INSTRUCTOR'S SOLUTIONS MANUAL

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EARLY TRANSCENDENTALS

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Chapter D2

Second-Order Differential Equations

D2.1 Basic Ideas

1. The *order* of a differential equation is the highest-order derivative that appears in the equation. Thus for example $y'(t) + y(t) = 0$ is a first-order equation, while $y''(t) + y(t) = 0$ is a second-order equation.

2. A differential equation is *linear* if each additive term of the equation either does not depend on the unknown function y (so that it is either constant or depends only on t), or is a multiple of y or one of its derivatives by a constant or by a function of t only. In other words, a differential equation is linear if it is of the form

$$y^{(n)}(t) + p_{n-1}(t)y^{(n-1)}(t) + \cdots + p_1(t)y'(t) + p_0(t)y(t) = f(t).$$

   A *nonlinear* differential equation is one that is not linear.

3. A differential equation $y''(t) + p(t)y'(t) + q(t)y(t) = f(t)$ is *homogeneous* if $f(t) = 0$ for t in the domain we are interested in. It is *nonhomogeneous* if this is not the case. Thus for example $y''(t) + 3ty(t) = 0$ is homogeneous, while $y''(t) + 3ty(t) = t^2$ is nonhomogeneous.

4. The general form is $y''(t) + p(t)y'(t) + q(t)y(t) = f(t)$. If the original form of the equation has a coefficient on $y''(t)$ other than 1, simply divide through by it to get an equation in this general form.

5. Two functions f and g are linearly dependent on an interval I if there is some nonzero constant c such that for each $x \in I$ we have $f(x) = cg(x)$. That is, they are linearly dependent if one is a nonzero constant multiple of another.

6. By Theorem 16.2, there are two linearly independent solutions to a second-order linear homogeneous differential equation.

7. The general solution of a second-order linear nonhomogeneous differential equation is the sum of (a) any single particular solution of the nonhomogeneous equation, and (b) the general solution of the homogeneous equation derived by setting $f(t) = 0$ in the nonhomogeneous equation. See Theorems 16.3 and 16.4.

8. If $y''(t) + p(t)y'(t) + q(t)y(t) = f(t)$ is a second-order linear nonhomogeneous differential equation with initial conditions $y(0) = A$ and $y'(0) = B$, we solve it as follows: first find the general solution of the corresponding homogeneous differential equation $y''(t) + p(t)y'(t) + q(t)y(t) = 0$; this will have the form $c_1y_1(t) + c_2y_2(t)$ where $y_1(t)$ and $y_2(t)$ are linearly independent solutions

to the homogeneous equation. Next, find any particular solution, say $y_3(t)$, to the original nonhomogeneous equation. By Theorem 16.4, the general solution to the nonhomogeneous equation is then $c_1y_1(t) + c_2y_2(t) + y_3(t)$. Now use the initial conditions to construct the two equations

$$\begin{aligned} c_1y_1(0) + c_2y_2(0) &= A - y_3(0) \\ c_1y_1'(0) + c_2y_2'(0) &= B - y_3'(0) \end{aligned}$$

and solve these for c_1 and c_2.

9. Since the highest order derivative appearing is the second derivative, this is a second-order differential equation. Since y and its derivatives only appear in terms by themselves, not with other derivatives of y, it is linear. Finally, since there is a nonzero term $(10t^2)$ that does not depend on y, it is nonhomogeneous.

10. Since the highest order derivative appearing is the first derivative, this is a first-order differential equation. Since there is a y^3 term, it is nonlinear. Finally, since the term $-4t$ is a nonzero term not depending on y, it is nonhomogeneous.

11. Since the highest order derivative appearing is the second derivative, this is a second-order differential equation. Since there is a term involving yy', it is nonlinear. Finally, since there is a nonzero term (e^t) that does not depend on y, it is nonhomogeneous.

12. Since the highest order derivative appearing is the second derivative, this is a second-order differential equation. Since z and its derivatives only appear in terms by themselves, not with other derivatives of z, it is linear. Finally, since every nonzero term depends on z, it is homogeneous.

13. Since $\frac{d^2}{dt^2}e^{kt} = \frac{d}{dt}(ke^{kt}) = k^2e^{kt}$, we have

$$y''(t) - 4y(t) = (3e^{2t} - 5e^{-2t})'' - 4(3e^{2t} - 5e^{-2t}) = 12e^{2t} - 20e^{-2t} - (12e^{2t} - 20e^{-2t}) = 0.$$

14. Since $(\sin at)'' = (a\cos at)' = -a^2 \sin at$ and $(\cos at)'' = (-a\sin at)' = -a^2\cos at$, we have

$$\begin{aligned} y''(t) + 16y(t) &= (10\sin 4t - 20\cos 4t)'' + 16(10\sin 4t - 20\cos 4t) \\ &= -160\sin 4t + 320\cos 4t + (160\sin 4t - 320\cos 4t) \\ &= 0. \end{aligned}$$

15. Since $\frac{d^2}{dt^2}e^{kt} = \frac{d}{dt}(ke^{kt}) = k^2e^{kt}$, we have

$$\begin{aligned} y''(t) - 9y(t) &= (4e^{3t} + 3e^{-3t} - 2t)'' - 9(4e^{3t} + 3e^{-3t} - 2t) \\ &= (36e^{3t} + 27e^{-3t}) - (36e^{3t} + 27e^{-3t} - 18t) \\ &= 18t. \end{aligned}$$

16. Using the derivatives of $\sin at$ and $\cos at$ from Exercise 14, we have

$$\begin{aligned} y''(t) + 25y(t) &= \left(2\sin 5t - 6\cos 5t + \frac{1}{2}\cos t\right)'' + 25\left(2\sin 5t - 6\cos 5t + \frac{1}{2}\cos t\right) \\ &= \left(-50\sin 5t + 150\cos 5t - \frac{1}{2}\cos t\right) + \left(50\sin 5t - 150\cos 5t + \frac{25}{2}\cos t\right) \\ &= 12\cos t. \end{aligned}$$

17. We have

$$\begin{aligned}
y''(t) - y'(t) - 2y(t) &= (C_1e^{-t} + C_2e^{2t})'' - (C_1e^{-t} + C_2e^{2t})' - 2(C_1e^{-t} + C_2e^{2t}) \\
&= (C_1e^{-t} + 4C_2e^{2t}) - (-C_1e^{-t} + 2C_2e^{2t}) - (2C_1e^{-t} + 2C_2e^{2t}) \\
&= 0.
\end{aligned}$$

18. We have

$$\begin{aligned}
y''(t) + 2y'(t) - 3y(t) &= (C_1e^{-3t} + C_2e^{t} + e^{2t})'' + 2(C_1e^{-3t} + C_2e^{t} + e^{2t})' - 3(C_1e^{-3t} + C_2e^{t} + e^{2t}) \\
&= (9C_1e^{-3t} + C_2e^{t} + 4e^{2t}) + (-6C_1e^{-3t} + 2C_2e^{t} + 4e^{2t}) \\
&\quad - (3C_1e^{-3t} + 3C_2e^{t} + 3e^{2t}) \\
&= 5e^{2t}.
\end{aligned}$$

19. We have

$$\begin{aligned}
y''(t) + 6y'(t) + 25y(t) &= \left(e^{-3t}(C_1 \sin 4t + C_2 \cos 4t)\right)'' + 6\left(e^{-3t}(C_1 \sin 4t + C_2 \cos 4t)\right)' \\
&\quad + 25\left(e^{-3t}(C_1 \sin 4t + C_2 \cos 4t)\right) \\
&= \left(-3e^{-3t}(C_1 \sin 4t + C_2 \cos 4t) + e^{-3t}(4C_1 \cos 4t - 4C_2 \sin 4t)\right)' \\
&\quad + 6\left(-3e^{-3t}(C_1 \sin 4t + C_2 \cos 4t) + e^{-3t}(4C_1 \cos 4t - 4C_2 \sin 4t)\right) \\
&\quad + 25\left(e^{-3t}(C_1 \sin 4t + C_2 \cos 4t)\right) \\
&= \left(e^{-3t}((-3C_1 - 4C_2) \sin 4t + (4C_1 - 3C_2) \cos 4t)\right)' \\
&\quad + 6\left(-3e^{-3t}(C_1 \sin 4t + C_2 \cos 4t) + e^{-3t}(4C_1 \cos 4t - 4C_2 \sin 4t)\right) \\
&\quad + 25\left(e^{-3t}(C_1 \sin 4t + C_2 \cos 4t)\right) \\
&= -3e^{-3t}((-3C_1 - 4C_2) \sin 4t + (4C_1 - 3C_2) \cos 4t) \\
&\quad + e^{-3t}((-12C_1 - 16C_2) \cos 4t + (-16C_1 + 12C_2) \sin 4t) \\
&\quad + 6\left(-3e^{-3t}(C_1 \sin 4t + C_2 \cos 4t) + e^{-3t}(4C_1 \cos 4t - 4C_2 \sin 4t)\right) \\
&\quad + 25\left(e^{-3t}(C_1 \sin 4t + C_2 \cos 4t)\right) \\
&= e^{-3t}((9C_1 + 12C_2) \sin 4t + (-12C_1 + 9C_2) \cos 4t) \\
&\quad + e^{-3t}((-16C_1 + 12C_2) \sin 4t + (-12C_1 - 16C_2) \cos 4t) \\
&\quad + e^{-3t}((-18C_1 - 24C_2) \sin 4t + (24C_1 - 18C_2) \cos 4t) \\
&\quad + e^{-3t}(25C_1 \sin 4t + 25C_2 \cos 4t) \\
&= 0.
\end{aligned}$$

20. We have

$$
\begin{aligned}
y''(t)+8y'(t)+25y(t) &= \left(e^{-4t}(C_1\sin 3t+C_2\cos 3t)+2\right)''+8\left(e^{-4t}(C_1\sin 3t+C_2\cos 3t)+2\right)'\\
&\quad+25\left(e^{-4t}(C_1\sin 3t+C_2\cos 3t)+2\right)\\
&= \left(-4e^{-4t}(C_1\sin 3t+C_2\cos 3t)+e^{-4t}(3C_1\cos 3t-3C_2\sin 3t)\right)'\\
&\quad+8(-4e^{-4t}(C_1\sin 3t+C_2\cos 3t)+e^{-4t}(3C_1\cos 3t-3C_2\sin 3t))\\
&\quad+25\left(e^{-4t}(C_1\sin 3t+C_2\cos 3t)+2\right)\\
&= \left(e^{-4t}((-4C_1-3C_2)\sin 3t+(3C_1-4C_2)\cos 3t)\right)'\\
&\quad e^{-4t}(-32C_1\sin 3t-32C_2\cos 3t)+e^{-4t}(24C_1\cos 3t-24C_2\sin 3t))\\
&\quad+e^{-4t}(25C_1\sin 3t+25C_2\cos 3t)+50\\
&= \left(-4e^{-4t}((-4C_1-3C_2)\sin 3t+(3C_1-4C_2)\cos 3t)\right)\\
&\quad+e^{-4t}((-12C_1-9C_2)\cos 3t+(-9C_1+12C_2)\sin 3t)\\
&\quad+e^{-4t}((-32C_1-24C_2)\sin 3t+(24C_1-32C_2)\cos 3t)\\
&\quad+e^{-4t}(25C_1\sin 3t+25C_2\cos 3t)+50\\
&= e^{-4t}((16C_1+12C_2)\sin 3t+(-12C_1+16C_2)\cos 3t)\\
&\quad+e^{-4t}((-9C_1+12C_2)\sin 3t+(-12C_1-9C_2)\cos 3t)\\
&\quad+e^{-4t}((-32C_1-24C_2)\sin 3t+(24C_1-32C_2)\cos 3t)\\
&\quad+e^{-4t}(25C_1\sin 3t+25C_2\cos 3t)+50\\
&= 50.
\end{aligned}
$$

