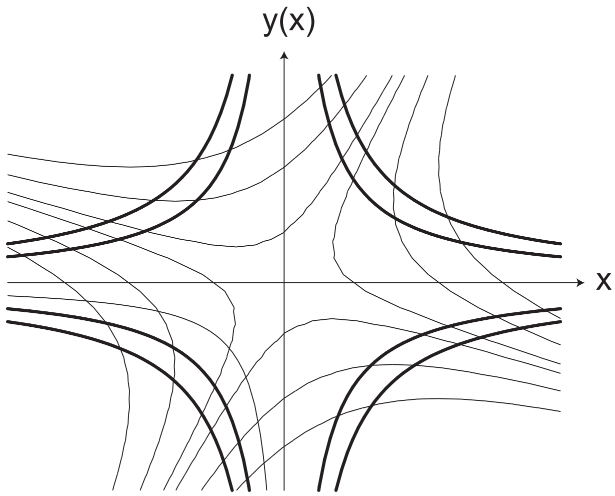

Figure 0.0.54: Figure for Problem 36(b)

37. **(a).** $x^2 + y^2 = c \Longrightarrow \dfrac{dy}{dx} = -\dfrac{x}{y}$. $m_1 = \dfrac{m_2 - \tan\alpha_0}{1 + m_2\tan\alpha_0} = \dfrac{-x/y - m}{1 - (x/y)m} = \dfrac{x + my}{mx - y}$. Let $y = vx$ so that $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$. Substituting these results into the last equation yields $v + x\dfrac{dv}{dx} = \dfrac{1 + mv}{m - v} \Longrightarrow x\dfrac{dv}{dx} = \dfrac{1 + v^2}{m - v} \Longrightarrow \dfrac{v - m}{1 + v^2}dv = -\dfrac{dx}{x} \Longrightarrow \int\left(\dfrac{v}{1 + v^2} - \dfrac{m}{1 + v^2}\right)dv = -\int\dfrac{dx}{x} \Longrightarrow \dfrac{1}{2}\ln(1 + v^2) - m\tan v = -\ln|x| + c_1$. In polar coordinates, $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1} y/x$, so this result becomes $\ln r - m\theta = c_1 \Longrightarrow r = e^{m\theta}$ where k is an arbitrary constant.

(b). See the accompanying figure.

38. $\dfrac{dy}{dx} - \dfrac{1}{x}y = 4x^2y^{-1}\cos x$. This is a Bernoulli equation. Multiplying both sides y results in $y\dfrac{dy}{dx} - \dfrac{1}{x}y^2 = 4x^2\cos x$. Let $u = y^2$ so $\dfrac{du}{dx} = 2y\dfrac{dy}{dx}$ or $y\dfrac{dy}{dx} = \dfrac{1}{2}\dfrac{du}{dx}$. Substituting these results into $y\dfrac{dy}{dx} - \dfrac{1}{x}y^2 = 4x^2\cos x$ yields $\dfrac{du}{dx} - \dfrac{2}{x}u = 8x^2\cos x$ which has an integrating factor $I(x) = x^{-2} \Longrightarrow \dfrac{d}{dx}(x^{-2}u) = 8\cos x \Longrightarrow x^{-2}u = 8\int\cos x dx + c \Longrightarrow x^{-2}u = 8\sin x + c \Longrightarrow u = x^2(8\sin x + c) \Longrightarrow y^2 = x^2(8\sin x + c)$.

39. $y^{-3}\dfrac{dy}{dx} + \dfrac{1}{2}y^{-2}\tan x = 2\sin x$. This is a Bernoulli equation. Let $u = y^{-2}$ so $\dfrac{du}{dx} = -2y^{-3}\dfrac{dy}{dx}$ or

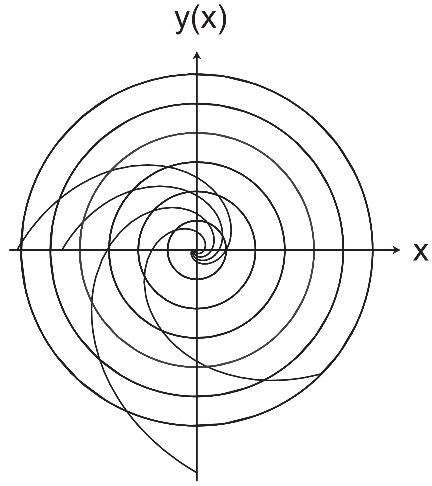

Figure 0.0.55: Figure for Problem 37(b)

$y^{-3}\dfrac{dy}{dx} = \dfrac{1}{2}\dfrac{du}{dx}$. Substituting these results into the last equation yields $\dfrac{du}{dx} - u\tan x = -4\sin x$. An integrating factor for this equation is $I(x) = \cos x$. Thus, $\dfrac{d}{dx}(u\cos x) = -4\cos x\sin x \Longrightarrow u\cos x = 4\int \cos x\sin x dx \Longrightarrow u(x) = \dfrac{1}{\cos x}(\cos^2 x + c) \Longrightarrow y^{-2} = 2\cos x + \dfrac{c}{\cos x}$.

40. $\dfrac{dy}{dx} - \dfrac{3}{2x}y = 6y^{1/3}x^2\ln x$ or $\dfrac{1}{y^{1/3}}\dfrac{dy}{dx} - \dfrac{3}{2x}y^{2/3} = 6x^2\ln x$. Let $u = y^{2/3} \Longrightarrow \dfrac{dy}{dx} = \dfrac{2}{3}y^{-1/3}\dfrac{dy}{dx}$. Substituting these results into $\dfrac{1}{y^{1/3}}\dfrac{dy}{dx} - \dfrac{3}{2x}y^{2/3} = 6x^2\ln x$ yields $\dfrac{du}{dx} - \dfrac{1}{x}u = 4x^2\ln x$. An integrating factor for this equation is $I(x) = \dfrac{1}{x}$ so $\dfrac{d}{dx}(x^{-1}u) = 4x\ln x \Longrightarrow x^{-1}u = 4\int x\ln x dx + c \Longrightarrow x^{-1}u = 2x^2\ln x - x^2 + c \Longrightarrow u(x) = x(2x^2\ln x - x^2 + c) \Longrightarrow y^{2/3} = x(2x^2\ln x - x^2 + c)$.

41. $\dfrac{dy}{dx} + \dfrac{2}{x}y = 6\sqrt{1+x^2}y^{1/2}$ or $y^{-1/2}\dfrac{dy}{dx} + \dfrac{2}{x}y^{1/2} = 6\sqrt{1+x^2}$. Let $u = y^{1/2} \Longrightarrow 2\dfrac{du}{dx} = y^{-1/2}\dfrac{dy}{dx}$. Substituting these results into $y^{-1/2}\dfrac{dy}{dx} + \dfrac{2}{x}y^{1/2} = 6\sqrt{1+x^2}$ yields $\dfrac{du}{dx} + \dfrac{1}{x}u = 3\sqrt{1+x^2}$. An integrating factor for this equation is $I(x) = x$ so $\dfrac{d}{dx}(xu) = 3x\sqrt{1+x^2} \Longrightarrow xu = \int x\sqrt{1+x^2}dx + c \Longrightarrow xu = (1+x^2)^{3/2} + c \Longrightarrow u = \dfrac{1}{x}(1+x^2)^{3/2} + \dfrac{c}{x} \Longrightarrow y^{1/2} = \dfrac{1}{x}(1+x^2)^{3/2} + \dfrac{c}{x}$.

42. $\dfrac{dy}{dx} + \dfrac{2}{x}y = 6y^2x^4$ or $y^{-2}\dfrac{dy}{dx} + \dfrac{2}{x}y^{-1} = 6x^4$. Let $u = y^{-1} \Longrightarrow -\dfrac{du}{dx} = y^{-2}\dfrac{dy}{dx}$. Substituting these results into $y^{-2}\dfrac{dy}{dx} + \dfrac{2}{x}y^{-1} = 6x^4$ yields $\dfrac{du}{dx} - \dfrac{2}{x}u = -6x^4$. An integrating factor for this equation is $I(x) = x^{-2}$ so $\dfrac{d}{dx}(x^{-2}u) = -6x^2 \Longrightarrow x^{-2}u = -2x^3 + c \Longrightarrow u = -2x^5 + cx^2 \Longrightarrow y^{-1} = -2x^5 + cx^2 \Longrightarrow y(x) = \dfrac{1}{x^2(c - 2x^3)}$.

43. $2x\left(\frac{dy}{dx}+y^3x^2\right)+y=0$ or $y^{-3}\frac{dy}{dx}+\frac{1}{2x}y^{-2}=-x^{-2}$. Let $u=y^{-2}\Longrightarrow -\frac{1}{2}\frac{du}{dx}=y^{-3}\frac{dy}{dx}$. Substituting these results into $y^{-3}\frac{dy}{dx}+\frac{1}{2x}y^{-2}=-x^{-2}$ yields $\frac{du}{dx}-\frac{1}{x}u=2x^2$. An integrating factor for this equation is $I(x)=\frac{1}{x}$ so $\frac{d}{dx}(x^{-1}u)=2x\Longrightarrow x^{-1}u=x^2+c\Longrightarrow u=x^3+cx\Longrightarrow y^{-2}=x^3+cx$.

44. $(x-a)(x-b)\left(\frac{dy}{dx}-y^{1/2}\right)=2(b-a)y$ or $y^{-1/2}\frac{dy}{dx}-\frac{2(b-a)}{(x-a)(x-b)}y^{1/2}=1$. Let $u=y^{1/2}\Longrightarrow 2\frac{du}{dx}=y^{-1/2}\frac{dy}{dx}$. Substituting these results into $y^{-1/2}\frac{dy}{dx}-\frac{2(b-a)}{(x-a)(x-b)}y^{1/2}=1$ yields $\frac{du}{dx}-\frac{(b-a)}{(x-1)(x-b)}u=\frac{1}{2}$. An integrating factor for this equation is $I(x)=\frac{x-a}{x-b}$ so $\frac{d}{dx}\left(\frac{x-a}{x-b}u\right)=\frac{x-a}{2(x-b)}\Longrightarrow\frac{x-a}{x-b}u=\frac{1}{2}[x+(b-a)\ln|x-b|+c]\Longrightarrow y^{1/2}=\frac{x-b}{2(x-a)}[x+(b-a)\ln|x-b|+c]\Longrightarrow y(x)=\frac{1}{4}\left(\frac{x-b}{x-a}\right)^2[x+(b-a)\ln|x-b|+c]^2$.

45. $\frac{dy}{dx}+\frac{6}{x}y=3y^{2/3}\frac{\cos x}{x}$ or $y^{-2/3}\frac{dy}{dx}+\frac{6}{x}y^{1/3}=3\frac{\cos x}{x}$. Let $u=y^{1/3}\Longrightarrow 3\frac{du}{dx}=y^{-2/3}\frac{dy}{dx}$. Substituting these results into $y^{-2/3}\frac{dy}{dx}+\frac{6}{x}y^{1/3}=3\frac{\cos x}{x}$ yields $\frac{du}{dx}+\frac{2}{x}u=\frac{\cos x}{x}$. An integrating factor for this equation is $I(x)=x^2$ so $\frac{d}{dx}(x^2u)=x\cos x\Longrightarrow x^2u=\cos x+x\sin x+c\Longrightarrow y^{1/3}=\frac{\cos x+x\sin x+c}{x^2}$.

46. $\frac{dy}{dx}+4xy=4x^3y^{1/2}$ or $y^{-1/2}\frac{dy}{dx}+4xy^{1/2}=4x^3$. Let $u=y^{1/2}\Longrightarrow 2\frac{du}{dx}=y^{-1/2}\frac{dy}{dx}$. Substituting these results into $y^{-1/2}\frac{dy}{dx}+4xy^{1/2}=4x^3$ yields $\frac{du}{dx}+2xu=2x^3$. An integrating factor for this equation is $I(x)=e^{x^2}$ so $\frac{d}{dx}(e^{x^2}u)=2e^{x^2}x^3\Longrightarrow e^{x^2}u=e^{x^2}(x^2-1)+c\Longrightarrow y^{1/2}=x^2-1+ce^{-x^2}\Longrightarrow y(x)=[(x^2-1)+ce^{-x^2}]^2$.

47. $\frac{dy}{dx}-\frac{1}{2x\ln x}=2xy^3$ or $y^{-3}\frac{dy}{dx}-\frac{1}{2x\ln x}y^{-2}=2x$. Let $u=y^{-2}\Longrightarrow\frac{1}{2}\frac{du}{dx}=y^{-3}\frac{dy}{dx}$. Substituting these results into $y^{-3}\frac{dy}{dx}-\frac{1}{2x\ln x}y^{-2}=2x$ yields $\frac{du}{dx}+\frac{1}{x\ln x}u=-4x$. An integrating factor for this equation is $I(x)=\ln x$ so $\frac{d}{dx}(u\ln x)=-4x\ln x\Longrightarrow u\ln x=x^2-2x^2\ln x+c\Longrightarrow y^2=\frac{\ln x}{x^2(1-2\ln x)+c}$.

48. $\frac{dy}{dx}-\frac{1}{(\pi-1)x}y=\frac{3}{(1-\pi)}xy^{\pi}$ or $y^{-\pi}\frac{dy}{dx}-\frac{1}{(\pi-1)x}y^{1-\pi}=\frac{3x}{1-\pi}$. Let $u=y^{1-\pi}\Longrightarrow\frac{1}{1-\pi}\frac{du}{dx}=y^{-\pi}\frac{dy}{dx}$. Substituting these results into $y^{-\pi}\frac{dy}{dx}-\frac{1}{(\pi-1)x}y^{1-\pi}=\frac{3x}{1-\pi}$ yields $\frac{du}{dx}+\frac{1}{x}u=3x$. An integrating factor for this equation is $I(x)=x$ so $\frac{d}{dx}(xu)=3x^2\Longrightarrow xu=x^3+c\Longrightarrow y^{1-\pi}=\frac{x^3+c}{x}\Longrightarrow y(x)=\left(\frac{x^3+c}{x}\right)^{1/(1-\pi)}$.

49. $2\frac{dy}{dx}+y\cot x=8y^{-1}\cos^3 x$ or $2y\frac{dy}{dx}+y^2\cot x=8\cos^2 x$. Let $u=y^2\Longrightarrow\frac{du}{dx}=2y\frac{dy}{dx}$. Substituting these results into $2y\frac{dy}{dx}+y^2\cot x=8\cos^2 x$ yields $\frac{du}{dx}+u\sec x=\sec x$. An integrating factor for this equation is $I(x)=\sin x$ so $\frac{d}{dx}(u\sin x)=8\cos^3 x\sin x\Longrightarrow u\sin x=-2\cos^4 x+c\Longrightarrow y^2=\frac{-2\cos^4 x+c}{\sin x}$.

50. $(1-\sqrt{3})\frac{dy}{dx}+y\sec x = y^{\sqrt{3}}\sec x$ or $(1-\sqrt{3})y^{-\sqrt{3}}\frac{dy}{dx}+y^{1-\sqrt{3}}\sec x = \sec x$. Let $u = y^{1-\sqrt{3}} \Longrightarrow \frac{du}{dx} = (1-\sqrt{3})y^{-\sqrt{3}}\frac{dy}{dx}$. Substituting these results into $(1-\sqrt{3})y^{-\sqrt{3}}\frac{dy}{dx}+y^{1-\sqrt{3}}\sec x = \sec x$ yields $\frac{du}{dx}+u\sec x = \sec x$. An integrating factor for this equation is $I(x) = \sec x+\tan x$ so $\frac{d}{dx}[(\sec x+\tan x)u] = \sec x(\sec x+\tan x) \Longrightarrow (\sec x+\tan x)u = \tan x+\sec x+c \Longrightarrow y^{1-\sqrt{3}} = 1+\frac{1}{\sec x+\tan x} \Longrightarrow y(x) = \left(1+\frac{c}{\sec x+\tan x}\right)^{1/(1-\sqrt{3})}$.

51. $\frac{dy}{dx}+\frac{2x}{1+x^2}y = xy^2$ or $\frac{1}{y^2}\frac{dy}{dx}+\frac{2x}{1+x^2}\frac{1}{y} = x$. Let $u = y^{-1}$ so $\frac{du}{dx} = -y^{-2}\frac{dy}{dx}$. Substituting these results into $\frac{1}{y^2}\frac{dy}{dx}+\frac{2x}{1+x^2}\frac{1}{y} = x$ yields $\frac{du}{dx}-\frac{2x}{1+x^2}u = -x$. An integrating factor for this equation is $I(x) = \frac{1}{1+x^2}$ so $\frac{d}{dx}\left(\frac{u}{1+x^2}\right) = -\frac{x}{1+x^2} \Longrightarrow \frac{u}{1+x^2} = -\int\frac{xdx}{1+x^2}+c \Longrightarrow \frac{u}{1+x^2} = -\frac{1}{2}\ln(1+x^2)+c \Longrightarrow u = (1+x^2)\left(-\frac{1}{2}\ln(1+x^2)+c\right) \Longrightarrow y^{-1} = (1+x^2)\left(-\frac{1}{2}\ln(1+x^2)+c\right)$. Since $y(0) = 1 \Longrightarrow c = 1$ so $\frac{1}{y} = (1+x^2)\left(-\frac{1}{2}\ln(1+x^2)+1\right)$.

52. $\frac{dy}{dx}+y\cot x = y^3\sin^3 x$ or $y^{-3}\frac{dy}{dx}+y^{-2}\cot x = \sin^3 x$. Let $u = y^{-2} \Longrightarrow -\frac{1}{2}\frac{du}{dx} = y^{-3}\frac{dy}{dx}$. Substituting these results into $y^{-3}\frac{dy}{dx}+y^{-2}\cot x = \sin^3 x$ yields $\frac{du}{dx}-2u\cot x = -2\sin^3 x$. An integrating factor for this equation is $I(x) = \csc^2 x$ so $\frac{d}{dx}(u\csc^2 x) = -\sin x \Longrightarrow u\csc^2 x = 2\cos x+c$. Since $y(\pi/2) = 1 \Longrightarrow c = 1$. Thus $y^2 = \frac{1}{\sin^2 x(2\cos x+1)}$.

53. $\frac{dy}{dx} = F(ax+by+c)$. Let $v = ax+by+c$ so that $\frac{dv}{dx} = a+b\frac{dy}{dx} \Longrightarrow b\frac{dy}{dx} = \frac{dv}{dx}-a \Longrightarrow \frac{dy}{dx} = \frac{1}{b}\left(\frac{dv}{dx}-a\right) = F(v) \Longrightarrow \frac{dv}{dx}-a = bF(v) \Longrightarrow \frac{dv}{dx} = bF(v)+a \Longrightarrow \frac{dv}{bf(v)+a} = dx$.

54. $\frac{dy}{dx} = (9x-y)^2$. Let $v = 9x-y$ so that $\frac{dy}{dx} = 9-\frac{dv}{dx} \Longrightarrow \frac{dv}{dx} = 9-v^2 \Longrightarrow \int\frac{dv}{9-v^2} = \int dx \Longrightarrow \frac{1}{3}\tanh^{-1}(v/3) = x+c_1$ but $y(0) = 0$ so $c = 0$. Thus, $\tanh^{-1}(3x-y/3) = 3x$ or $y(x) = 3(3x-\tanh 3x)$.

55. $\frac{dy}{dx} = (4x+y+2)^2$. Let $v = 4x+y+2$ so that $\frac{dv}{dx} = 4+\frac{dy}{dx} \Longrightarrow \frac{dv}{v^2+4} = dx \Longrightarrow \int\frac{dv}{v^2+4} = \int dx \Longrightarrow \frac{1}{2}\tan^{-1}v/2 = x+c_1 \Longrightarrow \tan^{-1}(2x+y/2+1) = 2x+c \Longrightarrow y(x) = 2[\tan(2x+c)-2x-1]$.

56. $\frac{dy}{dx} = \sin^2(3x-3y+1)$. Let $v = 3x-3y+1$ so that $\frac{dy}{dx}1-\frac{1}{3}\frac{dv}{dx} \Longrightarrow 1-\frac{1}{3}\frac{dv}{dx} = \sin^2 v \Longrightarrow \frac{dv}{dx} = 3\cos^2 v \Longrightarrow \int\sec^2 v dv = 3\int dx \Longrightarrow \tan v = 3x+c \Longrightarrow \tan(3x-3y+1) = 3x+c \Longrightarrow y(x) = \frac{1}{3}[3x-\tan^{-1}(3x+c)+1]$.

57. $V = xy \Longrightarrow V' = xy' + y \Longrightarrow y' = (V' - y)/x$. Substitution into the differential equation yields $(V' - y)/x = yF(V)/x \Longrightarrow V' = y[F(V) + 1] \Longrightarrow V' = V[F(V) + 1]/x$, so that $\dfrac{1}{V[F(V)+1]}\dfrac{dV}{dx} = \dfrac{1}{x}$.

58. Substituting into $\dfrac{1}{V[F(V)+1]}\dfrac{dV}{dx} = \dfrac{1}{x}$ for $F(V) = \ln V - 1$ yields $\dfrac{1}{V\ln V}dV = \dfrac{1}{x}dx \Longrightarrow \ln\ln V = \ln cx \Longrightarrow V = e^{cx} \Longrightarrow y(x) = \dfrac{1}{x}e^{cx}$.

59. (a). $y(x) = w(x) - x \Longrightarrow y' = w' - 1$. Substituting these results into (1.8.18) yields $w' - 1 = 2xw^2 - 1 \Longrightarrow w' = 2xw^2$.

(b). Separating the variables in the preceding differential equation and integrating yields

$$\int \frac{1}{w^2}\,dw = 2\int x\,dx + c \Longrightarrow -w^{-1} = x^2 + c \Longrightarrow w(x) = \frac{1}{c_1 - x^2},$$

where $c_1 = -c$. Hence, the general solution to (1.8.18) is $y(x) = \dfrac{1}{c_1 - x^2} - x$. Imposing the initial condition $y(0) = 1$ requires that $1 = \dfrac{1}{c_1} \Longrightarrow c_1 = 1$. Therefore, $y(x) = \dfrac{1}{1 - x^2} - x$.

60. (a). $x = u - 1, y = v + 1 \Longrightarrow \dfrac{dy}{dx} = \dfrac{dv}{du}$. Substitution into the given differential equation yields $\dfrac{dv}{du} = \dfrac{u + 2v}{2u - v}$.

(b). The differential equation obtained in (a) is first order homogeneous. We therefore let $W = v/u$, and substitute into the differential equation to obtain $W'u + W = \dfrac{1 + 2W}{2 - W} \Longrightarrow W'u = \dfrac{1 + W^2}{2 - W}$. Separating the variables yields $\left(\dfrac{2}{1 + W^2} - \dfrac{W}{1 + W^2}\right)dW = \dfrac{1}{u}du$. This can be integrated directly to obtain $2\tan^{-1}W - \dfrac{1}{2}\ln(1 + W^2) = \ln u + \ln c$. Simplifying we obtain $cu^2(1 + W^2) = e^{4\tan^{-1}W} \Longrightarrow c(u^2 + v^2) = e^{\tan^{-1}(v/u)}$. Substituting back in for x and y yields $c[(x + 1)^2 + (y - 1)^2] = e^{\tan^{-1}[(y-1)/(x+1)]}$.

61. (a). $y = Y(x) + v^{-1}(x) \Longrightarrow y' = Y'(x) - v^{-2}(x)v'(x)$. Now substitute into the given differential equation and simplify algebraically to obtain $Y'(x) + p(x)Y(x) + q(x)Y^2(x) - v^{-2}(x)v'(x) + v^{-1}(x)p(x) + q(x)[2Y(x)v^{-1}(x) + v^{-2}(x)] = r(x)$. We are told that $Y(x)$ is a particular solution to the given differential equation, and therefore $Y'(x) + p(x)Y(x) + q(x)Y^2(x) = r(x)$. Consequently the transformed differential equation reduces to $-v^{-2}(x)v'(x) + v^{-1}p(x) + q(x)[2Y(x)v^{-1}(x) + v^{-2}(x)] = 0$, or equivalently $v' - [p(x) + 2Y(x)q(x)]v = q(x)$.

(b). The given differential equation can be written as $y' - x^{-1}y - y^2 = x^{-2}$, which is a Riccati differential equation with $p(x) = -x^{-1}, q(x) = -1$, and $r(x) = x^{-2}$. Since $y(x) = -x^{-1}$ is a solution to the given differential equation, we make a substitution $y(x) = -x^{-1} + v^{-1}(x)$. According to the result from part (a), the given differential equation then reduces to $v' - (-x^{-1} + 2x^{-1})v = -1$, or equivalently $v' - x^{-1}v = -1$. This linear differential equation has an integrating factor $I(x) = x^{-1}$, so that v must satisfy $\dfrac{d}{dx}(x^{-1}v) = -x^{-1} \Longrightarrow v(x) = x(c - \ln x)$. Hence the solution to the original equation is $y(x) = -\dfrac{1}{x} + \dfrac{1}{x(c - \ln x)} = \dfrac{1}{x}\left(\dfrac{1}{c - \ln x} - 1\right)$.

62. (a). If $y = ax^r$, then $y' = arx^{r-1}$. Substituting these expressions into the given differential equation yields $arx^{r-1}+2ax^{r-1}-a^2x^{2r}=-2x^{-2}$. For this to hold for all $x>0$, the powers of x must match up on either side of the equation. Hence, $r=-1$. Then a is determined from the quadratic $-a+2a-a^2=-1 \iff a^2-a-2=0 \iff (a-2)(a+1)=0$. Consequently, $a=2,-1$ in order for us to have a solution to the given differential equation. Therefore, two solutions to the differential equation are $y_1(x)=2x^{-1}, y_2(x)=-x^{-1}$.

(b). Taking $Y(x)=2x^{-1}$ and using the result from Problem 61(a), we now substitute $y(x)=2x^{-1}+v^{-1}$ into the given Riccati equation. The result is $(-2x^{-2}-v^{-2}v')+2x^{-1}(2x^{-1}+v^{-1})-(4x^{-2}+4x^{-1}v^{-1}+v^{-2})=-2x^{-2}$. Simplifying this equation yields the linear equation $v'+2x^{-1}v=-1$. Multiplying by the integrating factor $I(x)=e^{\int 2x^{-1}dx}=x^2$ results in the integrable differential equation $\frac{d}{dx}(x^2v)=-x^2$. Integrating this differential equation we obtain $v(x)=x^2\left(-\frac{1}{3}x^3+c\right)=\frac{1}{3}x^2(c_1-x^3)$. Consequently, the general solution to the Riccati equation is $y(x)=\frac{2}{x}+\frac{3}{x^2(c_1-x^3)}$.

63. (a). $y=x^{-1}+w(x) \implies y'=-x^{-2}+w'$. Substituting into the given differential equation yields $(-x^{-2}+w')+7x^{-1}(x^{-1}+w)-3(x^{-2}+2x^{-1}w+w^2)=3x^{-2}$ which simplifies to $w'+x^{-1}w-3w^2=0$.

(b). The preceding equation can be written in the equivalent form $w^{-2}w'+x^{-1}w^{-1}=3$. We let $u=w^{-1}$, so that $u'=-w^{-2}w'$. Substitution into the differential equation gives, after simplification, $u'-x^{-1}u=-3$. An integrating factor for this linear differential equation is $I(x)=x^{-1}$, so that the differential equation can be written in the integrable form $\frac{d}{dx}(x^{-1}u)=-3x^{-1}$. Integrating we obtain $u(x)=x(-3\ln x+c)$, so that $w(x)=\frac{1}{x(c-3\ln x)}$. Consequently the solution to the original Riccati equation is $y(x)=\frac{1}{x}\left(1+\frac{1}{c-3\ln x}\right)$.

64. $y^{-1}\frac{dy}{dx}+p(x)\ln y=q(x)$. If we let $u=\ln y$, then $\frac{du}{dx}=\frac{1}{y}\frac{dy}{dx}$ and the given equation becomes $\frac{du}{dx}+p(x)u=q(x)$ which is a first order linear and has a solution of the form $u=e^{-\int p(x)dx}\left[\int e^{\int p(x)dx}q(x)dx+c\right]$. Substituting $\ln y=e^{-\int p(x)dx}\left[\int e^{\int p(x)dx}q(x)dx+c\right]$ into $u=\ln y$ we obtain $y(x)=e^{I^{-1}[\int I(t)q(t)dt+c]}$ where $I(x)=e^{\int p(t)dt}$ and c is an arbitrary constant.

65. $y^{-1}\frac{dy}{dx}-\frac{2}{x}\ln y=\frac{1-2\ln x}{x}$. Let $u=\ln y$ so using the technique of the preceding problem: $\frac{du}{dx}-\frac{2}{x}u=\frac{1-2\ln x}{x}$ and $u=e^{2\int\frac{dx}{x}}\left[\int e^{-2\int\frac{dx}{x}}\left(\frac{1-2\ln x}{x}\right)dx+c_1\right]=x^2\left[\int\left(\frac{1-2\ln x}{x^3}dx\right)+c_1\right]=\ln x+cx^2$, and since $u=\ln y, \ln y=\ln x+cx^2$. Now $y(1)=e$ so $c=1 \implies y(x)=xe^{x^2}$.

66. If $u=f(y)$, then $\frac{du}{dx}=f'(y)\frac{dy}{dx}$ and the given equation $f'(y)\frac{dy}{dx}+p(x)f(y)=q(x)$ becomes $\frac{du}{dx}+p(x)u=q(x)$ which has a solution of the form $u(x)=e^{-\int p(x)dx}\left[\int e^{\int p(x)dx}q(x)dx+c\right]$. Substituting $f(y)=e^{-\int p(x)dx}\left[\int e^{\int p(x)dx}q(x)dx+c\right]$ into $u=f(y)$ and using the fact that f is invertible, we obtain $y(x)=f^{-1}\left[I^{-1}\left(\int I(t)q(t)dt\right)+c\right]$ where $I(x)=e^{\int p(t)dt}$ and c is and arbitrary constant.

67. $\sec^2 y\frac{dy}{dx} + \frac{1}{2\sqrt{1+x}}\tan y = \frac{1}{2\sqrt{1+x}}$. Let $u = \tan y$ so that $\frac{du}{dx} = \sec^2 y\frac{dy}{dx}$ and the given equation becomes $\frac{du}{dx} + \frac{1}{2\sqrt{1+x}}u = \frac{1}{2\sqrt{1+x}}$ which is first order linear. An integrating factor for this equation is $I(x) = e^{\sqrt{1+x}} \Longrightarrow \frac{d}{dx}(e^{\sqrt{1+x}}u) = \frac{e^{\sqrt{1+x}}}{2\sqrt{1+x}} \Longrightarrow e^{\sqrt{1+x}}u = \int \frac{e^{\sqrt{1+x}}}{2\sqrt{1+x}} \Longrightarrow e^{\sqrt{1+x}}u = e^{\sqrt{1+x}} + c \Longrightarrow u = 1 + ce^{-\sqrt{1+x}}$. But $u = \tan y$ so $\tan y = 1 + ce^{-\sqrt{1+x}}$ or $y(x) = \tan^{-1}(1 + ce^{-\sqrt{1+x}})$.

Solutions to Section 1.9

True-False Review:

(a): FALSE. The requirement, as stated in Theorem 1.9.4, is that $M_y = N_x$, not $M_x = N_y$, as stated.

(b): FALSE. A potential function $\phi(x, y)$ is not an equation. The general solution to an exact differential equation takes the form $\phi(x, y,) = c$, where $\phi(x, y)$ is a potential function.

(c): FALSE. According to Definition 1.9.2, $M(x)dx + N(y)dy = 0$ is only exact if there exists a function $\phi(x, y)$ such that $\phi_x = M$ and $\phi_y = N$ for all (x, y) in a region R of the xy-plane.

(d): TRUE. This is the content of part 1 of Theorem 1.9.11.

(e): FALSE. If $\phi(x, y)$ is a potential function for $M(x, y)dx + N(x, y)dy = 0$, then so is $\phi(x, y) + c$ for any constant c.

(f): TRUE. We have

$$M_y = 2e^{2x} - \cos y \quad \text{and} \quad N_x = 2e^{2x} - \cos y,$$

and so since $M_y = N_x$, this equation is exact.

(g): FALSE. We have

$$M_y = \frac{(x^2 + y)^2(-2x) + 4xy(x^2 + y)}{(x^2 + y)^4}$$

and

$$N_x = \frac{(x^2 + y)^2(2x) - 2x^2(x^2 + y)(2x)}{(x^2 + y)^4}.$$

Thus, $M_y \neq N_x$, and so this equation is not exact.

(h): FALSE. We have

$$M_y = 2y \quad \text{and} \quad N_x = 2y^2,$$

and since $M_y \neq N_x$, we conclude that this equation is not exact.

(i): FALSE. We have

$$M_y = e^{x \sin y}\cos y + xe^{x \sin y}\cos y \quad \text{and} \quad N_x = \cos y \sin y e^{x \sin y},$$

and since $M_y \neq N_x$, we conclude that this equation is not exact.

Problems:

1. $ye^{xy}dx + (2y - xe^{-xy})dy = 0$. $M = ye^{xy}$ and $N = 2y - xe^{-xy} \Longrightarrow M_y = yxe^{xy} + e^{xy}$ and $N_x = xye^{-xy} - e^{-xy} \Longrightarrow M_y \neq N_x \Longrightarrow$ the differential equation is not exact.

2. $[\cos(xy) - xy\sin(xy)]dx - x^2\sin(xy)dy = 0 \Longrightarrow M = \cos(xy) - xy\sin(xy)$ and $N = -x^2\sin(xy) \Longrightarrow M_y = -2x\sin(xy) - x^2y\cos(xy)$ and $N_x = -2x\sin(xy) - x^2y\cos(xy) \Longrightarrow M_y = N_x \Longrightarrow$ the differential equation is exact.

3. $(y + 3x^2)dx + xdy = 0$. $M = y + 3x^2$ and $N = x \Longrightarrow M_y = 1$ and $N_x = 1 \Longrightarrow M_y = N_x \Longrightarrow$ the differential equation is exact.

4. $2xe^y\,dx + (3y^2 + x^2e^y)dy = 0$. $M = 2xe^y$ and $N = 3y^2 + x^2e^y \Longrightarrow M_y = 2xe^y$ and $N_x = 2xe^y \Longrightarrow M_y = N_x \Longrightarrow$ the differential equation is exact.

5. $2xydx + (x^2 + 1)dy = 0$. $M = 2xy$ and $N = x^2 + 1 \Longrightarrow M_y = 2x$ and $N_x = 2x \Longrightarrow M_y = N_x \Longrightarrow$ the differential equation is exact so there exists a potential function ϕ such that (a)$\dfrac{\partial\phi}{\partial x} = 2xy$ and (b)$\dfrac{\partial\phi}{\partial x} = 2xy + \dfrac{dh(x)}{dx}$ so from (a), $2xy = 2xy + \dfrac{dh(x)}{dx} \Longrightarrow \dfrac{dh(x)}{dx} = 0 \Longrightarrow h(x)$ is a constant. Since we need just one potential function, let $h(x) = 0$. Thus, $\phi(x, y) = (x^2 + 1)y$; hence, $(x^2 + 1)y = c$.

6. Given $(y^2 - 2x)dx + 2xydy = 0$ then $M_y = N_x = 2xy$ so the differential equation is exact and there exists a potential function ϕ such that (a) $\dfrac{\partial\phi}{\partial x} = y^2 - 2x$ and (b) $\dfrac{\partial\phi}{\partial y} = 2xy$. From (b) $\phi(x, y) = xy^2 + h(x) \Longrightarrow \dfrac{\partial\phi}{\partial x} = y^2 + \dfrac{dh(x)}{dx}$ so from (a) $y^2 + \dfrac{dh(x)}{dx} = y^2 - 2x \Longrightarrow \dfrac{dh(x)}{dx} = -2x \Longrightarrow h(x) = -2x$ where the constant of integration has been set to zero since we just need one potential function. $\phi(x, y) = xy^2 - x^2 \Longrightarrow xy^2 - x^2 = c$.

7. Given $(4e^{2x} + 2xy - y^2)dx + (x - y)^2dy = 0$ then $M_y = N_x = 2y$ so the differential equation is exact and there exists a potential function ϕ such that (a) $\dfrac{\partial\phi}{\partial x} = 4e^{2x} + 2xy - y^2$ and (b) $\dfrac{\partial\phi}{\partial y} = (x - y)^2$. From (b) $\phi(x, y) = x^2y - xy^2 + \dfrac{y^3}{3} + h(x) \Longrightarrow \dfrac{\partial\phi}{\partial x} = 2xy - y^2 + \dfrac{dh(x)}{dx}$ so from (a) $2xy - y^2 + \dfrac{dh(x)}{dx} = 4e^{2x} + 2xy - y^2 \Longrightarrow \dfrac{dh(x)}{dx} = 4e^{2x} \Longrightarrow h(x) = 2e^{2x}$ where the constant of integration has been set to zero since we need just one potential function. $\phi(x, y) = x^2y - xy^2 + \dfrac{y^3}{3} + 2e^{2x} \Longrightarrow x^2y - xy^2 + \dfrac{y^3}{3} + 2e^{2x} = c_1 \Longrightarrow 6e^{2x} + 3x^2y - 3xy^2 + y^3 = c$.

8. Given $\left(\dfrac{1}{x} - \dfrac{y}{x^2 + y^2}\right)dx + \dfrac{x}{x^2 + y^2}dy = 0$ then $M_y = N_x = \dfrac{y^2 - x^2}{(x^2 + y^2)^2}$ so the differential equation is exact and there exists a potential function ϕ such that (a) $\dfrac{\partial\phi}{\partial x} = \dfrac{1}{x} - \dfrac{y}{x^2 + y^2}$ and (b) $\dfrac{\partial\phi}{\partial y} = \dfrac{x}{x^2 + y^2}$. From (b) $\phi(x, y) = \tan^{-1}(y/x) + h(x) \Longrightarrow \dfrac{\partial\phi}{\partial x} = -\dfrac{y}{x^2 + y^2} + \dfrac{dh(x)}{dx}$ so from (a) $-\dfrac{y}{x^2 + y^2} + \dfrac{dh(x)}{dx} = \dfrac{1}{x} - \dfrac{y}{x^2 + y^2} \Longrightarrow \dfrac{dh}{dx} = x^{-1} \Longrightarrow h(x) = \ln|x|$ where the constant of integration is set to zero since we only need one potential function. $\phi(x, y) = \tan^{-1}(y/x) + \ln|x| \Longrightarrow \tan^{-1}(y/x) + \ln|x| = c$.

9. Given $[y\cos(xy) - \sin x]dx + x\cos(xy)dy = 0$ then $M_y = N_x = -xy\sin(xy) + \cos(xy)$ so the differential equation is exact so there exists a potential function ϕ such that (a) $\dfrac{\partial\phi}{\partial x} = y\cos(xy) - \sin x$ and (b) $\dfrac{\partial\phi}{\partial x} = x\cos(xy)$. From (b) $\phi(x, y) = \sin(xy) + h(x) \Longrightarrow \dfrac{\partial\phi}{\partial x} = y\cos(xy) + \dfrac{dh(x)}{dx}$ so from (a) $y\cos(xy) +$

$\frac{dh(x)}{dx} = y\cos(xy) - \sin x \Longrightarrow \frac{dh}{dx} = -\sin x \Longrightarrow h(x) = \cos x$ where the constant of integration is set to zero since we only need one potential function. $\phi(x,y) = \sin(xy) + \cos x \Longrightarrow \sin(xy) + \cos x = c$.

10. $(2y^2e^{2x} + 3x^2)dx + 2ye^{2x}dy = 0$. $M = 2y^2e^{2x} + 3x^2$ and $N = 2ye^{2x} \Longrightarrow M_y = 4ye^{2x}$ and $N_x = 4ye^{2x} \Longrightarrow M_y = N_x \Longrightarrow$ the differential equation is exact so there exists a potential function ϕ such that (a)$\frac{\partial\phi}{\partial x} = 2y^2e^{2x} + 3x^2$ and (b)$\frac{\partial\phi}{\partial y} = 2ye^{2x}$. From (a) $\phi(x,y) = y^2e^{2x} + x^3 + h(y) \Longrightarrow \frac{\partial\phi}{\partial y} = 2ye^{2x} + \frac{dh(y)}{dy}$ so from (b) $2ye^{2x} + \frac{dh(y)}{dy} = 2ye^{2x} \Longrightarrow \frac{dh}{dy} = 0 \Longrightarrow h(y) = c_1$. Since we only need one potential function we can set $c_1 = 0$. Then $\phi(x,y) = y^2e^{2x} + x^3 \Longrightarrow y^2e^{2x} + x^3 = c$.

11. $(y^2 + \cos x)dx + (2xy + \sin y)dy = 0$. $M = y^2 + \cos x$ and $N = 2xy + \sin y \Longrightarrow M_y = 2y$ and $N_x = 2y \Longrightarrow M_y = N_x \Longrightarrow$ the differential equation is exact so there exists a potential function ϕ such that (a)$\frac{\partial\phi}{\partial x} = y^2 + \cos x$ and (b)$\frac{\partial\phi}{\partial y} = 2xy + \sin y$. From (a) $\phi(x,y) = xy^2 + \sin x + h(y) \Longrightarrow \frac{\partial\phi}{\partial y} = 2xy + \frac{dh(y)}{dy}$ so from (b) $2xy + \frac{dh(y)}{dy} = 2xy + \sin y \Longrightarrow \frac{dh}{dy} = \sin y \Longrightarrow h(y) = -\cos y$ where the constant of integration has been set to zero since we just need one potential function. $\phi(x,y) = xy^2 + \sin x - \cos y \Longrightarrow xy^2 + \sin x - \cos y = c$.

12. $(\sin y + y\cos x)dx + (x\cos y + \sin x)dy = 0$. $M = \sin y + y\cos x$ and $N = x\cos y + \sin x \Longrightarrow M_y = \cos y + \cos x$ and $N_x = \cos y + \cos x \Longrightarrow M_y = N_x \Longrightarrow$ the differential equation is exact so there exists a potential function ϕ such that (a)$\frac{\partial\phi}{\partial x} = \sin y + y\cos x$ and (b)$\frac{\partial\phi}{\partial y} = x\cos y + \sin x$. From (a) $\phi(x,y) = x\sin y + y\sin x + h(y) \Longrightarrow \frac{\partial\phi}{\partial y} = x\cos y + \sin x + \frac{dh(y)}{dy}$ so from (b) $x\cos y + \sin x + \frac{dh(y)}{dy} = x\cos y + \sin x \Longrightarrow \frac{dh}{dy} = 0 \Longrightarrow h(y) = c_1$. Since we only need one potential function we can set $c_1 = 0$. $\phi(x,y) = x\sin y + y\sin x \Longrightarrow x\sin y + y\sin x = c$.

13. Given $[1 + \ln(xy)]dx + \frac{x}{y}dy = 0$ then $M_y = N_x = y^{-1}$ so the differential equation is exact and there exists a potential function ϕ such that (a) $\frac{\partial\phi}{\partial x} = 1 + \ln(xy)$ and (b) $\phi(x,y) = x\ln y + h(x) \Longrightarrow \frac{\partial\phi}{\partial x} = \ln y + \frac{dh(x)}{dx}$ so from (a) $\ln y + \frac{dh(x)}{dx} = 1\ln(xy) \Longrightarrow \frac{dh}{dx} = 1 + \ln x \Longrightarrow h(x) = c\ln x$ where the constant of integration is set to zero since we only need one potential function. $\phi(x,y) = x\ln y + x\ln x \Longrightarrow x\ln y + x\ln x = c \Longrightarrow x\ln(xy) = c$.

14. Given $\frac{xy-1}{x}dx + \frac{xy+1}{y}dy = 0$ then $M_y = N_x = 1 \Longrightarrow$ then the differential equation is exact so there exists a potential function ϕ such that (a) $\frac{\partial\phi}{\partial x} = \frac{xy-1}{x}$ and (b) $\frac{\partial\phi}{\partial y} = \frac{xy+1}{y}$. From (a) $\phi(x,y) = xy - \ln|x| + h(y) \Longrightarrow \frac{\partial\phi}{\partial y} = x + \frac{dh(y)}{dy}$ so from (b), $x + \frac{dh(y)}{dy} = \frac{xy+1}{y} \Longrightarrow \frac{dh(y)}{dy} = y^{-1} \Longrightarrow h(y) = \ln|y|$ where the constant of integration has been set to zero since we need just one potential function. $\phi(x,y) = xy + \ln|y/x| \Longrightarrow xy + \ln|x/y| = c$.

15. Given $(2xy + \cos y)dx + (x^2 - x\sin y - 2y)dy = 0$ then $M_y = N_x = 2x - \sin y$ so the differential equation is exact so there is a potential function ϕ such that (a) $\frac{\partial\phi}{\partial x} = 2xy + \cos y$ and (b) $\frac{\partial\phi}{\partial y} = x^2 - x\sin y - 2y$.

From (a) $\phi(x,y) = x^2y + x\cos y + h(y) \Longrightarrow \dfrac{\partial\phi}{\partial y} = x^2 - x\sin y + \dfrac{dh(y)}{dy}$ so from (b) $x^2 - x\sin y + \dfrac{dh(y)}{dy} = x^2 - x\sin y - 2y \Longrightarrow \dfrac{dh}{dy} = -2y \Longrightarrow h(y) = -y^2$ where the constant of integration has been set to zero since we only need one potential function. $\phi(x,y) = x^2y + x\cos y - y^2 \Longrightarrow x^2y + x\cos y - y^2 = c$.

16. Given $2x^2\dfrac{dx}{dy} + 4xy = 3\sin x \Longrightarrow (4xy - 3\sin x)dx + 2x^2dy = 0$ then $M_y = N_x = 4x$ so the differential equation is exact so there exists a potential function ϕ such that (a) $\dfrac{\partial\phi}{\partial x} = 4xy - 3\sin x$ and (b) $\dfrac{\partial\phi}{\partial y} = 2x^2$. From (b) $\phi(x,y) = 2x^2y + h(x) \Longrightarrow \dfrac{\partial\phi}{\partial x} = 4xy + \dfrac{dh(x)}{dx}$ so from (a) $4xy + \dfrac{dh(x)}{dx} = 4xy - 3\sin x \Longrightarrow \dfrac{dh(x)}{dx} = -3\sin x \Longrightarrow h(x) = 3\cos x$ where the constant of integration has been set to zero since we only need one potential function. $\phi(x,y) = 2x^2y + 3\cos x \Longrightarrow 2x^2y + 3\cos x = c$. Now since $y(2\pi) = 0, c = 3$; thus, $2x^2y + 3\cos x = 3$ or $y(x) = \dfrac{3 - 3\cos x}{2x^2}$.

17. Given $(3x^2\ln x + x^2 - y)dx - xdy = 0$ then $M_y = N_x = -1$ so the differential equation is exact so there exists a potential function ϕ such that (a) $\dfrac{\partial\phi}{\partial x} = 3x^2\ln x + x^2 - y$ and (b) $\dfrac{\partial\phi}{\partial y} = -x$. From (b) $\phi(x,y) = -xy + h(x) \Longrightarrow \dfrac{\partial\phi}{\partial x} = -y + \dfrac{dh(x)}{dx}$ so from (a) $-y + \dfrac{dh(x)}{dx} = 3x^2\ln x + x^2 - y \Longrightarrow \dfrac{dh(x)}{dx} = 3x^2\ln x + x^2 \Longrightarrow h(x) = x^3\ln x$ where the constant of integration has been set to zero since we only need one potential function. $\phi(x,y) = -xy + x^3\ln x \Longrightarrow -xy + x^3\ln x = c$. Now since $y(1) = 5, c = -5$; thus, $x^3\ln x - xy = -5$ or $y(x) = \dfrac{x^3\ln x + 5}{x}$.

18. Given $(ye^{xy} + \cos x)dx + xe^{xy}dy = 0$ then $M_y = N_x = xye^{xy} + e^{xy}$ so the differential equation is exact so there exists a potential function ϕ such that (a) $\dfrac{\partial\phi}{\partial x} = ye^{xy} + \cos x$ and (b) $\dfrac{\partial\phi}{\partial y} = xe^{xy}$. From (b) $\phi(x,y) = e^{xy} + h(x) \Longrightarrow \dfrac{\partial\phi}{\partial x} = ye^{xy} + \dfrac{dh(x)}{dx}$ so from (a) $ye^{xy} + \cos x \Longrightarrow \dfrac{dh(x)}{dx} = \cos x \Longrightarrow h(x) = \sin x$ where the constant of integration is set to zero since we only need one potential function. $\phi(x,y) = e^{xy} + \sin x \Longrightarrow e^{xy} + \sin x = c$. Now since $y(\pi/2) = 0, c = 2$; thus, $e^{xy} + \sin x = 2$ or $y(x) = \dfrac{\ln(2 - \sin x)}{x}$.

19. If $\phi(x,y)$ is a potential function for $Mdx + Ndy = 0 \Longrightarrow d(\phi(x,y)) = 0$ so $d(\phi(x,y) + c) = d(\phi(x,y)) + d(c) = 0 + 0 = 0 \Longrightarrow \phi(x,y) + c$ is also a potential function.

20. $M = \cos(xy)[\tan(xy) + xy]$ and $N = x^2\cos(xy) \Longrightarrow M_y = 2x\cos(xy) - x^2y\sin(xy) = N_x \Longrightarrow M_y = N_x \Longrightarrow Mdx = Ndy = 0$ is exact so $I(x,y) = \cos(xy)$ is an integrating factor for $[\tan(xy) + xy]dx + x^2dy = 0$.

21. $M = e^{-x/y}(x^2y^{-1} - 2x)$ and $N = -e^{-x/y}x^3y^{-2} \Longrightarrow M_y = e^{-x/y}(x^3y^{-3} - 3x^2y^{-2}) = N_x \Longrightarrow Mdx + Ndy = 0$ is exact so $I(x,y) = y^{-2}e^{-x/y}$ is an integrating factor for $y[x^2 - 2xy]dx - x^3dy = 0$.

22. $M = \sec x[2x - (x^2 + y^2)\tan x]$ and $N = 2y\sec x \Longrightarrow M_y = -2y\sec x\tan x$ and $N_x = 2y\sec x\tan x \Longrightarrow M_y \neq N_x \Longrightarrow Mdx + Ndy = 0$ is not exact so $I(x) = \sec x$ is not an integrating factor for $[2x - (x^2 + y^2)\tan x]dx + 2ydy = 0$.

23. Given $(y - x^2)dx + 2xdy = 0$ then $M = y - x^2$ and $N = 2x$. Thus $M_y = 1$ and $N_x = 2$ so $\dfrac{M_y - N_x}{N} =$

$-\dfrac{1}{2x} = f(x)$ is a function of x alone so $I(x) = e^{\int f(x)dx} = \dfrac{1}{\sqrt{x}}$ is an integrating factor for the given equation. Multiplying the given equation by $I(x)$ results in the exact equation $(x^{-1/2}y - x^{3/2})dx + 2x^{1/2}dy = 0$. We find that $\phi(x, y) = 2x^{1/2}y - \dfrac{2x^{5/2}}{5}$ and hence the general solution of our differential equation is $2x^{1/2}y - \dfrac{2x^{5/2}}{5} = c$ or $y(x) = \dfrac{c + 2x^{5/2}}{10\sqrt{x}}$.

24. Given $(3xy - 2y^{-1})dx + x(x + y^{-2})dy = 0$ then $M = 3xy - 2y^{-1}$ and $N = x(x + y^{-2})$. Thus $M_y = 3x + 2y^{-2}$ and $N_x = 2x + y^{-2}$ so $\dfrac{M_y - N_x}{N} = \dfrac{1}{x} = f(x)$ is a function of x alone so $I(x) = e^{\int f(x)dx} = x$ is an integrating factor for the given equation. Multiplying the given equation by $I(x)$ results in the exact equation $(3x^2y - 2xy^{-1})dx + x^2(x + y^{-2})dy = 0$. We find that $\phi(x, y) = x^3y - x^2y^{-1}$ and hence the general solution of our differential equation is $x^3y - x^2y^{-1} = c$.

25. Given $x^2ydx + y(x^3 + e^{-3y}\sin y)dy = 0$ then $M = x^2y$ and $N = y(x^3 + e^{-3y}\sin y)$. Thus $M_y = x^2$ and $N_x = 3x^2y$ so $\dfrac{M_y - N_x}{M} = y^{-1} - 3 = g(y)$ is a function of y alone so $I(y) = e^{\int g(y)dy} = e^{3y}/y$ is an integrating factor for the given equation. Multiplying the equation by $I(y)$ results in the exact equation $x^2e^{3y}dx + e^{3y}(x^3 + e^{-3y}\sin y)dy = 0$. We find that $\phi(x, y) = \dfrac{x^3e^{3y}}{3} - \cos y$ and hence the general solution of our differential equation is $\dfrac{x^3e^{3y}}{3} - \cos y = c$.

26. Given $(xy - 1)dx + x^2dy = 0$ then $M = xy - 1$ and $N = x^2$. Thus $M_y = x$ and $N_x = 2x$ so $\dfrac{M_y - N_x}{N} = -x^{-1} = f(x)$ is a function of x alone so $I(x) = e^{\int f(x)dx} = x^{-1}$ is an integrating factor for the given equation. Multiplying the given equation by $I(x)$ results in the exact equation $(y - x^{-1})dx + xdy = 0$. We find that $\phi(x, y) = xy - \ln|x|$ and hence, the general solution of our differential equation is $xy - \ln|x| = c$.

27. Given $\dfrac{dy}{dx} + \dfrac{2xy}{1+x^2} = \dfrac{1}{(1+x^2)^2} \Longrightarrow (2xy + 2x^3y - 1)dx + (1 + x^2)^2dy = 0$ then $M = 2xy + 2x^3y - 1$ and $N = (1 + x^2)^2$. Thus $M_y = 2x + 2x^3$ and $N_x = 4x(1 + x^2)$ so $\dfrac{M_y - N_x}{N} = -\dfrac{2x}{1+x^2} = f(x)$ is a function of x alone so $I(x) = e^{\int f(x)dx} = \dfrac{1}{1+x^2}$ is an integrating factor for the given equation. Multiplying the given equation by $I(x)$ yields the exact equation $\left(2xy - \dfrac{1}{1+x^2}\right)dx + (1 + x^2)dy = 0$. We find that $\phi(x, y) = (1 + x^2)y - \tan^{-1}x$ and hence the general solution of our differential equation is $(1 + x^2)y - \tan^{-1}x = c$ or $y(x) = \dfrac{\tan^{-1}x + c}{1+x^2}$.

