CHAPTER 1

Limits and Continuity

Section 1.1 Limits (An Intuitive Approach)

Suggested Time Allocation: 1 lecture

Slides Available on the Instructor Companion Site for this Text: Figures 1.1.9, 1.1.13, 1.1.15

Teaching Plan: The purpose of this section is to motivate the study of limits and to provide the student an intuitive feeling for the concept of a limit. Don't get too absorbed in a discussion of the tangent line and area problems. Cover these lightly (or as assigned reading) and emphasize the notion of limit, particularly from graphical or numerical evidence. Keep it informal; techniques for computing limits are given in the next section so don't worry about that here. You might consider using handouts or overhead transparencies when discussing numerical examples such as those in Examples 2 and 3, and in the opening discussion of infinite limits.

Key points to emphasize are:

- The idea that the limit of $f(x)$ at a finite point a is determined by the values of f *near* a but not the value of f *at* a. (See Figure 1.1.9.)
- Limit notation: $\lim_{x\to a} f(x) = L$ and $f(x) \to L$ as $x \to a$
- Limits can fail to exist because of "oscillation"; see the discussion at the top of page 72.
- The relationship between one-sided and two-sided limits. (See Figure 1.1.13.)
- Infinite limits, emphasizing that these describe particular ways in which the limit fails to exist. (See Figure 1.1.15.)
- The idea of a vertical asymptote and the notions conveyed by the notation $f(x) \to +\infty$ and $f(x) \to -\infty$.

Margin Notes:

p. 68: The quantity m_{sec} is the slope of the line through points P and Q. P and Q must be distinct so that they determine a unique line.

p. 71: (Technology Mastery) Use windows $[0.25, 1.75] \times [1.5, 2.5]$ and $[0.81, 1.19] \times [1.9, 2.1]$.

p. 72: (First margin note) Using degree measure,

x (degrees)	± 1.0	± 0.5	± 0.1	± 0.01
$\dfrac{\sin x}{x}$	0.0174524	0.0174531	0.0174533	0.0174533

$$\lim_{x\to 0} \frac{\sin x}{x} = \frac{\pi}{180} \approx 0.0174533$$

p. 76: The vertical asymptotes for $f(x) = \dfrac{x^2+2x}{x^2-1}$ are $x = -1$ and $x = 1$:

$$\lim_{x\to -1^-} f(x) = -\infty; \quad \lim_{x\to -1^+} f(x) = +\infty; \quad \lim_{x\to 1^-} f(x) = -\infty; \quad \lim_{x\to 1^+} f(x) = -\infty$$

Sample Assignment: Exercises 5, 8, 9, 10, 11, 13, 15, 21, 23, 25, 31

Section 1.2 Computing Limits

Suggested Time Allocation: 1 lecture

Slides Available on the Instructor Companion Site for this Text: Figures 1.2.1, 1.2.2, 1.2.5

Teaching Plan: This section is concerned with computing basic limits at a finite point a. It will open up more time for examples if you use handouts or transparencies to expedite the presentation of the main theorems.

Key points to emphasize are:

- The limits in Theorem 1.2.1. Students should be able to visualize these results graphically (as in Figure 1.2.1) and numerically (as in Table 1.2.1).
- Students usually find the verbal descriptions of Theorem 1.2.2 and Formula (1) (shaded in gray on page 81) helpful. You might want to emphasize these verbal formulations as you work through examples.
- The limit of a polynomial at a finite point a is obtained by evaluating the polynomial at a.
- The limit of a rational function at a finite point a is obtained by evaluating the function at a, if the evaluation is possible.
- The limit of a rational function at a point a where the denominator has a zero but the numerator does not fails to exist. Use Figure 1.2.2 to illustrate the ways in which limits can fail to exist.
- Example 8 or something comparable.
- Example 9 or something comparable.
- Example 10 revisits Section 1.1 Example 2, here using algebraic techniques (see Figure 1.1.9).
- One-sided limits and piecewise-defined functions. (See Figure 1.2.5.)