21. We have

$$
\begin{aligned}
ty''(t)-(t+1)y'(t)+y(t) &= t\left(C_1e^t+C_2(t+1)\right)''-(t+1)\left(C_1e^t+C_2(t+1)\right)'\\
&\quad+\left(C_1e^t+C_2(t+1)\right)\\
&= t\left(C_1e^t+C_2\right)'-(t+1)\left(C_1e^t+C_2\right)+\left(C_1e^t+C_2(t+1)\right)\\
&= tC_1e^t-tC_1e^t-tC_2-C_1e^t-C_2+C_1e^t+tC_2+C_2\\
&= 0.
\end{aligned}
$$

22. We have

$$
\begin{aligned}
t^2y''(t)+2ty'(t)-2y(t) &= t^2\left(C_1t^{-2}+C_2t+\frac{1}{2}t^3\right)''+2t\left(C_1t^{-2}+C_2t+\frac{1}{2}t^3\right)'\\
&\quad-2\left(C_1t^{-2}+C_2t+\frac{1}{2}t^3\right)\\
&= t^2\left(-2C_1t^{-3}+C_2+\frac{3}{2}t^2\right)'+2t\left(-2C_1t^{-3}+C_2+\frac{3}{2}t^2\right)\\
&\quad-2\left(C_1t^{-2}+C_2t+\frac{1}{2}t^3\right)\\
&= t^2\left(6C_1t^{-4}+3t\right)-4C_1t^{-2}+2C_2t+3t^3-2C_1t^{-2}-2C_2t-t^3\\
&= 6C_1t^{-2}+3t^3-4C_1t^{-2}+2C_2t+3t^3-2C_1t^{-2}-2C_2t-t^3\\
&= 5t^3.
\end{aligned}
$$

23. The two given solutions are linearly independent, since for example at $t = 0$, $\frac{1}{5} \cdot 5e^{-6\cdot 0} = e^{6\cdot 0}$ while at $t = 1$ we see that $\frac{1}{5} \cdot 5e^{-6} = e^{-6} \neq e^{6}$, so that the two solutions do not differ by a constant multiple. Since the two given solutions are linearly independent, the general solution is $y(t) = C_1 e^{6t} + C_2 e^{-6t}$. Note that the coefficient of 5 in the second solution has been subsumed into the constant C_2.

24. The two given solutions are linearly independent, since for example at $t = 0$, $\sin \sqrt{5}\, t = 0 \cdot \cos \sqrt{5}\, t$, but this is not true at $t = \frac{\pi}{2}$, so that the two solutions do not differ by a constant multiple. Since the two given solutions are linearly independent, the general solution is $y(t) = C_1 \cos \sqrt{5}\, t + C_2 \sin \sqrt{5}\, t$.

25. The two solutions are linearly independent, since for example at $t = 0$, $te^{-t} = 0 \cdot e^{-t}$, but this is not true at $t = 1$, so that the two solutions do not differ by a constant multiple. Since the two solutions are linearly independent, the general solution is $y(t) = C_1 e^{-t} + C_2 te^{-t}$.

26. The two solutions are linearly independent, since for example at $t = 1$, $t = 1 \cdot t^{-1}$, but this is not true at $t = 2$, so that the two solutions do not differ by a constant multiple. Since the two solutions are linearly independent, the general solution is $y(t) = C_1 t + C_2 t^{-1}$.

27. $y''(t) - y(t) = (e^{-3t})'' - e^{-3t} = 9e^{-3t} - e^{-3t} = 8e^{-3t}$.

28. Substituting gives

$$\begin{aligned} y''(t) + y(t) &= (2\sin t - \cos 2t)'' + (2\sin t - \cos 2t) \\ &= (2\cos t + 2\sin 2t)' + 2\sin t - \cos 2t \\ &= -2\sin t + 4\cos 2t + 2\sin t - \cos 2t \\ &= 3\cos 2t. \end{aligned}$$

29. Substituting gives

$$\begin{aligned} y''(t) - 4y'(t) + 4y(t) &= (t^2e^{2t})'' - 4(t^2e^{2t})' + 4(t^2e^{2t}) \\ &= (2te^{2t} + 2t^2e^{2t})' - 4(2te^{2t} + 2t^2e^{2t}) + 4t^2e^{2t} \\ &= 2e^{2t} + 4te^{2t} + 4te^{2t} + 4t^2e^{2t} - 8te^{2t} - 8t^2e^{2t} + 4t^2e^{2t} \\ &= 2e^{2t}. \end{aligned}$$

30. Substituting gives

$$\begin{aligned} t^2y''(t) + ty'(t) - 4y(t) &= t^2(-2t + t^2)'' + t(-2t + t^2)' - 4(-2t + t^2) \\ &= t^2(-2 + 2t)' + t(-2 + 2t) + 8t - 4t^2 \\ &= 2t^2 - 2t + 2t^2 + 8t - 4t^2 = 6t. \end{aligned}$$

31. Substituting $\frac{1}{2}e^{-t}$ for $y(t)$ gives

$$y''(t) - 49y(t) = \left(\frac{1}{2}e^{-t}\right)'' - 49\left(\frac{1}{2}e^{-t}\right) = \frac{1}{2}e^{-t} - \frac{49}{2}e^{-t} = -24e^{-t}.$$

Substituting $\frac{1}{2}e^{-t} + 3e^{7t}$ for $y(t)$ gives

$$y''(t) - 49y(t) = \left(\frac{1}{2}e^{-t} + 3e^{7t}\right)'' - 49\left(\frac{1}{2}e^{-t} + 3e^{7t}\right) = \frac{1}{2}e^{-t} + 147e^{7t} - \frac{49}{2}e^{-t} - 147e^{7t} = -24e^{-t}.$$

Thus both of the functions given are in fact particular solutions. Their difference is $3e^{7t}$; substituting this into the equation gives

$$y''(t) - 49y(t) = (3e^{7t})'' - 49(3e^{7t}) = 147e^{7t} - 147e^{7t} = 0,$$

so that the two particular solutions differ by a solution of the homogeneous equation.

32. Substituting $2\sin t$ for $y(t)$ gives

$$y''(t) + 16y(t) = (2\sin t)'' + 16(2\sin t) = -2\sin t + 32\sin t = 30\sin t.$$

Substituting $2\sin t - 8\cos 4t$ for $y(t)$ gives

$$\begin{aligned} y''(t) + 16y(t) &= (2\sin t - 8\cos 4t)'' + 16(2\sin t - 8\cos 4t) \\ &= -2\sin t + 128\cos 4t + 32\sin t - 128\cos 4t \\ &= 30\sin t. \end{aligned}$$

Thus both of the functions given are in fact particular solutions. Their difference is $8\cos 4t$; substituting this into the equation gives

$$y''(t) + 16y(t) = (8\cos 4t)'' + 16(8\cos 4t) = -128\cos 4t + 128\cos 4t = 0,$$

so that the two particular solutions differ by a solution of the homogeneous equation.

33. Substituting $-e^t$ for $y(t)$ gives

$$y''(t) - y'(t) - 12y(t) = (-e^t)'' - (-e^t)' - 12(-e^t) = -e^t + e^t + 12e^t = 12e^t.$$

Substituting $6e^{4t} - e^t$ for $y(t)$ gives

$$\begin{aligned} y''(t) - y'(t) - 12y(t) &= (6e^{4t} - e^t)'' - (6e^{4t} - e^t)' - 12(6e^{4t} - e^t) \\ &= 96e^{4t} - e^t - (24e^{4t} - e^t) - 72e^{4t} + 12e^t \\ &= 12e^t. \end{aligned}$$

Thus both of the functions given are in fact particular solutions. Their difference is $6e^{4t}$; substituting this into the equation gives

$$y''(t) - y'(t) - 12y(t) = (6e^{4t})'' - (6e^{4t})' - 12(6e^{4t}) = 96e^{4t} - 24e^{4t} - 72e^{4t} = 0,$$

so that the two particular solutions differ by a solution of the homogeneous equation.

34. Substituting $-\frac{t^2}{2}$ for $y(t)$ gives

$$\begin{aligned} t^2y''(t) + 2ty'(t) - 30y(t) &= t^2\left(-\frac{t^2}{2}\right)'' + 2t\left(-\frac{t^2}{2}\right)' - 30\left(-\frac{t^2}{2}\right) \\ &= -t^2 - 2t^2 + 15t^2 \\ &= 12t^2. \end{aligned}$$

Substituting $3t^5 - \frac{t^2}{2}$ for $y(t)$ gives

$$\begin{aligned} t^2y''(t) + 2ty'(t) - 30y(t) &= t^2\left(3t^5 - \frac{t^2}{2}\right)'' + 2t\left(3t^5 - \frac{t^2}{2}\right)' - 30\left(3t^5 - \frac{t^2}{2}\right) \\ &= t^2(60t^3 - 1) + 2t(15t^4 - t) - 90t^5 + 15t^2 \\ &= 60t^5 - t^2 + 30t^5 - 2t^2 - 90t^5 + 15t^2 \\ &= 12t^2. \end{aligned}$$

Thus both of the functions given are in fact particular solutions. Their difference is $3t^5$; substituting this into the equation gives

$$t^2y''(t) + 2ty'(t) - 30y(t) = t^2(3t^5)'' + 2t(3t^5)' - 30(3t^5) = 60t^5 + 30t^5 - 90t^5 = 0,$$

so that the two particular solutions differ by a solution of the homogeneous equation.

