Chapter 2

Limits

2.1 The Idea of Limits

2.1.1 The average velocity of the object between time t = a and t = b is the change in position divided by the elapsed time: vav =~~s(b)−s(a)~~

b−a

2.1.2 In order to compute the instantaneous velocity of the object at time t = a, we compute the average   
velocity over smaller and smaller time intervals of the form [a, t], using the formula: vav =~~s(t)−s(a)~~. We let t

t−a

approach a. If the quantity~~s(t)−s(a)~~

t−a approachesalimitast→a,thenthatlimitiscalledtheinstantaneous

velocity of the object at time t = a.

2.1.3 The slope of the secant line between points (a, f (a)) and (b, f (b)) is the ratio of the differences f (b) − f (a) and b − a. Thus msec =~~f(b)−f(a)~~

b−a

2.1.4 In order to compute the slope of the tangent line to the graph of y = f (t at (a, f (a)), we compute the   
slope of the secant line over smaller and smaller time intervals of the form [a, t]. Thus we consider~~f(t)−f(a)~~

t−a

and let t → a. If this quantity approaches a limit, then that limit is the slope of the tangent line to the curve y = f(t) at t = a.

2.1.5 Both problems involve the same mathematics, namely finding the limit as t → a of a quotient of differences of the form~~g(t)−g(a)~~

t−a forsomefunctiong.

2.1.6

y

20

Because f (x) = x2 is an even function, f (−a) =

f (a) for all a. Thus the slope of the secant line 15

between the points (a, f (a)) and (−a, f (−a)) is

0 10

msec =~~f(−a)−f(a)~~ −a−a

=

−2a =

0. The slope of the

tangent line at x = 0 is also zero.

2.1.7 The average velocity is~~s(3)−s(2)~~

H-a, f H-aLL

-4

= 156 − 136 = 20.

5

Ha, f HaLL

x

-2 2 4

3−2

2.1.8 The average velocity is~~s(4)−s(1)~~

4−1

=~~144−84~~ =~~60~~ = 20.

3 3

45

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2.1.9

a. Over [1, 4], we have vav =~~s(4)−s(1)~~

4−1

b. Over [1, 3], we have vav =~~s(3)−s(1)~~

3−1

c. Over [1, 2], we have vav =~~s(2)−s(1)~~

2−1

=~~256−112~~ = 48.

3

=~~240−112~~ = 64.

2

=~~192−112~~ = 80.

1

d. Over [1, 1 + h], we have vav =~~s(1+h)−s(1)~~

1+h−1

96 − 16h = 16(6 − h).

2.1.10

= ~~−~~~~16(1+h)2~~~~+~~~~128(1+h)−(112)~~   
 h

= ~~−~~~~16h2~~~~−~~~~32h+128h~~ = ~~h~~~~(−16h+96)~~ =

h h

a. Over [0, 3], we have vav =~~s(3)−s(0)~~

3−0

b. Over [0, 2], we have vav =~~s(2)−s(0)~~

2−0

c. Over [0, 1], we have vav =~~s(1)−s(0)~~

1−0

d. Over [0, h], we have vav =~~s(h)−s(0)~~

h−0

2.1.11   
 a.~~s(2)−s(0)~~ =~~72−0~~ = 36.

=~~65.9−20~~ = 15.3.

3

=~~60.4−20~~ = 20.2.

2

=~~45.1−20~~ = 25.1.

1

= ~~−~~~~4.9h2~~~~+~~~~30h+20−20~~ = ~~(~~~~h)(−4.9h+30)~~ = −4.9h + 30.

h h

2−0

b.~~s(1.5)−s(0)~~

1.5−0

2

=~~66−0~~ = 44.

1.5

c.~~s(1)−s(0)~~

1−0

d.~~s(.5)−s(0)~~

.5−0

2.1.12

=~~52−0~~ = 52.

1

=~~30−0~~ = 60.

.5

a.~~s(2.5)−s(.5)~~

2.5−.5

=~~150−46~~ = 52.

2

b.~~s(2)−s(.5)~~

2−.5

=~~136−46~~ = 60.

1.5

c.~~s(1.5)−s(.5)~~

1.5−.5

=~~114−46~~ = 68.

1

d.~~s(1)−s(.5)~~

1−.5

2.1.13

s

150

=~~84−46~~ = 76.

