STUDENT'S SOLUTIONS MANUAL

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Chapter D2

Second-Order Differential Equations

D2.1 Basic Ideas

1. The *order* of a differential equation is the highest-order derivative that appears in the equation. Thus for example $y'(t) + y(t) = 0$ is a first-order equation, while $y''(t) + y(t) = 0$ is a second-order equation.

3. A differential equation $y''(t) + p(t)y'(t) + q(t)y(t) = f(t)$ is *homogeneous* if $f(t) = 0$ for t in the domain we are interested in. It is *nonhomogeneous* if this is not the case. Thus for example $y''(t) + 3ty(t) = 0$ is homogeneous, while $y''(t) + 3ty(t) = t^2$ is nonhomogeneous.

5. Two functions f and g are linearly dependent on an interval I if there is some nonzero constant c such that for each $x \in I$ we have $f(x) = cg(x)$. That is, they are linearly dependent if one is a nonzero constant multiple of another.

7. The general solution of a second-order linear nonhomogeneous differential equation is the sum of (a) any single particular solution of the nonhomogeneous equation, and (b) the general solution of the homogeneous equation derived by setting $f(t) = 0$ in the nonhomogeneous equation. See Theorems 16.3 and 16.4.

9. Since the highest order derivative appearing is the second derivative, this is a second-order differential equation. Since y and its derivatives only appear in terms by themselves, not with other derivatives of y, it is linear. Finally, since there is a nonzero term ($10t^2$) that does not depend on y, it is nonhomogeneous.

11. Since the highest order derivative appearing is the second derivative, this is a second-order differential equation. Since there is a term involving yy', it is nonlinear. Finally, since there is a nonzero term (e^t) that does not depend on y, it is nonhomogeneous.

13. Since $\frac{d^2}{dt^2}e^{kt} = \frac{d}{dt}(ke^{kt}) = k^2e^{kt}$, we have

$$y''(t) - 4y(t) = (3e^{2t} - 5e^{-2t})'' - 4(3e^{2t} - 5e^{-2t}) = 12e^{2t} - 20e^{-2t} - (12e^{2t} - 20e^{-2t}) = 0.$$

15. Since $\frac{d^2}{dt^2}e^{kt} = \frac{d}{dt}(ke^{kt}) = k^2e^{kt}$, we have

$$\begin{aligned} y''(t) - 9y(t) &= (4e^{3t} + 3e^{-3t} - 2t)'' - 9(4e^{3t} + 3e^{-3t} - 2t) \\ &= (36e^{3t} + 27e^{-3t}) - (36e^{3t} + 27e^{-3t} - 18t) \\ &= 18t. \end{aligned}$$

17. We have

$$\begin{aligned}
y''(t) - y'(t) - 2y(t) &= (C_1e^{-t} + C_2e^{2t})'' - (C_1e^{-t} + C_2e^{2t})' - 2(C_1e^{-t} + C_2e^{2t}) \\
&= (C_1e^{-t} + 4C_2e^{2t}) - (-C_1e^{-t} + 2C_2e^{2t}) - (2C_1e^{-t} + 2C_2e^{2t}) \\
&= 0.
\end{aligned}$$

19. We have

$$\begin{aligned}
y''(t) + 6y'(t) + 25y(t) &= \left(e^{-3t}(C_1\sin 4t + C_2\cos 4t)\right)'' + 6\left(e^{-3t}(C_1\sin 4t + C_2\cos 4t)\right)' \\
&\quad + 25\left(e^{-3t}(C_1\sin 4t + C_2\cos 4t)\right) \\
&= \left(-3e^{-3t}(C_1\sin 4t + C_2\cos 4t) + e^{-3t}(4C_1\cos 4t - 4C_2\sin 4t)\right)' \\
&\quad + 6\left(-3e^{-3t}(C_1\sin 4t + C_2\cos 4t) + e^{-3t}(4C_1\cos 4t - 4C_2\sin 4t)\right) \\
&\quad + 25\left(e^{-3t}(C_1\sin 4t + C_2\cos 4t)\right) \\
&= \left(e^{-3t}((-3C_1 - 4C_2)\sin 4t + (4C_1 - 3C_2)\cos 4t)\right)' \\
&\quad + 6\left(-3e^{-3t}(C_1\sin 4t + C_2\cos 4t) + e^{-3t}(4C_1\cos 4t - 4C_2\sin 4t)\right) \\
&\quad + 25\left(e^{-3t}(C_1\sin 4t + C_2\cos 4t)\right) \\
&= -3e^{-3t}((-3C_1 - 4C_2)\sin 4t + (4C_1 - 3C_2)\cos 4t) \\
&\quad + e^{-3t}((-12C_1 - 16C_2)\cos 4t + (-16C_1 + 12C_2)\sin 4t) \\
&\quad + 6\left(-3e^{-3t}(C_1\sin 4t + C_2\cos 4t) + e^{-3t}(4C_1\cos 4t - 4C_2\sin 4t)\right) \\
&\quad + 25\left(e^{-3t}(C_1\sin 4t + C_2\cos 4t)\right) \\
&= e^{-3t}((9C_1 + 12C_2)\sin 4t + (-12C_1 + 9C_2)\cos 4t) \\
&\quad + e^{-3t}((-16C_1 + 12C_2)\sin 4t + (-12C_1 - 16C_2)\cos 4t) \\
&\quad + e^{-3t}((-18C_1 - 24C_2)\sin 4t + (24C_1 - 18C_2)\cos 4t) \\
&\quad + e^{-3t}(25C_1\sin 4t + 25C_2\cos 4t) \\
&= 0.
\end{aligned}$$

