Communication Networks

An Optimization, Control, and Stochastic Networks Perspective

Solution Manual

R. Srikant & Lei Ying

CHAPTER

2

where $B_l(y)$ is the cost of sending data at rate y on link l. The primal congestion control algorithm is a gradient ascent algorithm for solving the relaxed optimization problem:

$$\dot{x}_r = k_r(x_r)\left(U_r'(x_r) - \sum_{l:l\in r} f_l\left(\sum_{s:l\in s} x_s\right)\right),$$

where $f_l(y) = B_l'(y)$ can be regarded as the congestion price on link l.

- **Dual congestion control algorithm**: The dual solution is to consider the dual of the network utility maximization problem:

$$\min_{p\geq 0}\max_{\{x_r\geq 0\}} \sum_r U_r(x_r) - \sum_l p_l\left(\sum_{s:l\in s} x_s - c_l\right),$$

where the p_ls are the Lagrange multipliers. The following dual congestion control algorithm is the gradient descent solution of the dual problem:

$$x_r = U_r'^{-1}(q_r) \quad \text{and} \quad \dot{p}_l = h_l(\sum_{s:l\in s} x_s - c_l)^+_{p_l}.$$

- **The Vickrey-Clarke-Groves (VCG) mechanism**: The VCG mechanism is a pricing scheme to ensure that users do not have an incentive to lie about their true utility functions. Suppose that user r reveals its utility function as $\tilde{U}_r(\cdot)$, which may or may not be its true utility function $U_r(\cdot)$. The network planner allocates the optimal solution of $\max_{x\geq 0}\sum_r \tilde{U}_r(x_r)$ as the rates to the users. Then, it charges user r a price

$$q_r = \sum_{s\neq r}\tilde{U}(\bar{x}_s) - \sum_{s\neq r}\tilde{U}(\tilde{x}_s),$$

where $\tilde{x}_s$ is the optimal solution to $\max_{x\geq 0}\sum_{s:s\neq r}\tilde{U}_s(x_s)$. The price represents the decrease in the sum utility of other users due to the presence of user r. It can be shown that an optimal strategy for each user to maximize its payoff is to reveal its true utility function.

2.9 Problems

Exercise 2.1 (Bottleneck links and max-min fair rate allocation) Let x_r be the rate allocated to user r in a network where users' routes are fixed. Link l is called a bottleneck link for user r if $l \in r$, and

$$y_l = c_l \quad \text{and} \quad x_s \leq x_r \quad \forall s \text{ such that } \; l \in s,$$

i.e., link l is fully utilized and user r has the highest transmission rate among all users using link l. Show that $\{x_r\}$ is a max-min fair rate allocation *if* and *only if* every source has at least one bottleneck link.

Solution $\Rightarrow$: proof by contradiction

Assume we have max-min fairness allocation $\{\hat{x}_r\}$. Assume that there exists a user r that does not have a bottleneck link. Thus either $y_l < c_l$ $\forall l \in r$, or for all link $l \in r$ such that $y_l = c_l$, $\exists s$

such that $l \in s$ and $\hat{x}_s > \hat{x}_r$. In either cases, we can increase x_r by a small amount $\epsilon > 0$ either without changing the rates for any other sources (first case) or by decreasing the rates of only those users s such that they share a link with r and have $\hat{x}_s > \hat{x}_r$ (second case). Thus, $\{\hat{x}_r\}$ cannot be a max-min fair allocation.

$\Leftarrow$: Let $\hat{x}_r$ be an allocation such that each user has at least one bottleneck link. Thus, every user r has a link l such that $y_l = c_l$ and $\hat{x}_s \leq \hat{x}_r$ $\forall s$ s.t. $s \in l$. We increase the rate $\hat{x}_r$ and we look at the effect on its bottleneck link. There will be a user s s.t. $\hat{x}_s = \hat{x}_r$ and $x_s < \hat{x}_s$, where x_s is the new rate for user s. This is by definition max-min allocation.

□

Exercise 2.2 (A max-min fair resource allocation algorithm) Show that the allocation $\{x_r\}$ obtained from the algorithm below is a max-min fair allocation:

1. Let $\mathcal{S}^0$ be the set of all sources in the network, and $c_l^0 = c_l$, i.e., the capacity of link l.

2. Set $k = 0$.

3. Let $\mathcal{S}_l^k \subseteq \mathcal{S}^k$ be the set of sources whose routes include link l, and $|\mathcal{S}_l^k|$ be the cardinality of this set. Define $f_l^k = c_l^k / |\mathcal{S}_l^k|$, which is called the fair share on link l at the k^{th} iteration.

4. For each source $r \in \mathcal{S}^k$, let $z_r^i = \min_{l:l\in r} f_l^i$, which is the minimum of the fair shares on its route.

5. Let $\mathcal{T}^k$ be the set of sources such that $z_r^k = \min_{s\in\mathcal{S}^k} z_s^k$, and set $x_r = z_r^k$ $\forall r \in \mathcal{T}^k$. The sources in $\mathcal{T}^k$ are permanently allocated rate z_r^k.

6. Set $\mathcal{S}^{k+1} = \mathcal{S}^k \setminus \mathcal{T}^k$, and
$$c_l^{k+1} = c_l^k - \sum_{r:l\in r \text{and } \in\mathcal{T}^k} z_r^k$$
for all l. In other words, sources whose rate allocations are finalized are removed from from the set of sources under consideration and the capacity of each link is reduced by the total rate allocated to such sources.

7. Go to step (3).

Hint: Use the result in Exercise 2.1 above.

