

NOT FOR SALE

Complete Solutions Manual  
for  
**MULTIVARIABLE CALCULUS**  
SEVENTH EDITION

DAN CLEGG  
Palomar College

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## PREFACE

This *Complete Solutions Manual* contains detailed solutions to all exercises in the text *Multivariable Calculus*, Seventh Edition (Chapters 10–17 of *Calculus*, Seventh Edition, and *Calculus: Early Transcendentals*, Seventh Edition) by James Stewart. A *Student Solutions Manual* is also available, which contains solutions to the odd-numbered exercises in each chapter section, review section, True-False Quiz, and Problems Plus section as well as all solutions to the Concept Check questions. (It does not, however, include solutions to any of the projects.)

Because of differences between the regular version and the *Early Transcendentals* version of the text, some references are given in a dual format. In these cases, users of the *Early Transcendentals* text should use the references denoted by “ET.”

While we have extended every effort to ensure the accuracy of the solutions presented, we would appreciate correspondence regarding any errors that may exist. Other suggestions or comments are also welcome, and can be sent to dan clegg at [dclegg@palomar.edu](mailto:dclegg@palomar.edu) or in care of the publisher: Brooks/Cole, Cengage Learning, 20 Davis Drive, Belmont CA 94002-3098.

We would like to thank James Stewart for entrusting us with the writing of this manual and offering suggestions and Kathi Townes of TECH-arts for typesetting and producing this manual as well as creating the illustrations. We also thank Richard Stratton, Liz Covello, and Elizabeth Neustaetter of Brooks/Cole, Cengage Learning, for their trust, assistance, and patience.

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## ABBREVIATIONS AND SYMBOLS

CD	concave downward
CU	concave upward
D	the domain of $f$
FDT	First Derivative Test
HA	horizontal asymptote(s)
I	interval of convergence
I/D	Increasing/Decreasing Test
IP	inflection point(s)
R	radius of convergence
VA	vertical asymptote(s)
$\equiv^{\text{CAS}}$	indicates the use of a computer algebra system.
$\equiv^{\text{H}}$	indicates the use of l'Hospital's Rule.
$\equiv^j$	indicates the use of Formula $j$ in the Table of Integrals in the back endpapers.
$\equiv^s$	indicates the use of the substitution $\{u = \sin x, du = \cos x \, dx\}$ .
$\equiv^c$	indicates the use of the substitution $\{u = \cos x, du = -\sin x \, dx\}$ .

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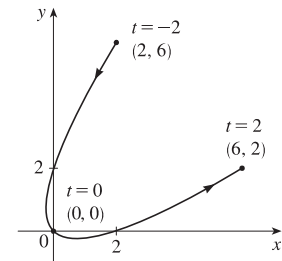
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## 10 □ PARAMETRIC EQUATIONS AND POLAR COORDINATES

### 10.1 Curves Defined by Parametric Equations

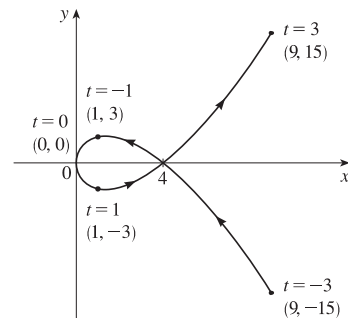
1.  $x = t^2 + t$ ,  $y = t^2 - t$ ,  $-2 \leq t \leq 2$

$t$	-2	-1	0	1	2
$x$	2	0	0	2	6
$y$	6	2	0	0	2



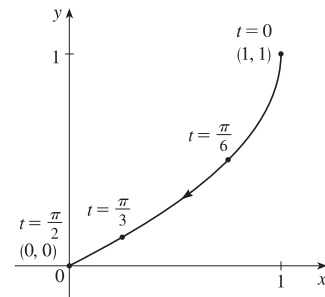
2.  $x = t^2$ ,  $y = t^3 - 4t$ ,  $-3 \leq t \leq 3$

$t$	±3	±2	±1	0
$x$	9	4	1	0
$y$	±15	0	∓3	0



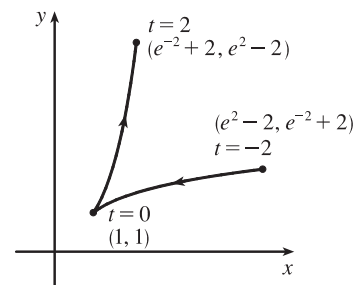
3.  $x = \cos^2 t$ ,  $y = 1 - \sin t$ ,  $0 \leq t \leq \pi/2$

$t$	0	$\pi/6$	$\pi/3$	$\pi/2$
$x$	1	3/4	1/4	0
$y$	1	1/2	$1 - \frac{\sqrt{3}}{2} \approx 0.13$	0



4.  $x = e^{-t} + t$ ,  $y = e^t - t$ ,  $-2 \leq t \leq 2$

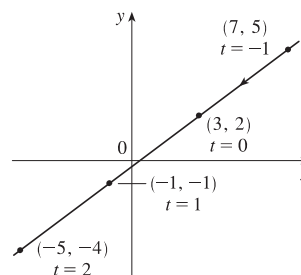
$t$	-2	-1	0	1	2
$x$	$e^2 - 2$ 5.39	$e - 1$ 1.72	1	$e^{-1} + 1$ 1.37	$e^{-2} + 2$ 2.14
$y$	$e^{-2} + 2$ 2.14	$e^{-1} + 1$ 1.37	1	$e - 1$ 1.72	$e^2 - 2$ 5.39



5.  $x = 3 - 4t$ ,  $y = 2 - 3t$

(a)

$t$	-1	0	1	2
$x$	7	3	-1	-5
$y$	5	2	-1	-4



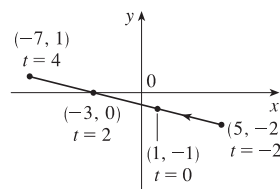
(b)  $x = 3 - 4t \Rightarrow 4t = -x + 3 \Rightarrow t = -\frac{1}{4}x + \frac{3}{4}$ , so

$$y = 2 - 3t = 2 - 3\left(-\frac{1}{4}x + \frac{3}{4}\right) = 2 + \frac{3}{4}x - \frac{9}{4} \Rightarrow y = \frac{3}{4}x - \frac{1}{4}$$

6.  $x = 1 - 2t$ ,  $y = \frac{1}{2}t - 1$ ,  $-2 \leq t \leq 4$

(a)

$t$	-2	0	2	4
$x$	5	1	-3	-7
$y$	-2	-1	0	1



(b)  $x = 1 - 2t \Rightarrow 2t = -x + 1 \Rightarrow t = -\frac{1}{2}x + \frac{1}{2}$ , so

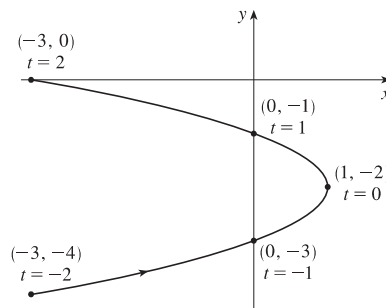
$$y = \frac{1}{2}t - 1 = \frac{1}{2}\left(-\frac{1}{2}x + \frac{1}{2}\right) - 1 = -\frac{1}{4}x + \frac{1}{4} - 1 \Rightarrow y = -\frac{1}{4}x - \frac{3}{4},$$

with  $-7 \leq x \leq 5$

7.  $x = 1 - t^2$ ,  $y = t - 2$ ,  $-2 \leq t \leq 2$

(a)

$t$	-2	-1	0	1	2
$x$	-3	0	1	0	-3
$y$	-4	-3	-2	-1	0



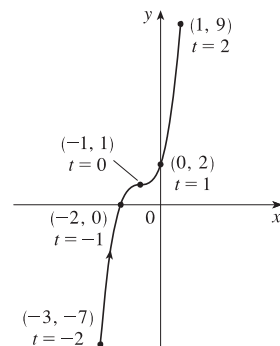
(b)  $y = t - 2 \Rightarrow t = y + 2$ , so  $x = 1 - t^2 = 1 - (y + 2)^2 \Rightarrow$

$$x = -(y + 2)^2 + 1, \text{ or } x = -y^2 - 4y - 3, \text{ with } -4 \leq y \leq 0$$

8.  $x = t - 1$ ,  $y = t^3 + 1$ ,  $-2 \leq t \leq 2$

(a)

$t$	-2	-1	0	1	2
$x$	-3	-2	-1	0	1
$y$	-7	0	1	2	9



(b)  $x = t - 1 \Rightarrow t = x + 1$ , so  $y = t^3 + 1 \Rightarrow y = (x + 1)^3 + 1$ ,

or  $y = x^3 + 3x^2 + 3x + 2$ , with  $-3 \leq x \leq 1$

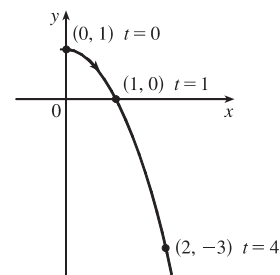
9.  $x = \sqrt{t}$ ,  $y = 1 - t$

(a)

$t$	0	1	2	3	4
$x$	0	1	1.414	1.732	2
$y$	1	0	-1	-2	-3

(b)  $x = \sqrt{t} \Rightarrow t = x^2 \Rightarrow y = 1 - t = 1 - x^2$ . Since  $t \geq 0$ ,  $x \geq 0$ .

So the curve is the right half of the parabola  $y = 1 - x^2$ .

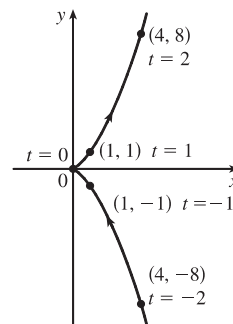


10.  $x = t^2$ ,  $y = t^3$

(a)

$t$	-2	-1	0	1	2
$x$	4	1	0	1	4
$y$	-8	-1	0	1	8

(b)  $y = t^3 \Rightarrow t = \sqrt[3]{y} \Rightarrow x = t^2 = (\sqrt[3]{y})^2 = y^{2/3}$ .  $t \in \mathbb{R}$ ,  $y \in \mathbb{R}$ ,  $x \geq 0$ .



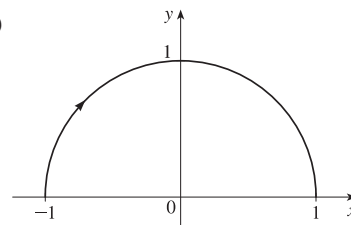
11. (a)  $x = \sin \frac{1}{2}\theta$ ,  $y = \cos \frac{1}{2}\theta$ ,  $-\pi \leq \theta \leq \pi$ .

$x^2 + y^2 = \sin^2 \frac{1}{2}\theta + \cos^2 \frac{1}{2}\theta = 1$ . For  $-\pi \leq \theta \leq 0$ , we have

$-1 \leq x \leq 0$  and  $0 \leq y \leq 1$ . For  $0 < \theta \leq \pi$ , we have  $0 < x \leq 1$

and  $1 > y \geq 0$ . The graph is a semicircle.

(b)



12. (a)  $x = \frac{1}{2} \cos \theta$ ,  $y = 2 \sin \theta$ ,  $0 \leq \theta \leq \pi$ .

$(2x)^2 + (\frac{1}{2}y)^2 = \cos^2 \theta + \sin^2 \theta = 1 \Rightarrow 4x^2 + \frac{1}{4}y^2 = 1 \Rightarrow$

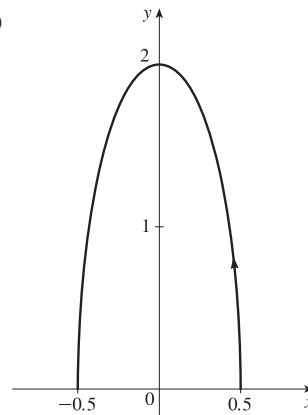
$\frac{x^2}{(1/2)^2} + \frac{y^2}{2^2} = 1$ , which is an equation of an ellipse with

$x$ -intercepts  $\pm \frac{1}{2}$  and  $y$ -intercepts  $\pm 2$ . For  $0 \leq \theta \leq \pi/2$ , we have

$\frac{1}{2} \geq x \geq 0$  and  $0 \leq y \leq 2$ . For  $\pi/2 < \theta \leq \pi$ , we have  $0 > x \geq -\frac{1}{2}$

and  $2 > y \geq 0$ . So the graph is the top half of the ellipse.

(b)

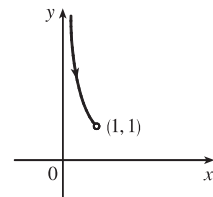


13. (a)  $x = \sin t$ ,  $y = \csc t$ ,  $0 < t < \frac{\pi}{2}$ .  $y = \csc t = \frac{1}{\sin t} = \frac{1}{x}$ .

For  $0 < t < \frac{\pi}{2}$ , we have  $0 < x < 1$  and  $y > 1$ . Thus, the curve is the

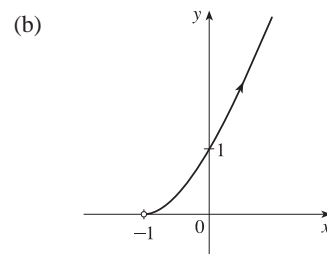
portion of the hyperbola  $y = 1/x$  with  $y > 1$ .

(b)



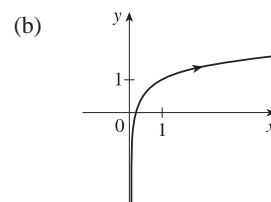
14. (a)  $x = e^t - 1, y = e^{2t}$ .

$y = (e^t)^2 = (x + 1)^2$  and since  $x > -1$ , we have the right side of the parabola  $y = (x + 1)^2$ .



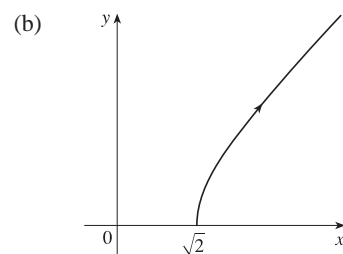
15. (a)  $x = e^{2t} \Rightarrow 2t = \ln x \Rightarrow t = \frac{1}{2} \ln x$ .

$y = t + 1 = \frac{1}{2} \ln x + 1$ .

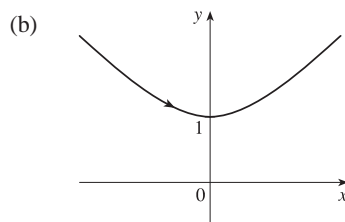


16. (a)  $x = \sqrt{t+1} \Rightarrow x^2 = t+1 \Rightarrow t = x^2 - 1$ .

$y = \sqrt{t-1} = \sqrt{(x^2 - 1) - 1} = \sqrt{x^2 - 2}$ . The curve is the part of the hyperbola  $x^2 - y^2 = 2$  with  $x \geq \sqrt{2}$  and  $y \geq 0$ .

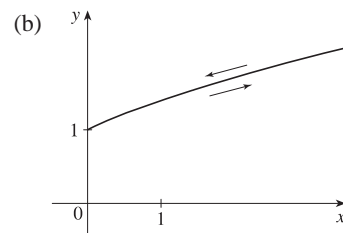


17. (a)  $x = \sinh t, y = \cosh t \Rightarrow y^2 - x^2 = \cosh^2 t - \sinh^2 t = 1$ . Since  $y = \cosh t \geq 1$ , we have the upper branch of the hyperbola  $y^2 - x^2 = 1$ .



18. (a)  $x = \tan^2 \theta, y = \sec \theta, -\pi/2 < \theta < \pi/2$ .

$1 + \tan^2 \theta = \sec^2 \theta \Rightarrow 1 + x = y^2 \Rightarrow x = y^2 - 1$ . For  $-\pi/2 < \theta \leq 0$ , we have  $x \geq 0$  and  $y \geq 1$ . For  $0 < \theta < \pi/2$ , we have  $0 < x$  and  $1 < y$ . Thus, the curve is the portion of the parabola  $x = y^2 - 1$  in the first quadrant. As  $\theta$  increases from  $-\pi/2$  to 0, the point  $(x, y)$  approaches  $(0, 1)$  along the parabola. As  $\theta$  increases from 0 to  $\pi/2$ , the point  $(x, y)$  retreats from  $(0, 1)$  along the parabola.

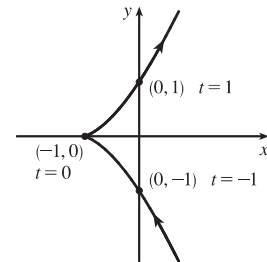


19.  $x = 3 + 2 \cos t, y = 1 + 2 \sin t, \pi/2 \leq t \leq 3\pi/2$ . By Example 4 with  $r = 2, h = 3$ , and  $k = 1$ , the motion of the particle takes place on a circle centered at  $(3, 1)$  with a radius of 2. As  $t$  goes from  $\pi/2$  to  $3\pi/2$ , the particle starts at the point  $(3, 3)$  and moves counterclockwise along the circle  $(x - 3)^2 + (y - 1)^2 = 4$  to  $(3, -1)$  [one-half of a circle].

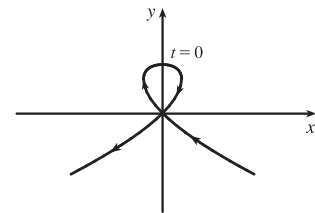
20.  $x = 2 \sin t, y = 4 + \cos t \Rightarrow \sin t = \frac{x}{2}, \cos t = y - 4, \sin^2 t + \cos^2 t = 1 \Rightarrow \left(\frac{x}{2}\right)^2 + (y - 4)^2 = 1$ . The motion of the particle takes place on an ellipse centered at  $(0, 4)$ . As  $t$  goes from 0 to  $3\pi/2$ , the particle starts at the point  $(0, 5)$  and moves clockwise to  $(-2, 4)$  [three-quarters of an ellipse].

21.  $x = 5 \sin t, y = 2 \cos t \Rightarrow \sin t = \frac{x}{5}, \cos t = \frac{y}{2}. \sin^2 t + \cos^2 t = 1 \Rightarrow \left(\frac{x}{5}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$ . The motion of the particle takes place on an ellipse centered at  $(0, 0)$ . As  $t$  goes from  $-\pi$  to  $5\pi$ , the particle starts at the point  $(0, -2)$  and moves clockwise around the ellipse 3 times.
22.  $y = \cos^2 t = 1 - \sin^2 t = 1 - x^2$ . The motion of the particle takes place on the parabola  $y = 1 - x^2$ . As  $t$  goes from  $-2\pi$  to  $-\pi$ , the particle starts at the point  $(0, 1)$ , moves to  $(1, 0)$ , and goes back to  $(0, 1)$ . As  $t$  goes from  $-\pi$  to  $0$ , the particle moves to  $(-1, 0)$  and goes back to  $(0, 1)$ . The particle repeats this motion as  $t$  goes from  $0$  to  $2\pi$ .
23. We must have  $1 \leq x \leq 4$  and  $2 \leq y \leq 3$ . So the graph of the curve must be contained in the rectangle  $[1, 4]$  by  $[2, 3]$ .
24. (a) From the first graph, we have  $1 \leq x \leq 2$ . From the second graph, we have  $-1 \leq y \leq 1$ . The only choice that satisfies either of those conditions is III.
- (b) From the first graph, the values of  $x$  cycle through the values from  $-2$  to  $2$  four times. From the second graph, the values of  $y$  cycle through the values from  $-2$  to  $2$  six times. Choice I satisfies these conditions.
- (c) From the first graph, the values of  $x$  cycle through the values from  $-2$  to  $2$  three times. From the second graph, we have  $0 \leq y \leq 2$ . Choice IV satisfies these conditions.
- (d) From the first graph, the values of  $x$  cycle through the values from  $-2$  to  $2$  two times. From the second graph, the values of  $y$  do the same thing. Choice II satisfies these conditions.

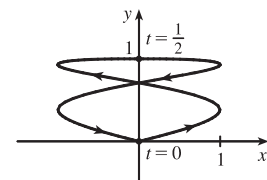
25. When  $t = -1$ ,  $(x, y) = (0, -1)$ . As  $t$  increases to  $0$ ,  $x$  decreases to  $-1$  and  $y$  increases to  $0$ . As  $t$  increases from  $0$  to  $1$ ,  $x$  increases to  $0$  and  $y$  increases to  $1$ . As  $t$  increases beyond  $1$ , both  $x$  and  $y$  increase. For  $t < -1$ ,  $x$  is positive and decreasing and  $y$  is negative and increasing. We could achieve greater accuracy by estimating  $x$ - and  $y$ -values for selected values of  $t$  from the given graphs and plotting the corresponding points.



26. For  $t < -1$ ,  $x$  is positive and decreasing, while  $y$  is negative and increasing (these points are in Quadrant IV). When  $t = -1$ ,  $(x, y) = (0, 0)$  and, as  $t$  increases from  $-1$  to  $0$ ,  $x$  becomes negative and  $y$  increases from  $0$  to  $1$ . At  $t = 0$ ,  $(x, y) = (0, 1)$  and, as  $t$  increases from  $0$  to  $1$ ,  $y$  decreases from  $1$  to  $0$  and  $x$  is positive. At  $t = 1$ ,  $(x, y) = (0, 0)$  again, so the loop is completed. For  $t > 1$ ,  $x$  and  $y$  both become large negative. This enables us to draw a rough sketch. We could achieve greater accuracy by estimating  $x$ - and  $y$ -values for selected values of  $t$  from the given graphs and plotting the corresponding points.

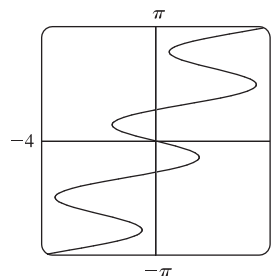


27. When  $t = 0$  we see that  $x = 0$  and  $y = 0$ , so the curve starts at the origin. As  $t$  increases from  $0$  to  $\frac{1}{2}$ , the graphs show that  $y$  increases from  $0$  to  $1$  while  $x$  increases from  $0$  to  $1$ , decreases to  $0$  and to  $-1$ , then increases back to  $0$ , so we arrive at the point  $(0, 1)$ . Similarly, as  $t$  increases from  $\frac{1}{2}$  to  $1$ ,  $y$  decreases from  $1$  to  $0$  while  $x$  repeats its pattern, and we arrive back at the origin. We could achieve greater accuracy by estimating  $x$ - and  $y$ -values for selected values of  $t$  from the given graphs and plotting the corresponding points.