35. Evaluating the differential expression $y''(t) + 2y(t)$ for the three values given, we get:

$$\begin{aligned}(\sin\sqrt{2}\,t)'' + 2\sin\sqrt{2}\,t &= -2\sin\sqrt{2}\,t + 2\sin\sqrt{2}\,t = 0\\ (e^t)'' + 2e^t = e^t + 2e^t &= 3e^t\\ (\cos\sqrt{2}\,t)'' + 2\cos\sqrt{2}\,t &= -2\cos\sqrt{2}\,t + 2\cos\sqrt{2}\,t = 0.\end{aligned}$$

Thus $\sin\sqrt{2}\,t$ and $\cos\sqrt{2}\,t$ are solutions of the homogeneous equation and e^t is a solution of the nonhomogeneous equation. Since $\sin\sqrt{2}\,t$ and $\cos\sqrt{2}\,t$ are linearly independent, the general solution of the nonhomogeneous equation is

$$y(t) = c_1\sin\sqrt{2}\,t + c_2\cos\sqrt{2}\,t + e^t.$$

36. Evaluating the differential expression $y''(t) - 4y(t)$ for the three values given, we get:

$$\begin{aligned}(5e^{2t})'' - 4(5e^{2t}) &= 20e^{2t} - 20e^{2t} = 0\\ (e^{-2t})'' - 4(e^{-2t}) &= 4e^{-2t} - 4e^{-2t} = 0\\ (-\cos t)'' - 4(-\cos t) &= \cos t + 4\cos t = 5\cos t.\end{aligned}$$

This $5e^{2t}$ and e^{-2t} are solutions of the homogeneous equation and $-\cos t$ is a solution of the nonhomogeneous equation. Since $5e^{2t}$ and e^{-2t} are linearly independent, the general solution of the nonhomogeneous equation is

$$y(t) = c_1e^{2t} + c_2e^{-2t} - \cos t.$$

Notice that the coefficient 5 of e^{2t} was subsumed into the constant c_1. Writing $y(t) = 5c_1e^{2t} + c_2e^{-2t} - \cos t$ is equally correct but unnecessarily complicated.

37. Evaluating the differential expression $y''(t) - 3y'(t) + \frac{25}{4}y(t)$ for the three values given, we get

$$\begin{aligned}&(e^{3t/2}\cos 2t)'' - 3(e^{3t/2}\cos 2t)' + \frac{25}{4}(e^{3t/2}\cos 2t)\\ &\qquad = \left(\frac{3}{2}e^{3t/2}\cos 2t - 2e^{3t/2}\sin 2t\right)' - 3\left(\frac{3}{2}e^{3t/2}\cos 2t - 2e^{3t/2}\sin t\right) + \frac{25}{4}(e^{3t/2}\cos 2t)\\ &\qquad = \left(e^{3t/2}\left(\frac{3}{2}\cos 2t - 2\sin 2t\right)\right)' - 3\left(\frac{3}{2}e^{3t/2}\cos 2t - 2e^{3t/2}\sin t\right) + \frac{25}{4}(e^{3t/2}\cos 2t)\\ &\qquad = \frac{3}{2}e^{3t/2}\left(\frac{3}{2}\cos 2t - 2\sin 2t\right) + e^{3t/2}(-3\sin 2t - 4\cos 2t)\\ &\qquad\qquad - \frac{9}{2}e^{3t/2}\cos 2t + 6e^{3t/2}\sin 2t + \frac{25}{4}e^{3t/2}\cos 2t\\ &\qquad = e^{3t/2}\left(\frac{9}{4}\cos 2t - 3\sin 2t - 3\sin 2t - 4\cos 2t - \frac{9}{2}\cos 2t + 6\sin 2t + \frac{25}{4}\cos 2t\right)\\ &\qquad = 0\end{aligned}$$

$$\begin{aligned}
&(e^{3t/2}\sin 2t)'' - 3(e^{3t/2}\sin 2t)' + \frac{25}{4}(e^{3t/2}\sin 2t)\\
&\qquad = \left(\frac{3}{2}e^{3t/2}\sin 2t + 2e^{3t/2}\cos 2t\right)' - 3\left(\frac{3}{2}e^{3t/2}\sin 2t + 2e^{3t/2}\cos t\right) + \frac{25}{4}(e^{3t/2}\sin 2t)\\
&\qquad = \left(e^{3t/2}\left(\frac{3}{2}\sin 2t + 2\cos 2t\right)\right)' - 3\left(\frac{3}{2}e^{3t/2}\sin 2t + 2e^{3t/2}\cos t\right) + \frac{25}{4}(e^{3t/2}\sin 2t)\\
&\qquad = \frac{3}{2}e^{3t/2}\left(\frac{3}{2}\sin 2t + 2\cos 2t\right) + e^{3t/2}(3\cos 2t - 4\sin 2t)\\
&\qquad\qquad - \frac{9}{2}e^{3t/2}\sin 2t - 6e^{3t/2}\cos 2t + \frac{25}{4}e^{3t/2}\sin 2t\\
&\qquad = e^{3t/2}\left(\frac{9}{4}\sin 2t + 3\cos 2t + 3\cos 2t - 4\sin 2t - \frac{9}{2}\sin 2t - 6\cos 2t + \frac{25}{4}\sin 2t\right)\\
&\qquad = 0\\
&(48+100t)'' - 3(48+100t)' + \frac{25}{4}(48+100t) = 0 - 300 + 300 + 625t = 625t.
\end{aligned}$$

Thus $e^{3t/2}\cos 2t$ and $e^{3t/2}\sin 2t$ are linearly independent solutions of the homogeneous equation, and $48 + 100t$ is a particular solution of the nonhomogeneous equation. Thus the general solution of the nonhomogeneous equation is

$$y(t) = c_1 e^{3t/2}\cos 2t + c_2 e^{3t/2}\sin 2t + 48 + 100t.$$

38. Evaluating the differential expression $t^2 y''(t) + 2ty'(t) - 6y(t)$ for the three values given, we get

$$\begin{aligned}
t^2(t^{-3})'' + 2t(t^{-3})' - 6(t^{-3}) &= 12t^{-3} - 6t^{-3} - 6t^{-3} = 0\\
t^2\left(\frac{t^4}{2}\right)'' + 2t\left(\frac{t^4}{2}\right)' - 6\left(\frac{t^4}{2}\right) &= 6t^4 + 4t^4 - 3t^4 = 7t^4\\
t^2(t^2)'' + 2t(t^2)' - 6(t^2) &= 2t^2 + 4t^2 - 6t^2 = 0.
\end{aligned}$$

Thus t^{-3} and t^2 are linearly independent solutions of the homogeneous equation, and $\frac{t^4}{2}$ is a particular solution of the nonhomogeneous equation. Thus the general solution of the nonhomogeneous equation is

$$y(t) = c_1 t^{-3} + c_2 t^2 + \frac{t^4}{2}.$$

39. Substituting the initial conditions into $y(t)$ gives the system of simultaneous equations

$$\begin{aligned}
c_1\sin 0 + c_2\cos 0 &= y(0) = 4\\
3c_1\cos 0 - 3c_2\sin 0 &= y'(0) = 0
\end{aligned}
\quad\text{so that}\quad
\begin{aligned}
c_2 &= 4\\
3c_1 \quad &= 0.
\end{aligned}$$

Thus $c_1 = 0$ and $c_2 = 4$, and the solution to the initial value problem is $y(t) = 4\cos 3t$.

40. Substituting the initial conditions into $y(t)$ gives the system of simultaneous equations

$$\begin{aligned}
c_1e^0 + c_2e^{-0} &= y(0) = 2\\
c_1e^0 - c_2e^{-0} &= y'(0) = -2
\end{aligned}
\quad\text{so that}\quad
\begin{aligned}
c_1 + c_2 &= 2\\
c_1 - c_2 &= -2.
\end{aligned}$$

Thus $c_1 = 0$ and $c_2 = 2$, and the solution to the initial value problem is $y(t) = 2e^{-t}$.

41. Substituting the initial conditions into $y(t)$ gives the system of simultaneous equations

$$\begin{aligned} c_1e^{5\cdot 0} + \ \ c_2e^{-4\cdot 0} &= y(0) = -3 \\ 5c_1e^{5\cdot 0} - 4c_2e^{-4\cdot 0} &= y'(0) = \ \ 3 \end{aligned} \quad \text{so that} \quad \begin{aligned} c_1 + \ \ c_2 &= -3 \\ 5c_1 - 4c_2 &= \ \ 3. \end{aligned}$$

Thus $c_1 = -1$ and $c_2 = -2$, and the solution to the initial value problem is $y(t) = -e^{5t} - 2e^{-4t}$.

42. Substituting the initial conditions into $y(t)$ gives the system of simultaneous equations

$$\begin{aligned} c_1 \sin 0 + \ \ c_2 \cos 0 - \ \ \cos 0 &= y(0) = 4 \\ 2c_1 \cos 0 - 2c_2 \sin 0 + 3 \sin 0 &= y'(0) = 2 \end{aligned} \quad \text{so that} \quad \begin{aligned} c_2 &= 5 \\ 2c_1 \quad &= 2. \end{aligned}$$

Thus $c_1 = 1$ and $c_2 = 5$, and the solution to the initial value problem is $y(t) = \sin 2t + 5\cos 2t - \cos 3t$.

43. Substituting the initial conditions into $y(t)$ gives the system of simultaneous equations

$$\begin{aligned} c_1e^{4\cdot 0} + \ \ c_2e^{-4\cdot 0} - \ \ 0^2 - \frac{1}{8} &= y(0) = 0 \\ 4c_1e^{4\cdot 0} - 4c_2e^{-4\cdot 0} - 2\cdot 0 \quad &= y'(0) = 0 \end{aligned} \quad \text{so that} \quad \begin{aligned} c_1 + \ \ c_2 &= \frac{1}{8} \\ 4c_1 - 4c_2 &= \ 0. \end{aligned}$$

Thus $c_1 = c_2 = \frac{1}{16}$, and the solution to the initial value problem is $y(t) = \frac{1}{16}e^{4t} + \frac{1}{16}e^{-4t} - t^2 - \frac{1}{8}$.

44. Substituting the initial conditions into $y(t)$ gives the system of simultaneous equations

$$\begin{aligned} c_1 \cdot 1^{-2} + c_2 \cdot 1 &= y(1) = 3 \\ -2c_1 \cdot 1^{-3} + \quad c_2 &= y'(1) = 0 \end{aligned} \quad \text{so that} \quad \begin{aligned} c_1 + c_2 &= \ 3 \\ -2c_1 + c_2 &= 0\,. \end{aligned}$$

Thus $c_1 = 1$ and $c_2 = 2$, and the solution to the initial value problem is $y(t) = t^{-2} + 2t$.

45. Substituting the initial conditions into $y(t)$ gives the system of simultaneous equations

$$\begin{aligned} c_1 \cdot 1^{-2} + c_2 \cdot 1^2 &= y(1) = \ \ 1 \\ -2c_1 \cdot 1^{-3} + 2c_2 \cdot 1 &= y'(1) = -1 \end{aligned} \quad \text{so that} \quad \begin{aligned} c_1 + \ \ c_2 &= 1 \\ -2c_1 + 2c_2 &= -1. \end{aligned}$$

Thus $c_1 = \frac{3}{4}$ and $c_2 = \frac{1}{4}$, and the solution to the initial value problem is $y(t) = \frac{3}{4}t^{-2} + \frac{1}{4}t^2$.