Solution First, we note the following simple fact. Consider a link with capacity c which is shared by n users. The fair share on this link is C/n. Now, suppose that users 1 through m (where $m < n$) are allocated rates x_1 through x_m, respectively such that each $x_i \leq C/n$. Next, remove these users from the network, and reduce the link capacity to $C - \sum_{i=1}^m x_i$. The new fair share for the remaining users is
$$\frac{C - \sum_{i=1}^m x_i}{n - m},$$
which is easily see to be greater than or equal to C/n. Thus, removing some users after allocating less than the fair share to them clearly increases the fair share for the remaining users. We will use this fact in the solution to this problem.

Each iteration in the algorithm serves the network users with the lowest fair share. At the end of the first iteration, users who receive the smallest fair share are assigned these rates and removed from the network, with the link capacities appropriately diminished. Thus, some links will have zero remaining capacity, and the users using these links are bottlenecked at these links.

Every subsequent iteration fixes the allocations of the users with the next lowest rates. Due to the fact mentioned in the first paragraph, these rates will be greater than or equal to the rates allocated to the users removed from the network in the previous iterations. Thus for ever whose rate is fixed at the current iteration has a bottleneck, since every user r whose rate is fixed in the current iteration because their fair share is the smallest in the network and is achieved at link l will have $x_s \leq x_r \ \forall \ s : l \in s$ and $y_l = c_l$. So, every user will have a bottleneck. The result then follows from Exercise 2.1.

□

Exercise 2.3 (Different notions of fairness in a simple network) Consider a two-link, three-source network as shown in Figure 2.10. Link A has a capacity of 2 (packets/time-slot) and link B has a capacity of 1 packet/time-slot. The route of source 0 consists of both links A and B, the route of source 1 consists of only link A, and the route of source 2 consists of only link B. Compute the resource allocations under the proportional fairness, minimum potential delay fairness, and max-min fairness.

Hint: For the max-min fair rate allocation, consider the algorithm in Exercise 2.2; and for the other two resource allocations, use Lagrange multipliers and the KKT theorem.

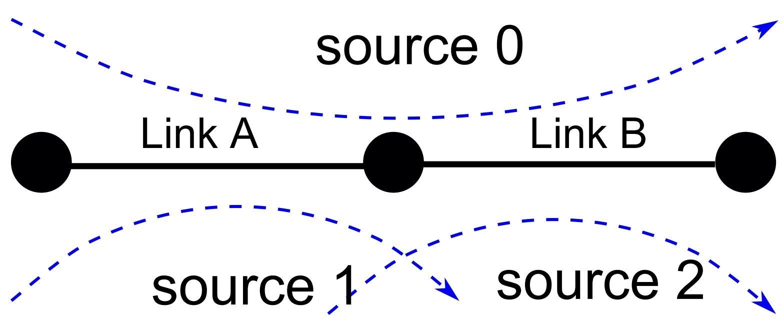

Figure 2.10: A two-link, three-source network

Solution The optimal solutions satisfy

$$x_0^* + x_1^* = 2 \quad \text{and} \quad x_0^* + x_2^* = 1. \tag{2.63}$$

If not, e.g., assuming that $x_0^* + x_1^* < 2$, we can increase x_1^* to increase $\sum_i U_i(x_i^*)$ without violating the capacity constraints, which contradicts with the fact that x^* is the optimal solution.

proportional fairness: Using the KKT condition, there exist λ_1^* and λ_2^* such that the optimal solution satisfies

$$\frac{1}{x_0^*} = \lambda_1^* + \lambda_2^*$$
$$\frac{1}{x_1^*} = \lambda_1^*$$
$$\frac{1}{x_2^*} = \lambda_2^*.$$

Substituting the equalities above into (2.63), we obtain

$$\frac{1}{\lambda_1^* + \lambda_2^*} + \frac{1}{\lambda_1^*} = 2$$
$$\frac{1}{\lambda_1^* + \lambda_2^*} + \frac{1}{\lambda_2^*} = 1.$$

Solving the equations above, we get

$$\lambda_1^* = \frac{3-\sqrt{3}}{2}$$
$$\lambda_2^* = \sqrt{3},$$

and

$$\begin{aligned} x_0^* &= \frac{3-\sqrt{3}}{3} \\ x_1^* &= \frac{3+\sqrt{3}}{3} \\ x_2^* &= \frac{\sqrt{3}}{3}. \end{aligned}$$

minimum potential delay fairness: Using the KKT condition, there exist λ_1^* and λ_2^* such that the optimal solution satisfies

$$\frac{1}{(x_0^*)^2} = \lambda_1^* + \lambda_2^*$$
$$\frac{1}{(x_1^*)^2} = \lambda_1^*$$
$$\frac{1}{(x_2^*)^2} = \lambda_2^*.$$

Substituting the equalities above into (2.63), we obtain

$$\frac{1}{\sqrt{\lambda_1^* + \lambda_2^*}} + \frac{1}{\sqrt{\lambda_1^*}} = 2$$
$$\frac{1}{\sqrt{\lambda_1^* + \lambda_2^*}} + \frac{1}{\sqrt{\lambda_2^*}} = 1.$$

Solving the equations above, we get

$$\lambda_1^* = 0.44$$
$$\lambda_2^* = 3.79,$$

and

$$\begin{aligned} x_0^* &= 0.4863 \\ x_1^* &= 1.5077 \\ x_2^* &= 0.5137. \end{aligned}$$

max-min fairness: $x_0^* = 0.5$, $x_1^* = 1.5$ and $x_2^* = 1$.

□

Exercise 2.4 (NUM in a simple network) Consider again the same two-link, three-user network shown in Figure 2.10. Now assume that the link capacities are $C_A = C_B = 1$. Suppose that the utility functions of the users are given as follows:

$$U_0(x_0) = \log(x_0)$$
$$U_1(x_1) = \log(1 + x_1)$$
$$U_2(x_2) = \log(1 + x_2)$$

Compute the data transmission rates of the three users, x_0, x_1, and x_2, which maximize the sum network utility.