28. Given $xy[2\ln(xy) + 1]dx + x^2dy = 0$ then $M = xy[2\ln(xy) + 1]$ and $N + x^2$. Thus $M_y = 3x + 2x\ln(xy)$ and $N_x = 2x$ so $\dfrac{M_Y - N_x}{M} = y^{-1} = g(y)$ is a function of y only so $I(y) = e^{\int g(y)dy} = \dfrac{1}{y}$ is an integrating factor for the given equation. Multiplying the given equation by $I(y)$ results in the exact equation $x[2\ln(xy) + 1]dx + x^2y^{-1}dy = 0$. We find that $\phi(x, y) = x^2\ln y + x^2\ln x$ and hence the general solution of our differential equation is $x^2\ln y + x^2\ln x = c$ or $y(x) = xe^{c/x^2}$.

29. Given $ydx - (2x + y^4)dy = 0$ then $M = y$ and $N = -(2x + y^4)$. Thus $M_y = 1$ and $N_x = -2$ so

$\dfrac{M_y - N_x}{M} = 3y^{-1} = g(y)$ is a function of y alone so $I(y) = e^{-\int g(y)dy} = 1/y^3$ is an integrating factor for the given differential equation. Multiplying the given equation by $I(y)$ results in the exact equation $y^{-2}dx - (2xy^{-3} + y)dy = 0$. We find that $\phi(x, y) = xy^{-2} - y^2/2$ and hence, the general solution of our differential equation is $xy^{-2} - y^2/2 = c_1 \implies 2x - y^4 = cy^2$.

30. Given $(y^{-1} - x^{-1})dx + (xy^{-2} - 2y^{-1})dy = 0 \implies x^r y^s(y^{-1} - x^{-1})dx + x^r y^s(xy^{-2} - 2y^{-1})dy = 0 \implies (x^r y^{s-1} - x^{r-1}y^s)dx + (x^{r+1}y^{s-2} - 2x^r y^{s-1})dy = 0$. Then $M = x^r y^{s-1} - x^{r-1}y^s$ and $N = x^{r+1}y^{s-2} - 2x^r y^{s-1}$ so $M_y = x^r(s-1)y^{s-2} - x^{r-1}sy^{s-1}$ and $N_x = (r+1)x^r y^{s-2} - 2rx^{r-1}y^{s-1}$. The equation is exact if and only if $M_y = N_x \implies x^r y^{s-1} - x^{r-1}y^s = (r+1)x^r y^{s-2} - 2rx^{r-1}y^{s-1} \implies \dfrac{s-1}{y^2} - \dfrac{s}{xy} = \dfrac{r+1}{y^2} - \dfrac{2r}{xy} \implies \dfrac{s-r-2}{y^2} = \dfrac{s-2r}{xy}$. From the last equation we require that $s - r - 2 = 0$ and $s - 2r = 0$. Solving this system yields $r = 2$ and $s = 4$.

31. Given $2y(y + 2x^2)dx + x(4y + 3x^2)dy = 0 \implies x^r y^s 2y(y + 2x^2)dx + x^r y^s x(4y + 3x^2)dy = 0$. Then $M = 2x^r y^{s+2} + 4x^{r+2}y^{s+1}$ and $N = 4x^{r+1}y^{s+1} + 3x^{r+3}y^s$ so $M_y = 2x^r(s+2)y^{s+1} + 4x^{r+2}(s+1)y^s$ and $N_x = 4(r+1)x^r y^{s+1} + 3(r+3)x^{r+2}y^s$. The equation is exact if and only if $M_y = N_x \implies 2x^r(s+2)y^{s+1} + 4x^{r+2}(s+1)y^s = 4(r+1)x^r y^{s+1} + 3(r+3)x^{r+2}y^s \implies 2(s+2)y + 4x^2(s+1) = 4(r+1)y + 3(r+3)x^2$. From this last equation we require that $2(s+2) = 4(r+1)$ and $4(s+1) = 3(r+3)$. Solving this system yields $r = 1$ and $s = 2$.

32. Given $y(5xy^2 + 4)dx + x(xy^2 - 1)dy = 0 \implies x^r y^s y(5xy^2 + 4)dx + x^r y^s x(xy^2 - 1)dy = 0$. Then $M = x^r y^{s+1}(5xy^2 + 4)$ and $N = x^{r+1}y^s(xy^2 - 1)$ so $M_y = 5(s+3)x^{r+1}y^{s+2} + 4(s+1)x^r y^s$ and $N_x = (r+2)x^{r+1}y^{s-2} - (r+1)x^r y^s$. The equation is exact if and only if $M_y = N_x \implies 5(s+3)x^{r+1}y^{s+2} + 4(s+1)x^r y^s = (r+2)x^{r+1}y^{s+2} - (r+1)x^r y^s \implies 5(s+3)xy^2 + 4(s+1) = (r+2)xy^2 - (r+1)$. From the last equation we require that $5(s+3) = r+2$ and $4(s+1) = -(r+1)$. Solving this system yields $r = 3$ and $s = -2$.

33. Suppose that $\dfrac{M_y - N_x}{M} = g(y)$ is a function of y only. Then dividing the equation (1.9.21) by M, it follows that I is an integrating factor for $M(x, y)dx + N(x, y)dy = 0$ if and only if it is a solution of $\dfrac{N}{M}\dfrac{\partial I}{\partial x} - \dfrac{\partial I}{\partial y} = Ig(y)$ (30.1). We must show that this differential equation has a solution $I = I(y)$. However, if $I = I(y)$, then (30.1) reduces to $\dfrac{dI}{dy} = -Ig(y)$, which is a separable equation with solution $I(y) = e^{-\int g(t)dt}$.

34. (a). Note $\dfrac{dy}{dx} + py = q$ can be written in the differential form as $(py - q)dx + dy = 0$ (34.1). This has $M = py - q$ and $N = 1$ so that $\dfrac{M_y - N_x}{N} = p(x)$. Consequently, an integrating factor for (34.1) is $I(x) = e^{\int^x p(t)dt}$.

(b). Multiplying (34.1) by $I(x) = e^{\int^x p(t)dt}$ yields the exact equation $e^{\int^x p(t)dt}(py - q)dx + e^{\int^x p(t)dt}dy = 0$. Hence, there exists a potential function ϕ such that (i) $\dfrac{\partial \phi}{\partial x} = e^{\int^x p(t)dt}(py - q)$ and (ii) $\dfrac{\partial \phi}{\partial y} = e^{\int^x p(t)dt}$. From (i), $p(x)ye^{\int^x p(t)dt} + \dfrac{dh(x)}{dx} = e^{\int^x p(t)dt}(py - q) \implies \dfrac{dh(x)}{dx} = -q(x)e^{\int^x p(t)dt} \implies h(x) = -\int q(x)e^{\int^x p(t)dt}dx$, where the constant of integration has been set to zero since we just need one potential function. Consequently, $\phi(x, y) = ye^{\int^x p(t)dt} - \int q(x)e^{\int^x p(t)dt}dx \implies y(x) = I^{-1}\left(\int^x Iq(t)dt + c\right)$, where $I(x) = e^{\int^x p(t)dt}$.

Solutions to Section 1.10

True-False Review:

(a): TRUE. This is well-illustrated by the calculations shown in Example 1.10.1.

(b): TRUE. The equation

$$y_1 = y_0 + f(x_0, y_0)(x_1 - x_0)$$

is the tangent line to the curve $\frac{dy}{dx} = f(x, y)$ at the point (x_0, y_0). Once the point (x_1, y_1) is determined, the procedure can be iterated over and over at the new points obtained to carry out Euler's method.

(c): FALSE. It is possible, depending on the circumstances, for the errors associated with Euler's method to decrease from one step to the next.

(d): TRUE. This is illustrated in Figure 1.10.3.

Problems:

1. Applying Euler's method with $y' = 4y - 1, x_0 = 0, y_0 = 1$, and $h = 0.05$ we have $y_{n+1} = y_n + 0.05(4y_n - 1)$. This generates the sequence of approximants given in the table below.

n	x_n	y_n
1	0.05	1.15
2	0.10	1.33
3	0.15	1.546
4	0.20	1.805
5	0.25	2.116
6	0.30	2.489
7	0.35	2.937
8	0.40	3.475
9	0.45	4.120
10	0.50	4.894

Consequently the Euler approximation to $y(0.5)$ is $y_{10} = 4.894$. (Actual value: $y(.05) = 5.792$ rounded to 3 decimal places).

2. Applying Euler's method with $y' = -\dfrac{2xy}{1 + x^2}, x_0 = 0, y_0 = 1$, and $h = 0.1$ we have $y_{n+1} = y_n - 0.2\dfrac{x_n y_n}{1 + x_n^2}$. This generates the sequence of approximants given in the table below.

n	x_n	y_n
1	0.1	1
2	0.2	0.980
3	0.3	0.942
4	0.4	0.891
5	0.5	0.829
6	0.6	0.763
7	0.7	0.696
8	0.8	0.610
9	0.9	0.569
10	1.0	0.512

Consequently the Euler approximation to $y(1)$ is $y_{10} = 0.512$. (Actual value: $y(1) = 0.5$).

3. Applying Euler's method with $y' = x - y^2, x_0 = 0, y_0 = 2$, and $h = 0.05$ we have $y_{n+1} = y_n + 0.05(x_n - y_n^2)$. This generates the sequence of approximants given in the table below.

n	x_n	y_n
1	0.05	1.80
2	0.10	1.641
3	0.15	1.511
4	0.20	1.404
5	0.25	1.316
6	0.30	1.242
7	0.35	1.180
8	0.40	1.127
9	0.45	1.084
10	0.50	1.048

Consequently the Euler approximation to $y(0.5)$ is $y_{10} = 1.048$. (Actual value: $y(.05) = 1.0477$ rounded to four decimal places).

4. Applying Euler's method with $y' = -x^2 y, x_0 = 0, y_0 = 1$, and $h = 0.2$ we have $y_{n+1} = y_n - 0.2x_n^2 y_n$. This generates the sequence of approximants given in the table below.

n	x_n	y_n
1	0.2	1
2	0.4	0.992
3	0.6	0.960
4	0.8	0.891
5	1.0	0.777

Consequently the Euler approximation to $y(1)$ is $y_5 = 0.777$. (Actual value: $y(1) = 0.717$ rounded to 3 decimal places).

5. Applying Euler's method with $y' = 2xy^2, x_0 = 0, y_0 = 1$, and $h = 0.1$ we have $y_{n+1} = y_n + 0.1x_n y_n^2$. This generates the sequence of approximants given in the table below.

n	x_n	y_n
1	0.1	0.5
2	0.2	0.505
3	0.3	0.515
4	0.4	0.531
5	0.5	0.554
6	0.6	0.584
7	0.7	0.625
8	0.8	0.680
9	0.9	0.754
10	1.0	0.858

Consequently the Euler approximation to $y(1)$ is $y_{10} = 0.856$. (Actual value: $y(1) = 1$).

6. Applying the modified Euler method with $y' = 4y - 1, x_0 = 0, y_0 = 1$, and $h = 0.05$ we have $y^*_{n+1} = y_n + 0.05(4y_n - 1)$
$y_{n+1} = y_n + 0.025(4y_n - 1 + 4y^*_{n+1} - 1)$. This generates the sequence of approximants given in the table below.

n	x_n	y_n
1	0.05	1.165
2	0.10	1.3663
3	0.15	1.6119
4	0.20	1.9115
5	0.25	2.2770
6	0.30	2.7230
7	0.35	3.2670
8	0.40	3.9308
9	0.45	4.7406
10	0.50	5.7285

Consequently the modified Euler approximation to $y(0.5)$ is $y_{10} = 5.7285$. (Actual value: $y(.05) = 5.7918$ rounded to 4 decimal places).

7. Applying the modified Euler method with $y' = -\dfrac{2xy}{1+x^2}, x_0 = 0, y_0 = 1$, and $h = 0.1$ we have $y^*_{n+1} = y_n - 0.2\dfrac{x_n y_n}{1+x_n^2}$
$y_{n+1} = y_n + 0.05\left[-\dfrac{x_n y_n}{1+x_n^2} - 2\dfrac{x_{n+1}y^*_{n+1}}{1+x_{n+1}^2}\right]$. This generates the sequence of approximants given in the table below.

n	x_n	y_n
1	0.1	0.9900
2	0.2	0.9616
3	0.3	0.9177
4	0.4	0.8625
5	0.5	0.8007
6	0.6	0.7163
7	0.7	0.6721
8	0.8	0.6108
9	0.9	0.5536
10	1.0	0.5012

Consequently the modified Euler approximation to $y(1)$ is $y_{10} = 0.5012$. (Actual value: $y(1) = 0.5$).

8. Applying the modified Euler method with $y' = x - y^2, x_0 = 0, y_0 = 2$, and $h = 0.05$ we have $y^*_{n+1} = y_n - 0.05(x_n - y_n^2)$
$y_{n+1} = y_n + 0.025(x_n - y_n^2 + x_{n+1} - (y^*_{n+1})^2)$. This generates the sequence of approximants given in the table below.

n	x_n	y_n
1	0.05	1.8203
2	0.10	1.6725
3	0.15	1.5497
4	0.20	1.4468
5	0.25	1.3600
6	0.30	1.2866
7	0.35	1.2243
8	0.40	1.1715
9	0.45	1.1269
10	0.50	1.0895

Consequently the modified Euler approximation to $y(0.5)$ is $y_{10} = 1.0895$. (Actual value: $y(.05) = 1.0878$ rounded to 4 decimal places).

9. Applying the modified Euler method with $y' = -x^2 y, x_0 = 0, y_0 = 1$, and $h = 0.2$ we have $y^*_{n+1} = y_n - 0.2x_n^2 y_n$
$y_{n+1} = y_n - 0.1[x_n^2 y_n + x_{n+1}^2 y^*_{n+1}]$. This generates the sequence of approximants given in the table below.

n	x_n	y_n
1	0.2	0.9960
2	0.4	0.9762
3	0.6	0.9266
4	0.8	0.8382
5	1.0	0.7114

Consequently the modified Euler approximation to $y(1)$ is $y_5 = 0.7114$. (Actual value: $y(1) = 0.7165$ rounded to 4 decimal places).

10. Applying the modified Euler method with $y' = 2xy^2, x_0 = 0, y_0 = 1$, and $h = 0.1$ we have $y^*_{n+1} = y_n + 0.1x_n y_n^2$
$y_{n+1} = y_n + 0.05[x_n y_n^2 + x_{n+1}(y^*_{n+1})^2]$. This generates the sequence of approximants given in the table below.

n	x_n	y_n
1	0.1	0.5025
2	0.2	0.5102
3	0.3	0.5235
4	0.4	0.5434
5	0.5	0.5713
6	0.6	0.6095
7	0.7	0.6617
8	0.8	0.7342
9	0.9	0.8379
10	1.0	0.9941

Consequently the modified Euler approximation to $y(1)$ is $y_{10} = 0.9941$. (Actual value: $y(1) = 1$).

11. We have $y' = 4y - 1, x_0 = 0, y_0 = 1$, and $h = 0.05$. So, $k_1 = 0.05(4y_n - 1), k_2 = 0.05[4(y_n + \frac{1}{2}k_1) - 1], k_3 = 0.05[4(y_n + \frac{1}{2}k_2) - 1], k_4 = 0.05[4(y_n + \frac{1}{2}k_3) - 1]$,

$y_{n+1} = y_n + \frac{1}{6}(k_1 + k_2 + k_3 + k_4)$. This generates the sequence of approximants given in the table below (computations rounded to five decimal places).

n	x_n	y_n
1	0.05	1.16605
2	0.10	1.36886
3	0.15	1.61658
4	0.20	1.91914
5	0.25	2.28868
6	0.30	2.74005
7	0.35	3.29135
8	0.40	3.96471
9	0.45	4.78714
10	0.50	5.79167

Consequently the Runge-Kutta approximation to $y(0.5)$ is $y_{10} = 5.79167$. (Actual value: $y(.05) = 5.79179$ rounded to 5 decimal places).

12. We have $y' = -2\dfrac{xy}{1+x^2}, x_0 = 0, y_0 = 1$, and $h = 0.1$. So, $k_1 = -0.2\dfrac{x_n y_n}{1+x_n^2}, k_2 = -0.2\dfrac{(x_n+0.05)(y_n+\frac{k_1}{2})}{[1+(x_n+0.05)^2]}), k_3 = -0.2\dfrac{(x_n+0.05)(y_n+\frac{k_2}{2})}{[1+(x_n+0.05)^2]}, k_4 = -0.2\dfrac{x_{n+1}(y_n+k_3)}{[1+(x_{n+1})^2]}$,
$y_{n+1} = y_n + \frac{1}{6}(k_1 + k_2 + k_3 + k_4)$. This generates the sequence of approximants given in the table below (computations rounded to seven decimal places).

n	x_n	y_n
1	0.1	0.9900990
2	0.2	0.9615383
3	0.3	0.9174309
4	0.4	0.8620686
5	0.5	0.7999996
6	0.6	0.7352937
7	0.7	0.6711406
8	0.8	0.6097558
9	0.9	0.5524860
10	1.0	0.4999999

Consequently the Runge-Kutta approximation to $y(1)$ is $y_{10} = 0.4999999$. (Actual value: $y(.05) = 0.5$).

13. We have $y' = x - y^2, x_0 = 0, y_0 = 2$, and $h = 0.05$. So, $k_1 = 0.05(x_n - y_n^2), k_2 = 0.05[x_n + 0.025 - (y_n + \frac{k_1}{2})^2], k_3 = 0.05[x_n + 0.025 - (y_n + \frac{k_2}{2})^2], k_4 = 0.05[x_{n+1} - (y_n + k_3)^2]]$,
$y_{n+1} = y_n + \frac{1}{6}(k_1 + k_2 + k_3 + k_4)$. This generates the sequence of approximants given in the table below (computations rounded to six decimal places).

n	x_n	y_n
1	0.05	1.1.81936
2	0.10	1.671135
3	0.15	1.548079
4	0.20	1.445025
5	0.25	1.358189
6	0.30	1.284738
7	0.35	1.222501
8	0.40	1.169789
9	0.45	1.125263
10	0.50	1.087845

Consequently the Runge-Kutta approximation to $y(0.5)$ is $y_{10} = 1.087845$. (Actual value: $y(0.5) = 1.087845$ rounded to 6 decimal places).

14. We have $y' = -x^2y, x_0 = 0, y_0 = 1$, and $h = 0.2$. So, $k_1 = -0.2x_n^2y_n, k_2 = -0.2(x_n+0.1)^2(y_n+\frac{k_1}{2}), k_3 = -0.2(x_n+0.1)^2(y_n+\frac{k_2}{2}), k_4 = -0.2(x_{n+1})^2(y_n+k_3)$,
$y_{n+1} = y_n + \frac{1}{6}(k_1+k_2+k_3+k_4)$. This generates the sequence of approximants given in the table below (computations rounded to six decimal places).

n	x_n	y_n
1	0.2	0.997337
2	0.4	0.978892
3	0.6	0.930530
4	0.8	0.843102
5	1.0	0.716530

Consequently the Runge-Kutta approximation to $y(1)$ is $y_{10} = 0.716530$. (Actual value: $y(1) = 0.716531$ rounded to 6 decimal places).

15. We have $y' = 2xy^2, x_0 = 0, y_0 = 1$, and $h = 0.1$. So, $k_1 = 0.2x_n - y_n^2, k_2 = 0.2(x_n+0.05)(y_n+\frac{k_1}{2})^2, k_3 = 0.2(x_n+0.05)(y_n+\frac{k_2}{2})^2, k_4 = 0.2x_{n+1}(y_n+k_3)^2]]$,
$y_{n+1} = y_n + \frac{1}{6}(k_1+k_2+k_3+k_4)$. This generates the sequence of approximants given in the table below (computations rounded to six decimal places).

n	x_n	y_n
1	0.1	0.502513
2	0.2	0.510204
3	0.3	0.523560
4	0.4	0.543478
5	0.5	0.571429
6	0.6	0.609756
7	0.7	0.662252
8	0.8	0.735295
9	0.9	0.840336
10	1.0	0.999996

Consequently the Runge-Kutta approximation to $y(1)$ is $y_{10} = 0.999996$. (Actual value: $y(1) = 1$).

16. We have $y'+\frac{1}{10}y = e^{-x/10}\cos x, x_0 = 0, y_0 = 0$, and $h = 0.5$ Hence, $k_1 = 0.5\left(-\frac{1}{10}y_n + e^{x_n/10}\cos x_n\right), k_2 = 0.5\left[-\frac{1}{10}\left(y_n + \frac{1}{2}k_1\right) + e^{(-x_n+0.25)/10}\cos(x_n + 0.25)\right], k_3 = 0.5\left[-\frac{1}{10}\left(y_n + \frac{1}{2}k_2\right) + e^{(-x_n+0.25)/10}\cos(x_n + 0.25)\right],$ $k_4 = 0.5\left[-\frac{1}{10}(y_n + k_3) + e^{-x_{n+1}/10}\cos x_{n+1}\right], y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$. This generates the sequence of approximants plotted in the accompanying figure. We see that the solution appears to be oscillating with a diminishing amplitude. Indeed, the exact solution to the initial value problem is $y(x) = e^{-x/10}\sin x$. The corresponding solution curve is also given in the figure.

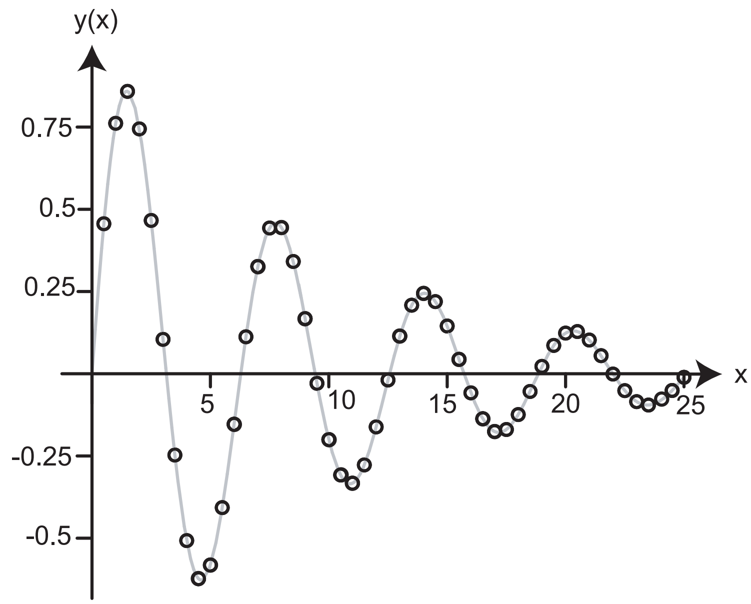

Figure 0.0.56: Figure for Problem 16

Solutions to Section 1.11

Problems:

1. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 6e^{3x}$. Let $u = \frac{dy}{dx}$ so that $\frac{du}{dx} = \frac{d^2y}{dx^2}$. Substituting these results into the first equation yields $\frac{du}{dx} - 2u = 6e^{3x}$. An appropriate integrating factor for this equation is $I(x) = e^{-2\int dx} = e^{-2x} \implies \frac{d}{dx}(e^{-2x}u) = 6e^x \implies e^{-2x}u = 6e^x + c_1 \implies u = 6e^{3x} + c_1e^{2x} \implies \frac{dy}{dx} = 6e^{3x} + c_1e^{2x} \implies y(x) = 2e^{3x} + c_1e^{2x} + c_2$, where we have redefined the constant c_1 in the last step.

2. $\frac{d^2y}{dx^2} = \frac{2}{x}\frac{dy}{dx} + 4x^2$. Let $u = \frac{dy}{dx}$ so that $\frac{du}{dx} = \frac{d^2y}{dx^2}$. Substituting these results into the first equation yields $\frac{du}{dx} = \frac{2}{x}u + 4x^2 \implies \frac{du}{dx} - \frac{2}{x}u = 4x^2$. An appropriate integrating factor for this equation is $I(x) =$

$e^{-2\int \frac{dx}{x}} = x^{-2} \Longrightarrow d(x^{-2}u) = 4 \Longrightarrow x^{-2}u = 4\int dx \Longrightarrow x^{-2}u = 4x + c_1 \Longrightarrow u = 4x^3 + c_1x^2 \Longrightarrow \frac{dy}{dx} = 4x^3 + c_1x^2 \Longrightarrow y(x) = c_1x^3 + x^4 + c_2$.

3. $\frac{d^2y}{dx^2} = \frac{1}{(x-1)(x-2)}\left[\frac{dy}{dx} - 1\right]$. Let $u = \frac{dy}{dx}$ so that $\frac{du}{dx} = \frac{d^2y}{dx^2}$. Substituting these results into the first equation yields $\frac{du}{dx} = \frac{1}{(x-1)(x-2)}(u-1) \Longrightarrow \frac{du}{dx} - \frac{1}{(x-1)(x-2)}u = -\frac{1}{(x-1)(x-2)}$. An appropriate integrating factor for this equation is $I(x) = e^{-\int \frac{1}{(x-1)(x-2)}dx} = \frac{x-1}{x-2} \Longrightarrow \frac{d}{dx}\left(\frac{x-1}{x-2}u\right) = \frac{1}{(x-2)^2} \Longrightarrow \frac{x-1}{x-2}u = \int (x-2)^{-2}dx \Longrightarrow u = -\frac{1}{x-1} + c_1 \Longrightarrow \frac{dy}{dx} = -\frac{1}{x-1} + c_1 \Longrightarrow y(x) = -\ln|x-1| + c_1x + c_2$.

4. $\frac{d^2y}{dx^2} + \frac{2}{y}\left(\frac{dy}{dx}\right)^2 = \frac{dy}{dx}$. Let $u = \frac{dy}{dx}$ so that $\frac{du}{dx} = u\frac{du}{dy} = \frac{d^2y}{dx^2}$. Substituting these results into the first equation yields $u\frac{du}{dy} + \frac{2}{y}u^2 = u \Longrightarrow u = 0$ or $\frac{du}{dy} - \frac{2}{y}u = 1$. An appropriate integrating factor for the last equation is $I(y) = e^{\int \frac{2}{y}dy} = y^2 \Longrightarrow \frac{d}{dy}(y^2u) = y^2 \Longrightarrow y^2u = \int y^2dy \Longrightarrow y^2u = \frac{y^3}{3} + c_1 \Longrightarrow \frac{dy}{dx} = \frac{y}{3} + \frac{c_1}{y^2} \Longrightarrow \ln|y^3 + c_2| = x + c_3 \Longrightarrow y(x) = \sqrt[3]{c_4e^x + c_5}$.

5. $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2 \tan y$. Let $u = \frac{dy}{dx}$ so that $\frac{du}{dx} = u\frac{du}{dy} = \frac{d^2y}{dx^2}$. Substituting these results into the first equation yields $u\frac{du}{dy} = u^2 \tan y$. If $u = 0$ then $\frac{du}{dx} = 0 \Longrightarrow y$ equals a constant and this is a solution to the equation. Now suppose that $u \neq 0$. Then $\frac{du}{dy} = u\tan y \Longrightarrow \int \frac{du}{u} = \int \tan y dy \Longrightarrow u = c_1 \sec y \Longrightarrow \frac{dy}{dx} = c_1 \sec y \Longrightarrow y(x) = \sin^{-1}(c_1x + c_2)$.

6. $\frac{d^2y}{dx^2} + \tan x\frac{dy}{dx} = \left(\frac{dy}{dx}\right)^2$. Let $u = \frac{dy}{dx}$ so that $\frac{du}{dx} = \frac{d^2y}{dx^2}$. Substituting these results into the first equation yields $\frac{du}{dx} + \tan xu = u^2$ which is a Bernoulli equation. Letting $z = u^{-1}$ gives $\frac{1}{u^2} = \frac{du}{dx} = -\frac{dz}{dx}$. Substituting these results into the last equation yields $\frac{dz}{dx} - \tan xz = -1$. Then an integrating factor for this equation is $I(x) = e^{-\int \tan x dx} = \cos x \Longrightarrow \frac{d}{dx}(z\cos x) = -\cos x \Longrightarrow z\cos x = -\int \cos x dx \Longrightarrow z = \frac{-\sin x + c_1}{\cos x} \Longrightarrow u = \frac{\cos x}{c_1 - \sin x} \Longrightarrow \frac{dy}{dx} = \frac{\cos x}{c_1 - \sin x} \Longrightarrow y(x) = c_2 - \ln|c_1 - \sin x|$.

7. $\frac{d^2x}{dt^2} = \left(\frac{dx}{dt}\right)^2 + 2\frac{dx}{dt}$. Let $u = \frac{dx}{dt}$ so that $\frac{du}{dt} = \frac{d^2x}{dt^2}$. Substituting these results into the first equation yields $\frac{du}{dt} = u^2 + 2u \Longrightarrow \frac{du}{dt} - 2u = u^2$ which is a Bernoulli equation. If $u = 0$ then x is a constant which satisfies the equation. Now suppose that $u \neq 0$. Let $z = u^{-1}$ so that $\frac{dz}{dt} = -\frac{1}{u^2}\frac{du}{dt}$. Substituting these results into the last equation yields $\frac{dz}{dt} + 2z = -1$. An integrating factor for this equation is $I(x) = e^{2t} \Longrightarrow \frac{d}{dt}(e^{2t}z) = -e^{2t} \Longrightarrow z = ce^{-2t} - \frac{1}{2} \Longrightarrow u = \frac{2e^{2t}}{2c - e^{2t}} \Longrightarrow x = \int \frac{2e^{2t}}{2c - e^{2t}}dt \Longrightarrow x(t) = c_2 - \ln|c_1 - e^{2t}|$.

8. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 6x^4$. Let $u = \frac{dy}{dx}$ so that $\frac{du}{dx} = \frac{d^2y}{dx^2}$. Substituting these results into the first equation yields $\frac{du}{dx} - \frac{2}{x}u = 6x^4$. An appropriate integrating factor for this equation is $I(x) = e^{-2\int \frac{dx}{x}} = x^{-2} \Longrightarrow \frac{d}{dx}(x^{-2}u) = 6x^2 \Longrightarrow x^{-2}u = 6\int x^2 dx \Longrightarrow u = 2x^5 + cx^2 \Longrightarrow \frac{dy}{dx} = 2x^5 + cx^2 \Longrightarrow y(x) = \frac{1}{3}x^6 + c_1x^3 + c_2$.

9. $t\frac{d^2x}{dt^2} = 2\left(t + \frac{dx}{dt}\right)$. Let $u = \frac{dx}{dt}$ so that $\frac{du}{dt} = \frac{d^2x}{dt^2}$. Substituting these results into the first equation yields $\frac{du}{dt} - \frac{2}{t}u = 2$. An integrating factor for this equation is $I(x) = t^{-2} \Longrightarrow \frac{d}{dt}(t^{-2}u) = 2t^{-2} \Longrightarrow u = -2t + ct^2 \Longrightarrow \frac{dx}{dt} = -2t + ct^2 \Longrightarrow x(t) = c_1t^3 - t^2 + c_2$.

10. $\frac{d^2y}{dx^2} - \alpha\left(\frac{dy}{dx}\right)^2 - \beta\frac{dy}{dx} = 0$. Let $u = \frac{dy}{dx}$ so that $\frac{du}{dx} = \frac{d^2y}{dx^2}$. Substituting these results into the first equation yields $\frac{du}{dx} - \beta u = \alpha u^2$ which is a Bernoulli equation. If $u = 0$ then y is a constant and satisfies the equation. Now suppose that $u \neq 0$. Let $z = u^{-1}$ so that $\frac{dz}{dx} = -u^{-2}\frac{du}{dx}$. Substituting these results into the last equation yields $\frac{dz}{dx} + \beta z = -\alpha$. The an integrating factor for this equation is $I(x) = e^{\beta\int dx} = e^{\beta x} \Longrightarrow e^{\beta x}z = -\alpha\int e^{\beta x}dx \Longrightarrow -\frac{\alpha}{\beta} + ce^{-\beta x} \Longrightarrow u = \frac{\beta e^{\beta x}}{c\beta - \alpha e^{\beta x}} \Longrightarrow \frac{dy}{dx} = \frac{\beta e^{\beta x}}{c\beta - \alpha e^{\beta x}} \Longrightarrow y = \int \frac{\beta e^{\beta x}}{c\beta - \alpha e^{\beta x}}dx \Longrightarrow y(x) = -\frac{1}{\alpha}\ln|c_1 + c_2e^{\beta x}|$.

11. $\frac{d^2y}{dx^2} - \frac{2}{x}\frac{dy}{dx} = 18x^4$. Let $u = \frac{dy}{dx}$ so that $\frac{du}{dx} = \frac{d^2y}{dx^2}$. Substituting these results into the first equation yields $\frac{du}{dx} - \frac{2}{x}u = 18x^4$ which has $I(x) = x^{-2}$ as an integrating factor so $\frac{d}{dx}(x^{-2}u) = 18x^2 \Longrightarrow u = 6x^5 + cx^2 \Longrightarrow \frac{dy}{dx} = 6x^5 + cx^2 \Longrightarrow y(x) = x^6 + c_1x^3 + c_2$.

12. $\frac{d^2y}{dx^2} = -\frac{2x}{1+x^2}\frac{dy}{dx}$. Let $u = \frac{dy}{dx}$ so that $\frac{du}{dx} = dfracd^2ydx^2$. If $u = 0$ then y is a constant and satisfies the equation. Now suppose that $u \neq 0$. Substituting these results into the first equation yields $\frac{du}{dx} = -\frac{2x}{1+x^2}u \Longrightarrow \ln|u| = -\ln(1+x^2) + c \Longrightarrow u = \frac{c_1}{1+x^2} \Longrightarrow \frac{dy}{dx} = \frac{c_1}{1+x^2} \Longrightarrow y(x) = c_1\tan^{-1}x + c_2$.

13. $\frac{d^2y}{dx^2} + \frac{1}{y}\left(\frac{dy}{dx}\right)^2 = ye^{-3}\left(\frac{dy}{dx}\right)^3$. Let $u = \frac{dy}{dx}$ so that $\frac{du}{dx} = u\frac{du}{dy} = \frac{d^2y}{dx^2}$. Substituting these results into the first equation yields $u\frac{du}{dx} + \frac{1}{y}u^2 = ye^{-y}u^3$. If $u = 0$ then y is a constant and satisfies the equation. Now suppose that $u \neq 0$. Substituting these results into the first equation yields $\frac{du}{dx} + \frac{u}{y} = ye^{-y}u^2$ which is a Bernoulli equation. Let $v = u^{-1}$ so that $\frac{dv}{dy} = -u^{-2}\frac{du}{dy}$. Substituting these results into the last equation yields $\frac{dv}{dy} - \frac{v}{y} = -ye^{-y}$. Then $I(y) = y^{-1}$ is an integrating factor for the equation thus $\frac{d}{dy}(y^{-1}v) = -e^{-1} \Longrightarrow v = y(e^{-1} + c) \Longrightarrow u = \frac{e^y}{y + cye^y} \Longrightarrow \frac{dy}{dx} = \frac{e^y}{y + cye^y} \Longrightarrow (ye^{-y} + cy)dy = dx \Longrightarrow e^{-y}(y+1) + c_1y^2 - x$.

14. $\frac{d^2y}{dx^2} - \tan x\frac{dy}{dx} = 1$. Let $u = \frac{dy}{dx}$ so that $u\frac{du}{dy} = \frac{d^2y}{dx^2}$. Substituting these results into the first equation yields $\frac{du}{dx} - u\tan x = 1$. An appropriate integrating factor for this equation is $I(x) = e^{-\int \tan x dx} = \cos x \Longrightarrow \frac{d}{dx}(u\cos x) = \cos x \Longrightarrow u\cos x = \sin x + c \Longrightarrow u(x) = \tan x + c\sec x \Longrightarrow \frac{dy}{dx} = \tan x + c\sec x \Longrightarrow y(x) = \ln\sec x + c_1\ln(\sec x + \tan x) + c_2$.

15. $y\frac{d^2y}{dx^2} = 2\left(\frac{dy}{dx}\right)^2 + y^2$. Let $u = \frac{dy}{dx}$ so that $\frac{du}{dx} = u\frac{du}{dy} = \frac{d^2y}{dx^2}$. Substituting these results into the first equation yields $u\frac{du}{dy} - \frac{2}{y}u^2 = y$, a Bernoulli equation. Let $z = u^2$ so that $u\frac{du}{dy} = \frac{1}{2}\frac{dz}{dy}$. Substituting these results into the last equation yields $\frac{dz}{dy} - \frac{4}{y}z = 2y$ which has $I(y) = y^{-4}$ as an integrating factor. Therefore, $\frac{d}{dy}(y^{-4}z) = 2y^{-3} \Longrightarrow z = c_1y^4 - y^2 \Longrightarrow u^2 = c_1y^4 - y^2 \Longrightarrow u = \pm\sqrt{c_1y^4 - y^2} \Longrightarrow \frac{dy}{dx} = \pm\sqrt{c_1y^4 - y^2} \Longrightarrow \cos^{-1}\left(\frac{1}{y\sqrt{c_1}}\right) = \pm x + c_2$. Using the facts that $f(0) = 1$ and $y'(0) = 0$ we find that $c_1 = 1$ and $c_2 = 0$; thus $y(x) = \sec x$.

16. $\frac{d^2y}{dx^2} = \omega^2 y$ where $\omega > 0$. Let $u = \frac{dy}{dx}$ so that $\frac{du}{dx} = u\frac{du}{dy} = \frac{d^2y}{dx^2}$. Substituting these results into the first equation yields $u\frac{du}{dy} = \omega^2 y \Longrightarrow u^2 = \omega^2y^2 + c_2$. Using the given that $y(0) = a$ and $y'(0) = 0$ we find that $c_2 = a^2\omega^2$. Then $\frac{dy}{dx} = \pm\omega\sqrt{y^2 - a^2} \Longrightarrow \frac{1}{\omega}\cosh^{-1}(y/a) = \pm x + c \Longrightarrow y(x) = a\cosh[\omega(c \pm x)] \Longrightarrow y' = \pm a\omega\sinh[\omega(c \pm x)]$ and since $y'(0) = 0, c = 0$; hence, $y(x) = a\cosh(\omega x)$.

17. Let $u = \frac{dy}{dx}$ so that $u\frac{du}{dx} = \frac{d^2y}{dx^2}$. Substituting these results into the differential equation yields $u\frac{du}{dy} = \frac{1}{a}\sqrt{1 + u^2}$. Separating the variables and integrating we obtain $\sqrt{1 + u^2} = \frac{1}{a}y + c$. Imposing the initial conditions $y(0) = a, \frac{dy}{dx}(0) = 0$ gives $c = 0$. Hence, $\sqrt{1 + u^2} = \frac{1}{a}y$ so that $1 + u^2 = \frac{1}{a^2}y^2$ or equivalently, $u = \pm\sqrt{y^2/a^2 - 1}$. Substituting $u = \frac{dy}{dx}$ and separating the variables gives $\frac{1}{\sqrt{y^2 - a^2}} = \pm\frac{1}{|a|}dx$ which can be integrated to obtain $\cosh^{-1}(y/a) = \pm x/a + c_1$ so that $y = a\cosh(\pm x/a + c_1)$. Imposing the initial conditions $y(0) = a$ gives $c_1 = 0$ so that $y(x) = a\cosh(x/a)$.

18. $\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} = q(x)$. Let $u = \frac{dy}{dx}$ so that $\frac{du}{dx} = \frac{d^2y}{dx^2}$. Substituting these results into the first equation gives us the equivalent system: $\frac{du}{dx}+p(x)u = q(x)$ which has a solution $u = e^{-\int p(x)dx}\left[\int e^{-\int p(x)dx}q(x)dx + c_1\right]$ so $\frac{dy}{dx} = e^{-\int p(x)dx}\left[\int e^{-\int p(x)dx}q(x)dx + c_1\right]$. Thus $y = \int\left\{e^{-\int p(x)dx}\left[\int e^{-\int p(x)dx}q(x)dx + c_1 dx\right]\right\} + c_2$ is a solution to the original equation.

19. (a). $u_1 = y \Longrightarrow u_2 = \frac{du_1}{dx} = \frac{dy}{dx} \Longrightarrow u_3 = \frac{du_2}{dx} = \frac{d^2y}{dx^2} \Longrightarrow \frac{du_3}{dx} = \frac{d^3y}{dx^3}$; thus $\frac{d^3y}{dx^3} = F\left(x, \frac{d^2y}{dx^2}\right)$ since the latter equation is equivalent to $\frac{du_3}{dx} = F(x, u_3)$.

(b). $\frac{d^3y}{dx^3} = \frac{1}{x}\left(\frac{d^2y}{dx^2} - 1\right)$. Replace this equation by the equivalent first order system: $\frac{du_1}{dx} = u_2, \frac{du_2}{dx} = u_3$, and $\frac{du_3}{dx} = \frac{1}{x}(u_3 - 1) \Longrightarrow \int \frac{du_3}{u_3 - 1} = \int \frac{dx}{x} \Longrightarrow u_3 = Kx + 1 \Longrightarrow \frac{du_2}{dx} = Kx + 1 \Longrightarrow u_2 = \frac{K}{2}x^2 + x + c_2 \Longrightarrow \frac{du_1}{dx} = \frac{K}{2}x^2 + x + c_2 \Longrightarrow u_1 = \frac{K}{6}x^3 + \frac{1}{2}x^2 + c_2x + c_3 \Longrightarrow y(x) = u_1 = c_1x^3 + \frac{1}{2}x^2 + c_2x + c_3$.

19. Given $\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0, \theta(0) = \theta_0$, and $\frac{d\theta}{dt}(0) = 0$.

(a). $\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$. Let $u = \frac{d\theta}{dt}$ so that $\frac{du}{dt} = \frac{d^2\theta}{dt^2} = \frac{du}{d\theta}\frac{d\theta}{dt} = u\frac{du}{d\theta}$. Substituting these results into the last equation yields $u\frac{du}{d\theta} + \frac{g}{L}\theta = 0 \Longrightarrow u^2 = -\frac{g}{L}\theta^2 + c_1^2$, but $\frac{d\theta}{dt}(0) = 0$ and $\theta(0) = \theta_0$ so $c_1^2 = \frac{g}{L}\theta_0^2 \Longrightarrow u^2 = \frac{g}{L}(\theta_0^2 - \theta^2) \Longrightarrow u = \pm\sqrt{\frac{g}{L}}\sqrt{\theta_0^2 - \theta^2} \Longrightarrow \sin^{-1}\left(\frac{\theta}{\theta_0}\right) = \pm\sqrt{\frac{g}{L}}t + c_2$, but $\theta(0) = \theta_0$ so $c_2 = \frac{\pi}{2} \Longrightarrow \sin^{-1}\left(\frac{\theta}{\theta_0}\right) = \frac{\pi}{2} \pm \sqrt{\frac{g}{L}}t \Longrightarrow \theta = \theta_0\sin\left(\frac{\pi}{2} \pm \sqrt{\frac{g}{L}}t\right) \Longrightarrow \theta = \theta_0\cos\left(\sqrt{\frac{g}{L}}t\right)$. Yes, the predicted motion is reasonable.

(b). $\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$. Let $u = \frac{d\theta}{dt}$ so that $\frac{du}{dt} = \frac{d^2\theta}{dt^2} = \frac{du}{d\theta}\frac{d\theta}{dt} = u\frac{du}{d\theta}$. Substituting these results into the last equation yields $u\frac{du}{d\theta} + \frac{g}{L}\sin\theta = 0 \Longrightarrow u^2 = \frac{2g}{L}\cos\theta + c$. Since $\theta(0) = \theta_0$ and $\frac{d\theta}{dt}(0) = 0$, then $c = -\frac{2g}{L}\cos\theta_0$ and so $u^2 = \frac{2g}{L}\cos\theta - \frac{2g}{L}\cos\theta_0 \Longrightarrow \frac{d\theta}{dt} = \pm\sqrt{\frac{2g}{L}\cos\theta - \frac{2g}{L}\cos\theta_0} \Longrightarrow \frac{d\theta}{dt} = \pm\sqrt{\frac{2g}{L}}[\cos\theta - \cos\theta_0]^{1/2}$.

(c). From part (b), $\sqrt{\frac{L}{2g}}\frac{d\theta}{[\cos\theta - \cos\theta_0]^{1/2}} = \pm dt$. When the pendulum goes from $\theta = \theta_0$ to $\theta = 0$ (which corresponds to one quarter of a period) $\frac{d\theta}{dt}$ is negative; hence, choose the negative sign. Thus, $T = -\sqrt{\frac{L}{2g}}\int_{\theta_0}^{0}\frac{d\theta}{[\cos\theta - \cos\theta_0]^{1/2}} \Longrightarrow T = \sqrt{\frac{L}{2g}}\int_0^{\theta_0}\frac{d\theta}{[\cos\theta - \cos\theta_0]^{1/2}}$.

(d). $T = \sqrt{\frac{L}{2g}}\int_0^{\theta_0}\frac{d\theta}{[\cos\theta - \cos\theta_0]^{1/2}} \Longrightarrow$

$$T = \sqrt{\frac{L}{2g}}\int_0^{\theta_0}\frac{d\theta}{\left[2\sin^2\left(\frac{\theta_0}{2}\right) - 2\sin^2\left(\frac{\theta}{2}\right)\right]^{1/2}} = \frac{1}{2}\sqrt{\frac{L}{2g}}\int_0^{\theta_0}\frac{d\theta}{\left[\sin^2\left(\frac{\theta_0}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)\right]^{1/2}}.$$

Let $k = \sin\left(\frac{\theta_0}{2}\right)$ so that

$$T = \frac{1}{2}\sqrt{\frac{L}{2g}}\int_0^{\theta_0} \frac{d\theta}{\left[k^2 - \sin^2\left(\frac{\theta}{2}\right)\right]^{1/2}}. \tag{0.0.7}$$

Now let $\sin\theta/2 = k\sin u$. When $\theta = 0, u = 0$ and when $\theta = \theta_0, u = \pi/2$; moreover, $d\theta = \frac{2k\cos(u)du}{\cos(\theta/2)} \Longrightarrow d\theta = \frac{2k\sqrt{1-\sin^2(u)du}}{\sqrt{1-\sin^2(\theta/2)}} \Longrightarrow d\theta = \frac{2\sqrt{k^2-(k\sin(u))^2du}}{\sqrt{1-k^2\sin^2(u)}} \Longrightarrow d\theta = \frac{2\sqrt{k^2-\sin^2(\theta/2)du}}{\sqrt{1-k^2\sin^2(u)}}$. Making this change of variables in equation (0.0.7) yields

$$T = \sqrt{\frac{L}{g}}\int_0^{\pi/2} \frac{du}{\sqrt{1-k^2\sin^2(u)}} \quad \text{where} \quad k = \sin\theta_0/2.$$

Solutions to Section 1.12

1. The acceleration of gravity is $a = 9.8$ meters/sec^2. Integrating, we find that the vertical component of the velocity of the rocket is $v(t) = 4.9t + c_1$. We are given that $v(0) = -10$, so that $c_1 = -10$. Thus, $v(t) = 4.9t - 10$. Integrating again, we find the position $s(t) = 2.495t^2 - 10t + c_2$. Setting $s = 0$ at two meters above the ground, we have $s(0) = 0$ so that $s(t) = 2.495t^2 - 10t$.

(a). The highest point above the ground is obtained when $v(t) = 0$. That is, $t = \frac{10}{4.9} \approx 2.04$ seconds. Thus, the highest point is approximately $s(2.04) = 2.495 \cdot (2.04)^2 - 10(2.04) \approx -10.02$, which is 12.02 meters above the ground.

(b). The rocket hits the ground when $s(t) = 2$. That is $2.495t^2 - 10t - 2 = 0$. Solving for t with the quadratic formula, we find that $t = -0.19$ or $t = 4.27$. Since we must report a positive answer, we conclude that the rocket hits the ground 4.27 seconds after launch.

2. The acceleration of gravity is $a = 32$ ft/sec^2. Integrating, we find that the vertical component of the velocity of the ball is $v(t) = 16t + c_1$. Since the ball is initially hit horizontally, we have $v(0) = 0$, so that $c_1 = 0$. Hence, $v(t) = 16t$. Integrating again, we find the position $s(t) = 8t^2 + c_2$. Setting $s = 0$ at two feet above the ground, we have $s(0) = 0$ so that $c_2 = 0$. Thus, $s(t) = 8t^2$. The ball hits the ground when $s(t) = 2$, so that $t^2 = \frac{1}{4}$. Therefore, $t = \frac{1}{2}$. Since 80 miles per hour equates to over 117 ft/sec. In one-half second, the horizontal change in position of the ball is therefore more than $\frac{117}{2} = 58.5$ feet, more than enough to span the necessary 40 feet for the ball to reach the front wall. Therefore, the ball does reach the front wall before hitting the ground.

3. We first determine the slope of the given family at the point (x, y). Differentiating

$$y = cx^3 \tag{0.0.8}$$

with respect to x yields

$$\frac{dy}{dx} = 3cx^2. \tag{0.0.9}$$

From (0.0.8) we have $c = \frac{y}{x^3}$ which, when substituted into Equation (0.0.9) yields

$$\frac{dy}{dx} = \frac{3y}{x}.$$

Consequently, the differential equation for the orthogonal trajectories is

$$\frac{dy}{dx} = -\frac{x}{3y}.$$

Separating the variables and integrating gives

$$\frac{3}{2}y^2 = -\frac{1}{2}x^2 + C,$$

which can be written in the equivalent form

$$x^2 + 3y^2 = k.$$

4. We first determine the slope of the given family at the point (x, y). Differentiating

$$y = \ln(cx) \tag{0.0.10}$$

with respect to x yields

$$\frac{dy}{dx} = \frac{1}{x}.$$

Consequently, the differential equation for the orthogonal trajectories is

$$\frac{dy}{dx} = -x.$$

which can be integrated directly to obtain

$$y = -\frac{1}{2}x^2 + k.$$

5. We first determine the slope of the given family at the point (x, y). Differentiating

$$y^2 = cx^3 \tag{0.0.11}$$

with respect to x yields

$$2y\frac{dy}{dx} = 3cx^2$$

so that

$$\frac{dy}{dx} = \frac{3cx^2}{2y}. \tag{0.0.12}$$

From (0.0.11) we have $c = \frac{y^2}{x^3}$ which, when substituted into Equation (0.0.12) yields

$$\frac{dy}{dx} = \frac{3y}{2x}.$$

Consequently, the differential equation for the orthogonal trajectories is

$$\frac{dy}{dx} = -\frac{2x}{3y}.$$

Separating the variables and integrating gives

$$\frac{3}{2}y^2 = -x^2 + C,$$

which can be written in the equivalent form

$$2x^2 + 3y^2 = k.$$

6. We first determine the slope of the given family at the point (x, y). Differentiating

$$x^4 + y^4 = c \tag{0.0.13}$$

with respect to x yields

$$4x^3 + 4y^3\frac{dy}{dx} = 0$$

so that

$$\frac{dy}{dx} = -\frac{x^3}{y^3}. \tag{0.0.14}$$

Consequently, the differential equation for the orthogonal trajectories is

$$\frac{dy}{dx} = \frac{y^3}{x^3}.$$

Separating the variables and integrating gives

$$-\frac{1}{2}y^{-2} = -\frac{1}{2}x^{-2} + C,$$

which can be written in the equivalent form

$$y^2 - x^2 = kx^2y^2.$$

7. (a). We first determine the slope of the given family at the point (x, y). Differentiating

$$x^2 + 3y^2 = 2cy \tag{0.0.15}$$

with respect to x yields

$$2x + 6y\frac{dy}{dx} = 2c\frac{dy}{dx}$$

so that

$$\frac{dy}{dx} = \frac{x}{c - 3y}. \tag{0.0.16}$$

From (0.0.15) we have $c = \frac{x^2+3y^2}{2y}$ which, when substituted into Equation (0.0.16) yields

$$\frac{dy}{dx} = \frac{x}{\frac{x^2+3y^2}{2y} - 3y} = \frac{2xy}{x^2 - 3y^2}, \tag{0.0.17}$$

as required.