.5

The slope of the secant line is given by~~s(2)−s(.5)~~ =

2−.5

100

136−46

1.5

= 60. This represents the average velocity

50

0.5 1.0 1.5

of the object over the time interval [.5, 2].

t

2.0 2.5

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2.1.14

s

1.2

1.0

0.8

The slope of the secant line is given by~~s(.5)−s(0)~~

.5−0

1

=

0.6

.5

= 2. This represents the average velocity of the

0.4

0.2

0.1 0.2 0.3 0.4 0.5

object over the time interval [0, .5].

t

0.6

2.1.15

Time Interval [1, 2]

Average Velocity 80

[1, 1.5] [1, 1.1] [1, 1.01] [1, 1.001]

88 94.4 95.84 95.984

The instantaneous velocity appears to be 96 ft/s.

2.1.16

Time Interval [2, 3]

Average Velocity 5.5

[2, 2.25] [2, 2.1] [2, 2.01] [2, 2.001]

9.175 9.91 10.351 10.395

The instantaneous velocity appears to be 10.4 m/s.

2.1.17

s(1.01)−s(1)   
 .01

= 47.84, while~~s(1.001)−s(1)~~

.001

= 47.984 and~~s(1.0001)−s(1)~~

.0001

= 47.9984. It appears that the

instantaneous velocity at t = 1 is approximately 48.

2.1.18

s(2.01)−s(2)   
 .01

= −4.16, while ~~s(2.001)−s(2)~~

.001

= −4.016 and ~~s(2.0001)−s(2)~~

.0001

= −4.0016. It appears that the

instantaneous velocity at t = 2 is approximately −4.

2.1.19

Time Interval [2, 3]

Average Velocity 20

[2.9, 3] [2.99, 3] [2.999, 3]

5.6 4.16 4.016

[2.9999, 3] [2.99999, 3]

4.0016 4.00016

The instantaneous velocity appears to be 4 ft/s.

2.1.20

Time Interval [π/2, π]

Average Velocity −1.90986

[π/2, π/2 + .1] [π/2, π/2 + .01]

−.149875 −.0149999

[π/2, π/2 + .001] [π/2, π/2 + .0001]

−.0015 −.00015

The instantaneous velocity appears to be 0 ft/s.

2.1.21

Time Interval [3, 3.1]

Average Velocity −17.6

[3, 3.01] [3, 3.001] [3, 3.0001]

−16.16 −16.016 −16.002

The instantaneous velocity appears to be −16 ft/s.

2.1.22

Time Interval [π/2, π/2 + .1]

Average Velocity −19.9667

[π/2, π/2 + .01] [π/2, π/2 + .001] [π/2, π/2 + .0001]

−19.9997 −20.0000 −20.0000

The instantaneous velocity appears to be −20 ft/s.

2.1.23

Time Interval [0, 0.1]

Average Velocity 79.4677

[0, 0.01] [0, 0.001] [0, 0.0001]

79.9947 79.9999 80.0000

The instantaneous velocity appears to be 80 ft/s.

2.1.24

Time Interval [0, 1]

Average Velocity −10

[0, 0.1] [0, 0.01] [0, 0.001]

−18.1818 −19.802 −19.98

The instantaneous velocity appears to be −20 ft/s.

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2.1.25

x Interval   
Slope of Secant Line

[2, 2.1] [2, 2.01] [2, 2.001] [2, 2.0001]

8.2 8.02 8.002 8.0002

The slope of the tangent line appears to be 8.

2.1.26

x Interval   
Slope of Secant Line

[π/2, π/2 + .1] [π/2, π/2 + .01]

−2.995 −2.99995

[π/2, π/2 + .001] [π/2, π/2 + .0001]

−3.0000 −3.0000

The slope of the tangent line appears to be −3.

2.1.27

x Interval   
Slope of the Secant Line

[−1, −.9] [−1, −.99] [−1, −.999] [−1, −.9999]

.524862 .5025 .50025 .500025

The slope of the tangent line appears to be .5.

2.1.28

x Interval   
Slope of the Secant Line

[1, 1.1] [1, 1.01] [1, 1.001] [1, 1.0001]

2.31 2.0301 2.003 2.0003

The slope of the tangent line appears to be 2.

2.1.29

y

8

a. Note that the graph is a parabola with ver-  
 tex (2, −1).

b. At (2, −1) the function has tangent line with   
 slope 0.