46. Using the facts that

$$\begin{aligned} (e^{-4t} \sin 3t)' &= -4e^{-4t} \sin 3t + 3e^{-4t} \cos 3t = e^{-4t}(3\cos 3t - 4\sin 3t) \\ (e^{-4t} \cos 3t)' &= -4e^{-4t} \cos 3t - 3e^{-4t} \sin 3t = -e^{-4t}(4\cos 3t + 3\sin 3t), \end{aligned}$$

substitute the initial conditions into $y(t)$ to get the system of simultaneous equations

$$\begin{aligned} c_1e^{-4\cdot 0} \sin 0 + \qquad\qquad\qquad c_2e^{-4\cdot 0} \cos 0 &= y(0) = \ \ 1 \\ c_1(e^{-4\cdot 0}(3\cos 0 - 4\sin 0)) - c_2(e^{-4\cdot 0}(4\cos 0 + 3\sin 0)) &= y'(0) = -1 \end{aligned}$$

so that

$$\begin{aligned} c_2 &= \ \ 1 \\ 3c_1 - 4c_2 &= -1. \end{aligned}$$

Thus $c_1 = c_2 = 1$, and the solution to the initial value problem is $y(t) = e^{-4t}(\sin 3t + \cos 3t)$.

47. (a) False. By Theorems 16.2 and 16.4, a second-order linear differential equation has two linearly independent solutions, so that the general solution must involve two terms with arbitrary constants. Note that 0 is linearly dependent with any nonzero function, so that these theorems imply that neither linearly independent solution is everywhere zero.

(b) True. Substituting $y_p + cy_h$ into the nonhomogeneous equation gives

$$\begin{aligned} y'' + py' + qy &= (y_p + cy_h)'' + p(y_p + cy_h)' + q(y_p + cy_h) \\ &= (y_p'' + py_p' + qy_p) + c(y_h'' + py_h' + qy_h) \\ &= f + 0 = f, \end{aligned}$$

so that $y_p + cy_h$ satisfies the nonhomogeneous equation. This is the content of Theorem 16.4.

(c) False. Since $1 - \cos^2 x = \sin^2 x$, this pair of function is $\{\sin^2 x, 5\sin^2 x\}$, which are obviously constant multiples of one another and thus linearly dependent.

(d) False. Substitute $y_1 + y_2$ into the formula to get

$$\begin{aligned} y'' + yy' &= (y_1 + y_2)'' + (y_1 + y_2)(y_1 + y_2)' \\ &= y_1'' + y_2'' + y_1y_1' + y_2y_2' + y_1y_2' + y_2y_1' \\ &= (y_1'' + y_1y_1') + (y_2'' + y_2y_2') + y_1y_2' + y_2y_1' \\ &= y_1y_2' + y_2y_1' \end{aligned}$$

since both y_1 and y_2 satisfy the differential equation. Since there is no reason to expect $y_1y_2' + y_2y_1'$ to be zero, we see that $y_1 + y_2$ need not be a solution of the equation. This does not violate Theorem 16.1 since the given equation is not linear.

(e) False. The general solution of this equation is $y(t) = c_1 \sin\sqrt{2}\,t + c_2 \cos\sqrt{2}\,t$. The condition $y(0) = 4$ means that $c_2 = 4$. We need a second condition in order to get a value for c_1. Thus there are multiple solutions.

48. Substitution gives

$$\begin{aligned} y''(t) - 12y'(t) + 36y(t) &= (C_1e^{6t} + C_2te^{6t})'' - 12(C_1e^{6t} + C_2te^{6t})' + 36(C_1e^{6t} + C_2te^{6t}) \\ &= 36C_1e^{6t} + C_2(e^{6t} + 6te^{6t})' - (72C_1e^{6t} + 12C_2e^{6t} + 72C_2te^{6t}) \\ &\qquad + 36C_1e^{6t} + 36C_2te^{6t} \\ &= 36C_1e^{6t} + 6C_2e^{6t} + 6C_2e^{6t} + 36C_2te^{6t} - 72C_1e^{6t} - 12C_2e^{6t} - 72C_2te^{6t} \\ &\qquad + 36C_1e^{6t} + 36C_2te^{6t} \\ &= 0. \end{aligned}$$

49. Substitution gives

$$
\begin{aligned}
y''(t) - 12y'(t) + 36y(t) &= (C_1e^{6t} + C_2te^{6t} + t^2e^{6t})'' - 12(C_1e^{6t} + C_2te^{6t} + t^2e^{6t})' \\
&\quad + 36(C_1e^{6t} + C_2te^{6t} + t^2e^{6t}) \\
&= 36C_1e^{6t} + (C_2e^{6t} + 6C_2te^{6t} + 2te^{6t} + 6t^2e^{6t})' \\
&\quad - 72C_1e^{6t} - 12C_2e^{6t} - 72C_2te^{6t} - 24te^{6t} - 72t^2e^{6t} \\
&\quad + 36C_1e^{6t} + 36C_2te^{6t} + 36t^2e^{6t} \\
&= 36C_1e^{6t} + 6C_2e^{6t} + 6C_2e^{6t} + 36C_2te^{6t} + 2e^{6t} + 12te^{6t} + 12te^{6t} + 36t^2e^{6t} \\
&\quad - 72C_1e^{6t} - 12C_2e^{6t} - 72C_2te^{6t} - 24te^{6t} - 72t^2e^{6t} \\
&\quad + 36C_1e^{6t} + 36C_2te^{6t} + 36t^2e^{6t} \\
&= 36C_1e^{6t} + 12C_2e^{6t} + 36C_2te^{6t} + 2e^{6t} + 24te^{6t} + 36t^2e^{6t} \\
&\quad - 72C_1e^{6t} - 12C_2e^{6t} - 72C_2te^{6t} - 24te^{6t} - 72t^2e^{6t} \\
&\quad + 36C_1e^{6t} + 36C_2te^{6t} + 36t^2e^{6t} \\
&= 2e^{6t}.
\end{aligned}
$$

50. Substitution gives

$$
\begin{aligned}
y''(t) + 4y(t) &= (C_1\sin 2t + C_2\cos 2t - 2t\cos 2t)'' + 4(C_1\sin 2t + C_2\cos 2t - 2t\cos 2t) \\
&= -4C_1\sin 2t - 4C_2\cos 2t - 2(\cos 2t - 2t\sin 2t)' + 4C_1\sin 2t + 4C_2\cos 2t - 8t\cos 2t \\
&= -2(-2\sin 2t - 2\sin 2t - 4t\cos 2t) - 8t\cos 2t \\
&= 8\sin 2t
\end{aligned}
$$

51. Substitution gives

$$
\begin{aligned}
t^2y''(t) - 3ty'(t) + 4y(t) &= t^2(C_1t^2 + C_2t^2\ln t)'' - 3t(C_1t^2 + C_2t^2\ln t)' + 4(C_1t^2 + C_2t^2\ln t) \\
&= t^2(2C_1) + t^2(2C_2t\ln t + C_2t)' - 6C_1t^2 - 3t(2C_2t\ln t + C_2t) \\
&\quad + 4(C_1t^2 + C_2t^2\ln t) \\
&= 2C_1t^2 + 2C_2t^2\ln t + 2C_2t^2 + C_2t^2 - 6C_1t^2 - 6C_2t^2\ln t - 3C_2t^2 \\
&\quad + 4C_1t^2 + 4C_2t^2\ln t \\
&= 0.
\end{aligned}
$$

52. Substitution gives

$$\begin{aligned}
t^2y''(t) - 3ty'(t) + 4y(t) &= t^2(C_1t^2 + C_2t^2\ln t + t^2\ln^2 t)'' - 3t(C_1t^2 + C_2t^2\ln t + t^2\ln^2 t)' \\
&\quad + 4(C_1t^2 + C_2t^2\ln t + t^2\ln^2 t) \\
&= t^2(2C_1t + 2C_2t\ln t + C_2t + 2t\ln^2 t + 2t\ln t)' \\
&\quad - 3t(2tC_1 + 2C_2t\ln t + C_2t + 2t\ln^2 t + 2t\ln t) \\
&\quad 4C_1t^2 + 4C_2t^2\ln t + 4t^2\ln^2 t \\
&= t^2(2C_1 + 2C_2\ln t + 2C_2 + C_2 + 2\ln^2 t + 4\ln t + 2\ln t + 2) \\
&\quad - 6C_1t^2 - 6C_2t^2\ln t - 3C_2t^2 - 6t^2\ln^2 t - 6t^2\ln t \\
&\quad 4C_1t^2 + 4C_2t^2\ln t + 4t^2\ln^2 t \\
&= 2C_1t^2 + 2C_2t^2\ln t + 2C_2t^2 + C_2t^2 + 2t^2\ln^2 t + 4t^2\ln t + 2t^2\ln t + 2t^2 \\
&\quad - 6C_1t^2 - 6C_2t^2\ln t - 3C_2t^2 - 6t^2\ln^2 t - 6t^2\ln t \\
&\quad 4C_1t^2 + 4C_2t^2\ln t + 4t^2\ln^2 t \\
&= 2t^2.
\end{aligned}$$

53. Substitution gives

$$\begin{aligned}
t^2y''(t) + ty'(t) + \left(t^2 - \frac{1}{4}\right)y(t) &= t^2\left(t^{-1/2}(C_1\cos t + C_2\sin t)\right)'' \\
&\quad + t\left(t^{-1/2}(C_1\cos t + C_2\sin t)\right)' + \left(t^2 - \frac{1}{4}\right)\left(t^{-1/2}(C_1\cos t + C_2\sin t)\right) \\
&= t^2\left(-\frac{1}{2}t^{-3/2}(C_1\cos t + C_2\sin t) + t^{-1/2}(-C_1\sin t + C_2\cos t)\right)' \\
&\quad + t\left(-\frac{1}{2}t^{-3/2}(C_1\cos t + C_2\sin t) + t^{-1/2}(-C_1\sin t + C_2\cos t)\right) \\
&\quad + C_1t^{3/2}\cos t + C_2t^{3/2}\sin t - \frac{1}{4}C_1t^{-1/2}\cos t - \frac{1}{4}C_2t^{-1/2}\sin t \\
&= t^2\left(\frac{3}{4}t^{-5/2}(C_1\cos t + C_2\sin t) - \frac{1}{2}t^{-3/2}(-C_1\sin t + C_2\cos t)\right) \\
&\quad + t^2\left(-\frac{1}{2}t^{-3/2}(-C_1\sin t + C_2\cos t) + t^{-1/2}(-C_1\cos t - C_2\sin t)\right) \\
&\quad - \frac{1}{2}C_1t^{-1/2}\cos t - \frac{1}{2}C_2t^{-1/2}\sin t - C_1t^{1/2}\sin t + C_2t^{1/2}\cos t \\
&\quad + C_1t^{3/2}\cos t + C_2t^{3/2}\sin t - \frac{1}{4}C_1t^{-1/2}\cos t - \frac{1}{4}C_2t^{-1/2}\sin t \\
&= \frac{3}{4}t^{-1/2}(C_1\cos t + C_2\sin t) - \frac{1}{2}t^{1/2}(-C_1\sin t + C_2\cos t) \\
&\quad - \frac{1}{2}t^{1/2}(-C_1\sin t + C_2\cos t) + t^{3/2}(-C_1\cos t - C_2\sin t) \\
&\quad - \frac{1}{2}C_1t^{-1/2}\cos t - \frac{1}{2}C_2t^{-1/2}\sin t - C_1t^{1/2}\sin t + C_2t^{1/2}\cos t \\
&\quad + C_1t^{3/2}\cos t + C_2t^{3/2}\sin t - \frac{1}{4}C_1t^{-1/2}\cos t - \frac{1}{4}C_2t^{-1/2}\sin t \\
&= 0.
\end{aligned}$$

54. (a) Substituting $y = \sin t$ and $y = \cos t$ into $y'' + y = 0$ gives

$$y'' + y = (\sin t)'' + \sin t = (\cos t)' + \sin t = -\sin t + \sin t = 0$$
$$y'' + y = (\cos t)'' + \cos t = (-\sin t)' + \cos t = -\cos t + \cos t = 0.$$

(b) Since we have found two linearly independent solutions, the general solution, by Theorem 16.2, is $y = C_1 \sin t + C_2 \cos t$.