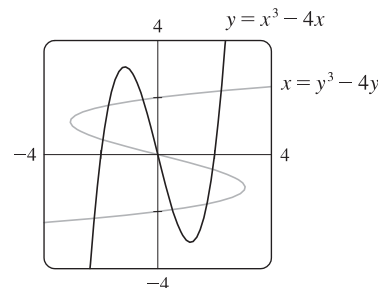


28. (a)  $x = t^4 - t + 1 = (t^4 + 1) - t > 0$  [think of the graphs of  $y = t^4 + 1$  and  $y = t$ ] and  $y = t^2 \geq 0$ , so these equations are matched with graph V.
- (b)  $y = \sqrt{t} \geq 0$ .  $x = t^2 - 2t = t(t - 2)$  is negative for  $0 < t < 2$ , so these equations are matched with graph I.
- (c)  $x = \sin 2t$  has period  $2\pi/2 = \pi$ . Note that  
 $y(t + 2\pi) = \sin[t + 2\pi + \sin 2(t + 2\pi)] = \sin(t + 2\pi + \sin 2t) = \sin(t + \sin 2t) = y(t)$ , so  $y$  has period  $2\pi$ .  
 These equations match graph II since  $x$  cycles through the values  $-1$  to  $1$  twice as  $y$  cycles through those values once.
- (d)  $x = \cos 5t$  has period  $2\pi/5$  and  $y = \sin 2t$  has period  $\pi$ , so  $x$  will take on the values  $-1$  to  $1$ , and then  $1$  to  $-1$ , before  $y$  takes on the values  $-1$  to  $1$ . Note that when  $t = 0$ ,  $(x, y) = (1, 0)$ . These equations are matched with graph VI.
- (e)  $x = t + \sin 4t$ ,  $y = t^2 + \cos 3t$ . As  $t$  becomes large,  $t$  and  $t^2$  become the dominant terms in the expressions for  $x$  and  $y$ , so the graph will look like the graph of  $y = x^2$ , but with oscillations. These equations are matched with graph IV.
- (f)  $x = \frac{\sin 2t}{4 + t^2}$ ,  $y = \frac{\cos 2t}{4 + t^2}$ . As  $t \rightarrow \infty$ ,  $x$  and  $y$  both approach 0. These equations are matched with graph III.

29. Use  $y = t$  and  $x = t - 2 \sin \pi t$  with a  $t$ -interval of  $[-\pi, \pi]$ .



30. Use  $x_1 = t$ ,  $y_1 = t^3 - 4t$  and  $x_2 = t^3 - 4t$ ,  $y_2 = t$  with a  $t$ -interval of  $[-3, 3]$ . There are 9 points of intersection;  $(0, 0)$  is fairly obvious. The point in quadrant I is approximately  $(2.2, 2.2)$ , and by symmetry, the point in quadrant III is approximately  $(-2.2, -2.2)$ . The other six points are approximately  $(\mp 1.9, \pm 0.5)$ ,  $(\mp 1.7, \pm 1.7)$ , and  $(\mp 0.5, \pm 1.9)$ .



31. (a)  $x = x_1 + (x_2 - x_1)t$ ,  $y = y_1 + (y_2 - y_1)t$ ,  $0 \leq t \leq 1$ . Clearly the curve passes through  $P_1(x_1, y_1)$  when  $t = 0$  and through  $P_2(x_2, y_2)$  when  $t = 1$ . For  $0 < t < 1$ ,  $x$  is strictly between  $x_1$  and  $x_2$  and  $y$  is strictly between  $y_1$  and  $y_2$ . For every value of  $t$ ,  $x$  and  $y$  satisfy the relation  $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$ , which is the equation of the line through  $P_1(x_1, y_1)$  and  $P_2(x_2, y_2)$ .

Finally, any point  $(x, y)$  on that line satisfies  $\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$ ; if we call that common value  $t$ , then the given

parametric equations yield the point  $(x, y)$ ; and any  $(x, y)$  on the line between  $P_1(x_1, y_1)$  and  $P_2(x_2, y_2)$  yields a value of  $t$  in  $[0, 1]$ . So the given parametric equations exactly specify the line segment from  $P_1(x_1, y_1)$  to  $P_2(x_2, y_2)$ .

- (b)  $x = -2 + [3 - (-2)]t = -2 + 5t$  and  $y = 7 + (-1 - 7)t = 7 - 8t$  for  $0 \leq t \leq 1$ .



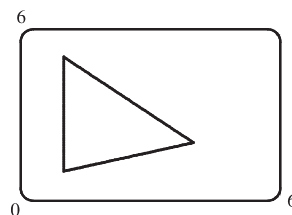
32. For the side of the triangle from  $A$  to  $B$ , use  $(x_1, y_1) = (1, 1)$  and  $(x_2, y_2) = (4, 2)$ .

Hence, the equations are

$$\begin{aligned}x &= x_1 + (x_2 - x_1)t = 1 + (4 - 1)t = 1 + 3t, \\y &= y_1 + (y_2 - y_1)t = 1 + (2 - 1)t = 1 + t.\end{aligned}$$

Graphing  $x = 1 + 3t$  and  $y = 1 + t$  with  $0 \leq t \leq 1$  gives us the side of the

triangle from  $A$  to  $B$ . Similarly, for the side  $BC$  we use  $x = 4 - 3t$  and  $y = 2 + 3t$ , and for the side  $AC$  we use  $x = 1$  and  $y = 1 + 4t$ .



33. The circle  $x^2 + (y - 1)^2 = 4$  has center  $(0, 1)$  and radius 2, so by Example 4 it can be represented by  $x = 2 \cos t$ ,  $y = 1 + 2 \sin t$ ,  $0 \leq t \leq 2\pi$ . This representation gives us the circle with a counterclockwise orientation starting at  $(2, 1)$ .

(a) To get a clockwise orientation, we could change the equations to  $x = 2 \cos t$ ,  $y = 1 - 2 \sin t$ ,  $0 \leq t \leq 2\pi$ .

(b) To get three times around in the counterclockwise direction, we use the original equations  $x = 2 \cos t$ ,  $y = 1 + 2 \sin t$  with the domain expanded to  $0 \leq t \leq 6\pi$ .

(c) To start at  $(0, 3)$  using the original equations, we must have  $x_1 = 0$ ; that is,  $2 \cos t = 0$ . Hence,  $t = \frac{\pi}{2}$ . So we use

$$x = 2 \cos t, \quad y = 1 + 2 \sin t, \quad \frac{\pi}{2} \leq t \leq \frac{3\pi}{2}.$$

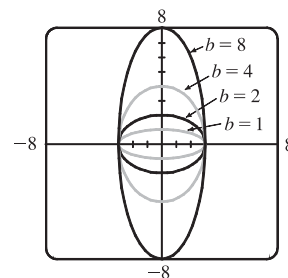
Alternatively, if we want  $t$  to start at 0, we could change the equations of the curve. For example, we could use

$$x = -2 \sin t, \quad y = 1 + 2 \cos t, \quad 0 \leq t \leq \pi.$$

34. (a) Let  $x^2/a^2 = \sin^2 t$  and  $y^2/b^2 = \cos^2 t$  to obtain  $x = a \sin t$  and  $y = b \cos t$  with  $0 \leq t \leq 2\pi$  as possible parametric equations for the ellipse  $x^2/a^2 + y^2/b^2 = 1$ .

(b) The equations are  $x = 3 \sin t$  and  $y = b \cos t$  for  $b \in \{1, 2, 4, 8\}$ .

(c) As  $b$  increases, the ellipse stretches vertically.



35. *Big circle*: It's centered at  $(2, 2)$  with a radius of 2, so by Example 4, parametric equations are

$$x = 2 + 2 \cos t, \quad y = 2 + 2 \sin t, \quad 0 \leq t \leq 2\pi$$

*Small circles*: They are centered at  $(1, 3)$  and  $(3, 3)$  with a radius of 0.1. By Example 4, parametric equations are

$$(left) \quad x = 1 + 0.1 \cos t, \quad y = 3 + 0.1 \sin t, \quad 0 \leq t \leq 2\pi$$

and

$$(right) \quad x = 3 + 0.1 \cos t, \quad y = 3 + 0.1 \sin t, \quad 0 \leq t \leq 2\pi$$

*Semicircle*: It's the lower half of a circle centered at  $(2, 2)$  with radius 1. By Example 4, parametric equations are

$$x = 2 + 1 \cos t, \quad y = 2 + 1 \sin t, \quad \pi \leq t \leq 2\pi$$

To get all four graphs on the same screen with a typical graphing calculator, we need to change the last  $t$ -interval to  $[0, 2\pi]$  in order to match the others. We can do this by changing  $t$  to  $0.5t$ . This change gives us the upper half. There are several ways to get the lower half—one is to change the “+” to a “−” in the  $y$ -assignment, giving us

$$x = 2 + 1 \cos(0.5t), \quad y = 2 - 1 \sin(0.5t), \quad 0 \leq t \leq 2\pi$$

36. If you are using a calculator or computer that can overlay graphs (using multiple  $t$ -intervals), the following is appropriate.

*Left side:*  $x = 1$  and  $y$  goes from 1.5 to 4, so use

$$x = 1, \quad y = t, \quad 1.5 \leq t \leq 4$$

*Right side:*  $x = 10$  and  $y$  goes from 1.5 to 4, so use

$$x = 10, \quad y = t, \quad 1.5 \leq t \leq 4$$

*Bottom:*  $x$  goes from 1 to 10 and  $y = 1.5$ , so use

$$x = t, \quad y = 1.5, \quad 1 \leq t \leq 10$$

*Handle:* It starts at  $(10, 4)$  and ends at  $(13, 7)$ , so use

$$x = 10 + t, \quad y = 4 + t, \quad 0 \leq t \leq 3$$

*Left wheel:* It's centered at  $(3, 1)$ , has a radius of 1, and appears to go about  $30^\circ$  above the horizontal, so use

$$x = 3 + 1 \cos t, \quad y = 1 + 1 \sin t, \quad \frac{5\pi}{6} \leq t \leq \frac{13\pi}{6}$$

*Right wheel:* Similar to the left wheel with center  $(8, 1)$ , so use

$$x = 8 + 1 \cos t, \quad y = 1 + 1 \sin t, \quad \frac{5\pi}{6} \leq t \leq \frac{13\pi}{6}$$

If you are using a calculator or computer that cannot overlay graphs (using one  $t$ -interval), the following is appropriate.

We'll start by picking the  $t$ -interval  $[0, 2.5]$  since it easily matches the  $t$ -values for the two sides. We now need to find parametric equations for all graphs with  $0 \leq t \leq 2.5$ .

*Left side:*  $x = 1$  and  $y$  goes from 1.5 to 4, so use

$$x = 1, \quad y = 1.5 + t, \quad 0 \leq t \leq 2.5$$

*Right side:*  $x = 10$  and  $y$  goes from 1.5 to 4, so use

$$x = 10, \quad y = 1.5 + t, \quad 0 \leq t \leq 2.5$$

*Bottom:*  $x$  goes from 1 to 10 and  $y = 1.5$ , so use

$$x = 1 + 3.6t, \quad y = 1.5, \quad 0 \leq t \leq 2.5$$

To get the  $x$ -assignment, think of creating a linear function such that when  $t = 0$ ,  $x = 1$  and when  $t = 2.5$ ,  $x = 10$ . We can use the point-slope form of a line with  $(t_1, x_1) = (0, 1)$  and  $(t_2, x_2) = (2.5, 10)$ .

$$x - 1 = \frac{10 - 1}{2.5 - 0}(t - 0) \Rightarrow x = 1 + 3.6t.$$

*Handle:* It starts at  $(10, 4)$  and ends at  $(13, 7)$ , so use

$$x = 10 + 1.2t, \quad y = 4 + 1.2t, \quad 0 \leq t \leq 2.5$$

$$(t_1, x_1) = (0, 10) \text{ and } (t_2, x_2) = (2.5, 13) \text{ gives us } x - 10 = \frac{13 - 10}{2.5 - 0}(t - 0) \Rightarrow x = 10 + 1.2t.$$

$$(t_1, y_1) = (0, 4) \text{ and } (t_2, y_2) = (2.5, 7) \text{ gives us } y - 4 = \frac{7 - 4}{2.5 - 0}(t - 0) \Rightarrow y = 4 + 1.2t.$$

*Left wheel:* It's centered at  $(3, 1)$ , has a radius of 1, and appears to go about  $30^\circ$  above the horizontal, so use

$$x = 3 + 1 \cos\left(\frac{8\pi}{15}t + \frac{5\pi}{6}\right), \quad y = 1 + 1 \sin\left(\frac{8\pi}{15}t + \frac{5\pi}{6}\right), \quad 0 \leq t \leq 2.5$$

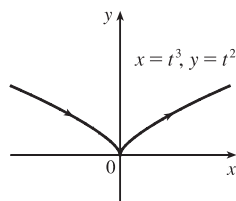
$$(t_1, \theta_1) = \left(0, \frac{5\pi}{6}\right) \text{ and } (t_2, \theta_2) = \left(\frac{5}{2}, \frac{13\pi}{6}\right) \text{ gives us } \theta - \frac{5\pi}{6} = \frac{\frac{13\pi}{6} - \frac{5\pi}{6}}{\frac{5}{2} - 0}(t - 0) \Rightarrow \theta = \frac{5\pi}{6} + \frac{8\pi}{15}t.$$

*Right wheel:* Similar to the left wheel with center  $(8, 1)$ , so use

$$x = 8 + 1 \cos\left(\frac{8\pi}{15}t + \frac{5\pi}{6}\right), \quad y = 1 + 1 \sin\left(\frac{8\pi}{15}t + \frac{5\pi}{6}\right), \quad 0 \leq t \leq 2.5$$

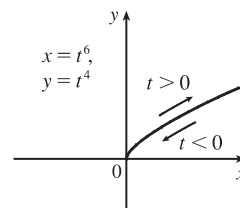
37. (a)  $x = t^3 \Rightarrow t = x^{1/3}$ , so  $y = t^2 = x^{2/3}$ .

We get the entire curve  $y = x^{2/3}$  traversed in a left to right direction.



(b)  $x = t^6 \Rightarrow t = x^{1/6}$ , so  $y = t^4 = x^{4/6} = x^{2/3}$ .

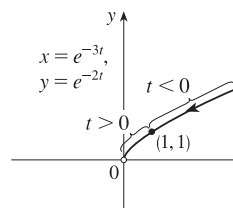
Since  $x = t^6 \geq 0$ , we only get the right half of the curve  $y = x^{2/3}$ .



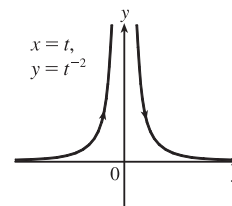
(c)  $x = e^{-3t} = (e^{-t})^3$  [so  $e^{-t} = x^{1/3}$ ],

$$y = e^{-2t} = (e^{-t})^2 = (x^{1/3})^2 = x^{2/3}.$$

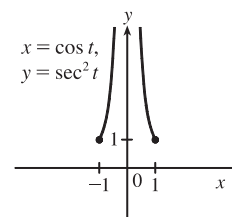
If  $t < 0$ , then  $x$  and  $y$  are both larger than 1. If  $t > 0$ , then  $x$  and  $y$  are between 0 and 1. Since  $x > 0$  and  $y > 0$ , the curve never quite reaches the origin.



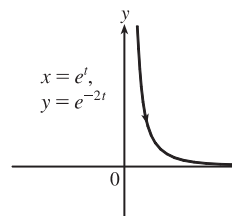
38. (a)  $x = t$ , so  $y = t^{-2} = x^{-2}$ . We get the entire curve  $y = 1/x^2$  traversed in a left-to-right direction.



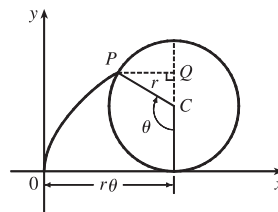
(b)  $x = \cos t$ ,  $y = \sec^2 t = \frac{1}{\cos^2 t} = \frac{1}{x^2}$ . Since  $\sec t \geq 1$ , we only get the parts of the curve  $y = 1/x^2$  with  $y \geq 1$ . We get the first quadrant portion of the curve when  $x > 0$ , that is,  $\cos t > 0$ , and we get the second quadrant portion of the curve when  $x < 0$ , that is,  $\cos t < 0$ .



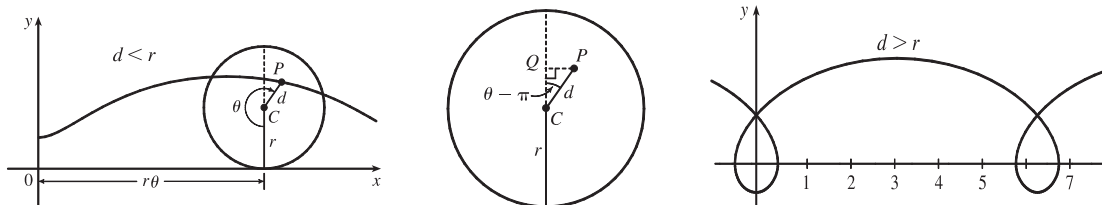
(c)  $x = e^t$ ,  $y = e^{-2t} = (e^t)^{-2} = x^{-2}$ . Since  $e^t$  and  $e^{-2t}$  are both positive, we only get the first quadrant portion of the curve  $y = 1/x^2$ .



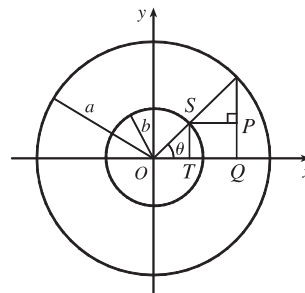
39. The case  $\frac{\pi}{2} < \theta < \pi$  is illustrated.  $C$  has coordinates  $(r\theta, r)$  as in Example 7, and  $Q$  has coordinates  $(r\theta, r + r \cos(\pi - \theta)) = (r\theta, r(1 - \cos \theta))$  [since  $\cos(\pi - \alpha) = \cos \pi \cos \alpha + \sin \pi \sin \alpha = -\cos \alpha$ ], so  $P$  has coordinates  $(r\theta - r \sin(\pi - \theta), r(1 - \cos \theta)) = (r(\theta - \sin \theta), r(1 - \cos \theta))$  [since  $\sin(\pi - \alpha) = \sin \pi \cos \alpha - \cos \pi \sin \alpha = \sin \alpha$ ]. Again we have the parametric equations  $x = r(\theta - \sin \theta)$ ,  $y = r(1 - \cos \theta)$ .



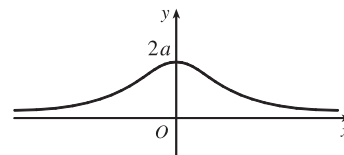
40. The first two diagrams depict the case  $\pi < \theta < \frac{3\pi}{2}$ ,  $d < r$ . As in Example 7,  $C$  has coordinates  $(r\theta, r)$ . Now  $Q$  (in the second diagram) has coordinates  $(r\theta, r + d \cos(\theta - \pi)) = (r\theta, r - d \cos \theta)$ , so a typical point  $P$  of the trochoid has coordinates  $(r\theta + d \sin(\theta - \pi), r - d \cos \theta)$ . That is,  $P$  has coordinates  $(x, y)$ , where  $x = r\theta - d \sin \theta$  and  $y = r - d \cos \theta$ . When  $d = r$ , these equations agree with those of the cycloid.



41. It is apparent that  $x = |OQ|$  and  $y = |QP| = |ST|$ . From the diagram,  $x = |OQ| = a \cos \theta$  and  $y = |ST| = b \sin \theta$ . Thus, the parametric equations are  $x = a \cos \theta$  and  $y = b \sin \theta$ . To eliminate  $\theta$  we rearrange:  $\sin \theta = y/b \Rightarrow \sin^2 \theta = (y/b)^2$  and  $\cos \theta = x/a \Rightarrow \cos^2 \theta = (x/a)^2$ . Adding the two equations:  $\sin^2 \theta + \cos^2 \theta = 1 = x^2/a^2 + y^2/b^2$ . Thus, we have an ellipse.



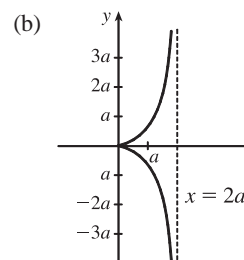
42.  $A$  has coordinates  $(a \cos \theta, a \sin \theta)$ . Since  $OA$  is perpendicular to  $AB$ ,  $\triangle OAB$  is a right triangle and  $B$  has coordinates  $(a \sec \theta, 0)$ . It follows that  $P$  has coordinates  $(a \sec \theta, b \sin \theta)$ . Thus, the parametric equations are  $x = a \sec \theta$ ,  $y = b \sin \theta$ .
43.  $C = (2a \cot \theta, 2a)$ , so the  $x$ -coordinate of  $P$  is  $x = 2a \cot \theta$ . Let  $B = (0, 2a)$ . Then  $\angle OAB$  is a right angle and  $\angle OBA = \theta$ , so  $|OA| = 2a \sin \theta$  and  $A = ((2a \sin \theta) \cos \theta, (2a \sin \theta) \sin \theta)$ . Thus, the  $y$ -coordinate of  $P$  is  $y = 2a \sin^2 \theta$ .



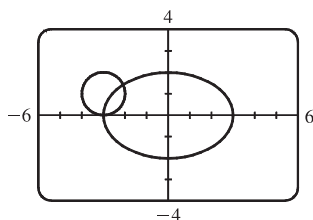
44. (a) Let  $\theta$  be the angle of inclination of segment  $OP$ . Then  $|OB| = \frac{2a}{\cos \theta}$ . Let  $C = (2a, 0)$ . Then by use of right triangle  $OAC$  we see that  $|OA| = 2a \cos \theta$ . Now

$$\begin{aligned} |OP| &= |AB| = |OB| - |OA| \\ &= 2a \left( \frac{1}{\cos \theta} - \cos \theta \right) = 2a \frac{1 - \cos^2 \theta}{\cos \theta} = 2a \frac{\sin^2 \theta}{\cos \theta} = 2a \sin \theta \tan \theta \end{aligned}$$

So  $P$  has coordinates  $x = 2a \sin \theta \tan \theta \cdot \cos \theta = 2a \sin^2 \theta$  and  $y = 2a \sin \theta \tan \theta \cdot \sin \theta = 2a \sin^2 \theta \tan \theta$ .



45. (a)



There are 2 points of intersection:

$(-3, 0)$  and approximately  $(-2.1, 1.4)$ .

(b) A collision point occurs when  $x_1 = x_2$  and  $y_1 = y_2$  for the same  $t$ . So solve the equations:

$$3 \sin t = -3 + \cos t \quad (1)$$

$$2 \cos t = 1 + \sin t \quad (2)$$

From (2),  $\sin t = 2 \cos t - 1$ . Substituting into (1), we get  $3(2 \cos t - 1) = -3 + \cos t \Rightarrow 5 \cos t = 0 \quad (*) \Rightarrow \cos t = 0 \Rightarrow t = \frac{\pi}{2}$  or  $\frac{3\pi}{2}$ . We check that  $t = \frac{3\pi}{2}$  satisfies (1) and (2) but  $t = \frac{\pi}{2}$  does not. So the only collision point occurs when  $t = \frac{3\pi}{2}$ , and this gives the point  $(-3, 0)$ . [We could check our work by graphing  $x_1$  and  $x_2$  together as functions of  $t$  and, on another plot,  $y_1$  and  $y_2$  as functions of  $t$ . If we do so, we see that the only value of  $t$  for which *both* pairs of graphs intersect is  $t = \frac{3\pi}{2}$ .]