6

4

2

x

-1 1 2 3 4 5

c.

x Interval   
Slope of the Secant Line

[2, 2.1] [2, 2.01] [2, 2.001] [2, 2.0001]

.1 .01 .001 .0001

The slope of the tangent line at (2, −1) appears to be 0.

2.1.30

a. Note that the graph is a parabola with ver-  
 tex (0, 4).

y

b. At (0, 4) the function has a tangent line with   
 slope 0.

c. This is true for this function - because the   
 function is symmetric about the y-axis and

we are taking pairs of points symmetrically -2 -1

about the y axis. Thus f (0 + h) = 4 − (0 +

h)2 = 4 − (−h)2 = f (0 − h). So the slope of   
any such secant line is~~4−h2~~ ~~−~~~~(4−h2~~ ~~)~~h−(−h) =~~0~~ 2h =   
0.

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4

3

2

1

x

1 2

-1

-2

2.1. THE IDEA OF LIMITS 49

2.1.31

y

400

a. Note that the graph is a parabola with ver-

tex (4, 448).

b. At (4, 448) the function has tangent line with   
 slope 0, so a = 4.

300

200

100

x

2 4 6 8

c.

x Interval   
Slope of the Secant Line

[4, 4.1] [4, 4.01] [4, 4.001] [4, 4.0001]

−1.6 −.16 −.016 −.0016

The slopes of the secant lines appear to be approaching zero.

d. On the interval [0, 4) the instantaneous ve-  
 locity of the projectile is positive.

e. On the interval (4, 9] the instantaneous ve-  
 locity of the projectile is negative.

2.1.32

a. The rock strikes the water when s(t) = 96. This occurs when 16t2 = 96, or t2 = 6, whose only positive√

b.

solution is t = 6 ≈ 2.45 seconds.

√ √

t Interval [ 6 − .1, 6]

Average Velocity 76.7837

√ √ √

[ 6 − .01, 6] [ 6 − .001,

78.2237 78.3677

√ √ √

6] [ 6 − .0001, 6]

78.3821

When the rock strikes the water, its instantaneous velocity is about 78.38 ft/s.

2.1.33 For line AD, we have

mAD =

For line AC, we have   
 yC − yA

yD − yA

xD − xA

1

= ~~f~~~~(π)−f(π/2)~~ = ≈ .63662.

π − (π/2) π/2

mAC =

For line AB, we have

xC − xA

= ~~f~~~~(π/2+.5)−f(π/2)~~   
 (π/2 + .5) − (π/2)

= ~~−~~~~cos(π/2+.5)~~ ≈ .958851.

.5

mAB =

yB − yA

xB − xA

= ~~f~~~~(π/2+.05)−f(π/2)~~   
 (π/2 + .05) − (π/2)

= ~~−~~~~cos(π/2+.05)~~ ≈ .999583.

.05

Computing one more slope of a secant line:

msec =

f (π/2 + .01) − f (π/2)   
 (π/2 + .01) − (π/2)

= ~~−~~~~cos(π/2+.01)~~ ≈ .999983.

.01

Conjecture: The slope of the tangent line to the graph of f at x = π/2 is 1.   
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2.2 Definitions of Limits

2.2.1 Suppose the function f is defined for all x near a except possibly at a. If f (x) is arbitrarily close to a number L whenever x is sufficiently close to (but not equal to) a, then we write lim f (x) = L.

x→a



2.2.2 False. For example, consider the function f (x) =

Then lim f (x) = 0, but f (0) = 4.

x→0

x2 if x = 0

4 if x = 0.

2.2.3 Suppose the function f is defined for all x near a but greater than a. If f (x) is arbitrarily close to L

for x sufficiently close to (but strictly greater than) a, then lim f (x) = L.

x→a+

2.2.4 Suppose the function f is defined for all x near a but less than a. If f (x) is arbitrarily close to L for

x sufficiently close to (but strictly less than) a, then lim f (x) = L.

x→a−

2.2.5 It must be true that L = M .

2.2.6 Because graphing utilities generally just plot a sampling of points and “connect the dots,” they can sometimes mislead the user investigating the subtleties of limits.