(b). It follows from Equation(0.0.17) that the differential equation for the orthogonal trajectories is

$$\frac{dy}{dx} = -\frac{3y^2 - x^2}{2xy}.$$

This differential equation is first-order homogeneous. Substituting $y = xV$ into the preceding differential equation gives

$$x\frac{dV}{dx} + V = \frac{3V^2 - 1}{2V}$$

which simplifies to

$$\frac{dV}{dx} = \frac{V^2 - 1}{2V}.$$

Separating the variables and integrating we obtain

$$\ln(V^2 - 1) = \ln x + C,$$

or, upon exponentiation,

$$V^2 - 1 = kx.$$

Inserting $V = y/x$ into the preceding equation yields

$$\frac{y^2}{x^2} - 1 = kx,$$

that is,

$$y^2 - x^2 = kx^3.$$

8. Slope field.

9. Slope field.

10. Slope field.

11. See accompanying figure.

12. Slope field.

13. **(a).** If $v(t) = 25$, then

$$\frac{dv}{dt} = 0 = \frac{1}{2}(25 - v).$$

(b). The accompanying figure suggests that

$$\lim_{t\to\infty} v(t) = 25.$$

14. **(a).** The equilibrium solutions are any constant values of m that satisfy

$$am^{3/4}\left[1 - \left(\frac{m}{M}\right)^{1/4}\right] = 0.$$

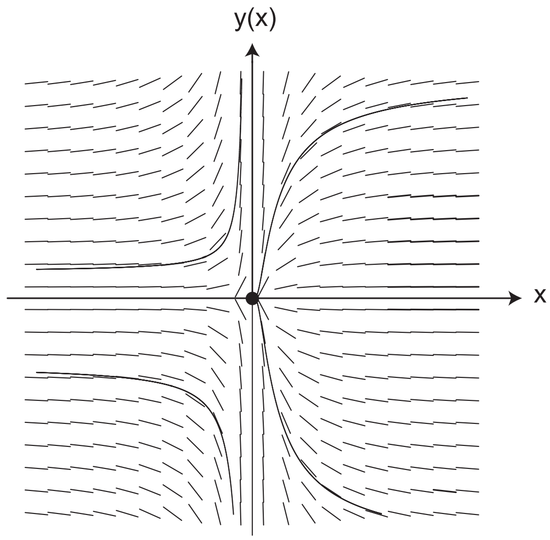

Figure 0.0.57: Figure for Problem 11

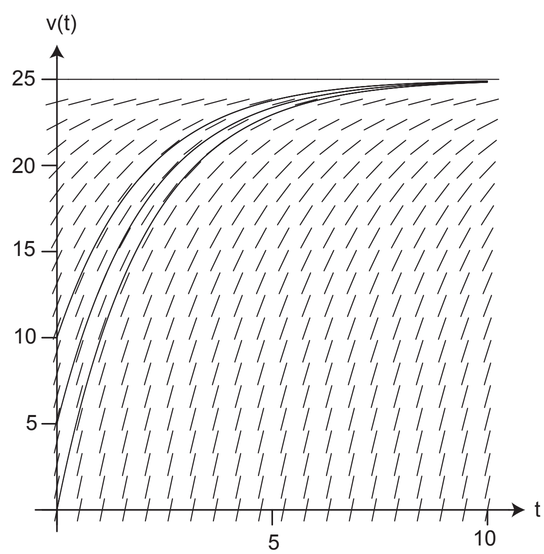

Figure 0.0.58: Figure for Problem 13

Hence, there are two equilibrium solutions, namely, $m = 0$ and $m = M$.

(b). This follows since $a > 0$, and $0 < m(t) < M$.

(c). The given differential equation can be written in the equivalent form

$$\frac{dm}{dt} = a\left(m^{3/4} - \frac{1}{M^{1/4}}m\right),$$

so that

$$\begin{aligned}\frac{d^2m}{dt^2} &= a\left(\frac{3}{4}m^{-1/4} - \frac{1}{M^{1/4}}\right)\frac{dm}{dt} \\ &= a^2m^{3/4}\left(m^{3/4} - \frac{1}{M^{1/4}}m\right)\left[1-\left(\frac{m}{M}\right)^{1/4}\right] \\ &= \frac{1}{4}a^2m^{1/2}\left[3-4\left(\frac{m}{M}\right)^{1/4}\right]\left[1-\left(\frac{m}{M}\right)^{1/4}\right].\end{aligned}$$

Since $0 < m < M$, the expression on the right-hand side of the preceding equation is positive when

$$3 - 4\left(\frac{m}{M}\right)^{1/4} > 0,$$

that is, $\left(\frac{m}{M}\right)^{1/4} < \frac{3}{4}$, or equivalently, $m < \frac{81}{256}M$. Consequently, the solution curves are concave up for $0 < m < \frac{81}{256}M$, and concave down for $\frac{81}{256}M < m < M$.

(d). From the results of (c), there is a change in concavity when $m = \frac{81}{256}M$. Substituting this value of m into the right-hand side of the given differential equation yields the following value for the slope of the solutions curves at the point of inflection:

$$a\left(\frac{81}{256}\right)^{3/4}\left[1-\left(\frac{81}{256}\right)^{1/4}\right] = \frac{27}{256}a.$$

(e). Slope field.

15. (a). Separating the variables in Equation (1.12.6) yields

$$\frac{mv}{mg - kv^2}\frac{dv}{dy} = 1$$

which can be integrated to obtain

$$-\frac{m}{2k}\ln(mg - kv^2) = y + c.$$

Multiplying both sides of this equation by -1 and exponentiating gives

$$mg - kv^2 = c_1e^{-\frac{2k}{m}y}.$$

The initial condition $v(0) = 0$ requires that $c_1 = mg$, which, when inserted into the preceding equation yields

$$mg - kv^2 = mge^{-\frac{2k}{m}y},$$

or equivalently,

$$v^2 = \frac{mg}{k}\left(1 - e^{-\frac{2k}{m}y}\right),$$

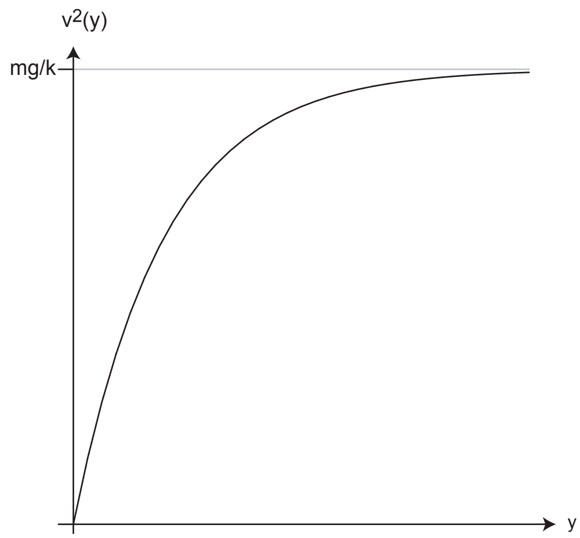

Figure 0.0.59: Figure for Problem 15

as required.

(b). See accompanying figure.

16. By inspection the differential equation is separable, but not first-order homogeneous. Further, the differential equation can be re-written as

$$\frac{dy}{dx} + x^2y = x^2y^2$$

which reveals that it is also a Bernoulli equation, but is not linear. Finally, a different rearrangement of terms yields

$$x^2\,dx - \frac{1}{y(y-1)}\,dy = 0,$$

which is an exact differential equation. Separating the variables in the given differential equation yields

$$\frac{1}{y(y-1)}\,dy = x^2\,dx \Longrightarrow \int \frac{1}{y(y-1)}\,dy = \frac{1}{3}x^3 + c$$

Using a partial fraction decomposition of the integrand on the left-hand side of the preceding equation we obtain

$$\int \left(\frac{1}{y-1} - \frac{1}{y}\right) dy = \frac{1}{3}x^3 + c \Longrightarrow \ln\left(\frac{y-1}{y}\right) = \frac{1}{3}x^3 + c \Longrightarrow \frac{y-1}{y} = c_1e^{x^3/3}$$

so that

$$y(x) = \frac{1}{1 - c_1e^{x^3/3}}.$$

41. Writing the differential equation in the form

$$\frac{dy}{dx} = xe^x \cdot \frac{e^{-y}}{1+y}$$

we see that it is separable, but not homogeneous. It is also neither a linear differential equation nor a Bernoulli differential equation. Rearranging the differential equation we have

$$xe^x\,dx - e^y(1+y)dy = 0$$

which is exact. Separating the variables in the given differential equation and integrating yields

$$\int e^y(1+y)dy = \int xe^x\,dx + c \Longrightarrow ye^y = e^x(x-1) + c.$$

17. The given differential equation is separable. Separating the variables gives

$$y\frac{dy}{dx} = 2\frac{\ln x}{x},$$

which can be integrated directly to obtain

$$\frac{1}{2}y^2 = (\ln x)^2 + c,$$

or, equivalently,

$$y^2 = 2(\ln x)^2 + c_1.$$

18. The given differential equation is first-order linear. We first divide by x to put the differential equation in standard form:

$$\frac{dy}{dx} - \frac{2}{x}y = 2x\ln x. \tag{0.0.18}$$

An integrating factor for this equation is $I = e^{\int(-2/x)dx} = x^{-2}$. Multiplying Equation (0.0.18) by x^{-2} reduces it to

$$\frac{d}{dx}(x^{-2}y) = 2x^{-1}\ln x,$$

which can be integrated to obtain

$$x^{-2}y = (\ln x)^2 + c$$

so that

$$y(x) = x^2[(\ln x)^2 + c].$$

19. We first re-write the given differential equation in the differential form

$$2xy\,dx + (x^2 + 2y)dy = 0. \tag{0.0.19}$$

Then

$$M_y = 2x = N_x$$

so that the differential equation is exact. Consequently, there exists a potential function ϕ satisfying

$$\frac{\partial\phi}{\partial x} = 2xy, \quad \frac{\partial\phi}{\partial y} = x^2 + 2y.$$

Integrating these two equations in the usual manner yields

$$\phi(x,y) = x^2y + y^2.$$

Therefore Equation (0.0.19) can be written in the equivalent form

$$d(x^2y + y^2) = 0$$

with general solution

$$x^2y + y^2 = c.$$

20. We first rewrite the given differential equation as

$$\frac{dy}{dx} = \frac{y^2 + 3xy + x^2}{x^2},$$

which is first order homogeneous. Substituting $y = xV$ into the preceding equation yields

$$x\frac{dV}{dx} + V = V^2 + 3V + 1$$

so that

$$x\frac{dV}{dx} = V^2 + 2V + 1 = (V+1)^2,$$

or, in separable form,

$$\frac{1}{(V+1)^2}\frac{dV}{dx} = \frac{1}{x}.$$

This equation can be integrated to obtain

$$-(V+1)^{-1} = \ln x + c$$

so that

$$V + 1 = \frac{1}{c_1 - \ln x}.$$

Inserting $V = y/x$ into the preceding equation yields

$$\frac{y}{x} + 1 = \frac{1}{c_1 - \ln x},$$

so that

$$y(x) = \frac{x}{c_1 - \ln x} - x.$$

21. We first rewrite the given differential equation in the equivalent form

$$\frac{dy}{dx} + y \cdot \tan x = -y^2 \sin x,$$

which is a Bernoulli equation. Dividing this equation by y^2 yields

$$y^{-2}\frac{dy}{dx} + y^{-1}\tan x = -\sin x. \tag{0.0.20}$$

Now make the change of variables $u = y^{-1}$ in which case $\frac{du}{dx} = -y^{-2}\frac{dy}{dx}$. Substituting these results into Equation (0.0.20) gives the linear differential equation

$$-\frac{du}{dx} + u \cdot \tan x = -\sin x$$

or, in standard form,

$$\frac{du}{dx} - u \cdot \tan x = \sin x. \tag{0.0.21}$$

An integrating factor for this differential equation is $I = e^{-\int \tan x\, dx} = \cos x$. Multiplying Equation (0.0.21) by $\cos x$ reduces it to

$$\frac{d}{dx}(u \cdot \cos x) = \sin x \cos x$$

which can be integrated directly to obtain

$$u \cdot \cos x = -\frac{1}{2}\cos^2 x + c_1,$$

so that

$$u = \frac{-\cos^2 x + c_2}{\cos x}.$$

Inserting $u = y^{-1}$ into the preceding equation and rearranging yields

$$y(x) = \frac{2\cos x}{-\cos^2 x + c_2} = \frac{-2\cos x}{\cos^2 x + c}.$$

22. The given differential equation is linear with integrating factor

$$I = e^{\int \frac{2e^{2x}}{1+e^{2x}}\, dx} = e^{\ln(1+e^{2x})} = 1 + e^{2x}.$$

Multiplying the given differential equation by $1 + e^{2x}$ yields

$$\frac{d}{dx}\left[(1+e^{2x})y\right] = \frac{e^{2x}+1}{e^{2x}-1} = -1 + \frac{2e^{2x}}{e^{2x}-1}$$

which can be integrated directly to obtain

$$(1+e^{2x})y = -x + \ln|e^{2x}-1| + c,$$

so that

$$y(x) = \frac{-x + \ln|e^{2x}-1| + c}{1+e^{2x}}.$$

23. We first rewrite the given differential equation in the equivalent form

$$\frac{dy}{dx} = \frac{y + \sqrt{x^2 - y^2}}{x},$$

which we recognize as being first order homogeneous. Inserting $y = xV$ into the preceding equation yields

$$x\frac{dV}{dx} + V = V + \frac{|x|}{x}\sqrt{1-V^2},$$

that is,

$$\frac{1}{\sqrt{1-V^2}}\frac{dV}{dx} = \pm\frac{1}{x}.$$

Integrating we obtain

$$\sin^{-1} V = \pm \ln|x| + c,$$

so that

$$V = \sin(c \pm \ln|x|).$$

Inserting $V = y/x$ into the preceding equation yields

$$y(x) = x\sin(c \pm \ln|x|).$$

24. We first rewrite the given differential equation in the equivalent form

$$(\sin y + y\cos x + 1)dx - (1 - x\cos y - \sin x)dy = 0.$$

Then

$$M_y = \cos y + \cos x = N_x$$

so that the differential equation is exact. Consequently, there is a potential function satisfying

$$\frac{\partial\phi}{\partial x} = \sin y + y\cos x + 1, \quad \frac{\partial\phi}{\partial y} = -(1 - x\cos y - \sin x).$$

Integrating these two equations in the usual manner yields

$$\phi(x, y) = x - y + x\sin y + y\sin x,$$

so that the differential equation can be written as

$$d(x - y + x\sin y + y\sin x) = 0,$$

and therefore has general solution

$$x - y + x\sin y + y\sin x = c.$$

25. Writing the given differential equation as

$$\frac{dy}{dx} + \frac{1}{x}y = \frac{25}{2}y^{-1}x^2\ln x,$$

we see that it is a Bernoulli equation with $n = -1$. We therefore divide the equation by y^{-1} to obtain

$$y\frac{dy}{dx} + \frac{1}{x}y^2 = \frac{25}{2}x^2\ln x.$$

We now make the change of variables $u = y^2$, in which case, $\frac{du}{dx} = 2y\frac{dy}{dx}$. Inserting these results into the preceding differential equation yields

$$\frac{1}{2}\frac{du}{dx} + \frac{1}{x}u = \frac{25}{2}x^2\ln x,$$

or, in standard form,

$$\frac{du}{dx} + \frac{2}{x}u = 25x^2\ln x.$$

An integrating factor for this linear differential equation is $I = e^{\int(2/x)dx} = x^2$. Multiplying the previous differential equation by x^2 reduces it to

$$\frac{d}{dx}(x^2u) = 25x^4\ln x$$

which can be integrated directly to obtain

$$x^2u = 25\left(\frac{1}{5}x^5\ln x - \frac{1}{25}x^5\right) + c$$

so that

$$u = x^3(5\ln x - 1) + cx^{-2}.$$

Making the replacement $u = y^2$ in this equation gives

$$y^2 = x^3(5\ln x - 1) + cx^{-2}.$$

26. The given differential equation can be written in the equivalent form

$$\frac{dy}{dx} = \frac{e^{x-y}}{e^{2x+y}} = e^{-x}e^{-2y},$$

which we recognize as being separable. Separating the variables gives

$$e^{2y}\frac{dy}{dx} = e^{-x}$$

which can be integrated to obtain

$$\frac{1}{2}e^{2y} = -e^{-x} + c$$

so that

$$y(x) = \frac{1}{2}\ln(c_1 - 2e^{-x}).$$

27. The given differential equation is linear with integrating factor $I = e^{\int \cot x\,dx} = \sin x$. Multiplying the given differential equation by $\sin x$ reduces it to

$$\frac{d}{dx}(y\sin x) = \frac{\sin x}{\cos x}$$

which can be integrated directly to obtain

$$y\sin x = -\ln(\cos x) + c,$$

so that

$$y(x) = \frac{c - \ln(\cos x)}{\sin x}.$$

28. Writing the given differential equation as

$$\frac{dy}{dx} + \frac{2e^x}{1+e^x}y = 2y^{\frac{1}{2}}e^{-x},$$

we see that it is a Bernoulli equation with $n = 1/2$. We therefore divide the equation by $y^{\frac{1}{2}}$ to obtain

$$y^{-\frac{1}{2}}\frac{dy}{dx} + \frac{2e^x}{1+e^x}y^{\frac{1}{2}} = 2e^{-x}.$$

We now make the change of variables $u = y^{\frac{1}{2}}$, in which case, $\frac{du}{dx} = \frac{1}{2}y^{-\frac{1}{2}}\frac{dy}{dx}$. Inserting these results into the preceding differential equation yields

$$2\frac{du}{dx} + \frac{2e^x}{1+e^x}u = 2e^{-x},$$

or, in standard form,

$$\frac{du}{dx} + \frac{e^x}{1+e^x}u = e^{-x}.$$

An integrating factor for this linear differential equation is

$$I = e^{\int \frac{e^x}{1+e^x}\,dx} = e^{\ln(1+e^x)} = 1 + e^x.$$

Multiplying the previous differential equation by $1 + e^x$ reduces it to

$$\frac{d}{dx}\left[(1+e^x)u\right] = e^{-x}(1+e^x) = e^{-x} + 1$$

which can be integrated directly to obtain

$$(1+e^x)u = -e^{-x} + x + c$$

so that

$$u = \frac{x - e^{-x} + c}{1+e^x}.$$

Making the replacement $u = y^{\frac{1}{2}}$ in this equation gives

$$y^{\frac{1}{2}} = \frac{x - e^{-x} + c}{1+e^x}.$$

29. We first rewrite the given differential equation in the equivalent form

$$\frac{dy}{dx} = \frac{y}{x}\left[\ln\left(\frac{y}{x}\right) + 1\right].$$

The function appearing on the right of this equation is homogeneous of degree zero, and therefore the differential equation itself is first order homogeneous. We therefore insert $y = xV$ into the differential equation to obtain

$$x\frac{dV}{x} + V = V(\ln V + 1),$$

so that

$$x\frac{dV}{dx} = V\ln V.$$

Separating the variables yields

$$\frac{1}{V\ln V}\frac{dV}{dx} = \frac{1}{x}$$

which can be integrated to obtain

$$\ln(\ln V) = \ln x + c.$$

Exponentiating both side of this equation gives

$$\ln V = c_1 x,$$

or equivalently,

$$V = e^{c_1 x}.$$

Inserting $V = y/x$ in the preceding equation yields

$$y = xe^{c_1 x}.$$

30. For the given differential equation we have

$$M(x, y) = 1 + 2xe^y, \quad N(x, y) = -(e^y + x),$$

so that

$$\frac{M_y - N_x}{M} = \frac{1 + 2xe^y}{1 + 2xe^y} = 1.$$

Consequently, an integrating factor for the given differential equation is

$$I = e^{-\int dy} = e^{-y}.$$

Multiplying the given differential equation by e^{-y} yields the exact differential equation

$$(2x + e^{-y})dx - (1 + xe^{-y})dy = 0. \tag{0.0.22}$$

Therefore, there exists a potential function ϕ satisfying

$$\frac{\partial \phi}{\partial x} = 2x + e^{-y}, \quad \frac{\partial \phi}{\partial y} = -(1 + xe^{-y}).$$

Integrating these two equations in the usual manner yields

$$\phi(x, y) = x^2 - y + xe^{-y}.$$

Therefore Equation (0.0.22) can be written in the equivalent form

$$d(x^2 - y + xe^{-y}) = 0$$

with general solution

$$x^2 - y + xe^{-y} = c.$$

31. The given differential equation is first-order linear. However, it can also e written in the equivalent form

$$\frac{dy}{dx} = (1 - y)\sin x$$

which is separable. Separating the variables and integrating yields

$$-\ln|1 - y| = -\cos x + c,$$

so that

$$1 - y = c_1 e^{\cos x}.$$

Hence,

$$y(x) = 1 - c_1 e^{\cos x}.$$

32. For the given differential equation we have

$$M(x,y) = 3y^2 + x^2, \quad N(x,y) = -2xy,$$

so that

$$\frac{M_y - N_x}{N} = -\frac{4}{x}.$$

Consequently, an integrating factor for the given differential equation is

$$I = e^{-\int \frac{4}{x}dx} = x^{-4}.$$

Multiplying the given differential equation by x^{-4} yields the exact differential equation

$$(3y^2x^{-4} + x^{-2})dx - 2yx^{-3}dy = 0. \tag{0.0.23}$$

Therefore, there exists a potential function ϕ satisfying

$$\frac{\partial \phi}{\partial x} = 3y^2x^{-4} + x^{-2}, \quad \frac{\partial \phi}{\partial y} = -2yx^{-3}.$$

Integrating these two equations in the usual manner yields

$$\phi(x,y) = -y^2x^{-3} - x^{-1}.$$

Therefore Equation (0.0.23) can be written in the equivalent form

$$d(-y^2x^{-3} - x^{-1}) = 0$$

with general solution

$$-y^2x^{-3} - x^{-1} = c,$$

or equivalently,

$$x^2 + y^2 = c_1x^3.$$

Notice that the given differential equation can be written in the equivalent form

$$\frac{dy}{dx} = \frac{3y^2 + x^2}{2xy},$$

which is first-order homogeneous. Another equivalent way of writing the given differential equation is

$$\frac{dy}{dx} - \frac{3}{2x}y = \frac{1}{2}xy^{-1},$$

which is a Bernoulli equation.

33. The given differential equation can be written in the equivalent form

$$\frac{dy}{dx} - \frac{1}{2x\ln x}y = -\frac{9}{2}x^2y^3,$$

which is a Bernoulli equation with $n = 3$. We therefore divide the equation by y^3 to obtain

$$y^{-3}\frac{dy}{dx} - \frac{1}{2x\ln x}y^{-2} = -\frac{9}{2}x^2.$$

We now make the change of variables $u = y^{-2}$, in which case, $\frac{du}{dx} = -2y^{-3}\frac{dy}{dx}$. Inserting these results into the preceding differential equation yields

$$-\frac{1}{2}\frac{du}{dx} - \frac{1}{2x\ln x}u = -\frac{9}{2}x^2,$$

or, in standard form,

$$\frac{du}{dx} + \frac{1}{x\ln x}u = 9x^2.$$

An integrating factor for this linear differential equation is

$$I = e^{\int \frac{1}{x\ln x}dx} = e^{\ln(\ln x)} = \ln x.$$

Multiplying the previous differential equation by $\ln x$ reduces it to

$$\frac{d}{dx}(\ln x \cdot u) = 9x^2 \ln x$$

which can be integrated to obtain

$$\ln x \cdot u = x^3(3\ln x - 1) + c$$

so that

$$u = \frac{x^3(3\ln x - 1) + c}{\ln x}.$$

Making the replacement $u = y^3$ in this equation gives

$$y^3 = \frac{x^3(3\ln x - 1) + c}{\ln x}.$$

34. Separating the variables in the given differential equation yields

$$\frac{1}{y}\frac{dy}{dx} = \frac{2+x}{1+x} = 1 + \frac{1}{1+x},$$

which can be integrated to obtain

$$\ln|y| = x + \ln|1+x| + c.$$

Exponentiating both sides of this equation gives

$$y(x) = c_1(1+x)e^x.$$

35. The given differential equation can be written in the equivalent form

$$\frac{dy}{dx} + \frac{2}{x^2-1}y = 1 \qquad (0.0.24)$$

which is first-order linear. An integrating factor is

$$I = e^{\int \frac{2}{x^2-1}dx} = e^{\int\left(\frac{1}{x-1} - \frac{1}{x+1}\right)dx} = e^{[\ln(x-1) - \ln(x+1)]} = \frac{x-1}{x+1}.$$

Multiplying (0.0.24) by $(x-1)/(x+1)$ reduces it to the integrable form

$$\frac{d}{dx}\left(\frac{x-1}{x+1}\cdot y\right) = \frac{x-1}{x+1} = 1 - \frac{2}{x+1}.$$

Integrating both sides of this differential equation yields

$$\left(\frac{x-1}{x+1}\cdot y\right) = x - 2\ln(x+1) + c$$

so that

$$y(x) = \left(\frac{x+1}{x-1}\right)[x - 2\ln(x+1) + c].$$

36. The given differential equation can be written in the equivalent form

$$[y\sec^2(xy) + 2x]dx + x\sec^2(xy)dy = 0$$

Then

$$M_y = \sec^2(xy) + 2xy\sec^2(x)\tan(xy) = N_x$$

so that the differential equation is exact. Consequently, there is a potential function satisfying

$$\frac{\partial\phi}{\partial x} = y\sec^2(xy) + 2x, \quad \frac{\partial\phi}{\partial y} = x\sec^2(xy).$$

Integrating these two equations in the usual manner yields

$$\phi(x,y) = x^2 + \tan(xy),$$

so that the differential equation can be written as

$$d(x^2 + \tan(xy)) = 0,$$

and therefore has general solution

$$x^2 + \tan(xy) = c,$$

or equivalently,

$$y(x) = \frac{\tan^{-1}(c - x^2)}{x}.$$

37. The given differential equation is first-order homogeneous. Inserting $y = xV$ into the given equation yields

$$x\frac{dV}{dx} + V = \frac{1}{1+V^2} + V,$$

that is,

$$(1+V^2)\frac{dV}{dx} = \frac{1}{x}.$$

Integrating we obtain

$$V + \frac{1}{3}V^3 = \ln|x| + c.$$

Inserting $V = y/x$ into the preceding equation yields

$$\frac{y}{x} + \frac{y^3}{3x^3} = \ln|x| + c,$$

or equivalently,

$$3x^2y + y^3 = 3x^3(\ln|x| + c).$$

38. The differential equation is first-order homogeneous. We therefore let $y = xv$, in which case $\frac{dy}{dx} = x\frac{dv}{dx} + v$. Substituting these results into the given differential equation yields

$$x\frac{dv}{dx} + v = \frac{1+v^2}{1-3v^2},$$

so that

$$x\frac{dv}{dx} = \frac{1+v^2-v+3v^3}{1-3v^2} = \frac{(v+1)(3v^2-2v+1)}{1-3v^2}.$$

Separating the variables gives

$$\frac{1-3v^2}{(v+1)(3v^2-2v+1)}dv = \frac{1}{x}dx.$$

Decomposing the left-hand side into partial fractions yields

$$\left[-\frac{1}{3(v+1)} - \frac{2(3v-2)}{3(3v^2-3v+1)}\right]dv = \frac{1}{x}dx,$$

or equivalently,

$$\left[-\frac{1}{3(v+1)} - \frac{2(3v-2)}{\left(v-\frac{1}{3}\right)^2+\frac{2}{9}}\right]dv = \frac{1}{x}dx.$$

This can be integrated to yield

$$2\left\{\frac{3}{\sqrt{2}}\arctan\left(\frac{3v-1}{\sqrt{2}}\right) - \frac{3}{2}\ln\left[\frac{1}{9}(3v-1)^2+\frac{2}{9}\right]\right\} - \frac{1}{3}\ln(v+1) = \ln x + c.$$

Therefore,

$$2\left\{\frac{3}{\sqrt{2}}\arctan\left(\frac{3y-x}{\sqrt{2}x}\right) - \frac{3}{2}\ln\left[\frac{(3y-x)^2}{9x^2}+\frac{2}{9}\right]\right\} - \frac{1}{3}\ln\left(\frac{y+x}{x}\right) = \ln x + c.$$

39. The given differential equation is a Bernoulli equation with $n = -1$. We therefore divide the equation by y^{-1} to obtain

$$y\frac{dy}{dx} + \frac{1}{x}y^2 = \frac{25\ln x}{2x^3}.$$

We now make the change of variables $u = y^2$, in which case, $\frac{du}{dx} = 2y\frac{dy}{dx}$. Inserting these results into the preceding differential equation yields

$$\frac{1}{2}\frac{du}{dx} + \frac{1}{x}u = \frac{25\ln x}{2x^3},$$

or, in standard form,

$$\frac{du}{dx} + \frac{2}{x}u = 25x^{-3}\ln x.$$

An integrating factor for this linear differential equation is

$$I = e^{\int \frac{2}{x}\,dx} = x^2.$$

Multiplying the previous differential equation by x^2 reduces it to

$$\frac{d}{dx}(x^2u) = 25x^{-1}\ln x,$$

which can be integrated directly to obtain

$$x^2u = \frac{25}{2}(\ln x)^2 + c$$

so that

$$u = \frac{25(\ln x)^2 + c}{2x^2}.$$

Making the replacement $u = y^2$ in this equation gives

$$y^2 = \frac{25(\ln x)^2 + c}{2x^2}.$$

40. The differential equation is first-order homogeneous. We therefore let $y = xv$, in which case $\frac{dy}{dx} = x\frac{dv}{dx} + v$. Substituting these results into the given differential equation yields

$$x\frac{dv}{dx} + v = \frac{1+v^2}{1-3v^2},$$

so that

$$x\frac{dv}{dx} = \frac{1+v^2-v+3v^3}{1-3v^2} = \frac{(v+1)(3v^2-2v+1)}{1-3v^2}.$$

Separating the variables gives

$$\frac{1-3v^2}{(v+1)(3v^2-2v+1)}dv = \frac{1}{x}dx.$$

Decomposing the left-hand side into partial fractions yields

$$\left[-\frac{1}{3(v+1)} - \frac{2(3v-2)}{3(3v^2-3v+1)}\right]dv = \frac{1}{x}dx,$$

or equivalently,

$$\left[-\frac{1}{3(v+1)} - \frac{2(3v-2)}{\left(v-\frac{1}{3}\right)^2+\frac{2}{9}}\right]dv = \frac{1}{x}dx.$$

This can be integrated to yield

$$2\left\{\frac{3}{\sqrt{2}}\arctan\left(\frac{3v-1}{\sqrt{2}}\right) - \frac{3}{2}\ln\left[\frac{1}{9}(3v-1)^2+\frac{2}{9}\right]\right\} - \frac{1}{3}\ln(v+1) = \ln x + c.$$

Therefore,

$$2\left\{\frac{3}{\sqrt{2}}\arctan\left(\frac{3y-x}{\sqrt{2}x}\right) - \frac{3}{2}\ln\left[\frac{(3y-x)^2}{9x^2}+\frac{2}{9}\right]\right\} - \frac{1}{3}\ln\left(\frac{y+x}{x}\right) = \ln x + c.$$

41. The given differential equation can be written in the equivalent form

$$e^y(1+y)\frac{dy}{dx} = xe^x$$

which is separable. Integrating both sides of this equation gives

$$ye^y = e^x(x-1) + c.$$

42. The given differential equation can be written in the equivalent form

$$\frac{dy}{dx} - \frac{\cos x}{\sin x}y = -\cos x$$

which is first order linear with integrating factor

$$I = e^{-\int \frac{\cos x}{\sin x}\,dx} = e^{-\ln(\sin x)} = \frac{1}{\sin x}.$$

Multiplying the preceding differential equation by $\frac{1}{\sin x}$ reduces it to

$$\frac{d}{dx}\left(\frac{1}{\sin x}\cdot y\right) = -\frac{\cos x}{\sin x}$$

which can be integrated directly to obtain

$$\frac{1}{\sin x}\cdot y = -\ln(\sin x) + c$$

so that

$$y(x) = \sin x[c - \ln(\sin x)].$$

43. The given differential equation is linear, and therefore can be solved using an appropriate integrating factor. However, if we rearrange the terms in the given differential equation then it can be written in the equivalent form

$$\frac{1}{1+y}\frac{dy}{dx} = x^2$$

which is separable. Integrating both sides of the preceding differential equation yields

$$\ln(1+y) = \frac{1}{3}x^3 + c$$

so that

$$y(x) = c_1 e^{\frac{1}{3}x^3} - 1.$$

Imposing the initial condition $y(0) = 5$ we find $c_1 = 6$. Therefore the solution to the initial-value problem is

$$y(x) = 6e^{\frac{1}{3}x^3} - 1.$$

44. The given differential equation can be written in the equivalent form

$$e^{-6y}\frac{dy}{dx} = -e^{-4x}$$

which is separable. Integrating both sides of the preceding equation yields

$$-\frac{1}{6}e^{-6y} = \frac{1}{4}e^{-4x} + c$$

so that

$$y(x) = -\frac{1}{6}\ln\left(c_1 - \frac{3}{2}e^{-4x}\right).$$

Imposing the initial condition $y(0) = 0$ requires that

$$0 = \ln\left(c_1 - \frac{3}{2}\right).$$

Hence, $c_1 = \frac{5}{2}$, and so

$$y(x) = -\frac{1}{6}\ln\left(\frac{5 - 3e^{-4x}}{2}\right).$$

45. For the given differential equation we have

$$M_y = 4xy = N_x$$

so that the differential equation is exact. Consequently, there is a potential function satisfying

$$\frac{\partial \phi}{\partial x} = 3x^2 + 2xy^2, \quad \frac{\partial \phi}{\partial y} = 2x^2 y.$$

Integrating these two equations in the usual manner yields

$$\phi(x, y) = x^2y^2 + x^3,$$

so that the differential equation can be written as

$$d(x^2y^2 + x^3) = 0,$$

and therefore has general solution

$$x^2y^2 + x^3 = c.$$

Imposing the initial condition $y(1) = 3$ yields $c = 10$. Therefore,

$$x^2y^2 + x^3 = 10$$

so that

$$y^2 = \frac{10 - x^3}{x^2}.$$

Note that the given differential equation can be written in the equivalent form

$$\frac{dy}{dx} + \frac{1}{x}y = -\frac{3}{2}y^{-1},$$

which is a Bernoulli equation with $n = -1$. Consequently, the Bernoulli technique could also have been used to solve the differential equation.

46. The given differential equation is linear with integrating factor

$$I = e^{-\int \sin x \, dx} = e^{\cos x}.$$

Multiplying the given differential equation by $e^{\cos x}$ reduces it to the integrable form

$$\frac{d}{dx}\left(e^{\cos x} \cdot y\right) = 1,$$

which can be integrated directly to obtain

$$e^{\cos x} \cdot y = x + c..$$

Hence,

$$y(x) = e^{-\cos x}(x + c).$$

Imposing the given initial condition $y(0) = \frac{1}{e}$ requires that $c = 1$. Consequently,

$$y(x) = e^{-\cos x}(x + 1).$$

47. (a). For the given differential equation we have

$$M_y = my^{m-1}, \qquad N_x = -nx^{n-1}y^3.$$

We see that the only values for m and n for which $M_y = N_x$ are $m = n0$. Consequently, these are the only values of m and n for which the differential equation is exact.

(b). We rewrite the given differential equation in the equivalent form

$$\frac{dy}{dx} = \frac{x^5 + y^m}{x^n y^3}, \tag{0.0.25}$$

from which we see that the differential equation is separable provided $m = 0$. In this case there are no restrictions on n.

(c). From Equation (0.0.25) we see that the only values of m and n for which the differential equation is first-order homogeneous are $m = 5$ and $n = 2$.

(d). We now rewrite the given differential equation in the equivalent form

$$\frac{dy}{dx} - x^{-n}y^{m-3} = x^{5-n}y^{-3}. \tag{0.0.26}$$

Due to the y^{-3} term on the right-hand side of the preceding differential equation, it follows that there are no values of m and n for which the equation is linear.

(e). From Equation (0.0.26) we see that the differential equation is a Bernoulli equation whenever $m = 4$. There are no constraints on n in this case.

48. In Newton's Law of Cooling we have

$$T_m = 180°\text{F}, \quad T(0) = 80°\text{F}, \quad T(3) = 100°\text{F}.$$

We need to determine the time, t_0 when $T(t_0) = 140°$F. The temperature of the sandals at time t is governed by the differential equation

$$\frac{dT}{dt} = -k(T - 180).$$

This separable differential equation is easily integrated to obtain

$$T(t) = 180 + ce^{-kt}.$$

Since $T(0) = 80$ we have

$$80 = 180 + c \implies c = -100.$$

Hence,

$$T(t) = 180 - 100e^{-kt}.$$

Imposing the condition $T(3) = 100$ requires

$$100 = 180 - 100e^{-3k}.$$

Solving for k we find $k = \frac{1}{3}\ln\left(\frac{5}{4}\right)$. Inserting this value for k into the preceding expression for $T(t)$ yields

$$T(t) = 180 - 100e^{-\frac{t}{3}\ln\left(\frac{5}{4}\right)}.$$

We need to find t_0 such that

$$140 = 180 - 100e^{-\frac{t_0}{3}\ln\left(\frac{5}{4}\right)}.$$

Solving for t_0 we find

$$t_0 = 3\frac{\ln\left(\frac{5}{2}\right)}{\ln\left(\frac{5}{4}\right)} \approx 12.32 \text{ min}.$$

49. In Newton's Law of Cooling we have

$$T_m = 70°\text{F}, \quad T(0) = 150°\text{F}, \quad T(10) = 125°\text{F}.$$

We need to determine the time, t_0 when $T(t_0) = 100°$F. The temperature of the plate at time t is governed by the differential equation

$$\frac{dT}{dt} = -k(T - 70).$$

This separable differential equation is easily integrated to obtain

$$T(t) = 70 + ce^{-kt}.$$

Since $T(0) = 150$ we have

$$150 = 70 + c \implies c = 80.$$

Hence,

$$T(t) = 70 + 80e^{-kt}.$$

Imposing the condition $T(10) = 125$ requires

$$125 = 70 + 80e^{-10k}.$$

Solving for k we find $k = \frac{1}{10}\ln\left(\frac{16}{11}\right)$. Inserting this value for k into the preceding expression for $T(t)$ yields

$$T(t) = 70 + 80e^{-\frac{t}{10}\ln\left(\frac{16}{11}\right)}.$$

We need to find t_0 such that

$$100 = 70 + 80e^{-\frac{t_0}{10}\ln\left(\frac{16}{11}\right)}.$$

Solving for t_0 we find

$$t_0 = 10\frac{\ln\left(\frac{8}{3}\right)}{\ln\left(\frac{16}{11}\right)} \approx 26.18 \text{ min.}$$

50. Let $T(t)$ denote the temperature of the object at time t, and let T_m denote the temperature of the surrounding medium. Then we must solve the initial-value problem

$$\frac{dT}{dt} = k(T - T_m)^2, \quad T(0) = T_0,$$

where k is a constant. The differential equation can be written in separated form as

$$\frac{1}{(T - T_m)^2}\frac{dT}{dt} = k.$$

Integrating both sides of this differential equation yields

$$-\frac{1}{T - T_m} = kt + c$$

so that

$$T(t) = T_m - \frac{1}{kt + c}.$$

Imposing the initial condition $T(0) = T_0$ we find that

$$c = \frac{1}{T_m - T_0}$$

which, when substituted back into the preceding expression for $T(t)$ yields

$$T(t) = T_m - \frac{1}{kt + \frac{1}{T_m - T_0}} = T_m - \frac{T_m - T_0}{k(T_m - T_0)t + 1}.$$

As $t \to \infty$, $T(t)$ approaches T_m.

51.(a). Since $\frac{dv}{dt}(0) = 2$, the velocity is increasing at the rate of 2 m/s^2 at $t = 0$.

(b). Evaluating the given differential equation at $t = 0$, using the given initial conditions yields

$$2 + 20k = 80k \Longrightarrow k = \frac{1}{3}.$$

(c). An integrating factor for the given differential equation is $I = e^{\int k\,dt} = e^{kt}$. Multiplying the given differential equation by this integrating factor reduces is to $\frac{d}{dt}(e^{kt} \cdot v) = 80k \Longrightarrow v(t) = e^{-kt}(80t + c)$. Imposing the initial condition $v(0) = 20$ yields $c = 20$, so that $v(t) = 20e^{-kt}(4kt + 1) \Longrightarrow v(t) = \frac{4}{3}e^{-t/30}(2t + 15)$.

(d). $v(t) = \frac{4}{3}e^{-t/30}(2t + 15) \Longrightarrow$ there is no finite $t > 0$ when $v(t) = 0$. Hence the object does not come to rest in a finite time.

(e). $\lim_{t\to\infty} v(t) = 0$.

52. We are given the differential equation

$$\frac{dT}{dt} = -k(T - 5\cos 2t) \tag{0.0.27}$$

together with the initial conditions

$$T(0) = 0; \quad \frac{dT}{dt}(0) = 5. \tag{0.0.28}$$

(a). Setting $t = 0$ in (0.0.27) and using (0.0.28) yields

$$5 = -k(0 - 5)$$

so that $k = 1$.

(b). Substituting $k = 1$ into the differential equation (0.0.27) and rearranging terms yields

$$\frac{dT}{dt} + T = 5\cos t.$$

An integrating factor for this linear differential equation is $I = e^{\int dt} = e^t$. Multiplying the preceding differential equation by e^t reduces it to

$$\frac{d}{dt}(e^t \cdot T) = 5e^t \cos 2t$$

which upon integration yields

$$e^t \cdot T = e^t(\cos 2t + 2\sin 2t) + c,$$

so that

$$T(t) = ce^{-t} + \cos 2t + 2\sin 2t.$$

Imposing the initial condition $T(0) = 0$ we find that $c = -1$. Hence,

$$T(t) = \cos 2t + 2\sin 2t - e^{-t}.$$

(c). For large values of t we have

$$T(t) \approx \cos 2t + 2\sin 2t,$$

which can be written in phase-amplitude form as

$$T(t) \approx \sqrt{5}\cos(2t - \phi),$$

where $\tan\phi = 2$. consequently, for large t, the temperature is approximately oscillatory with period π and amplitude $\sqrt{5}$.

53. If we let $C(t)$ denote the number of sandhill cranes in the Platte River valley t days after April 1, then $C(t)$ is governed by the differential equation

$$\frac{dC}{dt} = -kC \tag{0.0.29}$$

together with the auxiliary conditions

$$C(0) = 500,000; \quad C(15) = 100,000. \tag{0.0.30}$$

Separating the variables in the differential equation (0.0.29) yields

$$\frac{1}{C}\frac{dC}{dt} = -k,$$

which can be integrated directly to obtain

$$\ln C = -kt + c.$$

Exponentiation yields

$$C(t) = c_0 e^{-kt}.$$

The initial condition $C(0) = 500,000$ requires $c_0 = 500,000$, so that

$$C(t) = 500,000e^{-kt}. \tag{0.0.31}$$

Imposing the auxiliary condition $C(15) = 100,000$ yields

$$100,000 = 500,000e^{-15k}.$$

Taking the natural logarithm of both sides of the preceding equation and simplifying we find that $k = \frac{1}{15}\ln 5$. Substituting this value for k into (0.0.31) gives

$$C(t) = 500,000e^{-\frac{t}{15}\ln 5}. \tag{0.0.32}$$

(a). $C(3) = 500,000e^{-2\ln 5} = 500,000 \cdot \frac{1}{25} = 20,000$ sandhile cranes.

(b). $C(35) = 500,000e^{-\frac{35}{15}\ln 5} \approx 11696$ sandhile cranes.

(c). We need to determine t_0 such that

$$1000 = 500,000e^{-\frac{t_0}{15}\ln 5}$$

that is,

$$e^{-\frac{t_0}{15}\ln 5} = \frac{1}{500}.$$

Taking the natural logarithm of both sides of this equation and simplifying yields

$$t_0 = 15 \cdot \frac{\ln 500}{\ln 5} \approx 57.9 \text{ days after April 1.}$$

54. Substituting $P_0 = 200,000$ into Equation (1.5.3) in the text yields

$$P(t) = \frac{200,000C}{200,000 + (C - 200,000)e^{-rt}}. \tag{0.0.33}$$

We are given

$$P(3) = P(t_1) = 230,000, \quad P(6) = P(t_2) = 250,000.$$

Since $t_2 = 2t_1$ we can use the formulas (1.5.5) and (1.5.6) of the text to obtain r and C directly as follows:

$$r = \frac{1}{3}\ln\left[\frac{25(23-20)}{20(25-23)}\right] = \frac{1}{3}\ln\left(\frac{15}{8}\right) \approx 0.21.$$

$$C = \frac{230,000[(23)(45) - (40)(25)]}{(23)^2 - (20)(25)} = 277586.$$

Substituting these values for r and C into (0.0.33) yields

$$P(t) = \frac{55517200000}{200,000 + (77586)e^{-0.21t}}.$$

Therefore,

$$P(10) = \frac{55517200000}{200,000 + (77586)e^{-2.1}} \approx 264,997,$$

and

$$P(20) = \frac{55517200000}{200,000 + (77586)e^{-4.2}} \approx 275981.$$

55. The differential equation for determining $q(t)$ is

$$\frac{dq}{dt} + \frac{5}{4}q = \frac{3}{2}\cos 2t,$$

which has integrating factor $I = e^{\int \frac{5}{4}\,dt} = e^{\frac{5}{4}t}$. Multiplying the preceding differential equation by $e^{\frac{5}{4}t}$ reduces it to the integrable form

$$\frac{d}{dt}\left(e^{\frac{5}{4}t} \cdot q\right) = \frac{3}{2}e^{\frac{5}{4}t}\cos 2t.$$

Integrating and simplifying we find

$$q(t) = \frac{6}{89}(5\cos 2t + 8\sin 2t) + ce^{-\frac{5}{4}t}. \tag{0.0.34}$$

The initial condition $q(0) = 3$ requires

$$3 = \frac{30}{89} + c,$$

so that $c = \frac{237}{89}$. Making this replacement in (0.0.34) yields

$$q(t) = \frac{6}{89}(5\cos 2t + 8\sin 2t) + \frac{237}{89}e^{-\frac{5}{4}t}.$$

The current in the circuit is

$$i(t) = \frac{dq}{dt} = \frac{12}{89}(8\cos 2t - 5\sin 2t) - \frac{1185}{356}e^{-\frac{5}{4}t}.$$

56. The current in the circuit is governed by the differential equation

$$\frac{di}{dt} + 10i = \frac{100}{3},$$

which has integrating factor $I = e^{\int 10\,dt} = e^{10t}$. Multiplying the preceding differential equation by e^{10t} reduces it to the integrable form

$$\frac{d}{dt}\left(e^{10t} \cdot i\right) = \frac{100}{3}e^{10t}.$$

Integrating and simplifying we find

$$i(t) = \frac{10}{3} + ce^{-10t}. \tag{0.0.35}$$

The initial condition $i(0) = 3$ requires

$$3 = \frac{10}{3} + c,$$

so that $c = -\frac{1}{3}$. Making this replacement in (0.0.35) yields

$$i(t) = \frac{1}{3}(10 - e^{-10t}).$$

57. We are given:

$$r_1 = 6 \text{ L/min}, \quad c_1 = 3 \text{ g/L}, \quad r_2 = 4 \text{ L/min}, \quad V(0) = 30 \text{ L}, \quad A(0) = 0 \text{ g},$$

and we need to determine the amount of salt in the tank when $V(t) = 60$L. Consider a small time interval Δt. Using the preceding information we have:

$$\Delta V = 6\Delta t - 4\Delta t = 2\Delta t,$$

and

$$\Delta A \approx 18\Delta t - 4\frac{A}{V}\Delta t.$$

Dividing both of these equations by Δt and letting $\Delta t \to 0$ yields

$$\frac{dV}{dt} = 2. \tag{0.0.36}$$

$$\frac{dA}{dt} + 4\frac{A}{V} = 18. \tag{0.0.37}$$

Integrating (0.0.36) and imposing the initial condition $V(0) = 30$ yields

$$V(t) = 2(t + 15). \tag{0.0.38}$$

We now insert this expression for $V(t)$ into (0.0.37) to obtain

$$\frac{dA}{dt} + \frac{2}{t+15}A = 18.$$

An integrating factor for this differential equation is $I = e^{\int \frac{2}{t+15}dt} = (t + 15)^2$. Multiplying the preceding differential equation by $(t + 15)^2$ reduces it to the integrable form

$$\frac{d}{dt}\left[(t + 15)^2 A\right] = 18(t + 15)^2.$$

Integrating and simplifying we find

$$A(t) = \frac{6(t + 15)^3 + c}{(t + 15)^2}.$$

Imposing the initial condition $A(0) = 0$ requires

$$0 = \frac{6(15)^3 + c}{(15)^2},$$

so that $c = -20250$. Consequently,

$$A(t) = \frac{6(t+15)^3 - 20250}{(t+15)^2}.$$

We need to determine the time when the solution overflows. Since the tank can hold 60 L of solution, from (0.0.38) overflow will occur when

$$60 = 2(t+15) \implies t = 15.$$

The amount of chemical in the tank at this time is

$$A(15) = \frac{6(30)^3 - 20250}{(30)^2} \approx 157.5 \text{ g}.$$

58. Applying Euler's method with $y' = x^2 + 2y^2, x_0 = 0, y_0 = -3$, and $h = 0.1$ we have $y_{n+1} = y_n + 0.1(x_n^2 + 2y_n^2)$. This generates the sequence of approximants given in the table below.

n	x_n	y_n
1	0.1	−1.2
2	0.2	−0.911
3	0.3	−0.74102
4	0.4	−0.62219
5	0.5	−0.52877
6	0.6	−0.44785
7	0.7	−0.371736
8	0.8	−0.29510
9	0.9	−0.21368
10	1.0	−0.12355

Consequently the Euler approximation to $y(1)$ is $y_{10} = -0.12355$.

59. Applying Euler's method with $y' = \frac{3x}{y} + 2, x_0 = 1, y_0 = 2$, and $h = 0.05$ we have

$$y_{n+1} = y_n + 0.05\left(\frac{3x_n}{y_n} + 2\right).$$

This generates the sequence of approximants given in the table below.

n	x_n	y_n
1	1.05	2.1750
2	1.10	2.34741
3	1.15	2.51770
4	1.20	2.68622
5	1.25	2.85323
6	1.30	3.01894
7	1.35	3.18353
8	1.40	3.34714
9	1.45	3.50988
10	1.50	3.67185

Consequently, the Euler approximation to $y(1.5)$ is $y_{10} = 3.67185$.

60. Applying the modified Euler method with $y' = x^2 + 2y^2, x_0 = 0, y_0 = -3$, and $h = 0.1$ generates the sequence of approximants given in the table below.

n	x_n	y_n
1	0.1	−1.9555
2	0.2	−1.42906
3	0.3	−1.11499
4	0.4	−0.90466
5	0.5	−0.74976
6	0.6	−0.62555
7	0.7	−0.51778
8	0.8	−0.41723
9	0.9	−0.31719
10	1.0	−0.21196

Consequently, the modified Euler approximation to $y(1)$ is $y_{10} = -0.21196$. Comparing this to the corresponding Euler approximation from Problem 58 we have

$$|y_{\mathrm{ME}} - y_{\mathrm{E}}| = |0.21196 - 0.12355| = 0.8841.$$

61. Applying the modified Euler method with $y' = \dfrac{3x}{y} + 2, x_0 = 1, y_0 = 2$, and $h = 0.05$ generates the sequence of approximants given in the table below.

n	x_n	y_n
1	1.05	2.17371
2	1.10	2.34510
3	1.15	2.51457
4	1.20	2.68241
5	1.25	2.84886
6	1.30	3.01411
7	1.35	3.17831
8	1.40	3.34159
9	1.45	3.50404
10	1.50	3.66576

Consequently, the modified Euler approximation to $y(1.5)$ is $y_{10} = 3.66576$. Comparing this to the corresponding Euler approximation from Problem 59 we have

$$|y_{\mathrm{ME}} - y_{\mathrm{E}}| = |3.66576 - 3.67185| = 0.00609.$$

62. Applying the Runge-Kutta method with $y' = x^2 + 2y^2, x_0 = 0, y_0 = -3$, and $h = 0.1$ generates the sequence of approximants given in the table below.

n	x_n	y_n
1	0.1	−1.87392
2	0.2	−1.36127
3	0.3	−1.06476
4	0.4	−0.86734
5	0.5	−0.72143
6	0.6	−0.60353
7	0.7	−0.50028
8	0.8	−0.40303
9	0.9	−0.30541
10	1.0	−0.20195

Consequently the Runge-Kutta approximation to $y(1)$ is $y_{10} = -0.20195$. Comparing this to the corresponding Euler approximation from Problem 58 we have

$$|y_{\text{RK}} - y_{\text{E}}| = |0.20195 - 0.12355| = 0.07840.$$

63. Applying the Runge-Kutta method with $y' = \dfrac{3x}{y} + 2, x_0 = 1, y_0 = 2$, and $h = 0.05$ generates the sequence of approximants given in the table below.

n	x_n	y_n
1	1.05	2.17369
2	1.10	2.34506
3	1.15	2.51452
4	1.20	2.68235
5	1.25	2.84880
6	1.30	3.01404
7	1.35	3.17823
8	1.40	3.34151
9	1.45	3.50396
10	1.50	3.66568

Consequently the Runge-Kutta approximation to $y(1.5)$ is $y_{10} = 3.66568$. Comparing this to the corresponding Euler approximation from Problem 59 we have

$$|y_{\text{RK}} - y_{\text{E}}| = |3.66568 - 3.67185| = 0.00617.$$

Chapter 2 Solutions

Solutions to Section 2.1

True-False Review:

(a): TRUE. A diagonal matrix has no entries below the main diagonal, so it is upper triangular. Likewise, it has no entries above the main diagonal, so it is also lower triangular.

(b): FALSE. An $m \times n$ matrix has m row vectors and n column vectors.

(c): TRUE. This is a square matrix, and all entries off the main diagonal are zero, so it is a diagonal matrix (the entries *on* the diagonal also happen to be zero, but this is not required).

(d): FALSE. The main diagonal entries of a skew-symmetric matrix must be zero. In this case, $a_{11} = 4 \neq 0$, so this matrix is not skew-symmetric.

(e): FALSE. The form presented uses the same number along the entire main diagonal, but a symmetric matrix need not have identical entries on the main diagonal.

(f): TRUE. Since A is symmetric, $A = A^T$. Thus, $(A^T)^T = A = A^T$, so A^T is symmetric.

(g): FALSE. The trace of a matrix is the *sum* of the entries along the main diagonal.

(h): TRUE. If A is skew-symmetric, then $A^T = -A$. But A and A^T contain the same entries along the main diagonal, so for $A^T = -A$, both A and $-A$ must have the same main diagonal. This is only possible if all entries along the main diagonal are 0.

(i): TRUE. If A is both symmetric and skew-symmetric, then $A = A^T = -A$, and $A = -A$ is only possible if all entries of A are zero.

(j): TRUE. Both matrix functions are defined for values of t such that $t > 0$.

(k): FALSE. The $(3, 2)$-entry contains a function that is not defined for values of t with $t \leq 3$. So for example, this matrix functions is not defined for $t = 2$.

(l): TRUE. Each numerical entry of the matrix function is a constant function, which has domain $\mathbb{R}$.

(m): FALSE. For instance, the matrix function $A(t) = [t]$ and $B(t) = [t^2]$ satisfy $A(0) = B(0)$, but A and B are not the same matrix function.

Problems:

1(a). $a_{31} = 0, a_{24} = -1, a_{14} = 2, a_{32} = 2, a_{21} = 7, a_{34} = 4$.

