Chapter 2 Project

The Lunar Module

The Lunar Module (LM) was a small spacecraft that detached from the Apollo Command Module and was designed to land on the Moon. Fast and accurate computations were needed to bring the LM from an orbiting speed of about 5500 ft/s to a speed slow enough to land it within a few feet of a designated target on the Moon's surface. The LM carried a 70-lb computer to assist in guiding it successfully to its target. The approach to the target was split into three phases, each of which followed a reference trajectory specified by NASA engineers. The position and velocity of the LM were monitored by sensors that tracked its deviation from the preassigned path at each moment. Whenever the LM strayed from the reference trajectory, control thrusters were fired to reposition it. In other words, the LM's position and velocity were adjusted by changing its acceleration.

The reference trajectory for each phase was specified by the engineers to have the form

$$r_{\text{ref}}(t) = R_T + V_T t + \frac{1}{2}A_T t^2 + \frac{1}{6}J_T t^3 + \frac{1}{24}S_T t^4 \qquad (1)$$

The variable r_{ref} represents the intended position of the LM at time t before the end of the landing phase. The engineers specified the end of the landing phase to take place at $t = 0$, so that during the phase, t was always negative. Note that the LM was landing in three dimensions, so there were actually three equations like (1). Since each of those equations had this same form, we will work in one dimension, assuming, for example, that r represents the distance of the LM above the surface of the Moon.

1. If the LM follows the reference trajectory, what is the reference velocity $v_{\text{ref}}(t)$?
2. What is the reference acceleration $a_{\text{ref}}(t)$?
3. The rate of change of acceleration is called **jerk**. Find the reference jerk $J_{\text{ref}}(t)$.
4. The rate of change of jerk is called **snap**. Find the reference snap $S_{\text{ref}}(t)$.
5. Evaluate $r_{\text{ref}}(t)$, $v_{\text{ref}}(t)$, $a_{\text{ref}}(t)$, $J_{\text{ref}}(t)$, and $S_{\text{ref}}(t)$ when $t = 0$.

The reference trajectory given in equation (1) is a fourth-degree polynomial, the lowest degree polynomial that has enough free parameters to satisfy all the mission criteria. Now we see that the parameters $R_T = r_{\text{ref}}(0)$, $V_T = v_{\text{ref}}(0)$, $A_T = a_{\text{ref}}(0)$, $J_T = J_{\text{ref}}(0)$, $S_T = S_{\text{ref}}(0)$. The five parameters in equation (1) are referred to as the **target parameters** since they provide the path the LM should follow.

But small variations in propulsion, mass, and countless other variables caused the LM to deviate from the predetermined path. To correct the LM's position and velocity, NASA engineers applied a force to the LM using rocket thrusters. That is, they changed the acceleration. (Remember Newton's second law, $F = ma$.) Engineers modeled the actual trajectory of the LM by

$$r(t) = R_T + V_T t + \frac{1}{2} A_T t^2 + \frac{1}{6} J_A t^3 + \frac{1}{24} S_A t^4 \qquad (2)$$

We know the target parameters for position, velocity, and acceleration. We need to find the actual parameters for jerk and snap to know the proper force (acceleration) to apply.

6. Find the actual velocity $v = v(t)$ of the LM.
7. Find the actual acceleration $a = a(t)$ of the LM.
8. Use equation (2) and the actual velocity found in Problem 6 to express J_A and S_A in terms of R_T, V_T, A_T, $r(t)$, and $v(t)$.
9. Use the results of Problems 7 and 8 to express the actual acceleration $a = a(t)$ in terms of R_T, V_T, A_T, $r(t)$, and $v(t)$.

The result found in Problem 9 provides the acceleration (force) required to keep the LM in its reference trajectory.

10. When riding in an elevator, the sensation one feels just before the elevator stops at a floor is a jerk. Would you want jerk to be small or large in an elevator? Explain. Would you want jerk to be small or large on a roller coaster ride? Explain. How would you explain snap?

Solutions for Chapter 2 Project

The reference trajectory is necessary to solve many of the parts of this Project and is given in the problem set-up: $r_{ref}(t) = R_T + V_T t + \frac{1}{2}A_T t^2 + \frac{1}{6}J_T t^3 + \frac{1}{24}S_T t^4$. By definition, $t < 0$ until the LM lands at $t = 0$. Note, for parts 1–4, each answer is simply the derivative with respect to t of the previous polynomial form.