Solution

$$\begin{aligned} \max_{x \geq 0} \quad & \log x_0 + \log(1 + x_1) + \log(1 + x_2) \\ \text{s.t.} \quad & x_0 + x_1 \leq 1 \\ & x_0 + x_2 \leq 1 \end{aligned}$$

Let p_A and p_B denote the Lagrange multipliers corresponding to the capacity constraints of the links A and B respectively. Then the Lagrangian is

$$\max_{x \geq 0} \quad \log x_0 + \log(1 + x_1) + \log(1 + x_2) - p_A(x_0 + x_1 - 1) - p_B(x_0 + x_2 - 1)$$

Differentiating w.r.t x_0, x_1, x_2, p_A, p_B, and equating to zero, we get

$$x_0 = \frac{1}{p_A + p_B}$$
$$x_1 = \left(\frac{1}{p_A} - 1\right)^+$$
$$x_2 = \left(\frac{1}{p_B} - 1\right)^+$$
$$x_0 + x_1 = 1$$
$$x_0 + x_2 = 1$$

From these, we get

$$\frac{1}{p_A + p_B} + \frac{1}{p_A} - 1 = 1$$
$$\frac{1}{p_A + p_B} + \frac{1}{p_B} - 1 = 1$$

Thus, we have $p_A = p_B$. So,

$$\begin{aligned}\frac{1}{2p_A} + \frac{1}{p_A} &= 2\\ \frac{3}{2p_A} &= 2\\ p_A = p_B &= \frac{3}{4}\\ x_1 = \frac{4}{3} - 1 &= \frac{1}{3}\\ x_2 &= \frac{1}{3}\\ x_0 &= \frac{2}{3}\end{aligned}$$

□

Exercise 2.5 (The utility function of a primal congestion controller) Consider the following primal congestion control algorithm.

$$\dot{x_r} = k_r\left[(1 - q_r) - q_r x_r\right],$$

where q_r is the sum of the link prices on route r, x_r is the transmission rate of user r and $k_r > 0$ is some constant. Identify the utility function of user r.

Hint: Recall the form of the primal congestion control algorithm, and compare it to the above differential equation.

Solution

$$\begin{aligned}\dot{x}_r &= k_r(1 - q_r(1 + x_r))\\ &= (1 + x_r)k_r\left[\frac{1}{1 + x_r} - q_r\right]\\ U_r'(x_r) &= \frac{1}{1 + x_r}\end{aligned}$$

This means that $U_r(x_r) = \log(1 + x_r) +$ constant.

□

Exercise 2.6 (An alternative proof of the stability of the primal algorithm) Consider the primal congestion controller with $\kappa_r(x) = 1\ \forall x$ and $\forall r$. Using the Lyapunov function

$$\sum_r (x_r - \hat{x}_r)^2,$$

to prove that the controller is globally, asymptotically stable, where $\hat{x}$ is the global maximizer of $W(x) = \sum_r U_r(x_r) - \sum_l B_l(y_l)$.

Hint: Since $\hat{x}$ is the global maximizer, it has the properties presented in Result 2.1.9.

Solution First note that

$$\tilde{W}(x) \triangleq \sum_r (x_r - \hat{x}_r)^2 \begin{cases} > 0, & \forall x \neq \hat{x}; \\ = 0, & x = \hat{x}. \end{cases} \tag{2.64}$$

Recall that $W(x) = \sum_r U_r(x_r) - \sum_l B_l(y_l)$ and $W(\hat{x}) = \max_x W(x)$. Further, since $W(x)$ is concave, we know that:

$$W(\hat{x}) \leq W(x) + \nabla W(x)(\hat{x} - x),$$

which implies that

$$0 \leq W(\hat{x}) - W(x) \leq \nabla W(x)(\hat{x} - x), \tag{2.65}$$

where equality holds only at $x = \hat{x}$. Since $\dot{x}_r = \frac{\partial W}{\partial x_r}$, we get that

$$\dot{\tilde{W}} = \sum_r 2\dot{x}_r(x_r - \hat{x_r}) = -2\nabla W^T(x)(\hat{x} - x) \leq 0,$$

where the equality holds if and only if $x = \hat{x}$. This shows that the controller is globally asymptotically stable.

□

Exercise 2.7 (The primal congestion controller with non-negligible p_l) Assume link prices $p_l(y_l) \in [0, 1]$ and $q_r = 1 - \prod_{l:l \in r}(1 - p_l)$. For example, if p_l is the probability that a packet is *marked* on link l, then q_r is the probability that a packet is marked on route r. In this exercise, you will be asked to prove that the primal congestion controller is globally, asymptotically stable under this model, without the assumption that p_l's are really small.

1. First, show that the primal congestion controller in this case can be rewritten as

$$\dot{x}_r = k_r(x_r) \left(\prod_{l:l \in r} (1 - p_l) - (1 - U_r'(x_r)) \right).$$

2. Show that $W(x)$ given by

$$W(x) = \sum_l \int_0^{y_l} \log(1 - p_l(y))dy - \sum_r \int_0^{x_r} \log(1 - U_r'(x))dx$$

is strictly concave.

Hint: Assume $1 - U_r'(x_r) > 0$ so that $\log(1 - U_r'(x))$ is well defined. The following fact may be useful: $\log$ is an increasing function.

3. Use $W(\hat{x}) - W(x)$ as the Lyapunov function to show the global, asymptotic stability of the primal congestion controller, where $\hat{x}$ is the global maximizer of $W(x)$.

   Hint: Assume that U_r, k_r and p_l are such that $W(x) \to \infty$ as $||x|| \to \infty$, there exists a unique equilibrium point and $x_r(t)$ can never go below zero.