1(b). $(1, 4)$ and $(3, 2)$.

2(a). $b_{12} = -1$, $b_{33} = 4$, $b_{41} = 0$, $b_{43} = 8$, $b_{51} = -1$, and $b_{52} = 9$.

2(b). $(1, 2)$, $(1, 3)$, $(2, 1)$, $(3, 2)$, and $(5, 1)$.

3. $\begin{bmatrix} 1 & 5 \\ -1 & 3 \end{bmatrix}$; 2×2 matrix.

4. $\begin{bmatrix} 2 & 1 & -1 \\ 0 & 4 & -2 \end{bmatrix}$; 2×3 matrix.

5. $\begin{bmatrix} -1 \\ 1 \\ 1 \\ -5 \end{bmatrix}$; 4×1 matrix.

6. $\begin{bmatrix} 1 & -3 & -2 \\ 3 & 6 & 0 \\ 2 & 7 & 4 \\ -4 & -1 & 5 \end{bmatrix}$; 4×3 matrix.

7. $\begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}$; 3×3 matrix.

8. $\begin{bmatrix} 0 & -1 & -2 & -3 \\ 1 & 0 & -1 & -2 \\ 2 & 1 & 0 & -1 \\ 3 & 2 & 1 & -0 \end{bmatrix}$; 4×4 matrix.

9. $\begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 8 \end{bmatrix}$; 4×4 matrix.

10. $\text{tr}(A) = 1 + 3 = 4$.

11. $\text{tr}(A) = 1 + 2 + (-3) = 0$.

12. $\text{tr}(A) = 2 + 2 + (-5) = -1$.

13. Column vectors: $\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \end{bmatrix}$.
Row vectors: $[1 \ \ -1], [3 \ \ 5]$.

14. Column vectors: $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 6 \end{bmatrix}, \begin{bmatrix} -4 \\ 5 \\ 7 \end{bmatrix}$.
Row vectors: $[1 \ \ 3 \ \ -4], [-1 \ \ -2 \ \ 5], [2 \ \ 6 \ \ 7]$.

15. Column vectors: $\begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 10 \\ -1 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \end{bmatrix}$. Row vectors: $[2 \ \ 10 \ \ 6], [5 \ \ -1 \ \ 3]$.

16. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 1 \end{bmatrix}$. Column vectors: $\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$.

17. $A = \begin{bmatrix} -2 & 0 & 4 & -1 & -1 \\ 9 & -4 & -4 & 0 & 8 \end{bmatrix}$; column vectors: $\begin{bmatrix} -2 \\ 9 \end{bmatrix}, \begin{bmatrix} 0 \\ -4 \end{bmatrix}, \begin{bmatrix} 4 \\ -4 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 8 \end{bmatrix}$.

18. $B = \begin{bmatrix} -2 & -4 \\ -6 & -6 \\ 3 & 0 \\ -1 & 0 \\ -2 & 1 \end{bmatrix}$; row vectors: $\begin{bmatrix} -2 & -4 \end{bmatrix}, \begin{bmatrix} -6 & -6 \end{bmatrix}, \begin{bmatrix} 3 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \end{bmatrix}, \begin{bmatrix} -2 & 1 \end{bmatrix}$.

19. $B = \begin{bmatrix} 2 & 5 & 0 & 1 \\ -1 & 7 & 0 & 2 \\ 4 & -6 & 0 & 3 \end{bmatrix}$. Row vectors: $[2 \ \ 5 \ \ 0 \ \ 1], [-1 \ \ 7 \ \ 0 \ \ 2], [4 \ \ -6 \ \ 0 \ \ 3]$.

20. $A = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_p]$ has p columns and each column q-vector has q rows, so the resulting matrix has dimensions $q \times p$.

21. One example: $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

22. One example: $\begin{bmatrix} 2 & 3 & 1 & 2 \\ 0 & 5 & 6 & 2 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

23. One example: $\begin{bmatrix} 1 & 3 & -1 & 2 \\ -3 & 0 & 4 & -3 \\ 1 & -4 & 0 & 1 \\ -2 & 3 & -1 & 0 \end{bmatrix}$.

24. One example: $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$.

25. The only possibility here is the zero matrix: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

26. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

27. One example: $\begin{bmatrix} t^2 - t & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$.

28. One example: $\begin{bmatrix} \frac{1}{\sqrt{3-t}} & \sqrt{t+2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

29. One example: $\begin{bmatrix} \frac{1}{t^2+1} \\ 0 \end{bmatrix}$.

30. One example: $\begin{bmatrix} t^2+1 & 1 & 1 & 1 & 1 \end{bmatrix}$.

31. One example: Let A and B be 1×1 matrix functions given by

$$A(t) = [t] \qquad \text{and} \qquad B(t) = [t^2].$$

32. Let A be a symmetric upper triangular matrix. Then all elements below the main diagonal are zeros. Consequently, since A is symmetric, all elements above the main diagonal must also be zero. Hence, the only nonzero entries can occur along the main diagonal. That is, A is a diagonal matrix.

33. Since A is skew-symmetric, we know that $a_{ij} = -a_{ji}$ for all (i, j). But since A is symmetric, we know that $a_{ij} = a_{ji}$ for all (i, j). Thus, for all (i, j), we must have $-a_{ji} = a_{ji}$. That is, $a_{ji} = 0$ for all (i, j). That is, every element of A is zero.

Solutions to Section 2.2

True-False Review:

(a): FALSE. The correct statement is $(AB)C = A(BC)$, the associative law. A counterexample to the particular statement given in this review item can be found in Problem 5.

(b): TRUE. Multiplying from left to right, we note that AB is an $m \times p$ matrix, and right multiplying AB by the $p \times q$ matrix C, we see that ABC is an $m \times q$ matrix.

(c): TRUE. We have $(A + B)^T = A^T + B^T = A + B$, so $A + B$ is symmetric.

(d): FALSE. For example, let $A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ -3 & 0 & 0 \end{bmatrix}$. Then A and B are skew-symmetric, but $AB = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix}$ is not symmetric.

(e): FALSE. The correct equation is $(A+B)^2 = A^2+AB+BA+B^2$. The statement is false since $AB+BA$ does not necessarily equal $2AB$. For instance, if $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then $(A+B)^2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $A^2 + 2AB + B^2 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \neq (A+B)^2$.

(f): FALSE. For example, let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then $AB = 0$ even though $A \neq 0$ and $B \neq 0$.

(g): FALSE. For example, let $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and let $B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Then A is not upper triangular, despite the fact that AB is the zero matrix, hence automatically upper triangular.

(h): FALSE. For instance, the matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is neither the zero matrix nor the identity matrix, and yet $A^2 = A$.

(i): TRUE. The derivative of each entry of the matrix is zero, since in each entry, we take the derivative of a constant, thus obtaining zero for each entry of the derivative of the matrix.

(j): FALSE. The correct statement is given in Problem 45. The problem with the statement as given is that the second term should be $\frac{dA}{dt}B$, not $B\frac{dA}{dt}$.

(k): FALSE. For instance, the matrix function $A = \begin{bmatrix} 2e^t & 0 \\ 0 & 3e^t \end{bmatrix}$ satisfies $A = \frac{dA}{dt}$, but A does not have the form $\begin{bmatrix} ce^t & 0 \\ 0 & ce^t \end{bmatrix}$.

(l): TRUE. This follows by exactly the same proof as given in the text for matrices of numbers (see part 3 of Theorem 2.2.23).

Problems:

1(a). $5A = \begin{bmatrix} -10 & 30 & 5 \\ -5 & 0 & -15 \end{bmatrix}$.

1(b). $-3B = \begin{bmatrix} -6 & -3 & 3 \\ 0 & -12 & 12 \end{bmatrix}$.

1(c). $iC = \begin{bmatrix} -1+i & -1+2i \\ -1+3i & -1+4i \\ -1+5i & -1+6i \end{bmatrix}$.

1(d). $2A - B = \begin{bmatrix} -6 & 11 & 3 \\ -2 & -4 & -2 \end{bmatrix}$.

1(e). $A + 3C^T = \begin{bmatrix} 1+3i & 15+3i & 16+3i \\ 5+3i & 12+3i & 15+3i \end{bmatrix}$.

1(f). $3D - 2E = \begin{bmatrix} 8 & 10 & 7 \\ 1 & 4 & 9 \\ 1 & 7 & 12 \end{bmatrix}$.

1(g). $D + E + F = \begin{bmatrix} 12 & -3-3i & -1+i \\ 3+i & 3-2i & 8 \\ 6 & 4+2i & 2 \end{bmatrix}$.

1(h). Solving for G and simplifying, we have that

$$G = -\frac{3}{2}A - B = \begin{bmatrix} 1 & -10 & -1/2 \\ 3/2 & -4 & 17/2 \end{bmatrix}.$$

1(i). Solving for H and simplifying, we have that $H = 4E - D - 2F =$

$$\begin{bmatrix} 8 & -20 & -8 \\ 4 & 4 & 12 \\ 16 & -8 & -12 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 1 \\ 1 & 2 & 5 \\ 3 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 12 & 4-6i & 2i \\ 2+2i & -4i & 0 \\ -2 & 10+4i & 6 \end{bmatrix} = \begin{bmatrix} -8 & -24+6i & -9-2i \\ 1-2i & 2+4i & 7 \\ 15 & -19-4i & -20 \end{bmatrix}.$$

1(j). We have $K^T = 2B - 3A$, so that $K = (2B - 3A)^T = 2B^T - 3A^T$. Thus,

$$K = 2\begin{bmatrix} 2 & 0 \\ 1 & 4 \\ -1 & -4 \end{bmatrix} - 3\begin{bmatrix} -2 & -1 \\ 6 & 0 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 10 & 3 \\ -16 & 8 \\ -5 & 1 \end{bmatrix}.$$

2(a). $-D = \begin{bmatrix} -4 & 0 & -1 \\ -1 & -2 & -5 \\ -3 & -1 & -2 \end{bmatrix}$.

2(b). $4B^T = 4\begin{bmatrix} 2 & 0 \\ 1 & 4 \\ -1 & -4 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 4 & 16 \\ -4 & -16 \end{bmatrix}$.

2(c). $-2A^T + C = -2\begin{bmatrix} -2 & -1 \\ 6 & 0 \\ 1 & -3 \end{bmatrix} + \begin{bmatrix} 1+i & 2+i \\ 3+i & 4+i \\ 5+i & 6+i \end{bmatrix} = \begin{bmatrix} 5+i & 4+i \\ -9+i & 4+i \\ 3+i & 12+i \end{bmatrix}$.

2(d). $5E + D = \begin{bmatrix} 10 & -25 & -10 \\ 5 & 5 & 15 \\ 20 & -10 & -15 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 1 \\ 1 & 2 & 5 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 14 & -25 & -9 \\ 6 & 7 & 20 \\ 23 & -9 & -13 \end{bmatrix}$.

2(e). We have

$$4A^T - 2B^T + iC = 4\begin{bmatrix} -2 & -1 \\ 6 & 0 \\ 1 & -3 \end{bmatrix} - 2\begin{bmatrix} 2 & 0 \\ 1 & 4 \\ -1 & -4 \end{bmatrix} + i\begin{bmatrix} 1+i & 2+i \\ 3+i & 4+i \\ 5+i & 6+i \end{bmatrix} = \begin{bmatrix} -13+i & -5+2i \\ 21+3i & -9+4i \\ 5+5i & -5+6i \end{bmatrix}.$$

2(f). We have

$$4E - 3D^T = \begin{bmatrix} 8 & -20 & -8 \\ 4 & 4 & 12 \\ 16 & -8 & -12 \end{bmatrix} - \begin{bmatrix} 12 & 3 & 9 \\ 0 & 6 & 3 \\ 3 & 15 & 6 \end{bmatrix} = \begin{bmatrix} -4 & -23 & -17 \\ 4 & -2 & 9 \\ 13 & -23 & -18 \end{bmatrix}.$$

2(g). We have $(1-6i)F + iD =$

$$\begin{bmatrix} 6-36i & -16-15i & 6+i \\ 7-5i & -12-2i & 0 \\ -1+6i & 17-28i & 3-18i \end{bmatrix} + \begin{bmatrix} 4i & 0 & i \\ i & 2i & 5i \\ 3i & i & 2i \end{bmatrix} = \begin{bmatrix} 6-32i & -16-15i & 6+2i \\ 7-4i & -12 & 5i \\ -1+9i & 17-27i & 3-16i \end{bmatrix}.$$

2(h). Solving for G, we have

$$\begin{aligned} G = A + (1-i)C^T &= \begin{bmatrix} -2 & 6 & 1 \\ -1 & 0 & -3 \end{bmatrix} + (1-i)\begin{bmatrix} 1+i & 3+i & 5+i \\ 2+i & 4+i & 6+i \end{bmatrix} \\ &= \begin{bmatrix} -2 & 6 & 1 \\ -1 & 0 & -3 \end{bmatrix} + \begin{bmatrix} 2 & 4-2i & 6-4i \\ 3-i & 5-3i & 7-5i \end{bmatrix} \\ &= \begin{bmatrix} 0 & 10-2i & 7-4i \\ 2-i & 5-3i & 4-5i \end{bmatrix}. \end{aligned}$$

2(i). Solve for H, we have

$$\begin{aligned} H &= \frac{3}{2}D - \frac{3}{2}E + 3I_3 \\ &= \begin{bmatrix} 6 & 0 & 3/2 \\ 3/2 & 3 & 15/2 \\ 9/2 & 3/2 & 3 \end{bmatrix} - \begin{bmatrix} 3 & -15/2 & -3 \\ 3/2 & 3/2 & 9/2 \\ 6 & -3 & -9/2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 6 & 15/2 & 9/2 \\ 0 & 9/2 & 3 \\ -3/2 & 9/2 & 21/2 \end{bmatrix}. \end{aligned}$$

2(j). We have $K^T = D^T + E^T - F^T = (D+E-F)^T$, so that

$$K = D + E - F = \begin{bmatrix} 0 & -7+3i & -1-i \\ 1-i & 3+2i & 8 \\ 8 & -6-2i & -4 \end{bmatrix}.$$

3(a).

$$AB = \begin{bmatrix} 5 & 10 & -3 \\ 27 & 22 & 3 \end{bmatrix}$$

3(b).

$$BC = \begin{bmatrix} 9 \\ 8 \\ -6 \end{bmatrix}$$

3(c). CA cannot be computed.

3(d).

$$A^T E = \begin{bmatrix} 1 & 3 \\ -1 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2-i & 1+i \\ -i & 2+4i \end{bmatrix} = \begin{bmatrix} 2-4i & 7+13i \\ -2 & 1+3i \\ 4-6i & 10+18i \end{bmatrix}$$

3(e).

$$CD = \begin{bmatrix} 2 & -2 & 3 \\ -2 & 2 & -3 \\ 4 & -4 & 6 \end{bmatrix}.$$

3(f).

$$C^T A^T = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -1 & 1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 10 \end{bmatrix}$$

3(g).

$$F^2 = \begin{bmatrix} i & 1-3i \\ 0 & 4+i \end{bmatrix} \begin{bmatrix} i & 1-3i \\ 0 & 4+i \end{bmatrix} = \begin{bmatrix} -1 & 10-10i \\ 0 & 15+8i \end{bmatrix}$$

3(h).

$$BD^T = \begin{bmatrix} 2 & -1 & 3 \\ 5 & 1 & 2 \\ 4 & 6 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 15 \\ 14 \\ -10 \end{bmatrix}$$

3(i).

$$A^T A = \begin{bmatrix} 1 & 3 \\ -1 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 10 & 2 & 14 \\ 2 & 2 & 2 \\ 14 & 2 & 20 \end{bmatrix}$$

3(j).

$$FE = \begin{bmatrix} i & 1-3i \\ 0 & 4+i \end{bmatrix} \begin{bmatrix} 2-i & 1+i \\ -i & 2+4i \end{bmatrix} = \begin{bmatrix} -2+i & 13-i \\ 1-4i & 4+18i \end{bmatrix}$$

4(a).

$$AC = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \end{bmatrix}$$

4(b).

$$DC = [10]$$

4(c).

$$DB = [6 \;\; 14 \;\; -4]$$

4(d). AD cannot be computed.

4(e). $EF = \begin{bmatrix} 2-i & 1+i \\ -i & 2+4i \end{bmatrix} \begin{bmatrix} i & 1-3i \\ 0 & 4+i \end{bmatrix} = \begin{bmatrix} 1+2i & 2-2i \\ 1 & 1+17i \end{bmatrix}.$

4(f). Since A^T is a 3×2 matrix and B is a 3×3 matrix, the product $A^T B$ cannot be constructed.

4(g). Since C is a 3×1 matrix, it is impossible to form the product $C \cdot C = C^2$.

4(h). $E^2 = \begin{bmatrix} 2-i & 1+i \\ -i & 2+4i \end{bmatrix} \begin{bmatrix} 2-i & 1+i \\ -i & 2+4i \end{bmatrix} = \begin{bmatrix} 4-5i & 1+7i \\ 3-4i & -11+15i \end{bmatrix}.$

4(i). $AD^T = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 10 \\ 16 \end{bmatrix}$.

4(j). $E^T A = \begin{bmatrix} 2-i & -i \\ 1+i & 2+4i \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 2-4i & -2 & 4-6i \\ 7+13i & 1+3i & 10+18i \end{bmatrix}$.

5. We have

$$\begin{aligned} ABC = (AB)C &= \left(\begin{bmatrix} -3 & 2 & 7 & -1 \\ 6 & 0 & -3 & -5 \end{bmatrix} \begin{bmatrix} -2 & 8 \\ 8 & -3 \\ -1 & -9 \\ 0 & 2 \end{bmatrix} \right) \begin{bmatrix} -6 & 1 \\ 1 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 15 & -95 \\ -9 & 65 \end{bmatrix} \begin{bmatrix} -6 & 1 \\ 1 & 5 \end{bmatrix} \\ &= \begin{bmatrix} -185 & -460 \\ 119 & 316 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} CAB = C(AB) &= \begin{bmatrix} -6 & 1 \\ 1 & 5 \end{bmatrix} \left(\begin{bmatrix} -3 & 2 & 7 & -1 \\ 6 & 0 & -3 & -5 \end{bmatrix} \begin{bmatrix} -2 & 8 \\ 8 & -3 \\ -1 & -9 \\ 0 & 2 \end{bmatrix} \right) \\ &= \begin{bmatrix} -6 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 15 & -95 \\ -9 & 65 \end{bmatrix} \\ &= \begin{bmatrix} -99 & 635 \\ -30 & 230 \end{bmatrix}. \end{aligned}$$

6.

$$A\mathbf{c} = \begin{bmatrix} 1 & 3 \\ -5 & 4 \end{bmatrix} \begin{bmatrix} 6 \\ -2 \end{bmatrix} = 6 \begin{bmatrix} 1 \\ -5 \end{bmatrix} + (-2) \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ -38 \end{bmatrix}.$$

7.

$$A\mathbf{c} = \begin{bmatrix} 3 & -1 & 4 \\ 2 & 1 & 5 \\ 7 & -6 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} = 2 \begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 1 \\ -6 \end{bmatrix} + (-4) \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} -13 \\ -13 \\ -16 \end{bmatrix}.$$

8.

$$A\mathbf{c} = \begin{bmatrix} -1 & 2 \\ 4 & 7 \\ 5 & -4 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \end{bmatrix} = 5 \begin{bmatrix} -1 \\ 4 \\ 5 \end{bmatrix} + (-1) \begin{bmatrix} 2 \\ 7 \\ -4 \end{bmatrix} = \begin{bmatrix} -7 \\ 13 \\ 29 \end{bmatrix}.$$

9. We have

$$A\mathbf{c} = x \begin{bmatrix} a \\ e \end{bmatrix} + y \begin{bmatrix} b \\ f \end{bmatrix} + z \begin{bmatrix} c \\ g \end{bmatrix} + w \begin{bmatrix} d \\ h \end{bmatrix} = \begin{bmatrix} xa + yb + zc + wd \\ xe + yf + zg + wh \end{bmatrix}.$$

10(a). The dimensions of B should be $n \times r$ in order that ABC is defined.

10(b). The elements of the ith row of A are $a_{i1}, a_{i2}, \dots, a_{in}$ and the elements of the jth column of BC are

$$\sum_{m=1}^{r} b_{1m}c_{mj}, \quad \sum_{m=1}^{r} b_{2m}c_{mj}, \quad \dots \quad , \sum_{m=1}^{r} b_{nm}c_{mj},$$

so the element in the ith row and jth column of $ABC = A(BC)$ is

$$a_{i1}\sum_{m=1}^{r} b_{1m}c_{mj} + a_{i2}\sum_{m=1}^{r} b_{2m}c_{mj} + \cdots + a_{in}\sum_{m=1}^{r} b_{nm}c_{mj}$$
$$= \sum_{k=1}^{n} a_{ik}\left(\sum_{m=1}^{r} b_{km}c_{mj}\right) = \sum_{k=1}^{n}\left(\sum_{m=1}^{r} a_{ik}b_{km}\right)c_{mj}.$$

11(a).

$$A^2 = AA = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -1 & -4 \\ 8 & 7 \end{bmatrix}.$$
$$A^3 = A^2A = \begin{bmatrix} -1 & -4 \\ 8 & 7 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -9 & -11 \\ 22 & 13 \end{bmatrix}.$$
$$A^4 = A^3A = \begin{bmatrix} -9 & -11 \\ 22 & 13 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -31 & -24 \\ 48 & 17 \end{bmatrix}.$$

11(b).

$$A^2 = AA = \begin{bmatrix} 0 & 1 & 0 \\ -2 & 0 & 1 \\ 4 & -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ -2 & 0 & 1 \\ 4 & -1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 1 \\ 4 & -3 & 0 \\ 2 & 4 & -1 \end{bmatrix}.$$
$$A^3 = A^2A = \begin{bmatrix} -2 & 0 & 1 \\ 4 & -3 & 0 \\ 2 & 4 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ -2 & 0 & 1 \\ 4 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 4 & -3 & 0 \\ 6 & 4 & -3 \\ -12 & 3 & 4 \end{bmatrix}.$$
$$A^4 = A^3A = \begin{bmatrix} 4 & -3 & 0 \\ 6 & 4 & -3 \\ -12 & 3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ -2 & 0 & 1 \\ 4 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 4 & -3 \\ -20 & 9 & 4 \\ 10 & -16 & 3 \end{bmatrix}.$$

12(a). We apply the distributive property of matrix multiplication as follows:

$$(A+2B)^2 = (A+2B)(A+2B) = A(A+2B)+(2B)(A+2B) = (A^2+A(2B))+((2B)A+(2B)^2) = A^2+2AB+2BA+4B^2,$$

where scalar factors of 2 are moved in front of the terms since they commute with matrix multiplication.

12(b). We apply the distributive property of matrix multiplication as follows:

$$\begin{aligned}(A+B+C)^2 = (A+B+C)(A+B+C) &= A(A+B+C) + B(A+B+C) + C(A+B+C) \\ &= A^2 + AB + AC + BA + B^2 + BC + CA + CB + C^2 \\ &= A^2 + B^2 + C^2 + AB + BA + AC + CA + BC + CB,\end{aligned}$$

as required.

12(c). We can use the formula for $(A+B)^3$ found in Example 2.2.20 and substitute $-B$ for B throughout the expression:

$$\begin{aligned}(A-B)^3 &= A^3 + A(-B)A + (-B)A^2 + (-B)^2A + A^2(-B) + A(-B)^2 + (-B)A(-B) + (-B)^3 \\ &= A^3 - ABA - BA^2 + B^2A - A^2B + AB^2 + BAB - B^3,\end{aligned}$$

as needed.

13. We have

$$A^2 = \begin{bmatrix} 2 & -5 \\ 6 & -6 \end{bmatrix} \begin{bmatrix} 2 & -5 \\ 6 & -6 \end{bmatrix} = \begin{bmatrix} -26 & 20 \\ -24 & 6 \end{bmatrix},$$

so that

$$A^2 + 4A + 18I_2 = \begin{bmatrix} -26 & 20 \\ -24 & 6 \end{bmatrix} + \begin{bmatrix} 8 & -20 \\ 24 & -24 \end{bmatrix} + \begin{bmatrix} 18 & 0 \\ 0 & 18 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

14. We have

$$A^2 = \begin{bmatrix} -1 & 0 & 4 \\ 1 & 1 & 2 \\ -2 & 3 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 4 \\ 1 & 1 & 2 \\ -2 & 3 & 0 \end{bmatrix} = \begin{bmatrix} -7 & 12 & -4 \\ -4 & 7 & 6 \\ 5 & 3 & -2 \end{bmatrix}$$

and

$$A^3 = \begin{bmatrix} -1 & 0 & 4 \\ 1 & 1 & 2 \\ -2 & 3 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 4 \\ 1 & 1 & 2 \\ -2 & 3 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 4 \\ 1 & 1 & 2 \\ -2 & 3 & 0 \end{bmatrix} = \begin{bmatrix} -7 & 12 & -4 \\ -4 & 7 & 6 \\ 5 & 3 & -2 \end{bmatrix} \begin{bmatrix} -1 & 0 & 4 \\ 1 & 1 & 2 \\ -2 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 27 & 0 & -4 \\ -1 & 25 & -2 \\ 2 & -3 & 26 \end{bmatrix}.$$

Therefore, we have

$$A^3 + A - 26I_3 = \begin{bmatrix} 27 & 0 & -4 \\ -1 & 25 & -2 \\ 2 & -3 & 26 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 4 \\ 1 & 1 & 2 \\ -2 & 3 & 0 \end{bmatrix} - \begin{bmatrix} 26 & 0 & 0 \\ 0 & 26 & 0 \\ 0 & 0 & 26 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

15.

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Substituting $A = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}$ for A, we have

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

that is,

$$\begin{bmatrix} 1 & 2x & 2z + xy \\ 0 & 1 & 2y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since corresponding elements of equal matrices are equal, we obtain the following implications:

$$2y = 1 \Longrightarrow y = 1/2,$$
$$2x = 1 \Longrightarrow x = 1/2,$$
$$2z + xy = 0 \Longrightarrow 2z + (1/2)(1/2) = 0 \Longrightarrow z = -1/8.$$

Thus, $A = \begin{bmatrix} 1 & 1/2 & -1/8 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}$.

16. In order that $A^2 = A$, we require $\begin{bmatrix} x & 1 \\ -2 & y \end{bmatrix}\begin{bmatrix} x & 1 \\ -2 & y \end{bmatrix} = \begin{bmatrix} x & 1 \\ -2 & y \end{bmatrix}$, that is, $\begin{bmatrix} x^2-2 & x+y \\ -2x-2y & -2+y^2 \end{bmatrix} = \begin{bmatrix} x & 1 \\ -2 & y \end{bmatrix}$, or equivalently, $\begin{bmatrix} x^2-x-2 & x+y-1 \\ -2x-2y+2 & y^2-y-2 \end{bmatrix} = 0_2$. Since corresponding elements of equal matrices are equal, it follows that

$$x^2 - x - 2 = 0 \Longrightarrow x = -1 \text{ or } x = 2, \text{ and}$$
$$y^2 - y - 2 = 0 \Longrightarrow y = -1 \text{ or } y = 2.$$

Two cases arise from $x + y - 1 = 0$:

(a): If $x = -1$, then $y = 2$.

(b): If $x = 2$, then $y = -1$. Thus,

$$A = \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}.$$

17.

$$\sigma_1\sigma_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = i\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = i\sigma_3.$$
$$\sigma_2\sigma_3 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = i\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = i\sigma_1.$$
$$\sigma_3\sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = i\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = i\sigma_2.$$

18.

$$\begin{aligned} [A, B] &= AB - BA \\ &= \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & -1 \\ 10 & 4 \end{bmatrix} - \begin{bmatrix} 5 & -2 \\ 8 & -2 \end{bmatrix} \\ &= \begin{bmatrix} -6 & 1 \\ 2 & 6 \end{bmatrix} \neq 0_2. \end{aligned}$$

19.

$$\begin{aligned} [A_1, A_2] &= A_1A_2 - A_2A_1 \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 0_2, \quad \text{thus } A_1 \text{ and } A_2 \text{ commute.} \end{aligned}$$

$$\begin{aligned} [A_1, A_3] &= A_1A_3 - A_3A_1 \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = 0_2, \quad \text{thus } A_1 \text{ and } A_3 \text{ commute.} \end{aligned}$$

$$
\begin{aligned}
[A_2, A_3] &= A_2A_3 - A_3A_2 \\
&= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \neq 0_2.
\end{aligned}
$$

Then $[A_3, A_2] = -[A_2, A_3] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \neq 0_2$. Thus, A_2 and A_3 do not commute.

20.

$$
\begin{aligned}
[A_1, A_2] &= A_1A_2 - A_2A_1 \\
&= \frac{1}{4}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \\
&= \frac{1}{4}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} - \frac{1}{4}\begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \\
&= \frac{1}{4}\begin{bmatrix} 2i & 0 \\ 0 & -2i \end{bmatrix} = \frac{1}{2}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = A_3.
\end{aligned}
$$

$$
\begin{aligned}
[A_2, A_3] &= A_2A_3 - A_3A_2 \\
&= \frac{1}{4}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} - \frac{1}{4}\begin{bmatrix} i & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\
&= \frac{1}{4}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} \\
&= \frac{1}{4}\begin{bmatrix} 0 & 2i \\ 2i & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = A_1.
\end{aligned}
$$

$$
\begin{aligned}
[A_3, A_1] &= A_3A_1 - A_1A_3 \\
&= \frac{1}{4}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \\
&= \frac{1}{4}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
&= \frac{1}{4}\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = A_2.
\end{aligned}
$$

21.

$$
\begin{aligned}
&[A, [B, C]] + [B, [C, A]] + [C, [A, B]] \\
&= [A, BC - CB] + [B, CA - AC] + [C, AB - BA] \\
&= A(BC - CB) - (BC - CB)A + B(CA - AC) - (CA - AC)B + C(AB - BA) - (AB - BA)C \\
&= ABC - ACB - BCA + CBA + BCA - BAC - CAB + ACB + CAB - CBA - ABC + BAC \quad = 0.
\end{aligned}
$$

22.
Proof that $A(BC) = (AB)C$: Let $A = [a_{ij}]$ be of size $m \times n$, $B = [b_{jk}]$ be of size $n \times p$, and $C = [c_{kl}]$ be of size $p \times q$. Consider the (i, j)-element of $(AB)C$:

$$
[(AB)C]_{ij} = \sum_{k=1}^{p}\left(\sum_{h=1}^{n} a_{ih}b_{hk}\right)c_{kj} = \sum_{h=1}^{n} a_{ih}\left(\sum_{k=1}^{p} b_{hk}c_{kj}\right) = [A(BC)]_{ij}.
$$

Proof that $A(B+C) = AB + AC$**:** We have

$$\begin{aligned}[A(B+C)]_{ij} &= \sum_{k=1}^{n} a_{ik}(b_{kj} + c_{kj}) \\ &= \sum_{k=1}^{n}(a_{ij}b_{kj} + a_{ik}c_{kj}) \\ &= \sum_{k=1}^{n} a_{ik}b_{kj} + \sum_{k=1}^{n} a_{ik}c_{kj} \\ &= [AB + AC]_{ij}.\end{aligned}$$

23.
Proof that $(A^T)^T = A$**:** Let $A = [a_{ij}]$. Then $A^T = [a_{ji}]$, so $(A^T)^T = [a_{ji}]^T = a_{ij} = A$, as needed.
Proof that $(A+C)^T = A^T + C^T$**:** Let $A = [a_{ij}]$ and $C = [c_{ij}]$. Then $[(A+C)^T]_{ij} = [A+C]_{ji} = [A]_{ji} + [C]_{ji} = a_{ji} + c_{ji} = [A^T]_{ij} + [C^T]_{ij} = [A^T + C^T]_{ij}$. Hence, $(A+C)^T = A^T + C^T$.

24. We have

$$(IA)_{ij} = \sum_{k=1}^{m} \delta_{ik}a_{kj} = \delta_{ii}a_{ij} = a_{ij},$$

for $1 \le i \le m$ and $1 \le j \le p$. Thus, $I_m A_{m\times p} = A_{m\times p}$.

25. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $n \times n$ matrices. Then

$$\text{tr}(AB) = \sum_{k=1}^{n}\left(\sum_{i=1}^{n} a_{ki}b_{ik}\right) = \sum_{k=1}^{n}\left(\sum_{i=1}^{n} b_{ik}a_{ki}\right) = \sum_{i=1}^{n}\left(\sum_{k=1}^{n} b_{ik}a_{ki}\right) = \text{tr}(BA).$$

26(a). $B^TA^T = \begin{bmatrix} 0 & -7 & -1 \\ -4 & 1 & -3 \end{bmatrix}\begin{bmatrix} -3 \\ -1 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ -7 \end{bmatrix}.$

26(b). $C^TB^T = \begin{bmatrix} -9 & 1 \\ 0 & 1 \\ 3 & 5 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} 0 & -7 & -1 \\ -4 & 1 & -3 \end{bmatrix} = \begin{bmatrix} -4 & 64 & 6 \\ -4 & 1 & -3 \\ -20 & -16 & -18 \\ 8 & 12 & 8 \end{bmatrix}.$

26(c). Since D^T is a 3×3 matrix and A is a 1×3 matrix, it is not possible to compute the expression D^TA.

27(a). $AD^T = \begin{bmatrix} -3 & -1 & 6 \end{bmatrix}\begin{bmatrix} -2 & 0 & 1 \\ 1 & 0 & -2 \\ 5 & 7 & -1 \end{bmatrix} = \begin{bmatrix} 35 & 42 & -7 \end{bmatrix}.$

27(b). First note that $C^TC = \begin{bmatrix} -9 & 1 \\ 0 & 1 \\ 3 & 5 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} -9 & 0 & 3 & -2 \\ 1 & 1 & 5 & -2 \end{bmatrix} = \begin{bmatrix} 82 & 1 & -22 & 16 \\ 1 & 1 & 5 & -2 \\ -22 & 5 & 34 & -16 \\ 16 & -2 & -16 & 8 \end{bmatrix}$. Therefore,

$$(C^TC)^2 = \begin{bmatrix} 82 & 1 & -22 & 16 \\ 1 & 1 & 5 & -2 \\ -22 & 5 & 34 & -16 \\ 16 & -2 & -16 & 8 \end{bmatrix}\begin{bmatrix} 82 & 1 & -22 & 16 \\ 1 & 1 & 5 & -2 \\ -22 & 5 & 34 & -16 \\ 16 & -2 & -16 & 8 \end{bmatrix} = \begin{bmatrix} 7465 & -59 & -2803 & 1790 \\ -59 & 31 & 185 & -82 \\ -2803 & 185 & 1921 & -1034 \\ 1790 & -82 & -1034 & 580 \end{bmatrix}.$$

27(c). $D^TB = \begin{bmatrix} -2 & 0 & 1 \\ 1 & 0 & -2 \\ 5 & 7 & -1 \end{bmatrix} \begin{bmatrix} 0 & -4 \\ -7 & 1 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 2 & 2 \\ -48 & -10 \end{bmatrix}$.

28(a). We have

$$S = [\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3] = \begin{bmatrix} -x & -y & z \\ 0 & y & 2z \\ x & -y & z \end{bmatrix},$$

so

$$AS = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 5 & 2 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} -x & -y & z \\ 0 & y & 2z \\ x & -y & z \end{bmatrix} = \begin{bmatrix} -x & -y & 7z \\ 0 & y & 14z \\ x & -y & 7z \end{bmatrix} = [\mathbf{s}_1, \mathbf{s}_2, 7\mathbf{s}_3].$$

28(b).

$$S^TAS = S^T(AS) = \begin{bmatrix} -x & 0 & x \\ -y & y & -y \\ z & 2z & z \end{bmatrix} \begin{bmatrix} -x & -y & 7z \\ 0 & y & 14z \\ x & -y & 7z \end{bmatrix} = \begin{bmatrix} 2x^2 & 0 & 0 \\ 0 & 3y^2 & 0 \\ 0 & 0 & 42z^2 \end{bmatrix},$$

but $S^TAS = \text{diag}(1, 1, 7)$, so we have the following

$$2x^2 = 1 \Longrightarrow x = \pm\frac{\sqrt{2}}{2}$$
$$3y^2 = 1 \Longrightarrow y = \pm\frac{\sqrt{3}}{3}$$
$$6z^2 = 1 \Longrightarrow z = \pm\frac{\sqrt{6}}{6}.$$

29(a). We have

$$\begin{aligned} AS &= \begin{bmatrix} 1 & -4 & 0 \\ -4 & 7 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 0 & 2x & y \\ 0 & x & -2y \\ z & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -2x & 9y \\ 0 & -x & -18y \\ 5z & 0 & 0 \end{bmatrix} \\ &= [5\mathbf{s}_1, -\mathbf{s}_2, 9\mathbf{s}_3]. \end{aligned}$$

29(b). We have

$$S^TAS = \begin{bmatrix} 0 & 0 & z \\ 2x & x & 0 \\ y & -2y & 0 \end{bmatrix} \begin{bmatrix} 0 & -2x & 9y \\ 0 & -x & -18y \\ 5z & 0 & 0 \end{bmatrix} = \begin{bmatrix} 5z^2 & 0 & 0 \\ 0 & -5x^2 & 0 \\ 0 & 0 & 45y^2 \end{bmatrix},$$

so in order for this to be equal to $\text{diag}(5, -1, 9)$, we must have

$$5z^2 = 5, \qquad -5x^2 = -1, \qquad 45y^2 = 9.$$

Thus, we must have $z^2 = 1$, $x^2 = \frac{1}{5}$, and $y^2 = \frac{1}{5}$. Therefore, the values of x, y, and z that we are looking for are $x = \pm\sqrt{\frac{1}{5}}$, $y = \pm\sqrt{\frac{1}{5}}$, and $z = \pm 1$.

30(a). $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

30(b). $\begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$.

31. Suppose A is an $n \times n$ scalar matrix with trace k. If $A = aI_n$, then $\text{tr}(A) = na = k$, so we conclude that $a = k/n$. So $A = \frac{k}{n}I_n$, a uniquely determined matrix.

32. We have

$$B^T = \left[\frac{1}{2}(A + A^T)\right]^T = \frac{1}{2}(A + A^T)^T = \frac{1}{2}(A^T + A) = B$$

and

$$C^T = \left[\frac{1}{2}(A - A^T)\right]^T = \frac{1}{2}(A^T - A) = -\frac{1}{2}(A - A^T) = -C.$$

Thus, B is symmetric and C is skew-symmetric.

33. We have

$$B + C = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) = \frac{1}{2}A + \frac{1}{2}A^T + \frac{1}{2}A - \frac{1}{2}A^T = A.$$

34. We have

$$B = \frac{1}{2}(A + A^T) = \frac{1}{2}\left(\begin{bmatrix} 4 & -1 & 0 \\ 9 & -2 & 3 \\ 2 & 5 & 5 \end{bmatrix} + \begin{bmatrix} 4 & 9 & 2 \\ -1 & -2 & 5 \\ 0 & 3 & 5 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 8 & 8 & 2 \\ 8 & -4 & 8 \\ 2 & 8 & 10 \end{bmatrix} = \begin{bmatrix} 4 & 4 & 1 \\ 4 & -2 & 4 \\ 1 & 4 & 5 \end{bmatrix}$$

and

$$C = \frac{1}{2}(A - A^T) = \frac{1}{2}\left(\begin{bmatrix} 4 & -1 & 0 \\ 9 & -2 & 3 \\ 2 & 5 & 5 \end{bmatrix} - \begin{bmatrix} 4 & 9 & 2 \\ -1 & -2 & 5 \\ 0 & 3 & 5 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 0 & -10 & -2 \\ 10 & 0 & -2 \\ 2 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -5 & -1 \\ 5 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}.$$

35.

$$B = \frac{1}{2}\left(\begin{bmatrix} 1 & -5 & 3 \\ 3 & 2 & 4 \\ 7 & -2 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 7 \\ -5 & 2 & -2 \\ 3 & 4 & 6 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 2 & -2 & 10 \\ -2 & 4 & 2 \\ 10 & 2 & 12 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 5 \\ -1 & 2 & 1 \\ 5 & 1 & 6 \end{bmatrix}.$$

$$C = \frac{1}{2}\left(\begin{bmatrix} 1 & -5 & 3 \\ 3 & 2 & 4 \\ 7 & -2 & 6 \end{bmatrix} - \begin{bmatrix} 1 & 3 & 7 \\ -5 & 2 & -2 \\ 3 & 4 & 6 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 0 & -8 & -4 \\ 8 & 0 & 6 \\ 4 & -6 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -4 & -2 \\ 4 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}.$$

36(a). If A is symmetric, then $A^T = A$, so that

$$B = \frac{1}{2}(A + A^T) = \frac{1}{2}(A + A) = \frac{1}{2}(2A) = A$$

and

$$C = \frac{1}{2}(A - A^T) = \frac{1}{2}(A - A) = \frac{1}{2}(0_n) = 0_n.$$

36(b). If A is skew-symmetric, then $A^T = -A$, so that

$$B = \frac{1}{2}(A + A^T) = \frac{1}{2}(A + (-A)) = \frac{1}{2}(0_n) = 0_n$$

and

$$C = \frac{1}{2}(A - A^T) = \frac{1}{2}(A - (-A)) = \frac{1}{2}(2A) = A.$$

37. If $A = [a_{ij}]$ and $D = \text{diag}(d_1, d_2, \dots, d_n)$, then we must show that the (i,j)-entry of DA is $d_i a_{ij}$. In index notation, we have

$$(DA)_{ij} = \sum_{k=1}^{n} d_i \delta_{ik} a_{kj} = d_i \delta_{ii} a_{ij} = d_i a_{ij}.$$

Hence, DA is the matrix obtained by multiplying the ith row vector of A by d_i, where $1 \leq i \leq n$.

38. If $A = [a_{ij}]$ and $D = \text{diag}(d_1, d_2, \dots, d_n)$, then we must show that the (i,j)-entry of AD is $d_j a_{ij}$. In index notation, we have

$$(AD)_{ij} = \sum_{k=1}^{n} a_{ik} d_j \delta_{kj} = a_{ij} d_j \delta_{jj} = a_{ij} d_j.$$

Hence, AD is the matrix obtained by multiplying the jth column vector of A by d_j, where $1 \leq j \leq n$.

39. Since A and B are symmetric, we have that $A^T = A$ and $B^T = B$. Using properties of the transpose operation, we therefore have

$$(AB)^T = B^T A^T = BA = AB,$$

and this shows that AB is symmetric.

40(a). We have $(AA^T)^T = (A^T)^T A^T = AA^T$, so that AA^T is symmetric.

40(b). We have $(ABC)^T = [(AB)C]^T = C^T(AB)^T = C^T(B^T A^T) = C^T B^T A^T$, as needed.

41. $A'(t) = \begin{bmatrix} 1 & \cos t \\ -\sin t & 4 \end{bmatrix}$.

42. $A'(t) = \begin{bmatrix} -2e^{-2t} \\ \cos t \end{bmatrix}$.

43. $A'(t) = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 1 \\ 0 & 3 & 0 \end{bmatrix}$.

44. $A'(t) = \begin{bmatrix} e^t & 2e^{2t} & 2t \\ 2e^t & 8e^{2t} & 10t \end{bmatrix}$.

45. We show that the (i,j)-entry of both sides of the equation agree. First, recall that the (i,j)-entry of AB is $\sum_{k=1}^{n} a_{ik} b_{kj}$, and therefore, the (i,j)-entry of $\frac{d}{dt}(AB)$ is (by the product rule)

$$\sum_{k=1}^{n} a'_{ik} b_{kj} + a_{ik} b'_{kj} = \sum_{k=1}^{n} a'_{ik} b_{kj} + \sum_{k=1}^{n} a_{ik} b'_{kj}.$$

The former term is precise the (i,j)-entry of the matrix $\frac{dA}{dt}B$, while the latter term is precise the (i,j)-entry of the matrix $A\frac{dB}{dt}$. Thus, the (i,j)-entry of $\frac{d}{dt}(AB)$ is precisely the sum of the (i,j)-entry of $\frac{dA}{dt}B$ and the (i,j)-entry of $A\frac{dB}{dt}$. Thus, the equation we are proving follows immediately.

46. We have

$$\int_0^1 \begin{bmatrix} e^t & e^{-t} \\ 2e^t & 5e^{-t} \end{bmatrix} dt = \begin{bmatrix} e^t & -e^{-t} \\ 2e^t & -5e^{-t} \end{bmatrix} \Big|_0^1 = \begin{bmatrix} e & -1/e \\ 2e & -5/e \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ 2 & -5 \end{bmatrix} = \begin{bmatrix} e-1 & 1-1/e \\ 2e-2 & 5-5/e \end{bmatrix}.$$

47. We have

$$\int_0^{\pi/2} \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} dt = \begin{bmatrix} \sin t \\ -\cos t \end{bmatrix} \Big|_0^{\pi/2} = \begin{bmatrix} \sin(\pi/2) \\ -\cos(\pi/2) \end{bmatrix} - \begin{bmatrix} \sin 0 \\ -\cos 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

48. We have

$$\int_0^1 \begin{bmatrix} e^t & e^{2t} & t^2 \\ 2e^t & 4e^{2t} & 5t^2 \end{bmatrix} dt = = \begin{bmatrix} e^t & \frac{1}{2}e^{2t} & \frac{t^3}{3} \\ 2e^t & 2e^{2t} & \frac{5}{3}t^3 \end{bmatrix} \Big|_0^1$$

$$= \begin{bmatrix} e & e^2/2 & 1/3 \\ 2e & 2e^2 & 5/3 \end{bmatrix} - \begin{bmatrix} 1 & 1/2 & 0 \\ 2 & 2 & 0 \end{bmatrix} = \begin{bmatrix} e-1 & \frac{e^2-1}{2} & 1/3 \\ 2e-2 & 2e^2-2 & 5/3 \end{bmatrix}.$$

49. We have

$$\int_0^1 \begin{bmatrix} e^{2t} & \sin 2t \\ t^2-5 & te^t \\ \sec^2 t & 3t - \sin t \end{bmatrix} dt = \begin{bmatrix} \frac{1}{2}e^{2t} & -\frac{1}{2}\cos 2t \\ \frac{t^3}{3} - 5t & te^t - e^t \\ \tan t & \frac{3}{2}t^2 + \cos t \end{bmatrix} \Big|_0^1$$

$$= \begin{bmatrix} \frac{e^2}{2} & -\frac{\cos 2}{2} \\ -14/3 & 0 \\ \tan 1 & \frac{3}{2} + \cos 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{e^2-1}{2} & \frac{1-\cos 2}{2} \\ -14/3 & 1 \\ \tan 1 & \frac{1}{2} + \cos 1 \end{bmatrix}.$$

50. $\int A(t)dt = \begin{bmatrix} \int -5dt & \int \frac{1}{t^2+1}dt & \int e^{3t}dt \end{bmatrix} = \begin{bmatrix} -5t & \tan^{-1}(t) & \frac{1}{3}e^{3t} \end{bmatrix}$.

51. $\int \begin{bmatrix} 2t \\ 3t^2 \end{bmatrix} dt = \begin{bmatrix} t^2 \\ t^3 \end{bmatrix}$.

52. $\int \begin{bmatrix} \sin t & \cos t & 0 \\ -\cos t & \sin t & t \\ 0 & 3t & 1 \end{bmatrix} dt = \begin{bmatrix} -\cos t & \sin t & 0 \\ -\sin t & -\cos t & t^2/2 \\ 0 & 3t^2/2 & t \end{bmatrix}$.

53. $\int \begin{bmatrix} e^t & e^{-t} \\ 2e^t & 5e^{-t} \end{bmatrix} dt = \begin{bmatrix} e^t & -e^{-t} \\ 2e^t & -5e^{-t} \end{bmatrix}$.

54. $\int \begin{bmatrix} e^{2t} & \sin 2t \\ t^2-5 & te^t \\ \sec^2 t & 3t - \sin t \end{bmatrix} dt = \begin{bmatrix} \frac{1}{2}e^{2t} & -\frac{1}{2}\cos 2t \\ \frac{t^3}{3} - 5t & te^t - e^t \\ \tan t & \frac{3}{2}t^2 + \cos t \end{bmatrix}$.

Solutions to Section 2.3

True-False Review:

(a): FALSE. The last column of the augmented matrix corresponds to the constants on the right-hand side of the linear system, so if the augmented matrix has n columns, there are only $n-1$ unknowns under consideration in the system.

(b): FALSE. Three distinct planes can intersect in a line (e.g. Figure 2.3.1, lower right picture). For instance, the xy-plane, the xz-plane, and the plane $y = z$ intersect in the x-axis.

(c): FALSE. The right-hand side vector must have m components, not n components.

(d): TRUE. If a linear system has two distinct solutions $\mathbf{x}_1$ and $\mathbf{x}_2$, then any point on the line containing $\mathbf{x}_1$ and $\mathbf{x}_2$ is also a solution, giving us infinitely many solutions, not exactly two solutions.

(e): TRUE. The augmented matrix for a linear system has one additional column (containing the constants on the right-hand side of the equation) beyond the matrix of coefficients.

(f): FALSE. Because the vector $(x_1, x_2, x_3, 0, 0)$ has five entries, this vector belongs to $\mathbb{R}^5$. Vectors in $\mathbb{R}^3$ can only have three slots.

(g): FALSE. The two column vectors given have different numbers of components, so they are not the same vectors.

Problems:

1.

$$2 \cdot 1 - 3(-1) + 4 \cdot 2 = 13,$$
$$1 + (-1) - 2 = -2,$$
$$5 \cdot 1 + 4(-1) + 2 = 3.$$

2.

$$2 + (-3) - 2 \cdot 1 = -3,$$
$$3 \cdot 2 - (-3) - 7 \cdot 1 = 2,$$
$$2 + (-3) + 1 = 0,$$
$$2 \cdot 2 + 2(-3) - 4 \cdot 1 = -6.$$

3.

$$(1 - t) + (2 + 3t) + (3 - 2t) = 6,$$
$$(1 - t) - (2 + 3t) - 2(3 - 2t) = -7,$$
$$5(1 - t) + (2 + 3t) - (3 - 2t) = 4.$$

4.

$$s + (s - 2t) - (2s + 3t) + 5t = 0,$$
$$2(s - 2t) - (2s + 3t) + 7t = 0,$$
$$4s + 2(s - 2t) - 3(2s + 3t) + 13t = 0.$$

5. The two given lines are the same line. Therefore, since this line contains an infinite number of points, there must be an infinite number of solutions to this linear system.

6. These two lines are parallel and distinct, and therefore, there are no common points on these lines. In other words, there are no solutions to this linear system.

7. These two lines have different slopes, and therefore, they will intersect in exactly one point. Thus, this system of equations has exactly one solution.

8. The first and third equations describe lines that are parallel and distinct, and therefore, there are no common points on these lines. In other words, there are no solutions to this linear system.

9. $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -5 \\ 7 & 2 & -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $A^{\#} = \left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 2 & 4 & -5 & 2 \\ 7 & 2 & -1 & 3 \end{array}\right]$.

10. $A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 2 & 4 & -3 & 7 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $A^{\#} = \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 3 \\ 2 & 4 & -3 & 7 & 2 \end{array}\right]$.

11. $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & -2 \\ 5 & 6 & -5 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $A^{\#} = \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 2 & 3 & -2 & 0 \\ 5 & 6 & -5 & 0 \end{array}\right]$.

12. It is acceptable to use any variable names. We will use x_1, x_2, x_3, x_4:

$$\begin{aligned} x_1 - x_2 \quad +2x_3 + 3x_4 &= 1, \\ x_1 + x_2 \quad -2x_3 + 6x_4 &= -1, \\ 3x_1 + x_2 \quad +4x_3 + 2x_4 &= 2. \end{aligned}$$

13. It is acceptable to use any variable names. We will use x_1, x_2, x_3:

$$\begin{aligned} 2x_1 + \ \ x_2 \quad +3x_3 &= 3, \\ 4x_1 - \ \ x_2 \quad +2x_3 &= 1, \\ 7x_1 + 6x_2 \quad +3x_3 &= -5. \end{aligned}$$

14. The system of equations here only contains one equation: $4x_1 - 2x_2 - 2x_3 - 3x_5 = -9$.

15. This system of equations has three equations: $-3x_2 = -1, \quad 2x_1 - 7x_2 = 6, \quad 5x_1 + 5x_2 = 7$.

16. Given $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{y} = \mathbf{0}$, and an arbitrary constant c,

(a). we have

$$A\mathbf{z} = A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \mathbf{0} + \mathbf{0} = \mathbf{0}$$

and

$$A\mathbf{w} = A(c\mathbf{x}) = c(A\mathbf{x}) = c\mathbf{0} = \mathbf{0}.$$

(b). No, because

$$A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \mathbf{b} + \mathbf{b} = 2\mathbf{b} \neq \mathbf{b},$$

and

$$A(c\mathbf{x}) = c(A\mathbf{x}) = c\mathbf{b} \neq \mathbf{b}$$

in general.

17. $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -4 & 3 \\ 6 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 4t \\ t^2 \end{bmatrix}$.