1. $v_{ref}(t) = V_T + A_T t + \frac{1}{2}J_T t^2 + \frac{1}{6}S_T t^3$

2. $a_{ref}(t) = A_T + J_T t + \frac{1}{2}S_T t^2$

3. $J_{ref}(t) = J_T + S_T t$

4. $S_{ref}(t) = S_T$

5. When we evaluate each of these functions at $t = 0$ (the moment of LM landing), for each quantity, all terms except for the constant terms (the first terms listed) equal 0. So the so-called "target parameters" are given by $r_{ref}(0) = R_T$, $v_{ref}(0) = V_T$, $a_{ref}(0) = A_T$, $J_{ref}(0) = J_T$, and $S_{ref}(0) = S_T$

Note that parts 6– 9 require reference to the actual trajectory, $r(t) = R_T + V_T t + \frac{1}{2}A_T t^2 + \frac{1}{6}J_A t^3 + \frac{1}{24}S_A t^4$. The only difference between this and the reference trajectory is the subscripts on the jerk and snap parameters.

6. The actual velocity is simply the derivative of the actual trajectory with respect to t: $v(t) = V_T + A_T t + \frac{1}{2}J_A t^2 + \frac{1}{6}S_A t^3$.

7. Likewise, the actual acceleration is the derivative of the actual velocity with respect to t: $a(t) = A_T + J_A t + \frac{1}{2}S_A t^2$.

8. While this looks overly complicated, in fact you can think of the actual trajectory, given above as $r(t) = R_T + V_T t + \frac{1}{2}A_T t^2 + \frac{1}{6}J_A t^3 + \frac{1}{24}S_A t^4$, and the actual velocity, given in part 6 as $v(t) = V_T + A_T t + \frac{1}{2}J_A t^2 + \frac{1}{6}S_A t^3$, as linear equations in the variables J_A and S_A. Here, we will consider the setting as a snapshot in time; that is, we will assess the situation for a specific value of t, rather than as a variable. Two simple steps will make this system more manageable. First, multiply through $r(t)$ by 24, and multiply $v(t)$ by 6 to temporarily remove all fractions. The equations result in $24r(t) = 24R_T + 24V_T t + 12A_T t^2 + 4J_A t^3 + S_A t^4$ and $6v(t) = 6V_T + 6A_T t + 3J_A t^2 + S_A t^3$.

Next, we shuffle terms around with simple addition and subtraction to isolate J_A and S_A on the left, with all other terms on the right. We obtain $4J_At^3 + S_At^4 = 24r(t) - 24R_T - 24V_Tt - 12A_Tt^2$, and $3J_At^2 + S_At^3 = 6v(t) - 6V_T - 6A_Tt$.

This system can now be solved for J_A and S_A using substitution or elimination, but Kramer's Rule is probably more efficient. Kramer's Rule requires the determinant of the matrix made up of the coefficients of the left side of the equation:

$\begin{vmatrix} 4t^3 & t^4 \\ 3t^2 & t^3 \end{vmatrix} = (4t^3)(t^3) - (t^4)(3t^2) = t^6$. The value of J_A is the determinant of the matrix $\begin{vmatrix} 24r(t) - 24R_T - 24V_Tt - 12A_Tt^2 & t^4 \\ 6v(t) - 6V_T - 6A_Tt & t^3 \end{vmatrix}$ divided by the result from the coefficient matrix, t^6. This is an algebra-level skill, if a bit messy. The result is $J_A = \frac{24r(t)}{t^3} - \frac{24R_T}{t^3} - \frac{18V_T}{t^2} - \frac{6A_T}{t} - \frac{6v(t)}{t^2}$. Similar manipulations, dividing t^6 into the determinant of the matrix $\begin{vmatrix} 4t^3 & 24r(t) - 24R_T - 24V_Tt - 12A_Tt^2 \\ 3t^2 & 6v(t) - 6V_T - 6A_Tt \end{vmatrix}$, result in $S_A = \frac{24v(t)}{t^3} + \frac{48V_T}{t^3} + \frac{12A_T}{t^2} - \frac{72r(t)}{t^4} + \frac{72R_T}{t^4}$.

9. From answer 7, $a(t) = A_T + J_At + \frac{1}{2}S_At^2$. But answer 8 provided us with terms for J_A and S_A. Plugging these forms into the expression for $a(t)$ gives us the actual acceleration for the LM, independent of the actual jerk and snap. After simplifying algebra, $a(t) = A_T - \frac{12r(t)}{t^2} + \frac{12R_T}{t^2} + \frac{6V_T}{t} + \frac{6v(t)}{t}$.

10. Answers will vary.