21. We have

$$\begin{aligned}
ty''(t) - (t+1)y'(t) + y(t) &= t\left(C_1e^t + C_2(t+1)\right)'' - (t+1)\left(C_1e^t + C_2(t+1)\right)' \\
&\quad + \left(C_1e^t + C_2(t+1)\right) \\
&= t\left(C_1e^t + C_2\right)' - (t+1)\left(C_1e^t + C_2\right) + \left(C_1e^t + C_2(t+1)\right) \\
&= tC_1e^t - tC_1e^t - tC_2 - C_1e^t - C_2 + C_1e^t + tC_2 + C_2 \\
&= 0.
\end{aligned}$$

23. The two given solutions are linearly independent, since for example at $t = 0$, $\frac{1}{5} \cdot 5e^{-6\cdot 0} = e^{6\cdot 0}$ while at $t = 1$ we see that $\frac{1}{5} \cdot 5e^{-6} = e^{-6} \neq e^{6}$, so that the two solutions do not differ by a constant multiple. Since the two given solutions are linearly independent, the general solution is $y(t) = C_1e^{6t} + C_2e^{-6t}$. Note that the coefficient of 5 in the second solution has been subsumed into the constant C_2.

25. The two solutions are linearly independent, since for example at $t = 0$, $te^{-t} = 0 \cdot e^{-t}$, but this is not true at $t = 1$, so that the two solutions do not differ by a constant multiple. Since the two solutions are linearly independent, the general solution is $y(t) = C_1 e^{-t} + C_2 te^{-t}$.

27. $y''(t) - y(t) = (e^{-3t})'' - e^{-3t} = 9e^{-3t} - e^{-3t} = 8e^{-3t}$.

29. Substituting gives

$$\begin{aligned} y''(t) - 4y'(t) + 4y(t) &= (t^2e^{2t})'' - 4(t^2e^{2t})' + 4(t^2e^{2t}) \\ &= (2te^{2t} + 2t^2e^{2t})' - 4(2te^{2t} + 2t^2e^{2t}) + 4t^2e^{2t} \\ &= 2e^{2t} + 4te^{2t} + 4te^{2t} + 4t^2e^{2t} - 8te^{2t} - 8t^2e^{2t} + 4t^2e^{2t} \\ &= 2e^{2t}. \end{aligned}$$

31. Substituting $\frac{1}{2}e^{-t}$ for $y(t)$ gives

$$y''(t) - 49y(t) = \left(\frac{1}{2}e^{-t}\right)'' - 49\left(\frac{1}{2}e^{-t}\right) = \frac{1}{2}e^{-t} - \frac{49}{2}e^{-t} = -24e^{-t}.$$

Substituting $\frac{1}{2}e^{-t} + 3e^{7t}$ for $y(t)$ gives

$$y''(t) - 49y(t) = \left(\frac{1}{2}e^{-t} + 3e^{7t}\right)'' - 49\left(\frac{1}{2}e^{-t} + 3e^{7t}\right) = \frac{1}{2}e^{-t} + 147e^{7t} - \frac{49}{2}e^{-t} - 147e^{7t} = -24e^{-t}.$$

Thus both of the functions given are in fact particular solutions. Their difference is $3e^{7t}$; substituting this into the equation gives

$$y''(t) - 49y(t) = (3e^{7t})'' - 49(3e^{7t}) = 147e^{7t} - 147e^{7t} = 0,$$

so that the two particular solutions differ by a solution of the homogeneous equation.

33. Substituting $-e^t$ for $y(t)$ gives

$$y''(t) - y'(t) - 12y(t) = (-e^t)'' - (-e^t)' - 12(-e^t) = -e^t + e^t + 12e^t = 12e^t.$$

Substituting $6e^{4t} - e^t$ for $y(t)$ gives

$$\begin{aligned} y''(t) - y'(t) - 12y(t) &= (6e^{4t} - e^t)'' - (6e^{4t} - e^t)' - 12(6e^{4t} - e^t) \\ &= 96e^{4t} - e^t - (24e^{4t} - e^t) - 72e^{4t} + 12e^t \\ &= 12e^t. \end{aligned}$$

Thus both of the functions given are in fact particular solutions. Their difference is $6e^{4t}$; substituting this into the equation gives

$$y''(t) - y'(t) - 12y(t) = (6e^{4t})'' - (6e^{4t})' - 12(6e^{4t}) = 96e^{4t} - 24e^{4t} - 72e^{4t} = 0,$$

so that the two particular solutions differ by a solution of the homogeneous equation.