18. $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} t^2 & -t \\ -\sin t & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

19. $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & e^{2t} \\ -\sin t & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

20. $\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 0 & -\sin t & 1 \\ -e^t & 0 & t^2 \\ -t & t^2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} t \\ t^3 \\ 1 \end{bmatrix}.$

21. We have

$$\mathbf{x}'(t) = \begin{bmatrix} 4e^{4t} \\ -2(4e^{4t}) \end{bmatrix} = \begin{bmatrix} 4e^{4t} \\ -8e^{4t} \end{bmatrix}$$

and

$$A\mathbf{x} + \mathbf{b} = \begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} e^{4t} \\ -2e^{4t} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2e^{4t} + (-1)(-2e^{4t}) + 0 \\ -2e^{4t} + 3(-2e^{4t}) + 0 \end{bmatrix} = \begin{bmatrix} 4e^{4t} \\ -8e^{4t} \end{bmatrix}.$$

22. We have

$$\mathbf{x}'(t) = \begin{bmatrix} 4(-2e^{-2t}) + 2\cos t \\ 3(-2e^{-2t}) + \sin t \end{bmatrix} = \begin{bmatrix} -8e^{-2t} + 2\cos t \\ -6e^{-2t} + \sin t \end{bmatrix}$$

and

$$A\mathbf{x} + \mathbf{b} = \begin{bmatrix} 1 & -4 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 4e^{-2t} + 2\sin t \\ 3e^{-2t} - \cos t \end{bmatrix} + \begin{bmatrix} -2(\cos t + \sin t) \\ 7\sin t + 2\cos t \end{bmatrix}$$
$$= \begin{bmatrix} 4e^{-2t} + 2\sin t - 4(3e^{-2t} - \cos t) - 2(\cos t + \sin t) \\ -3(4e^{-2t} + 2\sin t) + 2(3e^{-2t} - \cos t) + 7\sin t + 2\cos t \end{bmatrix} = \begin{bmatrix} -8e^{-2t} + 2\cos t \\ -6e^{-2t} + \sin t \end{bmatrix}.$$

23. We compute

$$\mathbf{x}' = \begin{bmatrix} 3e^t + 2te^t \\ e^t + 2te^t \end{bmatrix}$$

and

$$A\mathbf{x} + \mathbf{b} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2te^t + e^t \\ 2te^t - e^t \end{bmatrix} + \begin{bmatrix} 0 \\ 4e^t \end{bmatrix} = \begin{bmatrix} 2(2te^t + e^t) - (2te^t - e^t) + 0 \\ -(2te^t + e^t) + 2(2te^t - e^t) + 4e^t \end{bmatrix} = \begin{bmatrix} 2te^t + 3e^t \\ 2te^t + e^t \end{bmatrix}.$$

Therefore, we see from these calculations that $\mathbf{x}' = A\mathbf{x} + \mathbf{b}$.

24. We compute

$$\mathbf{x}' = \begin{bmatrix} -te^t - e^t \\ -9e^{-t} \\ te^t + e^t - 6e^{-t} \end{bmatrix}$$

and

$$A\mathbf{x}+\mathbf{b} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -3 & 2 \\ 1 & -2 & 2 \end{bmatrix} \begin{bmatrix} -te^t \\ 9e^{-t} \\ te^t + 6e^{-t} \end{bmatrix} + \begin{bmatrix} -e^t \\ 6e^{-t} \\ e^t \end{bmatrix} = \begin{bmatrix} -te^t \\ 2(-te^t) - 3(9e^{-t}) + 2(te^t + 6e^{-t}) \\ -te^t - 2(9e^{-t}) + 2(te^t + 6e^{-t}) \end{bmatrix} + \begin{bmatrix} -e^t \\ 6e^{-t} \\ e^t \end{bmatrix} = \begin{bmatrix} -te^t - e^t \\ -9e^{-t} \\ te^t + e^t - 6e^{-t} \end{bmatrix}$$

Therefore, we see from these calculations that $\mathbf{x}' = A\mathbf{x} + \mathbf{b}$.

Solutions to Section 2.4

True-False Review:

(a): TRUE. The precise row-echelon form obtained for a matrix depends on the particular elementary row operations (and their order). However, Theorem 2.4.15 states that there is a unique reduced row-echelon form for a matrix.

(b): FALSE. Upper triangular matrices could have pivot entries that are not 1. For instance, the following matrix is upper triangular, but not in row echelon form: $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$.

(c): TRUE. The pivots in a row-echelon form of an $n \times n$ matrix must move down and to the right as we look from one row to the next beneath it. Thus, the pivots must occur on or to the right of the main diagonal of the matrix, and thus all entries below the main diagonal of the matrix are zero.

(d): FALSE. This would not be true, for example, if A was a zero matrix with 5 rows and B was a nonzero matrix with 4 rows.

(e): FALSE. If A is a nonzero matrix and $B = -A$, then $A + B = 0$, so $\text{rank}(A + B) = 0$, but $\text{rank}(A)$, $\text{rank}(B) \geq 1$ so $\text{rank}(A)+ \text{rank}(B) \geq 2$.

(f): FALSE. For example, if $A = B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then $AB = 0$, so $\text{rank}(AB) = 0$, but $\text{rank}(A)+ \text{rank}(B) = 1 + 1 = 2$.

(g): TRUE. A matrix of rank zero cannot have any pivots, hence no nonzero rows. It must be the zero matrix.

(h): TRUE. The matrices A and $2A$ have the same reduced row-echelon form, since we can move between the two matrices by multiplying the rows of one of them by 2 or 1/2, a matter of carrying out elementary row operations. If the two matrices have the same reduced row-echelon form, then they have the same rank.

(i): TRUE. The matrices A and $2A$ have the same reduced row-echelon form, since we can move between the two matrices by multiplying the rows of one of them by 2 or 1/2, a matter of carrying out elementary row operations.

Problems:

1. Neither.

2. Reduced row-echelon form.

3. Neither.

4. Row-echelon form.

5. Row-echelon form.

6. Reduced row-echelon form.

7. Reduced row-echelon form.

8. Reduced row-echelon form.

9.

$$\begin{bmatrix} 2 & -4 \\ -4 & 8 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & -2 \\ -4 & 8 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}, \text{Rank } (A) = 1.$$

1. $M_1(\frac{1}{2})$ **2.** $A_{12}(4)$

10.

$$\begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & -3 \\ 0 & 7 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}, \text{Rank } (A) = 2.$$

1. P_{12} **2.** $A_{12}(-2)$ **3.** $M_2(\frac{1}{7})$

11.

$$\begin{bmatrix} 0 & 1 & 3 \\ 0 & 1 & 4 \\ 0 & 3 & 5 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 4 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \text{Rank }(A) = 2.$$

1. $A_{12}(-1)$, $A_{13}(-3)$ **2.** $A_{23}(-4)$

12.

$$\begin{bmatrix} 2 & 1 & 4 \\ 2 & -3 & 4 \\ 3 & -2 & 6 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 3 & -2 & 6 \\ 2 & -3 & 4 \\ 2 & 1 & 4 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & 1 & 2 \\ 2 & -3 & 4 \\ 2 & 1 & 4 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -5 & 0 \\ 0 & -1 & 0 \end{bmatrix} \overset{4}{\sim} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 0 \\ 0 & -5 & 0 \end{bmatrix}$$

$$\overset{5}{\sim} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & -5 & 0 \end{bmatrix} \overset{6}{\sim} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{Rank }(A) = 2.$$

1. P_{13} **2.** $A_{21}(-1)$ **3.** $A_{12}(-2)$, $A_{13}(-3)$ **4.** P_{23} **5.** $M_2(-1)$ **6.** $A_{32}(5)$

13.

$$\begin{bmatrix} 2 & -1 & 3 \\ 3 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 3 & 1 & -2 \\ 2 & -1 & 3 \\ 2 & -2 & 1 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & 2 & -5 \\ 2 & -1 & 3 \\ 0 & -1 & -2 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & 2 & -5 \\ 0 & -5 & 13 \\ 0 & -1 & -2 \end{bmatrix} \overset{4}{\sim} \begin{bmatrix} 1 & 2 & -5 \\ 0 & -1 & -2 \\ 0 & -5 & 13 \end{bmatrix}$$

$$\overset{5}{\sim} \begin{bmatrix} 1 & 2 & -5 \\ 0 & 1 & 2 \\ 0 & -5 & 13 \end{bmatrix} \overset{6}{\sim} \begin{bmatrix} 1 & 2 & -5 \\ 0 & 1 & 2 \\ 0 & 0 & 23 \end{bmatrix} \overset{7}{\sim} \begin{bmatrix} 1 & 2 & -5 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \text{Rank }(A) = 3.$$

1. P_{12} **2.** $A_{21}(-1)$, $A_{23}(-1)$ **3.** $A_{12}(-2)$ **4.** P_{23} **5.** $M_2(-1)$ **6.** $A_{23}(5)$ **7.** $M_3(1/23)$.

14.

$$\begin{bmatrix} 2 & -1 \\ 3 & 2 \\ 2 & 5 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 3 & 2 \\ 2 & -1 \\ 2 & 5 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & 3 \\ 2 & -1 \\ 2 & 5 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & 3 \\ 0 & -7 \\ 0 & -1 \end{bmatrix} \overset{4}{\sim} \begin{bmatrix} 1 & 3 \\ 0 & -1 \\ 0 & -7 \end{bmatrix} \overset{5}{\sim} \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \text{Rank }(A) = 2.$$

1. P_{12} **2.** $A_{21}(-1)$ **3.** $A_{12}(-2)$, $A_{13}(-2)$ **4.** P_{23} **5.** $M_2(-1)$, $A_{23}(7)$.

15.

$$\begin{bmatrix} 2 & -2 & -1 & 3 \\ 3 & -2 & 3 & 1 \\ 1 & -1 & 1 & 0 \\ 2 & -1 & 2 & 2 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & -1 & 1 & 0 \\ 3 & -2 & 3 & 1 \\ 2 & -2 & -1 & 3 \\ 2 & -1 & 2 & 2 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -3 & 3 \\ 0 & 1 & 0 & 2 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -3 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\overset{4}{\sim} \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{Rank }(A) = 4.$$

1. P_{13} **2.** $A_{12}(-3)$, $A_{13}(-2)$, $A_{14}(-2)$ **3.** $A_{24}(-1)$ **4.** $M_3(1/3)$

16.

$$\begin{bmatrix} 2 & -1 & 3 & 4 \\ 1 & -2 & 1 & 3 \\ 1 & -5 & 0 & 5 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} 1 & -2 & 1 & 3 \\ 2 & -1 & 3 & 4 \\ 1 & -5 & 0 & 5 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \stackrel{3}{\sim} \begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 1 & \frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{Rank }(A) = 2.$$

1. P_{12} **2.** $A_{12}(-2)$, $A_{13}(-1)$ **3.** $M_2(1/3)$

17.

$$\begin{bmatrix} 2 & 1 & 3 & 4 & 2 \\ 1 & 0 & 2 & 1 & 3 \\ 2 & 3 & 1 & 5 & 7 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} 1 & 0 & 2 & 1 & 3 \\ 2 & 1 & 3 & 4 & 2 \\ 2 & 3 & 1 & 5 & 7 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 1 & 0 & 2 & 1 & 3 \\ 0 & 1 & -1 & 2 & -4 \\ 0 & 3 & -3 & 3 & 1 \end{bmatrix} \stackrel{3}{\sim} \begin{bmatrix} 1 & 0 & 2 & 1 & 3 \\ 0 & 1 & -1 & 2 & -4 \\ 0 & 0 & 0 & -3 & 1 \end{bmatrix}$$

$$\stackrel{4}{\sim} \begin{bmatrix} 1 & 0 & 2 & 1 & 3 \\ 0 & 1 & -1 & 2 & -4 \\ 0 & 0 & 0 & 1 & -\frac{1}{3} \end{bmatrix}, \text{Rank }(A) = 3.$$

1. P_{12} **2.** $A_{12}(-2)$, $A_{13}(-2)$, **3.** $A_{23}(-3)$ **4.** $M_3(-\frac{1}{3})$

18.

$$\begin{bmatrix} 4 & 7 & 4 & 7 \\ 3 & 5 & 3 & 5 \\ 2 & -2 & 2 & -2 \\ 5 & -2 & 5 & -2 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 3 & 5 & 3 & 5 \\ 2 & -2 & 2 & -2 \\ 5 & -2 & 5 & -2 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -1 & 0 & -1 \\ 0 & -6 & 0 & -6 \\ 0 & -12 & 0 & -12 \end{bmatrix} \stackrel{3}{\sim} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & -6 & 0 & -6 \\ 0 & -12 & 0 & -12 \end{bmatrix}$$

$$\stackrel{4}{\sim} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{Rank }(A) = 2.$$

1. $A_{21}(-1)$ **2.** $A_{12}(-3)$, $A_{13}(-2)$, $A_{14}(-5)$ **3.** $M_2(-1)$ **4.** $A_{23}(6)$, $A_{24}(12)$

19.

$$\begin{bmatrix} -4 & 2 \\ -6 & 3 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} 1 & -1/2 \\ -6 & 3 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}, \quad \text{Rank}(A) = 1.$$

1. $M_1(-\frac{1}{4})$ **2.** $A_{12}(6)$

20.

$$\begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 1 & -1 \\ 0 & 5 \end{bmatrix} \stackrel{3}{\sim} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \stackrel{4}{\sim} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2, \text{Rank }(A) = 2.$$

1. P_{12} **2.** $A_{12}(-3)$ **3.** $M_2(\frac{1}{5})$ **4.** $A_{21}(1)$

21.

$$\begin{bmatrix} 3 & 7 & 10 \\ 2 & 3 & -1 \\ 1 & 2 & 1 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & -1 \\ 3 & 7 & 10 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -3 \\ 0 & 1 & 7 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 1 & 7 \end{bmatrix} \overset{4}{\sim} \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\overset{5}{\sim} \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \overset{6}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3, \text{Rank } (A) = 3.$$

1. P_{13} **2.** $A_{12}(-2)$, $A_{13}(-3)$ **3.** $M_2(-1)$ **4.** $A_{21}(-2)$, $A_{23}(-1)$ **5.** $M_3(\frac{1}{4})$ **6.** $A_{31}(5)$, $A_{32}(-3)$

22.

$$\begin{bmatrix} 3 & -3 & 6 \\ 2 & -2 & 4 \\ 6 & -6 & 12 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{Rank } (A) = 1.$$

1. $M_1(\frac{1}{3})$, $A_{12}(-2)$, $A_{13}(-6)$

23.

$$\begin{bmatrix} 3 & 5 & -12 \\ 2 & 3 & -7 \\ -2 & -1 & 1 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & 2 & -5 \\ 0 & -1 & 3 \\ 0 & 3 & -9 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & 2 & -5 \\ 0 & 1 & -3 \\ 0 & 3 & -9 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}, \text{Rank } (A) = 2.$$

1. $A_{21}(-1)$, $A_{12}(-2)$, $A_{13}(2)$ **2.** $M_2(-1)$ **3.** $A_{21}(-2)$, $A_{23}(-3)$

24.

$$\begin{bmatrix} 1 & -1 & -1 & 2 \\ 3 & -2 & 0 & 7 \\ 2 & -1 & 2 & 4 \\ 4 & -2 & 3 & 8 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & -1 & -1 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 4 & 0 \\ 0 & 2 & 7 & 0 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\overset{4}{\sim} \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \overset{5}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_4, \text{Rank } (A) = 4.$$

1. $A_{12}(-3)$, $A_{13}(-2)$, $A_{14}(-4)$ **2.** $A_{21}(1)$, $A_{23}(-1)$, $A_{24}(-2)$ **3.** $A_{31}(-2)$, $A_{32}(-3)$, $A_{34}(-1)$
4. $M_4(-1)$ **5.** $A_{41}(-5)$, $A_{42}(-4)$, $A_{43}(1)$

25.

$$\begin{bmatrix} 1 & -2 & 1 & 3 \\ 3 & -6 & 2 & 7 \\ 4 & -8 & 3 & 10 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & -1 & -2 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -1 & -2 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{Rank } (A) = 2.$$

1. $A_{12}(-3)$, $A_{13}(-4)$ **2.** $M_2(-1)$ **3.** $A_{21}(-1)$, $A_{23}(1)$

26.

$$\begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 3 & 1 & 2 \\ 0 & 2 & 0 & 1 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 0 & -6 & -2 \\ 0 & 0 & -4 & -1 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1/3 \\ 0 & 0 & -4 & -1 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 1/3 \end{bmatrix}$$

$$\overset{4}{\sim} \begin{bmatrix} 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \overset{5}{\sim} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{Rank } (A) = 3.$$

1. $A_{12}(-3)$, $A_{13}(-2)$ **2.** $M_2(-\frac{1}{6})$ **3.** $A_{21}(-2)$, $A_{23}(4)$ **4.** $M_3(3)$ **5.** $A_{32}(-\frac{1}{3})$, $A_{31}(-\frac{1}{3})$

Solutions to Section 2.5

True-False Review:

(a): FALSE. This process is known as Gaussian elimination. Gauss-Jordan elimination is the process by which a matrix is brought to *reduced* row echelon form via elementary row operations.

(b): TRUE. A homogeneous linear system always has the trivial solution $\mathbf{x} = \mathbf{0}$, hence it is consistent.

(c): TRUE. The columns of the row-echelon form that contain leading 1s correspond to leading variables, while columns of the row-echelon form that do not contain leading 1s correspond to free variables.

(d): TRUE. If the last column of the row-reduced augmented matrix for the system does not contain a pivot, then the system can be solved by back-substitution. On the other hand, if this column does contain a pivot, then that row of the row-reduced matrix containing the pivot in the last column corresponds to the impossible equation $0 = 1$.

(e): FALSE. The linear system $x = 0$, $y = 0$, $z = 0$ has a solution in $(0, 0, 0)$ even though none of the variables here is free.

(f): FALSE. The columns containing the leading 1s correspond to the *leading* variables, not the free variables.

Problems:

For the problems of this section, A will denote the coefficient matrix of the given system, and $A^{\#}$ will denote the augmented matrix of the given system.

1. Converting the given system of equations to an augmented matrix and using Gaussian elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{cc|c} 1 & -5 & 3 \\ 3 & -9 & 15 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{cc|c} 1 & -5 & 3 \\ 0 & 6 & 6 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{cc|c} 1 & -5 & 3 \\ 0 & 1 & 1 \end{array}\right].$$

1. $A_{12}(-3)$ **2.** $M_2(\frac{1}{6})$

By back substitution, we find that $x_2 = 1$, and then $x_1 = 8$. Therefore, the solution is $(8, 1)$.

2. Converting the given system of equations to an augmented matrix and using Gaussian elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{cc|c} 4 & -1 & 8 \\ 2 & 1 & 1 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{cc|c} 1 & -\frac{1}{4} & 2 \\ 2 & 1 & 1 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{cc|c} 1 & -\frac{1}{4} & 2 \\ 0 & \frac{3}{2} & -3 \end{array}\right] \overset{3}{\sim} \left[\begin{array}{cc|c} 1 & -\frac{1}{4} & 2 \\ 0 & 1 & -2 \end{array}\right].$$

1. $M_1(\frac{1}{4})$ 2. $A_{12}(-2)$ 3. $M_2(\frac{2}{3})$

By back substitution, we find that $x_2 = -2$, and then $x_1 = \dfrac{3}{2}$. Therefore, the solution is $(\frac{3}{2}, -2)$.

3. Converting the given system of equations to an augmented matrix and using Gaussian elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{cc|c} 7 & -3 & 5 \\ 14 & -6 & 10 \end{array}\right] \stackrel{1}{\sim} \left[\begin{array}{cc|c} 7 & -3 & 5 \\ 0 & 0 & 0 \end{array}\right] \stackrel{2}{\sim} \left[\begin{array}{cc|c} 1 & -\frac{3}{7} & \frac{5}{7} \\ 0 & 0 & 0 \end{array}\right].$$

1. $A_{12}(-2)$ 2. $M_2(\frac{1}{7})$

Observe that x_2 is a free variable, so we set $x_2 = t$. Then by back substitution, we have $x_1 = \frac{3}{7}t + \frac{5}{7}$. Therefore, the solution set to this system is

$$\left\{ \left(\frac{3}{7}t + \frac{5}{7}, t \right) : t \in \mathbb{R} \right\}.$$

4. Converting the given system of equations to an augmented matrix and using Gaussian elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 3 & 5 & 1 & 3 \\ 2 & 6 & 7 & 1 \end{array}\right] \stackrel{1}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -1 & -2 & 0 \\ 0 & 2 & 5 & -1 \end{array}\right] \stackrel{2}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 5 & -1 \end{array}\right] \stackrel{3}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right].$$

1. $A_{12}(-3)$, $A_{13}(-2)$ 2. $M_2(-1)$ 3. $A_{23}(-2)$

The last augmented matrix results in the system:

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1, \\ x_2 + 2x_3 &= 0, \\ x_3 &= -1. \end{aligned}$$

By back substitution we obtain the solution $(-2, 2, -1)$.

5. Converting the given system of equations to an augmented matrix and using Gaussian elimination, we obtain the following equivalent matrices:

$$\left[\begin{array}{ccc|c} 3 & -1 & 0 & 1 \\ 2 & 1 & 5 & 4 \\ 7 & -5 & -8 & -3 \end{array}\right] \stackrel{1}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & -5 & -3 \\ 2 & 1 & 5 & 4 \\ 7 & -5 & -8 & -3 \end{array}\right] \stackrel{2}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & -5 & -3 \\ 0 & 5 & 15 & 10 \\ 0 & 9 & 27 & 18 \end{array}\right]$$

$$\stackrel{3}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & -5 & -3 \\ 0 & 1 & 3 & 2 \\ 0 & 9 & 27 & 18 \end{array}\right] \stackrel{4}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

1. $A_{21}(-1)$ 2. $A_{12}(-2)$, $A_{13}(-7)$ 3. $M_2(\frac{1}{5})$ 4. $A_{21}(2)$, $A_{23}(-9)$

The last augmented matrix results in the system:

$$\begin{aligned} x_1 \qquad + \ x_3 &= 1, \\ x_2 + 3x_3 &= 2. \end{aligned}$$

Let the free variable $x_3 = t$, a real number. By back substitution we find that the system has the solution set $\{(1-t, 2-3t, t) : \text{ for all real numbers } t\}$.

6. Converting the given system of equations to an augmented matrix and using Gaussian elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{ccc|c} 3 & 5 & -1 & 14 \\ 1 & 2 & 1 & 3 \\ 2 & 5 & 6 & 2 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 3 & 5 & -1 & 4 \\ 2 & 5 & 6 & 2 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & -4 & -5 \\ 0 & 1 & 4 & -4 \end{array}\right] \overset{3}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 1 & 4 & -4 \end{array}\right]$$

$$\overset{4}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & -9 \end{array}\right] \overset{5}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

1. P_{12} **2.** $A_{12}(-3)$, $A_{13}(-2)$ **3.** $M_2(-1)$ **4.** $A_{23}(-1)$ **5.** $M_4(-\frac{1}{9})$

This system of equations is inconsistent since $2 = \text{rank}(A) < \text{rank}(A^{\#}) = 3$.

7. Converting the given system of equations to an augmented matrix and using Gaussian elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{ccc|c} 6 & -3 & 3 & 12 \\ 2 & -1 & 1 & 4 \\ -4 & 2 & -2 & -8 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & -\frac{1}{2} & 2 \\ 2 & -1 & 1 & 4 \\ -4 & 2 & -2 & -8 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{1}{2} & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

1. $M_1(\frac{1}{6})$ **2.** $A_{12}(-2)$, $A_{13}(4)$

Since x_2 and x_3 are free variables, let $x_2 = s$ and $x_3 = t$. The single equation obtained from the augmented matrix is given by $x_1 - \frac{1}{2}x_2 + \frac{1}{2}x_3 = 2$. Thus, the solution set of our system is given by

$$\{(2 + \frac{s}{2} - \frac{t}{2}, s, t) : s, t \text{ any real numbers } \}.$$

8. Converting the given system of equations to an augmented matrix and using Gaussian elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 14 \\ 3 & 1 & -2 & -1 \\ 7 & 2 & -3 & 3 \\ 5 & -1 & -2 & 5 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccc|c} 3 & 1 & -2 & -1 \\ 2 & -1 & 3 & 14 \\ 7 & 2 & -3 & 3 \\ 5 & -1 & -2 & 5 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & -5 & -15 \\ 2 & -1 & 3 & -14 \\ 7 & 2 & -3 & 3 \\ 5 & -1 & -2 & 5 \end{array}\right] \overset{3}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & -5 & -15 \\ 0 & -5 & 13 & 44 \\ 0 & -12 & 32 & 108 \\ 0 & -11 & 23 & 80 \end{array}\right]$$

$$\overset{4}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & -5 & -15 \\ 0 & -12 & 32 & 108 \\ 0 & -5 & 13 & 44 \\ 0 & -11 & 23 & 80 \end{array}\right] \overset{5}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & -5 & -15 \\ 0 & -1 & 9 & 28 \\ 0 & -5 & 13 & 44 \\ 0 & -11 & 23 & 80 \end{array}\right] \overset{6}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & -5 & -15 \\ 0 & 1 & -9 & -28 \\ 0 & -5 & 13 & 44 \\ 0 & -11 & 23 & 80 \end{array}\right]$$

$$\overset{7}{\sim}\left[\begin{array}{ccc|c}1&2&-5&-15\\0&1&-9&-28\\0&0&-32&-96\\0&0&-76&-228\end{array}\right]\overset{8}{\sim}\left[\begin{array}{ccc|c}1&2&-5&-15\\0&1&-9&-28\\0&0&32&96\\0&0&-76&-228\end{array}\right]\overset{9}{\sim}\left[\begin{array}{ccc|c}1&2&-5&-15\\0&1&-9&-28\\0&0&1&3\\0&0&0&0\end{array}\right].$$

1. P_{12} **2.** $A_{21}(-1)$ **3.** $A_{12}(-2)$, $A_{13}(-7)$, $A_{14}(-5)$ **4.** P_{23}
5. $A_{42}(-1)$ **6.** $M_2(-1)$ **7.** $A_{23}(5)$, $A_{24}(11)$ **8.** $M_3(-1)$ **9.** $M_3(\frac{1}{32})$, $A_{34}(76)$.

The last augmented matrix results in the system of equations:

$$\begin{aligned} x_1 - 2x_2 - 5x_3 &= -15, \\ x_2 - 9x_3 &= -28, \\ x_3 &= \quad 3. \end{aligned}$$

Thus, using back substitution, the solution set for our system is given by $\{(2, -1, 3)\}$.

9. Converting the given system of equations to an augmented matrix and using Gaussian elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{ccc|c}2&-1&-4&5\\3&2&-5&8\\5&6&-6&20\\1&1&-3&-3\end{array}\right]\overset{1}{\sim}\left[\begin{array}{ccc|c}1&1&-3&-3\\3&2&-5&8\\5&6&-6&20\\2&-1&-4&-5\end{array}\right]\overset{2}{\sim}\left[\begin{array}{ccc|c}1&1&-3&-3\\0&-1&4&17\\0&1&9&35\\0&-3&2&11\end{array}\right]\overset{3}{\sim}\left[\begin{array}{ccc|c}1&1&-3&-3\\0&1&-4&-17\\0&1&9&35\\0&-3&2&11\end{array}\right]$$

$$\overset{4}{\sim}\left[\begin{array}{ccc|c}1&1&-3&-3\\0&1&-4&-17\\0&0&13&52\\0&0&-10&-40\end{array}\right]\overset{5}{\sim}\left[\begin{array}{ccc|c}1&1&-3&-3\\0&1&-4&-17\\0&0&1&4\\0&0&-10&-40\end{array}\right]\overset{6}{\sim}\left[\begin{array}{ccc|c}1&1&-3&-3\\0&1&-4&-17\\0&0&1&4\\0&0&0&0\end{array}\right].$$

1. P_{14} **2.** $A_{12}(-3)$, $A_{13}(-5)$, $A_{14}(-2)$ **3.** $M_2(-1)$ **4.** $A_{23}(-1)$, $A_{24}(3)$ **5.** $M_3(\frac{1}{13})$ **6.** $A_{34}(10)$

The last augmented matrix results in the system of equations:

$$\begin{aligned} x_1 + x_2 - 3x_3 &= -\ 3, \\ x_2 - 4x_3 &= -17, \\ x_3 &= \quad 4. \end{aligned}$$

By back substitution, we obtain the solution set $\{(10, -1, 4)\}$.

10. Converting the given system of equations to an augmented matrix and using Gaussian elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{cccc|c}1&2&-1&1&1\\2&4&-2&2&2\\5&10&-5&5&5\end{array}\right]\overset{1}{\sim}\left[\begin{array}{cccc|c}1&2&-1&1&1\\0&0&0&0&0\\0&0&0&0&0\end{array}\right].$$

1. $A_{12}(-2)$, $A_{13}(-5)$

The last augmented matrix results in the equation $x_1 + 2x_3 - x_3 + x_4 = 1$. Now x_2, x_3, and x_4 are free variables, so we let $x_2 = r$, $x_3 = s$, and $x_4 = t$. It follows that $x_1 = 1 - 2r + s - t$. Consequently, the solution set of the system is given by $\{(1 - 2r + s - t, r, s, t) : r, s, t \text{ and real numbers }\}$.

11. Converting the given system of equations to an augmented matrix and using Gaussian elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{rrrr|r} 1 & 2 & -1 & 1 & 1 \\ 2 & -3 & 1 & -1 & 2 \\ 1 & -5 & 2 & -2 & 1 \\ 4 & 1 & -1 & 1 & 3 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{rrrr|r} 1 & 2 & -1 & 1 & 1 \\ 0 & -7 & 3 & -3 & 0 \\ 0 & -7 & 3 & -3 & 0 \\ 0 & -7 & 3 & -3 & -1 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{rrrr|r} 1 & 2 & -1 & 1 & 1 \\ 0 & 1 & -\frac{3}{7} & \frac{3}{7} & 0 \\ 0 & -7 & 3 & -3 & 0 \\ 0 & -7 & 3 & -3 & -1 \end{array}\right]$$

$$\overset{3}{\sim} \left[\begin{array}{rrrr|r} 1 & 2 & -1 & 1 & 1 \\ 0 & 1 & -\frac{3}{7} & \frac{3}{7} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{array}\right] \overset{4}{\sim} \left[\begin{array}{rrrr|r} 1 & 2 & -1 & 1 & 1 \\ 0 & 1 & -\frac{3}{7} & \frac{3}{7} & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \overset{5}{\sim} \left[\begin{array}{rrrr|r} 1 & 2 & -1 & 1 & 1 \\ 0 & 1 & -\frac{3}{7} & \frac{3}{7} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

1. $A_{12}(-2)$, $A_{13}(-1)$, $A_{14}(-4)$ **2.** $M_2(-\frac{1}{7})$ **3.** $A_{23}(7)$, $A_{24}(7)$ **4.** P_{34} **5.** $M_3(-1)$

The given system of equations is inconsistent since $2 = \text{rank}(A) < \text{rank}(A^{\#}) = 3$.

12. Converting the given system of equations to an augmented matrix and using Gauss-Jordan elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{rrrrr|r} 1 & 2 & 1 & 1 & -2 & 3 \\ 0 & 0 & 1 & 4 & -3 & 2 \\ 2 & 4 & -1 & -10 & 5 & 0 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{rrrrr|r} 1 & 2 & 1 & 1 & -2 & 3 \\ 0 & 0 & 1 & 4 & -3 & 2 \\ 0 & 0 & -3 & -12 & 9 & -6 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{rrrrr|r} 1 & 2 & 1 & 1 & -2 & 3 \\ 0 & 0 & 1 & 4 & -3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

1. $A_{13}(-2)$ **2.** $A_{23}(3)$

The last augmented matrix indicates that the first two equations of the initial system completely determine its solution. We see that x_4 and x_5 are free variables, so let $x_4 = s$ and $x_5 = t$. Then $x_3 = 2 - 4x_4 + 3x_5 = 2-4s+3t$. Moreover, x_2 is a free variable, say $x_2 = r$, so then $x_1 = 3-2r-(2-4s+3t)-s+2t = 1-2r+3s-t$. Hence, the solution set for the system is

$$\{(1 - 2r + 3s - t, r, 2 - 4s + 3t, s, t) : r, s, t \text{ any real numbers }\}.$$

13. Converting the given system of equations to an augmented matrix and using Gauss-Jordan elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{rrr|r} 2 & -1 & -2 & 2 \\ 4 & 3 & -2 & -1 \\ 1 & 4 & 1 & 4 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{rrr|r} 1 & 4 & 1 & 4 \\ 4 & 3 & -2 & -1 \\ 2 & -1 & -1 & 2 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{rrr|r} 1 & 4 & 1 & 4 \\ 0 & -13 & -6 & -17 \\ 0 & -9 & -3 & -6 \end{array}\right] \overset{3}{\sim} \left[\begin{array}{rrr|r} 1 & 4 & 1 & 4 \\ 0 & -9 & -3 & -6 \\ 0 & -13 & -6 & -17 \end{array}\right]$$

$$\overset{4}{\sim} \left[\begin{array}{rrr|r} 1 & 4 & 1 & 4 \\ 0 & 12 & 4 & 8 \\ 0 & -13 & -6 & -17 \end{array}\right] \overset{5}{\sim} \left[\begin{array}{rrr|r} 1 & 4 & 1 & 4 \\ 0 & 12 & 4 & 8 \\ 0 & -1 & -2 & -9 \end{array}\right] \overset{6}{\sim} \left[\begin{array}{rrr|r} 1 & 4 & 1 & 4 \\ 0 & -1 & -2 & -9 \\ 0 & 12 & 4 & 8 \end{array}\right] \overset{7}{\sim} \left[\begin{array}{rrr|r} 1 & 4 & 1 & 4 \\ 0 & 1 & 2 & 9 \\ 0 & 12 & 4 & 8 \end{array}\right]$$

$$\overset{8}{\sim} \left[\begin{array}{rrr|r} 1 & 0 & -7 & -32 \\ 0 & 1 & 2 & 9 \\ 0 & 0 & -20 & -100 \end{array}\right] \overset{9}{\sim} \left[\begin{array}{rrr|r} 1 & 0 & -7 & -32 \\ 0 & 1 & 2 & 9 \\ 0 & 0 & 1 & 5 \end{array}\right] \overset{10}{\sim} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 5 \end{array}\right].$$

1. P_{13} **2.** $A_{12}(-4)$, $A_{13}(-2)$ **3.** P_{23} **4.** $M_2(-\frac{4}{3})$ **5.** $A_{23}(1)$ **6.** P_{23} **7.** $M_2(-1)$ **8.** $A_{21}(-4)$, $A_{23}(-12)$ **9.** $M_3(-\frac{1}{20})$ **10.** $A_{31}(7)$, $A_{32}(-2)$

The last augmented matrix results in the solution $(3, -1, 5)$.

14. Converting the given system of equations to an augmented matrix and using Gauss-Jordan elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{ccc|c} 3 & 1 & 5 & 2 \\ 1 & 1 & -1 & 1 \\ 2 & 1 & 2 & 3 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 3 & 1 & 5 & 2 \\ 2 & 1 & 2 & 3 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & -2 & 8 & -1 \\ 0 & -1 & 4 & 1 \end{array}\right]$$

$$\overset{3}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & -4 & \frac{1}{2} \\ 0 & -1 & 4 & 1 \end{array}\right] \overset{4}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & -4 & 1/2 \\ 0 & 0 & 0 & 3/2 \end{array}\right].$$

We can stop here, since we see from this last augmented matrix that the system is inconsistent. In particular, $2 = \text{rank}(A) < \text{rank}(A^{\#}) = 3$.

1. P_{12} **2.** $A_{12}(-3)$, $A_{13}(-2)$ **3.** $M_2(-\frac{1}{2})$ **4.** $A_{23}(1)$

15. Converting the given system of equations to an augmented matrix and using Gauss-Jordan elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & -3 \\ 3 & -2 & 4 & -9 \\ 1 & -4 & 2 & -3 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -2 & -3 \\ 0 & -2 & 2 & 0 \\ 0 & -4 & 4 & 0 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -2 & -3 \\ 0 & 1 & -1 & 0 \\ 0 & -4 & 4 & 0 \end{array}\right] \overset{3}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -2 & -3 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

.

1. $A_{12}(-3)$, $A_{13}(-1)$ **2.** $M_2(-\frac{1}{2})$ **3.** $A_{23}(4)$

The last augmented matrix results in the following system of equations:

$$x_1 - 2x_3 = -3 \quad \text{and} \quad x_2 - x_3 = 0.$$

Since x_3 is free, let $x_3 = t$. Thus, from the system we obtain the solutions $\{(2t-3, t, t) : t \text{ any real number }\}$.

16. Converting the given system of equations to an augmented matrix and using Gauss-Jordan elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{cccc|c} 2 & -1 & 3 & -1 & 3 \\ 3 & 2 & 1 & -5 & -6 \\ 1 & -2 & 3 & 1 & 6 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{cccc|c} 1 & -2 & 3 & 1 & 6 \\ 3 & 2 & 1 & -5 & -6 \\ 2 & -1 & 3 & -1 & 3 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{cccc|c} 1 & -2 & 3 & 1 & 6 \\ 0 & 8 & -8 & -8 & -24 \\ 0 & 3 & -3 & -3 & -9 \end{array}\right]$$

$$\overset{3}{\sim} \left[\begin{array}{cccc|c} 1 & -2 & 3 & 1 & 6 \\ 0 & 1 & -1 & -1 & -3 \\ 0 & 3 & -3 & -3 & -9 \end{array}\right] \overset{4}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

1. P_{13} **2.** $A_{12}(-3)$, $A_{13}(-2)$ **3.** $M_2(\frac{1}{8})$ **4.** $A_{21}(2)$, $A_{23}(-3)$

The last augmented matrix results in the following system of equations:

$$x_1 + x_3 - x_4 = 0 \quad \text{and} \quad x_2 - x_3 - x_4 = -3.$$

Since x_3 and x_4 are free variables, we can let $x_3 = s$ and $x_4 = t$, where s and t are real numbers. It follows that the solution set of the system is given by $\{(t - s, s + t - 3, s, t) : s, t \text{ any real numbers }\}$.

17. Converting the given system of equations to an augmented matrix and using Gauss-Jordan elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 4 \\ 1 & -1 & -1 & -1 & 2 \\ 1 & 1 & -1 & 1 & -2 \\ 1 & -1 & 1 & 1 & -8 \end{array}\right] \stackrel{1}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 4 \\ 0 & -2 & -2 & 0 & -2 \\ 0 & 0 & -2 & 2 & -6 \\ 0 & -2 & 0 & 2 & -12 \end{array}\right] \stackrel{2}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 4 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 3 \\ 0 & 1 & 0 & -1 & 6 \end{array}\right]$$

$$\stackrel{3}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 3 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & -1 & -1 & 5 \end{array}\right] \stackrel{4}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & -2 & 8 \end{array}\right] \stackrel{5}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 1 & -4 \end{array}\right] \stackrel{6}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -4 \end{array}\right].$$

1. $A_{12}(-1)$, $A_{13}(-1)$, $A_{14}(-1)$ **2.** $M_2(-\frac{1}{2})$, $M_3(-\frac{1}{2})$, $M_4(-\frac{1}{2})$ **3.** $A_{24}(-1)$
4. $A_{32}(-1)$, $A_{34}(1)$ **5.** $M_4(-\frac{1}{2})$ **6.** $A_{41}(1)$, $A_{42}(-1)$, $A_{43}(1)$

It follows from the last augmented matrix that the solution to the system is given by $(-1, 2, -1, -4)$.

18. Converting the given system of equations to an augmented matrix and using Gauss-Jordan elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{ccccc|c} 2 & -1 & 3 & 1 & -1 & 11 \\ 1 & -3 & -2 & -1 & -2 & 2 \\ 3 & 1 & -2 & -1 & 1 & -2 \\ 1 & 2 & 1 & 2 & 3 & -3 \\ 5 & -3 & -3 & 1 & 2 & 2 \end{array}\right] \stackrel{1}{\sim} \left[\begin{array}{ccccc|c} 1 & -3 & -2 & -1 & -2 & 2 \\ 2 & -1 & 3 & 1 & -1 & 11 \\ 3 & 1 & -2 & -1 & 1 & -2 \\ 1 & 2 & 1 & 2 & 3 & -3 \\ 5 & -3 & -3 & 1 & 2 & 2 \end{array}\right] \stackrel{2}{\sim} \left[\begin{array}{ccccc|c} 1 & -3 & -2 & -1 & -2 & 2 \\ 0 & 5 & 7 & 3 & 3 & 7 \\ 0 & 10 & 4 & 2 & 7 & -8 \\ 0 & 5 & 3 & 3 & 5 & -5 \\ 0 & 12 & 7 & 6 & 12 & -8 \end{array}\right]$$

$$\stackrel{3}{\sim} \left[\begin{array}{ccccc|c} 1 & -3 & -2 & -1 & -2 & 2 \\ 0 & 1 & \frac{7}{5} & \frac{3}{5} & \frac{3}{5} & \frac{7}{5} \\ 0 & 10 & 4 & 2 & 7 & -8 \\ 0 & 5 & 3 & 3 & 5 & -5 \\ 0 & 12 & 7 & 6 & 12 & -8 \end{array}\right] \stackrel{4}{\sim} \left[\begin{array}{ccccc|c} 1 & 0 & \frac{11}{5} & \frac{4}{5} & -\frac{1}{5} & \frac{31}{5} \\ 0 & 1 & \frac{7}{5} & \frac{3}{5} & \frac{3}{5} & \frac{7}{5} \\ 0 & 0 & -10 & -4 & 1 & -22 \\ 0 & 0 & -4 & 0 & 2 & -12 \\ 0 & 0 & -\frac{49}{5} & -\frac{6}{5} & \frac{24}{5} & -\frac{124}{5} \end{array}\right] \stackrel{5}{\sim} \left[\begin{array}{ccccc|c} 1 & 0 & \frac{11}{5} & \frac{4}{5} & -\frac{1}{5} & \frac{31}{5} \\ 0 & 1 & \frac{7}{5} & \frac{3}{5} & \frac{3}{5} & \frac{7}{5} \\ 0 & 0 & 1 & \frac{2}{5} & -\frac{1}{10} & \frac{11}{5} \\ 0 & 0 & -4 & 0 & 2 & -12 \\ 0 & 0 & -\frac{49}{5} & -\frac{6}{5} & \frac{24}{5} & -\frac{124}{5} \end{array}\right]$$

$$\stackrel{6}{\sim} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & -\frac{2}{25} & \frac{1}{50} & \frac{34}{25} \\ 0 & 1 & 0 & \frac{1}{25} & \frac{37}{50} & -\frac{42}{25} \\ 0 & 0 & 1 & \frac{2}{5} & -\frac{1}{10} & \frac{11}{5} \\ 0 & 0 & 0 & \frac{8}{5} & \frac{8}{5} & -\frac{16}{5} \\ 0 & 0 & 0 & \frac{68}{25} & \frac{191}{50} & -\frac{81}{25} \end{array}\right] \stackrel{7}{\sim} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & -\frac{2}{25} & \frac{1}{50} & \frac{34}{25} \\ 0 & 1 & 0 & \frac{1}{25} & \frac{37}{50} & -\frac{42}{25} \\ 0 & 0 & 1 & \frac{2}{5} & -\frac{1}{10} & \frac{11}{5} \\ 0 & 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & \frac{68}{25} & \frac{191}{50} & -\frac{81}{25} \end{array}\right] \stackrel{8}{\sim} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & \frac{1}{10} & \frac{6}{5} \\ 0 & 1 & 0 & 0 & \frac{7}{10} & -\frac{8}{5} \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & 3 \\ 0 & 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & \frac{11}{10} & \frac{11}{5} \end{array}\right]$$

$$\stackrel{9}{\sim} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & \frac{1}{10} & \frac{6}{5} \\ 0 & 1 & 0 & 0 & \frac{7}{10} & -\frac{8}{5} \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & 3 \\ 0 & 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right] \stackrel{10}{\sim} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

1. P_{12} **2.** $A_{12}(-2), A_{13}(-3), A_{14}(-1), A_{15}(-5)$ **3.** $M_2(\frac{1}{5})$ **4.** $A_{21}(3), A_{23}(-10), A_{24}(-5), A_{25}(-12)$
5. $M_3(-\frac{1}{10})$ **6.** $A_{31}(-\frac{11}{5}), A_{32}(-\frac{7}{5}), A_{34}(4), A_{35}(\frac{49}{5})$ **7.** $M_4(\frac{5}{8})$
8. $A_{41}(\frac{2}{25}), A_{42}(-\frac{1}{25}), A_{43}(-\frac{2}{5}), A_{45}(-\frac{68}{25})$ **9.** $M_5(\frac{10}{11})$ **10.** $A_{51}(-\frac{1}{10}), A_{52}(-\frac{7}{10}), A_{53}(\frac{1}{2}), A_{54}(-1)$

It follows from the last augmented matrix that the solution to the system is given by $(1, -3, 4, -4, 2)$.

19. The equation $A\mathbf{x} = \mathbf{b}$ reads

$$\begin{bmatrix} 1 & -3 & 1 \\ 5 & -4 & 1 \\ 2 & 4 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 15 \\ -4 \end{bmatrix}.$$

Converting the given system of equations to an augmented matrix and using Gauss-Jordan elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 8 \\ 5 & -4 & 1 & 15 \\ 2 & 4 & -3 & -4 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccc|c} 1 & -3 & 1 & 8 \\ 0 & 11 & -4 & -25 \\ 0 & 10 & -5 & -20 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{ccc|c} 1 & -3 & 1 & 8 \\ 0 & 1 & 1 & -5 \\ 0 & 10 & -5 & -20 \end{array}\right]$$

$$\overset{3}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 4 & -7 \\ 0 & 1 & 1 & -5 \\ 0 & 0 & -15 & 30 \end{array}\right] \overset{4}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 4 & -7 \\ 0 & 1 & 1 & -5 \\ 0 & 0 & 1 & -2 \end{array}\right] \overset{5}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{array}\right].$$

1. $A_{12}(-5), A_{13}(-2)$ **2.** $A_{32}(-1)$ **3.** $A_{21}(3), A_{23}(-10)$ **4.** $M_3(-\frac{1}{15})$ **5.** $A_{31}(-4), A_{32}(-1)$

Thus, from the last augmented matrix, we see that $x_1 = 1$, $x_2 = -3$, and $x_3 = -2$.

20. The equation $A\mathbf{x} = \mathbf{b}$ reads

$$\begin{bmatrix} 1 & 0 & 5 \\ 3 & -2 & 11 \\ 2 & -2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}.$$

Converting the given system of equations to an augmented matrix and using Gauss-Jordan elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 3 & -2 & 11 & 2 \\ 2 & -2 & 6 & 2 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & -2 & -4 & 2 \\ 0 & -2 & -4 & 2 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & -2 & -4 & 2 \end{array}\right]$$

$$\overset{3}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

1. $A_{12}(-3), A_{13}(-2)$ **2.** $M_2(-1/2)$ **3.** $A_{23}(2)$

Hence, we have $x_1 + 5x_3 = 0$ and $x_2 + 2x_3 = -1$. Since x_3 is a free variable, we can let $x_3 = t$, where t is any real number. It follows that the solution set for the given system is given by $\{(-5t, -2t - 1, t) : t \in \mathbb{R}\}$.

21. The equation $A\mathbf{x} = \mathbf{b}$ reads

$$\begin{bmatrix} 0 & 1 & -1 \\ 0 & 5 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \\ 5 \end{bmatrix}.$$

Converting the given system of equations to an augmented matrix using Gauss-Jordan elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{ccc|c} 0 & 1 & -1 & -2 \\ 0 & 5 & 1 & 8 \\ 0 & 2 & 1 & 5 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccc|c} 0 & 1 & -1 & -2 \\ 0 & 0 & 6 & 18 \\ 0 & 0 & 3 & 9 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{ccc|c} 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 3 & 9 \end{array}\right] \overset{3}{\sim} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

1. $A_{12}(-5)$, $A_{13}(-2)$ **2.** $M_2(1/6)$ **3.** $A_{21}(1)$, $A_{23}(-3)$

Consequently, from the last augmented matrix it follows that the solution set for the matrix equation is given by $\{(t, 1, 3) : t \in \mathbb{R}\}$.

22. The equation $A\mathbf{x} = \mathbf{b}$ reads

$$\begin{bmatrix} 1 & -1 & 0 & -1 \\ 2 & 1 & 3 & 7 \\ 3 & -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}.$$

Converting the given system of equations to an augmented matrix and using Gauss-Jordan elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & -1 & 2 \\ 2 & 1 & 3 & 7 & 2 \\ 3 & -2 & 1 & 0 & 4 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{cccc|c} 1 & -1 & 0 & -1 & 2 \\ 0 & 3 & 3 & 9 & -2 \\ 0 & 1 & 1 & 3 & -2 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{cccc|c} 1 & -1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 3 & -2 \\ 0 & 3 & 3 & 9 & -2 \end{array}\right] \overset{3}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 & 4 \end{array}\right].$$

1. $A_{12}(-2)$, $A_{13}(-3)$ **2.** P_{23} **3.** $A_{21}(1)$, $A_{23}(-3)$

From the last row of the last augmented matrix, it is clear that the given system is inconsistent.

23. The equation $A\mathbf{x} = \mathbf{b}$ reads

$$\begin{bmatrix} 1 & 1 & 0 & -1 \\ 3 & 1 & -2 & 3 \\ 2 & 3 & 1 & 1 \\ -2 & 3 & 5 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 3 \\ -9 \end{bmatrix}.$$

Converting the given system of equations to an augmented matrix and using Gauss-Jordan elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 2 \\ 3 & 1 & -2 & 3 & 8 \\ 2 & 3 & 1 & 2 & 3 \\ -2 & 3 & 5 & -2 & -9 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 2 \\ 0 & -2 & -2 & 0 & 2 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 5 & 5 & 0 & -5 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & -2 & -2 & 0 & 2 \\ 0 & 5 & 5 & 0 & -5 \end{array}\right] \overset{3}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 3 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

1. $A_{12}(-3)$, $A_{13}(-2)$, $A_{14}(2)$ **2.** P_{23} **3.** $A_{21}(-1)$, $A_{23}(2)$, $A_{24}(-5)$

From the last augmented matrix, we obtain the system of equations: $x_1 - x_3 + x_4 = 3$, $x_2 + x_3 = -1$. Since both x_3 and x_4 are free variables, we may let $x_3 = r$ and $x_4 = t$, where r and t are real numbers. The solution set for the system is given by $\{(3 + r - t, -r - 1, r, t) : r, t \in \mathbb{R}\}$.

24. Converting the given system of equations to an augmented matrix and using Gauss-Jordan elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & 5 & 1 & 7 \\ 1 & 1 & -k^2 & -k \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & -1 & 1 - k^2 & -3 - k \end{array}\right] \overset{2}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 4 - k^2 & -2 - k \end{array}\right].$$

1. $A_{12}(-2), A_{13}(-1)$ **2.** $A_{23}(1)$

(a). If $k = 2$, then the last row of the last augmented matrix reveals an inconsistency; hence the system has no solutions in this case.

(b). If $k = -2$, then the last row of the last augmented matrix consists entirely of zeros, and hence we have only two pivots (first two columns) and a free variable x_3; hence the system has infinitely many solutions.

(c). If $k \neq \pm 2$, then the last augmented matrix above contains a pivot for each variable x_1, x_2, and x_3, and can be solved for a unique solution by back-substitution.

25. Converting the given system of equations to an augmented matrix and using Gauss-Jordan elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{cccc|c} 2 & 1 & -1 & 1 & 0 \\ 1 & 1 & 1 & -1 & 0 \\ 4 & 2 & -1 & 1 & 0 \\ 3 & -1 & 1 & k & 0 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 2 & 1 & -1 & 1 & 0 \\ 4 & 2 & -1 & 1 & 0 \\ 3 & -1 & 1 & k & 0 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & -1 & -3 & 3 & 0 \\ 0 & -2 & -5 & 5 & 0 \\ 0 & -4 & -2 & k+3 & 0 \end{array}\right]$$

$$\overset{3}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 3 & -3 & 0 \\ 0 & -2 & -5 & 5 & 0 \\ 0 & -4 & -2 & k+3 & 0 \end{array}\right] \overset{4}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 3 & -3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 10 & k-9 & 0 \end{array}\right] \overset{5}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 3 & -3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & k+1 & 0 \end{array}\right].$$

1. P_{12} **2.** $A_{12}(-2), A_{13}(-4), A_{14}(-3)$ **3.** $M_2(-1)$ **4.** $A_{23}(2), A_{24}(4)$ **5.** $A_{34}(-10)$

(a). Note that the trivial solution $(0, 0, 0, 0)$ exists under all circumstances, so there are no values of k for which there is no solution.

(b). From the last row of the last augmented matrix, we see that if $k = -1$, then the variable x_4 corresponds to an unpivoted column, and hence it is a free variable. In this case, therefore, we have infinitely solutions.

(c). Provided that $k \neq -1$, then each variable in the system corresponds to a pivoted column of the last augmented matrix above. Therefore, we can solve the system by back-substitution. The conclusion from this is that there is a unique solution, $(0, 0, 0, 0)$.

26. Converting the given system of equations to an augmented matrix and using Gauss-Jordan elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 4 \\ 3 & 5 & -4 & 16 \\ 2 & 3 & -a & b \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 4 \\ 0 & 2 & 2 & 4 \\ 0 & 1 & 4-a & b-8 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 4-a & b-8 \end{array}\right] \overset{3}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 3-a & b-10 \end{array}\right].$$

1. $A_{12}(-3), A_{13}(-2)$ **2.** $M_2(\frac{1}{2})$ **3.** $A_{21}(-1), A_{23}(-1)$

(a). From the last row of the last augmented matrix above, we see that there is no solution if $a = 3$ and $b \neq 10$.

(b). From the last row of the augmented matrix above, we see that there are infinitely many solutions if $a = 3$ and $b = 10$, because in that case, there is no pivot in the column of the last augmented matrix corresponding to the third variable x_3.

(c). From the last row of the augmented matrix above, we see that if $a \neq 3$, then regardless of the value of b, there is a pivot corresponding to each variable x_1, x_2, and x_3. Therefore, we can uniquely solve the corresponding system by back-substitution.

27. Converting the given system of equations to an augmented matrix and using Gauss-Jordan elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{cc|c} 1 & -a & 3 \\ 2 & 1 & 6 \\ -3 & a+b & 1 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{cc|c} 1 & -a & 3 \\ 0 & 1+2a & 0 \\ 0 & b-2a & 10 \end{array}\right].$$

From the middle row, we see that if $a \neq -\frac{1}{2}$, then we must have $x_2 = 0$, but this leads to an inconsistency in solving for x_1 (the first equation would require $x_1 = 3$ while the last equation would require $x_1 = -\frac{1}{3}$. Now suppose that $a = -\frac{1}{2}$. Then the augmented matrix on the right reduces to $\left[\begin{array}{cc|c} 1 & -1/2 & 3 \\ 0 & b+1 & 10 \end{array}\right]$. If $b = -1$, then once more we have an inconsistency in the last row. However, if $b \neq -1$, then the row-echelon form obtained has full rank, and there is a unique solution. Therefore, we draw the following conclusions:

(a). There is no solution to the system if $a \neq -\frac{1}{2}$ or if $a = -\frac{1}{2}$ and $b = -1$.