(c) The circle is centered at  $(3, 1)$  instead of  $(-3, 1)$ . There are still 2 intersection points:  $(3, 0)$  and  $(2.1, 1.4)$ , but there are no collision points, since  $(*)$  in part (b) becomes  $5 \cos t = 6 \Rightarrow \cos t = \frac{6}{5} > 1$ .

46. (a) If  $\alpha = 30^\circ$  and  $v_0 = 500$  m/s, then the equations become  $x = (500 \cos 30^\circ)t = 250\sqrt{3}t$  and

$$y = (500 \sin 30^\circ)t - \frac{1}{2}(9.8)t^2 = 250t - 4.9t^2. \quad y = 0 \text{ when } t = 0 \text{ (when the gun is fired) and again when}$$

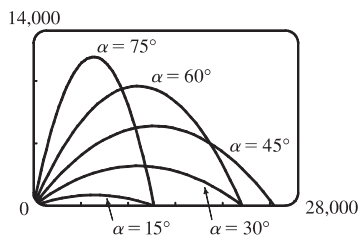
$$t = \frac{250}{4.9} \approx 51 \text{ s. Then } x = (250\sqrt{3})\left(\frac{250}{4.9}\right) \approx 22,092 \text{ m, so the bullet hits the ground about 22 km from the gun.}$$

The formula for  $y$  is quadratic in  $t$ . To find the maximum  $y$ -value, we will complete the square:

$$y = -4.9\left(t^2 - \frac{250}{4.9}t\right) = -4.9\left[t^2 - \frac{250}{4.9}t + \left(\frac{125}{4.9}\right)^2\right] + \frac{125^2}{4.9} = -4.9\left(t - \frac{125}{4.9}\right)^2 + \frac{125^2}{4.9} \leq \frac{125^2}{4.9}$$

with equality when  $t = \frac{125}{4.9}$  s, so the maximum height attained is  $\frac{125^2}{4.9} \approx 3189$  m.

(b)



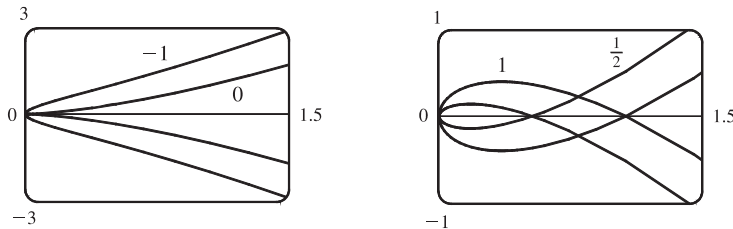
As  $\alpha$  ( $0^\circ < \alpha < 90^\circ$ ) increases up to  $45^\circ$ , the projectile attains a greater height and a greater range. As  $\alpha$  increases past  $45^\circ$ , the projectile attains a greater height, but its range decreases.

$$(c) \quad x = (v_0 \cos \alpha)t \Rightarrow t = \frac{x}{v_0 \cos \alpha}.$$

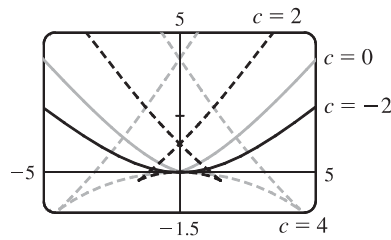
$$y = (v_0 \sin \alpha)t - \frac{1}{2}gt^2 \Rightarrow y = (v_0 \sin \alpha) \frac{x}{v_0 \cos \alpha} - \frac{g}{2} \left( \frac{x}{v_0 \cos \alpha} \right)^2 = (\tan \alpha)x - \left( \frac{g}{2v_0^2 \cos^2 \alpha} \right)x^2,$$

which is the equation of a parabola (quadratic in  $x$ ).

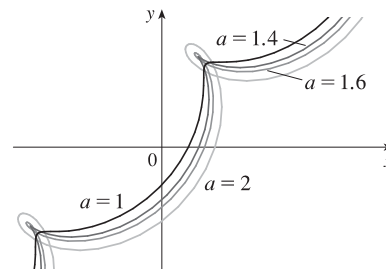
47.  $x = t^2, y = t^3 - ct$ . We use a graphing device to produce the graphs for various values of  $c$  with  $-\pi \leq t \leq \pi$ . Note that all the members of the family are symmetric about the  $x$ -axis. For  $c < 0$ , the graph does not cross itself, but for  $c = 0$  it has a cusp at  $(0, 0)$  and for  $c > 0$  the graph crosses itself at  $x = c$ , so the loop grows larger as  $c$  increases.



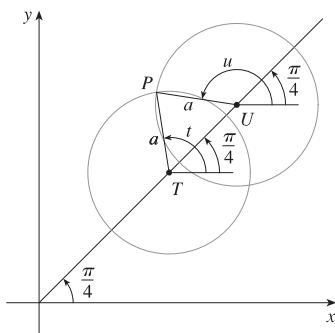
48.  $x = 2ct - 4t^3, y = -ct^2 + 3t^4$ . We use a graphing device to produce the graphs for various values of  $c$  with  $-\pi \leq t \leq \pi$ . Note that all the members of the family are symmetric about the  $y$ -axis. When  $c < 0$ , the graph resembles that of a polynomial of even degree, but when  $c = 0$  there is a corner at the origin, and when  $c > 0$ , the graph crosses itself at the origin, and has two cusps below the  $x$ -axis. The size of the “swallowtail” increases as  $c$  increases.



49.  $x = t + a \cos t, y = t + a \sin t, a > 0$ . From the first figure, we see that curves roughly follow the line  $y = x$ , and they start having loops when  $a$  is between 1.4 and 1.6. The loops increase in size as  $a$  increases.



While not required, the following is a solution to determine the *exact* values for which the curve has a loop, that is, we seek the values of  $a$  for which there exist parameter values  $t$  and  $u$  such that  $t < u$  and  $(t + a \cos t, t + a \sin t) = (u + a \cos u, u + a \sin u)$ .



In the diagram at the left,  $T$  denotes the point  $(t, t)$ ,  $U$  the point  $(u, u)$ , and  $P$  the point  $(t + a \cos t, t + a \sin t) = (u + a \cos u, u + a \sin u)$ .

Since  $\overline{PT} = \overline{PU} = a$ , the triangle  $PTU$  is isosceles. Therefore its base angles,  $\alpha = \angle PTU$  and  $\beta = \angle PUT$  are equal. Since  $\alpha = t - \frac{\pi}{4}$  and

$\beta = 2\pi - \frac{3\pi}{4} - u = \frac{5\pi}{4} - u$ , the relation  $\alpha = \beta$  implies that

$$u + t = \frac{3\pi}{2} \quad (1).$$

Since  $\overline{TU} = \text{distance}((t, t), (u, u)) = \sqrt{2(u-t)^2} = \sqrt{2}(u-t)$ , we see that

$$\cos \alpha = \frac{\frac{1}{2}\overline{TU}}{\overline{PT}} = \frac{(u-t)/\sqrt{2}}{a}, \text{ so } u-t = \sqrt{2}a \cos \alpha, \text{ that is,}$$

$$u-t = \sqrt{2}a \cos\left(t - \frac{\pi}{4}\right) \quad (2). \text{ Now } \cos\left(t - \frac{\pi}{4}\right) = \sin\left[\frac{\pi}{2} - \left(t - \frac{\pi}{4}\right)\right] = \sin\left(\frac{3\pi}{4} - t\right),$$

so we can rewrite (2) as  $u-t = \sqrt{2}a \sin\left(\frac{3\pi}{4} - t\right)$  (2'). Subtracting (2') from (1) and

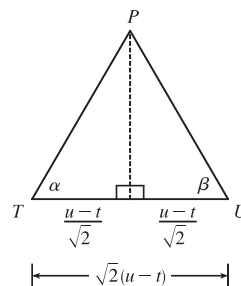
dividing by 2, we obtain  $t = \frac{3\pi}{4} - \frac{\sqrt{2}}{2}a \sin\left(\frac{3\pi}{4} - t\right)$ , or  $\frac{3\pi}{4} - t = \frac{a}{\sqrt{2}} \sin\left(\frac{3\pi}{4} - t\right)$  (3).

Since  $a > 0$  and  $t < u$ , it follows from (2') that  $\sin\left(\frac{3\pi}{4} - t\right) > 0$ . Thus from (3) we see that  $t < \frac{3\pi}{4}$ . [We have

implicitly assumed that  $0 < t < \pi$  by the way we drew our diagram, but we lost no generality by doing so since replacing  $t$  by  $t + 2\pi$  merely increases  $x$  and  $y$  by  $2\pi$ . The curve's basic shape repeats every time we change  $t$  by  $2\pi$ .] Solving for  $a$  in

(3), we get  $a = \frac{\sqrt{2}\left(\frac{3\pi}{4} - t\right)}{\sin\left(\frac{3\pi}{4} - t\right)}$ . Write  $z = \frac{3\pi}{4} - t$ . Then  $a = \frac{\sqrt{2}z}{\sin z}$ , where  $z > 0$ . Now  $\sin z < z$  for  $z > 0$ , so  $a > \sqrt{2}$ .

[As  $z \rightarrow 0^+$ , that is, as  $t \rightarrow \left(\frac{3\pi}{4}\right)^-$ ,  $a \rightarrow \sqrt{2}$ .]

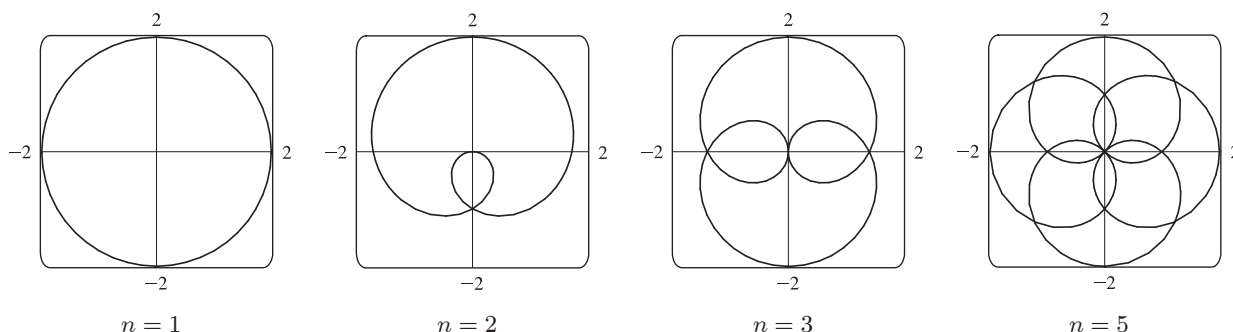


50. Consider the curves  $x = \sin t + \sin nt$ ,  $y = \cos t + \cos nt$ , where  $n$  is a positive integer. For  $n = 1$ , we get a circle of radius 2 centered at the origin. For  $n > 1$ , we get a curve lying on or inside that circle that traces out  $n - 1$  loops as  $t$  ranges from 0 to  $2\pi$ .

Note:

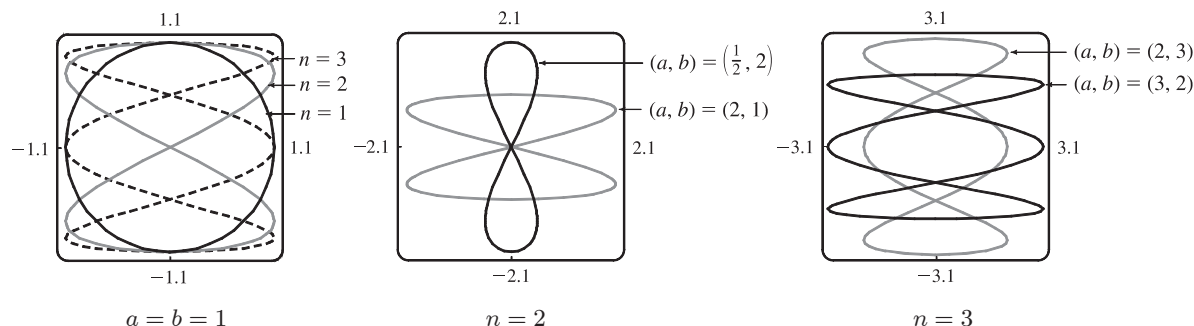
$$\begin{aligned} x^2 + y^2 &= (\sin t + \sin nt)^2 + (\cos t + \cos nt)^2 \\ &= \sin^2 t + 2 \sin t \sin nt + \sin^2 nt + \cos^2 t + 2 \cos t \cos nt + \cos^2 nt \\ &= (\sin^2 t + \cos^2 t) + (\sin^2 nt + \cos^2 nt) + 2(\cos t \cos nt + \sin t \sin nt) \\ &= 1 + 1 + 2 \cos(t - nt) = 2 + 2 \cos((1 - n)t) \leq 4 = 2^2, \end{aligned}$$

with equality for  $n = 1$ . This shows that each curve lies on or inside the curve for  $n = 1$ , which is a circle of radius 2 centered at the origin.

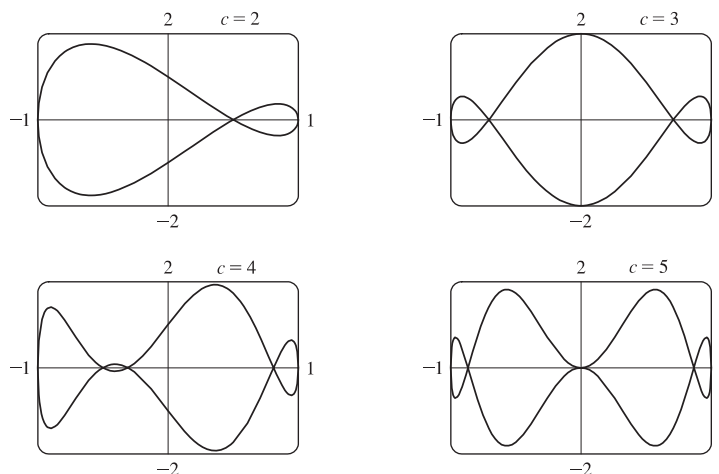


51. Note that all the Lissajous figures are symmetric about the  $x$ -axis. The parameters  $a$  and  $b$  simply stretch the graph in the  $x$ - and  $y$ -directions respectively. For  $a = b = n = 1$  the graph is simply a circle with radius 1. For  $n = 2$  the graph crosses

itself at the origin and there are loops above and below the  $x$ -axis. In general, the figures have  $n - 1$  points of intersection, all of which are on the  $y$ -axis, and a total of  $n$  closed loops.



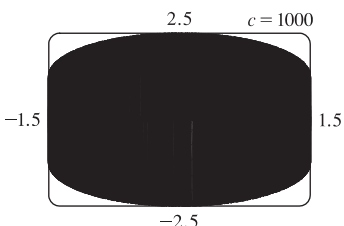
52.  $x = \cos t$ ,  $y = \sin t - \sin ct$ . If  $c = 1$ , then  $y = 0$ , and the curve is simply the line segment from  $(-1, 0)$  to  $(1, 0)$ . The graphs are shown for  $c = 2, 3, 4$  and  $5$ .



It is easy to see that all the curves lie in the rectangle  $[-1, 1]$  by  $[-2, 2]$ . When  $c$  is an integer,  $x(t + 2\pi) = x(t)$  and  $y(t + 2\pi) = y(t)$ , so the curve is closed. When  $c$  is a positive integer greater than 1, the curve intersects the  $x$ -axis  $c + 1$  times and has  $c$  loops (one of which degenerates to a tangency at the origin when  $c$  is an odd integer of the form  $4k + 1$ ).

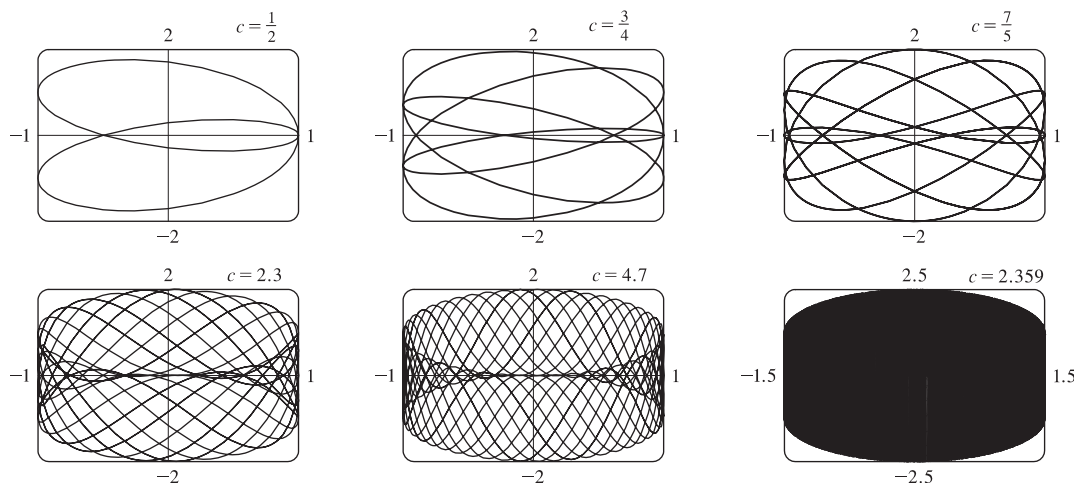
As  $c$  increases, the curve's loops become thinner, but stay in the region bounded by the semicircles  $y = \pm(1 + \sqrt{1 - x^2})$  and the line segments from  $(-1, -1)$  to  $(-1, 1)$  and from  $(1, -1)$  to  $(1, 1)$ . This is true because

$|y| = |\sin t - \sin ct| \leq |\sin t| + |\sin ct| \leq \sqrt{1 - x^2} + 1$ . This curve appears to fill the entire region when  $c$  is very large, as shown in the figure for  $c = 1000$ .





When  $c$  is a fraction, we get a variety of shapes with multiple loops, but always within the same region. For some fractional values, such as  $c = 2.359$ , the curve again appears to fill the region.



## LABORATORY PROJECT Running Circles Around Circles

1. The center  $Q$  of the smaller circle has coordinates  $((a - b)\cos \theta, (a - b)\sin \theta)$ .

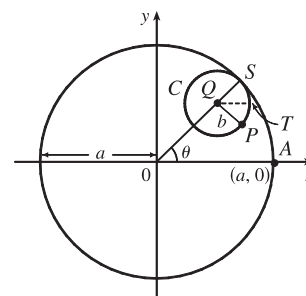
Arc  $PS$  on circle  $C$  has length  $a\theta$  since it is equal in length to arc  $AS$

(the smaller circle rolls without slipping against the larger.)

Thus,  $\angle PQS = \frac{a}{b}\theta$  and  $\angle PQT = \frac{a}{b}\theta - \theta$ , so  $P$  has coordinates

$$x = (a - b)\cos \theta + b \cos(\angle PQT) = (a - b)\cos \theta + b \cos\left(\frac{a - b}{b}\theta\right)$$

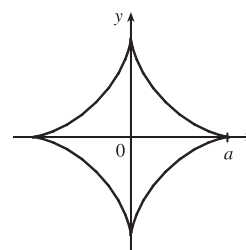
$$\text{and } y = (a - b)\sin \theta - b \sin(\angle PQT) = (a - b)\sin \theta - b \sin\left(\frac{a - b}{b}\theta\right).$$



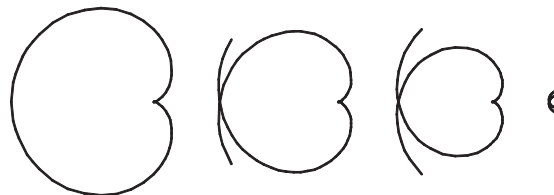
2. With  $b = 1$  and  $a$  a positive integer greater than 2, we obtain a hypocycloid of  $a$  cusps. Shown in the figure is the graph for  $a = 4$ . Let  $a = 4$  and  $b = 1$ . Using the sum identities to expand  $\cos 3\theta$  and  $\sin 3\theta$ , we obtain

$$x = 3 \cos \theta + \cos 3\theta = 3 \cos \theta + (4 \cos^3 \theta - 3 \cos \theta) = 4 \cos^3 \theta$$

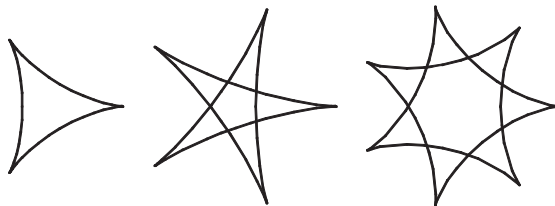
$$\text{and } y = 3 \sin \theta - \sin 3\theta = 3 \sin \theta - (3 \sin \theta - 4 \sin^3 \theta) = 4 \sin^3 \theta.$$



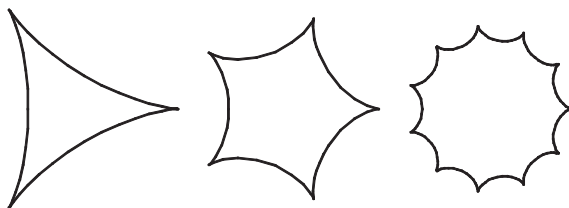
3. The graphs at the right are obtained with  $b = 1$  and  $a = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ , and  $\frac{1}{10}$  with  $-2\pi \leq \theta \leq 2\pi$ . We conclude that as the denominator  $d$  increases, the graph gets smaller, but maintains the basic shape shown.



Letting  $d = 2$  and  $n = 3, 5$ , and  $7$  with  $-2\pi \leq \theta \leq 2\pi$  gives us the following:



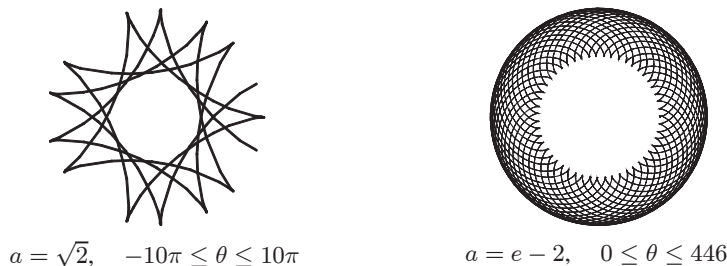
So if  $d$  is held constant and  $n$  varies, we get a graph with  $n$  cusps (assuming  $n/d$  is in lowest form). When  $n = d + 1$ , we obtain a hypocycloid of  $n$  cusps. As  $n$  increases, we must expand the range of  $\theta$  in order to get a closed curve. The following graphs have  $a = \frac{3}{2}$ ,  $\frac{5}{4}$ , and  $\frac{11}{10}$ .