Solution 1. Recalling the definition of the primal congestion controller, we obtain that

$$\begin{aligned}
\dot{x} &= k_r(x_r)(U_r^{'}(x_r) - q_r) \\
&= k_r(x_r)(U_r^{'}(x_r) - 1 + 1 - q_r) \\
&= k_r(x_r)\left(1 - q_r - \left(1 - U_r^{'}(x_r)\right)\right) \\
&= k_r(x_r)\left(\prod_{l:l\in r}(1-p_l) - (1 - U_r^{'}(x_r))\right).
\end{aligned}$$

2. Let

$$W(x) = \sum_l \int_0^{y_l} \log(1 - p_l(y))dy - \sum_r \int_0^{x_r} \log(1 - U_r^{'}(x))dx$$

   Since $p_l(y)$ is increasing in y, $\log(1 - p_l(y))$ is decreasing in y. Thus, the first term of the summation is concave in x as it is a composition of a concave function with a linear function of x. Similarly, $U_r^{'}(x)$ is decreasing in x and hence $\log(1 - U_r^{'}(x))$ is increasing in x. This implies that second term (with negative sign) is strictly concave in x.

3. Let $\hat{x}$ be the global maximizer of $W(x)$. Define $V(x) = W(\hat{x}) - W(x)$. Thus, $V(x) > 0$ for $x \neq \hat{x}$ and zero when $x = \hat{x}$. We use $V(x)$ as Lyapunov function. With this, $\frac{\partial V(x)}{\partial x_r} = -\frac{\partial W(x)}{\partial x_r}$.

   Now,

$$\begin{aligned}
\frac{\partial W(x)}{\partial x_r} &= \sum_{l\in r} \log(1-p_l) - \log\left(1 - U_r^{'}(x_r)\right) \\
&= \log\left(\prod_{l\in r}(1-p_l)\right) - \log\left(1 - U_r^{'}(x_r)\right),
\end{aligned}$$

   implying that at $\hat{x}$, $\prod_{l\in r}(1-p_l) = 1 - U_r^{'}(\hat{x}_r)$, and hence $\hat{x}$ is the stable point of the state dynamics too.

   This implies that,

$$\dot{V} = -\sum_r k_r(x_r)\left(\log\left(\prod_{l\in r}(1-p_l)\right) - \log\left(1 - U_r^{'}(x_r)\right)\right)\left(\prod_{l\in r}(1-p_l) - \left(1 - U_r^{'}(x_r)\right)\right) \leq 0,$$

   since for any $a > 0, b > 0,\ (a-b)(\log(a) - \log(b)) \geq 0$, with equality holding only when $a = b$. Thus, $\dot{V} = 0$ only if $\prod_{l\in r}(1-p_l) = \left(1 - U_r^{'}(x_r)\right)$, which is the equilibrium condition. Thus, this system is asymptotically stable.

For the analysis to make sense, we have to assume that $1 - U_r^{'}(x_r) > 0$. Also, $k_r(x_r)$, $U_r^{'}(x_r)$ and p_l are such that if $x > 0$, then $x_r^(t) \neq 0\ \forall r, t$, and that $V(x) \to \infty$ as $\|x\| \to \infty$.

□

Exercise 2.8 (Multi-path routing) In this problem, we expand the scope of the utility maximization problem to include adaptive, multi-path routing. Let s denote a source and let $\mathcal{R}(s)$ denote the set of routes used by source s. Each source s is allowed to split its packets along multiple routes. Let z_s denote the rate at which source s generates data and x_r denote the rate on route r. Thus, the penalty function formulation of the utility maximization problem becomes

$$\max_x \quad \sum_s U_s(z_s) - \sum_l \int_0^{y_l} f_l(y)dy + \epsilon \sum_r \log x_r,$$

where

$$z_s = \sum_{r \in \mathcal{R}(s)} x_r, \qquad y_l = \sum_{r:l \in r} x_r$$

and $\epsilon > 0$ is a small number.

1. Even when U_s is a strictly concave function, argue that the above objective need not be strictly concave if $\epsilon = 0$. (Thus, we have introduced the ϵ term only to ensure strict concavity. But the impact of this term on the optimal solution will be small if ϵ is chosen to be small.)

2. Derive a congestion control (and rate-splitting across routes) algorithm and prove that it asymptotically achieves the optimal rates which solve the above utility maximization problem.

   Hint: Use the approach used to derive the primal congestion control algorithm.

Solution 1. We establish this by showing that the composition of a strictly concave function with a linear function needs not be strictly concave.

Lemma 2.9.1 *Let $G(t) : \mathcal{D} \subseteq \mathcal{R} \mapsto \mathcal{R}$ be a strictly concave function. For $t = (t_1, t_2, \ldots, t_n)$ such that $\sum_{i=1}^n t_i \in \mathcal{D}$, define $H(t) = \sum_{i=1}^n t_i$. Then $G \circ H(t) = G(H(t))$ is concave with respect to t but not strictly concave.*

Proof Concavity of $G \circ H(t)$ with respect to t follows from the standard result that composition of a concave function with an affine function is still concave. We just need to show that this is not strictly concave. Consider vectors t and s such that $t \neq s$, but $\sum_{i=1}^n t_i = \sum_{i=1}^n s_i \in \mathcal{D}$. Clearly, we can find such t, s. Then $G \circ H(t) = G \circ H(s)$. Let $\lambda \in (0, 1)$. Then,

$$\begin{aligned} G \circ H(\lambda t + (1-\lambda)s) &= G\left(\sum_{i=1}^n (\lambda t_i + (1-\lambda)s_i)\right), \\ &= G\left(\lambda\left(\sum_{i=1}^n t_i\right) + (1-\lambda)\left(\sum_{i=1}^n s_i\right)\right), \\ &= G\left(\sum_{i=1}^n t_i\right) = G\left(\sum_{i=1}^n s_i\right) = G \circ H(t) = G \circ H(s). \end{aligned}$$

Thus, $G \circ H(\lambda t + (1-\lambda)s) = \lambda G \circ H(t) + (1-\lambda) G \circ H(s)$. But, for strict concavity, we need $G \circ H(\lambda t + (1-\lambda)s) > \lambda G \circ H(t) + (1-\lambda) G \circ H(s),\ \forall \lambda \in (0,1)$. Thus, $G \circ H(t)$ is not a strict concave function of t.