(c) Substituting $y = \sin 2t$ and $y = \cos 2t$ into $y'' + 4y = 0$ gives

$$y'' + 4y = (\sin 2t)'' + 4\sin 2t = (2\cos 2t)' + 4\sin 2t = -4\sin 2t + 4\sin 2t = 0$$
$$y'' + 4y = (\cos 2t)'' + 4\cos 2t = (-2\sin 2t)' + 4\cos 2t = -4\cos 2t + 4\cos 2t = 0.$$

(d) Since we have found two linearly independent solutions, the general solution, by Theorem 16.2, is $y = C_1 \sin 2t + C_2 \cos 2t$.

(e) Both $y = \sin kt$ and $y = \cos kt$ are solutions, since substitution gives

$$y'' + ky = (\sin kt)'' + k^2 \sin kt = (k\cos kt)' + k^2 \sin kt = -k^2 \sin kt + k^2 \sin kt = 0$$
$$y'' + ky = (\cos kt)'' + k^2 \cos kt = (-k\sin kt)' + k^2 \cos kt = -k^2 \cos kt + k^2 \cos kt = 0.$$

Since those solutions are linearly independent, the general solution, by Theorem 16.2, is $y = C_1 \sin kt + C_2 \cos kt$.

55. (a) Substitution gives

$$y'' - y = (e^t)'' - e^t = (e^t)' - e^t = e^t - e^t = 0$$
$$y'' - y = (e^{-t})'' - e^{-t} = (-e^{-t})' - e^{-t} = e^{-t} - e^{-t} = 0.$$

(b) $\sinh t$ and $\cosh t$ are each linear combinations of the solutions e^t and e^{-t}, so they are both solutions. They are linearly independent since if $a \sinh t + b \cosh t = 0$, then

$$a\left(\frac{e^t - e^{-t}}{2}\right) + b\left(\frac{e^t + e^{-t}}{2}\right) = \frac{a+b}{2}e^t + \frac{b-a}{2}e^{-t} = 0.$$

Since e^t and e^{-t} are linearly independent, we must have $a + b = b - a = 0$, so that $a = b = 0$. This proves that $\sinh t$ and $\cosh t$ are linearly independent as well.

(c) Since $\sinh' = \cosh$ and $\cosh' = \sinh$, substitution gives

$$y'' - y = (\sinh t)'' - \sinh t = (\cosh t)' - \sinh t = \sinh t - \sinh t = 0$$
$$y'' - y = (\cosh t)'' - \cosh t = (\sinh t)' - \cosh t = \cosh t - \cosh t = 0.$$

(d) From part (a), the general solution is $C_1 e^t + C_2 e^{-t}$. From part (c), the general solution is $C_1 \sinh t + C_2 \cosh t$.

(e) Substitution gives

$$y'' - k^2 y = (e^{kt})'' - k^2 e^{kt} = (ke^{kt})' - k^2 e^{kt} = k^2 e^{kt} - k^2 e^{kt} = 0$$
$$y'' - k^2 y = (e^{-kt})'' - k^2 e^{-kt} = (-ke^{-kt})' - k^2 e^{-kt} = k^2 e^{-kt} - k^2 e^{-kt} = 0.$$

(f) In terms of exponentials, from part (e), the general solution is $C_1 e^{kt} + C_2 e^{-kt}$. Since $\cosh kt = \frac{e^{kt}+e^{-kt}}{2}$ and $\sinh kt = \frac{e^{kt}-e^{-kt}}{2}$, an identical argument to that in part (b) shows that $\cosh(kt)$ and $\sinh(kt)$ are also solutions to $y'' - k^2 y$ and that they are linearly independent. So in terms of hyperbolic functions, the general solution is $C_1 \sinh kt + C_2 \cosh kt$.

56. With $y = C_1e^{-2t} + C_2e^{-t} + C_3e^t$, we have

$$\begin{aligned} y' &= -2C_1e^{-2t} - C_2e^{-t} + C_3e^t \\ y'' &= 4C_1e^{-2t} + C_2e^{-t} + C_3e^t \\ y''' &= -8C_1e^{-2t} - C_2e^{-t} + C_3e^t \end{aligned}$$

so that

$$\begin{aligned} y'''(t) + 2y''(t) - y'(t) - 2y(t) &= -8C_1e^{-2t} - C_2e^{-t} + C_3e^t + 2(4C_1e^{-2t} + C_2e^{-t} + C_3e^t) \\ &\quad - (-2C_1e^{-2t} - C_2e^{-t} + C_3e^t) - 2(C_1e^{-2t} + C_2e^{-t} + C_3e^t) \\ &= 0. \end{aligned}$$

57. Note that $(e^{kt})^{(\text{iv})} = k^4e^{kt}$, $(\sin kt)^{(\text{iv})} = k^4 \sin kt$, and $(\cos kt)^{(\text{iv})} = k^4 \cos kt$. Thus with $y = C_1e^{-2t} + C_2e^{2t} + C_3 \sin 2t + C_4 \cos 2t$, we have

$$y^{(\text{iv})} = 16C_1e^{-2t} + 16C_2e^{2t} + 16C_3 \sin 2t + 16C_4 \cos 2t = 16y(t).$$

58. (a) $\frac{d}{dt}\left(y(t)^2\right) = 2y(t)\frac{d}{dt}(y(t)) = 2y(t)y'(t)$.

(b) Since $2yy' = \frac{d}{dt}(y^2)$, we can substitute in $y''(t) - 2y(t)y'(t) = 0$ to get $y''(t) - (y(t)^2)' = 0$.

(c) Integrating with respect to t gives $\int \left(y''(t) - (y(t)^2)\right)\, dt = \int 0\, dt$, or $y'(t) - y(t)^2 = C$. This is a separable equation; rearranging gives $\frac{y'(t)}{y(t)^2+C} = 1$, or $\frac{dy}{y^2+C} = dt$.

(d) There are now three cases.

- If $C > 0$ then integrating gives $\frac{1}{\sqrt{C}} \tan^{-1}\left(\frac{y}{\sqrt{C}}\right) = t + C'$, so that $\frac{y}{\sqrt{C}} = \tan(\sqrt{C}\,t + C'\sqrt{C})$ and $y = \sqrt{C}\, \tan(\sqrt{C}\,t + C'\sqrt{C})$. Renaming constants gives $y = C_1 \tan(C_2 + C_1t)$.
- If $C = 0$ then integrating $\frac{y'}{y^2} = dt$ gives $-y^{-1} = t + C_1$ so that $y = -\frac{1}{t+C_1}$.
- If $C < 0$ then integrating gives

$$\frac{1}{2\sqrt{-C}} \ln\left|\frac{y - \sqrt{-C}}{y + \sqrt{-C}}\right| = t + C',$$

so that, writing $D = \sqrt{-C}$,

$$\frac{y-D}{y+D} = \pm e^{2D(t+C')}, \quad \text{and solving for } y \text{ gives} \quad y = D\frac{1 \pm e^{2D(t+C')}}{1 \mp e^{2D(t+C')}}.$$

Finally, rename constants to get

$$y = D\frac{1 \pm e^{2Dt+2DC'}}{1 \mp e^{2Dt+2DC'}} = C_1\frac{1 - C_2e^{2C_1t}}{1 + C_2e^{2C_1t}}.$$

(Note that in the final equation, the sign of C_2 may be chosen appropriately so that we can force the $-$ sign in the numerator and the $+$ sign in the denominator).

59. (a) $\frac{d}{dt}\left(y'(t)^2\right) = 2y'(t)\frac{d}{dt}(y'(t)) = 2y'(t)y''(t)$.

(b) From part (a), $y''(t)y'(t) = \frac{1}{2} \cdot \frac{d}{dt}(y'(t)^2) = 1$, so that $(y'(t)^2)' = 2$.

(c) Integrating both sides with respect to t gives $\int (y'(t)^2)'\, dt = \int 2\, dt$, or $y'(t)^2 = 2t + C_1$ where C_1 is an arbitrary constant. Thus $y'(t) = \pm\sqrt{2t + C_1}$.

(d) Solving this equation simply involves integrating the right-hand side:

$$y(t) = \int \pm\sqrt{2t + C_1}\, dt = \int \pm(2t + C_1)^{1/2}\, dt = \pm\frac{1}{3}(2t + C_1)^{3/2} + C_2.$$

Thus there are two families of solutions.

60. (a) With $v = y'$, we have $v' = 2v$, or $v' - 2v = 0$. The integrating factor is $e^{\int -2\,dt} = e^{-2t}$; this gives $e^{-2t}v' - 2e^{-2t}v = 0$, or $(e^{-2t}v)' = 0$. Thus $e^{-2t}v = C_1$, so that $v = y' = C_1e^{2t}$.

(b) Integrating once again gives $y = \frac{C_1}{2}e^{2t} + C_3 = C_2e^{2t} + C_3$. As a check, note that

$$y'' = (C_2e^{2t} + C_3)'' = (2C_2e^{2t})' = 4C_2e^{2t} = 2(2C_2e^{2t}) = 2y'.$$

61. (a) With $v = y'$, we have $v' = 3v + 4$, or $v' - 3v = 4$. The integrating factor is $e^{\int -3\,dt} = e^{-3t}$; this gives $e^{-3t}v' - 3e^{-3t}v = (e^{-3t}v)' = 4e^{-3t}$. Integrate both sides to get $e^{-3t}v = -\frac{4}{3}e^{-3t} + C_1$, so that $v = -\frac{4}{3} + C_1e^{3t}$. This is the same as $y' = -\frac{4}{3} + C_1e^{3t}$.