Margin Notes:

p. 85: Applying the statement "zero divided by anything is zero" assumes "anything" isn't zero, since this would run counter to the statement "you can't divide by zero." Also, "zero divided by anything is zero" assumes that the numerator is exactly zero (not just near zero) when x is near (but not equal to) a. This is rarely the case in a limit problem that gives rise to an indeterminate form of type 0/0.

Sample Assignment: Exercises 1–41 (odd)

Section 1.3 Limits at Inþnity; End-Behavior of a Function

Suggested Time Allocation: 1 lecture

Slides Available on the Instructor Companion Site for this Text: Figures 1.3.2, 1.3.4

Teaching Plan: This section is concerned with horizontal asymptotes and the idea of a limit at infinity. There are two goals: students should acquire an intuitive feel for the paradoxical notion of "approaching infinity," and they should accumulate computational skills and familiarity with the end-behavior of basic functions.

Key points to emphasize are:

- The idea expressed in the notation $x \to +\infty$ (or $x \to -\infty$), the informal view of limits at infinity expressed in Statement 1.3.1, and the numerical implications of a limit at infinity (see, for example, Table 1.3.1).
- Horizontal asymptotes as a graphical consequence of a limit at infinity; see Figure 1.3.2.
- Examples 1, 2, and 3 (end behavior for $1/x$, $\tan^{-1} x$, and $\left(1+\frac{1}{x}\right)^x$) are important examples that students should internalize. (Figure 1.3.4 illustrates the last of these.)
- Briefly point out that rules for limits at infinity analogous to those listed in Theorem 1.2.2 of the previous section hold.
- Infinite limits at infinity (Statement 1.3.2) and end-behavior of power functions (Formulas (15) and (16)). You can initiate this discussion by graphing a power function (say, $y = x^3$) and asking students to develop a notational description of the end behavior of the function ("$\lim_{x\to+\infty} x^3 = +\infty$" and "$\lim_{x\to-\infty} x^3 = -\infty$"). Then invite students to formulate their own definitions for infinite limits at infinity.
- The end-behavior of a polynomial matches that of its highest degree term.
- The algebraic technique of dividing numerator and denominator by the highest power of the variable in the denominator of a rational function to resolve limits at infinity of the rational function.
- Once students understand the computations for rational functions in Examples 7 and 8, observe the short-cut in the gray box at the bottom of page 93 and its companion Example 9.
- Issues arising in limits involving radicals, especially the resolution of $\sqrt{x^2} = |x|$ relative to $x \to +\infty$ versus $x \to -\infty$. See Example 10.
- End-behavior of trigonometric functions; limits can fail to exist because of "oscillation."
- End-behavior of logarithmic and exponential functions, particularly Formulas (20)–(25). Quick Check Exercise 4 can be used to initiate this discussion.

Margin Notes:

p. 94: (Technology Mastery) Note that the graph of $f(x) = (\sqrt{x^2+2})/(3x-6)$ also has a vertical asymptote at $x = 2$, and *crosses* its horizontal asymptote $y = -\frac{1}{3}$ at $x = \frac{1}{2}$.

Sample Assignment: Exercises 1, 3, 5, 7, 9, 11, 13, 15, 17, 29, 33, 46, 55, 59, 62, 66, 69

Section 1.4 Limits (Discussed More Rigorously)

Suggested Time Allocation: 1 to 2 lectures

Slides Available on the Instructor Companion Site for this Text: Figures 1.4.2, 1.4.4, 1.4.5

Teaching Plan: If you allot a single lecture for this section, we suggest that you focus on finite limits; the material is difficult and this will allow more time to focus on the key ideas. Since many students will be unclear about the purpose of this section, you might begin by explaining that mathematical precision is needed to prove new results about limits, and you might point out that even though terms like "closer and closer" or "as close as we please" help to explain the concept of a limit, they are informal and lack the precision required for mathematical proof.

Key points to emphasize are:

- The idea in Figure 1.4.1 and the related italicized material that follows on page 101.