□

Now, let $V(x) = \sum_s U_s(z_s) - \sum_l \int_0^{y_l} f_l(y)dy$, where x is the vector of all routes for all users. Thus, we need to show that $V(x)$ need not be a strict concave function. Now, for each s, $U_s(z_s) = U_s(\sum_{r \in R(s)} x_r)$ is a composition of $U_s(.)$ with a linear function of x, and hence not a strict concave function of x. Similarly, for each l, if $F_l(t) = \int_0^t f_l(s)ds$ then $\int_0^{y_l} f_l(s)ds = F_l(\sum_{r:l \in r} x_r)$ is a composition of a convex function $F(.)$ with a linear function of x, and hence not a strict convex function of x. Thus, $V(x)$ being a sum of concave functions of x that are not strictly concave need not be a strict concave function.

2. Let

$$V(x) = \sum_s U_s(z_s) - \sum_l \int_0^{y_l} f_l(y)dy + \epsilon \sum_r \log x_r,$$

where $z_s = \sum_{r \in R(s)} x_r$ and $y_l = \sum_{r:l \in r} x_r$. $V(x)$ is strictly concave. Thus, there exists a unique maximizer $\hat{x}$. Since $\hat{x}$ is the maximizer of $V(x)$, it must satisfy

$$\frac{\partial V(x)}{\partial x_r} = 0 \text{ at } x = \hat{x},\ \forall r.$$

Now,

$$\frac{\partial V(x)}{\partial x_r} = U_s^{'}(z_s) - \sum_{l:l \in r} f_l(y_l) + \frac{\epsilon}{x_r}.$$

For each source s and each $r \in R(s)$, let the state dynamics be

$$\dot{x_r} = k_r(x_r)\left(U_s^{'}(z_s) - \sum_{l:l \in r} f_l(y_l) + \frac{\epsilon}{x_r}\right), \tag{2.66}$$

where $k_r(x_r) > 0$. Let $W(x) \triangleq V(\hat{x}) - V(x)$. Thus, $W(x) > 0$ for all $x \neq \hat{x}$, and is equal to zero at $x = \hat{x}$. We use $W(x)$ as a Lyapunov function for showing that state dynamics (2.66) converges to $\hat{x}$. We assume that $U_s(.), k_r(.)$ and $f_l(.)$ are such that $W(x) \to \infty$ as $\|x\| \to \infty$.

$$\begin{aligned}
\dot{W} &= \sum_r \left(\frac{\partial W(x)}{\partial x_r} k_r(x_r)\left(U_s^{'}(z_s) - \sum_{l:l \in r} f_l(y_l) + \frac{\epsilon}{x_r}\right)\right), \\
&= \sum_r \left(-k_r(x_r)\frac{\partial V(x)}{\partial x_r}\frac{\partial V(x)}{\partial x_r}\right), \\
&= -\sum_r k_r(x_r)\left(\frac{\partial V(x)}{\partial x_r}\right)^2 \leq 0
\end{aligned}$$

where, $\dot{W} = 0$ implies $\frac{\partial V(x)}{\partial x_r} = 0$, which is true only when $x = \hat{x}$. Thus, system dynamics given by (2.66) converges to $\hat{x}$.

□

Exercise 2.9 (The global stability of the dual algorithm) Recall the dual algorithm

$$\begin{aligned} x_r &= U_r'^{-1}(q_r) \\ \dot{p}_l &= h_l(y_l - c_l)^+_{p_l}. \end{aligned}$$

Prove that the dual algorithm is globally, asymptotically stable when the routing matrix R has full row rank, i.e., given q, there exists a unique p which satisfies $q = R^T p$.

Solution The equilibrium point of the system satisfies

$$\begin{aligned} (\hat{y}_l - c_l)^+ &= 0 \\ U_r'(\hat{x}_r) &= \hat{q}_r. \end{aligned}$$

Since $q = R^T p$, we have a unique $\hat{p}$ when R has full row rank. Now consider Lyapunov function

$$V(t) = D(p(t)) - D(\hat{p}),$$

so

$$\begin{aligned} \dot{V} &= \dot{D} = \sum_l \frac{\partial D}{\partial p_l} \dot{p}_l \\ &= \sum_l \left(\frac{\partial D}{\partial p_l}\right) \left(-\frac{\partial D}{\partial p_l}\right)^+_{p_l} \\ &= \sum_l (c_l - y_l)(y_l - c_l)^+_{p_l} \\ &\leq 0. \end{aligned}$$

The equality holds if $y_l = c_l$, or $p_l = 0$ and $y_l \leq c_l$, which is the equilibrium point of the system.

Note that if $q_r = 0$, then $x_r = \infty$. So we also have to show that $q_r(t) \neq 0$ for all t if $q_r(0) > 0$. This can be shown by observing that if $q_r \to 0$ and $x_r \to \infty$, then p_l becomes large at all $l \in r$, so $q_r(t)$ cannot approach 0.