(b). Under no circumstances are there an infinite number of solutions to the linear system.

(c). There is a unique solution if $a = -\frac{1}{2}$ and $b \neq -1$.

28. The corresponding augmented matrix for this linear system can be reduced to row-echelon form via

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & y_1 \\ 2 & 3 & 1 & y_2 \\ 3 & 5 & 1 & y_3 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 1 & y_1 \\ 0 & 1 & -1 & y_2 - 2y_1 \\ 0 & 2 & -2 & y_3 - 3y_1 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 1 & y_1 \\ 0 & 1 & -1 & y_2 - 2y_1 \\ 0 & 0 & 0 & y_1 - 2y_2 + y_3 \end{array}\right].$$

1. $A_{12}(-2)$, $A_{13}(-3)$ **2.** $A_{23}(-2)$

For consistency, we must have $\text{rank}(A) = \text{rank}(A^\#)$, which requires (y_1, y_2, y_3) to satisfy $y_1 - 2y_2 + y_3 = 0$. If this holds, then the system has an infinite number of solutions, because the column of the augmented matrix corresponding to y_3 will be unpivoted, indicating that y_3 is a free variable in the solution set.

29. Converting the given system of equations to an augmented matrix and using Gaussian elimination we obtain the following row-equivalent matrices. Since $a_{11} \neq 0$:

$$\left[\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{cc|c} 1 & \frac{a_{12}}{a_{11}} & \frac{b_1}{a_{11}} \\ 0 & \frac{a_{22}a_{11} - a_{21}a_{12}}{a_{11}} & \frac{a_{11}b_2 - a_{21}b_1}{a_{11}} \end{array}\right] \overset{2}{\sim} \left[\begin{array}{cc|c} 1 & \frac{a_{12}}{a_{11}} & \frac{b_1}{a_{11}} \\ 0 & \frac{\Delta}{a_{11}} & \frac{\Delta_2}{a_{11}} \end{array}\right].$$

1. $M_1(1/a_{11})$, $A_{12}(-a_{21})$ **2.** Definition of Δ and Δ_2

(a). If $\Delta \neq 0$, then $\text{rank}(A) = \text{rank}(A^\#) = 2$, so the system has a unique solution (of course, we are assuming $a_{11} \neq 0$ here). Using the last augmented matrix above, $\left(\frac{\Delta}{a_{11}}\right) x_2 = \frac{\Delta_2}{a_{11}}$, so that $x_2 = \frac{\Delta_2}{\Delta}$. Using this, we can solve $x_1 + \frac{a_{12}}{a_{11}} x_2 = \frac{b_1}{a_{11}}$ for x_1 to obtain $x_1 = \frac{\Delta_1}{\Delta}$, where we have used the fact that $\Delta_1 = a_{22}b_1 - a_{12}b_2$.

(b). If $\Delta = 0$ and $a_{11} \neq 0$, then the augmented matrix of the system is $\left[\begin{array}{cc|c} 1 & \frac{a_{12}}{a_{11}} & \frac{b_1}{a_{11}} \\ 0 & 0 & \Delta_2 \end{array}\right]$, so it follows that the system has (i) no solution if $\Delta_2 \neq 0$, since $\text{rank}(A) < \text{rank}(A^\#) = 2$, and (ii) an infinite number of solutions if $\Delta_2 = 0$, since $\text{rank}(A^\#) < 2$.

(c). An infinite number of solutions would be represented as one line. No solution would be two parallel lines. A unique solution would be the intersection of two distinct lines at one point.

30. We first use the partial pivoting algorithm to reduce the augmented matrix of the system:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 3 & 5 & 1 & 3 \\ 2 & 6 & 7 & 1 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccc|c} 3 & 5 & 1 & 3 \\ 1 & 2 & 1 & 1 \\ 2 & 6 & 7 & 1 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{ccc|c} 3 & 5 & 1 & 3 \\ 0 & 1/3 & 2/3 & 0 \\ 0 & 8/3 & 19/3 & -1 \end{array}\right]$$

$$\overset{3}{\sim} \left[\begin{array}{ccc|c} 3 & 5 & 1 & 3 \\ 0 & 8/3 & 19/3 & -1 \\ 0 & 1/3 & 2/3 & 0 \end{array}\right] \overset{4}{\sim} \left[\begin{array}{ccc|c} 3 & 5 & 1 & 3 \\ 0 & 8/3 & 19/3 & -1 \\ 0 & 0 & -1/8 & 1/8 \end{array}\right].$$

1. P_{12} **2.** $A_{12}(-1/3)$, $A_{13}(-2/3)$ **3.** P_{23} **4.** $A_{23}(-1/8)$

Using back substitution to solve the equivalent system yields the unique solution $(-2, 2, -1)$.

31. We first use the partial pivoting algorithm to reduce the augmented matrix of the system:

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 14 \\ 3 & 1 & -2 & -1 \\ 7 & 2 & -3 & 3 \\ 5 & -1 & -2 & 5 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccc|c} 7 & 2 & -3 & 3 \\ 3 & 1 & -2 & -1 \\ 2 & -1 & 3 & 14 \\ 5 & -1 & -2 & 5 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{ccc|c} 7 & 2 & -3 & 3 \\ 0 & 1/7 & -5/7 & -16/7 \\ 0 & -11/7 & 27/7 & 92/7 \\ 0 & -17/7 & 1/7 & 20/7 \end{array}\right]$$

$$\overset{3}{\sim} \left[\begin{array}{ccc|c} 7 & 2 & -3 & 3 \\ 0 & -17/7 & 1/7 & 20/7 \\ 0 & -11/7 & 27/7 & 92/7 \\ 0 & 1/7 & -5/7 & -16/7 \end{array}\right] \overset{4}{\sim} \left[\begin{array}{ccc|c} 7 & 2 & -3 & 3 \\ 0 & -17/7 & 1/7 & 20/7 \\ 0 & 0 & 64/17 & 192/17 \\ 0 & 0 & -12/17 & -36/17 \end{array}\right]$$

$$\overset{5}{\sim} \left[\begin{array}{ccc|c} 7 & 2 & -3 & 3 \\ 0 & -17/7 & 1/7 & 20/7 \\ 0 & 0 & 64/17 & 192/17 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

1. P_{13} **2.** $A_{12}(-3/7)$, $A_{13}(-2/7)$, $A_{14}(-5/7)$ **3.** P_{24}
4. $A_{23}(-11/17)$, $A_{24}(1/17)$ **5.** $A_{34}(3/16)$

Using back substitution to solve the equivalent system yields the unique solution $(2, -1, 3)$.

32. We first use the partial pivoting algorithm to reduce the augmented matrix of the system:

$$\left[\begin{array}{ccc|c} 2 & -1 & -4 & 5 \\ 3 & 2 & -5 & 8 \\ 5 & 6 & -6 & 20 \\ 1 & 1 & -3 & -3 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccc|c} 5 & 6 & -6 & -20 \\ 3 & 2 & -5 & 8 \\ 2 & -1 & -4 & 5 \\ 1 & 1 & -3 & -3 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{ccc|c} 5 & 6 & -6 & 20 \\ 0 & -8/5 & -7/5 & -4 \\ 0 & -17/5 & -8/5 & -3 \\ 0 & -1/5 & -9/5 & -7 \end{array}\right]$$

$$\overset{3}{\sim} \left[\begin{array}{ccc|c} 5 & 6 & -6 & 20 \\ 0 & -17/5 & -8/5 & -3 \\ 0 & -8/5 & -7/5 & -4 \\ 0 & -1/5 & -9/5 & -7 \end{array}\right] \overset{4}{\sim} \left[\begin{array}{ccc|c} 5 & 6 & -6 & 20 \\ 0 & -17/5 & -8/5 & -3 \\ 0 & 0 & -11/17 & -44/17 \\ 0 & 0 & -29/17 & -116/17 \end{array}\right]$$

$$\overset{5}{\sim} \left[\begin{array}{ccc|c} 5 & 6 & -6 & 20 \\ 0 & -17/5 & -8/5 & -3 \\ 0 & 0 & -29/17 & -116/17 \\ 0 & 0 & -11/17 & -44/17 \end{array}\right] \overset{6}{\sim} \left[\begin{array}{ccc|c} 5 & 6 & -6 & 20 \\ 0 & -17/5 & -8/5 & -3 \\ 0 & 0 & -29/17 & -116/17 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

1. P_{13} **2.** $A_{12}(-3/5)$, $A_{13}(-2/5)$, $A_{14}(-1/5)$ **3.** P_{23} **4.** $A_{23}(-8/17)$, $A_{24}(-1/17)$ **5.** P_{34} **6.** $A_{34}(-11/29)$

Using back substitution to solve the equivalent system yields the unique solution $(10, -1, 4)$.

33. We first use the partial pivoting algorithm to reduce the augmented matrix of the system:

$$\left[\begin{array}{ccc|c} 2 & -1 & -1 & 2 \\ 4 & 3 & -2 & -1 \\ 1 & 4 & 1 & 4 \end{array}\right] \stackrel{1}{\sim} \left[\begin{array}{ccc|c} 4 & 3 & -2 & -1 \\ 2 & -1 & -1 & 2 \\ 1 & 4 & 1 & 4 \end{array}\right] \stackrel{2}{\sim} \left[\begin{array}{ccc|c} 4 & 3 & -2 & -1 \\ 0 & -5/2 & 0 & 5/2 \\ 0 & 13/4 & 3/2 & 17/4 \end{array}\right]$$

$$\stackrel{3}{\sim} \left[\begin{array}{ccc|c} 4 & 3 & -2 & -1 \\ 0 & 13/4 & 3/2 & 17/4 \\ 0 & -5/2 & 0 & 5/2 \end{array}\right] \stackrel{4}{\sim} \left[\begin{array}{ccc|c} 4 & 3 & -2 & -1 \\ 0 & 13/4 & 3/2 & 17/4 \\ 0 & 0 & 15/13 & 75/13 \end{array}\right].$$

1. P_{12} **2.** $A_{12}(-1/2)$, $A_{13}(-1/4)$ **3.** P_{23} **4.** $A_{23}(10/13)$

Using back substitution to solve the equivalent system yields the unique solution $(3, -1, 5)$.

34.

(a). Let

$$A^{\#} = \left[\begin{array}{ccccc|c} a_{11} & 0 & 0 & \dots & 0 & b_1 \\ a_{21} & a_{22} & 0 & \dots & 0 & b_2 \\ a_{31} & a_{32} & a_{33} & \dots & 0 & b_3 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} & b_n \end{array}\right]$$

represent the corresponding augmented matrix of the given system. Since $a_{11}x_1 = b_1$, we can solve for x_1 easily:

$$x_1 = \frac{b_1}{a_{11}}, \qquad (a_{11} \neq 0).$$

Now since $a_{21}x_1 + a_{22}x_2 = b_2$, by using the expression for x_1 we just obtained, we can solve for x_2:

$$x_2 = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22}}.$$

In a similar manner, we can solve for $x_3, x_4, \dots, x_n$.

(b). We solve instantly for x_1 from the first equation: $x_1 = 2$. Substituting this into the middle equation, we obtain $2 \cdot 2 - 3 \cdot x_2 = 1$, from which it quickly follows that $x_2 = 1$. Substituting for x_1 and x_2 in the bottom equation yields $3 \cdot 2 + 1 - x_3 = 8$, from which it quickly follows that $x_3 = -1$. Consequently, the solution of the given system is $(2, 1, -1)$.

35. This system of equations is not linear in x_1, x_2, and x_3; however, the system *is* linear in x_1^3, x_2^2, and x_3, so we can first solve for x_1^3, x_2^2, and x_3. Converting the given system of equations to an augmented matrix and using Gauss-Jordan elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{ccc|c} 4 & 2 & 3 & 12 \\ 1 & -1 & 1 & 2 \\ 3 & 1 & -1 & 2 \end{array}\right] \stackrel{1}{\sim} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 4 & 2 & 3 & 12 \\ 3 & 1 & -1 & 2 \end{array}\right] \stackrel{2}{\sim} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 6 & -1 & 4 \\ 0 & 4 & -4 & -4 \end{array}\right]$$

$$\overset{3}{\sim}\left[\begin{array}{rrr|r} 1 & -1 & 1 & 2 \\ 0 & 4 & -4 & -4 \\ 0 & 6 & -1 & 4 \end{array}\right] \overset{4}{\sim}\left[\begin{array}{rrr|r} 1 & -1 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 6 & -1 & 4 \end{array}\right] \overset{5}{\sim}\left[\begin{array}{rrr|r} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 5 & 10 \end{array}\right]$$

$$\overset{6}{\sim}\left[\begin{array}{rrr|r} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right] \overset{7}{\sim}\left[\begin{array}{rrr|r} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right].$$

1. P_{12} **2.** $A_{12}(-4)$, $A_{13}(-3)$ **3.** P_{23} **4.** $M_2(1/4)$
5. $A_{21}(1)$, $A_{23}(-6)$ **6.** $M_2(1/5)$ **7.** $A_{32}(1)$

Thus, taking only real solutions, we have $x_1^3 = 1$, $x_2^2 = 1$, and $x_3 = 2$. Therefore, $x_1 = 1$, $x_2 = \pm 1$, and $x_3 = 2$, leading to the *two* solutions $(1, 1, 2)$ and $(1, -1, 2)$ to the original system of equations. There is no contradiction of Theorem 2.5.9 here since, as mentioned above, this system is *not linear* in x_1, x_2, and x_3.

36. Reduce the augmented matrix of the system:

$$\left[\begin{array}{rrr|r} 3 & 2 & -1 & 0 \\ 2 & 1 & 1 & 0 \\ 5 & -4 & 1 & 0 \end{array}\right] \overset{1}{\sim}\left[\begin{array}{rrr|r} 1 & 1 & -2 & 0 \\ 0 & -1 & 5 & 0 \\ 0 & -9 & 11 & 0 \end{array}\right] \overset{2}{\sim}\left[\begin{array}{rrr|r} 1 & 1 & -2 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & -9 & 11 & 0 \end{array}\right] \overset{3}{\sim}\left[\begin{array}{rrr|r} 1 & 0 & 3 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & -34 & 0 \end{array}\right]$$

$$\overset{4}{\sim}\left[\begin{array}{rrr|r} 1 & 0 & 3 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \overset{5}{\sim}\left[\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

1. $A_{21}(-1)$, $A_{12}(-2)$, $A_{13}(-5)$ **2.** $M_2(-1)$ **3.** $A_{21}(-1)$, $A_{23}(9)$
4. $M_3(-1/34)$ **5.** $A_{31}(-3)$, $A_{32}(5)$

Therefore, the unique solution to this system is $x_1 = x_2 = x_3 = 0$: $(0, 0, 0)$.

37. Reduce the augmented matrix of the system:

$$\left[\begin{array}{rrr|r} 2 & 1 & -1 & 0 \\ 3 & -1 & 2 & 0 \\ 1 & -1 & -1 & 0 \\ 5 & 2 & -2 & 0 \end{array}\right] \overset{1}{\sim}\left[\begin{array}{rrr|r} 1 & -1 & -1 & 0 \\ 3 & -1 & 2 & 0 \\ 2 & 1 & -1 & 0 \\ 5 & 2 & -2 & 0 \end{array}\right] \overset{2}{\sim}\left[\begin{array}{rrr|r} 1 & -1 & -1 & 0 \\ 0 & 2 & 5 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 7 & 3 & 0 \end{array}\right] \overset{3}{\sim}\left[\begin{array}{rrr|r} 1 & -1 & -1 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 2 & 5 & 0 \\ 0 & 7 & 3 & 0 \end{array}\right]$$

$$\overset{4}{\sim}\left[\begin{array}{rrr|r} 1 & -1 & -1 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 2 & 5 & 0 \\ 0 & 7 & 3 & 0 \end{array}\right] \overset{5}{\sim}\left[\begin{array}{rrr|r} 1 & 0 & -5 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 13 & 0 \\ 0 & 0 & 31 & 0 \end{array}\right] \overset{6}{\sim}\left[\begin{array}{rrr|r} 1 & 0 & -5 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 31 & 0 \end{array}\right] \overset{7}{\sim}\left[\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

1. P_{13} **2.** $A_{12}(-3)$, $A_{13}(-2)$, $A_{14}(-5)$ **3.** P_{23} **4.** $A_{32}(-1)$
5. $A_{21}(1)$, $A_{23}(-2)$, $A_{24}(-7)$ **6.** $M_3(1/13)$ **7.** $A_{31}(5)$, $A_{32}(4)$, $A_{34}(-31)$

Therefore, the unique solution to this system is $x_1 = x_2 = x_3 = 0$: $(0, 0, 0)$.

38. Reduce the augmented matrix of the system:

$$\left[\begin{array}{rrr|r} 2 & -1 & -1 & 0 \\ 5 & -1 & 2 & 0 \\ 1 & 1 & 4 & 0 \end{array}\right] \overset{1}{\sim}\left[\begin{array}{rrr|r} 1 & 1 & 4 & 0 \\ 5 & -1 & 2 & 0 \\ 2 & -1 & -1 & 0 \end{array}\right] \overset{2}{\sim}\left[\begin{array}{rrr|r} 1 & 1 & 4 & 0 \\ 0 & -6 & -18 & 0 \\ 0 & -3 & -9 & 0 \end{array}\right]$$

$$\stackrel{3}{\sim}\left[\begin{array}{ccc|c} 1 & 1 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & -3 & -9 & 0 \end{array}\right] \stackrel{4}{\sim}\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

1. P_{13} **2.** $A_{12}(-5)$, $A_{13}(-2)$ **3.** $M_2(-1/6)$ **4.** $A_{21}(-1)$, $A_{23}(3)$

It follows that $x_1 + x_3 = 0$ and $x_2 + 3x_3 = 0$. Setting $x_3 = t$, where t is a free variable, we get $x_2 = -3t$ and $x_1 = -t$. Thus we have that the solution set of the system is $\{(-t, -3t, t) : t \in \mathbb{R}\}$.

39. Reduce the augmented matrix of the system:

$$\left[\begin{array}{ccc|c} 1+2i & 1-i & 1 & 0 \\ i & 1+i & -i & 0 \\ 2i & 1 & 1+3i & 0 \end{array}\right] \stackrel{1}{\sim}\left[\begin{array}{ccc|c} i & 1+i & -i & 0 \\ 1+2i & 1-i & 1 & 0 \\ 2i & 1 & 1+3i & 0 \end{array}\right] \stackrel{2}{\sim}\left[\begin{array}{ccc|c} 1 & 1-i & -1 & 0 \\ 1+2i & 1-i & 1 & 0 \\ 2i & 1 & 1+3i & 0 \end{array}\right]$$

$$\stackrel{3}{\sim}\left[\begin{array}{ccc|c} 1 & 1-i & -1 & 0 \\ 0 & -2-2i & 1+2i & 0 \\ 0 & -1-2i & 1+5i & 0 \end{array}\right] \stackrel{4}{\sim}\left[\begin{array}{ccc|c} 1 & 1-i & -1 & 0 \\ 0 & -2-2i & 1+2i & 0 \\ 0 & 1 & 3i & 0 \end{array}\right] \stackrel{5}{\sim}\left[\begin{array}{ccc|c} 1 & 1-i & -1 & 0 \\ 0 & 0 & -5+8i & 0 \\ 0 & 1 & 3i & 0 \end{array}\right]$$

$$\stackrel{6}{\sim}\left[\begin{array}{ccc|c} 1 & 1-i & -1 & 0 \\ 0 & 1 & 3i & 0 \\ 0 & 0 & -5+8i & 0 \end{array}\right] \stackrel{7}{\sim}\left[\begin{array}{ccc|c} 1 & 1-i & -1 & 0 \\ 0 & 1 & 3i & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \stackrel{8}{\sim}\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

1. P_{12} **2.** $M_1(-i)$ **3.** $A_{12}(-1-2i)$, $A_{13}(-2i)$ **4.** $A_{23}(-1)$ **5.** $A_{32}(2+2i)$
6. P_{23} **7.** $M_3(\frac{1}{-5+8i})$ **8.** $A_{21}(-1+i)$, $A_{31}(1)$, $A_{32}(-3i)$

Therefore, the unique solution to this system is $x_1 = x_2 = x_3 = 0$: $(0, 0, 0)$.

40. Reduce the augmented matrix of the system:

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 0 \\ 6 & -1 & 2 & 0 \\ 12 & 6 & 4 & 0 \end{array}\right] \stackrel{1}{\sim}\left[\begin{array}{ccc|c} 1 & \frac{2}{3} & \frac{1}{3} & 0 \\ 6 & -1 & 2 & 0 \\ 12 & 6 & 4 & 0 \end{array}\right] \stackrel{2}{\sim}\left[\begin{array}{ccc|c} 1 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & -5 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{array}\right]$$

$$\stackrel{3}{\sim}\left[\begin{array}{ccc|c} 1 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{array}\right] \stackrel{4}{\sim}\left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

1. $M_1(1/3)$ **2.** $A_{12}(-6)$, $A_{13}(-12)$ **3.** $M_2(-1/5)$ **4.** $A_{21}(-2/3)$, $A_{23}(2)$

From the last augmented matrix, we have $x_1 + \frac{1}{3}x_3 = 0$ and $x_2 = 0$. Since x_3 is a free variable, we let $x_3 = t$, where t is a real number. It follows that the solution set for the given system is given by $\{(t, 0, -3t) : t \in \mathbb{R}\}$.

41. Reduce the augmented matrix of the system:

$$\left[\begin{array}{ccc|c} 2 & 1 & -8 & 0 \\ 3 & -2 & -5 & 0 \\ 5 & -6 & -3 & 0 \\ 3 & -5 & 1 & 0 \end{array}\right] \stackrel{1}{\sim}\left[\begin{array}{ccc|c} 3 & -2 & -5 & 0 \\ 2 & 1 & -8 & 0 \\ 5 & -6 & -3 & 0 \\ 3 & -5 & 1 & 0 \end{array}\right] \stackrel{2}{\sim}\left[\begin{array}{ccc|c} 1 & -3 & 3 & 0 \\ 2 & 1 & -8 & 0 \\ 5 & -6 & -3 & 0 \\ 3 & -5 & 1 & 0 \end{array}\right]$$

$$\stackrel{3}{\sim}\left[\begin{array}{ccc|c} 1 & -3 & 3 & 0 \\ 0 & 7 & -14 & 0 \\ 0 & 9 & -18 & 0 \\ 0 & 4 & -8 & 0 \end{array}\right] \stackrel{4}{\sim}\left[\begin{array}{ccc|c} 1 & -3 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 9 & -18 & 0 \\ 0 & 4 & -8 & 0 \end{array}\right] \stackrel{5}{\sim}\left[\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

1. P_{12} **2.** $A_{21}(-1)$ **3.** $A_{12}(-2), A_{13}(-5), A_{14}(-3)$ **4.** $M_2(1/7)$ **5.** $A_{21}(3), A_{23}(-9), A_{24}(-4)$

From the last augmented matrix we have: $x_1 - 3x_3 = 0$ and $x_2 - 2x_3 = 0$. Since x_3 is a free variable, we let $x_3 = t$, where t is a real number. It follows that $x_2 = 2t$ and $x_1 = 3t$. Thus, the solution set for the given system is given by $\{(3t, 2t, t) : t \in \mathbb{R}\}$.

42. Reduce the augmented matrix of the system:

$$\left[\begin{array}{ccc|c} 1 & 1+i & 1-i & 0 \\ i & 1 & i & 0 \\ 1-2i & -1+i & 1-3i & 0 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccc|c} 1 & 1+i & 1-i & 0 \\ 0 & 2-i & -1 & 0 \\ 0 & -4+2i & 2 & 0 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{ccc|c} 1 & 1+i & 1-i & 0 \\ 0 & 2-i & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\overset{3}{\sim} \left[\begin{array}{ccc|c} 1 & 1+i & 1-i & 0 \\ 0 & 1 & \frac{-2-i}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \overset{4}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & \frac{6-2i}{5} & 0 \\ 0 & 1 & \frac{-2-i}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

1. $A_{12}(-i), A_{13}(-1+2i)$ **2.** $A_{23}(2)$ **3.** $M_2(\frac{1}{2-i})$ **4.** $A_{21}(-1-i)$

From the last augmented matrix we see that x_3 is a free variable. We set $x_3 = 5s$, where $s \in \mathbb{C}$. Then $x_1 = 2(i-3)s$ and $x_2 = (2+i)s$. Thus, the solution set of the system is $\{(2(i-3)s, (2+i)s, 5s) : s \in \mathbb{C}\}$.

43. Reduce the augmented matrix of the system:

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 3 & 2 & 0 \\ 3 & 0 & -1 & 0 \\ 5 & 1 & -1 & 0 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 3 & -4 & 0 \\ 0 & 6 & -6 & 0 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2/3 & 0 \\ 0 & 3 & -4 & 0 \\ 0 & 6 & -6 & 0 \end{array}\right]$$

$$\overset{3}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 5/3 & 0 \\ 0 & 1 & 2/3 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & -10 & 0 \end{array}\right] \overset{4}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 5/3 & 0 \\ 0 & 1 & 2/3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -10 & 0 \end{array}\right] \overset{5}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

1. $A_{13}(-3), A_{14}(-5)$ **2.** $M_2(1/3)$ **3.** $A_{21}(1), A_{23}(-3), A_{24}(-6)$
4. $M_3(-1/6)$ **5.** $A_{31}(-5/3), A_{32}(-2/3), A_{34}(10)$

Therefore, the unique solution to this system is $x_1 = x_2 = x_3 = 0$: $(0, 0, 0)$.

44. Reduce the augmented matrix of the system:

$$\left[\begin{array}{ccc|c} 2 & -4 & 6 & 0 \\ 3 & -6 & 9 & 0 \\ 1 & -2 & 3 & 0 \\ 5 & -10 & 15 & 0 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 3 & -6 & 9 & 0 \\ 2 & -4 & 6 & 0 \\ 5 & -10 & 15 & 0 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

1. $M_1(1/2)$ **2.** $A_{12}(-3), A_{13}(-2), A_{14}(-5)$

From the last matrix we have that $x_1 - 2x_3 + 3x_3 = 0$. Since x_2 and x_3 are free variables, let $x_2 = s$ and let $x_3 = t$, where s and t are real numbers. The solution set of the given system is therefore $\{(2s - 3t, s, t) : s, t \in \mathbb{R}\}$.

45. Reduce the augmented matrix of the system:

$$\left[\begin{array}{rrrr|r} 4 & -2 & -1 & -1 & 0 \\ 3 & 1 & -2 & 3 & 0 \\ 5 & -1 & -2 & 1 & 0 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{rrrr|r} 1 & -3 & 1 & -4 & 0 \\ 3 & 1 & -2 & 3 & 0 \\ 5 & -1 & -2 & 1 & 0 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{rrrr|r} 1 & -3 & 1 & -4 & 0 \\ 0 & 10 & -5 & 15 & 0 \\ 0 & 14 & -7 & 21 & 0 \end{array}\right]$$

$$\overset{3}{\sim} \left[\begin{array}{rrrr|r} 1 & -3 & 1 & -4 & 0 \\ 0 & 2 & -1 & 3 & 0 \\ 0 & 2 & -1 & 3 & 0 \end{array}\right] \overset{4}{\sim} \left[\begin{array}{rrrr|r} 1 & -3 & 1 & -4 & 0 \\ 0 & 2 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \overset{5}{\sim} \left[\begin{array}{rrrr|r} 1 & -3 & 1 & -4 & 0 \\ 0 & 1 & -1/2 & 3/2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

1. $A_{21}(-1)$ **2.** $A_{12}(-3)$, $A_{13}(-5)$ **3.** $M_2(1/5)$, $M_3(1/7)$
4. $A_{23}(-1)$ **5.** $M_2(1/2)$

From the last augmented matrix above we have that $x_2 - \frac{1}{2}x_3 + \frac{3}{2}x_4 = 0$ and $x_1 - 3x_2 + x_3 - 4x_4 = 0$. Since x_3 and x_4 are free variables, we can set $x_3 = 2s$ and $x_4 = 2t$, where s and t are real numbers. Then $x_2 = s - 3t$ and $x_1 = s - t$. It follows that the solution set of the given system is $\{(s - t, s - 3t, 2s, 2t) : s, t \in \mathbb{R}\}$.

46. Reduce the augmented matrix of the system:

$$\left[\begin{array}{rrrr|r} 2 & 1 & -1 & 1 & 0 \\ 1 & 1 & 1 & -1 & 0 \\ 3 & -1 & 1 & -2 & 0 \\ 4 & 2 & -1 & 1 & 0 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{rrrr|r} 1 & 1 & 1 & -1 & 0 \\ 2 & 1 & -1 & 1 & 0 \\ 3 & -1 & 1 & -2 & 0 \\ 4 & 2 & -1 & 1 & 0 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{rrrr|r} 1 & 1 & 1 & -1 & 0 \\ 0 & -1 & -3 & 3 & 0 \\ 0 & -4 & -2 & 1 & 0 \\ 0 & -2 & -5 & 5 & 0 \end{array}\right]$$

$$\overset{3}{\sim} \left[\begin{array}{rrrr|r} 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 3 & -3 & 0 \\ 0 & -4 & -2 & 1 & 0 \\ 0 & -2 & -5 & 5 & 0 \end{array}\right] \overset{4}{\sim} \left[\begin{array}{rrrr|r} 1 & 0 & -2 & 2 & 0 \\ 0 & 1 & 3 & -3 & 0 \\ 0 & 0 & 10 & -11 & 0 \\ 0 & 0 & -3 & 3 & 0 \end{array}\right] \overset{5}{\sim} \left[\begin{array}{rrrr|r} 1 & 0 & -2 & 2 & 0 \\ 0 & 1 & 3 & -3 & 0 \\ 0 & 0 & -3 & 3 & 0 \\ 0 & 0 & 10 & -11 & 0 \end{array}\right]$$

$$\overset{6}{\sim} \left[\begin{array}{rrrr|r} 1 & 0 & -2 & 2 & 0 \\ 0 & 1 & 3 & -3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 10 & -11 & 0 \end{array}\right] \overset{7}{\sim} \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{array}\right] \overset{8}{\sim} \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \overset{9}{\sim} \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

1. P_{12} **2.** $A_{12}(-2)$, $A_{13}(-3)$, $A_{14}(-4)$ **3.** $M_2(-1)$ **4.** $A_{21}(-1)$, $A_{23}(4)$, $A_{24}(2)$
5. P_{34} **6.** $M_3(-1/3)$ **7.** $A_{31}(2)$, $A_{32}(-3)$, $A_{34}(-10)$ **8.** $M_4(-1)$ **9.** $A_{43}(1)$

From the last augmented matrix, it follows that the solution set to the system is given by $\{(0, 0, 0, 0)\}$.

47. The equation $A\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Reduce the augmented matrix of the system:

$$\left[\begin{array}{rr|r} 2 & -1 & 0 \\ 3 & 4 & 0 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{rr|r} 1 & -\frac{1}{2} & 0 \\ 3 & 4 & 0 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{rr|r} 1 & -\frac{1}{2} & 0 \\ 0 & \frac{11}{2} & 0 \end{array}\right] \overset{3}{\sim} \left[\begin{array}{rr|r} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 \end{array}\right] \overset{4}{\sim} \left[\begin{array}{rr|r} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right].$$

1. $M_1(1/2)$ **2.** $A_{12}(-3)$ **3.** $M_2(2/11)$ **4.** $A_{21}(1/2)$

From the last augmented matrix, we see that $x_1 = x_2 = 0$. Hence, the solution set is $\{(0,0)\}$.

48. The equation $A\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} 1-i & 2i \\ 1+i & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Reduce the augmented matrix of the system:

$$\left[\begin{array}{cc|c} 1-i & 2i & 0 \\ 1+i & -2 & 0 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{cc|c} 1 & -1+i & 0 \\ 1+i & -2 & 0 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{cc|c} 1 & -1+i & 0 \\ 0 & 0 & 0 \end{array}\right].$$

1. $M_1(\frac{1+i}{2})$ **2.** $A_{12}(-1-i)$

It follows that $x_1 + (-1+i)x_2 = 0$. Since x_2 is a free variable, we can let $x_2 = t$, where t is a complex number. The solution set to the system is then given by $\{(t(1-i), t) : t \in \mathbb{C}\}$.

49. The equation $A\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} 1+i & 1-2i \\ -1+i & 2+i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Reduce the augmented matrix of the system:

$$\left[\begin{array}{cc|c} 1+i & 1-2i & 0 \\ -1+i & 2+i & 0 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{cc|c} 1 & -\frac{1+3i}{2} & 0 \\ -1+i & 2+i & 0 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{cc|c} 1 & -\frac{1+3i}{2} & 0 \\ 0 & 0 & 0 \end{array}\right].$$

1. $M_1(\frac{1-i}{2})$ **2.** $A_{12}(1-i)$

It follows that $x_1 - \frac{1+3i}{2}x_2 = 0$. Since x_2 is a free variable, we can let $x_2 = r$, where r is any complex number. Thus, the solution set to the given system is $\{(\frac{1+3i}{2}r, r) : r \in \mathbb{C}\}$.

50. The equation $A\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Reduce the augmented matrix of the system:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -5 & -6 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & -5 & -6 & 0 \end{array}\right] \overset{3}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -5 & -6 & 0 \end{array}\right]$$

$$\overset{4}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right] \overset{5}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \overset{6}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

1. $A_{12}(-2)$, $A_{13}(-1)$ **2.** P_{23} **3.** $M_2(-1)$ **4.** $A_{21}(-2)$, $A_{23}(5)$ **5.** $M_3(1/4)$ **6.** $A_{31}(1)$, $A_{32}(-2)$

From the last augmented matrix, we see that the only solution to the given system is $x_1 = x_2 = x_3 = 0$: $\{(0,0,0)\}$.

51. The equation $A\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} 1 & 1 & 1 & -1 \\ -1 & 0 & -1 & 2 \\ 1 & 3 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Reduce the augmented matrix of the system:

$$\left[\begin{array}{rrrr|r} 1 & 1 & 1 & -1 & 0 \\ -1 & 0 & -1 & 2 & 0 \\ 1 & 3 & 2 & 2 & 0 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{rrrr|r} 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 3 & 0 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{rrrr|r} 1 & 0 & 1 & -2 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \overset{3}{\sim} \left[\begin{array}{rrrr|r} 1 & 0 & 0 & -3 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right].$$

1. $A_{12}(1)$, $A_{13}(-1)$ **2.** $A_{21}(-1)$, $A_{23}(-2)$ **3.** $A_{31}(-1)$

From the last augmented matrix, we see that x_4 is a free variable. We set $x_4 = t$, where t is a real number. The last row of the reduced row echelon form above corresponds to the equation $x_3 + x_4 = 0$. Therefore, $x_3 = -t$. The second row corresponds to the equation $x_2 + x_4 = 0$, so we likewise find that $x_2 = -t$. Finally, from the first equation we have $x_1 - 3x_4 = 0$, so that $x_1 = 3t$. Consequently, the solution set of the original system is given by $\{(3t, -t, -t, t) : t \in \mathbb{R}\}$.

52. The equation $A\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} 2-3i & 1+i & i-1 \\ 3+2i & -1+i & -1-i \\ 5-i & 2i & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Reduce the augmented matrix of this system:

$$\left[\begin{array}{ccc|c} 2-3i & 1+i & i-1 & 0 \\ 3+2i & -1+i & -1-i & 0 \\ 5-i & 2i & -2 & 0 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccc|c} 1 & \frac{-1+5i}{13} & \frac{-5-i}{13} & 0 \\ 3+2i & -1+i & -1-i & 0 \\ 5-i & 2i & -2 & 0 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{ccc|c} 1 & \frac{-1+5i}{13} & \frac{-5-i}{13} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

1. $M_1(\frac{2+3i}{13})$ **2.** $A_{12}(-3-2i)$, $A_{13}(-5+i)$

From the last augmented matrix, we see that $x_1 + \frac{-1+5i}{13}x_2 + \frac{-5-i}{13}x_3 = 0$. Since x_2 and x_3 are free variables, we can let $x_2 = 13r$ and $x_3 = 13s$, where r and s are complex numbers. It follows that the solution set of the system is $\{(r(1-5i) + s(5+i), 13r, 13s) : r, s \in \mathbb{C}\}$.

53. The equation $A\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} 1 & 3 & 0 \\ -2 & -3 & 0 \\ 1 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Reduce the augmented matrix of the system:

$$\left[\begin{array}{rrr|r} 1 & 3 & 0 & 0 \\ -2 & -3 & 0 & 0 \\ 1 & 4 & 0 & 0 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{rrr|r} 1 & 3 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{rrr|r} 1 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{array}\right] \overset{3}{\sim} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

1. $A_{12}(2)$, $A_{13}(-1)$ **2.** P_{23} **3.** $A_{21}(-3)$, $A_{23}(-3)$

From the last augmented matrix we see that the solution set of the system is $\{(0, 0, t) : t \in \mathbb{R}\}$.

54. The equation $A\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} 1 & 0 & 3 \\ 3 & -1 & 7 \\ 2 & 1 & 8 \\ 1 & 1 & 5 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Reduce the augmented matrix of the system:

$$\left[\begin{array}{rrr|r} 1 & 0 & 3 & 0 \\ 3 & -1 & 7 & 0 \\ 2 & 1 & 8 & 0 \\ 1 & 1 & 5 & 0 \\ -1 & 1 & -1 & 0 \end{array}\right] \stackrel{1}{\sim} \left[\begin{array}{rrr|r} 1 & 0 & 3 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \stackrel{2}{\sim} \left[\begin{array}{rrr|r} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \stackrel{3}{\sim} \left[\begin{array}{rrr|r} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

1. $A_{12}(-3)$, $A_{13}(-2)$, $A_{14}(-1)$, $A_{15}(1)$ **2.** $M_2(-1)$ **3.** $A_{23}(-1)$, $A_{24}(-1)$, $A_{25}(-1)$

From the last augmented matrix, we obtain the equations $x_1 + 3x_3 = 0$ and $x_2 + 2x_3 = 0$. Since x_3 is a free variable, we let $x_3 = t$, where t is a real number. The solution set for the given system is then given by $\{(-3t, -2t, t) : t \in \mathbb{R}\}$.

55. The equation $A\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} 1 & -1 & 0 & 1 \\ 3 & -2 & 0 & 5 \\ -1 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Reduce the augmented matrix of the system:

$$\left[\begin{array}{rrrr|r} 1 & -1 & 0 & 1 & 0 \\ 3 & -2 & 0 & 5 & 0 \\ -1 & 2 & 0 & 1 & 0 \end{array}\right] \stackrel{1}{\sim} \left[\begin{array}{rrrr|r} 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 & 0 \end{array}\right] \stackrel{2}{\sim} \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 3 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

1. $A_{12}(-3)$, $A_{13}(1)$ **2.** $A_{21}(1)$, $A_{23}(-1)$

From the last augmented matrix we obtain the equations $x_1 + 3x_4 = 0$ and $x_2 + 2x_4 = 0$. Because x_3 and x_4 are free, we let $x_3 = t$ and $x_4 = s$, where s and t are real numbers. It follows that the solution set of the system is $\{(-3s, -2s, t, s) : s, t \in \mathbb{R}\}$.

56. The equation $A\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} 1 & 0 & -3 & 0 \\ 3 & 0 & -9 & 0 \\ -2 & 0 & 6 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Reduce the augmented matrix of the system:

$$\left[\begin{array}{rrrr|r} 1 & 0 & -3 & 0 & 0 \\ 3 & 0 & -9 & 0 & 0 \\ -2 & 0 & 6 & 0 & 0 \end{array}\right] \stackrel{1}{\sim} \left[\begin{array}{rrrr|r} 1 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

1. $A_{12}(-3)$, $A_{13}(2)$

From the last augmented matrix we obtain $x_1 - 3x_3 = 0$. Therefore, x_2, x_3, and x_4 are free variables, so we let $x_2 = r$, $x_3 = s$, and $x_4 = t$, where r, s, t are real numbers. The solution set of the given system is therefore $\{(3s, r, s, t) : r, s, t \in \mathbb{R}\}$.

57. The equation $A\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} 2+i & i & 3-2i \\ i & 1-i & 4+3i \\ 3-i & 1+i & 1+5i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Reduce the augmented matrix of the system:

$$\left[\begin{array}{ccc|c} 2+i & i & 3-2i & 0 \\ i & 1-i & 4+3i & 0 \\ 3-i & 1+i & 1+5i & 0 \end{array}\right] \stackrel{1}{\sim} \left[\begin{array}{ccc|c} i & 1-i & 4+3i & 0 \\ 2+i & i & 3-2i & 0 \\ 3-i & 1+i & 1+5i & 0 \end{array}\right] \stackrel{2}{\sim} \left[\begin{array}{ccc|c} 1 & -1-i & 3-3i & 0 \\ 2+i & i & 3-2i & 0 \\ 3-i & 1+i & 1+5i & 0 \end{array}\right]$$

$$\stackrel{3}{\sim} \left[\begin{array}{ccc|c} 1 & -1-i & 3-4i & 0 \\ 0 & 1+4i & -7+3i & 0 \\ 0 & 5+3i & -4+20i & 0 \end{array}\right] \stackrel{4}{\sim} \left[\begin{array}{ccc|c} 1 & -1-i & 3-4i & 0 \\ 0 & 1 & \frac{5+31i}{17} & 0 \\ 0 & 5+3i & -4+20i & 0 \end{array}\right] \stackrel{5}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & \frac{25-32i}{17} & 0 \\ 0 & 1 & \frac{5+31i}{17} & 0 \\ 0 & 0 & 10i & 0 \end{array}\right]$$

$$\stackrel{6}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & \frac{25-32i}{17} & 0 \\ 0 & 1 & \frac{5+31i}{17} & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \stackrel{7}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

1. P_{12} **2.** $M_1(-i)$ **3.** $A_{12}(-2-i)$, $A_{13}(-3+i)$ **4.** $M_2(\frac{1-4i}{17})$ **5.** $A_{21}(1+i)$, $A_{23}(-5-3i)$ **6.** $M_3(-i/10)$ **7.** $A_{31}(\frac{-25+32i}{17})$, $A_{32}(\frac{-5-31i}{17})$

From the last augmented matrix above, we see that the only solution to this system is the trivial solution.

Solutions to Section 2.6

True-False Review:

(a): FALSE. An invertible matrix is also known as a *nonsingular* matrix.

(b): FALSE. For instance, the matrix $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ does not contain a row of zeros, but fails to be invertible.

(c): TRUE. If A is invertible, then the unique solution to $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = A^{-1}\mathbf{b}$.

(d): FALSE. For instance, if $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$, then $AB = I_2$, but A is not even a square matrix, hence certainly not invertible.

(e): FALSE. For instance, if $A = I_n$ and $B = -I_n$, then A and B are both invertible, but $A + B = 0_n$ is not invertible.

(f): TRUE. We have

$$(AB)B^{-1}A^{-1} = I_n \quad \text{and} \quad B^{-1}A^{-1}(AB) = I_n,$$

and therefore, AB is invertible, with inverse $B^{-1}A^{-1}$.

(g): TRUE. From $A^2 = A$, we subtract to obtain $A(A-I) = 0$. Left multiplying both sides of this equation by A^{-1} (since A is invertible, A^{-1} exists), we have $A - I = A^{-1}0 = 0$. Therefore, $A = I$, the identity matrix.

(h): TRUE. From $AB = AC$, we left-multiply both sides by A^{-1} (since A is invertible, A^{-1} exists) to obtain $A^{-1}AB = A^{-1}AC$. Since $A^{-1}A = I$, we obtain $IB = IC$, or $B = C$.

(i): TRUE. Any 5×5 invertible matrix must have rank 5, not rank 4 (Theorem 2.6.6).

(j): TRUE. Any 6×6 matrix of rank 6 is invertible (Theorem 2.6.6).

Problems:

1. We have

$$AA^{-1} = \begin{bmatrix} 4 & 9 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} 7 & -9 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} (4)(7)+(9)(-3) & (4)(-9)+(9)(4) \\ (3)(7)+(7)(-3) & (3)(-9)+(7)(4) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.$$

2. We have

$$AA^{-1} = \begin{bmatrix} 2 & -1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} (2)(-1)+(-1)(-3) & (2)(1)+(-1)(2) \\ (3)(-1)+(-1)(-3) & (3)(1)+(-1)(2) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.$$

3. We have

$$\begin{aligned} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \left(\frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right) &= \frac{1}{ad-bc} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{ad-bc} \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= I_2, \end{aligned}$$

and

$$\begin{aligned} \left(\frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right) \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= \frac{1}{ad-bc} \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= I_2. \end{aligned}$$

4. We have

$$\begin{aligned} AA^{-1} &= \begin{bmatrix} 3 & 5 & 1 \\ 1 & 2 & 1 \\ 2 & 6 & 7 \end{bmatrix} \begin{bmatrix} 8 & -29 & 3 \\ -5 & 19 & -2 \\ 2 & -8 & 1 \end{bmatrix} \\ &= \begin{bmatrix} (3)(8)+(5)(-5)+(1)(2) & (3)(-29)+(5)(19)+(1)(-8) & (3)(3)+(5)(-2)+(1)(1) \\ (1)(8)+(2)(-5)+(1)(2) & (1)(-29)+(2)(19)+(1)(-8) & (1)(3)+(2)(-2)+(1)(1) \\ (2)(8)+(6)(-5)+(7)(2) & (2)(-29)+(6)(19)+(7)(-8) & (2)(3)+(6)(-2)+(7)(1) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3. \end{aligned}$$

5. We have

$$[A|I_2] = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{cc|cc} 1 & 0 & 3 & -2 \\ 0 & 1 & -1 & 1 \end{array}\right] = [I_2|A^{-1}].$$

Therefore,

$$A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}.$$

1. $A_{12}(-1)$ **2.** $A_{21}(-2)$

6. We have

$$[A|I_2] = \left[\begin{array}{cc|cc} 1 & 1+i & 1 & 0 \\ 1-i & 1 & 0 & 1 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{cc|cc} 1 & 1+i & 1 & 0 \\ 0 & -1 & -1+i & 1 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{cc|cc} 1 & 1+i & 1 & 0 \\ 0 & 1 & 1-i & -1 \end{array}\right]$$

$$\overset{3}{\sim} \left[\begin{array}{cc|cc} 1 & 0 & -1 & 1+i \\ 0 & 1 & 1-i & -1 \end{array}\right] = [I_2|A^{-1}].$$

Thus,

$$A^{-1} = \begin{bmatrix} -1 & 1+i \\ 1-i & -1 \end{bmatrix}.$$

1. $A_{12}(-1+i)$ **2.** $M_2(-1)$ **3.** $A_{21}(-1-i)$

7. We have

$$[A|I_2] = \left[\begin{array}{cc|cc} 1 & -i & 1 & 0 \\ i-1 & 2 & 0 & 1 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{cc|cc} 1 & -i & 1 & 0 \\ 0 & 1-i & 1-i & 1 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{cc|cc} 1 & -i & 1 & 0 \\ 0 & 1 & 1 & \frac{1+i}{2} \end{array}\right]$$

$$\overset{3}{\sim} \left[\begin{array}{cc|cc} 1 & 0 & 1+i & \frac{-1+i}{2} \\ 0 & 1 & 1 & \frac{1+i}{2} \end{array}\right] = [I_2|A^{-1}].$$

Thus,

$$A^{-1} = \begin{bmatrix} 1+i & \frac{-1+i}{2} \\ 1 & \frac{1+i}{2} \end{bmatrix}.$$

1. $A_{12}(1-i)$ **2.** $M_2(1/(1-i))$ **3.** $A_{21}(i)$

8. Note that $AB = 0_2$ for all 2×2 matrices B. Therefore, A is not invertible.

9. We have

$$[A|I_3] = \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 2 & 1 & 11 & 0 & 1 & 0 \\ 4 & -3 & 10 & 0 & 0 & 1 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 3 & 7 & -2 & 1 & 0 \\ 0 & 1 & 2 & -4 & 0 & 1 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & -4 & 0 & 1 \\ 0 & 3 & 7 & -2 & 1 & 0 \end{array}\right]$$

$$\overset{3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & -3 & 0 & 1 \\ 0 & 1 & 2 & -4 & 0 & 1 \\ 0 & 0 & 1 & 10 & 1 & -3 \end{array}\right] \overset{4}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -43 & -4 & 13 \\ 0 & 1 & 0 & -24 & -2 & 7 \\ 0 & 0 & 1 & 10 & 1 & -3 \end{array}\right] = [I_3|A^{-1}].$$

Thus,

$$A^{-1} = \begin{bmatrix} -43 & -4 & 13 \\ -24 & -2 & 7 \\ 10 & 1 & -3 \end{bmatrix}.$$

1. $A_{12}(-2)$, $A_{13}(-4)$ **2.** P_{23} **3.** $A_{21}(1)$, $A_{23}(-3)$ **4.** $A_{31}(-4)$, $A_{32}(-2)$

10. We have

$$[A|I_3] = \left[\begin{array}{ccc|ccc} 3 & 5 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & 6 & 7 & 0 & 0 & 1 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 3 & 5 & 1 & 1 & 0 & 0 \\ 2 & 6 & 7 & 0 & 0 & 1 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & -1 & -2 & 1 & -3 & 0 \\ 0 & 2 & 5 & 0 & -2 & 1 \end{array}\right]$$

$$\overset{3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 & 3 & 0 \\ 0 & 2 & 5 & 0 & -2 & 1 \end{array}\right] \overset{4}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 2 & -5 & 0 \\ 0 & 1 & 2 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -8 & 1 \end{array}\right] \overset{5}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & -29 & 3 \\ 0 & 1 & 0 & -5 & 19 & -2 \\ 0 & 0 & 1 & 2 & -8 & 1 \end{array}\right] = [I_3|A^{-1}].$$

Thus,

$$A^{-1} = \begin{bmatrix} 8 & -29 & 3 \\ -5 & 19 & -2 \\ 2 & -8 & 1 \end{bmatrix}.$$

1. P_{12} **2.** $A_{12}(-3)$, $A_{13}(-2)$ **3.** $M_2(-1)$ **4.** $A_{21}(-2)$, $A_{23}(-2)$ **5.** $A_{31}(3)$, $A_{32}(-2)$

11. This matrix is not invertible, because the column of zeros guarantees that the rank of the matrix is less than three.

12. We have

$$[A|I_3] = \left[\begin{array}{ccc|ccc} 4 & 2 & -13 & 1 & 0 & 0 \\ 2 & 1 & -7 & 0 & 1 & 0 \\ 3 & 2 & 4 & 0 & 0 & 1 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccc|ccc} 3 & 2 & 4 & 0 & 0 & 1 \\ 2 & 1 & -7 & 0 & 1 & 0 \\ 4 & 2 & -13 & 1 & 0 & 0 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 1 & 11 & 0 & -1 & 1 \\ 2 & 1 & -7 & 0 & 1 & 0 \\ 4 & 2 & -13 & 1 & 0 & 0 \end{array}\right]$$

$$\overset{3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 1 & 11 & 0 & -1 & 1 \\ 0 & -1 & -29 & 0 & 3 & -2 \\ 0 & -2 & -57 & 1 & 4 & -4 \end{array}\right] \overset{4}{\sim} \left[\begin{array}{ccc|ccc} 1 & 1 & 11 & 0 & -1 & 1 \\ 0 & 1 & 29 & 0 & -3 & 2 \\ 0 & -2 & -57 & 1 & 4 & -4 \end{array}\right] \overset{5}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & -18 & 0 & 2 & -1 \\ 0 & 1 & 29 & 0 & -3 & 2 \\ 0 & 0 & 1 & 1 & -2 & 0 \end{array}\right]$$

$$\overset{6}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 18 & -34 & -1 & \\ 0 & 1 & 0 & -29 & 55 & 2 \\ 0 & 0 & 1 & 1 & -2 & 0 \end{array}\right] = [I_3|A^{-1}].$$

Thus,

$$A^{-1} = \begin{bmatrix} 18 & -34 & -1 \\ -29 & 55 & 2 \\ 1 & -2 & 0 \end{bmatrix}.$$

1. P_{13} **2.** $A_{21}(-1)$ **3.** $A_{12}(-2)$, $A_{13}(-4)$ **4.** $M_2(-1)$
5. $A_{21}(-1)$, $A_{23}(2)$ **6.** $A_{31}(18)$, $A_{32}(-29)$

13. We have

$$[A|I_3] = \left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 2 & 6 & -2 & 0 & 1 & 0 \\ -1 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & 2 & 4 & -2 & 1 & 0 \\ 0 & 3 & 1 & 1 & 0 & 1 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & \frac{1}{2} & 0 \\ 0 & 3 & 1 & 1 & 0 & 1 \end{array}\right]$$

$$\overset{3}{\sim}\left[\begin{array}{ccc|ccc}1&0&-7&3&-1&0\\0&1&2&-1&\frac{1}{2}&0\\0&0&-5&4&-\frac{3}{2}&1\end{array}\right]\overset{4}{\sim}\left[\begin{array}{ccc|ccc}1&0&-7&3&-1&0\\0&1&2&-1&\frac{1}{2}&0\\0&0&1&-\frac{4}{5}&\frac{3}{10}&-\frac{1}{5}\end{array}\right]$$

$$\overset{5}{\sim}\left[\begin{array}{ccc|ccc}1&0&0&-\frac{13}{5}&\frac{11}{10}&-\frac{7}{5}\\0&1&0&\frac{3}{5}&-\frac{1}{10}&\frac{2}{5}\\0&0&1&-\frac{4}{5}&\frac{3}{10}&-\frac{1}{5}\end{array}\right]=[I_3|A^{-1}].$$

Thus,

$$A^{-1}=\begin{bmatrix}-\frac{13}{5}&\frac{11}{10}&-\frac{7}{5}\\\frac{3}{5}&-\frac{1}{10}&\frac{2}{5}\\-\frac{4}{5}&\frac{3}{10}&-\frac{1}{5}\end{bmatrix}.$$

1. $A_{12}(-2), A_{13}(1)$ **2.** $M_2(\frac{1}{2})$ **3.** $A_{21}(-2), A_{23}(-3)$ **4.** $M_3(-\frac{1}{5})$ **5.** $A_{31}(7), A_{32}(-2)$

14. We have

$$[A|I_3]=\left[\begin{array}{ccc|ccc}1&i&2&1&0&0\\1+i&-1&2i&0&1&0\\2&2i&5&0&0&1\end{array}\right]\overset{1}{\sim}\left[\begin{array}{ccc|ccc}1&i&2&1&0&0\\0&-i&-2&-1-i&1&0\\0&0&1&-2&0&1\end{array}\right]\overset{2}{\sim}\left[\begin{array}{ccc|ccc}1&i&2&1&0&0\\0&1&-2i&1-i&i&0\\0&0&1&-2&0&1\end{array}\right]$$

$$\overset{3}{\sim}\left[\begin{array}{ccc|ccc}1&0&0&-i&1&0\\0&1&-2i&1-i&i&0\\0&0&1&-2&0&1\end{array}\right]\overset{4}{\sim}\left[\begin{array}{ccc|ccc}1&0&0&-i&1&0\\0&1&0&1-5i&i&2i\\0&0&1&-2&0&1\end{array}\right]=[I_3|A^{-1}].$$

Thus,

$$A^{-1}=\begin{bmatrix}-i&1&0\\1-5i&i&2i\\-2&0&1\end{bmatrix}.$$

1. $A_{12}(-1-i), A_{13}(-2)$ **2.** $M_2(i)$ **3.** $A_{21}(-i)$ **4.** $A_{32}(2i)$

15. We have

$$[A|I_3]=\left[\begin{array}{ccc|ccc}2&1&3&1&0&0\\1&-1&2&0&1&0\\3&3&4&0&0&1\end{array}\right]\overset{1}{\sim}\left[\begin{array}{ccc|ccc}1&-1&2&0&1&0\\2&1&3&1&0&0\\3&3&4&0&0&1\end{array}\right]\overset{2}{\sim}\left[\begin{array}{ccc|ccc}1&-1&2&0&1&0\\0&3&-1&1&-2&0\\0&6&-2&0&-3&1\end{array}\right]$$

$$\overset{3}{\sim}\left[\begin{array}{ccc|ccc}1&-1&2&0&1&0\\0&3&-1&1&-2&0\\0&0&0&-2&1&1\end{array}\right]$$

Since $2=\text{rank}(A)<\text{rank}(A^{\#})=3$, we know that A^{-1} does not exist (we have obtained a row of zeros in the block matrix on the left.