- The idea that once an interval containing $x = a$ has been found for which the ϵ condition is satisfied, any smaller open interval containing a will also work. Then tell students that it is usually desirable to make the interval symmetric and use an interval of the form $(a - \delta, a + \delta)$. That the choice of δ isn't unique can be illustrated using Figure 1.4.2, and reinforced algebraically after discussion of an example illustrating Definition 1.4.1.
- Definition 1.4.1 (illustrated in Figure 1.4.2) and an example of the limit of a linear function, such as Example 1.
- Examples 2 and 3, or something comparable.
- Consider working with transparencies or handouts containing Definitions 1.4.2 and 1.4.3, and Figure 1.4.4 to convey the definition of a limit at infinity. Time permitting, cover a simple example, comparable to Example 4.
- An overview of Definitions 1.4.4 and 1.4.5 for infinite limits can be accomplished using Figure 1.4.5.

Margin Notes:

p. 101: For a limit from the left, the ϵ condition of Definition 1.4.1 should be met on an open interval $a - \delta < x < a$ extending to the left of a.

p. 105: $\lim_{x\to a^+} f(x) = +\infty$: Given any positive number M, we can find a number $\delta > 0$ such that $a < x < a + \delta$ guarantees that $f(x) > M$.

$\lim_{x\to a^+} f(x) = -\infty$: Given any negative number M, we can find a number $\delta > 0$ such that $a < x < a + \delta$ guarantees that $f(x) < M$.

$\lim_{x\to a^-} f(x) = +\infty$: Given any positive number M, we can find a number $\delta > 0$ such that $a - \delta < x < a$ guarantees that $f(x) > M$.

$\lim_{x\to a^-} f(x) = -\infty$: Given any negative number M, we can find a number $\delta > 0$ such that $a - \delta < x < a$ guarantees that $f(x) < M$.

$\lim_{x\to +\infty} f(x) = +\infty$: Given any positive number M, we can find a positive number N such that $x > N$ guarantees that $f(x) > M$.

$\lim_{x\to +\infty} f(x) = -\infty$: Given any negative number M, we can find a positive number N such that $x > N$ guarantees that $f(x) < M$.

$\lim_{x\to -\infty} f(x) = +\infty$: Given any positive number M, we can find a negative number N such that $x < N$ guarantees that $f(x) > M$.

$\lim_{x\to -\infty} f(x) = -\infty$: Given any negative number M, we can find a negative number N such that $x < N$ guarantees that $f(x) < M$.

Sample Assignment: Exercises 1, 3, 5, 8, 9, 12, 19, 21, 22, 29, 31, 40

Section 1.5 Continuity

Suggested Time Allocation: 1 lecture

Slides Available on the Instructor Companion Site for this Text: Figures 1.5.1, 1.5.8

Teaching Plan: In ordinary conversation, the word "continuous" has a variety of meanings, among them "unending" or "unbroken." As is often the case, mathematics takes a common word and applies a very specific meaning to it; here the nuance of "continuous" we want is better conveyed by the association with "unbroken." The section opens by developing a definition of continuity starting from what it is not (compare "unbroken" versus "broken"); see Figure 1.5.1. Note that the exposition always connects "continuous" to a location: continuous *at a point* or continuous *on an interval.* The vague statement "f is a continuous function" is avoided.

The goal in this section is to enter the terms "continuous" and "discontinuous" into the student's mathematical vocabulary, explore graphically and algebraically where functions are or are not continuous, and introduce the Intermediate-Value Theorem, an important consequence of continuity. There are quite a few concepts in this section, so plan your time carefully.

Key points to emphasize are:

- The formal definition of continuity in Definition 1.5.1.
- The margin note on page 110 that the third continuity condition implies the first two.
- Example 1 or something comparable.
- Definition 1.5.2, continuity on a closed interval.
- Example 2 or something comparable.
- Algebraic properties of continuity, summarized in Theorem 1.5.3.
- Continuity of polynomials and the fact that a rational function is continuous except where the denominator is zero, at which point there is a discontinuity. (Note that these results are just restatements of results in Section 1.2 using the new vocabulary.)
- Theorem 1.5.5, emphasizing that it allows a limit symbol to be moved through the function sign when the function is continuous at the limit (see the margin note beside the theorem).
- Composition of continuous functions is continuous. In particular, the absolute value of a continuous function is continuous.
- The Intermediate-Value Theorem. (See Figure 1.5.8.)
- Example 5 or something comparable, applying the Intermediate-Value Theorem to finding a root of a polynomial.