□

Exercise 2.10 (The primal-dual algorithm for congestion control) Consider the following congestion control algorithm:

$$\begin{aligned} \dot{x}_r = \kappa_r \left(\frac{w_r}{x_r} - q_r\right), \\ \dot{p}_l = h_l \left(y_l - c_l\right)^+_{p_l}, \end{aligned}$$

where $q_r = \sum_{l:l\in r} p_l$, $y_l = \sum_{r:l\in r} x_r$, and κ_r and h_l are positive constants. This algorithm is called the primal-dual algorithm for congestion control.

1. Show that the equilibrium point of the above congestion control algorithm solves a utility maximization problem, which allocates rates in a weighted proportionally-fair manner.

2. Assume that the equilibrium point is unique and show that the congestion controller is globally asymptotically stable by using the Lyapunov function

$$V(x,p) = \sum_r \frac{(x_r - \hat{x}_r)^2}{\kappa_r} + \frac{\sum_l (p_l - \hat{p}_l)^2}{h_l},$$

where $(\hat{x}, \hat{p})$ denotes the equilibrium point. To do this, show that (i) $\dot{V} \leq 0$ and (ii) that $\dot{V} = 0$ implies $(x(t), p(t)) = (\hat{x}, \hat{p})$. The result then follows from LaSalle's invariance principle (see Section 2.3, Theorem 2.3.3).

Note: In this problem, we have derived a third type of congestion control algorithm, called the primal-dual algorithm. As in the case of the primal algorithm and the dual algorithm, one can design window flow control algorithms that mimic the behavior of this algorithm. The question of which one of these algorithms is best is debatable. Clearly all of the algorithms lead to the same steady-state rate allocation.

Solution 1. At the equilibrium:

$$\dot{x_r} = 0 \Rightarrow x_r = w_r/q_r$$

and

$$\dot{p_l} = 0$$

which yields

(i) $p_l = 0$ if $y_l \leq c_l$

(ii) $p_l > 0$ if $y_l = c_l$.

If we choose $U_r(x_r) = w_r \log x_r$, then, from the first condition, we get

$$U'(x_r) = q_r,$$

and the second condition is equivalent to

$$p_l(y_l - c_l) = 0, \text{ for all } l.$$

Therefore, the pair (x, p) satisfies the KKT conditions for the utility maximization problem with $U_r(x_r) = w_r \log x_r$, which is a proportional fairness resource allocation.

2. i)

$$\begin{aligned}
\dot{V} &= \sum_r \frac{\partial V}{\partial x_r}\dot{x_r} + \sum_l \frac{\partial V}{\partial p_l}\dot{p_l} \\
&= \sum_r 2(x_r - \hat{x_r})(\frac{w_r}{x_r} - q_r) + \sum_l 2(p_l - \hat{p_l})(y_l - c_l)^+_{p_l} \\
&\leq \sum_r 2(x_r - \hat{x_r})(\frac{w_r}{x_r} - q_r) + \sum_l 2(p_l - \hat{p_l})(y_l - c_l)
\end{aligned}$$

But, if $\hat{p}_l > 0$, $\hat{y}_l = c_l$, and therefore $(p_l - \hat{p}_l)(y_l - c_l) = (p_l - \hat{p}_l)(y_l - \hat{y}_l)$, and if $\hat{p}_l = 0$, $\hat{y}_l \leq c_l$ and hence $(p_l - \hat{p}_l)(y_l - c_l) \leq (p_l - \hat{p}_l)(y_l - \hat{y}_l)$. So

$$\begin{aligned}
\dot{V} &\leq \sum_r 2(x_r - \hat{x_r})(\frac{w_r}{x_r} - q_r) + \sum_l 2(p_l - \hat{p}_l)(y_l - \hat{y}_l) \\
&= \sum_r 2(x_r - \hat{x_r})(\frac{w_r}{x_r} - q_r) + \sum_r 2(q_r - \hat{q_r})(x_r - \hat{x}_r) \\
&= \sum_r 2(x_r - \hat{x_r})(\frac{w_r}{x_r} - \hat{q_r}) \\
&= \sum_r 2(x_r - \hat{x_r})(\frac{w_r}{x_r} - \frac{w_r}{\hat{x}_r}) \\
&= 2\sum_r w_r \frac{-(x_r - \hat{x_r})^2}{x_r \hat{x}_r} < 0.
\end{aligned}$$

ii) Note that $\dot{V} < 0$ (strictly negative) for $x_r \neq \hat{x}_r$, and $\dot{V} = 0$ iff $x_r(t) = \hat{x}_r$ which results in $y_l(t) = \hat{y}_l$, and consequently $p_l(t) = \hat{p}_l$.

□

Exercise 2.11 (A discrete-time version of the dual algorithm) Consider the following discrete-time version of the dual congestion control algorithm: at each time slot k, each source chooses a transmission rate $x_r(k)$ which is the solution to

$$\max_{0 \leq x_r \leq X_{\max}} U_r(x_r) - q_r(k)x_r,$$

where $X_{\max}$ is the maximum rate at which any user can transmit. Each link l computes its price $p_l(k)$ according to the following update rule which is a discretization of the continuous-time algorithm:

$$p_l(k+1) = (p_l(k) + \epsilon(y_l - c_l))^+ ,$$

where $\epsilon > 0$ is a small step-size parameter. The variables y_l and q_r are defined as usual:

$$q_r(k) = \sum_{l:l \in r} p_l(k), \qquad y_l(k) = \sum_{r:l \in r} x_r(k).$$

We will show that, on average, the above discrete-time algorithm is nearly optimal in the sense that it approximately solves the utility maximization problem.

1. Consider the Lyapunov function

$$V(k) = \frac{1}{2}\sum_l p_l^2(k).$$

Show that

$$V(k+1) - V(k) \leq K\epsilon^2 + \epsilon \sum_r q_r(k)(x_r(k) - \hat{x}_r),$$

for some constant $K > 0$, where $\hat{x}$ is an optimal solution to the utility maximization problem

$$\max_{x\geq 0} \sum_r U_r(x_r), \qquad \text{subject to} \qquad \sum_{r:l\in r} x_r \leq c_l.$$

Assume that $X_{\max} > \max_r \hat{x}_r$.