(b) Integrating once again gives $y = -\frac{4}{3}t + \frac{C_1}{3}e^{3t} + C_2 = C_2 - \frac{4}{3}t + C_3e^{3t}$. As a check, note that

$$\begin{aligned} y'' &= (C_2 - \frac{4}{3}t + C_3e^{3t})'' = 9C_3e^{3t} \\ 3y' + 4 &= 3(C_2 - \frac{4}{3}t + C_3e^{3t})' + 4 = -4 + 9C_3e^{3t} + 4 = 9C_3e^{3t}. \end{aligned}$$

62. (a) With $v = y'$, we have $v' = e^{-v}$. This is separable: $e^v v' = 1$; integrating both sides gives $e^v = t + C$, so that $v = \ln(t + C_1)$. Substituting back gives $y' = \ln(t + C_1)$.

(b) Integrating once again gives $y = (t + C_1)\ln(t + C_1) - t + C_2$. As a check, note that

$$\begin{aligned} y'' &= ((t + C_1)\ln(t + C_1))'' + (-t + C_2)'' = (\ln(t + C_1) + 1)' = \frac{1}{t + C_1} \\ e^{-y'} &= e^{-((t+C_1)\ln(t+C_1)-t+C_2)'} = e^{-1-\ln(t+C_1)+1} = e^{-\ln(t+C_1)} = \frac{1}{t + C_1}. \end{aligned}$$

63. (a) With $v = y'$ we get $v' = 2tv^2$, so that $v^{-2}v' = 2t$. Integrating both sides gives $-v^{-1} = t^2 + C_1$, so that $v = -\frac{1}{t^2+C_1}$. Substituting back gives $y' = -\frac{1}{t^2+C_1}$.

(b) Integrating once again gives if $C_1 > 0$

$$y = -\frac{1}{\sqrt{C_1}}\arctan\left(\frac{t}{\sqrt{C_1}}\right) + C_2.$$

and if $C_1 < 0$

$$y = \frac{1}{2\sqrt{|C_1|}}\ln\left|\frac{t + \sqrt{|C_1|}}{t - \sqrt{|C_1|}}\right| + C_2.$$

As a check, note that for $C_1 > 0$,

$$\begin{aligned} y' &= -\frac{1}{\sqrt{C_1}} \cdot \frac{\sqrt{C_1}}{t^2 + C_1} = -(t^2 + C_1)^{-1} \\ y'' &= 2t(t^2 + C_1)^{-2} \end{aligned}$$

so that indeed $y'' = 2t(y')^2$, and for $C_1 < 0$, regardless of the sign of $\frac{t+\sqrt{|C_1|}}{t-\sqrt{|C_1|}}$,

$$\begin{aligned} y' &= \frac{-1}{t^2 - |C_1|} = \frac{-1}{t^2 + C_1} \\ y'' &= \frac{2t}{(t^2 + C_1)^2} \end{aligned}$$

and again $y'' = 2t(y')^2$.

64. (a) Since

$$(\sin 4t)'' + 16\sin 4t = (4\cos 4t)' + 16\sin 4t = -16\sin 4t + 16\sin 4t = 0$$
$$(\cos 4t)'' + 16\cos 4t = (-4\sin 4t)' + 16\cos 4t = -16\cos 4t + 16\cos 4t = 0$$

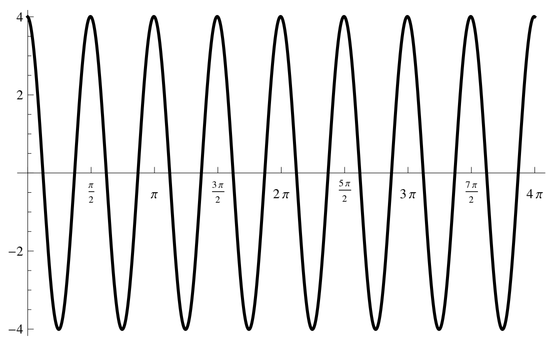
we see that $\sin 4t$ and $\cos 4t$ are linearly independent solutions, so that $y(t) = C_1 \sin 4t + C_2 \cos 4t$ is the general solution.

(b) Substituting the initial conditions into $y(t)$ gives the system of simultaneous equations

$$\begin{aligned} C_1 \cdot \sin(4\cdot 0) + C_2 \cdot \cos(4 \cdot 0) &= y(0) = 4 \\ 4C_1 \cdot \cos(4\cdot 0) - 4C_2 \cdot \sin(4\cdot 0) &= y'(0) = -1 \end{aligned} \quad \text{so that} \quad \begin{aligned} C_2 &= 4 \\ 4C_1 &= -1. \end{aligned}$$

Thus $C_1 = -\frac{1}{4}$ and $C_2 = 4$, and the solution to the initial value problem is $y(t) = -\frac{1}{4}\sin 4t + 4\cos 4t$.

(c) A graph of the solution for $0 \le t \le 4\pi$ is



65. (a) Computing derivatives gives

$$\begin{aligned}
(e^{-3t/2}\sin 2t)' &= -\frac{3}{2}e^{-3t/2}\sin 2t + 2e^{-3t/2}\cos 2t = e^{-3t/2}\left(2\cos 2t - \frac{3}{2}\sin 2t\right) \\
(e^{-3t/2}\sin 2t)'' &= \left(-\frac{3}{2}e^{-3t/2}\sin 2t + 2e^{-3t/2}\cos 2t\right)' \\
&= \frac{9}{4}e^{-3t/2}\sin 2t - 3e^{-3t/2}\cos 2t - 3e^{-3t/2}\cos 2t - 4e^{-3t/2}\sin 2t \\
&= e^{-3t/2}\left(-6\cos 2t - \frac{7}{4}\sin 2t\right) \\
(e^{-3t/2}\cos 2t)' &= -\frac{3}{2}e^{-3t/2}\cos 2t - 2e^{-3t/2}\sin 2t = e^{-3t/2}\left(-\frac{3}{2}\cos 2t - 2\sin 2t\right) \\
(e^{-3t/2}\cos 2t)'' &= \left(-\frac{3}{2}e^{-3t/2}\cos 2t - 2e^{-3t/2}\sin 2t\right)' \\
&= \frac{9}{4}e^{-3t/2}\cos 2t + 3e^{-3t/2}\sin 2t + 3e^{-3t/2}\sin 2t - 4e^{-3t/2}\cos 2t \\
&= e^{-3t/2}\left(-\frac{7}{4}\cos 2t + 6\sin 2t\right)
\end{aligned}$$

Substituting $e^{-3t/2}\sin 2t$ and $e^{-3t/2}\cos 2t$ gives

$$\begin{aligned}
y'' + 3y' + \frac{25}{4}y &= (e^{-3t/2}\sin 2t)'' + 3(e^{-3t/2}\sin 2t)' + \frac{25}{4}(e^{-3t/2}\sin 2t) \\
&= e^{-3t/2}\left(-6\cos 2t - \frac{7}{4}\sin 2t\right) + 3e^{-3t/2}\left(2\cos 2t - \frac{3}{2}\sin 2t\right) \\
&\quad + \frac{25}{4}(e^{-3t/2}\sin 2t) \\
&= e^{-3t/2}\left(-6\cos 2t - \frac{7}{4}\sin 2t + 6\cos 2t - \frac{9}{2}\sin 2t + \frac{25}{4}\sin 2t\right) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
y'' + 3y' + \frac{25}{4}y &= (e^{-3t/2}\cos 2t)'' + 3(e^{-3t/2}\cos 2t)' + \frac{25}{4}(e^{-3t/2}\cos 2t) \\
&= e^{-3t/2}\left(-\frac{7}{4}\cos 2t + 6\sin 2t\right) + 3e^{-3t/2}\left(-\frac{3}{2}\cos 2t - 2\sin 2t\right) \\
&\quad + \frac{25}{4}(e^{-3t/2}\cos 2t) \\
&= e^{-3t/2}\left(-\frac{7}{4}\cos 2t + 6\sin 2t - \frac{9}{2}\cos 2t - 6\sin 2t + \frac{25}{4}\cos 2t\right) \\
&= 0
\end{aligned}$$

so that $e^{-3t/2}\sin 2t$ and $e^{-3t/2}\cos 2t$ are linearly independent solutions, so that the general solution is $y(t) = e^{-3t/2}(C_1 \sin 2t + C_2 \cos 2t)$.

(b) Since

$$\begin{aligned}
y'(t) &= -\frac{3}{2}e^{-3t/2}(C_1\sin 2t + C_2\cos 2t) + e^{-3t/2}(2C_1\cos 2t - 2C_2\sin 2t) \\
&= e^{-3t/2}\left(\left(2\cos 2t - \frac{3}{2}\sin 2t\right)C_1 + \left(-2\sin 2t - \frac{3}{2}\cos 2t\right)C_2\right),
\end{aligned}$$

substituting the initial conditions into $y(t)$ gives the system of simultaneous equations

$$\begin{aligned}
e^{-3\cdot 0/2}(\sin(2\cdot 0)C_1 + \qquad\qquad \cos(2\cdot 0)C_2) &= y(0) = 4 \\
e^{-3\cdot 0/2}\left(\left(2\cos 0 - \frac{3}{2}\sin 0\right)C_1 + \left(-2\sin 0 - \frac{3}{2}\cos 0\right)C_2\right) &= y'(0) = 0
\end{aligned}$$

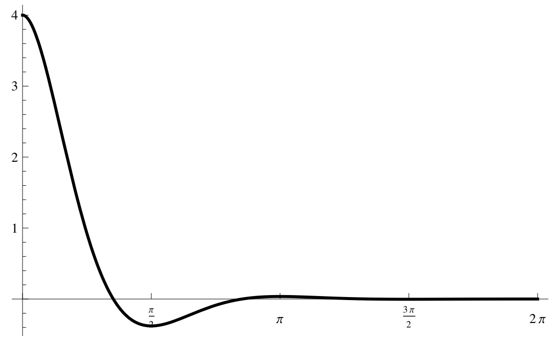
so that

$$\begin{aligned}
C_2 &= 4 \\
2C_1 - \frac{3}{2}C_2 &= 0.
\end{aligned}$$

Thus $C_1 = 3$ and $C_2 = 4$, and the solution to the initial value problem is

$$y(t) = e^{-3t/2}(3\sin 2t + 4\cos 2t).$$

(c) A plot of the solution for $0 \le t \le 2\pi$ is



66. (a) Since $(\sin 3t)' = 3\cos 3t$, $(\sin 3t)'' = -9\sin 3t$, $(\cos 3t)' = -3\sin 3t$, and $(\cos 3t)'' = -9\cos 3t$, have, substituting $y = \sin 3t$ and $y = \cos 3t$,

$$\begin{aligned} y'' + 9y &= -9\sin 3t + 9\sin 3t = 0 \\ y'' + 9y &= -9\cos 3t + 9\cos 3t = 0, \end{aligned}$$

so that $\sin 3t$ and $\cos 3t$ are two linearly independent solutions to the homogeneous problem. Also, substituting $y = \sin t$, we get

$$y'' + 9y = -\sin t + 9\sin t = 8\sin t,$$

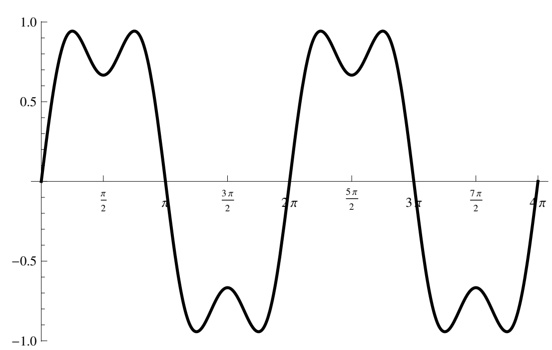
so that $\sin t$ is a solution to the nonhomogeneous problem. Thus the general solution to the nonhomogeneous problem is $y(t) = C_1 \sin 3t + C_2 \cos 3t + \sin t$.