35. Evaluating the differential expression $y''(t) + 2y(t)$ for the three values given, we get:

$$\begin{aligned} (\sin\sqrt{2}\,t)'' + 2\sin\sqrt{2}\,t &= -2\sin\sqrt{2}\,t + 2\sin\sqrt{2}\,t = 0 \\ (e^t)'' + 2e^t &= e^t + 2e^t = 3e^t \\ (\cos\sqrt{2}\,t)'' + 2\cos\sqrt{2}\,t &= -2\cos\sqrt{2}\,t + 2\cos\sqrt{2}\,t = 0. \end{aligned}$$

Thus $\sin\sqrt{2}\,t$ and $\cos\sqrt{2}\,t$ are solutions of the homogeneous equation and e^t is a solution of the nonhomogeneous equation. Since $\sin\sqrt{2}\,t$ and $\cos\sqrt{2}\,t$ are linearly independent, the general solution of the nonhomogeneous equation is

$$y(t) = c_1 \sin\sqrt{2}\,t + c_2\cos\sqrt{2}\,t + e^t.$$

37. Evaluating the differential expression $y''(t) - 3y'(t) + \frac{25}{4}y(t)$ for the three values given, we get

$$\begin{aligned}
&(e^{3t/2}\cos 2t)'' - 3(e^{3t/2}\cos 2t)' + \frac{25}{4}(e^{3t/2}\cos 2t)\\
&\quad= \left(\frac{3}{2}e^{3t/2}\cos 2t - 2e^{3t/2}\sin 2t\right)' - 3\left(\frac{3}{2}e^{3t/2}\cos 2t - 2e^{3t/2}\sin t\right) + \frac{25}{4}(e^{3t/2}\cos 2t)\\
&\quad= \left(e^{3t/2}\left(\frac{3}{2}\cos 2t - 2\sin 2t\right)\right)' - 3\left(\frac{3}{2}e^{3t/2}\cos 2t - 2e^{3t/2}\sin t\right) + \frac{25}{4}(e^{3t/2}\cos 2t)\\
&\quad= \frac{3}{2}e^{3t/2}\left(\frac{3}{2}\cos 2t - 2\sin 2t\right) + e^{3t/2}(-3\sin 2t - 4\cos 2t)\\
&\qquad - \frac{9}{2}e^{3t/2}\cos 2t + 6e^{3t/2}\sin 2t + \frac{25}{4}e^{3t/2}\cos 2t\\
&\quad= e^{3t/2}\left(\frac{9}{4}\cos 2t - 3\sin 2t - 3\sin 2t - 4\cos 2t - \frac{9}{2}\cos 2t + 6\sin 2t + \frac{25}{4}\cos 2t\right)\\
&\quad= 0
\end{aligned}$$

$$\begin{aligned}
&(e^{3t/2}\sin 2t)'' - 3(e^{3t/2}\sin 2t)' + \frac{25}{4}(e^{3t/2}\sin 2t)\\
&\quad= \left(\frac{3}{2}e^{3t/2}\sin 2t + 2e^{3t/2}\cos 2t\right)' - 3\left(\frac{3}{2}e^{3t/2}\sin 2t + 2e^{3t/2}\cos t\right) + \frac{25}{4}(e^{3t/2}\sin 2t)\\
&\quad= \left(e^{3t/2}\left(\frac{3}{2}\sin 2t + 2\cos 2t\right)\right)' - 3\left(\frac{3}{2}e^{3t/2}\sin 2t + 2e^{3t/2}\cos t\right) + \frac{25}{4}(e^{3t/2}\sin 2t)\\
&\quad= \frac{3}{2}e^{3t/2}\left(\frac{3}{2}\sin 2t + 2\cos 2t\right) + e^{3t/2}(3\cos 2t - 4\sin 2t)\\
&\qquad - \frac{9}{2}e^{3t/2}\sin 2t - 6e^{3t/2}\cos 2t + \frac{25}{4}e^{3t/2}\sin 2t\\
&\quad= e^{3t/2}\left(\frac{9}{4}\sin 2t + 3\cos 2t + 3\cos 2t - 4\sin 2t - \frac{9}{2}\sin 2t - 6\cos 2t + \frac{25}{4}\sin 2t\right)\\
&\quad= 0
\end{aligned}$$

$$(48+100t)'' - 3(48+100t)' + \frac{25}{4}(48+100t) = 0 - 300 + 300 + 625t = 625t.$$