1. P_{12} **2.** $A_{12}(-2), A_{13}(-3)$ **3.** $A_{23}(-2)$

16. We have

$$[A|I_4]=\left[\begin{array}{cccc|cccc}1&-1&2&3&1&0&0&0\\2&0&3&-4&0&1&0&0\\3&-1&7&8&0&0&1&0\\1&0&3&5&0&0&0&1\end{array}\right]\overset{1}{\sim}\left[\begin{array}{cccc|cccc}1&-1&2&3&1&0&0&0\\0&2&-1&-10&-2&1&0&0\\0&2&1&-1&-3&0&1&0\\0&1&1&2&-1&0&0&1\end{array}\right]$$

$$\overset{2}{\sim}\left[\begin{array}{cccc|cccc} 1 & -1 & 2 & 3 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 & 0 & 1 \\ 0 & 2 & 1 & -1 & -3 & 0 & 1 & 0 \\ 0 & 2 & -1 & -10 & -2 & 1 & 0 & 0 \end{array}\right] \overset{3}{\sim}\left[\begin{array}{cccc|cccc} 1 & 0 & 3 & 5 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & -5 & -1 & 0 & 1 & -2 \\ 0 & 0 & -3 & -14 & 0 & 1 & 0 & -2 \end{array}\right]$$

$$\overset{4}{\sim}\left[\begin{array}{cccc|cccc} 1 & 0 & 3 & 5 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 5 & 1 & 0 & -1 & 2 \\ 0 & 0 & -3 & -14 & 0 & 1 & 0 & -2 \end{array}\right] \overset{5}{\sim}\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -10 & -3 & 0 & 3 & -5 \\ 0 & 1 & 0 & -3 & -2 & 0 & 1 & -1 \\ 0 & 0 & 1 & 5 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 3 & 1 & -3 & 4 \end{array}\right]$$

$$\overset{6}{\sim}\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 27 & 10 & -27 & 35 \\ 0 & 1 & 0 & 0 & 7 & 3 & -8 & 11 \\ 0 & 0 & 1 & 0 & -14 & -5 & 14 & -18 \\ 0 & 0 & 0 & 1 & 3 & 1 & -3 & 4 \end{array}\right] = [I_4|A^{-1}].$$

Thus,

$$A^{-1} = \left[\begin{array}{cccc} 27 & 10 & -27 & 35 \\ 7 & 3 & -8 & 11 \\ -14 & -5 & 14 & -18 \\ 3 & 1 & -3 & 4 \end{array}\right].$$

1. $A_{12}(-2)$, $A_{13}(-3)$, $A_{14}(-1)$	**2.** P_{13}	**3.** $A_{21}(1)$, $A_{23}(-2)$, $A_{24}(-2)$
4. $M_3(-1)$	**5.** $A_{31}(-3)$, $A_{32}(-1)$, $A_{34}(3)$	**6.** $A_{41}(10)$, $A_{42}(3)$, $A_{43}(5)$

17. We have

$$[A|I_4] = \left[\begin{array}{cccc|cccc} 0 & -2 & -1 & -3 & 1 & 0 & 0 & 0 \\ 2 & 0 & 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & -2 & 0 & 2 & 0 & 0 & 1 & 0 \\ 3 & -1 & -2 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \overset{1}{\sim}\left[\begin{array}{cccc|cccc} 1 & -2 & 0 & 2 & 0 & 0 & 1 & 0 \\ 2 & 0 & 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & -2 & -1 & -3 & 1 & 0 & 0 & 0 \\ 3 & -1 & -2 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\overset{2}{\sim}\left[\begin{array}{cccc|cccc} 1 & -2 & 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 4 & 2 & -3 & 0 & 1 & -2 & 0 \\ 0 & -2 & -1 & -3 & 1 & 0 & 0 & 0 \\ 0 & 5 & -2 & -6 & 0 & 0 & -3 & 1 \end{array}\right] \overset{3}{\sim}\left[\begin{array}{cccc|cccc} 1 & -2 & 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{3}{4} & 0 & \frac{1}{4} & -\frac{1}{2} & 0 \\ 0 & -2 & -1 & -3 & 1 & 0 & 0 & 0 \\ 0 & 5 & -2 & -6 & 0 & 0 & -3 & 1 \end{array}\right]$$

$$\overset{4}{\sim}\left[\begin{array}{cccc|cccc} 1 & 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{3}{4} & 0 & \frac{1}{4} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{9}{2} & 1 & \frac{1}{2} & -1 & 0 \\ 0 & 0 & -\frac{9}{2} & -\frac{9}{4} & 0 & -\frac{5}{4} & -\frac{1}{2} & 1 \end{array}\right] \overset{5}{\sim}\left[\begin{array}{cccc|cccc} 1 & 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{3}{4} & 0 & \frac{1}{4} & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{9}{2} & -\frac{9}{4} & 0 & -\frac{5}{4} & -\frac{1}{2} & 1 \\ 0 & 0 & 0 & -\frac{9}{2} & 1 & \frac{1}{2} & -1 & 0 \end{array}\right]$$

$$\overset{6}{\sim}\left[\begin{array}{cccc|cccc} 1 & 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{3}{4} & 0 & \frac{1}{4} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & \frac{5}{18} & \frac{1}{9} & -\frac{2}{9} \\ 0 & 0 & 0 & -\frac{9}{2} & 1 & \frac{1}{2} & -1 & 0 \end{array}\right] \overset{7}{\sim}\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & \frac{2}{9} & -\frac{1}{9} & \frac{2}{9} \\ 0 & 1 & 0 & -1 & 0 & \frac{1}{9} & -\frac{5}{9} & \frac{1}{9} \\ 0 & 0 & 1 & \frac{1}{2} & 0 & \frac{5}{18} & \frac{1}{9} & -\frac{2}{9} \\ 0 & 0 & 0 & -\frac{9}{2} & 1 & \frac{1}{2} & -1 & 0 \end{array}\right]$$

$$\overset{8}{\sim}\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & \frac{2}{9} & -\frac{1}{9} & \frac{2}{9} \\ 0 & 1 & 0 & -1 & 0 & \frac{1}{9} & -\frac{5}{9} & \frac{1}{9} \\ 0 & 0 & 1 & \frac{1}{2} & 0 & \frac{5}{18} & \frac{1}{9} & -\frac{2}{9} \\ 0 & 0 & 0 & 1 & -\frac{2}{9} & -\frac{1}{9} & \frac{2}{9} & 0 \end{array}\right] \overset{9}{\sim}\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & \frac{2}{9} & -\frac{1}{9} & \frac{2}{9} \\ 0 & 1 & 0 & 0 & -\frac{2}{9} & 0 & -\frac{1}{3} & \frac{1}{9} \\ 0 & 0 & 1 & 0 & \frac{1}{9} & \frac{1}{3} & 0 & -\frac{2}{9} \\ 0 & 0 & 0 & 1 & -\frac{2}{9} & -\frac{1}{9} & \frac{2}{9} & 0 \end{array}\right] = [I_4|A^{-1}].$$

Thus,

$$A^{-1} = \begin{bmatrix} 0 & \frac{2}{9} & -\frac{1}{9} & \frac{2}{9} \\ -\frac{2}{9} & 0 & -\frac{1}{3} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{3} & 0 & -\frac{2}{9} \\ -\frac{2}{9} & -\frac{1}{9} & \frac{2}{9} & 0 \end{bmatrix}.$$

1. P_{13} **2.** $A_{12}(-2)$, $A_{14}(-3)$ **3.** $M_2(\frac{1}{4})$ **4.** $A_{21}(2)$, $A_{23}(2)$, $A_{24}(-5)$
5. P_{34} **6.** $M_3(-\frac{2}{9})$ **7.** $A_{31}(-1)$, $A_{32}(-\frac{1}{2})$ **8.** $M_4(-\frac{2}{9})$ **9.** $A_{42}(1)$, $A_{43}(-\frac{1}{2})$

18. We have

$$= \left[\begin{array}{cccc|cccc} 1 & 2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 6 & 0 & 0 & 1 & 0 \\ 0 & 0 & 7 & 8 & 0 & 0 & 0 & 1 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{cccc|cccc} 1 & 2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & -3 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{6}{5} & 0 & 0 & \frac{1}{5} & 0 \\ 0 & 0 & 7 & 8 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\overset{2}{\sim} \left[\begin{array}{cccc|cccc} 1 & 2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & -3 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{6}{5} & 0 & 0 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & -\frac{2}{5} & 0 & 0 & -\frac{7}{5} & 1 \end{array}\right] \overset{3}{\sim} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & -3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -4 & 3 \\ 0 & 0 & 0 & -\frac{2}{5} & 0 & 0 & -\frac{7}{5} & 1 \end{array}\right]$$

$$\overset{4}{\sim} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{3}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -4 & -3 \\ 0 & 0 & 0 & 1 & 0 & 0 & \frac{7}{2} & -\frac{5}{2} \end{array}\right] = [I_4|A^{-1}].$$

Thus,

$$A^{-1} = \begin{bmatrix} -2 & 1 & 0 & 0 \\ \frac{3}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -4 & 3 \\ 0 & 0 & \frac{7}{2} & -\frac{5}{2} \end{bmatrix}.$$

1. $A_{12}(-3)$, $M_3(\frac{1}{5})$ **2.** $A_{34}(-7)$ **3.** $A_{21}(1)$, $A_{13}(3)$ **4.** $M_2(-\frac{1}{2})$, $M_4(-\frac{5}{2})$

19. To determine the third column vector of A^{-1} without determining the whole inverse, we solve the system $\begin{bmatrix} -1 & -2 & 3 \\ -1 & 1 & 1 \\ -1 & -2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. The corresponding augmented matrix $\left[\begin{array}{ccc|c} -1 & -2 & 3 & 0 \\ -1 & 1 & 1 & 0 \\ -1 & -2 & -1 & 1 \end{array}\right]$ can be row-reduced to $\left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 1 & -\frac{1}{4} \end{array}\right]$. Thus, back substitution yields $z = -\frac{1}{4}$, $y = -\frac{1}{6}$, and $x = -\frac{5}{12}$. Thus, the third column vector of A^{-1} is $\begin{bmatrix} -5/12 \\ -1/6 \\ -1/4 \end{bmatrix}$.

20. To determine the second column vector of A^{-1} without determining the whole inverse, we solve the linear system $\begin{bmatrix} 2 & -1 & 4 \\ 5 & 1 & 2 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. The corresponding augmented matrix $\left[\begin{array}{ccc|c} 2 & -1 & 4 & 0 \\ 5 & 1 & 2 & 1 \\ 1 & -1 & 3 & 0 \end{array}\right]$ can be row-reduced to $\left[\begin{array}{rrr|r} 1 & -1 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right]$. Thus, back-substitution yields $z = -1$, $y = -2$, and $x = 1$. Thus, the second column vector of A^{-1} is $\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$.

21. We have $A = \begin{bmatrix} 6 & 20 \\ 2 & 7 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -8 \\ 2 \end{bmatrix}$, and the Gauss-Jordan method yields $A^{-1} = \begin{bmatrix} \frac{7}{2} & -10 \\ -1 & 3 \end{bmatrix}$. Therefore, we have

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} \frac{7}{2} & -10 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} -8 \\ 2 \end{bmatrix} = \begin{bmatrix} -48 \\ 14 \end{bmatrix}.$$

Hence, we have $x_1 = -48$ and $x_2 = 14$.

22. We have $A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, and the Gauss-Jordan method yields $A^{-1} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$. Therefore, we have

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}.$$

So we have $x_1 = 4$ and $x_2 = -1$.

23. We have $A = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \\ 2 & 4 & -3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$, and the Gauss-Jordan method yields $A^{-1} = \begin{bmatrix} 7 & 5 & -3 \\ -2 & -1 & 1 \\ 2 & 2 & -1 \end{bmatrix}$. Therefore, we have

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 7 & 5 & -3 \\ -2 & -1 & 1 \\ 2 & 2 & -1 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}.$$

Hence, we have $x_1 = -2$, $x_2 = 2$, and $x_3 = 1$.

24. We have $A = \begin{bmatrix} 1 & -2i \\ 2-i & 4i \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 2 \\ -i \end{bmatrix}$, and the Gauss-Jordan method yields $A^{-1} = \frac{1}{2+8i}\begin{bmatrix} 4i & 2i \\ -2+i & 1 \end{bmatrix}$. Therefore, we have

$$\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{2+8i}\begin{bmatrix} 4i & 2i \\ -2+i & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -i \end{bmatrix} = \frac{1}{2+8i}\begin{bmatrix} 2+8i \\ -4+i \end{bmatrix}.$$

Hence, we have $x_1 = 1$ and $x_2 = \frac{-4+i}{2+8i}$.

25. We have $A = \begin{bmatrix} 3 & 4 & 5 \\ 2 & 10 & 1 \\ 4 & 1 & 8 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and the Gauss-Jordan method yields $A^{-1} = \begin{bmatrix} -79 & 27 & 46 \\ 12 & -4 & -7 \\ 38 & -13 & -22 \end{bmatrix}$. Therefore, we have

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -79 & 27 & 46 \\ 12 & -4 & -7 \\ 38 & -13 & -22 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ 1 \\ 3 \end{bmatrix}.$$

Hence, we have $x_1 = -6$, $x_2 = 1$, and $x_3 = 3$.

26. We have $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & -1 \\ 2 & -1 & 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 12 \\ 24 \\ -36 \end{bmatrix}$, and the Gauss-Jordan method yields $A^{-1} = \frac{1}{12}\begin{bmatrix} -1 & 3 & 5 \\ 3 & 3 & -3 \\ 5 & -3 & -1 \end{bmatrix}$. Therefore, we have

$$\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{12}\begin{bmatrix} -1 & 3 & 5 \\ 3 & 3 & -3 \\ 5 & -3 & -1 \end{bmatrix}\begin{bmatrix} 12 \\ 24 \\ -36 \end{bmatrix} = \begin{bmatrix} -10 \\ 18 \\ 2 \end{bmatrix}.$$

Hence, $x_1 = -10$, $x_2 = 18$, and $x_3 = 2$.

27. We have

$$AA^T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} (0)(0)+(1)(1) & (0)(-1)+(1)(0) \\ (-1)(0)+(0)(1) & (-1)(-1)+(0)(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2,$$

so $A^T = A^{-1}$.

28. We have

$$\begin{aligned} AA^T &= \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}\begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix} \\ &= \begin{bmatrix} (\sqrt{3}/2)(\sqrt{3}/2)+(1/2)(1/2) & (\sqrt{3}/2)(-1/2)+(1/2)(\sqrt{3}/2) \\ (-1/2)(\sqrt{3}/2)+(\sqrt{3}/2)(1/2) & (-1/2)(-1/2)+(\sqrt{3}/2)(\sqrt{3}/2) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2, \end{aligned}$$

so $A^T = A^{-1}$.

29. We have

$$\begin{aligned} AA^T &= \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \\ &= \begin{bmatrix} \cos^2\alpha+\sin^2\alpha & (\cos\alpha)(-\sin\alpha)+(\sin\alpha)(\cos\alpha) \\ (-\sin\alpha)(\cos\alpha)+(\cos\alpha)(\sin\alpha) & (-\sin\alpha)^2+\cos^2\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2, \end{aligned}$$

so $A^T = A^{-1}$.

30. We have

$$\begin{aligned} AA^T &= \left(\frac{1}{1+2x^2}\right)\begin{bmatrix} 1 & -2x & 2x^2 \\ 2x & 1-2x^2 & -2x \\ 2x^2 & 2x & 1 \end{bmatrix}\left(\frac{1}{1+2x^2}\right)\begin{bmatrix} 1 & 2x & 2x^2 \\ -2x & 1-2x^2 & 2x \\ 2x^2 & -2x & 1 \end{bmatrix} \\ &= \left(\frac{1}{1+4x^2+4x^4}\right)\begin{bmatrix} 1+4x^2+4x^4 & 0 & 0 \\ 0 & 1+4x^2+4x^4 & 0 \\ 0 & 0 & 1+4x^2+4x^4 \end{bmatrix} = I_3, \end{aligned}$$

so $A^T = A^{-1}$.

31. For part 2, we have

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}I_nB = B^{-1}B = I_n,$$

and for part 3, we have

$$(A^{-1})^TA^T = (AA^{-1})^T = I_n^T = I_n.$$

32. We prove this by induction on k, with $k = 1$ trivial and $k = 2$ proven in part 2 of Theorem 2.6.10. Assuming the statement is true for a product involving $k-1$ matrices, we may proceed as follows:

$$\begin{aligned}(A_1A_2\cdots A_k)^{-1} = ((A_1A_2\cdots A_{k-1})A_k)^{-1} &= A_k^{-1}(A_1A_2\cdots A_{k-1})^{-1} \\ &= A_k^{-1}(A_{k-1}^{-1}\cdots A_2^{-1}A_1^{-1}) = A_k^{-1}A_{k-1}^{-1}\cdots A_2^{-1}A_1^{-1}.\end{aligned}$$

In the second equality, we have applied part 2 of Theorem 2.6.10 to the two matrices $A_1A_2\cdots A_{k-1}$ and A_k, and in the third equality, we have assumed that the desired property is true for products of $k-1$ matrices.

33. Since A is skew-symmetric, we know that $A^T = -A$. We wish to show that $(A^{-1})^T = -A^{-1}$. We have

$$(A^{-1})^T = (A^T)^{-1} = (-A)^{-1} = -(A^{-1}),$$

which shows that A^{-1} is skew-symmetric. The first equality follows from part 3 of Theorem 2.6.10, and the second equality results from the assumption that A^{-1} is skew-symmetric.

34. Since A is symmetric, we know that $A^T = A$. We wish to show that $(A^{-1})^T = A^{-1}$. We have

$$(A^{-1})^T = (A^T)^{-1} = A^{-1},$$

which shows that A^{-1} is symmetric. The first equality follows from part 3 of Theorem 2.6.10, and the second equality results from the assumption that A is symmetric.

35. We have

$$\begin{aligned}(I_n - A^3)(I_n + A^3 + A^6 + A^9) &= I_n(I_n + A^3 + A^6 + A^9) - A^3(I_n + A^3 + A^6 + A^9) \\ &= I_n + A^3 + A^6 + A^9 - A^3 - A^6 - A^9 - A^{12} = I_n - A^{12} = I_n,\end{aligned}$$

where the last equality uses the assumption that $A^{12} = 0$. This calculation shows that $I_n - A^3$ and $I_n + A^3 + A^6 + A^9$ are inverses of one another.

36. We have

$$\begin{aligned}(I_n - A)(I_n + A + A^2 + A^3) &= I_n(I_n + A + A^2 + A^3) - A(I_n + A + A^2 + A^3) \\ &= I_n + A + A^2 + A^3 - A - A^2 - A^3 - A^4 = I_n - A^4 = I_n,\end{aligned}$$

where the last equality uses the assumption that $A^4 = 0$. This calculation shows that $I_n - A$ and $I_n + A + A^2 + A^3$ are inverses of one another.

37. We claim that the inverse of A^{15} is B^9. To verify this, use the fact that $A^5B^3 = I$ to observe that

$$A^{15}B^9 = A^5(A^5(A^5B^3)B^3)B^3 = A^5(A^5IB^3)B^3 = A^5(A^5B^3)B^3 = A^5IB^3 = A^5B^3 = I.$$

This calculation shows that the inverse of A^{15} is B^9.

38. We claim that the inverse of A^9 is B^{-3}. To verify this, use the fact that $A^3B^{-1} = I$ to observe that

$$A^9B^{-3} = A^3(A^3(A^3B^{-1})B^{-1})B^{-1} = A^3(A^3IB^{-1})B^{-1} = A^3(A^3B^{-1})B^{-1} = A^3IB^{-1} = A^3B^{-1} = I.$$

This calculation shows that the inverse of A^9 is B^{-3}.

39. We have

$$B = BI_n = B(AC) = (BA)C = I_nC = C.$$

40. YES. Since $BA = I_n$, we know that $A^{-1} = B$ (see Theorem 2.6.12). Likewise, since $CA = I_n$, $A^{-1} = C$. Since the inverse of A is unique, it must follow that $B = C$.

41. We can simply compute

$$\frac{1}{\Delta}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} a_{22}a_{11} - a_{12}a_{21} & a_{22}a_{12} - a_{12}a_{22} \\ -a_{21}a_{11} + a_{11}a_{21} & -a_{21}a_{12} + a_{11}a_{22} \end{bmatrix}$$
$$= \frac{1}{\Delta}\begin{bmatrix} a_{11}a_{22} - a_{12}a_{21} & 0 \\ 0 & a_{11}a_{22} - a_{12}a_{21} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.$$

Therefore,

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{\Delta}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$

42. Assume that A is an invertible matrix and that $A\mathbf{x}_i = \mathbf{b}_i$ for $i = 1, 2, \dots, p$ (where each $\mathbf{b}_i$ is given). Use elementary row operations on the augmented matrix of the system to obtain the equivalence

$$[A|\mathbf{b}_1\ \mathbf{b}_2\ \mathbf{b}_3\ \dots\ \mathbf{b}_p] \sim [I_n|\mathbf{c}_1\ \mathbf{c}_2\ \mathbf{c}_3\ \dots\ \mathbf{c}_p].$$

The solutions to the system can be read from the last matrix: $\mathbf{x}_i = \mathbf{c}_i$ for each $i = 1, 2, \dots, p$.

43. We have

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & -1 & 2 \\ 2 & -1 & 4 & 1 & 2 & 3 \\ 1 & 1 & 6 & -1 & 5 & 2 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & -1 & 4 & -1 \\ 0 & 2 & 5 & -2 & 6 & 0 \end{array}\right]$$
$$\overset{2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 3 & 1 \\ 0 & 1 & 2 & -1 & 4 & -1 \\ 0 & 0 & 1 & 0 & -2 & 2 \end{array}\right] \overset{3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 9 & -5 \\ 0 & 1 & 0 & -1 & 8 & -5 \\ 0 & 0 & 1 & 0 & -2 & 2 \end{array}\right].$$

Hence,

$$\mathbf{x}_1 = (0, -1, 0), \qquad \mathbf{x}_2 = (9, 8, -2), \qquad \mathbf{x}_3 = (-5, -5, 2).$$

1. $A_{12}(-2)$, $A_{13}(-1)$ **2.** $A_{21}(1)$, $A_{23}(-2)$ **3.** $A_{31}(-3)$, $A_{32}(-2)$

44.

(a). Let $\mathbf{e}_i$ denote the ith column vector of the identity matrix I_m, and consider the m linear systems of equations

$$A\mathbf{x}_i = \mathbf{e}_i$$

for $i = 1, 2, \dots, m$. Since $\text{rank}(A) = m$ and each $\mathbf{e}_i$ is a column m-vector, it follows that

$$\text{rank}(A^{\#}) = m = \text{rank}(A)$$

and so each of the systems $A\mathbf{x}_i = \mathbf{e}_i$ above has a solution (Note that if $m < n$, then there will be an infinite number of solutions). If we let $B = [\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m]$, then

$$AB = A[\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m] = [A\mathbf{x}_1, A\mathbf{x}_2, \dots, A\mathbf{x}_m] = [\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_m] = I_n.$$

(b). A right inverse for A in this case is a 3×2 matrix $\begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$ such that

$$\begin{bmatrix} a + 3b + c & d + 3e + f \\ 2a + 7b + 4c & 2d + 7e + 4f \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus, we must have

$$a+3b+c=1, \quad d+3e+f=0, \quad 2a+7b+4c=0, \quad 2d+7e+4f=1.$$

The first and third equation comprise a linear system with augmented matrix $\left[\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ 2 & 7 & 4 & 0 \end{array}\right]$ for a, b, and c. The row-echelon form of this augmented matrix is $\left[\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ 0 & 1 & 2 & -2 \end{array}\right]$. Setting $c = t$, we have $b = -2 - 2t$ and $a = 7+5t$. Next, the second and fourth equation above comprise a linear system with augmented matrix $\left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 2 & 7 & 4 & 1 \end{array}\right]$ for d, e, and f. The row-echelon form of this augmented matrix is $\left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{array}\right]$. Setting $f = s$, we have $e = 1 - 2s$ and $d = -3 + 5s$. Thus, right inverses of A are precisely the matrices of the form $\begin{bmatrix} 7+5t & -3+5s \\ -2-2t & 1-2s \\ t & s \end{bmatrix}$.

Solutions to Section 2.7

True-False Review:

(a): TRUE. Since every elementary matrix corresponds to a (reversible) elementary row operation, the reverse elementary row operation will correspond to an elementary matrix that is the inverse of the original elementary matrix.

(b): FALSE. For instance, the matrices $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ are both elementary matrices, but their product, $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, is not.

(c): FALSE. Every *invertible* matrix can be expressed as a product of elementary matrices. Since every elementary matrix is invertible and products of invertible matrices are invertible, any product of elementary matrices must be an invertible matrix.

(d): TRUE. Performing an elementary row operation on a matrix does not alter its rank, and the matrix EA is obtained from A by performing the elementary row operation associated with the elementary matrix E. Therefore, A and EA have the same rank.

(e): FALSE. If P_{ij} is a permutation matrix, then $P_{ij}^2 = I_n$, since permuting the ith and jth rows of I_n twice yields I_n. Alternatively, we can observe that $P_{ij}^2 = I_n$ from the fact that $P_{ij}^{-1} = P_{ij}$.

(f): FALSE. For example, consider the elementary matrices $E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix}$ and $E_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Then we have $E_1E_2 = \begin{bmatrix} 1 & 1 \\ 0 & 7 \end{bmatrix}$ and $E_2E_1 = \begin{bmatrix} 1 & 7 \\ 0 & 7 \end{bmatrix}$.

(g): FALSE. For example, consider the elementary matrices $E_1 = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$. Then we have $E_1E_2 = \begin{bmatrix} 1 & 3 & 6 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ and $E_2E_1 = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

(h): FALSE. The only matrices we perform an LU factorization for are invertible matrices for which the reduction to upper triangular form can be accomplished without permuting rows.

(i): FALSE. The matrix U need not be a *unit* upper triangular matrix.

(j): FALSE. As can be seen in Example 2.7.8, a 4×4 matrix with LU factorization will have 6 multipliers, not 10 multipliers.

Problems:

1.

Permutation Matrices: $P_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad P_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$

Scaling Matrices: $M_1(k) = \begin{bmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad M_2(k) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad M_3(k) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & k \end{bmatrix}.$

Row Combinations:

$$A_{12}(k) = \begin{bmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_{13}(k) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k & 0 & 1 \end{bmatrix}, \quad A_{23}(k) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{bmatrix},$$

$$A_{21}(k) = \begin{bmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_{31}(k) = \begin{bmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_{32}(k) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix}.$$

2. We have

$$\begin{bmatrix} -4 & -1 \\ 0 & 3 \\ -3 & 7 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} -1 & -8 \\ 0 & 3 \\ -3 & 7 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & 8 \\ 0 & 3 \\ -3 & 7 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & 8 \\ 0 & 3 \\ 0 & 31 \end{bmatrix} \overset{4}{\sim} \begin{bmatrix} 1 & 8 \\ 0 & 1 \\ 0 & 31 \end{bmatrix} \overset{5}{\sim} \begin{bmatrix} 1 & 8 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

1. $A_{31}(-1)$ **2.** $M_1(-1)$ **3.** $A_{13}(3)$ **4.** $M_2(\frac{1}{3})$ **5.** $A_{23}(-31)$

Elementary Matrices: $A_{23}(31)$, $M_2(\frac{1}{3})$, $A_{13}(3)$, $M_1(-1)$, $A_{31}(-1)$.

3. We have

$$\begin{bmatrix} 3 & 5 \\ 1 & -2 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & -2 \\ 0 & 11 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}.$$

1. P_{12} **2.** $A_{12}(-3)$ **3.** $M_2(\frac{1}{11})$

Elementary Matrices: $M_2(\frac{1}{11})$, $A_{12}(-3)$, P_{12}.

4. We have

$$\begin{bmatrix} 5 & 8 & 2 \\ 1 & 3 & -1 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & 3 & -1 \\ 5 & 8 & 2 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & 3 & -1 \\ 0 & -7 & 7 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & -1 \end{bmatrix}.$$

1. P_{12} **2.** $A_{12}(-5)$ **3.** $M_2(-\frac{1}{7})$

Elementary Matrices: $M_2(-\frac{1}{7})$, $A_{12}(-5)$, P_{12}.

5. We have

$$\begin{bmatrix} 3 & -1 & 4 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & -1 & 4 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & 3 & 2 \\ 0 & -5 & -1 \\ 0 & -10 & -2 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & 3 & 2 \\ 0 & -5 & -1 \\ 0 & 0 & 0 \end{bmatrix} \overset{4}{\sim} \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & \frac{1}{5} \\ 0 & 0 & 0 \end{bmatrix}.$$

1. P_{13} **2.** $A_{12}(-2), A_{13}(-3)$ **3.** $A_{23}(-2)$ **4.** $M_2(-\frac{1}{5})$

Elementary Matrices: $M_2(-\frac{1}{5})$, $A_{23}(-2)$, $A_{13}(-3)$, $A_{12}(-2)$, P_{13}.

6. We have

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & -2 & -4 & -6 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

1. $A_{12}(-2), A_{13}(-3)$ **2.** $M_2(-1)$ **3.** $A_{23}(2)$

Elementary Matrices: $A_{23}(2)$, $M_2(-1)$, $A_{13}(-3)$, $A_{12}(-2)$.

7. We reduce A to the identity matrix:

$$\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

1. $A_{12}(-1)$ **2.** $A_{21}(-2)$

The elementary matrices corresponding to these row operations are $E_1 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ and $E_2 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$.

We have $E_2E_1A = I_2$, so that

$$A = E_1^{-1}E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix},$$

which is the desired expression since E_1^{-1} and E_2^{-1} are elementary matrices.

8. We reduce A to the identity matrix:

$$\begin{bmatrix} -2 & -3 \\ 5 & 7 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} -2 & -3 \\ 1 & 1 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & 1 \\ -2 & -3 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \overset{4}{\sim} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \overset{5}{\sim} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

1. $A_{12}(2)$ **2.** P_{12} **3.** $A_{12}(2)$ **4.** $A_{21}(1)$ **5.** $M_2(-1)$

The elementary matrices corresponding to these row operations are

$$E_1 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad E_5 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

We have $E_5E_4E_3E_2E_1A = I_2$, so

$$A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

which is the desired expression since each E_i^{-1} is an elementary matrix.

9. We reduce A to the identity matrix:

$$\begin{bmatrix} 3 & -4 \\ -1 & 2 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & -2 \\ 0 & 2 \end{bmatrix} \overset{4}{\sim} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \overset{5}{\sim} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

1. P_{12} **2.** $M_1(-1)$ **3.** $A_{12}(-3)$ **4.** $M_2(\frac{1}{2})$ **5.** $A_{21}(2)$

The elementary matrices corresponding to these row operations are

$$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \quad E_5 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

We have $E_5E_4E_3E_2E_1A = I_2$, so

$$A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix},$$

which is the desired expression since each E_i^{-1} is an elementary matrix.

10. We reduce A to the identity matrix:

$$\begin{bmatrix} 4 & -5 \\ 1 & 4 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & 4 \\ 4 & -5 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & 4 \\ 0 & -21 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \overset{4}{\sim} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

1. P_{12} **2.** $A_{12}(-4)$ **3.** $M_2(-\frac{1}{21})$ **4.** $A_{21}(-4)$

The elementary matrices corresponding to these row operations are

$$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{21} \end{bmatrix}, \quad E_4 = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}.$$

We have $E_4E_3E_2E_1A = I_2$, so

$$A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -21 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix},$$

which is the desired expression since each E_i^{-1} is an elementary matrix.

11. We reduce A to the identity matrix:

$$\begin{bmatrix} 1 & -1 & 0 \\ 2 & 2 & 2 \\ 3 & 1 & 3 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 4 & 2 \\ 3 & 1 & 3 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 4 & 2 \\ 0 & 4 & 3 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\overset{4}{\sim} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \overset{5}{\sim} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \overset{6}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

1. $A_{12}(-2)$ **2.** $A_{13}(-3)$ **3.** $A_{23}(-1)$ **4.** $M_2(\frac{1}{4})$ **5.** $A_{32}(-\frac{1}{2})$ **6.** $A_{21}(1)$

The elementary matrices corresponding to these row operations are

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix},$$

$$E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}, \quad E_6 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We have $E_6E_5E_4E_3E_2E_1A = I_3$, so

$$A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}E_6^{-1}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which is the desired expression since each E_i^{-1} is an elementary matrix.

12. We reduce A to the identity matrix:

$$\begin{bmatrix} 0 & -4 & -2 \\ 1 & -1 & 3 \\ -2 & 2 & 2 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & -1 & 3 \\ 0 & -4 & -2 \\ -2 & 2 & 2 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & -1 & 3 \\ 0 & -4 & -2 \\ 0 & 0 & 8 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & -1 & 3 \\ 0 & -4 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\overset{4}{\sim} \begin{bmatrix} 1 & -1 & 3 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \overset{5}{\sim} \begin{bmatrix} 1 & -1 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \overset{6}{\sim} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \overset{7}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

1. P_{12} **2.** $A_{13}(2)$ **3.** $M_3(\frac{1}{8})$ **4.** $A_{32}(2)$ **5.** $A_{31}(-3)$ **6.** $M_2(-\frac{1}{4})$ **7.** $A_{21}(1)$

The elementary matrices corresponding to these row operations are

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{8} \end{bmatrix}, \quad E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix},$$

$$E_5 = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_7 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We have $E_7E_6E_5E_4E_3E_2E_1A = I_3$, so

$$A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}E_6^{-1}E_7^{-1}$$
$$= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which is the desired expression since each E_i^{-1} is an elementary matrix.

13. We reduce A to the identity matrix:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 8 & 0 \\ 3 & 4 & 5 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 3 & 4 & 5 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & -2 & -4 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & -2 & -4 \end{bmatrix}$$

$$\overset{4}{\sim} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix} \overset{5}{\sim} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \overset{6}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

1. $M_2(\frac{1}{8})$ **2.** $A_{13}(-3)$ **3.** $A_{21}(-2)$ **4.** $A_{23}(2)$ **5.** $M_3(-\frac{1}{4})$ **6.** $A_{31}(-3)$

The elementary matrices corresponding to these row operations are

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{8} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}, \quad E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{4} \end{bmatrix}, \quad E_6 = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We have $E_6E_5E_4E_3E_2E_1A = I_3$, so

$$\begin{aligned} A &= E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}E_6^{-1} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \end{aligned}$$

which is the desired expression since each E_i^{-1} is an elementary matrix.

14. We reduce A to the identity matrix:

$$\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & 3 \\ 0 & -7 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \overset{4}{\sim} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

1. P_{12} **2.** $A_{12}(-2)$ **3.** $M_2(-\frac{1}{7})$ **4.** $A_{21}(-3)$

The elementary matrices corresponding to these row operations are

$$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{7} \end{bmatrix}, \quad E_4 = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}.$$

Direct multiplication verifies that $E_4E_3E_2E_1A = I_2$.

15. We have

$$\begin{bmatrix} 3 & -2 \\ -1 & 5 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 3 & -2 \\ 0 & \frac{13}{3} \end{bmatrix} = U.$$

1. $A_{12}(\frac{1}{3})$

Hence, $E_1 = A_{12}(\frac{1}{3})$. Then Equation (2.7.3) reads $L = E_1^{-1} = A_{12}(-\frac{1}{3}) = \begin{bmatrix} 1 & 0 \\ -\frac{1}{3} & 1 \end{bmatrix}$. Verifying Equation (2.7.2):

$$LU = \begin{bmatrix} 1 & 0 \\ -\frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 0 & \frac{13}{3} \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -1 & 5 \end{bmatrix} = A.$$

16. We have

$$\begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 2 & 3 \\ 0 & -\frac{13}{2} \end{bmatrix} = U \Longrightarrow m_{21} = \frac{5}{2} \Longrightarrow L = \begin{bmatrix} 1 & 0 \\ \frac{5}{2} & 1 \end{bmatrix}.$$

Then

$$LU = \begin{bmatrix} 1 & 0 \\ \frac{5}{2} & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & -\frac{13}{2} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix} = A.$$

1. $A_{12}(-\frac{5}{2})$

17. We have

$$\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 3 & 1 \\ 0 & \frac{1}{3} \end{bmatrix} = U \Longrightarrow m_{21} = \frac{5}{3} \Longrightarrow L = \begin{bmatrix} 1 & 0 \\ \frac{5}{3} & 1 \end{bmatrix}.$$

Then

$$LU = \begin{bmatrix} 1 & 0 \\ \frac{5}{3} & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} = A.$$

1. $A_{12}(-\frac{5}{3})$

18. We have

$$\begin{bmatrix} 3 & -1 & 2 \\ 6 & -1 & 1 \\ -3 & 5 & 2 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & -3 \\ 0 & 4 & 4 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 16 \end{bmatrix} = U \Longrightarrow m_{21} = 2, m_{31} = -1, m_{32} = 4.$$

Hence,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix} \quad \text{and} \quad LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 16 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 2 \\ 6 & -1 & 1 \\ -3 & 5 & 2 \end{bmatrix} = A.$$

1. $A_{12}(-2)$, $A_{13}(1)$ **2.** $A_{23}(-4)$

19. We have

$$\begin{bmatrix} 5 & 2 & 1 \\ -10 & -2 & 3 \\ 15 & 2 & -3 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 5 & 2 & 1 \\ 0 & 2 & 5 \\ 0 & -4 & -6 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 5 & 2 & 1 \\ 0 & 2 & 5 \\ 0 & 0 & 4 \end{bmatrix} = U \Longrightarrow m_{21} = -2, m_{31} = 3, m_{32} = -2.$$

Hence,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \quad \text{and} \quad LU = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 5 & 2 & 1 \\ 0 & 2 & 5 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 1 \\ -10 & -2 & 3 \\ 15 & 2 & -3 \end{bmatrix} = A.$$

1. $A_{12}(2)$, $A_{13}(-3)$ **2.** $A_{23}(2)$

20. We have

$$\begin{bmatrix} 1 & -1 & 2 & 3 \\ 2 & 0 & 3 & -4 \\ 3 & -1 & 7 & 8 \\ 1 & 3 & 4 & 5 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 2 & -1 & -10 \\ 0 & 2 & 1 & -1 \\ 0 & 4 & 2 & 2 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 2 & -1 & -10 \\ 0 & 0 & 2 & 9 \\ 0 & 0 & 4 & 22 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 2 & -1 & -10 \\ 0 & 0 & 2 & 9 \\ 0 & 0 & 0 & 4 \end{bmatrix} = U.$$

1. $A_{12}(-2)$, $A_{13}(-3)$, $A_{14}(-1)$ **2.** $A_{23}(-1)$, $A_{24}(-2)$ **3.** $A_{34}(-2)$

Hence,

$$m_{21} = 2, \quad m_{31} = 3, m_{41} = 1, m_{32} = 1, m_{42} = 2, m_{43} = 2.$$

Hence,

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 \\ 1 & 2 & 2 & 1 \end{bmatrix} \quad \text{and} \quad LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 \\ 1 & 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 2 & -1 & -10 \\ 0 & 0 & 2 & 9 \\ 0 & 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 & 3 \\ 2 & 0 & 3 & -4 \\ 3 & -1 & 7 & 8 \\ 1 & 3 & 4 & 5 \end{bmatrix} = A.$$

21. We have

$$\begin{bmatrix} 2 & -3 & 1 & 2 \\ 4 & -1 & 1 & 1 \\ -8 & 2 & 2 & -5 \\ 6 & 1 & 5 & 2 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} 2 & -3 & 1 & 2 \\ 0 & 5 & -1 & -3 \\ 0 & -10 & 6 & 3 \\ 0 & 10 & 2 & -4 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 2 & -3 & 1 & 2 \\ 0 & 5 & -1 & -3 \\ 0 & 0 & 4 & -3 \\ 0 & 0 & 4 & 2 \end{bmatrix} \stackrel{3}{\sim} \begin{bmatrix} 2 & -3 & 1 & 2 \\ 0 & 5 & -1 & -3 \\ 0 & 0 & 4 & -3 \\ 0 & 0 & 0 & 5 \end{bmatrix} = U.$$

1. $A_{12}(-2)$, $A_{13}(4)$, $A_{14}(-3)$ **2.** $A_{23}(2)$, $A_{24}(-2)$ **3.** $A_{34}(-1)$

Hence,

$$m_{21} = 2, \quad m_{31} = -4, \quad m_{41} = 3, \quad m_{32} = -2, \quad m_{42} = 2, \quad m_{43} = 1.$$

Hence,

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -4 & -2 & 1 & 0 \\ 3 & 2 & 1 & 1 \end{bmatrix} \text{ and } LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -4 & -2 & 1 & 0 \\ 3 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 & 1 & 2 \\ 0 & 5 & -1 & -3 \\ 0 & 0 & 4 & -3 \\ 0 & 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 1 & 2 \\ 4 & -1 & 1 & 1 \\ -8 & 2 & 2 & -5 \\ 6 & 1 & 5 & 2 \end{bmatrix} = A.$$

22. We have

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} = U \Longrightarrow m_{21} = 2 \Longrightarrow L = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}.$$

1. $A_{12}(-2)$

We now solve the triangular systems $L\mathbf{y} = \mathbf{b}$ and $U\mathbf{x} = \mathbf{y}$. From $L\mathbf{y} = \mathbf{b}$, we obtain $\mathbf{y} = \begin{bmatrix} 3 \\ -7 \end{bmatrix}$. Then $U\mathbf{x} = \mathbf{y}$ yields $\mathbf{x} = \begin{bmatrix} -11 \\ 7 \end{bmatrix}$.

23. We have

$$\begin{bmatrix} 1 & -3 & 5 \\ 3 & 2 & 2 \\ 2 & 5 & 2 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} 1 & -3 & 5 \\ 0 & 11 & -13 \\ 0 & 11 & -8 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 1 & -3 & 5 \\ 0 & 11 & -13 \\ 0 & 0 & 5 \end{bmatrix} = U \Longrightarrow m_{21} = 3, m_{31} = 2, m_{32} = 1.$$

1. $A_{12}(-3)$, $A_{13}(-2)$ **2.** $A_{23}(-1)$

Hence, $L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$. We now solve the triangular systems $L\mathbf{y} = \mathbf{b}$ and $U\mathbf{x} = \mathbf{y}$. From $L\mathbf{y} = \mathbf{b}$, we obtain $\mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}$. Then $U\mathbf{x} = \mathbf{y}$ yields $\mathbf{x} = \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}$.

24. We have

$$\begin{bmatrix} 2 & 2 & 1 \\ 6 & 3 & -1 \\ -4 & 2 & 2 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 2 & 2 & 1 \\ 0 & -3 & -4 \\ 0 & 0 & -4 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 2 & 2 & 1 \\ 0 & -3 & -4 \\ 0 & 0 & -4 \end{bmatrix} = U \Longrightarrow m_{21} = 3, m_{31} = -2, m_{32} = -2.$$

1. $A_{12}(-3)$, $A_{13}(2)$ **2.** $A_{23}(2)$

Hence, $L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & -2 & 1 \end{bmatrix}$. We now solve the triangular systems $L\mathbf{y} = \mathbf{b}$ and $U\mathbf{x} = \mathbf{y}$. From $L\mathbf{y} = \mathbf{b}$, we obtain $\mathbf{y} = \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}$. Then $U\mathbf{x} = \mathbf{y}$ yields $\mathbf{x} = \begin{bmatrix} -1/12 \\ 1/3 \\ 1/2 \end{bmatrix}$.

25. We have

$$\begin{bmatrix} 4 & 3 & 0 & 0 \\ 8 & 1 & 2 & 0 \\ 0 & 5 & 3 & 6 \\ 0 & 0 & -5 & 7 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 4 & 3 & 0 & 0 \\ 0 & -5 & 2 & 0 \\ 0 & 5 & 3 & 6 \\ 0 & 0 & -5 & 7 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 4 & 3 & 0 & 0 \\ 0 & -5 & 2 & 0 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & -5 & 7 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 4 & 3 & 0 & 0 \\ 0 & -5 & 2 & 0 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 0 & 13 \end{bmatrix} = U.$$

1. $A_{12}(-2)$ **2.** $A_{23}(1)$ **3.** $A_{34}(1)$

The only nonzero multipliers are $m_{21} = 2, m_{32} = -1$, and $m_{43} = -1$. Hence, $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$. We now solve the triangular systems $L\mathbf{y} = \mathbf{b}$ and $U\mathbf{x} = \mathbf{y}$. From $L\mathbf{y} = \mathbf{b}$, we obtain $\mathbf{y} = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 4 \end{bmatrix}$. Then $U\mathbf{x} = \mathbf{y}$ yields $\mathbf{x} = \begin{bmatrix} 677/1300 \\ -9/325 \\ -37/65 \\ 4/13 \end{bmatrix}$.

26. We have

$$\begin{bmatrix} 2 & -1 \\ -8 & 3 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 2 & -1 \\ 0 & -1 \end{bmatrix} = U \Longrightarrow m_{21} = -4 \Longrightarrow L = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}.$$

1. $A_{12}(4)$

We now solve the triangular systems

$$L\mathbf{y}_i = \mathbf{b}_i, \qquad U\mathbf{x}_i = \mathbf{y}_i$$

for $i = 1, 2, 3$. We have

$$L\mathbf{y}_1 = \mathbf{b}_1 \Longrightarrow \mathbf{y}_1 = \begin{bmatrix} 3 \\ 11 \end{bmatrix}. \text{ Then } U\mathbf{x}_1 = \mathbf{y}_1 \Longrightarrow \mathbf{x}_1 = \begin{bmatrix} -4 \\ -11 \end{bmatrix};$$

$$L\mathbf{y}_2 = \mathbf{b}_2 \Longrightarrow \mathbf{y}_2 = \begin{bmatrix} 2 \\ 15 \end{bmatrix}. \text{ Then } U\mathbf{x}_2 = \mathbf{y}_2 \Longrightarrow \mathbf{x}_2 = \begin{bmatrix} -6.5 \\ -15 \end{bmatrix};$$

$$L\mathbf{y}_3 = \mathbf{b}_3 \Longrightarrow \mathbf{y}_3 = \begin{bmatrix} 5 \\ 11 \end{bmatrix}. \text{ Then } U\mathbf{x}_3 = \mathbf{y}_3 \Longrightarrow \mathbf{x}_3 = \begin{bmatrix} -3 \\ -11 \end{bmatrix}.$$

27. We have

$$\begin{bmatrix} -1 & 4 & 2 \\ 3 & 1 & 4 \\ 5 & -7 & 1 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} -1 & 4 & 2 \\ 0 & 13 & 10 \\ 0 & 13 & 11 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} -1 & 4 & 2 \\ 0 & 13 & 10 \\ 0 & 0 & 1 \end{bmatrix} = U.$$

1. $A_{12}(3)$, $A_{13}(5)$ **2.** $A_{23}(-1)$

Thus, $m_{21} = -3$, $m_{31} = -5$, and $m_{32} = 1$. We now solve the triangular systems

$$L\mathbf{y}_i = \mathbf{b}_i, \qquad U\mathbf{x}_i = \mathbf{y}_i$$

for $i = 1, 2, 3$. We have

$$L\mathbf{y}_1 = \mathbf{e}_1 \Longrightarrow \mathbf{y}_1 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}. \text{ Then } U\mathbf{x}_1 = \mathbf{y}_1 \Longrightarrow \mathbf{x}_1 = \begin{bmatrix} -29/13 \\ -17/13 \\ 2 \end{bmatrix};$$

$$L\mathbf{y}_2 = \mathbf{e}_2 \Longrightarrow \mathbf{y}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}. \text{ Then } U\mathbf{x}_2 = \mathbf{y}_2 \Longrightarrow \mathbf{x}_2 = \begin{bmatrix} 18/13 \\ 11/13 \\ -1 \end{bmatrix};$$

$$L\mathbf{y}_3 = \mathbf{e}_3 \Longrightarrow \mathbf{y}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \text{ Then } U\mathbf{x}_3 = \mathbf{y}_3 \Longrightarrow \mathbf{x}_3 = \begin{bmatrix} -14/13 \\ -10/13 \\ 1 \end{bmatrix}.$$

28. Observe that if P_i is an elementary permutation matrix, then $P_i^{-1} = P_i = P_i^T$. Therefore, we have

$$P^{-1} = (P_1P_2 \dots P_k)^{-1} = P_k^{-1}P_{k-1}^{-1} \dots P_2^{-1}P_1^{-1} = P_k^T P_{k-1}^T \dots P_2^T \dots P_1^T = (P_1P_2 \dots P_k)^T = P^T.$$

29.

(a). Let A be an invertible upper triangular matrix with inverse B. Therefore, we have $AB = I_n$. Write $A = [a_{ij}]$ and $B = [b_{ij}]$. We will show that $b_{ij} = 0$ for all $i > j$, which shows that B is upper triangular. We have

$$\sum_{k=1}^{n} a_{ik}b_{kj} = \delta_{ij}.$$

Since A is upper triangular, $a_{ik} = 0$ whenever $i > k$. Therefore, we can reduce the above summation to

$$\sum_{k=i}^{n} a_{ik}b_{ij} = \delta_{ij}.$$

Let $i = n$. Then the above summation reduces to $a_{nn}b_{nj} = \delta_{nj}$. If $j = n$, we have $a_{nn}b_{nn} = 1$, so $a_{nn} \neq 0$. For $j < n$, we have $a_{nn}b_{nj} = 0$, and therefore $b_{nj} = 0$ for all $j < n$.

Next let $i = n - 1$. Then we have

$$a_{n-1,n-1}b_{n-1,j} + a_{n-1,n}b_{nj} = \delta_{n-1,j}.$$

Setting $j = n-1$ and using the fact that $b_{n,n-1} = 0$ by the above calculation, we obtain $a_{n-1,n-1}b_{n-1,n-1} = 1$, so $a_{n-1,n-1} \neq 0$. For $j < n-1$, we have $a_{n-1,n-1}b_{n-1,j} = 0$ so that $b_{n-1,j} = 0$.

Next let $i = n-2$. Then we have $a_{n-2,n-2}b_{n-2,j} + a_{n-2,n-1}b_{n-1,j} + a_{n-2,n}b_{nj} = \delta_{n-2,j}$. Setting $j = n-2$ and using the fact that $b_{n-1,n-2} = 0$ and $b_{n,n-2} = 0$, we have $a_{n-2,n-2}b_{n-2,n-2} = 1$, so that $a_{n-2,n-2} \neq 0$. For $j < n-2$, we have $a_{n-2,n-2}b_{n-2,j} = 0$ so that $b_{n-2,j} = 0$.

Proceeding in this way, we eventually show that $b_{ij} = 0$ for all $i > j$.

For an invertible lower triangular matrix A with inverse B, we can either modify the preceding argument, or we can proceed more briefly as follows: Note that A^T is an invertible upper triangular matrix with inverse B^T. By the preceding argument, B^T is upper triangular. Therefore, B is lower triangular, as required.

(b). Let A be an invertible unit upper triangular matrix with inverse B. Use the notations from (a). By (a), we know that B is upper triangular. We simply must show that $b_{jj} = 0$ for all j. From $a_{nn}b_{nn} = 1$ (see proof of (a)), we see that if $a_{nn} = 1$, then $b_{nn} = 1$. Moreover, from $a_{n-1,n-1}b_{n-1,n-1} = 1$, the fact that $a_{n-1,n-1} = 1$ proves that $b_{n-1,n-1} = 1$. Likewise, the fact that $a_{n-2,n-2}b_{n-2,n-2} = 1$ implies that if $a_{n-2,n-2} = 1$, then $b_{n-2,n-2} = 1$. Continuing in this fashion, we prove that $b_{jj} = 1$ for all j.