Margin Notes:

p. 112: f is continuous on $(a, b]$ if f is continuous on (a, b) and f is continuous from the left at b. (Continuity on $(-\infty, b]$ is defined similarly.)

f is continuous on $[a, b)$ if f is continuous on (a, b) and f is continuous from the right at a. (Continuity on $[a, +\infty)$ is defined similarly.)

p. 114: (Technology Mastery) The equation of the curve simplifies to $y = \dfrac{x+3}{x-2}$, $x \neq 3$, so the discontinuity at $x = 3$ is similar to that in Figure 1.5.1*a*. This is a *removable discontinuity*; see Exercises 33–36.

p. 115: Yes, for example, the function

$$f(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$$

is not continuous at $x = 0$, but $|f(x)|$ is the constant function 1, so $|f(x)|$ is continuous everywhere. (Draw the graphs of $y = f(x)$ and $y = |f(x)|$.)

Sample Assignment: Exercises 1, 3, 5, 7, 11, 13, 15, 17, 21, 23, 35, 44, 47

Section 1.6 Continuity of Trigonometric, Exponential, and Inverse Functions

Suggested Time Allocation: 1 lecture

Slides Available on the Instructor Companion Site for this Text: Figures 1.6.5, 1.6.6

Teaching Plan: This section summarizes results on the continuity of trigonometric functions, exponential functions, and the inverse of a continuous one-to-one function. It also covers the Squeezing Theorem for limits and two important limits involving sine and cosine.

Key points to emphasize are:

- A review of the definition of sine and cosine from the unit circle. Note the continuity of $\sin x$ and $\cos x$ and the continuity of the remaining trigonometric functions at points where they are defined.
- Example 1 or something comparable.
- The continuity of the inverse of a one-to-one continuous function (Theorem 1.6.2).
- You may want to review the relationship between exponential and logarithmic functions, but should discuss their continuity (Theorem 1.6.3)
- Explain that the Squeezing Theorem is an indirect procedure for using known limits to find an unknown limit that is difficult to obtain directly. Students will have little difficulty grasping the idea from Figure 1.6.2, but will generally have a difficult time knowing when to apply the theorem. Tell them that this will come from experience and by paying attention to situations where it is applied in the text.
- The limits in Theorem 1.6.5. These results will be needed in Section 2.5 when we derive the derivatives of sine and cosine. The proof is important because it illustrates the Squeezing Theorem, reviews the area of a sector, and uses some trigonometric inequalities with which students should be familiar.
- Example 4 or something comparable, as an application of the limits in Theorem 1.6.5.
- Example 5 (see Figures 1.6.5 and 1.6.6) and the remark that follows.

Margin Notes:

p. 123: For a limit from the left: if f, g, and h are functions satisfying $g(x) \leq f(x) \leq h(x)$ on an open interval (a, c), and if $\lim_{x\to c^-} g(x) = \lim_{x\to c^-} h(x) = L$, then $\lim_{x\to c^-} f(x) = L$.

For a limit at $+\infty$: if f, g, and h are functions satisfying $g(x) \leq f(x) \leq h(x)$ on an open interval $(a, +\infty)$, and if $\lim_{x\to+\infty} g(x) = \lim_{x\to+\infty} h(x) = L$, then $\lim_{x\to+\infty} f(x) = L$.

For a limit from the right: if f, g, and h are functions satisfying $g(x) \leq f(x) \leq h(x)$ on an open interval (c, b), and if $\lim_{x\to c^+} g(x) = \lim_{x\to c^+} h(x) = L$, then $\lim_{x\to c^+} f(x) = L$.

For a limit at $+\infty$: if f, g, and h are functions satisfying $g(x) \leq f(x) \leq h(x)$ on an open interval $(-\infty, b)$, and if $\lim_{x\to-\infty} g(x) = \lim_{x\to-\infty} h(x) = L$, then $\lim_{x\to-\infty} f(x) = L$.

Relating to the versions of the Squeezing Theorem at $\pm\infty$, see Exercise 66.

Sample Assignment: Exercises 1, 3, 5, 9, 11, 17, 19, 21, 23, 26, 27, 35, 41, 49, 61, 67