Thus $e^{3t/2}\cos 2t$ and $e^{3t/2}\sin 2t$ are linearly independent solutions of the homogeneous equation, and $48 + 100t$ is a particular solution of the nonhomogeneous equation. Thus the general solution of the nonhomogeneous equation is

$$y(t) = c_1 e^{3t/2}\cos 2t + c_2 e^{3t/2}\sin 2t + 48 + 100t.$$

39. Substituting the initial conditions into $y(t)$ gives the system of simultaneous equations

$$\begin{aligned} c_1\sin 0 + c_2\cos 0 &= y(0) = 4\\ 3c_1\cos 0 - 3c_2\sin 0 &= y'(0) = 0 \end{aligned} \qquad \text{so that} \qquad \begin{aligned} c_2 &= 4\\ 3c_1 \quad &= 0. \end{aligned}$$

Thus $c_1 = 0$ and $c_2 = 4$, and the solution to the initial value problem is $y(t) = 4\cos 3t$.

41. Substituting the initial conditions into $y(t)$ gives the system of simultaneous equations

$$\begin{aligned} c_1 e^{5\cdot 0} + \ \ c_2 e^{-4\cdot 0} &= \ y(0) = -3 \\ 5c_1 e^{5\cdot 0} - 4c_2 e^{-4\cdot 0} &= y'(0) = \ \ \ 3 \end{aligned} \quad \text{so that} \quad \begin{aligned} c_1 + \ \ c_2 &= -3 \\ 5c_1 - 4c_2 &= \ \ \ 3. \end{aligned}$$

Thus $c_1 = -1$ and $c_2 = -2$, and the solution to the initial value problem is $y(t) = -e^{5t} - 2e^{-4t}$.

43. Substituting the initial conditions into $y(t)$ gives the system of simultaneous equations

$$\begin{aligned} c_1 e^{4\cdot 0} + \ \ c_2 e^{-4\cdot 0} - \ \ 0^2 - \frac{1}{8} &= \ y(0) = 0 \\ 4c_1 e^{4\cdot 0} - 4c_2 e^{-4\cdot 0} - 2\cdot 0 \qquad &= y'(0) = 0 \end{aligned} \quad \text{so that} \quad \begin{aligned} c_1 + \ \ c_2 &= \frac{1}{8} \\ 4c_1 - 4c_2 &= \ 0. \end{aligned}$$

Thus $c_1 = c_2 = \frac{1}{16}$, and the solution to the initial value problem is $y(t) = \frac{1}{16}e^{4t} + \frac{1}{16}e^{-4t} - t^2 - \frac{1}{8}$.

45. Substituting the initial conditions into $y(t)$ gives the system of simultaneous equations

$$\begin{aligned} c_1 \cdot 1^{-2} + c_2 \cdot 1^2 &= \ y(1) = \ \ \ 1 \\ -2c_1 \cdot 1^{-3} + 2c_2 \cdot 1 &= y'(1) = -1 \end{aligned} \quad \text{so that} \quad \begin{aligned} c_1 + \ \ c_2 &= 1 \\ -2c_1 + 2c_2 &= \ -1. \end{aligned}$$

Thus $c_1 = \frac{3}{4}$ and $c_2 = \frac{1}{4}$, and the solution to the initial value problem is $y(t) = \frac{3}{4}t^{-2} + \frac{1}{4}t^2$.

47. (a) False. By Theorems 16.2 and 16.4, a second-order linear differential equation has two linearly independent solutions, so that the general solution must involve two terms with arbitrary constants. Note that 0 is linearly dependent with any nonzero function, so that these theorems imply that neither linearly independent solution is everywhere zero.

(b) True. Substituting $y_p + cy_h$ into the nonhomogeneous equation gives

$$\begin{aligned} y'' + py' + qy &= (y_p + cy_h)'' + p(y_p + cy_h)' + q(y_p + cy_h) \\ &= (y_p'' + py_p' + qy_p) + c(y_h'' + py_h' + qy_h) \\ &= f + 0 = f, \end{aligned}$$

so that $y_p + cy_h$ satisfies the nonhomogeneous equation. This is the content of Theorem 16.4.

(c) False. Since $1 - \cos^2 x = \sin^2 x$, this pair of function is $\{\sin^2 x, 5\sin^2 x\}$, which are obviously constant multiples of one another and thus linearly dependent.

(d) False. Substitute $y_1 + y_2$ into the formula to get

$$\begin{aligned} y'' + yy' &= (y_1 + y_2)'' + (y_1 + y_2)(y_1 + y_2)' \\ &= y_1'' + y_2'' + y_1y_1' + y_2y_2' + y_1y_2' + y_2y_1' \\ &= (y_1'' + y_1y_1') + (y_2'' + y_2y_2') + y_1y_2' + y_2y_1' \\ &= y_1y_2' + y_2y_1' \end{aligned}$$

since both y_1 and y_2 satisfy the differential equation. Since there is no reason to expect $y_1y_2' + y_2y_1'$ to be zero, we see that $y_1 + y_2$ need not be a solution of the equation. This does not violate Theorem 16.1 since the given equation is not linear.