4. If  $b = 1$ , the equations for the hypocycloid are

$$x = (a - 1) \cos \theta + \cos((a - 1)\theta) \quad y = (a - 1) \sin \theta - \sin((a - 1)\theta)$$

which is a hypocycloid of  $a$  cusps (from Problem 2). In general, if  $a > 1$ , we get a figure with cusps on the “outside ring” and if  $a < 1$ , the cusps are on the “inside ring”. In any case, as the values of  $\theta$  get larger, we get a figure that looks more and more like a washer. If we were to graph the hypocycloid for all values of  $\theta$ , every point on the washer would eventually be arbitrarily close to a point on the curve.



5. The center  $Q$  of the smaller circle has coordinates  $((a + b) \cos \theta, (a + b) \sin \theta)$ .

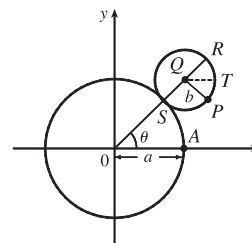
Arc  $PS$  has length  $a\theta$  (as in Problem 1), so that  $\angle PQS = \frac{a\theta}{b}$ ,  $\angle PQR = \pi - \frac{a\theta}{b}$ ,

and  $\angle PQT = \pi - \frac{a\theta}{b} - \theta = \pi - \left(\frac{a+b}{b}\right)\theta$  since  $\angle RQT = \theta$ .

Thus, the coordinates of  $P$  are

$$x = (a + b) \cos \theta + b \cos\left(\pi - \frac{a+b}{b}\theta\right) = (a + b) \cos \theta - b \cos\left(\frac{a+b}{b}\theta\right)$$

and  $y = (a + b) \sin \theta - b \sin\left(\pi - \frac{a+b}{b}\theta\right) = (a + b) \sin \theta - b \sin\left(\frac{a+b}{b}\theta\right).$

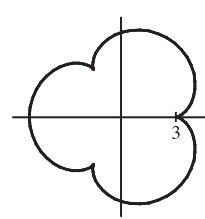


6. Let  $b = 1$  and the equations become

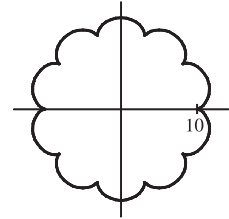
$$x = (a + 1) \cos \theta - \cos((a + 1)\theta)$$

$$y = (a + 1) \sin \theta - \sin((a + 1)\theta)$$

If  $a = 1$ , we have a cardioid. If  $a$  is a positive integer greater than 1, we get the graph of an “ $a$ -leafed clover”, with cusps that are  $a$  units from the origin. (Some of the pairs of figures are not to scale.)

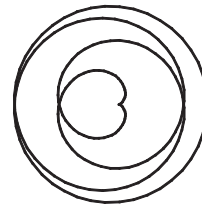


$$a = 3, -2\pi \leq \theta \leq 2\pi$$

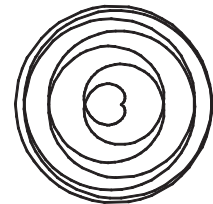


$$a = 10, -2\pi \leq \theta \leq 2\pi$$

If  $a = n/d$  with  $n = 1$ , we obtain a figure that does not increase in size and requires  $-d\pi \leq \theta \leq d\pi$  to be a closed curve traced exactly once.

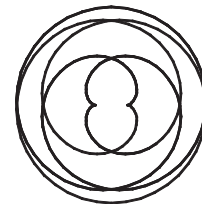


$$a = \frac{1}{4}, -4\pi \leq \theta \leq 4\pi$$

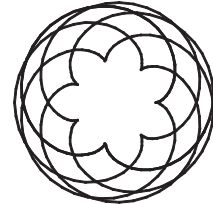


$$a = \frac{1}{7}, -7\pi \leq \theta \leq 7\pi$$

Next, we keep  $d$  constant and let  $n$  vary. As  $n$  increases, so does the size of the figure. There is an  $n$ -pointed star in the middle.

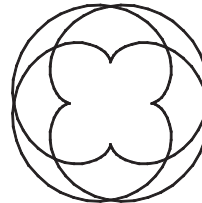


$$a = \frac{2}{5}, -5\pi \leq \theta \leq 5\pi$$

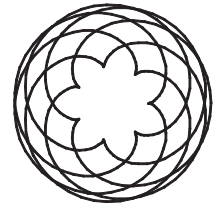


$$a = \frac{7}{5}, -5\pi \leq \theta \leq 5\pi$$

Now if  $n = d + 1$  we obtain figures similar to the previous ones, but the size of the figure does not increase.

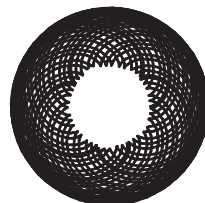


$$a = \frac{4}{3}, -3\pi \leq \theta \leq 3\pi$$

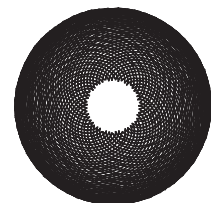


$$a = \frac{7}{6}, -6\pi \leq \theta \leq 6\pi$$

If  $a$  is irrational, we get washers that increase in size as  $a$  increases.



$$a = \sqrt{2}, 0 \leq \theta \leq 200$$



$$a = e - 2, 0 \leq \theta \leq 446$$

## 10.2 Calculus with Parametric Curves

1.  $x = t \sin t$ ,  $y = t^2 + t \Rightarrow \frac{dy}{dt} = 2t + 1$ ,  $\frac{dx}{dt} = t \cos t + \sin t$ , and  $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t + 1}{t \cos t + \sin t}$ .

2.  $x = \frac{1}{t}$ ,  $y = \sqrt{t} e^{-t} \Rightarrow \frac{dy}{dt} = t^{1/2}(-e^{-t}) + e^{-t} \left( \frac{1}{2} t^{-1/2} \right) = \frac{1}{2} t^{-1/2} e^{-t} (-2t + 1) = \frac{-2t + 1}{2t^{1/2} e^t}$ ,  $\frac{dx}{dt} = -\frac{1}{t^2}$ , and

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-2t + 1}{2t^{1/2} e^t} \left( -\frac{t^2}{1} \right) = \frac{(2t - 1)t^{3/2}}{2e^t}.$$

3.  $x = 1 + 4t - t^2$ ,  $y = 2 - t^3$ ;  $t = 1$ .  $\frac{dy}{dt} = -3t^2$ ,  $\frac{dx}{dt} = 4 - 2t$ , and  $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-3t^2}{4 - 2t}$ . When  $t = 1$ ,

$(x, y) = (4, 1)$  and  $dy/dx = -\frac{3}{2}$ , so an equation of the tangent to the curve at the point corresponding to  $t = 1$  is

$$y - 1 = -\frac{3}{2}(x - 4), \text{ or } y = -\frac{3}{2}x + 7.$$

4.  $x = t - t^{-1}$ ,  $y = 1 + t^2$ ;  $t = 1$ .  $\frac{dy}{dt} = 2t$ ,  $\frac{dx}{dt} = 1 + t^{-2} = \frac{t^2 + 1}{t^2}$ , and  $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = 2t \left( \frac{t^2}{t^2 + 1} \right) = \frac{2t^3}{t^2 + 1}$ .

When  $t = 1$ ,  $(x, y) = (0, 2)$  and  $dy/dx = \frac{2}{2} = 1$ , so an equation of the tangent to the curve at the point corresponding to  $t = 1$  is  $y - 2 = 1(x - 0)$ , or  $y = x + 2$ .

5.  $x = t \cos t$ ,  $y = t \sin t$ ;  $t = \pi$ .  $\frac{dy}{dt} = t \cos t + \sin t$ ,  $\frac{dx}{dt} = t(-\sin t) + \cos t$ , and  $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{t \cos t + \sin t}{-t \sin t + \cos t}$ .

When  $t = \pi$ ,  $(x, y) = (-\pi, 0)$  and  $dy/dx = -\pi/(-1) = \pi$ , so an equation of the tangent to the curve at the point corresponding to  $t = \pi$  is  $y - 0 = \pi[x - (-\pi)]$ , or  $y = \pi x + \pi^2$ .

6.  $x = \sin^3 \theta$ ,  $y = \cos^3 \theta$ ,  $\theta = \pi/6$ .  $\frac{dy}{d\theta} = 3 \cos^2 \theta (-\sin \theta)$ ,  $\frac{dx}{d\theta} = 3 \sin^2 \theta \cos \theta$ , and

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-3 \cos^2 \theta \sin \theta}{3 \sin^2 \theta \cos \theta} = -\cot \theta. \text{ When } \theta = \pi/6, (x, y) = \left( \frac{1}{8}, \frac{3}{8}\sqrt{3} \right) \text{ and } dy/dx = -\cot(\pi/6) = -\sqrt{3},$$

so an equation of the tangent line to the curve at the point corresponding to  $\theta = \pi/6$  is  $y - \frac{3}{8}\sqrt{3} = -\sqrt{3}(x - \frac{1}{8})$ ,

$$\text{or } y = -\sqrt{3}x + \frac{1}{2}\sqrt{3}.$$

7. (a)  $x = 1 + \ln t$ ,  $y = t^2 + 2$ ;  $(1, 3)$ .  $\frac{dy}{dt} = 2t$ ,  $\frac{dx}{dt} = \frac{1}{t}$ , and  $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{1/t} = 2t^2$ . At  $(1, 3)$ ,

$$x = 1 + \ln t = 1 \Rightarrow \ln t = 0 \Rightarrow t = 1 \text{ and } \frac{dy}{dx} = 2, \text{ so an equation of the tangent is } y - 3 = 2(x - 1),$$

$$\text{or } y = 2x + 1.$$

(b)  $x = 1 + \ln t \Rightarrow \ln t = x - 1 \Rightarrow t = e^{x-1}$ , so  $y = t^2 + 2 = (e^{x-1})^2 + 2 = e^{2x-2} + 2$ , and  $y' = e^{2x-2} \cdot 2$ .

$$\text{At } (1, 3), y' = e^{2(1)-2} \cdot 2 = 2, \text{ so an equation of the tangent is } y - 3 = 2(x - 1), \text{ or } y = 2x + 1.$$

8. (a)  $x = 1 + \sqrt{t}$ ,  $y = e^{t^2}$ ;  $(2, e)$ .  $\frac{dy}{dt} = e^{t^2} \cdot 2t$ ,  $\frac{dx}{dt} = \frac{1}{2\sqrt{t}}$ , and  $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2te^{t^2}}{1/(2\sqrt{t})} = 4t^{3/2}e^{t^2}$ . At  $(2, e)$ ,

$$x = 1 + \sqrt{t} = 2 \Rightarrow \sqrt{t} = 1 \Rightarrow t = 1 \text{ and } \frac{dy}{dx} = 4e, \text{ so an equation of the tangent is } y - e = 4e(x - 2),$$

$$\text{or } y = 4ex - 7e.$$

(b)  $x = 1 + \sqrt{t} \Rightarrow \sqrt{t} = x - 1 \Rightarrow t = (x - 1)^2$ , so  $y = e^{t^2} = e^{(x-1)^4}$ , and  $y' = e^{(x-1)^4} \cdot 4(x - 1)^3$ .

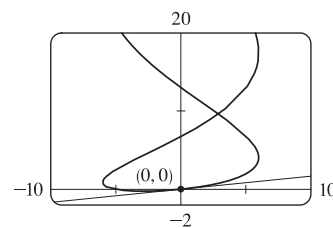
At  $(2, e)$ ,  $y' = e \cdot 4 = 4e$ , so an equation of the tangent is  $y - e = 4e(x - 2)$ , or  $y = 4ex - 7e$ .

9.  $x = 6 \sin t$ ,  $y = t^2 + t$ ;  $(0, 0)$ .

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t + 1}{6 \cos t}. \text{ The point } (0, 0) \text{ corresponds to } t = 0, \text{ so the}$$

slope of the tangent at that point is  $\frac{1}{6}$ . An equation of the tangent is therefore

$$y - 0 = \frac{1}{6}(x - 0), \text{ or } y = \frac{1}{6}x.$$



10.  $x = \cos t + \cos 2t$ ,  $y = \sin t + \sin 2t$ ;  $(-1, 1)$ .

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos t + 2 \cos 2t}{-\sin t - 2 \sin 2t}. \text{ To find the value of } t \text{ corresponding to}$$

the point  $(-1, 1)$ , solve  $x = -1 \Rightarrow \cos t + \cos 2t = -1 \Rightarrow$

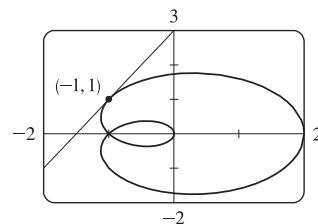
$$\cos t + 2 \cos^2 t - 1 = -1 \Rightarrow \cos t (1 + 2 \cos t) = 0 \Rightarrow \cos t = 0 \text{ or}$$

$\cos t = -\frac{1}{2}$ . The interval  $[0, 2\pi]$  gives the complete curve, so we need only find

the values of  $t$  in this interval. Thus,  $t = \frac{\pi}{2}$  or  $t = \frac{3\pi}{2}$  or  $t = \frac{4\pi}{3}$ . Checking  $t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{2\pi}{3}$ , and  $\frac{4\pi}{3}$  in the equation for  $y$ ,

we find that  $t = \frac{\pi}{2}$  corresponds to  $(-1, 1)$ . The slope of the tangent at  $(-1, 1)$  with  $t = \frac{\pi}{2}$  is  $\frac{0 - 2}{-1 - 0} = 2$ . An equation

of the tangent is therefore  $y - 1 = 2(x + 1)$ , or  $y = 2x + 3$ .



11.  $x = t^2 + 1$ ,  $y = t^2 + t \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t + 1}{2t} = 1 + \frac{1}{2t} \Rightarrow \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \frac{-1/(2t^2)}{2t} = -\frac{1}{4t^3}.$

The curve is CU when  $\frac{d^2y}{dx^2} > 0$ , that is, when  $t < 0$ .

12.  $x = t^3 + 1$ ,  $y = t^2 - t \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t - 1}{3t^2} = \frac{2}{3t} - \frac{1}{3t^2} \Rightarrow$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \frac{-\frac{2}{3t^2} + \frac{2}{3t^3}}{3t^2} = \frac{2 - 2t}{9t^5} = \frac{2(1 - t)}{9t^5}. \text{ The curve is CU when } \frac{d^2y}{dx^2} > 0, \text{ that is, when } 0 < t < 1.$$

13.  $x = e^t$ ,  $y = te^{-t} \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-te^{-t} + e^{-t}}{e^t} = \frac{e^{-t}(1 - t)}{e^t} = e^{-2t}(1 - t) \Rightarrow$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \frac{e^{-2t}(-1) + (1 - t)(-2e^{-2t})}{e^t} = \frac{e^{-2t}(-1 - 2 + 2t)}{e^t} = e^{-3t}(2t - 3). \text{ The curve is CU when}$$

$\frac{d^2y}{dx^2} > 0$ , that is, when  $t > \frac{3}{2}$ .

14.  $x = t^2 + 1$ ,  $y = e^t - 1 \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{e^t}{2t} \Rightarrow \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \frac{\frac{2te^t - e^t \cdot 2}{(2t)^2}}{2t} = \frac{2e^t(t - 1)}{(2t)^3} = \frac{e^t(t - 1)}{4t^3}.$

The curve is CU when  $\frac{d^2y}{dx^2} > 0$ , that is, when  $t < 0$  or  $t > 1$ .

15.  $x = 2 \sin t$ ,  $y = 3 \cos t$ ,  $0 < t < 2\pi$ .

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-3 \sin t}{2 \cos t} = -\frac{3}{2} \tan t, \text{ so } \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \frac{-\frac{3}{2} \sec^2 t}{2 \cos t} = -\frac{3}{4} \sec^3 t.$$

The curve is CU when  $\sec^3 t < 0 \Rightarrow \sec t < 0 \Rightarrow \cos t < 0 \Rightarrow \frac{\pi}{2} < t < \frac{3\pi}{2}$ .

16.  $x = \cos 2t$ ,  $y = \cos t$ ,  $0 < t < \pi$ .

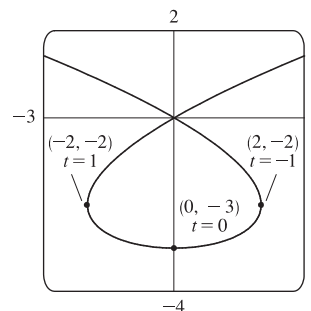
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\sin t}{-2 \sin 2t} = \frac{\sin t}{2 \cdot 2 \sin t \cos t} = \frac{1}{4 \cos t} = \frac{1}{4} \sec t, \text{ so } \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \frac{\frac{1}{4} \sec t \tan t}{-4 \sin t \cos t} = -\frac{1}{16} \sec^3 t.$$

The curve is CU when  $\sec^3 t < 0 \Rightarrow \sec t < 0 \Rightarrow \cos t < 0 \Rightarrow \frac{\pi}{2} < t < \pi$ .

17.  $x = t^3 - 3t$ ,  $y = t^2 - 3$ .  $\frac{dy}{dt} = 2t$ , so  $\frac{dy}{dt} = 0 \Leftrightarrow t = 0 \Leftrightarrow$

$$(x, y) = (0, -3). \quad \frac{dx}{dt} = 3t^2 - 3 = 3(t+1)(t-1), \text{ so } \frac{dx}{dt} = 0 \Leftrightarrow$$

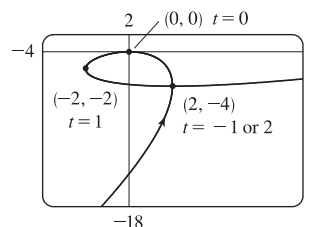
$t = -1 \text{ or } 1 \Leftrightarrow (x, y) = (2, -2) \text{ or } (-2, -2)$ . The curve has a horizontal tangent at  $(0, -3)$  and vertical tangents at  $(2, -2)$  and  $(-2, -2)$ .



18.  $x = t^3 - 3t$ ,  $y = t^3 - 3t^2$ .  $\frac{dy}{dt} = 3t^2 - 6t = 3t(t-2)$ , so  $\frac{dy}{dt} = 0 \Leftrightarrow$

$$t = 0 \text{ or } 2 \Leftrightarrow (x, y) = (0, 0) \text{ or } (2, -4). \quad \frac{dx}{dt} = 3t^2 - 3 = 3(t+1)(t-1),$$

so  $\frac{dx}{dt} = 0 \Leftrightarrow t = -1 \text{ or } 1 \Leftrightarrow (x, y) = (2, -4) \text{ or } (-2, -2)$ . The curve has horizontal tangents at  $(0, 0)$  and  $(2, -4)$ , and vertical tangents at  $(2, -4)$  and  $(-2, -2)$ .



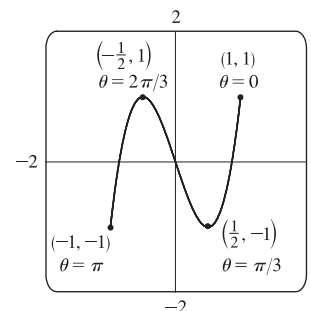
19.  $x = \cos \theta$ ,  $y = \cos 3\theta$ . The whole curve is traced out for  $0 \leq \theta \leq \pi$ .

$$\frac{dy}{d\theta} = -3 \sin 3\theta, \text{ so } \frac{dy}{d\theta} = 0 \Leftrightarrow \sin 3\theta = 0 \Leftrightarrow 3\theta = 0, \pi, 2\pi, \text{ or } 3\pi \Leftrightarrow$$

$$\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \text{ or } \pi \Leftrightarrow (x, y) = (1, 1), \left(\frac{1}{2}, -1\right), \left(-\frac{1}{2}, 1\right), \text{ or } (-1, -1).$$

$$\frac{dx}{d\theta} = -\sin \theta, \text{ so } \frac{dx}{d\theta} = 0 \Leftrightarrow \sin \theta = 0 \Leftrightarrow \theta = 0 \text{ or } \pi \Leftrightarrow$$

$(x, y) = (1, 1) \text{ or } (-1, -1)$ . Both  $\frac{dy}{d\theta}$  and  $\frac{dx}{d\theta}$  equal 0 when  $\theta = 0$  and  $\pi$ .



To find the slope when  $\theta = 0$ , we find  $\lim_{\theta \rightarrow 0} \frac{dy}{dx} = \lim_{\theta \rightarrow 0} \frac{-3 \sin 3\theta}{-\sin \theta} \stackrel{H}{=} \lim_{\theta \rightarrow 0} \frac{-9 \cos 3\theta}{-\cos \theta} = 9$ , which is the same slope when  $\theta = \pi$ .

Thus, the curve has horizontal tangents at  $(\frac{1}{2}, -1)$  and  $(-\frac{1}{2}, 1)$ , and there are no vertical tangents.