2.2.7

a. h(2) = 5.

b. lim h(x) = 3.

x→2

c. h(4) does not exist.

d. lim f (x) = 1.

x→4

e. lim h(x) = 2.

x→5

2.2.9

a. f (1) = −1.

b. lim f (x) = 1.

x→1

c. f (0) = 2.

d. lim f (x) = 2.

x→0

2.2.11

2.2.8

a. g(0) = 0.

b. lim g(x) = 1.

x→0

c. g(1) = 2.

d. lim g(x) = 2.

x→1

2.2.10

a. f (2) = 2.

b. lim f (x) = 4.

x→2

c. lim f (x) = 4.

x→4

d. lim f (x) = 2.

x→5

a.

x

f (x) =~~x2~~ ~~−~~~~4~~

x−2

1.9 1.99 1.999 1.9999

3.9 3.99 3.999 3.9999

2 2.0001 2.001 2.01 2.1

undefined 4.0001 4.001 4.01 4.1

b. lim f (x) = 4.

x→2

2.2.12

a.

x

f (x) =~~x3~~ ~~−~~~~1~~

x−1

.9 .99 .999 .9999

2.71 2.9701 2.997 2.9997

1 1.0001 1.001 1.01 1.1

undefined 3.0003 3.003 3.0301 3.31

b. lim

x3 − 1

=3

x→1 x − 1

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2.2.13

a.

g(t)

t

√

t−3

8.9 8.99 8.999

5.98329 5.99833 5.99983

9 9.001 9.01 9.1

undefined 6.00017 6.00167 6.01662

b. lim

t→9

2.2.14

t−9

√ = 6.

t−3

a.

x .01

f (x) = (1 + x)1/x 2.70481

.001 .0001 .00001

2.71692 2.71815 2.71827

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| x | −.01 | −.001 | −.0001 | −.00001 |
| f (x) = (1 + x)1/x | 2.732 | 2.71964 | 2.71842 | 2.71830 |

b. lim (1 + x)1/x ≈ 2.718.

x→0

c. lim (1 + x)1/x = e.

x→0

2.2.15

a.

y

2.0

1.5

1.0

0.5

x

0 1 2 3 4

b.

x 1.8 1.9

f (x) 1.0067 1.00167

1.99 2.01 2.1 2.2

1.00002 1.00002 1.00167 1.0067

From both the graph and the table, the limit appears to be 1.

2.2.16

a.

y

2.5

2.0

1.5

1.0

0.5

x

-1.0 -0.5 0.5 1.0

b.

x −.2 −0.1

f (x) 2.03336 2.00834

−0.01 .01 0.1 0.2

2.00008 2.00008 2.00834 2.03336

From both the graph and the table, the limit appears to be 2.

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=t−9

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2.2.17

a.

y

2.0

1.5

1.0

0.5

x

-1 1 2 3

b.

x 0.9 0.99

f (x) 1.993342 1.999933

0.999 1.001 1.01 1.1

1.999999 1.999999 1.999933 1.993342

From both the graph and the table, the limit appears to be 2.

2.2.18

a.

y

3.5

3.0

2.5

2.0

1.5

1.0

0.5

x

-1 1 2 3

b.

x −0.1 −0.01

f (x) 2.8951 2.99

−0.001 0.001 0.01 0.1

2.999 3.001 3.0099 3.0949

From both the graph and the table, the limit appears to be 3.

2.2.19

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| x | 4.9 | 4.99 | 4.999 | 4.9999 | 5 | 5.0001 | 5.001 | 5.01 | 5.1 |
| f (x) =~~x2~~ ~~−~~~~25~~  x−5 | 9.9 | 9.99 | 9.999 | 9.9999 | undefined | 10.0001 | 10.001 | 10.01 | 10.1 |

lim

x2 − 25 x2 − 25

= 10, lim

x2 − 25   
= 10, and thus lim = 10.

x→5+ x − 5 x→5− x − 5 x→5 x−5

2.2.20

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| x | 99.9 | 99.99 | 99.999 | 99.9999 | 100 | 100.0001 | 100.001 | 100.01 | 100.1 |
| f (x) =~~√−100~~  x−10 | 19.995 | 19.9995 | 19.99995 | ≈ 20 | undefined | ≈ 20 | 20.0005 | 20.00005 | 20.005 |

lim

x→100+

2.2.21

x − 100 x − 100

= 20, lim   
√x − 10 x→100− √x − 10

x − 100

= 20, and thus lim = 20.

x→100 √x − 10

a. f (1) = 0. b. lim f (x) = 1. c. lim f (x) = 0.

x→1− x→1+

d. lim f (x) does not exist, since the two one-sided limits aren’t equal.

x→1

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