(b) Substituting the initial conditions into $y(t)$ gives the system of simultaneous equations

$$\begin{aligned} C_1 \sin(3\cdot 0) + \ C_2 \cos(3\cdot 0) + \sin 0 &= \ y(0) = 0 \\ 3C_1 \cos(3\cdot 0) - 3C_2 \sin(3\cdot 0) + \cos 0 &= y'(0) = 2 \end{aligned} \quad \text{so that} \quad \begin{aligned} C_2 &= 0 \\ 3C_1 \quad &= 1. \end{aligned}$$

Thus $C_1 = \frac{1}{3}$ and $C_2 = 0$, and the solution to the initial value problem is $y(t) = \frac{1}{3}\sin 3t + \sin t$.

(c) A plot of the solution for $0 \le t \le 4\pi$ is



67. (a) Computing derivatives gives

$$\begin{aligned}
(e^{-3t}\sin 4t)' &= -3e^{-3t}\sin 4t + 4e^{-3t}\cos 4t = e^{-3t}(-3\sin 4t + 4\cos 4t)\\
(e^{-3t}\sin 4t)'' &= (e^{-3t}(-3\sin 4t + 4\cos 4t))'\\
&= -3e^{-3t}(-3\sin 4t + 4\cos 4t) + e^{-3t}(-12\cos 4t - 16\sin 4t)\\
&= e^{-3t}(-7\sin 4t - 24\cos 4t)\\
(e^{-3t}\cos 4t)' &= -3e^{-3t}\cos 4t - 4e^{-3t}\sin 4t = e^{-3t}(-3\cos 4t - 4\sin 4t)\\
(e^{-3t}\cos 4t)'' &= (e^{-3t}(-3\cos 4t - 4\sin 4t))'\\
&= -3e^{-3t}(-3\cos 4t - 4\sin 4t) + e^{-3t}(12\sin 4t - 16\cos 4t)\\
&= e^{-3t}(24\sin 4t - 7\cos 4t)
\end{aligned}$$

Then, substituting $y = e^{-3t}\sin 4t$ and $y = e^{-3t}\cos 4t$ gives

$$\begin{aligned}
y'' + 6y' + 25y &= (e^{-3t}\sin 4t)'' + 6(e^{-3t}\sin 4t)' + 25(e^{-3t}\sin 4t)\\
&= e^{-3t}(-7\sin 4t - 24\cos 4t) + 6e^{-3t}(-3\sin 4t + 4\cos 4t) + 25e^{-3t}\sin 4t\\
&= 0\\
y'' + 6y' + 25y &= (e^{-3t}\cos 4t)'' + 6(e^{-3t}\cos 4t)' + 25(e^{-3t}\cos 4t)\\
&= e^{-3t}(24\sin 4t - 7\cos 4t) + 6e^{-3t}(-3\cos 4t - 4\sin 4t) + 25e^{-3t}\cos 4t\\
&= 0
\end{aligned}$$

so that $e^{-3t}\sin 4t$ and $e^{-3t}\cos 4t$ are two linearly independent solutions to the homogeneous problem. Also, substituting $y = e^{-t}$, we get

$$y'' + 6y' + 25y = e^{-t} - 6e^{-t} + 25e^{-t} = 20e^{-t},$$

so that e^{-t} is a solution to the nonhomogeneous problem. Thus the general solution to the nonhomogeneous problem is $y(t) = e^{-3t}(C_1\sin 4t + C_2\cos 4t) + e^{-t}$.

(b) Since

$$\begin{aligned}
y'(t) &= -3e^{-3t}(C_1\sin 4t + C_2\cos 4t) + e^{-3t}(4C_1\cos 4t - 4C_2\sin 4t) - e^{-t}\\
&= e^{-3t}((-3\sin 4t + 4\cos 4t)C_1 + (-4\sin 4t - 3\cos 4t)C_2) - e^{-t},
\end{aligned}$$

substituting the initial conditions into $y(t)$ gives the system of simultaneous equations

$$\begin{aligned}
e^{-3\cdot 0}(\sin(0)C_1 + \qquad\qquad \cos(0)C_2) + e^{-0} &= y(0) = 2\\
e^{-3\cdot 0}((-3\sin 0 + 4\cos 0)C_1 + (-4\sin 0 - 3\cos 0)C_2 - e^{-0} &= y'(0) = 0
\end{aligned}$$

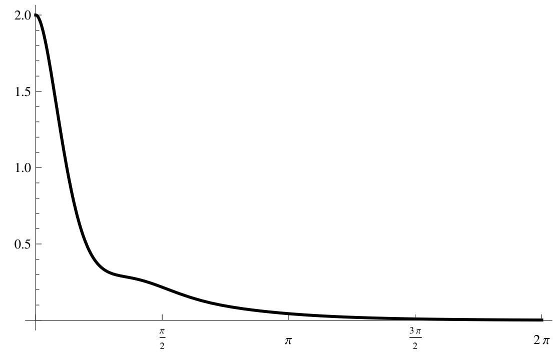
so that

$$\begin{aligned}
C_2 &= 1\\
4C_1 - 3C_2 &= 1.
\end{aligned}$$

Thus $C_1 = C_2 = 1$, and the solution to the initial value problem is

$$y(t) = e^{-3t}(\sin 4t + \cos 4t) + e^{-t}.$$

(c) A plot of the solution for $0 \le t \le 2\pi$ is



68. (a) Let $u = y'$; then the differential equation becomes

$$u'(x) = \frac{\sqrt{1+u(x)^2}}{sx}.$$

The initial condition $y(1) = 0$ is irrelevant to this problem, while $y'(1) = 0$ becomes the initial condition $u(1) = 0$.

(b) This equation is separable: divide both sides by $\sqrt{1+u^2}$ to obtain

$$\frac{du}{\sqrt{1+u^2}} = \frac{dx}{sx}.$$

Integrating both sides gives

$$\ln\left|u + \sqrt{1+u^2}\right| = \frac{1}{s}\ln(sx) + C,$$

so that exponentiating both sides gives

$$u + \sqrt{1+u^2} = \pm e^C \cdot e^{(1/s)\ln(sx)} = \pm e^C \cdot (sx)^{1/s} = \pm e^C s^{1/s} x^{1/s} = C_1 x^{1/s}.$$

Since $u(1) = y'(1) = 0$, we get

$$0 + \sqrt{1+0^2} = C_1 \cdot 1^{1/s}, \quad \text{so that} \quad C_1 = 1.$$

Thus $u + \sqrt{1+u^2} = x^{1/s}$. To simplify, subtract u from both sides and square to get

$$1 + u^2 = x^{2/s} - 2ux^{1/s} + u^2, \quad \text{so that} \quad u = \frac{1}{2}x^{-1/s}(x^{2/s} - 1) = \frac{1}{2}(x^{1/s} - x^{-1/s}).$$

(c) Since $u = y'$, we have the equation $y' = \frac{1}{2}(x^{1/s} - x^{-1/s})$; solve this by integrating both sides with respect to x. (Recall that $s > 1$, so that we need not worry about integrating $x^{\pm 1/s}$ and getting logarithmic functions):

$$y = \frac{1}{2}\left(\frac{s}{s+1}x^{(s+1)/s} - \frac{s}{s-1}x^{(s-1)/s}\right) + C_2. \qquad \text{(D2.1)}$$

Since $y(1) = 0$, we get

$$0 = \frac{1}{2}\left(\frac{s}{s+1} - \frac{s}{s-1}\right) + C_2 = -\frac{s}{s^2-1} + C_2,$$

so that $C_2 = \frac{s}{s^2-1}$.

(d) Replacing C_2 in (D2.1) by its value, and factoring out sx, we get

$$y = \frac{sx}{2}\left(\frac{x^{1/s}}{s+1} - \frac{x^{-1/s}}{s-1}\right) + \frac{s}{s^2-1}.$$

69. (a) Multiplying $mx''(t) = F(x)$ by $x'(t)$ gives

$$mx''(t)x'(t) = F(x)x'(t). \tag{D2.2}$$

Now, by the Chain Rule (or, see Exercise 59(a)), $2x''(t)x'(t) = \frac{d}{dt}(x'(t)^2)$, so that $x''(t)x'(t) = \frac{1}{2}\frac{d}{dt}(x'(t)^2)$. Further, with $\varphi'(x) = -F(x)$, differentiating with respect to time gives, again by the Chain Rule, $\frac{d}{dt}(\varphi(x)) = \varphi'(x)x'(t) = -F(x)x'(t)$. Making these substitutions in (D2.2) gives

$$\frac{d}{dt}\left(m\cdot\frac{1}{2}(x'(t))^2\right) = -\frac{d}{dt}(\varphi(x)), \qquad \text{so} \qquad \frac{d}{dt}\left[\frac{1}{2}m(x'(t))^2 + \varphi(x)\right] = 0.$$

(b) With $E = \frac{1}{2}mv^2 + \varphi$, since the time derivative of E is zero, it follows that E is conserved in time.

70. (a) With $y = t$, we have $y' = 1$ and $y'' = 0$. Substituting gives

$$0 - \frac{1}{t}\cdot 1 + \frac{1}{t^2}t = 0,$$

so that $y_1 = t$ is a solution.

(b) Let $y_2 = v(t)y_1(t) = tv(t)$ be any other solution. Now, $y_2' = v(t) + tv'(t)$ and $y_2'' = 2v'(t) + tv''(t)$, substituting y_2 in the equation gives

$$\begin{aligned} y_2'' - \frac{1}{t}y_2' + \frac{1}{t^2}y_2 &= 2v'(t) + tv''(t) - \frac{1}{t}\left(v(t) + tv'(t)\right) + \frac{1}{t^2}(tv(t)) \\ &= 2v'(t) + tv''(t) - \frac{1}{t}v(t) - v'(t) + \frac{1}{t}v(t) \\ &= tv''(t) + v'(t). \end{aligned}$$

Since y_2 is a solution, we have $tv''(t) + v'(t) = 0$, or $v'' = -\frac{v'}{t}$.

(c) Letting $w = v'$ gives the differential equation $w' = -\frac{w}{t}$.