(e) False. The general solution of this equation is $y(t) = c_1 \sin\sqrt{2}\,t + c_2 \cos\sqrt{2}\,t$. The condition $y(0) = 4$ means that $c_2 = 4$. We need a second condition in order to get a value for c_1. Thus there are multiple solutions.

49. Substitution gives

$$
\begin{aligned}
y''(t) - 12y'(t) + 36y(t) &= (C_1e^{6t} + C_2te^{6t} + t^2e^{6t})'' - 12(C_1e^{6t} + C_2te^{6t} + t^2e^{6t})' \\
&\quad + 36(C_1e^{6t} + C_2te^{6t} + t^2e^{6t}) \\
&= 36C_1e^{6t} + (C_2e^{6t} + 6C_2te^{6t} + 2te^{6t} + 6t^2e^{6t})' \\
&\quad - 72C_1e^{6t} - 12C_2e^{6t} - 72C_2te^{6t} - 24te^{6t} - 72t^2e^{6t} \\
&\quad + 36C_1e^{6t} + 36C_2te^{6t} + 36t^2e^{6t} \\
&= 36C_1e^{6t} + 6C_2e^{6t} + 6C_2e^{6t} + 36C_2te^{6t} + 2e^{6t} + 12te^{6t} + 12te^{6t} + 36t^2e^{6t} \\
&\quad - 72C_1e^{6t} - 12C_2e^{6t} - 72C_2te^{6t} - 24te^{6t} - 72t^2e^{6t} \\
&\quad + 36C_1e^{6t} + 36C_2te^{6t} + 36t^2e^{6t} \\
&= 36C_1e^{6t} + 12C_2e^{6t} + 36C_2te^{6t} + 2e^{6t} + 24te^{6t} + 36t^2e^{6t} \\
&\quad - 72C_1e^{6t} - 12C_2e^{6t} - 72C_2te^{6t} - 24te^{6t} - 72t^2e^{6t} \\
&\quad + 36C_1e^{6t} + 36C_2te^{6t} + 36t^2e^{6t} \\
&= 2e^{6t}.
\end{aligned}
$$

51. Substitution gives

$$
\begin{aligned}
t^2y''(t) - 3ty'(t) + 4y(t) &= t^2(C_1t^2 + C_2t^2\ln t)'' - 3t(C_1t^2 + C_2t^2\ln t)' + 4(C_1t^2 + C_2t^2\ln t) \\
&= t^2(2C_1) + t^2(2C_2t\ln t + C_2t)' - 6C_1t^2 - 3t(2C_2t\ln t + C_2t) \\
&\quad + 4(C_1t^2 + C_2t^2\ln t) \\
&= 2C_1t^2 + 2C_2t^2\ln t + 2C_2t^2 + C_2t^2 - 6C_1t^2 - 6C_2t^2\ln t - 3C_2t^2 \\
&\quad + 4C_1t^2 + 4C_2t^2\ln t \\
&= 0.
\end{aligned}
$$

53. Substitution gives

$$
\begin{aligned}
t^2y''(t)+ty'(t)+\left(t^2-\frac{1}{4}\right)y(t) &= t^2\left(t^{-1/2}(C_1\cos t+C_2\sin t)\right)'' \\
&\quad +t\left(t^{-1/2}(C_1\cos t+C_2\sin t)\right)'+\left(t^2-\frac{1}{4}\right)\left(t^{-1/2}(C_1\cos t+C_2\sin t)\right) \\
&= t^2\left(-\frac{1}{2}t^{-3/2}(C_1\cos t+C_2\sin t)+t^{-1/2}(-C_1\sin t+C_2\cos t)\right)' \\
&\quad +t\left(-\frac{1}{2}t^{-3/2}(C_1\cos t+C_2\sin t)+t^{-1/2}(-C_1\sin t+C_2\cos t)\right) \\
&\quad +C_1t^{3/2}\cos t+C_2t^{3/2}\sin t-\frac{1}{4}C_1t^{-1/2}\cos t-\frac{1}{4}C_2t^{-1/2}\sin t \\
&= t^2\left(\frac{3}{4}t^{-5/2}(C_1\cos t+C_2\sin t)-\frac{1}{2}t^{-3/2}(-C_1\sin t+C_2\cos t)\right) \\
&\quad +t^2\left(-\frac{1}{2}t^{-3/2}(-C_1\sin t+C_2\cos t)+t^{-1/2}(-C_1\cos t-C_2\sin t)\right) \\
&\quad -\frac{1}{2}C_1t^{-1/2}\cos t-\frac{1}{2}C_2t^{-1/2}\sin t-C_1t^{1/2}\sin t+C_2t^{1/2}\cos t \\
&\quad +C_1t^{3/2}\cos t+C_2t^{3/2}\sin t-\frac{1}{4}C_1t^{-1/2}\cos t-\frac{1}{4}C_2t^{-1/2}\sin t \\
&= \frac{3}{4}t^{-1/2}(C_1\cos t+C_2\sin t)-\frac{1}{2}t^{1/2}(-C_1\sin t+C_2\cos t) \\
&\quad -\frac{1}{2}t^{1/2}(-C_1\sin t+C_2\cos t)+t^{3/2}(-C_1\cos t-C_2\sin t) \\
&\quad -\frac{1}{2}C_1t^{-1/2}\cos t-\frac{1}{2}C_2t^{-1/2}\sin t-C_1t^{1/2}\sin t+C_2t^{1/2}\cos t \\
&\quad +C_1t^{3/2}\cos t+C_2t^{3/2}\sin t-\frac{1}{4}C_1t^{-1/2}\cos t-\frac{1}{4}C_2t^{-1/2}\sin t \\
&= 0.
\end{aligned}
$$