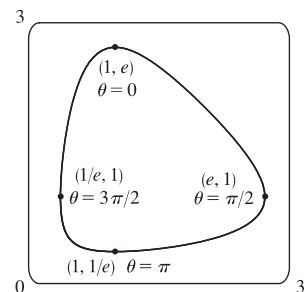
20.  $x = e^{\sin \theta}$ ,  $y = e^{\cos \theta}$ . The whole curve is traced out for  $0 \leq \theta < 2\pi$ .

$$\frac{dy}{d\theta} = -\sin \theta e^{\cos \theta}, \text{ so } \frac{dy}{d\theta} = 0 \Leftrightarrow \sin \theta = 0 \Leftrightarrow \theta = 0 \text{ or } \pi \Leftrightarrow$$

$$(x, y) = (1, e) \text{ or } (1, 1/e). \quad \frac{dx}{d\theta} = \cos \theta e^{\sin \theta}, \text{ so } \frac{dx}{d\theta} = 0 \Leftrightarrow \cos \theta = 0 \Leftrightarrow$$

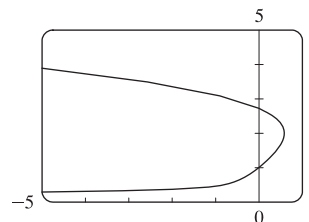
$$\theta = \frac{\pi}{2} \text{ or } \frac{3\pi}{2} \Leftrightarrow (x, y) = (e, 1) \text{ or } (1/e, 1). \text{ The curve has horizontal tangents}$$

$$\text{at } (1, e) \text{ and } (1, 1/e), \text{ and vertical tangents at } (e, 1) \text{ and } (1/e, 1).$$



21. From the graph, it appears that the rightmost point on the curve  $x = t - t^6$ ,  $y = e^t$  is about  $(0.6, 2)$ . To find the exact coordinates, we find the value of  $t$  for which the graph has a vertical tangent, that is,  $0 = dx/dt = 1 - 6t^5 \Leftrightarrow t = 1/\sqrt[5]{6}$ . Hence, the rightmost point is

$$\left(1/\sqrt[5]{6} - 1/(6\sqrt[5]{6}), e^{1/\sqrt[5]{6}}\right) = \left(5 \cdot 6^{-6/5}, e^{6^{-1/5}}\right) \approx (0.58, 2.01).$$



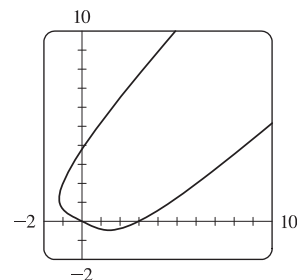
22. From the graph, it appears that the lowest point and the leftmost point on the curve  $x = t^4 - 2t$ ,  $y = t + t^4$  are  $(1.5, -0.5)$  and  $(-1.2, 1.2)$ , respectively. To find the exact coordinates, we solve  $dy/dt = 0$  (horizontal tangents) and  $dx/dt = 0$  (vertical tangents).

$$\frac{dy}{dt} = 0 \Leftrightarrow 1 + 4t^3 = 0 \Leftrightarrow t = -\frac{1}{\sqrt[3]{4}}, \text{ so the lowest point is}$$

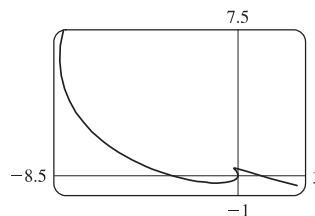
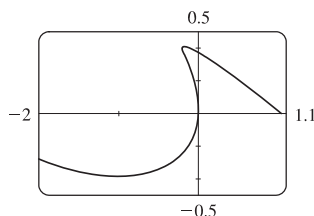
$$\left(\frac{1}{\sqrt[3]{256}} + \frac{2}{\sqrt[3]{4}}, -\frac{1}{\sqrt[3]{4}} + \frac{1}{\sqrt[3]{256}}\right) = \left(\frac{9}{\sqrt[3]{256}}, -\frac{3}{\sqrt[3]{256}}\right) \approx (1.42, -0.47).$$

$$\frac{dx}{dt} = 0 \Leftrightarrow 4t^3 - 2 = 0 \Leftrightarrow t = \frac{1}{\sqrt[3]{2}}, \text{ so the leftmost point is}$$

$$\left(\frac{1}{\sqrt[3]{16}} - \frac{2}{\sqrt[3]{2}}, \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{16}}\right) = \left(-\frac{3}{\sqrt[3]{16}}, \frac{3}{\sqrt[3]{16}}\right) \approx (-1.19, 1.19).$$



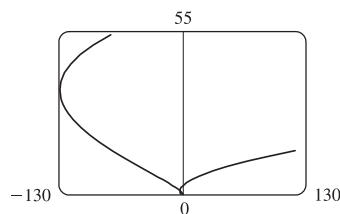
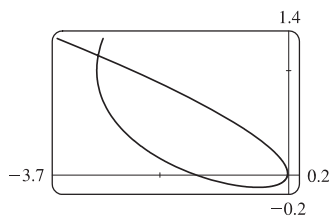
23. We graph the curve  $x = t^4 - 2t^3 - 2t^2$ ,  $y = t^3 - t$  in the viewing rectangle  $[-2, 1.1]$  by  $[-0.5, 0.5]$ . This rectangle corresponds approximately to  $t \in [-1, 0.8]$ .



We estimate that the curve has horizontal tangents at about  $(-1, -0.4)$  and  $(-0.17, 0.39)$  and vertical tangents at about  $(0, 0)$  and  $(-0.19, 0.37)$ . We calculate  $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 1}{4t^3 - 6t^2 - 4t}$ . The horizontal tangents occur when  $dy/dt = 3t^2 - 1 = 0 \Leftrightarrow t = \pm \frac{1}{\sqrt{3}}$ , so both horizontal tangents are shown in our graph. The vertical tangents occur when

$dx/dt = 2t(2t^2 - 3t - 2) = 0 \Leftrightarrow 2t(2t+1)(t-2) = 0 \Leftrightarrow t = 0, -\frac{1}{2}$  or  $2$ . It seems that we have missed one vertical tangent, and indeed if we plot the curve on the  $t$ -interval  $[-1.2, 2.2]$  we see that there is another vertical tangent at  $(-8, 6)$ .

24. We graph the curve  $x = t^4 + 4t^3 - 8t^2$ ,  $y = 2t^2 - t$  in the viewing rectangle  $[-3.7, 0.2]$  by  $[-0.2, 1.4]$ . It appears that there is a horizontal tangent at about  $(-0.4, -0.1)$ , and vertical tangents at about  $(-3, 1)$  and  $(0, 0)$ .



We calculate  $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4t-1}{4t^3+12t^2-16t}$ , so there is a horizontal tangent where  $dy/dt = 4t-1 = 0 \Leftrightarrow t = \frac{1}{4}$ .

This point (the lowest point) is shown in the first graph. There are vertical tangents where  $dx/dt = 4t^3 + 12t^2 - 16t = 0 \Leftrightarrow 4t(t^2 + 3t - 4) = 0 \Leftrightarrow 4t(t+4)(t-1) = 0$ . We have missed one vertical tangent corresponding to  $t = -4$ , and if we plot the graph for  $t \in [-5, 3]$ , we see that the curve has another vertical tangent line at approximately  $(-128, 36)$ .

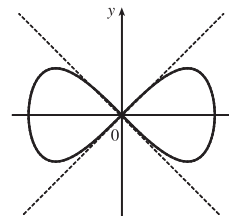
25.  $x = \cos t$ ,  $y = \sin t \cos t$ .  $dx/dt = -\sin t$ ,  $dy/dt = -\sin^2 t + \cos^2 t = \cos 2t$ .

$(x, y) = (0, 0) \Leftrightarrow \cos t = 0 \Leftrightarrow t$  is an odd multiple of  $\frac{\pi}{2}$ . When  $t = \frac{\pi}{2}$ ,

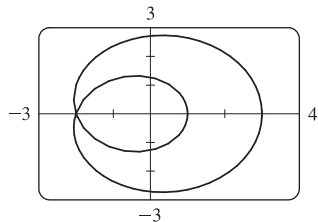
$dx/dt = -1$  and  $dy/dt = -1$ , so  $dy/dx = 1$ . When  $t = \frac{3\pi}{2}$ ,  $dx/dt = 1$  and

$dy/dt = -1$ . So  $dy/dx = -1$ . Thus,  $y = x$  and  $y = -x$  are both tangent to the

curve at  $(0, 0)$ .



- 26.



From the graph, we discover that the graph of the curve  $x = \cos t + 2 \cos 2t$ ,

$y = \sin t + 2 \sin 2t$  crosses itself at the point  $(-2, 0)$ . To find  $t$  at  $(-2, 0)$ ,

solve  $y = 0 \Leftrightarrow \sin t + 2 \sin 2t = 0 \Leftrightarrow \sin t + 4 \sin t \cos t = 0 \Leftrightarrow$

$\sin t(1 + 4 \cos t) = 0 \Leftrightarrow \sin t = 0$  or  $\cos t = -\frac{1}{4}$ . We find that

$t = \pm \arccos(-\frac{1}{4})$  corresponds to  $(-2, 0)$ .

Now  $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos t + 4 \cos 2t}{-\sin t - 4 \sin 2t} = -\frac{\cos t + 8 \cos^2 t - 4}{\sin t + 8 \sin t \cos t}$ . When  $t = \arccos(-\frac{1}{4})$ ,  $\cos t = -\frac{1}{4}$ ,  $\sin t = \frac{\sqrt{15}}{4}$ ,

and  $\frac{dy}{dx} = -\frac{-\frac{1}{4} + \frac{1}{2} - 4}{\frac{\sqrt{15}}{4} - \frac{\sqrt{15}}{2}} = -\frac{-\frac{15}{4}}{-\frac{\sqrt{15}}{4}} = -\sqrt{15}$ . By symmetry,  $t = -\arccos(-\frac{1}{4}) \Rightarrow \frac{dy}{dx} = \sqrt{15}$ .

The tangent lines are  $y - 0 = \pm\sqrt{15}(x + 2)$ , or  $y = \sqrt{15}x + 2\sqrt{15}$  and  $y = -\sqrt{15}x - 2\sqrt{15}$ .

27.  $x = r\theta - d \sin \theta$ ,  $y = r - d \cos \theta$ .

(a)  $\frac{dx}{d\theta} = r - d \cos \theta$ ,  $\frac{dy}{d\theta} = d \sin \theta$ , so  $\frac{dy}{dx} = \frac{d \sin \theta}{r - d \cos \theta}$ .

(b) If  $0 < d < r$ , then  $|d \cos \theta| \leq d < r$ , so  $r - d \cos \theta \geq r - d > 0$ . This shows that  $dx/d\theta$  never vanishes,

so the trochoid can have no vertical tangent if  $d < r$ .



28.  $x = a \cos^3 \theta$ ,  $y = a \sin^3 \theta$ .

(a)  $\frac{dx}{d\theta} = -3a \cos^2 \theta \sin \theta$ ,  $\frac{dy}{d\theta} = 3a \sin^2 \theta \cos \theta$ , so  $\frac{dy}{dx} = -\frac{\sin \theta}{\cos \theta} = -\tan \theta$ .

(b) The tangent is horizontal  $\Leftrightarrow dy/dx = 0 \Leftrightarrow \tan \theta = 0 \Leftrightarrow \theta = n\pi \Leftrightarrow (x, y) = (\pm a, 0)$ .

The tangent is vertical  $\Leftrightarrow \cos \theta = 0 \Leftrightarrow \theta$  is an odd multiple of  $\frac{\pi}{2} \Leftrightarrow (x, y) = (0, \pm a)$ .

(c)  $dy/dx = \pm 1 \Leftrightarrow \tan \theta = \pm 1 \Leftrightarrow \theta$  is an odd multiple of  $\frac{\pi}{4} \Leftrightarrow (x, y) = \left(\pm \frac{\sqrt{2}}{4}a, \pm \frac{\sqrt{2}}{4}a\right)$

[All sign choices are valid.]

29.  $x = 2t^3$ ,  $y = 1 + 4t - t^2 \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4-2t}{6t^2}$ . Now solve  $\frac{dy}{dx} = 1 \Leftrightarrow \frac{4-2t}{6t^2} = 1 \Leftrightarrow$

$6t^2 + 2t - 4 = 0 \Leftrightarrow 2(3t-2)(t+1) = 0 \Leftrightarrow t = \frac{2}{3}$  or  $t = -1$ . If  $t = \frac{2}{3}$ , the point is  $(\frac{16}{27}, \frac{29}{9})$ , and if  $t = -1$ , the point is  $(-2, -4)$ .

30.  $x = 3t^2 + 1$ ,  $y = 2t^3 + 1$ ,  $\frac{dx}{dt} = 6t$ ,  $\frac{dy}{dt} = 6t^2$ , so  $\frac{dy}{dx} = \frac{6t^2}{6t} = t$  [even where  $t = 0$ ].

So at the point corresponding to parameter value  $t$ , an equation of the tangent line is  $y - (2t^3 + 1) = t[x - (3t^2 + 1)]$ .

If this line is to pass through  $(4, 3)$ , we must have  $3 - (2t^3 + 1) = t[4 - (3t^2 + 1)] \Leftrightarrow 2t^3 - 2 = 3t^3 - 3t \Leftrightarrow$

$t^3 - 3t + 2 = 0 \Leftrightarrow (t-1)^2(t+2) = 0 \Leftrightarrow t = 1$  or  $-2$ . Hence, the desired equations are  $y - 3 = x - 4$ , or

$y = x - 1$ , tangent to the curve at  $(4, 3)$ , and  $y - (-15) = -2(x - 13)$ , or  $y = -2x + 11$ , tangent to the curve at  $(13, -15)$ .

31. By symmetry of the ellipse about the  $x$ - and  $y$ -axes,

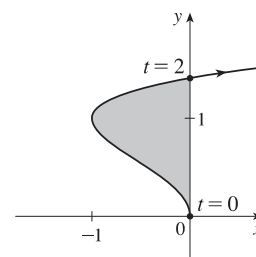
$$\begin{aligned} A &= 4 \int_0^a y \, dx = 4 \int_{\pi/2}^0 b \sin \theta (-a \sin \theta) \, d\theta = 4ab \int_0^{\pi/2} \sin^2 \theta \, d\theta = 4ab \int_0^{\pi/2} \frac{1}{2}(1 - \cos 2\theta) \, d\theta \\ &= 2ab \left[ \theta - \frac{1}{2} \sin 2\theta \right]_0^{\pi/2} = 2ab \left( \frac{\pi}{2} \right) = \pi ab \end{aligned}$$

32. The curve  $x = t^2 - 2t = t(t-2)$ ,  $y = \sqrt{t}$  intersects the  $y$ -axis when  $x = 0$ ,

that is, when  $t = 0$  and  $t = 2$ . The corresponding values of  $y$  are 0 and  $\sqrt{2}$ .

The shaded area is given by

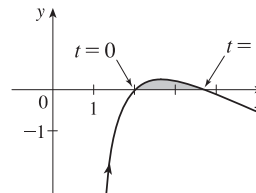
$$\begin{aligned} \int_{y=0}^{y=\sqrt{2}} (x_R - x_L) \, dy &= \int_{t=0}^{t=2} [0 - x(t)] y'(t) \, dt = - \int_0^2 (t^2 - 2t) \left( \frac{1}{2\sqrt{t}} \, dt \right) \\ &= - \int_0^2 \left( \frac{1}{2} t^{3/2} - t^{1/2} \right) \, dt = - \left[ \frac{1}{5} t^{5/2} - \frac{2}{3} t^{3/2} \right]_0^2 \\ &= - \left( \frac{1}{5} \cdot 2^{5/2} - \frac{2}{3} \cdot 2^{3/2} \right) = -2^{1/2} \left( \frac{4}{5} - \frac{4}{3} \right) \\ &= -\sqrt{2} \left( -\frac{8}{15} \right) = \frac{8}{15} \sqrt{2} \end{aligned}$$



33. The curve  $x = 1 + e^t$ ,  $y = t - t^2 = t(1-t)$  intersects the  $x$ -axis when  $y = 0$ ,

that is, when  $t = 0$  and  $t = 1$ . The corresponding values of  $x$  are 2 and  $1 + e$ .

The shaded area is given by



$$\begin{aligned}
 \int_{x=2}^{x=1+e} (y_T - y_B) dx &= \int_{t=0}^{t=1} [y(t) - 0] x'(t) dt = \int_0^1 (t - t^2) e^t dt \\
 &= \int_0^1 t e^t dt - \int_0^1 t^2 e^t dt = \int_0^1 t e^t dt - [t^2 e^t]_0^1 + 2 \int_0^1 t e^t dt \quad [\text{Formula 97 or parts}] \\
 &= 3 \int_0^1 t e^t dt - (e - 0) = 3 [(t - 1) e^t]_0^1 - e \quad [\text{Formula 96 or parts}] \\
 &= 3[0 - (-1)] - e = 3 - e
 \end{aligned}$$

34. By symmetry,  $A = 4 \int_0^a y dx = 4 \int_{\pi/2}^0 a \sin^3 \theta (-3a \cos^2 \theta \sin \theta) d\theta = 12a^2 \int_0^{\pi/2} \sin^4 \theta \cos^2 \theta d\theta$ . Now

$$\begin{aligned}
 \int \sin^4 \theta \cos^2 \theta d\theta &= \int \sin^2 \theta \left( \frac{1}{4} \sin^2 2\theta \right) d\theta = \frac{1}{8} \int (1 - \cos 2\theta) \sin^2 2\theta d\theta \\
 &= \frac{1}{8} \int \left[ \frac{1}{2} (1 - \cos 4\theta) - \sin^2 2\theta \cos 2\theta \right] d\theta = \frac{1}{16} \theta - \frac{1}{64} \sin 4\theta - \frac{1}{48} \sin^3 2\theta + C
 \end{aligned}$$

$$\text{so } \int_0^{\pi/2} \sin^4 \theta \cos^2 \theta d\theta = \left[ \frac{1}{16} \theta - \frac{1}{64} \sin 4\theta - \frac{1}{48} \sin^3 2\theta \right]_0^{\pi/2} = \frac{\pi}{32}. \text{ Thus, } A = 12a^2 \left( \frac{\pi}{32} \right) = \frac{3}{8} \pi a^2.$$

35.  $x = r\theta - d \sin \theta$ ,  $y = r - d \cos \theta$ .

$$\begin{aligned}
 A &= \int_0^{2\pi} y dx = \int_0^{2\pi} (r - d \cos \theta)(r - d \cos \theta) d\theta = \int_0^{2\pi} (r^2 - 2dr \cos \theta + d^2 \cos^2 \theta) d\theta \\
 &= \left[ r^2 \theta - 2dr \sin \theta + \frac{1}{2} d^2 \left( \theta + \frac{1}{2} \sin 2\theta \right) \right]_0^{2\pi} = 2\pi r^2 + \pi d^2
 \end{aligned}$$

36. (a) By symmetry, the area of  $\mathcal{R}$  is twice the area inside  $\mathcal{R}$  above the  $x$ -axis. The top half of the loop is described by

$$x = t^2, y = t^3 - 3t, -\sqrt{3} \leq t \leq 0, \text{ so, using the Substitution Rule with } y = t^3 - 3t \text{ and } dx = 2t dt, \text{ we find that}$$

$$\begin{aligned}
 \text{area} &= 2 \int_0^3 y dx = 2 \int_0^{-\sqrt{3}} (t^3 - 3t) 2t dt = 2 \int_0^{-\sqrt{3}} (2t^4 - 6t^2) dt = 2 \left[ \frac{2}{5} t^5 - 2t^3 \right]_0^{-\sqrt{3}} \\
 &= 2 \left[ \frac{2}{5} (-3^{1/2})^5 - 2(-3^{1/2})^3 \right] = 2 \left[ \frac{2}{5} (-9\sqrt{3}) - 2(-3\sqrt{3}) \right] = \frac{24}{5} \sqrt{3}
 \end{aligned}$$

(b) Here we use the formula for disks and use the Substitution Rule as in part (a):

$$\begin{aligned}
 \text{volume} &= \pi \int_0^3 y^2 dx = \pi \int_0^{-\sqrt{3}} (t^3 - 3t)^2 2t dt = 2\pi \int_0^{-\sqrt{3}} (t^6 - 6t^4 + 9t^2) t dt = 2\pi \left[ \frac{1}{8} t^8 - t^6 + \frac{9}{4} t^4 \right]_0^{-\sqrt{3}} \\
 &= 2\pi \left[ \frac{1}{8} (-3^{1/2})^8 - (-3^{1/2})^6 + \frac{9}{4} (-3^{1/2})^4 \right] = 2\pi \left[ \frac{81}{8} - 27 + \frac{81}{4} \right] = \frac{27}{4} \pi
 \end{aligned}$$

(c) By symmetry, the  $y$ -coordinate of the centroid is 0. To find the  $x$ -coordinate, we note that it is the same as the  $x$ -coordinate of the centroid of the top half of  $\mathcal{R}$ , the area of which is  $\frac{1}{2} \cdot \frac{24}{5} \sqrt{3} = \frac{12}{5} \sqrt{3}$ . So, using Formula 8.3.8 with  $A = \frac{12}{5} \sqrt{3}$ , we get

$$\begin{aligned}
 \bar{x} &= \frac{5}{12\sqrt{3}} \int_0^3 xy dx = \frac{5}{12\sqrt{3}} \int_0^{-\sqrt{3}} t^2 (t^3 - 3t) 2t dt = \frac{5}{6\sqrt{3}} \left[ \frac{1}{7} t^7 - \frac{3}{5} t^5 \right]_0^{-\sqrt{3}} \\
 &= \frac{5}{6\sqrt{3}} \left[ \frac{1}{7} (-3^{1/2})^7 - \frac{3}{5} (-3^{1/2})^5 \right] = \frac{5}{6\sqrt{3}} \left[ -\frac{27}{7} \sqrt{3} + \frac{27}{5} \sqrt{3} \right] = \frac{9}{7}
 \end{aligned}$$

So the coordinates of the centroid of  $\mathcal{R}$  are  $(x, y) = \left( \frac{9}{7}, 0 \right)$ .