(d) This is a separable equation; rearranging gives $\frac{w'}{w} = -\frac{1}{t}$, and integrating both sides gives $\ln|w| = C - \ln t = C + \ln\frac{1}{t}$. Thus $w = \pm e^{C+\ln(1/t)} = \frac{C_1}{t}$. Note that since $t > 0$, we do not need absolute value signs around t.

(e) Substituting back gives $v' = \frac{C_1}{t}$. Integrating both sides gives $v = C_1\ln t + C_2$. Note that since $t > 0$, we do not need absolute value signs here.

(f) Since $y_2(t) = tv(t)$, we get for a general solution $C_2t + C_1t\ln t$.

D2.2 Linear Homogeneous Equations

1. We assume that all derivatives of the function are multiples of the function itself, so we start with e^{rt}.

2. The characteristic polynomial is $r^2 - 3r + 10 = 0$; see the discussion in the first few paragraphs of this section.

3. The characteristic polynomial is a quadratic equation, so it can have two distinct real roots, one repeated real root, or two conjugate complex roots.

4. The general solution when the characteristic polynomial has distinct real roots is $y(t) = c_1e^{r_1t} + c_2e^{r_2t}$, where $r_1 \neq r_2$ are the real roots.

5. When the characteristic polynomial has a repeated real root r, the general solution is $y(t) = c_1e^{rt} + c_2te^{rt}$.

6. If the characteristic polynomial $r^2 + pr + q = 0$ has two conjugate complex roots, the general solution is $y(t) = c_1e^{at}\cos bt + c_2e^{at}\sin bt$, where $a = -\frac{p}{2}$ and $b = 4q - p^2$.

7. Since the roots are $-2 \pm 3i$, we have $a = -2$ and $b = 3$ in Case 3, so that the general solution is $y(t) = c_1e^{-2t}\cos 3t + c_2e^{-2t}\sin 3t$.

8. The trial solution for a second-order Cauchy-Euler equation is $y(t) = t^p$.

9. The characteristic polynomial is $r^2 - 25 = (r+5)(r-5) = 0$, with roots ± 5. Since the roots are real and distinct, the general solution is $y(t) = c_1e^{5t} + c_2e^{-5t}$.

10. The characteristic polynomial is $r^2 - 2r - 15 = (r-5)(r+3) = 0$, with roots 5 and -3. Since the roots are real and distinct, the general solution is $y(t) = c_1e^{5t} + c_2e^{-3t}$.

11. The characteristic polynomial is $r^2 - 3r = r(r-3) = 0$, with roots 0 and 3. Since the roots are real and distinct, the general solution is $y(t) = c_1e^{0t} + c_2e^{3t} = c_1 + c_2e^{3t}$.

12. The characteristic polynomial is $r^2 - r - \frac{3}{4} = \left(r - \frac{3}{2}\right)\left(r + \frac{1}{2}\right) = 0$, with roots $\frac{3}{2}$ and $-\frac{1}{2}$. Since the roots are real and distinct, the general solution is $y(t) = c_1e^{3t/2} + c_2e^{-t/2}$.

13. The characteristic polynomial is $2r^2 + 6r - 20 = (2r-4)(r+5)$, with roots 2 and -5. Since the roots are real and distinct, the general solution is $y(t) = c_1e^{2t} + c_2e^{-5t}$.

14. The characteristic polynomial is $r^2 - \frac{5}{2}r + 1 = \frac{1}{2}(x-2)(2x-1)$, with roots 2 and $\frac{1}{2}$. Since the roots are real and distinct, the general solution is $y(t) = c_1e^{2t} + c_2e^{t/2}$.

15. The characteristic polynomial $r^2 - 36 = (r+6)(r-6)$ has distinct real roots ± 6, so the general solution is $y(t) = c_1e^{6t} + c_2e^{-6t}$. Then $y'(t) = 6c_1e^{6t} - 6c_2e^{-6t}$. Substituting the initial conditions gives
$$\begin{aligned} c_1e^{6\cdot 0} + \ \ c_2e^{-6\cdot 0} &= y(0) = 3 \\ 6c_1e^{6\cdot 0} - 6c_2e^{-6\cdot 0} &= y'(0) = 0 \end{aligned} \qquad \text{so that} \qquad \begin{aligned} c_1 + \ \ c_2 &= 3 \\ 6c_1 - 6c_2 &= 0. \end{aligned}$$
Thus $c_1 = c_2 = \frac{3}{2}$, and the solution is $y(t) = \frac{3}{2}(e^{6t} + e^{-6t})$.

16. The characteristic polynomial $r^2 - 6r = r(r-6)$ has distinct real roots 0 and 6, so the general solution is $y(t) = c_1e^{0t} + c_2e^{6t} = c_1 + c_2e^{6t}$. Then $y'(t) = 6c_2e^{-6t}$. Substituting the initial conditions gives
$$\begin{aligned} c_1 + \ \ c_2e^{6\cdot 0} &= y(0) = -1 \\ 6c_2e^{6\cdot 0} &= y'(0) = \ \ 2 \end{aligned} \qquad \text{so that} \qquad \begin{aligned} c_1 + \ \ c_2 &= -1 \\ 6c_2 &= \ \ 2. \end{aligned}$$
Thus $c_1 = -\frac{4}{3}$ and $c_2 = \frac{1}{3}$, so that the solution is $y(t) = -\frac{4}{3} + \frac{1}{3}e^{6t}$.

17. The characteristic polynomial $r^2 - 3r - 18 = (r-6)(r+3)$ has distinct real roots -3 and 6, so the general solution is $y(t) = c_1e^{6t} + c_2e^{-3t}$. Then $y'(t) = 6c_1e^{6t} - 3c_2e^{-3t}$. Substituting the initial conditions gives

$$\begin{aligned} c_1e^{6\cdot 0} + c_2e^{-3\cdot 0} &= y(0) = 0 \\ 6c_1e^{6\cdot 0} - 3c_2e^{-3\cdot 0} &= y'(0) = 4 \end{aligned} \quad \text{so that} \quad \begin{aligned} c_1 + c_2 &= 0 \\ 6c_1 - 3c_2 &= 4. \end{aligned}$$

Thus $c_1 = \frac{4}{9}$ and $c_2 = -\frac{4}{9}$. The general solution is $y(t) = \frac{4}{9}(e^{6t} - e^{-3t})$.

18. The characteristic polynomial $r^2 + 8r + 15 = (r+3)(r+5)$ has distinct real roots -3 and -5, so the general solution is $y(t) = c_1e^{-3t} + c_2e^{-5t}$. Then $y'(t) = -3c_1e^{-3t} - 5c_2e^{-5t}$. Substituting the initial conditions gives

$$\begin{aligned} c_1e^{-3\cdot 0} + c_2e^{-5\cdot 0} &= y(0) = 2 \\ -3c_1e^{-3\cdot 0} - 5c_2e^{-5\cdot 0} &= y'(0) = 4 \end{aligned} \quad \text{so that} \quad \begin{aligned} c_1 + c_2 &= 2 \\ -3c_1 - 5c_2 &= 4. \end{aligned}$$

Thus $c_1 = 7$ and $c_2 = -5$. The solution is $y(t) = 7e^{-3t} - 5e^{-5t}$.

19. The characteristic polynomial $r^2 - 2r - \frac{5}{4} = \frac{1}{4}(2r-5)(2r+1)$ has distinct real roots $-\frac{1}{2}$ and $\frac{5}{2}$, so the general solution is $y(t) = c_1e^{5t/2} + c_2e^{-t/2}$. Then $y'(t) = \frac{5}{2}c_1e^{5t/2} - \frac{1}{2}c_2e^{-t/2}$. Substituting the initial conditions gives

$$\begin{aligned} c_1e^{5\cdot 0/2} + \phantom{\frac{1}{2}}c_2e^{-0/2} &= y(0) = 3 \\ \frac{5}{2}c_1e^{5\cdot 0/2} - \frac{1}{2}c_2e^{-0/2} &= y'(0) = 0 \end{aligned} \quad \text{so that} \quad \begin{aligned} c_1 + \phantom{\frac{1}{2}}c_2 &= 3 \\ \frac{5}{2}c_1 - \frac{1}{2}c_2 &= 0. \end{aligned}$$

Thus $c_1 = \frac{1}{2}$ and $c_2 = \frac{5}{2}$. The solution is $y(t) = \frac{1}{2}e^{5t/2} + \frac{5}{2}e^{-t/2}$.

20. The characteristic polynomial $r^2 - 10r + 21 = (r-3)(r-7)$ has distinct real roots 3 and 7, so the general solution is $y(t) = c_1e^{3t} + c_2e^{7t}$. Then $y'(t) = 3c_1e^{3t} + 7c_2e^{7t}$. Substituting the initial conditions gives

$$\begin{aligned} c_1e^{3\cdot 0} + c_2e^{7\cdot 0} &= y(0) = -3 \\ 3c_1e^{3\cdot 0} + 7c_2e^{7\cdot 0} &= y'(0) = -1 \end{aligned} \quad \text{so that} \quad \begin{aligned} c_1 + c_2 &= -3 \\ 3c_1 + 7c_2 &= -1. \end{aligned}$$

Thus $c_1 = -5$ and $c_2 = 2$. The solution is $y(t) = -5e^{3t} + 2e^{7t}$.

21. The characteristic polynomial $r^2 - 2r + 1 = (r-1)^2$ has the repeated real root 1, so the general solution is $y(t) = c_1e^t + c_2te^t$. Then $y'(t) = c_1e^t + c_2(e^t + te^t)$. Substituting the initial conditions gives

$$\begin{aligned} c_1e^0 + \qquad\quad c_2 \cdot 0 \cdot e^0 &= y(0) = 4 \\ c_1e^0 + c_2(e^0 + 0 \cdot e^0)e^0 &= y'(0) = 0 \end{aligned} \quad \text{so that} \quad \begin{aligned} c_1 \qquad &= 4 \\ c_1 + c_2 &= 0. \end{aligned}$$

Thus $c_1 = 4$ and $c_2 = -4$, and the solution is $y(t) = 4e^t - 4te^t$.

22. The characteristic polynomial $r^2 + 6r + 9 = (r+3)^2$ has the repeated real root -3, so the general solution is $y(t) = c_1e^{-3t} + c_2te^{-3t}$. Then $y'(t) = -3c_1e^{-3t} + c_2(e^{-3t} - 3te^{-3t})$. Substituting for the initial conditions gives

$$\begin{aligned} c_1e^{-3\cdot 0} + \qquad\qquad\quad c_2 \cdot 0 \cdot e^{-3\cdot 0} &= y(0) = 0 \\ -3c_1e^{-3\cdot 0} + c_2(e^{-3\cdot 0} - 3 \cdot 0 \cdot e^{-3\cdot 0}) &= y'(0) = -1 \end{aligned} \quad \text{so that} \quad \begin{aligned} c_1 \qquad &= 0 \\ -3c_1 + c_2 &= -1. \end{aligned}$$

Thus $c_1 = 0$ and $c_2 = -1$, so that the solution is $y(t) = -te^{-3t}$.