55. (a) Substitution gives

$$
\begin{aligned}
y''-y &= (e^t)''-e^t=(e^t)'-e^t=e^t-e^t=0 \\
y''-y &= (e^{-t})''-e^{-t}=(-e^{-t})'-e^{-t}=e^{-t}-e^{-t}=0.
\end{aligned}
$$

(b) $\sinh t$ and $\cosh t$ are each linear combinations of the solutions e^t and e^{-t}, so they are both solutions. They are linearly independent since if $a\sinh t+b\cosh t=0$, then

$$
a\left(\frac{e^t-e^{-t}}{2}\right)+b\left(\frac{e^t+e^{-t}}{2}\right)=\frac{a+b}{2}e^t+\frac{b-a}{2}e^{-t}=0.
$$

Since e^t and e^{-t} are linearly independent, we must have $a+b=b-a=0$, so that $a=b=0$. This proves that $\sinh t$ and $\cosh t$ are linearly independent as well.

(c) Since $\sinh'=\cosh$ and $\cosh'=\sinh$, substitution gives

$$
\begin{aligned}
y''-y &= (\sinh t)''-\sinh t=(\cosh t)'-\sinh t=\sinh t-\sinh t=0 \\
y''-y &= (\cosh t)''-\cosh t=(\sinh t)'-\cosh t=\cosh t-\cosh t=0.
\end{aligned}
$$

(d) From part (a), the general solution is $C_1e^t+C_2e^{-t}$. From part (c), the general solution is $C_1\sinh t+C_2\cosh t$.

(e) Substitution gives

$$\begin{aligned} y'' - k^2 y &= (e^{kt})'' - k^2 e^{kt} = (ke^{kt})' - k^2 e^{kt} = k^2 e^{kt} - k^2 e^{kt} = 0 \\ y'' - k^2 y &= (e^{-kt})'' - k^2 e^{-kt} = (-ke^{-kt})' - k^2 e^{-kt} = k^2 e^{-kt} - k^2 e^{-kt} = 0. \end{aligned}$$

(f) In terms of exponentials, from part (e), the general solution is $C_1 e^{kt} + C_2 e^{-kt}$. Since $\cosh kt = \frac{e^{kt}+e^{-kt}}{2}$ and $\sinh kt = \frac{e^{kt}-e^{-kt}}{2}$, an identical argument to that in part (b) shows that $\cosh(kt)$ and $\sinh(kt)$ are also solutions to $y'' - k^2 y$ and that they are linearly independent. So in terms of hyperbolic functions, the general solution is $C_1 \sinh kt + C_2 \cosh kt$.

57. Note that $(e^{kt})^{(\mathrm{iv})} = k^4 e^{kt}$, $(\sin kt)^{(\mathrm{iv})} = k^4 \sin kt$, and $(\cos kt)^{(\mathrm{iv})} = k^4 \cos kt$. Thus with $y = C_1 e^{-2t} + C_2 e^{2t} + C_3 \sin 2t + C_4 \cos 2t$, we have

$$y^{(\mathrm{iv})} = 16C_1 e^{-2t} + 16C_2 e^{2t} + 16C_3 \sin 2t + 16C_4 \cos 2t = 16y(t).$$

59. (a) $\frac{d}{dt}\left(y'(t)^2\right) = 2y'(t)\frac{d}{dt}(y'(t)) = 2y'(t)y''(t)$.

(b) From part (a), $y''(t)y'(t) = \frac{1}{2} \cdot \frac{d}{dt}(y'(t)^2) = 1$, so that $(y'(t)^2)' = 2$.

(c) Integrating both sides with respect to t gives $\int (y'(t)^2)'\,dt = \int 2\,dt$, or $y'(t)^2 = 2t + C_1$ where C_1 is an arbitrary constant. Thus $y'(t) = \pm\sqrt{2t + C_1}$.