37.  $x = t + e^{-t}$ ,  $y = t - e^{-t}$ ,  $0 \leq t \leq 2$ .  $dx/dt = 1 - e^{-t}$  and  $dy/dt = 1 + e^{-t}$ , so

$$(dx/dt)^2 + (dy/dt)^2 = (1 - e^{-t})^2 + (1 + e^{-t})^2 = 1 - 2e^{-t} + e^{-2t} + 1 + 2e^{-t} + e^{-2t} = 2 + 2e^{-2t}.$$

$$\text{Thus, } L = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2} dt = \int_0^2 \sqrt{2 + 2e^{-2t}} dt \approx 3.1416.$$

38.  $x = t^2 - t$ ,  $y = t^4$ ,  $1 \leq t \leq 4$ .  $dx/dt = 2t - 1$  and  $dy/dt = 4t^3$ , so

$$(dx/dt)^2 + (dy/dt)^2 = (2t - 1)^2 + (4t^3)^2 = 4t^2 - 4t + 1 + 16t^6.$$

$$\text{Thus, } L = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2} dt = \int_1^4 \sqrt{16t^6 + 4t^2 - 4t + 1} dt \approx 255.3756.$$

39.  $x = t - 2 \sin t$ ,  $y = 1 - 2 \cos t$ ,  $0 \leq t \leq 4\pi$ .  $dx/dt = 1 - 2 \cos t$  and  $dy/dt = 2 \sin t$ , so

$$(dx/dt)^2 + (dy/dt)^2 = (1 - 2 \cos t)^2 + (2 \sin t)^2 = 1 - 4 \cos t + 4 \cos^2 t + 4 \sin^2 t = 5 - 4 \cos t.$$

$$\text{Thus, } L = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2} dt = \int_0^{4\pi} \sqrt{5 - 4 \cos t} dt \approx 26.7298.$$

40.  $x = t + \sqrt{t}$ ,  $y = t - \sqrt{t}$ ,  $0 \leq t \leq 1$ .  $\frac{dx}{dt} = 1 + \frac{1}{2\sqrt{t}}$  and  $\frac{dy}{dt} = 1 - \frac{1}{2\sqrt{t}}$ , so

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(1 + \frac{1}{2\sqrt{t}}\right)^2 + \left(1 - \frac{1}{2\sqrt{t}}\right)^2 = 1 + \frac{1}{\sqrt{t}} + \frac{1}{4t} + 1 - \frac{1}{\sqrt{t}} + \frac{1}{4t} = 2 + \frac{1}{2t}.$$

$$\text{Thus, } L = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2} dt = \int_0^1 \sqrt{2 + \frac{1}{2t}} dt = \lim_{t \rightarrow 0^+} \int_t^1 \sqrt{2 + \frac{1}{2t}} dt \approx 2.0915.$$

41.  $x = 1 + 3t^2$ ,  $y = 4 + 2t^3$ ,  $0 \leq t \leq 1$ .  $dx/dt = 6t$  and  $dy/dt = 6t^2$ , so  $(dx/dt)^2 + (dy/dt)^2 = 36t^2 + 36t^4$

$$\begin{aligned} \text{Thus, } L &= \int_0^1 \sqrt{36t^2 + 36t^4} dt = \int_0^1 6t \sqrt{1 + t^2} dt = 6 \int_1^2 \sqrt{u} \left(\frac{1}{2} du\right) \quad [u = 1 + t^2, du = 2t dt] \\ &= 3 \left[ \frac{2}{3} u^{3/2} \right]_1^2 = 2(2^{3/2} - 1) = 2(2\sqrt{2} - 1) \end{aligned}$$

42.  $x = e^t + e^{-t}$ ,  $y = 5 - 2t$ ,  $0 \leq t \leq 3$ .  $dx/dt = e^t - e^{-t}$  and  $dy/dt = -2$ , so

$$(dx/dt)^2 + (dy/dt)^2 = e^{2t} - 2 + e^{-2t} + 4 = e^{2t} + 2 + e^{-2t} = (e^t + e^{-t})^2.$$

$$\text{Thus, } L = \int_0^3 (e^t + e^{-t}) dt = [e^t - e^{-t}]_0^3 = e^3 - e^{-3} - (1 - 1) = e^3 - e^{-3}.$$

43.  $x = t \sin t$ ,  $y = t \cos t$ ,  $0 \leq t \leq 1$ .  $\frac{dx}{dt} = t \cos t + \sin t$  and  $\frac{dy}{dt} = -t \sin t + \cos t$ , so

$$\begin{aligned} \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= t^2 \cos^2 t + 2t \sin t \cos t + \sin^2 t + t^2 \sin^2 t - 2t \sin t \cos t + \cos^2 t \\ &= t^2(\cos^2 t + \sin^2 t) + \sin^2 t + \cos^2 t = t^2 + 1. \end{aligned}$$

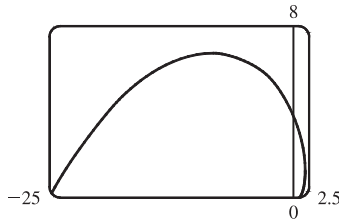
$$\text{Thus, } L = \int_0^1 \sqrt{t^2 + 1} dt \stackrel{21}{=} \left[ \frac{1}{2} t \sqrt{t^2 + 1} + \frac{1}{2} \ln(t + \sqrt{t^2 + 1}) \right]_0^1 = \frac{1}{2} \sqrt{2} + \frac{1}{2} \ln(1 + \sqrt{2}).$$

44.  $x = 3 \cos t - \cos 3t$ ,  $y = 3 \sin t - \sin 3t$ ,  $0 \leq t \leq \pi$ .  $\frac{dx}{dt} = -3 \sin t + 3 \sin 3t$  and  $\frac{dy}{dt} = 3 \cos t - 3 \cos 3t$ , so

$$\begin{aligned} \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= 9 \sin^2 t - 18 \sin t \sin 3t + 9 \sin^2(3t) + 9 \cos^2 t - 18 \cos t \cos 3t + 9 \cos^2(3t) \\ &= 9(\cos^2 t + \sin^2 t) - 18(\cos t \cos 3t + \sin t \sin 3t) + 9[\cos^2(3t) + \sin^2(3t)] \\ &= 9(1) - 18 \cos(t - 3t) + 9(1) = 18 - 18 \cos(-2t) = 18(1 - \cos 2t) \\ &= 18[1 - (1 - 2 \sin^2 t)] = 36 \sin^2 t. \end{aligned}$$

$$\text{Thus, } L = \int_0^\pi \sqrt{36 \sin^2 t} dt = 6 \int_0^\pi |\sin t| dt = 6 \int_0^\pi \sin t dt = -6[\cos t]_0^\pi = -6(-1 - 1) = 12.$$

45.



$$x = e^t \cos t, \quad y = e^t \sin t, \quad 0 \leq t \leq \pi.$$

$$\begin{aligned} \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= [e^t(\cos t - \sin t)]^2 + [e^t(\sin t + \cos t)]^2 \\ &= (e^t)^2(\cos^2 t - 2\cos t \sin t + \sin^2 t) \\ &\quad + (e^t)^2(\sin^2 t + 2\sin t \cos t + \cos^2 t) \\ &= e^{2t}(2\cos^2 t + 2\sin^2 t) = 2e^{2t} \end{aligned}$$

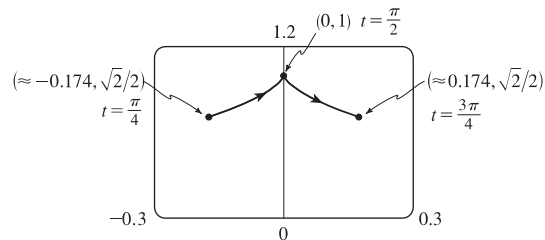
$$\text{Thus, } L = \int_0^\pi \sqrt{2e^{2t}} dt = \int_0^\pi \sqrt{2} e^t dt = \sqrt{2} [e^t]_0^\pi = \sqrt{2}(e^\pi - 1).$$

46.  $x = \cos t + \ln(\tan \frac{1}{2}t), \quad y = \sin t, \quad \pi/4 \leq t \leq 3\pi/4.$

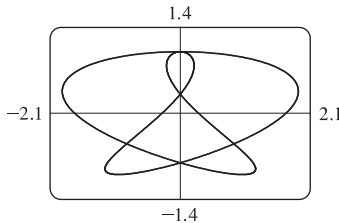
$$\frac{dx}{dt} = -\sin t + \frac{\frac{1}{2} \sec^2(t/2)}{\tan(t/2)} = -\sin t + \frac{1}{2 \sin(t/2) \cos(t/2)} = -\sin t + \frac{1}{\sin t} \text{ and } \frac{dy}{dt} = \cos t, \text{ so}$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \sin^2 t - 2 + \frac{1}{\sin^2 t} + \cos^2 t = 1 - 2 + \csc^2 t = \cot^2 t. \text{ Thus,}$$

$$\begin{aligned} L &= \int_{\pi/4}^{3\pi/4} |\cot t| dt = 2 \int_{\pi/4}^{\pi/2} \cot t dt \\ &= 2 \left[ \ln |\sin t| \right]_{\pi/4}^{\pi/2} = 2 \left( \ln 1 - \ln \frac{1}{\sqrt{2}} \right) \\ &= 2(0 + \ln \sqrt{2}) = 2(\tfrac{1}{2} \ln 2) = \ln 2. \end{aligned}$$



47.



The figure shows the curve  $x = \sin t + \sin 1.5t, y = \cos t$  for  $0 \leq t \leq 4\pi$ .

$$dx/dt = \cos t + 1.5 \cos 1.5t \text{ and } dy/dt = -\sin t, \text{ so}$$

$$(dx/dt)^2 + (dy/dt)^2 = \cos^2 t + 3 \cos t \cos 1.5t + 2.25 \cos^2 1.5t + \sin^2 t.$$

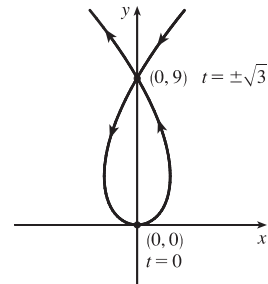
$$\text{Thus, } L = \int_0^{4\pi} \sqrt{1 + 3 \cos t \cos 1.5t + 2.25 \cos^2 1.5t} dt \approx 16.7102.$$

48.  $x = 3t - t^3, y = 3t^2. \quad dx/dt = 3 - 3t^2 \text{ and } dy/dt = 6t, \text{ so}$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (3 - 3t^2)^2 + (6t)^2 = (3 + 3t^2)^2$$

and the length of the loop is given by

$$\begin{aligned} L &= \int_{-\sqrt{3}}^{\sqrt{3}} (3 + 3t^2) dt = 2 \int_0^{\sqrt{3}} (3 + 3t^2) dt = 2[3t + t^3]_0^{\sqrt{3}} \\ &= 2(3\sqrt{3} + 3\sqrt{3}) = 12\sqrt{3}. \end{aligned}$$



49.  $x = t - e^t, y = t + e^t, -6 \leq t \leq 6.$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (1 - e^t)^2 + (1 + e^t)^2 = (1 - 2e^t + e^{2t}) + (1 + 2e^t + e^{2t}) = 2 + 2e^{2t}, \text{ so } L = \int_{-6}^6 \sqrt{2 + 2e^{2t}} dt.$$

Set  $f(t) = \sqrt{2 + 2e^{2t}}$ . Then by Simpson's Rule with  $n = 6$  and  $\Delta t = \frac{6 - (-6)}{6} = 2$ , we get

$$L \approx \frac{2}{3}[f(-6) + 4f(-4) + 2f(-2) + 4f(0) + 2f(2) + 4f(4) + f(6)] \approx 612.3053.$$

$$50. \quad x = 2a \cot \theta \Rightarrow dx/dt = -2a \csc^2 \theta \text{ and } y = 2a \sin^2 \theta \Rightarrow dy/dt = 4a \sin \theta \cos \theta = 2a \sin 2\theta.$$

So  $L = \int_{\pi/4}^{\pi/2} \sqrt{4a^2 \csc^4 \theta + 4a^2 \sin^2 2\theta} d\theta = 2a \int_{\pi/4}^{\pi/2} \sqrt{\csc^4 \theta + \sin^2 2\theta} d\theta$ . Using Simpson's Rule with

$$n = 4, \Delta\theta = \frac{\pi/2 - \pi/4}{4} = \frac{\pi}{16}, \text{ and } f(\theta) = \sqrt{\csc^4 \theta + \sin^2 2\theta}, \text{ we get}$$

$$L \approx 2a \cdot S_4 = (2a) \frac{\pi}{16 \cdot 3} \left[ f\left(\frac{\pi}{4}\right) + 4f\left(\frac{5\pi}{16}\right) + 2f\left(\frac{3\pi}{8}\right) + 4f\left(\frac{7\pi}{16}\right) + f\left(\frac{\pi}{2}\right) \right] \approx 2.2605a.$$

$$51. \quad x = \sin^2 t, y = \cos^2 t, 0 \leq t \leq 3\pi.$$

$$(dx/dt)^2 + (dy/dt)^2 = (2 \sin t \cos t)^2 + (-2 \cos t \sin t)^2 = 8 \sin^2 t \cos^2 t = 2 \sin^2 2t \Rightarrow$$

$$\text{Distance} = \int_0^{3\pi} \sqrt{2} |\sin 2t| dt = 6\sqrt{2} \int_0^{\pi/2} \sin 2t dt \quad [\text{by symmetry}] = -3\sqrt{2} \left[ \cos 2t \right]_0^{\pi/2} = -3\sqrt{2}(-1 - 1) = 6\sqrt{2}.$$

The full curve is traversed as  $t$  goes from 0 to  $\frac{\pi}{2}$ , because the curve is the segment of  $x + y = 1$  that lies in the first quadrant (since  $x, y \geq 0$ ), and this segment is completely traversed as  $t$  goes from 0 to  $\frac{\pi}{2}$ . Thus,  $L = \int_0^{\pi/2} \sin 2t dt = \sqrt{2}$ , as above.

$$52. \quad x = \cos^2 t, y = \cos t, 0 \leq t \leq 4\pi. \quad \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (-2 \cos t \sin t)^2 + (-\sin t)^2 = \sin^2 t (4 \cos^2 t + 1)$$

$$\begin{aligned} \text{Distance} &= \int_0^{4\pi} |\sin t| \sqrt{4 \cos^2 t + 1} dt = 4 \int_0^{\pi} \sin t \sqrt{4 \cos^2 t + 1} dt \\ &= -4 \int_1^{-1} \sqrt{4u^2 + 1} du \quad [u = \cos t, du = -\sin t dt] = 4 \int_{-1}^1 \sqrt{4u^2 + 1} du \\ &= 8 \int_0^1 \sqrt{4u^2 + 1} du = 8 \int_0^{\tan^{-1} 2} \sec \theta \cdot \frac{1}{2} \sec^2 \theta d\theta \quad [2u = \tan \theta, 2 du = \sec^2 \theta d\theta] \\ &= 4 \int_0^{\tan^{-1} 2} \sec^3 \theta d\theta \stackrel{71}{=} \left[ 2 \sec \theta \tan \theta + 2 \ln |\sec \theta + \tan \theta| \right]_0^{\tan^{-1} 2} = 4\sqrt{5} + 2 \ln(\sqrt{5} + 2) \end{aligned}$$

$$\text{Thus, } L = \int_0^{\pi} |\sin t| \sqrt{4 \cos^2 t + 1} dt = \sqrt{5} + \frac{1}{2} \ln(\sqrt{5} + 2).$$

$$53. \quad x = a \sin \theta, y = b \cos \theta, 0 \leq \theta \leq 2\pi.$$

$$\begin{aligned} \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= (a \cos \theta)^2 + (-b \sin \theta)^2 = a^2 \cos^2 \theta + b^2 \sin^2 \theta = a^2(1 - \sin^2 \theta) + b^2 \sin^2 \theta \\ &= a^2 - (a^2 - b^2) \sin^2 \theta = a^2 - c^2 \sin^2 \theta = a^2 \left( 1 - \frac{c^2}{a^2} \sin^2 \theta \right) = a^2(1 - e^2 \sin^2 \theta) \end{aligned}$$

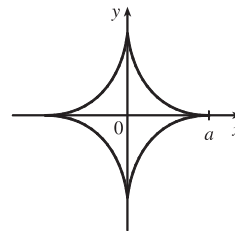
$$\text{So } L = 4 \int_0^{\pi/2} \sqrt{a^2(1 - e^2 \sin^2 \theta)} d\theta \quad [\text{by symmetry}] = 4a \int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 \theta} d\theta.$$

$$54. \quad x = a \cos^3 \theta, y = a \sin^3 \theta.$$

$$\begin{aligned} \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= (-3a \cos^2 \theta \sin \theta)^2 + (3a \sin^2 \theta \cos \theta)^2 \\ &= 9a^2 \cos^4 \theta \sin^2 \theta + 9a^2 \sin^4 \theta \cos^2 \theta \\ &= 9a^2 \sin^2 \theta \cos^2 \theta (\cos^2 \theta + \sin^2 \theta) = 9a^2 \sin^2 \theta \cos^2 \theta. \end{aligned}$$

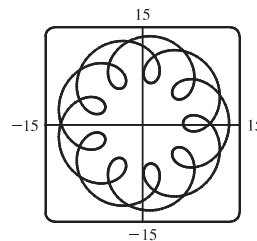
The graph has four-fold symmetry and the curve in the first quadrant corresponds to  $0 \leq \theta \leq \pi/2$ . Thus,

$$\begin{aligned} L &= 4 \int_0^{\pi/2} 3a \sin \theta \cos \theta d\theta \quad [\text{since } a > 0 \text{ and } \sin \theta \text{ and } \cos \theta \text{ are positive for } 0 \leq \theta \leq \pi/2] \\ &= 12a \left[ \frac{1}{2} \sin^2 \theta \right]_0^{\pi/2} = 12a \left( \frac{1}{2} - 0 \right) = 6a \end{aligned}$$



55. (a)  $x = 11 \cos t - 4 \cos(11t/2)$ ,  $y = 11 \sin t - 4 \sin(11t/2)$ .

Notice that  $0 \leq t \leq 2\pi$  does not give the complete curve because  $x(0) \neq x(2\pi)$ . In fact, we must take  $t \in [0, 4\pi]$  in order to obtain the complete curve, since the first term in each of the parametric equations has period  $2\pi$  and the second has period  $\frac{2\pi}{11/2} = \frac{4\pi}{11}$ , and the least common integer multiple of these two numbers is  $4\pi$ .



- (b) We use the CAS to find the derivatives  $dx/dt$  and  $dy/dt$ , and then use Theorem 6 to find the arc length. Recent versions of Maple express the integral  $\int_0^{4\pi} \sqrt{(dx/dt)^2 + (dy/dt)^2} dt$  as  $88E(2\sqrt{2}i)$ , where  $E(x)$  is the elliptic integral

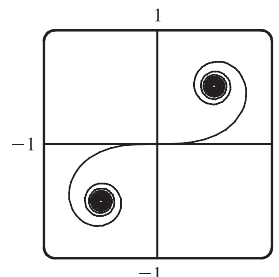
$$\int_0^1 \frac{\sqrt{1-x^2t^2}}{\sqrt{1-t^2}} dt \text{ and } i \text{ is the imaginary number } \sqrt{-1}.$$

Some earlier versions of Maple (as well as Mathematica) cannot do the integral exactly, so we use the command `evalf(Int(sqrt(diff(x,t)^2+diff(y,t)^2),t=0..4*Pi))` to estimate the length, and find that the arc length is approximately 294.03. Derive's `Para_arc_length` function in the utility file `Int_apps` simplifies the integral to  $11 \int_0^{4\pi} \sqrt{-4 \cos t \cos(\frac{11t}{2}) - 4 \sin t \sin(\frac{11t}{2}) + 5} dt$ .

56. (a) It appears that as  $t \rightarrow \infty$ ,  $(x, y) \rightarrow (\frac{1}{2}, \frac{1}{2})$ , and as  $t \rightarrow -\infty$ ,  $(x, y) \rightarrow (-\frac{1}{2}, -\frac{1}{2})$ .

- (b) By the Fundamental Theorem of Calculus,  $dx/dt = \cos(\frac{\pi}{2}t^2)$  and  $dy/dt = \sin(\frac{\pi}{2}t^2)$ , so by Formula 4, the length of the curve from the origin to the point with parameter value  $t$  is

$$\begin{aligned} L &= \int_0^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2} du = \int_0^t \sqrt{\cos^2\left(\frac{\pi}{2}u^2\right) + \sin^2\left(\frac{\pi}{2}u^2\right)} du \\ &= \int_0^t 1 du = t \quad [\text{or } -t \text{ if } t < 0] \end{aligned}$$



We have used  $u$  as the dummy variable so as not to confuse it with the upper limit of integration.

57.  $x = t \sin t$ ,  $y = t \cos t$ ,  $0 \leq t \leq \pi/2$ .  $dx/dt = t \cos t + \sin t$  and  $dy/dt = -t \sin t + \cos t$ , so  $(dx/dt)^2 + (dy/dt)^2 = t^2 \cos^2 t + 2t \sin t \cos t + \sin^2 t + t^2 \sin^2 t - 2t \sin t \cos t + \cos^2 t = t^2(\cos^2 t + \sin^2 t) + \sin^2 t + \cos^2 t = t^2 + 1$   
 $S = \int 2\pi y ds = \int_0^{\pi/2} 2\pi t \cos t \sqrt{t^2 + 1} dt \approx 4.7394$ .
58.  $x = \sin t$ ,  $y = \sin 2t$ ,  $0 \leq t \leq \pi/2$ .  $dx/dt = \cos t$  and  $dy/dt = 2 \cos 2t$ , so  $(dx/dt)^2 + (dy/dt)^2 = \cos^2 t + 4 \cos^2 2t$ .  
 $S = \int 2\pi y ds = \int_0^{\pi/2} 2\pi \sin 2t \sqrt{\cos^2 t + 4 \cos^2 2t} dt \approx 8.0285$ .
59.  $x = 1 + te^t$ ,  $y = (t^2 + 1)e^t$ ,  $0 \leq t \leq 1$ .

$$\begin{aligned} \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= (te^t + e^t)^2 + [(t^2 + 1)e^t + e^t(2t)]^2 = [e^t(t + 1)]^2 + [e^t(t^2 + 2t + 1)]^2 \\ &= e^{2t}(t + 1)^2 + e^{2t}(t + 1)^4 = e^{2t}(t + 1)^2[1 + (t + 1)^2], \quad \text{so} \end{aligned}$$

$$S = \int 2\pi y ds = \int_0^1 2\pi(t^2 + 1)e^t \sqrt{e^{2t}(t + 1)^2(t^2 + 2t + 2)} dt = \int_0^1 2\pi(t^2 + 1)e^{2t}(t + 1) \sqrt{t^2 + 2t + 2} dt \approx 103.5999.$$

60.  $x = t^2 - t^3$ ,  $y = t + t^4$ ,  $0 \leq t \leq 1$ .

$$(dx/dt)^2 + (dy/dt)^2 = (2t - 3t^2)^2 + (1 + 4t^3)^2 = 4t^2 - 12t^3 + 9t^4 + 1 + 8t^3 + 16t^6, \text{ so}$$

$$S = \int 2\pi y \, ds = \int_0^1 2\pi(t + t^4)\sqrt{16t^6 + 9t^4 - 4t^3 + 4t^2 + 1} \, dt \approx 12.7176.$$

61.  $x = t^3$ ,  $y = t^2$ ,  $0 \leq t \leq 1$ .  $(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 = (3t^2)^2 + (2t)^2 = 9t^4 + 4t^2$ .

$$\begin{aligned} S &= \int_0^1 2\pi y \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} \, dt = \int_0^1 2\pi t^2 \sqrt{9t^4 + 4t^2} \, dt = 2\pi \int_0^1 t^2 \sqrt{t^2(9t^2 + 4)} \, dt \\ &= 2\pi \int_4^{13} \left(\frac{u-4}{9}\right) \sqrt{u} \left(\frac{1}{18} du\right) \left[ \begin{array}{l} u = 9t^2 + 4, \, t^2 = (u-4)/9, \\ du = 18t \, dt, \text{ so } t \, dt = \frac{1}{18} du \end{array} \right] = \frac{2\pi}{9 \cdot 18} \int_4^{13} (u^{3/2} - 4u^{1/2}) \, du \\ &= \frac{\pi}{81} \left[ \frac{2}{5} u^{5/2} - \frac{8}{3} u^{3/2} \right]_4^{13} = \frac{\pi}{81} \cdot \frac{2}{15} \left[ 3u^{5/2} - 20u^{3/2} \right]_4^{13} \\ &= \frac{2\pi}{1215} [(3 \cdot 13^2 \sqrt{13} - 20 \cdot 13 \sqrt{13}) - (3 \cdot 32 - 20 \cdot 8)] = \frac{2\pi}{1215} (247\sqrt{13} + 64) \end{aligned}$$

62.  $x = 3t - t^3$ ,  $y = 3t^2$ ,  $0 \leq t \leq 1$ .  $(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 = (3 - 3t^2)^2 + (6t)^2 = 9(1 + 2t^2 + t^4) = [3(1 + t^2)]^2$ .

$$S = \int_0^1 2\pi \cdot 3t^2 \cdot 3(1 + t^2) \, dt = 18\pi \int_0^1 (t^2 + t^4) \, dt = 18\pi \left[ \frac{1}{3}t^3 + \frac{1}{5}t^5 \right]_0^1 = \frac{48}{5}\pi$$