For the last part, if A is an invertible unit lower triangular matrix with inverse B, then A^T is an invertible unit upper triangular matrix with inverse B^T, and by the preceding argument, B^T is a unit upper triangular matrix. This implies that B is a unit lower triangular matrix, as desired.

30.

(a). Since A is invertible, Corollary 2.6.13 implies that both L_2 and U_1 are invertible. Since $L_1U_1 = L_2U_2$, we can left-multiply by L_2^{-1} and right-multiply by U_1^{-1} to obtain $L_2^{-1}L_1 = U_2U_1^{-1}$.

(b). By Problem 29, we know that L_2^{-1} is a unit lower triangular matrix and U_1^{-1} is an upper triangular matrix. Therefore, $L_2^{-1}L_1$ is a unit lower triangular matrix and $U_2U_1^{-1}$ is an upper triangular matrix. Since these two matrices are equal, we must have $L_2^{-1}L_1 = I_n$ and $U_2U_1^{-1} = I_n$. Therefore, $L_1 = L_2$ and $U_1 = U_2$.

31. The system $A\mathbf{x} = \mathbf{b}$ can be written as $QR\mathbf{x} = \mathbf{b}$. If we can solve $Q\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$ and then solve $R\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$, then $QR\mathbf{x} = \mathbf{b}$ as desired. Multiplying $Q\mathbf{y} = \mathbf{b}$ by Q^T and using the fact that $Q^TQ = I_n$, we obtain $\mathbf{y} = Q^T\mathbf{b}$. Therefore, $R\mathbf{x} = \mathbf{y}$ can be replaced by $R\mathbf{x} = Q^T\mathbf{b}$. Therefore, to solve $A\mathbf{x} = \mathbf{b}$, we first determine $\mathbf{y} = Q^T\mathbf{b}$ and then solve the upper triangular system $R\mathbf{x} = Q^T\mathbf{b}$ by back-substitution.

Solutions to Section 2.8

True-False Review:

(a): FALSE. According to the given information, part (c) of the Invertible Matrix Theorem fails, while part (e) holds. This is impossible.

(b): TRUE. This holds by the equivalence of parts (d) and (f) of the Invertible Matrix Theorem.

(c): FALSE. Part (d) of the Invertible Matrix Theorem fails according to the given information, and therefore part (b) also fails. Hence, the equation $A\mathbf{x} = \mathbf{b}$ does not have a unique solution. But it is not valid to conclude that the equation has infinitely many solutions; it could have no solutions. For instance, if $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, there are no solutions to $A\mathbf{x} = \mathbf{b}$, although $\text{rank}(A) = 2$.

(d): FALSE. An easy counterexample is the matrix 0_n, which fails to be invertible even though it is upper triangular. Since it fails to be invertible, it cannot e row-equivalent to I_n, by the Invertible Matrix Theorem.

Problems:

1. Since A is an invertible matrix, the only solution to $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$. However, if we assume that $AB = AC$, then $A(B - C) = 0$. If $\mathbf{x}_i$ denotes the ith column of $B - C$, then $\mathbf{x}_i = \mathbf{0}$ for each i. That is, $B - C = 0$, or $B = C$, as required.

2. If $\text{rank}(A) = n$, then the augmented matrix $A^{\#}$ for the system $A\mathbf{x} = \mathbf{0}$ can be reduced to REF such that each column contains a pivot except for the right-most column of all-zeros. Solving the system by back-substitution, we find that $\mathbf{x} = \mathbf{0}$, as claimed.

3. Since $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, REF(A) contains a pivot in every column. Therefore, the linear system $A\mathbf{x} = \mathbf{b}$ can be solved by back-substitution for every $\mathbf{b}$ in $\mathbb{R}^n$. Therefore, $A\mathbf{x} = \mathbf{b}$ does have a solution.

Now suppose there are two solutions $\mathbf{y}$ and $\mathbf{z}$ to the system $A\mathbf{x} = \mathbf{b}$. That is, $A\mathbf{y} = \mathbf{b}$ and $A\mathbf{z} = \mathbf{b}$. Subtracting, we find

$$A(\mathbf{y} - \mathbf{z}) = \mathbf{0},$$

and so by assumption, $\mathbf{y} - \mathbf{z} = \mathbf{0}$. That is, $\mathbf{y} = \mathbf{z}$. Therefore, there is only one solution to the linear system $A\mathbf{x} = \mathbf{b}$.

4. If A and B are each invertible matrices, then A and B can each be expressed as a product of elementary matrices, say

$$A = E_1E_2 \dots E_k \qquad \text{and} \qquad B = E_1'E_2' \dots E_l'.$$

Then

$$AB = E_1E_2 \dots E_kE_1'E_2' \dots E_l',$$

so AB can be expressed as a product of elementary matrices. Thus, by the equivalence of (a) and (e) in the Invertible Matrix Theorem, AB is invertible.

5. We are assuming that the equations $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ each have only the trivial solution $\mathbf{x} = \mathbf{0}$. Now consider the linear system

$$(AB)\mathbf{x} = \mathbf{0}.$$

Viewing this equation as

$$A(B\mathbf{x}) = \mathbf{0},$$

we conclude that $B\mathbf{x} = \mathbf{0}$. Thus, $\mathbf{x} = \mathbf{0}$. Hence, the linear equation $(AB)\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Solutions to Section 2.9

Problems:

1. $A^T - 5B = \begin{bmatrix} -2 & -1 \\ 4 & -1 \\ 2 & 5 \\ 6 & 0 \end{bmatrix} - \begin{bmatrix} -15 & 0 \\ 10 & 10 \\ 5 & -15 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 13 & -1 \\ -6 & -11 \\ -3 & 20 \\ 6 & -5 \end{bmatrix}.$

2. $C^TB = \begin{bmatrix} -5 & -6 & 3 & 1 \end{bmatrix} \begin{bmatrix} -3 & 0 \\ 2 & 2 \\ 1 & -3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -20 \end{bmatrix}.$

3. Since A is not a square matrix, it is not possible to compute A^2.

4. $-4A - B^T = \begin{bmatrix} 8 & -16 & -8 & -24 \\ 4 & 4 & -20 & 0 \end{bmatrix} - \begin{bmatrix} -3 & 2 & 1 & 0 \\ 0 & 2 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 11 & -18 & -9 & -24 \\ 4 & 2 & -17 & -1 \end{bmatrix}.$

5. We have

$$AB = \begin{bmatrix} -2 & 4 & 2 & 6 \\ -1 & -1 & 5 & 0 \end{bmatrix} \begin{bmatrix} -3 & 0 \\ 2 & 2 \\ 1 & -3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 16 & 8 \\ 6 & -17 \end{bmatrix}.$$

Moreover,

$$\text{tr}(AB) = -1.$$

6. We have

$$(AC)(AC)^T = \begin{bmatrix} -2 \\ 26 \end{bmatrix} \begin{bmatrix} -2 & 26 \end{bmatrix} = \begin{bmatrix} 4 & -52 \\ -52 & 676 \end{bmatrix}.$$

7. $(-4B)A = \begin{bmatrix} 12 & 0 \\ -8 & -8 \\ -4 & 12 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} -2 & 4 & 2 & 6 \\ -1 & -1 & 5 & 0 \end{bmatrix} = \begin{bmatrix} -24 & 48 & 24 & 72 \\ 24 & -24 & -56 & -48 \\ -4 & -28 & 52 & -24 \\ 4 & 4 & -20 & 0 \end{bmatrix}.$

8. Using Problem 5, we find that

$$(AB)^{-1} = \begin{bmatrix} 16 & 8 \\ 6 & -17 \end{bmatrix}^{-1} = -\frac{1}{320} \begin{bmatrix} -17 & -8 \\ -6 & 16 \end{bmatrix}.$$

9. We have

$$C^T C = \begin{bmatrix} -5 & -6 & 3 & 1 \end{bmatrix} \begin{bmatrix} -5 \\ -6 \\ 3 \\ 1 \end{bmatrix} = [71],$$

and

$$\text{tr}(C^T C) = 71.$$

10.

(a). We have

$$AB = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \end{bmatrix} \begin{bmatrix} 3 & b \\ -4 & a \\ a & b \end{bmatrix} = \begin{bmatrix} 3a-5 & 2a+4b \\ 7a-14 & 5a+9b \end{bmatrix}.$$

In order for this product to equal I_2, we require

$$3a - 5 = 1, \quad 2a + 4b = 0, \quad 7a - 14 = 0, \quad 5a + 9b = 1.$$

We quickly solve this for the unique solution: $a = 2$ and $b = -1$.

(b). We have

$$BA = \begin{bmatrix} 3 & -1 \\ -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & -1 & -1 \end{bmatrix}.$$

11. We compute the (i, j)-entry of each side of the equation. We will denote the entries of A^T by a^T_{ij}, which equals a_{ji}. On the left side, note that the (i, j)-entry of $(AB^T)^T$ is the same as the (j, i)-entry of AB^T, and

$$(j, i)\text{-entry of } AB^T = \sum_{k=0}^{n} a_{jk} b^T_{ki} = \sum_{k=0}^{n} a_{jk} b_{ik} = \sum_{k=0}^{n} b_{ik} a^T_{kj},$$

and the latter expression is the (i, j)-entry of BA^T. Therefore, the (i, j)-entries of $(AB^T)^T$ and BA^T are the same, as required.

12.

(a). The (i, j)-entry of A^2 is

$$\sum_{k=1}^{n} a_{ik}a_{kj}.$$

(b). Assume that A is symmetric. That means that $A^T = A$. We claim that A^2 is symmetric. To see this, note that

$$(A^2)^T = (AA)^T = A^T A^T = AA = A^2.$$

Thus, $(A^2)^T = A^2$, and so A^2 is symmetric.

13. We are assuming that A is skew-symmetric, so $A^T = -A$. To show that $B^T AB$ is skew-symmetric, we observe that

$$(B^T AB)^T = B^T A^T (B^T)^T = B^T A^T B = B^T(-A)B = -(B^T AB),$$

as required.

14. We have

$$A^2 = \begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

so A is nilpotent.

15. We have

$$A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$A^3 = A^2 A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

so A is nilpotent.

16. We have

$$A'(t) = \begin{bmatrix} -3e^{-3t} & -2\sec^2 t \tan t \\ 6t^2 & -\sin t \\ 6/t & -5 \end{bmatrix}.$$

17. We have

$$\int_0^1 B(t)\,dt = \begin{bmatrix} -7t & t^3/3 \\ 6t - t^2/2 & 3t^4/4 + 2t^3 \\ t + t^2/2 & \frac{2}{\pi}\sin(\pi t/2) \\ e^t & t - t^4/4 \end{bmatrix}\Bigg|_0^1 = \begin{bmatrix} -7 & 1/3 \\ 11/2 & 11/4 \\ 3/2 & 2/\pi \\ e-1 & 3/4 \end{bmatrix}.$$

18. Since $A(t)$ is 3×2 and $B(t)$ is 4×2, it is impossible to perform the indicated subtraction.

19. Since $A(t)$ is 3×2 and $B(t)$ is 4×2, it is impossible to perform the indicated subtraction.

20. From the last equation, we see that $x_3 = 0$. Substituting this into the middle equation, we find that $x_2 = 0.5$. Finally, putting the values of x_2 and x_3 into the first equation, we find $x_1 = -6 - 2.5 = -8.5$. Thus, there is a unique solution to the linear system, and the solution set is

$$\{(-8.5, 0.5, 0)\}.$$

21. To solve this system, we need to reduce the corresponding augmented matrix for the linear system to row-echelon form. This gives us

$$\left[\begin{array}{ccc|c} 5 & -1 & 2 & 7 \\ -2 & 6 & 9 & 0 \\ -7 & 5 & -3 & -7 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccc|c} 1 & 11 & 20 & 7 \\ -2 & 6 & 9 & 0 \\ -7 & 5 & -3 & -7 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{ccc|c} 1 & 11 & 20 & 7 \\ 0 & 28 & 49 & 14 \\ 0 & 82 & 137 & 42 \end{array}\right] \overset{3}{\sim} \left[\begin{array}{ccc|c} 1 & 11 & 20 & 7 \\ 0 & 1 & 7/4 & 1/2 \\ 0 & 82 & 137 & 42 \end{array}\right]$$

$$\overset{4}{\sim} \left[\begin{array}{ccc|c} 1 & 11 & 20 & 7 \\ 0 & 1 & 7/4 & 1/2 \\ 0 & 0 & -13/2 & 1 \end{array}\right] \overset{5}{\sim} \left[\begin{array}{ccc|c} 1 & 11 & 20 & 7 \\ 0 & 1 & 7/4 & 1/2 \\ 0 & 0 & 1 & -2/13 \end{array}\right].$$

From the last row, we conclude that $x_3 = -2/13$, and using the middle row, we can solve for x_2: we have $x_2 + \frac{7}{4} \cdot \left(-\frac{2}{13}\right) = \frac{1}{2}$, so $x_2 = \frac{20}{26} = \frac{10}{13}$. Finally, from the first row we can get x_1: we have $x_1 + 11 \cdot \frac{10}{13} + 20 \cdot \left(-\frac{2}{13}\right) = 7$, and so $x_1 = \frac{21}{13}$. So there is a unique solution:

$$\left\{\left(\frac{21}{13}, \frac{10}{13}, -\frac{2}{13}\right)\right\}.$$

1. $A_{21}(2)$	**2.** $A_{12}(2)$, $A_{13}(7)$	**3.** $M_2(1/28)$	**4.** $A_{23}(-82)$	**5.** $M_3(-2/13)$

22. To solve this system, we need to reduce the corresponding augmented matrix for the linear system to row-echelon form. This gives us

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 1 & 0 & 1 & 5 \\ 4 & 4 & 0 & 12 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -2 & 2 & 4 \\ 0 & -4 & 4 & 8 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & -4 & 4 & 8 \end{array}\right] \overset{3}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

From this row-echelon form, we see that z is a free variable. Set $z = t$. Then from the middle row of the matrix, $y = t - 2$, and from the top row, $x + 2(t-2) - t = 1$ or $x = -t + 5$. So the solution set is

$$\{(-t+5, t-2, t) : t \in \mathbb{R}\} = \{(5, -2, 0) + t(-1, 1, 1) : t \in \mathbb{R}\}.$$

1. $A_{12}(-1)$, $A_{13}(-4)$	**2.** $M_2(-1/2)$	**3.** $A_{23}(4)$

23. To solve this system, we need to reduce the corresponding augmented matrix for the linear system to row-echelon form. This gives us

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 0 \\ -2 & 4 & 5 & -5 & 3 \\ 3 & -6 & -6 & 8 & 2 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & -3 & -1 & 2 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & 0 & 0 & 5 \end{array}\right] \overset{3}{\sim} \left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 1 & 1/3 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

The bottom row of this matrix shows that this system has no solutions.

1. $A_{12}(2)$, $A_{13}(-3)$ **2.** $A_{23}(1)$ **3.** $M_2(1/3)$, $M_3(1/3)$

24. To solve this system, we need to reduce the corresponding augmented matrix for the linear system to row-echelon form. This gives us

$$\left[\begin{array}{ccccc|c} 3 & 0 & -1 & 2 & -1 & 1 \\ 1 & 3 & 1 & -3 & 2 & -1 \\ 4 & -2 & -3 & 6 & -1 & 5 \\ 0 & 0 & 0 & 1 & 4 & -2 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccccc|c} 1 & 3 & 1 & -3 & 2 & -1 \\ 3 & 0 & -1 & 2 & -1 & 1 \\ 4 & -2 & -3 & 6 & -1 & 5 \\ 0 & 0 & 0 & 1 & 4 & -2 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{ccccc|c} 1 & 3 & 1 & -3 & 2 & -1 \\ 0 & -9 & -4 & 11 & -7 & 4 \\ 0 & -14 & -7 & 18 & -9 & 9 \\ 0 & 0 & 0 & 1 & 4 & -2 \end{array}\right]$$

$$\overset{3}{\sim} \left[\begin{array}{ccccc|c} 1 & 3 & 1 & -3 & 2 & -1 \\ 0 & -27 & -12 & 33 & -21 & 12 \\ 0 & 28 & 14 & -36 & 18 & -18 \\ 0 & 0 & 0 & 1 & 4 & -2 \end{array}\right] \overset{4}{\sim} \left[\begin{array}{ccccc|c} 1 & 3 & 1 & -3 & 2 & -1 \\ 0 & -27 & -12 & 33 & -21 & 12 \\ 0 & 1 & 2 & -3 & -3 & -6 \\ 0 & 0 & 0 & 1 & 4 & -2 \end{array}\right]$$

$$\overset{5}{\sim} \left[\begin{array}{ccccc|c} 1 & 3 & 1 & -3 & 2 & -1 \\ 0 & 1 & 2 & -3 & -3 & -6 \\ 0 & -27 & -12 & 33 & -21 & 12 \\ 0 & 0 & 0 & 1 & 4 & -2 \end{array}\right] \overset{6}{\sim} \left[\begin{array}{ccccc|c} 1 & 3 & 1 & -3 & 2 & -1 \\ 0 & 1 & 2 & -3 & -3 & -6 \\ 0 & 0 & 42 & -48 & -102 & -150 \\ 0 & 0 & 0 & 1 & 4 & -2 \end{array}\right] \overset{7}{\sim} \left[\begin{array}{ccccc|c} 1 & 3 & 1 & -3 & 2 & -1 \\ 0 & 1 & 2 & -3 & -3 & -6 \\ 0 & 0 & 1 & -\frac{8}{7} & -\frac{17}{7} & -\frac{25}{7} \\ 0 & 0 & 0 & 1 & 4 & -2 \end{array}\right].$$

We see that $x_5 = t$ is the only free variable. Back substitution yields the remaining values:

$$x_5 = t, \quad x_4 = -4t - 2, \quad x_3 = -\frac{41}{7} - \frac{15}{7}t, \quad x_2 = -\frac{2}{7} - \frac{33}{7}t, \quad x_1 = -\frac{2}{7} + \frac{16}{7}t.$$

So the solution set is

$$\left\{\left(-\frac{2}{7} + \frac{16}{7}t, -\frac{2}{7} - \frac{33}{7}t, -\frac{41}{7} - \frac{15}{7}t, -4t - 2, t\right) : t \in \mathbb{R}\right\}$$

$$= \left\{t\left(\frac{16}{7}, -\frac{33}{7}, -\frac{15}{7}, -4, 1\right) + \left(-\frac{2}{7}, -\frac{2}{7}, -\frac{41}{7}, -2, 0\right) : t \in \mathbb{R}\right\}.$$

1. P_{12} **2.** $A_{12}(-3)$, $A_{13}(-4)$ **3.** $M_2(3)$, $M_3(-2)$ **4.** $A_{23}(1)$ **5.** P_{23} **6.** $A_{23}(27)$ **7.** $M_3(1/42)$

25. To solve this system, we need to reduce the corresponding augmented matrix for the linear system to row-echelon form. This gives us

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & -3 & 6 \\ 1 & 1 & 1 & 2 & -5 & 8 \\ 2 & 3 & 1 & 4 & -9 & 17 \\ 2 & 2 & 2 & 3 & -8 & 14 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & -3 & 6 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 1 & -1 & 2 & -3 & 5 \\ 0 & 0 & 0 & -1 & 2 & -2 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & -3 & 6 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 1 & -1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\overset{3}{\sim} \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & -3 & 6 \\ 0 & 1 & -1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

From this row-echelon form, we see that $x_5 = t$ and $x_3 = s$ are free variables. Furthermore, solving this system by back-substitution, we see that

$$x_5 = t, \quad x_4 = 2t + 2, \quad x_3 = s, \quad x_2 = s - t + 1, \quad x_1 = 2t - 2s + 3.$$

So the solution set is

$$\{(2t-2s+3, s-t+1, s, 2t+2, t) : s,t \in \mathbb{R}\} = \{t(2,-1,0,2,1) + s(-2,1,1,0,0) + (3,1,0,2,0) : s,t \in \mathbb{R}\}.$$

1. $A_{12}(-1)$, $A_{13}(-2)$, $A_{14}(-2)$ **2.** $A_{24}(1)$ **3.** P_{23}

26. To solve this system, we need to reduce the corresponding augmented matrix for the linear system to row-echelon form. This gives us

$$\left[\begin{array}{rrr|r} 1 & -3 & 2i & 1 \\ -2i & 6 & 2 & -2 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{rrr|r} 1 & -3 & 2i & 1 \\ 0 & 6-6i & -2 & -2+2i \end{array}\right] \overset{2}{\sim} \left[\begin{array}{rrr|r} 1 & -3 & 2i & 1 \\ 0 & 1 & -\frac{1}{6}(1+i) & -\frac{1}{3} \end{array}\right].$$

1. $A_{12}(2i)$ **2.** $M_2(\frac{1}{6-6i})$

From the last augmented matrix above, we see that x_3 is a free variable. Let us set $x_3 = t$, where t is a complex number. Then we can solve for x_2 using the equation corresponding to the second row of the row-echelon form: $x_2 = -\frac{1}{3} + \frac{1}{6}(1+i)t$. Finally, using the first row of the row-echelon form, we can determine that $x_1 = \frac{1}{2}t(1-3i)$. Therefore, the solution set for this linear system of equations is

$$\{(\frac{1}{2}t(1-3i), -\frac{1}{3} + \frac{1}{6}(1+i)t, t) : t \in \mathbb{C}\}.$$

27. We reduce the corresponding linear system as follows:

$$\left[\begin{array}{rr|r} 1 & -k & 6 \\ 2 & 3 & k \end{array}\right] \overset{1}{\sim} \left[\begin{array}{rr|r} 1 & -k & 6 \\ 0 & 3+2k & k-12 \end{array}\right].$$

If $k \neq -\frac{3}{2}$, then each column of the row-reduced coefficient matrix will contain a pivot, and hence, the linear system will have a unique solution. If, on the other hand, $k = -\frac{3}{2}$, then the system is inconsistent, because the last row of the row-echelon form will have a pivot in the right-most column. Under no circumstances will the linear system have infinitely many solutions.

28. First observe that if $k = 0$, then the second equation requires that $x_3 = 2$, and then the first equation requires $x_2 = 2$. However, x_1 is a free variable in this case, so there are infinitely many solutions.

Now suppose that $k \neq 0$. Then multiplying each row of the corresponding augmented matrix for the linear system by $1/k$ yields a row-echelon form with pivots in the first two columns only. Therefore, the third variable, x_3, is free in this case. So once again, there are infinitely many solutions to the system.

We conclude that the system has infinitely many solutions for all values of k.

29. Since this linear system is homogeneous, it already has at least one solution: $(0,0,0)$. Therefore, it only remains to determine the values of k for which this will be the only solution. We reduce the corresponding matrix as follows:

$$\left[\begin{array}{rrr|r} 10 & k & -1 & 0 \\ k & 1 & -1 & 0 \\ 2 & 1 & -1 & 0 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{rrr|r} 10k & k^2 & -k & 0 \\ 10k & 10 & -10 & 0 \\ 1 & 1/2 & -1/2 & 0 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{rrr|r} 1 & 1/2 & -1/2 & 0 \\ 10k & 10 & -10 & 0 \\ 10k & k^2 & -k & 0 \end{array}\right]$$

$$\overset{3}{\sim} \left[\begin{array}{rrr|r} 1 & 1/2 & -1/2 & 0 \\ 0 & 10-5k & 5k-10 & 0 \\ 0 & k^2-5k & 4k & 0 \end{array}\right] \overset{4}{\sim} \left[\begin{array}{rrr|r} 1 & 1/2 & -1/2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & k^2-5k & 4k & 0 \end{array}\right] \overset{5}{\sim} \left[\begin{array}{rrr|r} 1 & 1/2 & -1/2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & k^2-k & 0 \end{array}\right].$$

1. $M_1(k)$, $M_2(10)$, $M_3(1/2)$ **2.** P_{13} **3.** $A_{12}(-10k)$, $A_{13}(-10k)$ **4.** $M_2(\frac{1}{10-5k})$ **5.** $A_{23}(5k-k^2)$

Note that the steps above are not valid if $k = 0$ or $k = 2$ (because Step 1 is not valid with $k = 0$ and Step 4 is not valid if $k = 2$). We will discuss those special cases individually in a moment. However if $k \neq 0, 2$, then the steps are valid, and we see from the last row of the last matrix that if $k = 1$, we have infinitely many solutions. Otherwise, if $k \neq 0, 1, 2$, then the matrix has full rank, and so there is a unique solution to the linear system.

If $k = 2$, then the last two rows of the original matrix are the same, and so the matrix of coefficients of the linear system is not invertible. Therefore, the linear system must have infinitely many solutions.

If $k = 0$, we reduce the original linear system as follows:

$$\left[\begin{array}{ccc|c} 10 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 2 & 1 & -1 & 0 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -1/10 & 0 \\ 0 & 1 & -1 & 0 \\ 2 & 1 & -1 & 0 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -1/10 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -4/5 & 0 \end{array}\right] \overset{3}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -1/10 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1/5 & 0 \end{array}\right].$$

The last matrix has full rank, so there will be a unique solution in this case.

1. $M_1(1/10)$ **2.** $A_{13}(-2)$ **3.** $A_{23}(-1)$

To summarize: The linear system has infinitely many solutions if and only if $k = 1$ or $k = 2$. Otherwise, the system has a unique solution.

30. To solve this system, we need to reduce the corresponding augmented matrix for the linear system to row-echelon form. This gives us

$$\left[\begin{array}{ccc|c} 1 & -k & k^2 & 0 \\ 1 & 0 & k & 0 \\ 0 & 1 & -1 & 1 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccc|c} 1 & -k & k^2 & 0 \\ 0 & k & k-k^2 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{ccc|c} 1 & -k & k^2 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & k & k-k^2 & 0 \end{array}\right] \overset{3}{\sim} \left[\begin{array}{ccc|c} 1 & -k & k^2 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2k-k^2 & -k \end{array}\right].$$

1. $A_{12}(-1)$ **2.** P_{23} **3.** $A_{23}(-k)$

Now provided that $2k - k^2 \neq 0$, the system can be solved without free variables via back-substitution, and therefore, there is a unique solution. Consider now what happens if $2k - k^2 = 0$. Then either $k = 0$ or $k = 2$. If $k = 0$, then only the first two columns of the last augmented matrix above are pivoted, and we have a free variable corresponding to x_3. Therefore, there are infinitely many solutions in this case. On the other hand, if $k = 2$, then the last row of the last matrix above reflects an inconsistency in the linear system, and there are no solutions.

To summarize, the system has no solutions if $k = 2$, a unique solution if $k \neq 0$ and $k \neq 2$, and infinitely many solutions if $k = 0$.

31. No, there are no common points of intersection. A common point of intersection would be indicated by a solution to the linear system consisting of the equations of the three planes. However, the corresponding augmented matrix can be row-reduced as follows:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 1 & 3 & 0 & 0 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -4 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -5 \end{array}\right].$$

The last row of this matrix shows that the linear system is inconsistent, and so there are no points common to all three planes.

1. $A_{13}(-1)$ **2.** $A_{23}(-1)$

32.

(a). We have

$$\begin{bmatrix} 4 & 7 \\ -2 & 5 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & 7/4 \\ -2 & 5 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & 7/4 \\ 0 & 17/2 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & 7/4 \\ 0 & 1 \end{bmatrix}.$$

1. $M_1(1/4)$ **2.** $A_{12}(2)$ **3.** $M_2(2/17)$

(b). We have: $\text{rank}(A) = 2$, since the row-echelon form of A in (a) consists two nonzero rows.

(c). We have

$$\left[\begin{array}{cc|cc} 4 & 7 & 1 & 0 \\ -2 & 5 & 0 & 1 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{cc|cc} 1 & 7/4 & 1/4 & 0 \\ -2 & 5 & 0 & 1 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{cc|cc} 1 & 7/4 & 1/4 & 0 \\ 0 & 17/2 & 1/2 & 1 \end{array}\right] \overset{3}{\sim} \left[\begin{array}{cc|cc} 1 & 7/4 & 1/4 & 0 \\ 0 & 1 & 1/17 & 2/17 \end{array}\right]$$

$$\overset{4}{\sim} \left[\begin{array}{cc|cc} 1 & 0 & 5/34 & -7/34 \\ 0 & 1 & 1/17 & 2/17 \end{array}\right].$$

1. $M_1(1/4)$ **2.** $A_{12}(2)$ **3.** $M_2(2/17)$ **4.** $A_{21}(-7/4)$

Thus,

$$A^{-1} = \begin{bmatrix} \frac{5}{34} & -\frac{7}{34} \\ \frac{1}{17} & \frac{2}{17} \end{bmatrix}.$$

33.

(a). We have

$$\begin{bmatrix} 2 & -7 \\ -4 & 14 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 2 & -7 \\ 0 & 0 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & -7/2 \\ 0 & 0 \end{bmatrix}.$$

1. $A_{12}(2)$ **2.** $M_1(1/2)$

(b). We have: $\text{rank}(A) = 1$, since the row-echelon form of A in (a) has one nonzero row.

(c). Since $\text{rank}(A) < 2$, A is not invertible.

34.

(a). We have

$$\begin{bmatrix} 3 & -1 & 6 \\ 0 & 2 & 3 \\ 3 & -5 & 0 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & -1/3 & 2 \\ 0 & 2 & 3 \\ 1 & -5/3 & 0 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & -1/3 & 2 \\ 0 & 2 & 3 \\ 0 & -4/3 & -2 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & -1/3 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix} \overset{4}{\sim} \begin{bmatrix} 1 & -1/3 & 2 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \end{bmatrix}.$$

1. $M_1(1/3)$, $M_3(1/3)$ **2.** $A_{13}(-1)$ **3.** $A_{23}(2/3)$ **4.** $M_2(1/2)$

(b). We have: rank$(A) = 2$, since the row-echelon form of A in (a) consists of two nonzero rows.

(c). Since rank$(A) < 3$, A is not invertible.

35.

(a). We have

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 4 & 3 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 4 & 3 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 3 & 4 \end{bmatrix}$$

$$\overset{4}{\sim} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 7 \end{bmatrix} \overset{5}{\sim} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

1. P_{12} **2.** $\mathrm{A}_{12}(-2)$, $\mathrm{A}_{34}(-1)$ **3.** P_{34} **4.** $\mathrm{M}_2(-1/3)$, $\mathrm{A}_{34}(-3)$ **5.** $\mathrm{M}_4(1/7)$

(b). We have: rank$(A) = 4$, since the row-echelon form of A in (a) consists of four nonzero rows.

(c). We have

$$\left[\begin{array}{cccc|cccc} 2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 3 & 0 & 0 & 0 & 1 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{cccc|cccc} 1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 3 & 0 & 0 & 0 & 1 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{cccc|cccc} 1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -3 & 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 3 & 4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \end{array}\right]$$

$$\overset{3}{\sim} \left[\begin{array}{cccc|cccc} 1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -3 & 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 3 & 4 & 0 & 0 & 1 & 0 \end{array}\right] \overset{4}{\sim} \left[\begin{array}{cccc|cccc} 1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/3 & 2/3 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 7 & 0 & 0 & 4 & -3 \end{array}\right]$$

$$\overset{5}{\sim} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 2/3 & -1/3 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/3 & 2/3 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 4/7 & -3/7 \end{array}\right] \overset{6}{\sim} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 2/3 & -1/3 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/3 & 2/3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -3/7 & 4/7 \\ 0 & 0 & 0 & 1 & 0 & 0 & 4/7 & -3/7 \end{array}\right].$$

1. P_{12} **2.** $\mathrm{A}_{12}(-2)$, $\mathrm{A}_{34}(-1)$ **3.** P_{34} **4.** $\mathrm{A}_{34}(-3)$, $\mathrm{M}_2(-1/3)$ **5.** $\mathrm{M}_4(1/7)$, $\mathrm{A}_{21}(-2)$ **6.** $\mathrm{A}_{43}(1)$

Thus,

$$A^{-1} = \begin{bmatrix} 2/3 & -1/3 & 0 & 0 \\ -1/3 & 2/3 & 0 & 0 \\ 0 & 0 & -3/7 & 4/7 \\ 0 & 0 & 4/7 & -3/7 \end{bmatrix}.$$

36.

(a). We have

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 2 & -1 \end{bmatrix} \overset{4}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 3 \end{bmatrix} \overset{5}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

1. $M_1(1/3)$ **2.** $A_{13}(-1)$ **3.** P_{23} **4.** $A_{23}(2)$ **5.** $M_2(-1)$, $M_3(1/3)$

(b). We have: rank$(A) = 3$, since the row-echelon form of A in (a) has 3 nonzero rows.

(c). We have

$$\left[\begin{array}{ccc|ccc} 3 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 1 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 1 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & -1/3 & 0 & 1 \end{array}\right]$$

$$\overset{3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & -1 & 2 & -1/3 & 0 & 1 \\ 0 & 2 & -1 & 0 & 1 & 0 \end{array}\right] \overset{4}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & -1 & 2 & -1/3 & 0 & 1 \\ 0 & 0 & 3 & -2/3 & 1 & 2 \end{array}\right]$$

$$\overset{5}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 1 & -2 & 1/3 & 0 & -1 \\ 0 & 0 & 1 & -2/9 & 1/3 & 2/3 \end{array}\right] \overset{6}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 1 & 0 & -1/9 & 2/3 & 1/3 \\ 0 & 0 & 1 & -2/9 & 1/3 & 2/3 \end{array}\right].$$

1. $M_1(1/3)$ **2.** $A_{13}(-1)$ **3.** P_{23} **4.** $A_{23}(2)$ **5.** $M_2(-1)$, $M_3(1/3)$ **6.** $A_{32}(2)$

Hence,

$$A^{-1} = \begin{bmatrix} 1/3 & 0 & 0 \\ -1/9 & 2/3 & 1/3 \\ -2/9 & 1/3 & 2/3 \end{bmatrix}.$$

37.

(a). We have

$$\begin{bmatrix} -2 & -3 & 1 \\ 1 & 4 & 2 \\ 0 & 5 & 3 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & 4 & 2 \\ -2 & -3 & 1 \\ 0 & 5 & 3 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & 4 & 2 \\ 0 & 5 & 5 \\ 0 & 5 & 3 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & 4 & 2 \\ 0 & 5 & 5 \\ 0 & 0 & -2 \end{bmatrix} \overset{4}{\sim} \begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

1. P_{12} **2.** $A_{12}(2)$ **3.** $A_{23}(-1)$ **4.** $M_2(1/5)$, $M_3(-1/2)$

(b). We have: rank$(A) = 3$, since the row-echelon form of A in (a) consists of 3 nonzero rows.

(c). We have

$$\left[\begin{array}{ccc|ccc} -2 & -3 & 1 & 1 & 0 & 0 \\ 1 & 4 & 2 & 0 & 1 & 0 \\ 0 & 5 & 3 & 0 & 0 & 1 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccc|ccc} 1 & 4 & 2 & 0 & 1 & 0 \\ -2 & -3 & 1 & 1 & 0 & 0 \\ 0 & 5 & 3 & 0 & 0 & 1 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 4 & 2 & 0 & 1 & 0 \\ 0 & 5 & 5 & 1 & 2 & 0 \\ 0 & 5 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\overset{3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 4 & 2 & 0 & 1 & 0 \\ 0 & 5 & 5 & 1 & 2 & 0 \\ 0 & 0 & -2 & -1 & -2 & 1 \end{array}\right] \overset{4}{\sim} \left[\begin{array}{ccc|ccc} 1 & 4 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1/5 & 2/5 & 0 \\ 0 & 0 & 1 & 1/2 & 1 & -1/2 \end{array}\right]$$

$$\overset{5}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & -4/5 & -3/5 & 0 \\ 0 & 1 & 1 & 1/5 & 2/5 & 0 \\ 0 & 0 & 1 & 1/2 & 1 & -1/2 \end{array}\right] \overset{6}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/5 & 7/5 & -1 \\ 0 & 1 & 0 & -3/10 & -3/5 & 1/2 \\ 0 & 0 & 1 & 1/2 & 1 & -1/2 \end{array}\right].$$

1. P_{12}	**2.** $A_{12}(2)$	**3.** $A_{23}(-1)$	**4.** $M_2(1/5)$, $M_3(-1/2)$	**5.** $A_{21}(-4)$	**6.** $A_{31}(2)$, $A_{32}(-1)$

Thus,

$$A^{-1} = \begin{bmatrix} 1/5 & 7/5 & -1 \\ -3/10 & -3/5 & 1/2 \\ 1/2 & 1 & -1/2 \end{bmatrix}.$$

38. We use the Gauss-Jordan method to find A^{-1}:

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 4 & -3 & 13 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & -4 & 1 & 0 \\ 0 & 2 & 1 & -1 & 0 & 1 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & -4 & 1 & 0 \\ 0 & 0 & -1 & 7 & -2 & 1 \end{array}\right]$$

$$\overset{3}{\sim} \left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & -4 & 1 & 0 \\ 0 & 0 & 1 & -7 & 2 & -1 \end{array}\right] \overset{4}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & -3 & 1 & 0 \\ 0 & 1 & 1 & -4 & 1 & 0 \\ 0 & 0 & 1 & -7 & 2 & -1 \end{array}\right] \overset{5}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 25 & -7 & 4 \\ 0 & 1 & 0 & 3 & -1 & 1 \\ 0 & 0 & 1 & -7 & 2 & -1 \end{array}\right].$$

1. $A_{12}(-4)$, $A_{13}(-1)$	**2.** $A_{23}(-2)$	**3.** $M_3(-1)$	**4.** $A_{21}(1)$	**5.** $A_{31}(-4)$, $A_{32}(-1)$

Thus,

$$A^{-1} = \begin{bmatrix} 25 & -7 & 4 \\ 3 & -1 & 1 \\ -7 & 2 & -1 \end{bmatrix}.$$

Now $\mathbf{x}_i = A^{-1}\mathbf{e}_i$ for each i. So

$$\mathbf{x}_1 = A^{-1}\mathbf{e}_1 = \begin{bmatrix} 25 \\ 3 \\ -7 \end{bmatrix}, \quad \mathbf{x}_2 = A^{-1}\mathbf{e}_2 = \begin{bmatrix} -7 \\ -1 \\ 2 \end{bmatrix}, \quad \mathbf{x}_3 = A^{-1}\mathbf{e}_3 = \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}.$$

39. We have $\mathbf{x}_i = A^{-1}\mathbf{b}_i$, where

$$A^{-1} = -\frac{1}{39}\begin{bmatrix} -2 & -5 \\ -7 & 2 \end{bmatrix}.$$

Therefore,

$$\mathbf{x}_1 = A^{-1}\mathbf{b}_1 = -\frac{1}{39}\begin{bmatrix} -2 & -5 \\ -7 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = -\frac{1}{39}\begin{bmatrix} -12 \\ -3 \end{bmatrix} = \frac{1}{39}\begin{bmatrix} 12 \\ 3 \end{bmatrix} = \frac{1}{13}\begin{bmatrix} 4 \\ 1 \end{bmatrix},$$

$$\mathbf{x}_2 = A^{-1}\mathbf{b}_2 = -\frac{1}{39}\begin{bmatrix} -2 & -5 \\ -7 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = -\frac{1}{39}\begin{bmatrix} -23 \\ -22 \end{bmatrix} = \frac{1}{39}\begin{bmatrix} 23 \\ 22 \end{bmatrix},$$

and

$$\mathbf{x}_3 = A^{-1}\mathbf{b}_3 = -\frac{1}{39}\begin{bmatrix} -2 & -5 \\ -7 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ 5 \end{bmatrix} = -\frac{1}{39}\begin{bmatrix} -21 \\ 24 \end{bmatrix} = \frac{1}{39}\begin{bmatrix} 21 \\ -24 \end{bmatrix} = \frac{1}{13}\begin{bmatrix} 7 \\ -8 \end{bmatrix}.$$

40.

(a). We have

$$(A^{-1}B)(B^{-1}A) = A^{-1}(BB^{-1})A = A^{-1}I_nA = A^{-1}A = I_n$$

and

$$(B^{-1}A)(A^{-1}B) = B^{-1}(AA^{-1})B = B^{-1}I_nB = B^{-1}B = I_n.$$

Therefore,

$$(B^{-1}A)^{-1} = A^{-1}B.$$

(b). We have

$$(A^{-1}B)^{-1} = B^{-1}(A^{-1})^{-1} = B^{-1}A,$$

as required.

41(a). We have $B^4 = (S^{-1}AS)(S^{-1}AS)(S^{-1}AS)(S^{-1}AS) = S^{-1}A(SS^{-1})A(SS^{-1})A(SS^{-1})AS = S^{-1}AIAIAIAS = S^{-1}A^4S$, as required.

41(b). We can prove this by induction on k. For $k = 1$, the result is $B = S^{-1}AS$, which was already given. Now assume that $B^k = S^{-1}A^kS$. Then $B^{k+1} = BB^k = S^{-1}AS(S^{-1}A^kS) = S^{-1}A(SS^{-1})A^kS = S^{-1}AIA^kS = S^{-1}A^{k+1}S$, which completes the induction step.

42.

(a). We reduce A to the identity matrix:

$$\begin{bmatrix} 4 & 7 \\ -2 & 5 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & \frac{7}{4} \\ -2 & 5 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & \frac{7}{4} \\ 0 & \frac{17}{2} \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & \frac{7}{4} \\ 0 & 1 \end{bmatrix} \overset{4}{\sim} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

1. $M_1(\frac{1}{4})$ **2.** $A_{12}(2)$ **3.** $M_2(\frac{2}{17})$ **4.** $A_{21}(-\frac{7}{4})$

The elementary matrices corresponding to these row operations are

$$E_1 = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{2}{17} \end{bmatrix}, \quad E_4 = \begin{bmatrix} 1 & -\frac{7}{4} \\ 0 & 1 \end{bmatrix}.$$

We have $E_4E_3E_2E_1A = I_2$, so that

$$A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{17}{2} \end{bmatrix} \begin{bmatrix} 1 & \frac{7}{4} \\ 0 & 1 \end{bmatrix},$$

which is the desired expression since E_i^{-1} is an elementary matrix for each i.

(b). We can reduce A to upper triangular form by the following elementary row operation:

$$\begin{bmatrix} 4 & 7 \\ -2 & 5 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 4 & 7 \\ 0 & \frac{17}{2} \end{bmatrix}.$$

1. $A_{12}(\frac{1}{2})$

Therefore we have the multiplier $m_{12} = -\frac{1}{2}$. Hence, setting

$$L = \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 4 & 7 \\ 0 & \frac{17}{2} \end{bmatrix},$$

we have the LU factorization $A = LU$, which can be easily verified by direct multiplication.

43.

(a). We reduce A to the identity matrix:

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 4 & 3 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 4 & 3 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 4 & 3 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 4 & 3 \end{bmatrix} \overset{4}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 4 & 3 \end{bmatrix}$$

$$\overset{5}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{4}{3} \\ 0 & 0 & 4 & 3 \end{bmatrix} \overset{6}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{4}{3} \\ 0 & 0 & 0 & -\frac{7}{3} \end{bmatrix} \overset{7}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{4}{3} \\ 0 & 0 & 0 & 1 \end{bmatrix} \overset{8}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

1. P_{12} **2.** $A_{12}(-2)$ **3.** $M_2(-\frac{1}{3})$ **4.** $A_{21}(-2)$ **5.** $M_3(\frac{1}{3})$
6. $A_{34}(-4)$ **7.** $M_4(-\frac{3}{7})$ **8.** $A_{43}(-\frac{4}{3})$

The elementary matrices corresponding to these row operations are

$$E_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$E_5 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E_6 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -4 & 1 \end{bmatrix}, \quad E_7 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{3}{7} \end{bmatrix}, \quad E_8 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{4}{3} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We have

$$E_8E_7E_6E_5E_4E_3E_2E_1A = I_4$$

so that

$$\begin{aligned} A &= E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}E_6^{-1}E_7^{-1}E_8^{-1} \\ &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdots \\ &\quad \cdots \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{7}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{4}{3} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \end{aligned}$$

which is the desired expression since E_i^{-1} is an elementary matrix for each i.

(b). We can reduce A to upper triangular form by the following elementary row operations:

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 4 & 3 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 4 & 3 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & -\frac{7}{3} \end{bmatrix}.$$

1. $A_{12}(-\frac{1}{2})$ 2. $A_{34}(-\frac{4}{3})$

Therefore, the nonzero multipliers are $m_{12} = \frac{1}{2}$ and $m_{34} = \frac{4}{3}$. Hence, setting

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{4}{3} & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & -\frac{7}{3} \end{bmatrix},$$

we have the LU factorization $A = LU$, which can be easily verified by direct multiplication.

44.

(a). We reduce A to the identity matrix:

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -1 \\ 3 & 0 & 0 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -1 \\ 0 & 3 & -6 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 3 & -6 \end{bmatrix} \overset{4}{\sim} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & -\frac{9}{2} \end{bmatrix}$$

$$\overset{5}{\sim} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \overset{6}{\sim} \begin{bmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \overset{7}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \overset{8}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

1. P_{13} 2. $A_{13}(-3)$ 3. $M_2(\frac{1}{2})$ 4. $A_{23}(-3)$ 5. $M_3(-\frac{2}{9})$ 6. $A_{21}(1)$ 7. $A_{31}(-\frac{3}{2})$ 8. $A_{32}(\frac{1}{2})$

The elementary matrices corresponding to these row operations are

$$E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

$$E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{2}{9} \end{bmatrix}, \quad E_6 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_7 = \begin{bmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_8 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}.$$

We have

$$E_8E_7E_6E_5E_4E_3E_2E_1A = I_3$$

so that

$$\begin{aligned} A &= E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}E_6^{-1}E_7^{-1}E_8^{-1} \\ &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \cdots \\ &\cdots \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{9}{2} \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}, \end{aligned}$$

which is the desired expression since E_i^{-1} is an elementary matrix for each i.

(b). We can reduce A to upper triangular form by the following elementary row operations:

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & \frac{3}{2} \end{bmatrix}.$$

1. $A_{13}(-\frac{1}{3})$ **2.** $A_{23}(\frac{1}{2})$

Therefore, the nonzero multipliers are $m_{13} = \frac{1}{3}$ and $m_{23} = -\frac{1}{2}$. Hence, setting

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{3} & -\frac{1}{2} & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & \frac{3}{2} \end{bmatrix},$$

we have the LU factorization $A = LU$, which can be verified by direct multiplication.

45.

(a). We reduce A to the identity matrix:

$$\begin{bmatrix} -2 & -3 & 1 \\ 1 & 4 & 2 \\ 0 & 5 & 3 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & 4 & 2 \\ -2 & -3 & 1 \\ 0 & 5 & 3 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & 4 & 2 \\ 0 & 5 & 5 \\ 0 & 5 & -3 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & 4 & 2 \\ 0 & 5 & 5 \\ 0 & 1 & -8 \end{bmatrix} \overset{4}{\sim} \begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & -8 \\ 0 & 5 & 5 \end{bmatrix}$$

$$\overset{5}{\sim} \begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & -8 \\ 0 & 0 & 45 \end{bmatrix} \overset{6}{\sim} \begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & -8 \\ 0 & 0 & 1 \end{bmatrix} \overset{7}{\sim} \begin{bmatrix} 1 & 0 & 34 \\ 0 & 1 & -8 \\ 0 & 0 & 1 \end{bmatrix} \overset{8}{\sim} \begin{bmatrix} 1 & 0 & 34 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \overset{9}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

1. P_{12} **2.** $A_{12}(2)$ **3.** $A_{23}(-1)$ **4.** P_{23} **5.** $A_{23}(-5)$ **6.** $M_3(\frac{1}{45})$ **7.** $A_{21}(-4)$ **8.** $A_{32}(8)$ **9.** $A_{31}(-34)$

The elementary matrices corresponding to these row operations are

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix},$$

$$E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix}, \quad E_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{45} \end{bmatrix},$$

$$E_7 = \begin{bmatrix} 1 & -4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_8 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 8 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_9 = \begin{bmatrix} 1 & 0 & -34 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We have

$$E_9E_8E_7E_6E_5E_4E_3E_2E_1A = I_3$$

so that

$$\begin{aligned} A &= E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}E_6^{-1}E_7^{-1}E_8^{-1}E_9^{-1} \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdots \\ &\quad \cdots \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 45 \end{bmatrix} \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -8 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 34 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \end{aligned}$$

which is the desired expression since E_i^{-1} is an elementary matrix for each i.

(b). We can reduce A to upper triangular form by the following elementary row operations:

$$\begin{bmatrix} -2 & -3 & 1 \\ 1 & 4 & 2 \\ 0 & 5 & 3 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} -2 & -3 & 1 \\ 0 & \frac{5}{2} & \frac{5}{2} \\ 0 & 5 & 3 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} -2 & -3 & 1 \\ 0 & \frac{5}{2} & \frac{5}{2} \\ 0 & 0 & -2 \end{bmatrix}.$$

Therefore, the nonzero multipliers are $m_{12} = -\frac{1}{2}$ and $m_{23} = 2$. Hence, setting

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} -2 & -3 & 1 \\ 0 & \frac{5}{2} & \frac{5}{2} \\ 0 & 0 & -2 \end{bmatrix},$$

we have the LU factorization $A = LU$, which can be verified by direct multiplication.

46(a). Using the distributive laws of matrix multiplication, first note that

$$(A+2B)^2 = (A+2B)(A+2B) = A(A+2B)+2B(A+2B) = A^2+A(2B)+(2B)A+(2B)^2 = A^2+2AB+2BA+4B^2.$$

Thus, we have

$$\begin{aligned} (A+2B)^3 &= (A+2B)(A+2B)^2 \\ &= A(A+2B)^2 + 2B(A+2B)^2 \\ &= A(A^2+2AB+2BA+4B^2) + 2B(A^2+2AB+2BA+4B^2) \\ &= A^3 + 2A^2B + 2ABA + 4AB^2 + 2BA^2 + 4BAB + 4B^2A + 8B^3, \end{aligned}$$

as needed.

46(b). Each occurrence of B in the answer to part (a) must now be accompanied by a minus sign. Therefore, all terms containing an odd number of Bs will experience a sign change. The answer is

$$(A-2B)^3 = A^3 - 2A^2B - 2ABA - 2BA^2 + 4AB^2 + 4BAB + 4B^2A - 8B^3.$$

47. The answer is 2^k, because each term in the expansion of $(A+B)^k$ consists of a string of k matrices, each of which is either A or B (2 possibilities for each matrix in the string). Multiplying the possibilities for each position in the string of length k, we get 2^k different strings, and hence 2^k different terms in the expansion of $(A+B)^k$. So, for instance, if $k=4$, we expect 16 terms, corresponding to the 16 strings $AAAA$, $AAAB$, $AABA$, $ABAA$, $BAAA$, $AABB$, $ABAB$, $ABBA$, $BAAB$, $BABA$, $BBAA$, $ABBB$, $BABB$, $BBAB$, $BBBA$, and $BBBB$. Indeed, one can verify that the expansion of $(A+B)^4$ is precisely the sum of the 16 terms we just wrote down.

48. We claim that

$$\begin{pmatrix} A & 0 \\ 0 & B^{-1} \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & 0 \\ 0 & B \end{pmatrix}.$$

To see this, simply note that

$$\begin{pmatrix} A & 0 \\ 0 & B^{-1} \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ 0 & I_m \end{pmatrix} = I_{n+m}$$

and

$$\begin{pmatrix} A^{-1} & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & B^{-1} \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ 0 & I_m \end{pmatrix} = I_{n+m}.$$

49. For a 2×4 matrix, the leading ones can occur in 6 different positions:

$$\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \end{bmatrix}, \begin{bmatrix} 1 & * & * & * \\ 0 & 0 & 1 & * \end{bmatrix}, \begin{bmatrix} 1 & * & * & * \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{bmatrix}, \begin{bmatrix} 0 & 1 & * & * \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For a 3×4 matrix, the leading ones can occur in 4 different positions:

$$\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{bmatrix}, \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & * & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For a 4×6 matrix, the leading ones can occur in 15 different positions:

$$\begin{bmatrix} 1 & * & * & * & * & * \\ 0 & 1 & * & * & * & * \\ 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 1 & * & * \end{bmatrix}, \begin{bmatrix} 1 & * & * & * & * & * \\ 0 & 1 & * & * & * & * \\ 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}, \begin{bmatrix} 1 & * & * & * & * & * \\ 0 & 1 & * & * & * & * \\ 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & * & * & * & * & * \\ 0 & 1 & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 1 & * \end{bmatrix},$$

$$\begin{bmatrix} 1 & * & * & * & * & * \\ 0 & 1 & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & * & * & * & * & * \\ 0 & 1 & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & * & * & * & * & * \\ 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}, \begin{bmatrix} 1 & * & * & * & * & * \\ 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & * & * & * & * & * \\ 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & * & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * & * & * & * \\ 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}, \begin{bmatrix} 0 & 1 & * & * & * & * \\ 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 & * & * & * & * \\ 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

For an $m \times n$ matrix with $m \leq n$, the answer is the binomial coefficient

$$C(n, m) = \binom{n}{m} = \frac{n!}{m!(n-m)!}.$$

This represents n "choose" m, which is the number of ways to choose m columns from the n columns of the matrix in which to put the leading ones. This choice then determines the structure of the matrix.

50. We claim that the inverse of A^{10} is B^5. To prove this, use the fact that $A^2B = I$ to observe that

$$A^{10}B^5 = A^2A^2A^2A^2(A^2B)BBBB = A^2A^2A^2A^2IBBBB = A^2A^2A^2(A^2B)BBB$$
$$= A^2A^2A^2IBBB = A^2A^2(A^2B)BB = A^2A^2IBB = A^2(A^2B)B = A^2IB = A^2B = I,$$

as required.

51. We claim that the inverse of A^9 is B^6. To prove this, use the fact that $A^3B^2 = I$ to observe that

$$A^9B^6 = A^3A^3(A^3B^2)B^2B^2 = A^3A^3IB^2B^2 = A^3(A^3B^2)B^2 = A^3IB^2 = A^3B^2 = I,$$

as required.