(d) Solving this equation simply involves integrating the right-hand side:

$$y(t) = \int \pm\sqrt{2t + C_1}\,dt = \int \pm(2t + C_1)^{1/2}\,dt = \pm\frac{1}{3}(2t + C_1)^{3/2} + C_2.$$

Thus there are two families of solutions.

61. (a) With $v = y'$, we have $v' = 3v + 4$, or $v' - 3v = 4$. The integrating factor is $e^{\int -3\,dt} = e^{-3t}$; this gives $e^{-3t}v' - 3e^{-3t}v = (e^{-3t}v)' = 4e^{-3t}$. Integrate both sides to get $e^{-3t}v = -\frac{4}{3}e^{-3t} + C_1$, so that $v = -\frac{4}{3} + C_1 e^{3t}$. This is the same as $y' = -\frac{4}{3} + C_1 e^{3t}$.

(b) Integrating once again gives $y = -\frac{4}{3}t + \frac{C_1}{3}e^{3t} + C_2 = C_2 - \frac{4}{3}t + C_3 e^{3t}$. As a check, note that

$$\begin{aligned} y'' &= (C_2 - \frac{4}{3}t + C_3 e^{3t})'' = 9C_3 e^{3t} \\ 3y' + 4 &= 3(C_2 - \frac{4}{3}t + C_3 e^{3t})' + 4 = -4 + 9C_3 e^{3t} + 4 = 9C_3 e^{3t}. \end{aligned}$$

63. (a) With $v = y'$ we get $v' = 2tv^2$, so that $v^{-2}v' = 2t$. Integrating both sides gives $-v^{-1} = t^2 + C_1$, so that $v = -\frac{1}{t^2+C_1}$. Substituting back gives $y' = -\frac{1}{t^2+C_1}$.

(b) Integrating once again gives if $C_1 > 0$

$$y = -\frac{1}{\sqrt{C_1}} \arctan\left(\frac{t}{\sqrt{C_1}}\right) + C_2.$$

and if $C_1 < 0$

$$y = \frac{1}{2\sqrt{|C_1|}} \ln\left|\frac{t + \sqrt{|C_1|}}{t - \sqrt{|C_1|}}\right| + C_2.$$

As a check, note that for $C_1 > 0$,

$$\begin{aligned} y' &= -\frac{1}{\sqrt{C_1}} \cdot \frac{\sqrt{C_1}}{t^2 + C_1} = -(t^2 + C_1)^{-1} \\ y'' &= 2t(t^2 + C_1)^{-2} \end{aligned}$$

so that indeed $y'' = 2t(y')^2$, and for $C_1 < 0$, regardless of the sign of $\frac{t+\sqrt{|C_1|}}{t-\sqrt{|C_1|}}$,

$$\begin{aligned}
y' &= \frac{-1}{t^2 - |C_1|} = \frac{-1}{t^2 + C_1} \\
y'' &= \frac{2t}{(t^2 + C_1)^2}
\end{aligned}$$

and again $y'' = 2t(y')^2$.

65. (a) Computing derivatives gives

$$\begin{aligned}
(e^{-3t/2}\sin 2t)' &= -\frac{3}{2}e^{-3t/2}\sin 2t + 2e^{-3t/2}\cos 2t = e^{-3t/2}\left(2\cos 2t - \frac{3}{2}\sin 2t\right) \\
(e^{-3t/2}\sin 2t)'' &= \left(-\frac{3}{2}e^{-3t/2}\sin 2t + 2e^{-3t/2}\cos 2t\right)' \\
&= \frac{9}{4}e^{-3t/2}\sin 2t - 3e^{-3t/2}\cos 2t - 3e^{-3t/2}\cos 2t - 4e^{-3t/2}\sin 2t \\
&= e^{-3t/2}\left(-6\cos 2t - \frac{7}{4}\sin 2t\right) \\
(e^{-3t/2}\cos 2t)' &= -\frac{3}{2}e^{-3t/2}\cos 2t - 2e^{-3t/2}\sin 2t = e^{-3t/2}\left(-\frac{3}{2}\cos 2t - 2\sin 2t\right) \\
(e^{-3t/2}\cos 2t)'' &= \left(-\frac{3}{2}e^{-3t/2}\cos 2t - 2e^{-3t/2}\sin 2t\right)' \\
&= \frac{9}{4}e^{-3t/2}\cos 2t + 3e^{-3t/2}\sin 2t + 3e^{-3t/2}\sin 2t - 4e^{-3t/2}\cos 2t \\
&= e^{-3t/2}\left(-\frac{7}{4}\cos 2t + 6\sin 2t\right)
\end{aligned}$$