63.  $x = a \cos^3 \theta$ ,  $y = a \sin^3 \theta$ ,  $0 \leq \theta \leq \frac{\pi}{2}$ .  $(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2 = (-3a \cos^2 \theta \sin \theta)^2 + (3a \sin^2 \theta \cos \theta)^2 = 9a^2 \sin^2 \theta \cos^2 \theta$ .

$$S = \int_0^{\pi/2} 2\pi \cdot a \sin^3 \theta \cdot 3a \sin \theta \cos \theta \, d\theta = 6\pi a^2 \int_0^{\pi/2} \sin^4 \theta \cos \theta \, d\theta = \frac{6}{5}\pi a^2 [\sin^5 \theta]_0^{\pi/2} = \frac{6}{5}\pi a^2$$

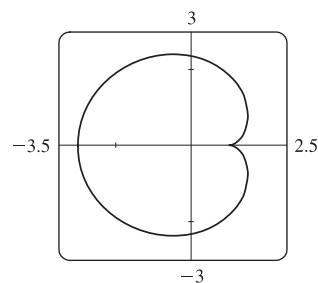
64.  $(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2 = (-2 \sin \theta + 2 \sin 2\theta)^2 + (2 \cos \theta - 2 \cos 2\theta)^2$   
 $= 4[(\sin^2 \theta - 2 \sin \theta \sin 2\theta + \sin^2 2\theta) + (\cos^2 \theta - 2 \cos \theta \cos 2\theta + \cos^2 2\theta)]$   
 $= 4[1 + 1 - 2(\cos 2\theta \cos \theta + \sin 2\theta \sin \theta)] = 8[1 - \cos(2\theta - \theta)] = 8(1 - \cos \theta)$

We plot the graph with parameter interval  $[0, 2\pi]$ , and see that we should only

integrate between 0 and  $\pi$ . (If the interval  $[0, 2\pi]$  were taken, the surface of revolution would be generated twice.) Also note that

$$y = 2 \sin \theta - \sin 2\theta = 2 \sin \theta (1 - \cos \theta). \text{ So}$$

$$\begin{aligned} S &= \int_0^\pi 2\pi \cdot 2 \sin \theta (1 - \cos \theta) 2 \sqrt{2} \sqrt{1 - \cos \theta} \, d\theta \\ &= 8\sqrt{2}\pi \int_0^\pi (1 - \cos \theta)^{3/2} \sin \theta \, d\theta = 8\sqrt{2}\pi \int_0^2 \sqrt{u^3} \, du \quad \left[ \begin{array}{l} u = 1 - \cos \theta, \\ du = \sin \theta \, d\theta \end{array} \right] \\ &= 8\sqrt{2}\pi \left[ \left(\frac{2}{5}\right) u^{5/2} \right]_0^2 = \frac{16}{5}\sqrt{2}\pi (2^{5/2}) = \frac{128}{5}\pi \end{aligned}$$



65.  $x = 3t^2$ ,  $y = 2t^3$ ,  $0 \leq t \leq 5 \Rightarrow (\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 = (6t)^2 + (6t^2)^2 = 36t^2(1 + t^2) \Rightarrow$

$$\begin{aligned} S &= \int_0^5 2\pi x \sqrt{(dx/dt)^2 + (dy/dt)^2} \, dt = \int_0^5 2\pi(3t^2)6t \sqrt{1 + t^2} \, dt = 18\pi \int_0^5 t^2 \sqrt{1 + t^2} \, dt \\ &= 18\pi \int_1^{26} (u - 1) \sqrt{u} \, du \quad \left[ \begin{array}{l} u = 1 + t^2, \\ du = 2t \, dt \end{array} \right] = 18\pi \int_1^{26} (u^{3/2} - u^{1/2}) \, du = 18\pi \left[ \frac{2}{5} u^{5/2} - \frac{2}{3} u^{3/2} \right]_1^{26} \\ &= 18\pi \left[ \left(\frac{2}{5} \cdot 676 \sqrt{26} - \frac{2}{3} \cdot 26 \sqrt{26}\right) - \left(\frac{2}{5} - \frac{2}{3}\right) \right] = \frac{24}{5}\pi (949\sqrt{26} + 1) \end{aligned}$$

66.  $x = e^t - t, y = 4e^{t/2}, 0 \leq t \leq 1. \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (e^t - 1)^2 + (2e^{t/2})^2 = e^{2t} + 2e^t + 1 = (e^t + 1)^2.$

$$\begin{aligned} S &= \int_0^1 2\pi(e^t - t)\sqrt{(e^t - 1)^2 + (2e^{t/2})^2} dt = \int_0^1 2\pi(e^t - t)(e^t + 1) dt \\ &= 2\pi\left[\frac{1}{2}e^{2t} + e^t - (t-1)e^t - \frac{1}{2}t^2\right]_0^1 = \pi(e^2 + 2e - 6) \end{aligned}$$

67. If  $f'$  is continuous and  $f'(t) \neq 0$  for  $a \leq t \leq b$ , then either  $f'(t) > 0$  for all  $t$  in  $[a, b]$  or  $f'(t) < 0$  for all  $t$  in  $[a, b]$ . Thus,  $f$  is monotonic (in fact, strictly increasing or strictly decreasing) on  $[a, b]$ . It follows that  $f$  has an inverse. Set  $F = g \circ f^{-1}$ , that is, define  $F$  by  $F(x) = g(f^{-1}(x))$ . Then  $x = f(t) \Rightarrow f^{-1}(x) = t$ , so  $y = g(t) = g(f^{-1}(x)) = F(x)$ .

68. By Formula 8.2.5 with  $y = F(x)$ ,  $S = \int_a^b 2\pi F(x)\sqrt{1 + [F'(x)]^2} dx$ . But by Formula 10.2.1,

$$1 + [F'(x)]^2 = 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(\frac{dy/dt}{dx/dt}\right)^2 = \frac{(dx/dt)^2 + (dy/dt)^2}{(dx/dt)^2}. \text{ Using the Substitution Rule with } x = x(t),$$

where  $a = x(\alpha)$  and  $b = x(\beta)$ , we have  $\left[\text{since } dx = \frac{dx}{dt} dt\right]$

$$S = \int_\alpha^\beta 2\pi F(x(t))\sqrt{\frac{(dx/dt)^2 + (dy/dt)^2}{(dx/dt)^2}} \frac{dx}{dt} dt = \int_\alpha^\beta 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt, \text{ which is Formula 10.2.6.}$$

69. (a)  $\phi = \tan^{-1}\left(\frac{dy}{dx}\right) \Rightarrow \frac{d\phi}{dt} = \frac{d}{dt} \tan^{-1}\left(\frac{dy}{dx}\right) = \frac{1}{1 + (dy/dx)^2} \left[\frac{d}{dt}\left(\frac{dy}{dx}\right)\right]$ . But  $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\dot{y}}{\dot{x}} \Rightarrow$

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{\dot{y}}{\dot{x}}\right) = \frac{\ddot{y}\dot{x} - \dot{y}\ddot{x}}{\dot{x}^2} \Rightarrow \frac{d\phi}{dt} = \frac{1}{1 + (\dot{y}/\dot{x})^2} \left(\frac{\ddot{y}\dot{x} - \dot{y}\ddot{x}}{\dot{x}^2}\right) = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^2 + \dot{y}^2}.$$
 Using the Chain Rule, and the

fact that  $s = \int_0^t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \Rightarrow \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = (\dot{x}^2 + \dot{y}^2)^{1/2}$ , we have that

$$\frac{d\phi}{ds} = \frac{d\phi/dt}{ds/dt} = \left(\frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^2 + \dot{y}^2}\right) \frac{1}{(\dot{x}^2 + \dot{y}^2)^{1/2}} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x}^2 + \dot{y}^2)^{3/2}}. \text{ So } \kappa = \left|\frac{d\phi}{ds}\right| = \left|\frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x}^2 + \dot{y}^2)^{3/2}}\right| = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}}.$$

(b)  $x = x$  and  $y = f(x) \Rightarrow \dot{x} = 1, \ddot{x} = 0$  and  $\dot{y} = \frac{dy}{dx}, \ddot{y} = \frac{d^2y}{dx^2}$ .

$$\text{So } \kappa = \frac{|1 \cdot (d^2y/dx^2) - 0 \cdot (dy/dx)|}{[1 + (dy/dx)^2]^{3/2}} = \frac{|d^2y/dx^2|}{[1 + (dy/dx)^2]^{3/2}}.$$

70. (a)  $y = x^2 \Rightarrow \frac{dy}{dx} = 2x \Rightarrow \frac{d^2y}{dx^2} = 2$ . So  $\kappa = \frac{|d^2y/dx^2|}{[1 + (dy/dx)^2]^{3/2}} = \frac{2}{(1 + 4x^2)^{3/2}}$ , and at  $(1, 1)$ ,

$$\kappa = \frac{2}{5^{3/2}} = \frac{2}{5\sqrt{5}}.$$

(b)  $\kappa' = \frac{d\kappa}{dx} = -3(1 + 4x^2)^{-5/2}(8x) = 0 \Leftrightarrow x = 0 \Rightarrow y = 0$ . This is a maximum since  $\kappa' > 0$  for  $x < 0$  and

$\kappa' < 0$  for  $x > 0$ . So the parabola  $y = x^2$  has maximum curvature at the origin.



71.  $x = \theta - \sin \theta \Rightarrow \dot{x} = 1 - \cos \theta \Rightarrow \ddot{x} = \sin \theta$ , and  $y = 1 - \cos \theta \Rightarrow \dot{y} = \sin \theta \Rightarrow \ddot{y} = \cos \theta$ . Therefore,

$$\kappa = \frac{|\cos \theta - \cos^2 \theta - \sin^2 \theta|}{[(1 - \cos \theta)^2 + \sin^2 \theta]^{3/2}} = \frac{|\cos \theta - (\cos^2 \theta + \sin^2 \theta)|}{(1 - 2 \cos \theta + \cos^2 \theta + \sin^2 \theta)^{3/2}} = \frac{|\cos \theta - 1|}{(2 - 2 \cos \theta)^{3/2}}.$$

The top of the arch is characterized by a horizontal tangent, and from Example 2(b) in Section 10.2, the tangent is horizontal when  $\theta = (2n - 1)\pi$ ,

so take  $n = 1$  and substitute  $\theta = \pi$  into the expression for  $\kappa$ :  $\kappa = \frac{|\cos \pi - 1|}{(2 - 2 \cos \pi)^{3/2}} = \frac{|-1 - 1|}{[2 - 2(-1)]^{3/2}} = \frac{1}{4}.$

72. (a) Every straight line has parametrizations of the form  $x = a + vt$ ,  $y = b + wt$ , where  $a, b$  are arbitrary and  $v, w \neq 0$ .

For example, a straight line passing through distinct points  $(a, b)$  and  $(c, d)$  can be described as the parametrized curve

$$x = a + (c - a)t, y = b + (d - b)t. \text{ Starting with } x = a + vt, y = b + wt, \text{ we compute } \dot{x} = v, \dot{y} = w, \ddot{x} = \ddot{y} = 0,$$

$$\text{and } \kappa = \frac{|v \cdot 0 - w \cdot 0|}{(v^2 + w^2)^{3/2}} = 0.$$

(b) Parametric equations for a circle of radius  $r$  are  $x = r \cos \theta$  and  $y = r \sin \theta$ . We can take the center to be the origin.

$$\text{So } \dot{x} = -r \sin \theta \Rightarrow \ddot{x} = -r \cos \theta \text{ and } \dot{y} = r \cos \theta \Rightarrow \ddot{y} = -r \sin \theta. \text{ Therefore,}$$

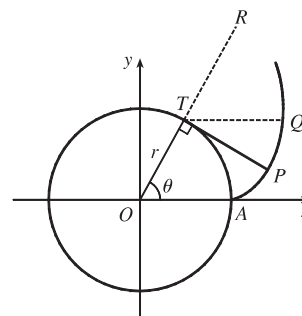
$$\kappa = \frac{|r^2 \sin^2 \theta + r^2 \cos^2 \theta|}{(r^2 \sin^2 \theta + r^2 \cos^2 \theta)^{3/2}} = \frac{r^2}{r^3} = \frac{1}{r}. \text{ And so for any } \theta \text{ (and thus any point), } \kappa = \frac{1}{r}.$$

73. The coordinates of  $T$  are  $(r \cos \theta, r \sin \theta)$ . Since  $TP$  was unwound from

arc  $TA$ ,  $TP$  has length  $r\theta$ . Also  $\angle PTQ = \angle PTR - \angle QTR = \frac{1}{2}\pi - \theta$ ,

so  $P$  has coordinates  $x = r \cos \theta + r\theta \cos(\frac{1}{2}\pi - \theta) = r(\cos \theta + \theta \sin \theta)$ ,

$$y = r \sin \theta - r\theta \sin(\frac{1}{2}\pi - \theta) = r(\sin \theta - \theta \cos \theta).$$



74. If the cow walks with the rope taut, it traces out the portion of the

involute in Exercise 73 corresponding to the range  $0 \leq \theta \leq \pi$ , arriving at

the point  $(-r, \pi r)$  when  $\theta = \pi$ . With the rope now fully extended, the

cow walks in a semicircle of radius  $\pi r$ , arriving at  $(-r, -\pi r)$ . Finally,

the cow traces out another portion of the involute, namely the reflection

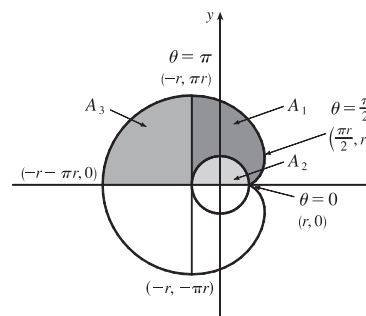
about the  $x$ -axis of the initial involute path. (This corresponds to the

range  $-\pi \leq \theta \leq 0$ .) Referring to the figure, we see that the total grazing

area is  $2(A_1 + A_3)$ .  $A_3$  is one-quarter of the area of a circle of radius  $\pi r$ , so  $A_3 = \frac{1}{4}\pi(\pi r)^2 = \frac{1}{4}\pi^3 r^2$ . We will compute

$A_1 + A_2$  and then subtract  $A_2 = \frac{1}{2}\pi r^2$  to obtain  $A_1$ .

To find  $A_1 + A_2$ , first note that the rightmost point of the involute is  $(\frac{\pi}{2}r, r)$ . [To see this, note that  $dx/d\theta = 0$  when  $\theta = 0$  or  $\frac{\pi}{2}$ .  $\theta = 0$  corresponds to the cusp at  $(r, 0)$  and  $\theta = \frac{\pi}{2}$  corresponds to  $(\frac{\pi}{2}r, r)$ .] The leftmost point of the involute is



$(-r, \pi r)$ . Thus,  $A_1 + A_2 = \int_{\theta=\pi}^{\pi/2} y \, dx - \int_{\theta=0}^{\pi/2} y \, dx = \int_{\theta=\pi}^0 y \, dx$ .

Now  $y \, dx = r(\sin \theta - \theta \cos \theta) r \theta \cos \theta \, d\theta = r^2(\theta \sin \theta \cos \theta - \theta^2 \cos^2 \theta) d\theta$ . Integrate:

$(1/r^2) \int y \, dx = -\theta \cos^2 \theta - \frac{1}{2}(\theta^2 - 1) \sin \theta \cos \theta - \frac{1}{6}\theta^3 + \frac{1}{2}\theta + C$ . This enables us to compute

$$A_1 + A_2 = r^2 \left[ -\theta \cos^2 \theta - \frac{1}{2}(\theta^2 - 1) \sin \theta \cos \theta - \frac{1}{6}\theta^3 + \frac{1}{2}\theta \right]_{\pi}^0 = r^2 \left[ 0 - \left( -\pi - \frac{\pi^3}{6} + \frac{\pi}{2} \right) \right] = r^2 \left( \frac{\pi}{2} + \frac{\pi^3}{6} \right)$$

Therefore,  $A_1 = (A_1 + A_2) - A_2 = \frac{1}{6}\pi^3 r^2$ , so the grazing area is  $2(A_1 + A_3) = 2\left(\frac{1}{6}\pi^3 r^2 + \frac{1}{4}\pi^3 r^2\right) = \frac{5}{6}\pi^3 r^2$ .

## LABORATORY PROJECT Bézier Curves

- The parametric equations for a cubic Bézier curve are

$$x = x_0(1-t)^3 + 3x_1t(1-t)^2 + 3x_2t^2(1-t) + x_3t^3$$

$$y = y_0(1-t)^3 + 3y_1t(1-t)^2 + 3y_2t^2(1-t) + y_3t^3$$

where  $0 \leq t \leq 1$ . We are given the points  $P_0(x_0, y_0) = (4, 1)$ ,  $P_1(x_1, y_1) = (28, 48)$ ,  $P_2(x_2, y_2) = (50, 42)$ , and  $P_3(x_3, y_3) = (40, 5)$ . The curve is then given by

$$x(t) = 4(1-t)^3 + 3 \cdot 28t(1-t)^2 + 3 \cdot 50t^2(1-t) + 40t^3$$

$$y(t) = 1(1-t)^3 + 3 \cdot 48t(1-t)^2 + 3 \cdot 42t^2(1-t) + 5t^3$$

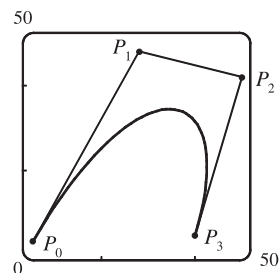
where  $0 \leq t \leq 1$ . The line segments are of the form  $x = x_0 + (x_1 - x_0)t$ ,

$y = y_0 + (y_1 - y_0)t$ :

$$P_0P_1 \quad x = 4 + 24t, \quad y = 1 + 47t$$

$$P_1P_2 \quad x = 28 + 22t, \quad y = 48 - 6t$$

$$P_2P_3 \quad x = 50 - 10t, \quad y = 42 - 37t$$



- It suffices to show that the slope of the tangent at  $P_0$  is the same as that of line segment  $P_0P_1$ , namely  $\frac{y_1 - y_0}{x_1 - x_0}$ .

We calculate the slope of the tangent to the Bézier curve:

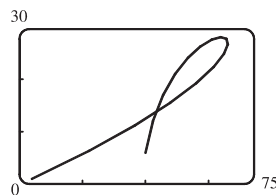
$$\frac{dy/dt}{dx/dt} = \frac{-3y_0(1-t)^2 + 3y_1[-2t(1-t) + (1-t)^2] + 3y_2[-t^2 + (2t)(1-t)] + 3y_3t^2}{-3x_0^2(1-t) + 3x_1[-2t(1-t) + (1-t)^2] + 3x_2[-t^2 + (2t)(1-t)] + 3x_3t^2}$$

At point  $P_0$ ,  $t = 0$ , so the slope of the tangent is  $\frac{-3y_0 + 3y_1}{-3x_0 + 3x_1} = \frac{y_1 - y_0}{x_1 - x_0}$ . So the tangent to the curve at  $P_0$  passes

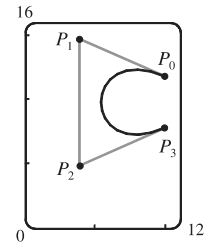
through  $P_1$ . Similarly, the slope of the tangent at point  $P_3$  [where  $t = 1$ ] is  $\frac{-3y_2 + 3y_3}{-3x_2 + 3x_3} = \frac{y_3 - y_2}{x_3 - x_2}$ , which is also the slope of line  $P_2P_3$ .

- It seems that if  $P_1$  were to the right of  $P_2$ , a loop would appear.

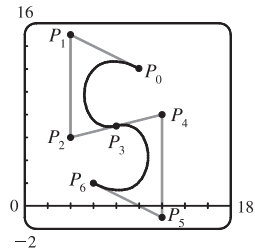
We try setting  $P_1 = (110, 30)$ , and the resulting curve does indeed have a loop.



4. Based on the behavior of the Bézier curve in Problems 1–3, we suspect that the four control points should be in an exaggerated C shape. We try  $P_0(10, 12)$ ,  $P_1(4, 15)$ ,  $P_2(4, 5)$ , and  $P_3(10, 8)$ , and these produce a decent C. If you are using a CAS, it may be necessary to instruct it to make the  $x$ - and  $y$ -scales the same so as not to distort the figure (this is called a “constrained projection” in Maple.)

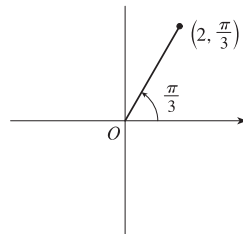


5. We use the same  $P_0$  and  $P_1$  as in Problem 4, and use part of our C as the top of an S. To prevent the center line from slanting up too much, we move  $P_2$  up to  $(4, 6)$  and  $P_3$  down and to the left, to  $(8, 7)$ . In order to have a smooth joint between the top and bottom halves of the S (and a symmetric S), we determine points  $P_4$ ,  $P_5$ , and  $P_6$  by rotating points  $P_2$ ,  $P_1$ , and  $P_0$  about the center of the letter (point  $P_3$ ). The points are therefore  $P_4(12, 8)$ ,  $P_5(12, -1)$ , and  $P_6(6, 2)$ .



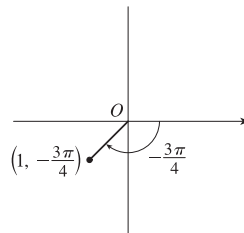
## 10.3 Polar Coordinates

1. (a)  $(2, \frac{\pi}{3})$



By adding  $2\pi$  to  $\frac{\pi}{3}$ , we obtain the point  $(2, \frac{7\pi}{3})$ . The direction opposite  $\frac{\pi}{3}$  is  $\frac{4\pi}{3}$ , so  $(-2, \frac{4\pi}{3})$  is a point that satisfies the  $r < 0$  requirement.