2. Next, show that

$$V(k+1) - V(k) \leq K\epsilon^2 + \epsilon \sum_r (U_r(x_r) - U_r(\hat{x}_r)).$$

3. Finally, show that

$$\sum_r U_r(\hat{x}_r) \leq \sum_r U_r(\bar{x}_r) + K\epsilon,$$

where

$$\bar{x}_r := \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} x_r(k).$$

Note: For this problem, we assume U_r is concave, but it does not have to be strictly concave for the results of this problem to hold. If U_r is not strictly concave, then there may be multiple optimal solutions $\hat{x}$. In this case, $X_{\max}$ is assumed to be greater than $\max_r \hat{x}_r$ for all possible $\hat{x}$.

Solution 1.

$$\begin{aligned}
V(k+1) - V(k) &= \frac{1}{2}\sum_l p_l^2(k+1) - p_l^2(k) \\
&= \frac{1}{2}\sum_l ((p_l(k) + \epsilon(y_l - c_l))^+)^2 - p_l^2(k) \\
&\leq \frac{1}{2}\sum_l (p_l(k) + \epsilon(y_l - c_l))^2 - p_l^2(k) \\
&= \frac{1}{2}\sum_l 2p_l(k)\epsilon(y_l - c_l) + \epsilon^2(y_l - c_l)^2 \\
&= \frac{\epsilon^2}{2}\sum_l (y_l - c_l)^2 + \epsilon \sum_l p_l(k)(y_l - c_l)
\end{aligned}$$

Noting that $x_r \leq X_{max}$, and $y^* \leq c_l$, for any feasible solution x_r^*, yields

$$\begin{aligned}
V(k+1) - V(k) &\leq K\epsilon^2 + \epsilon \sum_l p_l(k)(y_l - y_l^*) \\
&= K\epsilon^2 + \epsilon \sum_r q_r(k)(x_r - x_r^*)
\end{aligned}$$

where K is chosen to be a constant greater than $\sum_l (y_{l max} - c_l)^2$, where $y_{l max} = \sum_{r:l\in r} X_{max}$.

2. Choose $x^* = \hat{x}$. Since x_r is the maximizer of $U_r(x_r) - q_r(k)x_r$,

$$U_r(x_r) - q_r(k)x_r \geq U_r(\hat{x}_r) - q_r(k)\hat{x}_r,$$

or equivalently

$$q_r(k)(x_r - \hat{x}_r) \leq U_r(x_r) - U_r(\hat{x}_r).$$

Replacing the above expression in the result of part (a) yields

$$V(k+1) - V(k) \leq K\epsilon^2 + \epsilon \sum_r U_r(x_r) - U_r(\hat{x}_r).$$

3. Summing the inequality of part (b) for $k = 0, 1, ..., N$ yields

$$V(N) - V(0) \leq NK\epsilon^2 + \epsilon \sum_{k=1}^{N} \sum_r U_r(x_r) - U_r(\hat{x}_r)$$

Dividing the both sides by N, we get

$$\frac{V(N) - V(0)}{N} \leq K\epsilon^2 + \epsilon \frac{1}{N} \sum_{k=1}^{N} \sum_r U_r(x_r) - U_r(\hat{x}_r)$$

But $V(0)$ is finite and $V(N) \geq 0$, so $\lim_{N\to\infty} \frac{V(N)-V(0)}{N} \geq 0$, and thus

$$0 \leq K\epsilon + \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \sum_r U_r(x_r) - U_r(\hat{x}_r)$$

Therefore

$$\begin{aligned}
\sum_r U_r(\hat{x}_r) &\leq K\epsilon + \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \sum_r U_r(x_r(k)) \\
&\leq K\epsilon + \sum_r U_r(\lim_{N\to\infty} \sum_{k=1}^{N} x_r(k)) \\
&= K\epsilon + \sum_r U_r(\bar{x_r})
\end{aligned}$$

where the last inequality follows from concavity of $U_r(.)$.

□

Exercise 2.12 (An example illustrating the VCG algorithm) Consider the network shown in Figure 2.11, where the four links are owned by four different players. Suppose that the network wants to establish a communication path from node 1 to node 3. If a link is selected for the transmission, then it incurs a cost of p_l, where $l \in \{a, b, c, d\}$. The social welfare is to find the minimum cost path. Define x_l to be a variable such that $x_l = 0$ if link l is selected and $x_l = 1$ otherwise. Therefore, the utility function associated with player l is

$$U_l(x_l) = \begin{cases} -p_l, & \text{if } x_l = 0 \quad \text{i.e., link } l \text{ is selected} \\ 0, & \text{if } x_l = 1 \quad \text{i.e., link } l \text{ is not selected} \end{cases}.$$

The utility function revealed by player l is

$$\tilde{U}_l(x_l) = \begin{cases} -\tilde{p}_l, & \text{if } x_l = 0 \\ 0, & \text{if } x_l = 1 \end{cases},$$

where $\tilde{p}_l$ is the cost claimed by player l. The objective of the network is

$$\begin{aligned} & \max \sum_l U_l(x_l) \\ \text{subject to} \quad & x_a + x_c = 1 \qquad (2.67) \\ & x_b + x_d = 1 \qquad (2.68) \\ & x_a, x_b, x_c, x_d \in \{0, 1\}, \end{aligned}$$

where equalities (2.67) and (2.68) guarantee that there is a feasible path from node 1 to node 3.