Substituting $e^{-3t/2}\sin 2t$ and $e^{-3t/2}\cos 2t$ gives

$$\begin{aligned}
y'' + 3y' + \frac{25}{4}y &= (e^{-3t/2}\sin 2t)'' + 3(e^{-3t/2}\sin 2t)' + \frac{25}{4}(e^{-3t/2}\sin 2t) \\
&= e^{-3t/2}\left(-6\cos 2t - \frac{7}{4}\sin 2t\right) + 3e^{-3t/2}\left(2\cos 2t - \frac{3}{2}\sin 2t\right) \\
&\quad + \frac{25}{4}(e^{-3t/2}\sin 2t) \\
&= e^{-3t/2}\left(-6\cos 2t - \frac{7}{4}\sin 2t + 6\cos 2t - \frac{9}{2}\sin 2t + \frac{25}{4}\sin 2t\right) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
y'' + 3y' + \frac{25}{4}y &= (e^{-3t/2}\cos 2t)'' + 3(e^{-3t/2}\cos 2t)' + \frac{25}{4}(e^{-3t/2}\cos 2t) \\
&= e^{-3t/2}\left(-\frac{7}{4}\cos 2t + 6\sin 2t\right) + 3e^{-3t/2}\left(-\frac{3}{2}\cos 2t - 2\sin 2t\right) \\
&\quad + \frac{25}{4}(e^{-3t/2}\cos 2t) \\
&= e^{-3t/2}\left(-\frac{7}{4}\cos 2t + 6\sin 2t - \frac{9}{2}\cos 2t - 6\sin 2t + \frac{25}{4}\cos 2t\right) \\
&= 0
\end{aligned}$$

so that $e^{-3t/2}\sin 2t$ and $e^{-3t/2}\cos 2t$ are linearly independent solutions, so that the general solution is $y(t) = e^{-3t/2}(C_1 \sin 2t + C_2 \cos 2t)$.

(b) Since

$$\begin{aligned} y'(t) &= -\frac{3}{2}e^{-3t/2}(C_1 \sin 2t + C_2 \cos 2t) + e^{-3t/2}(2C_1 \cos 2t - 2C_2 \sin 2t) \\ &= e^{-3t/2}\left(\left(2\cos 2t - \frac{3}{2}\sin 2t\right)C_1 + \left(-2\sin 2t - \frac{3}{2}\cos 2t\right)C_2\right), \end{aligned}$$

substituting the initial conditions into $y(t)$ gives the system of simultaneous equations

$$\begin{aligned} e^{-3\cdot 0/2}(\sin(2\cdot 0)C_1 + \qquad\qquad \cos(2\cdot 0)C_2) &= y(0) = 4 \\ e^{-3\cdot 0/2}\left(\left(2\cos 0 - \frac{3}{2}\sin 0\right)C_1 + \left(-2\sin 0 - \frac{3}{2}\cos 0\right)C_2\right) &= y'(0) = 0 \end{aligned}$$

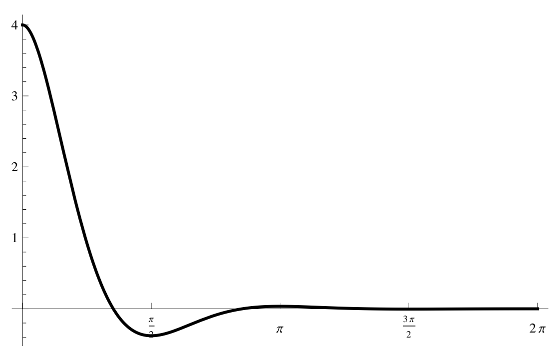
so that

$$\begin{aligned} C_2 &= 4 \\ 2C_1 - \frac{3}{2}C_2 &= 0. \end{aligned}$$

Thus $C_1 = 3$ and $C_2 = 4$, and the solution to the initial value problem is

$$y(t) = e^{-3t/2}(3\sin 2t + 4\cos 2t).$$

(c) A plot of the solution for $0 \le t \le 2\pi$ is



67. (a) Computing derivatives gives

$$\begin{aligned} (e^{-3t}\sin 4t)' &= -3e^{-3t}\sin 4t + 4e^{-3t}\cos 4t = e^{-3t}(-3\sin 4t + 4\cos 4t) \\ (e^{-3t}\sin 4t)'' &= (e^{-3t}(-3\sin 4t + 4\cos 4t))' \\ &= -3e^{-3t}(-3\sin 4t + 4\cos 4t) + e^{-3t}(-12\cos 4t - 16\sin 4t) \\ &= e^{-3t}(-7\sin 4t - 24\cos 4t) \\ (e^{-3t}\cos 4t)' &= -3e^{-3t}\cos 4t - 4e^{-3t}\sin 4t = e^{-3t}(-3\cos 4t - 4\sin 4t) \\ (e^{-3t}\cos 4t)'' &= (e^{-3t}(-3\cos 4t - 4\sin 4t))' \\ &= -3e^{-3t}(-3\cos 4t - 4\sin 4t) + e^{-3t}(12\sin 4t - 16\cos 4t) \\ &= e^{-3t}(24\sin 4t - 7\cos 4t) \end{aligned}$$