Assume $p_a > p_c$ and $p_b > p_d$. Let w_l denote the price charged to link l. Write the value of w_l under the VCG pricing mechanism.

Note: You will find that the prices are nonpositive. In fact, $-w_l$ can be interpreted as the payment link l receives when it is selected by the network.

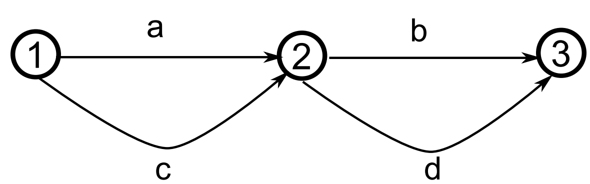

Figure 2.11: A simple network

Solution Since the VCG pricing mechanism is a truthful, all players reveal their true utilities. Further recall that

$$\begin{aligned} w_a &= \sum_{s \neq a} U_s(\bar{x}_s) - \sum_{s \neq a} U_s(\tilde{x}_s) = p_c + p_d - p_c - p_d = 0 \\ w_b &= \sum_{s \neq b} U_s(\bar{x}_s) - \sum_{s \neq b} U_s(\tilde{x}_s) = p_c + p_d - p_c - p_d = 0 \\ w_c &= \sum_{s \neq c} U_s(\bar{x}_s) - \sum_{s \neq c} U_s(\tilde{x}_s) = p_d - p_a - p_d = -p_a \\ w_d &= \sum_{s \neq d} U_s(\bar{x}_s) - \sum_{s \neq d} U_s(\tilde{x}_s) = p_c - p_c - p_b = -p_b. \end{aligned}$$

□

Exercise 2.13 (An example illustrating the PoA) Consider a network with two strategic users sharing a same link with capacity $c = 1$. The utility function of user i is $\alpha_i - \frac{\alpha_i}{(x_i+1)}$, where x_i is the rate allocated to user i. Each user bids an amount that it is willing to pay, say w_i for user i, and user i is allocated a data rate given by $x_i = w_i/(w_1 + w_2)$. Thus, the payoff to user i is $\alpha_i - \frac{\alpha_i}{x_i} - w_i$. We assume $\alpha_1 = 1$ and $\alpha_2 = 2$.

1. Write down the NE for the bids (w_1, w_2) for these strategic users.
2. Compute the PoA.

Solution 1. Following the definition of NE, we have

$$\frac{1}{\left(\frac{w_1}{w_1+w_2}+1\right)^2}\left(1-\frac{w_1}{w_1+w_2}\right) = w_1 + w_2$$
$$\frac{2}{\left(\frac{w_2}{w_1+w_2}+1\right)^2}\left(1-\frac{w_2}{w_1+w_2}\right) = w_1 + w_2,$$

which can be simplified to

$$w_2 = (2w_1 + w_2)^2$$
$$2w_1 = (w_1 + 2w_2)^2 .$$

Solving the equalities above, we obtain

$$w_1 = 0.1227 \quad \text{and} \quad w_2 = 0.1863.$$

2. Under the NE,

$$x_1 = 0.3970 \quad \text{and} \quad x_2 = 0.6030.$$

The sum utility is

$$1 - \frac{1}{x_1+1} + 2 - \frac{2}{x_2+1} = 3 - \frac{1}{1.3970} - \frac{2}{1.6030} = 1.0365.$$

Solving the following utility maximization problem

$$\max_{x_1+x_2\leq 1} 1 - \frac{1}{x_1+1} + 2 - \frac{2}{x_2+1} \tag{2.69}$$

yields

$$x_1 = 0.2426 \quad \text{and} \quad x_2 = 0.7574.$$

The sum utility is

$$1 - \frac{1}{x_1+1} + 2 - \frac{2}{x_2+1} = 3 - \frac{1}{1.3970} - \frac{2}{1.6030} = 1.0572.$$

So the PoA is $1.0365/1.0572 = 0.9804$.

□

2.10 Notes

The utility maximization framework for studying resource allocation in communication networks was introduced in [?]. Max-min fairness was originally developed in the context of political philosophy [?], and was extensively studied in the context of communication networks in [?,?]. Log utilities were introduced in the solution of a game where players bargain over the allocation of a common resource [?]. It is called the Nash bargaining solution in economics. It was studied under the name proportional fairness in the context of communication networks in [?]. Minimum potential delay fairness was introduced in [?]. The α-fair utility functions have been studied by economists under the name isoelastic utility functions [?]. They were proposed as a common framework to study many notions of fairness in communication networks in [?, ?].

The primal and dual algorithms for solving the network utility maximization problem were presented in [?]. The version of the dual algorithm presented in this chapter is a continuous-time version of the algorithms proposed in [?, ?]. The multi-path algorithm was also proposed in [?], while the addition of the ϵ term was proposed in [?].

The primal-dual algorithm for Internet congestion control was introduced in [?] and its convergence was proved in [?] although the algorithm at the nodes to compute the Lagrange multipliers is different from the computations presented in the Problems section in this chapter. The version of the primal-dual algorithm presented here is in [?, ?]. The block diagram view of the relationships between the primal and dual variables was suggested in [?].

One bit feedback for congestion control was proposed in [?, ?]. The idea of using exponential functions to convert price information to one-bit information is in [?]. The crude approximation for the DropTail price function was proposed in [?], as a limit of more accurate queueing-theoretic models.

A game-theoretic view of network resource allocation was presented in [?]; see also [?]. The price of anarchy result presented here is due to [?]. A survey of game theory in networks can be found in [?].

Several surveys of resource allocation in the Internet using the utility function framework are available in [?, ?, ?, ?].

Excellent sources for the background material on optimization include [?, ?, ?]. An introduction to differential equation models of dynamical systems and their stability can be found in [?]. Delay differential equations and stability are treated in [?].