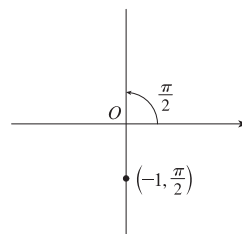
- (b)  $(1, -\frac{3\pi}{4})$



$$r > 0: (1, -\frac{3\pi}{4} + 2\pi) = (1, \frac{5\pi}{4})$$

$$r < 0: (-1, -\frac{3\pi}{4} + \pi) = (-1, \frac{\pi}{4})$$

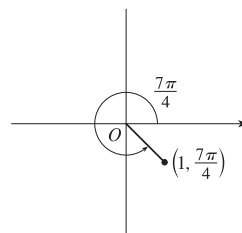
- (c)  $(-1, \frac{\pi}{2})$



$$r > 0: (-(-1), \frac{\pi}{2} + \pi) = (1, \frac{3\pi}{2})$$

$$r < 0: (-1, \frac{\pi}{2} + 2\pi) = (-1, \frac{5\pi}{2})$$

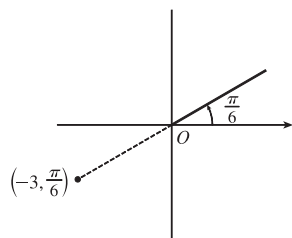
2. (a)  $(1, \frac{7\pi}{4})$



$$r > 0: (1, \frac{7\pi}{4} - 2\pi) = (1, -\frac{\pi}{4})$$

$$r < 0: (-1, \frac{7\pi}{4} - \pi) = (-1, \frac{3\pi}{4})$$

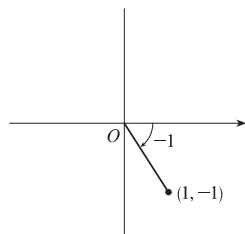
(b)  $(-3, \frac{\pi}{6})$



$$r > 0: (-(-3), \frac{\pi}{6} + \pi) = (3, \frac{7\pi}{6})$$

$$r < 0: (-3, \frac{\pi}{6} + 2\pi) = (-3, \frac{13\pi}{6})$$

(c)  $(1, -1)$

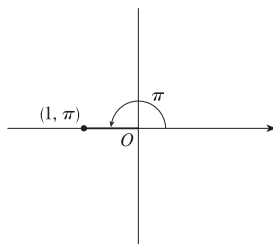


$$\theta = -1 \text{ radian} \approx -57.3^\circ$$

$$r > 0: (1, -1 + 2\pi)$$

$$r < 0: (-1, -1 + \pi)$$

3. (a)

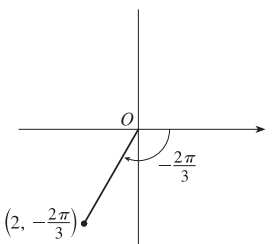


$$x = 1 \cos \pi = 1(-1) = -1 \text{ and}$$

$$y = 1 \sin \pi = 1(0) = 0 \text{ give us}$$

the Cartesian coordinates  $(-1, 0)$ .

(b)

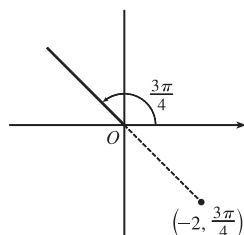


$$x = 2 \cos(-\frac{2\pi}{3}) = 2(-\frac{1}{2}) = -1 \text{ and}$$

$$y = 2 \sin(-\frac{2\pi}{3}) = 2(-\frac{\sqrt{3}}{2}) = -\sqrt{3}$$

give us  $(-1, -\sqrt{3})$ .

(c)

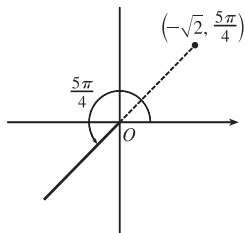


$$x = -2 \cos \frac{3\pi}{4} = -2(-\frac{\sqrt{2}}{2}) = \sqrt{2} \text{ and}$$

$$y = -2 \sin \frac{3\pi}{4} = -2(\frac{\sqrt{2}}{2}) = -\sqrt{2}$$

gives us  $(\sqrt{2}, -\sqrt{2})$ .

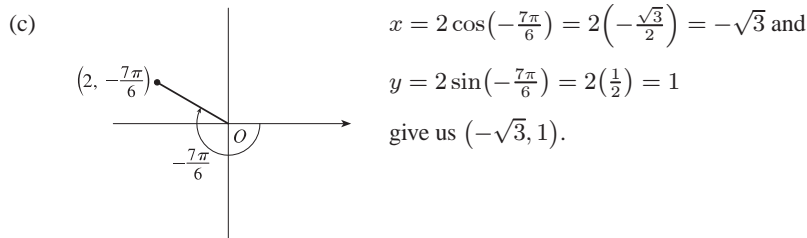
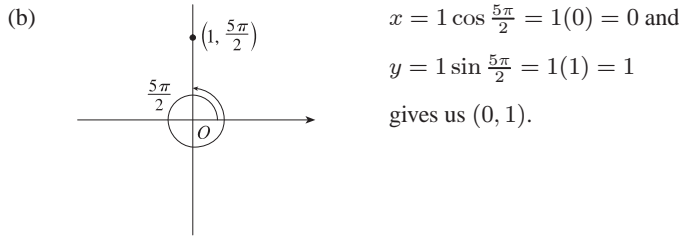
4. (a)



$$x = -\sqrt{2} \cos \frac{5\pi}{4} = -\sqrt{2}(-\frac{\sqrt{2}}{2}) = 1 \text{ and}$$

$$y = -\sqrt{2} \sin \frac{5\pi}{4} = -\sqrt{2}(-\frac{\sqrt{2}}{2}) = 1$$

gives us  $(1, 1)$ .



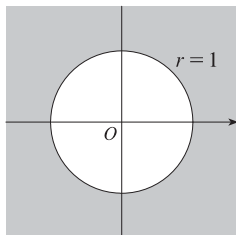
5. (a)  $x = 2$  and  $y = -2 \Rightarrow r = \sqrt{2^2 + (-2)^2} = 2\sqrt{2}$  and  $\theta = \tan^{-1}(\frac{-2}{2}) = -\frac{\pi}{4}$ . Since  $(2, -2)$  is in the fourth quadrant, the polar coordinates are (i)  $(2\sqrt{2}, \frac{7\pi}{4})$  and (ii)  $(-2\sqrt{2}, \frac{3\pi}{4})$ .

(b)  $x = -1$  and  $y = \sqrt{3} \Rightarrow r = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2$  and  $\theta = \tan^{-1}(\frac{\sqrt{3}}{-1}) = \frac{2\pi}{3}$ . Since  $(-1, \sqrt{3})$  is in the second quadrant, the polar coordinates are (i)  $(2, \frac{2\pi}{3})$  and (ii)  $(-2, \frac{5\pi}{3})$ .

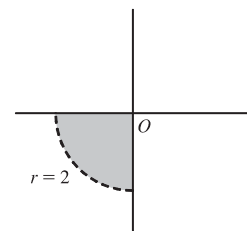
6. (a)  $x = 3\sqrt{3}$  and  $y = 3 \Rightarrow r = \sqrt{(3\sqrt{3})^2 + 3^2} = \sqrt{27+9} = 6$  and  $\theta = \tan^{-1}(\frac{3}{3\sqrt{3}}) = \tan^{-1}(\frac{1}{\sqrt{3}}) = \frac{\pi}{6}$ . Since  $(3\sqrt{3}, 3)$  is in the first quadrant, the polar coordinates are (i)  $(6, \frac{\pi}{6})$  and (ii)  $(-6, \frac{7\pi}{6})$ .

(b)  $x = 1$  and  $y = -2 \Rightarrow r = \sqrt{1^2 + (-2)^2} = \sqrt{5}$  and  $\theta = \tan^{-1}(\frac{-2}{1}) = -\tan^{-1} 2$ . Since  $(1, -2)$  is in the fourth quadrant, the polar coordinates are (i)  $(\sqrt{5}, 2\pi - \tan^{-1} 2)$  and (ii)  $(-\sqrt{5}, \pi - \tan^{-1} 2)$ .

7.  $r \geq 1$ . The curve  $r = 1$  represents a circle with center  $O$  and radius 1. So  $r \geq 1$  represents the region on or outside the circle. Note that  $\theta$  can take on any value.

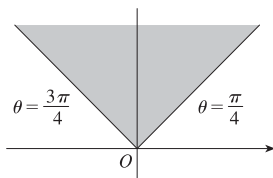


8.  $0 \leq r < 2$ ,  $\pi \leq \theta \leq 3\pi/2$ . This is the region inside the circle  $r = 2$  in the third quadrant.

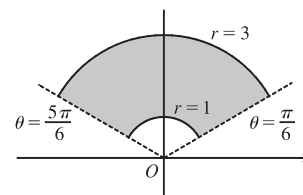


9.  $r \geq 0$ ,  $\pi/4 \leq \theta \leq 3\pi/4$ .

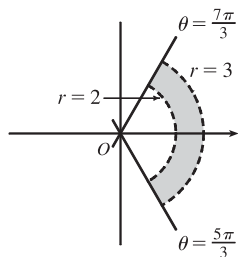
$\theta = k$  represents a line through  $O$ .



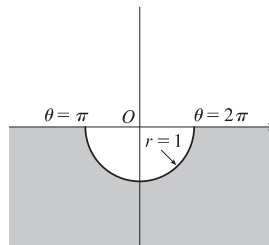
10.  $1 \leq r \leq 3$ ,  $\pi/6 < \theta < 5\pi/6$



11.  $2 < r < 3, \frac{5\pi}{3} \leq \theta \leq \frac{7\pi}{3}$



12.  $r \geq 1, \pi \leq \theta \leq 2\pi$



13. Converting the polar coordinates  $(2, \pi/3)$  and  $(4, 2\pi/3)$  to Cartesian coordinates gives us  $(2 \cos \frac{\pi}{3}, 2 \sin \frac{\pi}{3}) = (1, \sqrt{3})$  and  $(4 \cos \frac{2\pi}{3}, 4 \sin \frac{2\pi}{3}) = (-2, 2\sqrt{3})$ . Now use the distance formula.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(-2 - 1)^2 + (2\sqrt{3} - \sqrt{3})^2} = \sqrt{9 + 3} = \sqrt{12} = 2\sqrt{3}$$

14. The points  $(r_1, \theta_1)$  and  $(r_2, \theta_2)$  in Cartesian coordinates are  $(r_1 \cos \theta_1, r_1 \sin \theta_1)$  and  $(r_2 \cos \theta_2, r_2 \sin \theta_2)$ , respectively.

The *square* of the distance between them is

$$\begin{aligned} & (r_2 \cos \theta_2 - r_1 \cos \theta_1)^2 + (r_2 \sin \theta_2 - r_1 \sin \theta_1)^2 \\ &= (r_2^2 \cos^2 \theta_2 - 2r_1 r_2 \cos \theta_1 \cos \theta_2 + r_1^2 \cos^2 \theta_1) + (r_2^2 \sin^2 \theta_2 - 2r_1 r_2 \sin \theta_1 \sin \theta_2 + r_1^2 \sin^2 \theta_1) \\ &= r_1^2 (\sin^2 \theta_1 + \cos^2 \theta_1) + r_2^2 (\sin^2 \theta_2 + \cos^2 \theta_2) - 2r_1 r_2 (\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2) \\ &= r_1^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2) + r_2^2, \end{aligned}$$

so the distance between them is  $\sqrt{r_1^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2) + r_2^2}$ .

15.  $r^2 = 5 \Leftrightarrow x^2 + y^2 = 5$ , a circle of radius  $\sqrt{5}$  centered at the origin.

16.  $r = 4 \sec \theta \Leftrightarrow \frac{r}{\sec \theta} = 4 \Leftrightarrow r \cos \theta = 4 \Leftrightarrow x = 4$ , a vertical line.

17.  $r = 2 \cos \theta \Rightarrow r^2 = 2r \cos \theta \Leftrightarrow x^2 + y^2 = 2x \Leftrightarrow x^2 - 2x + 1 + y^2 = 1 \Leftrightarrow (x - 1)^2 + y^2 = 1$ , a circle of radius 1 centered at  $(1, 0)$ . The first two equations are actually equivalent since  $r^2 = 2r \cos \theta \Rightarrow r(r - 2 \cos \theta) = 0 \Rightarrow r = 0$  or  $r = 2 \cos \theta$ . But  $r = 2 \cos \theta$  gives the point  $r = 0$  (the pole) when  $\theta = 0$ . Thus, the equation  $r = 2 \cos \theta$  is equivalent to the compound condition  $(r = 0 \text{ or } r = 2 \cos \theta)$ .

18.  $\theta = \frac{\pi}{3} \Rightarrow \tan \theta = \tan \frac{\pi}{3} \Rightarrow \frac{y}{x} = \sqrt{3} \Leftrightarrow y = \sqrt{3}x$ , a line through the origin.

19.  $r^2 \cos 2\theta = 1 \Leftrightarrow r^2 (\cos^2 \theta - \sin^2 \theta) = 1 \Leftrightarrow (r \cos \theta)^2 - (r \sin \theta)^2 = 1 \Leftrightarrow x^2 - y^2 = 1$ , a hyperbola centered at the origin with foci on the  $x$ -axis.

20.  $r = \tan \theta \sec \theta = \frac{\sin \theta}{\cos^2 \theta} \Rightarrow r \cos^2 \theta = \sin \theta \Leftrightarrow (r \cos \theta)^2 = r \sin \theta \Leftrightarrow x^2 = y$ , a parabola with vertex at the origin opening upward. The first implication is reversible since  $\cos \theta = 0$  would imply  $\sin \theta = r \cos^2 \theta = 0$ , contradicting the fact that  $\cos^2 \theta + \sin^2 \theta = 1$ .

21.  $y = 2 \Leftrightarrow r \sin \theta = 2 \Leftrightarrow r = \frac{2}{\sin \theta} \Leftrightarrow r = 2 \csc \theta$

22.  $y = x \Rightarrow \frac{y}{x} = 1 \ [x \neq 0] \Rightarrow \tan \theta = 1 \Rightarrow \theta = \tan^{-1} 1 \Rightarrow \theta = \frac{\pi}{4} \text{ or } \theta = \frac{5\pi}{4}$  [either includes the pole]

23.  $y = 1 + 3x \Leftrightarrow r \sin \theta = 1 + 3r \cos \theta \Leftrightarrow r \sin \theta - 3r \cos \theta = 1 \Leftrightarrow r(\sin \theta - 3 \cos \theta) = 1 \Leftrightarrow$   
 $r = \frac{1}{\sin \theta - 3 \cos \theta}$

24.  $4y^2 = x \Leftrightarrow 4(r \sin \theta)^2 = r \cos \theta \Leftrightarrow 4r^2 \sin^2 \theta - r \cos \theta = 0 \Leftrightarrow r(4r \sin^2 \theta - \cos \theta) = 0 \Leftrightarrow r = 0$  or  
 $r = \frac{\cos \theta}{4 \sin^2 \theta} \Leftrightarrow r = 0$  or  $r = \frac{1}{4} \cot \theta \csc \theta$ .  $r = 0$  is included in  $r = \frac{1}{4} \cot \theta \csc \theta$  when  $\theta = \frac{\pi}{2}$ , so the curve is  
 represented by the single equation  $r = \frac{1}{4} \cot \theta \csc \theta$ .

25.  $x^2 + y^2 = 2cx \Leftrightarrow r^2 = 2cr \cos \theta \Leftrightarrow r^2 - 2cr \cos \theta = 0 \Leftrightarrow r(r - 2c \cos \theta) = 0 \Leftrightarrow r = 0$  or  $r = 2c \cos \theta$ .  
 $r = 0$  is included in  $r = 2c \cos \theta$  when  $\theta = \frac{\pi}{2} + n\pi$ , so the curve is represented by the single equation  $r = 2c \cos \theta$ .

26.  $xy = 4 \Leftrightarrow (r \cos \theta)(r \sin \theta) = 4 \Leftrightarrow r^2 \left(\frac{1}{2} \cdot 2 \sin \theta \cos \theta\right) = 4 \Leftrightarrow r^2 \sin 2\theta = 8 \Rightarrow r^2 = 8 \csc 2\theta$

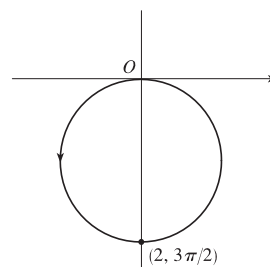
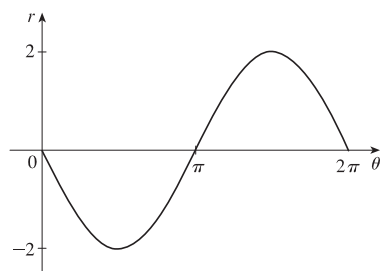
27. (a) The description leads immediately to the polar equation  $\theta = \frac{\pi}{6}$ , and the Cartesian equation  $y = \tan\left(\frac{\pi}{6}\right)x = \frac{1}{\sqrt{3}}x$  is slightly more difficult to derive.

(b) The easier description here is the Cartesian equation  $x = 3$ .

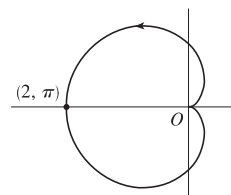
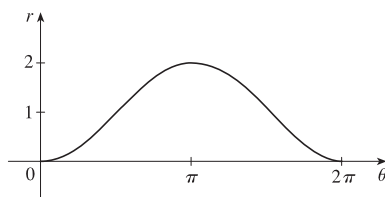
28. (a) Because its center is not at the origin, it is more easily described by its Cartesian equation,  $(x - 2)^2 + (y - 3)^2 = 5^2$ .

(b) This circle is more easily given in polar coordinates:  $r = 4$ . The Cartesian equation is also simple:  $x^2 + y^2 = 16$ .

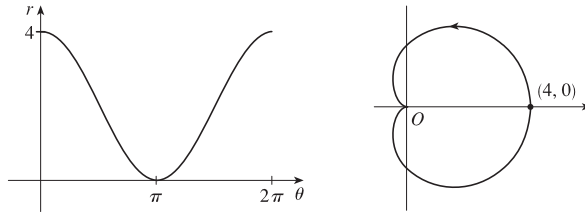
29.  $r = -2 \sin \theta$



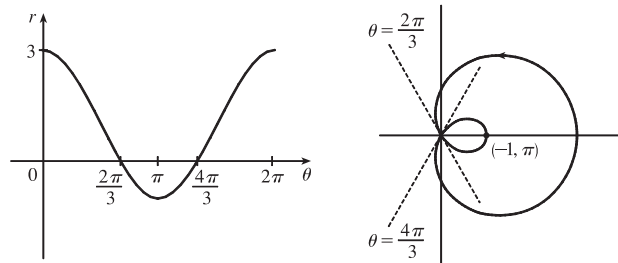
30.  $r = 1 - \cos \theta$



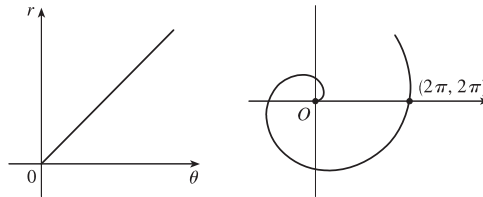
31.  $r = 2(1 + \cos \theta)$



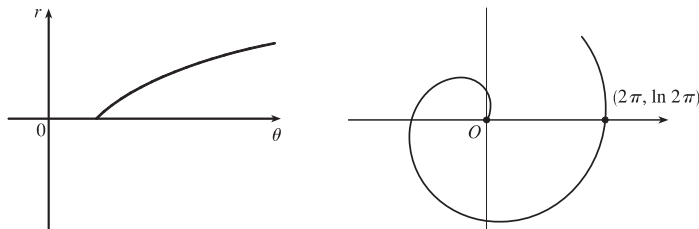
32.  $r = 1 + 2 \cos \theta$



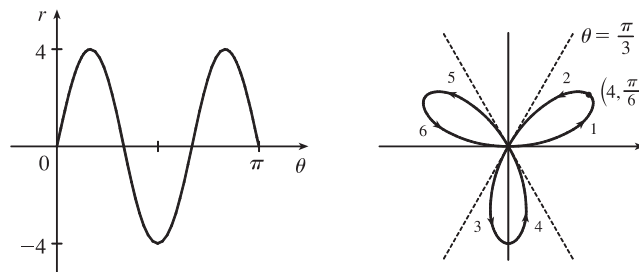
33.  $r = \theta, \theta \geq 0$



34.  $r = \ln \theta, \theta \geq 1$

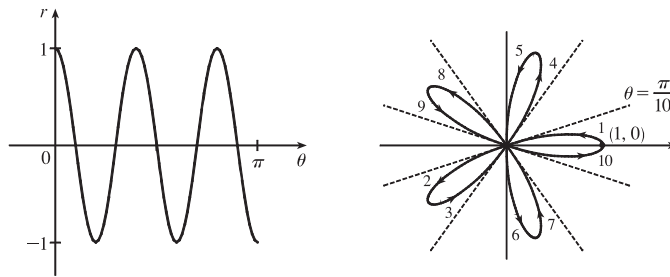


35.  $r = 4 \sin 3\theta$

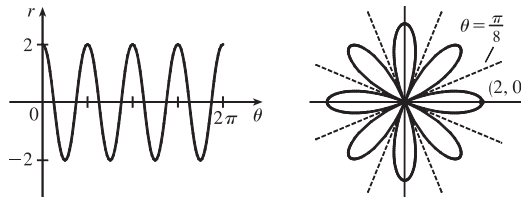




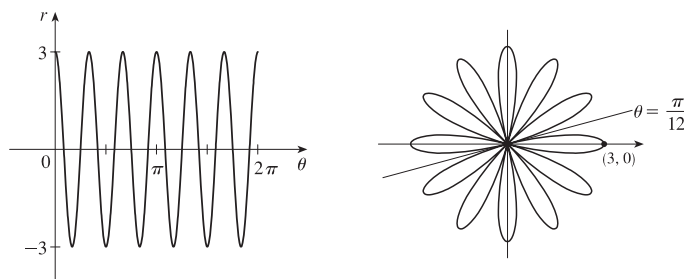
36.  $r = \cos 5\theta$



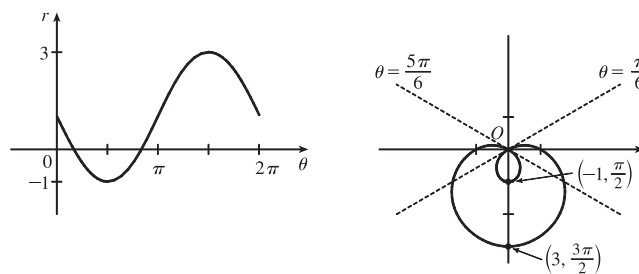
37.  $r = 2 \cos 4\theta$



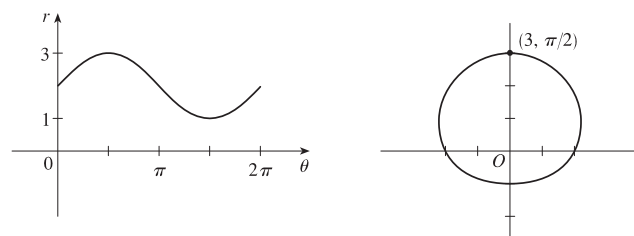
38.  $r = 3 \cos 6\theta$



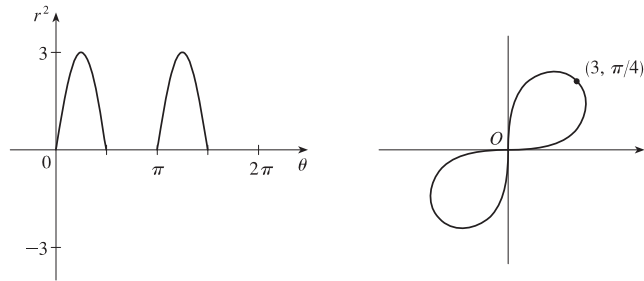
39.  $r = 1 - 2 \sin \theta$



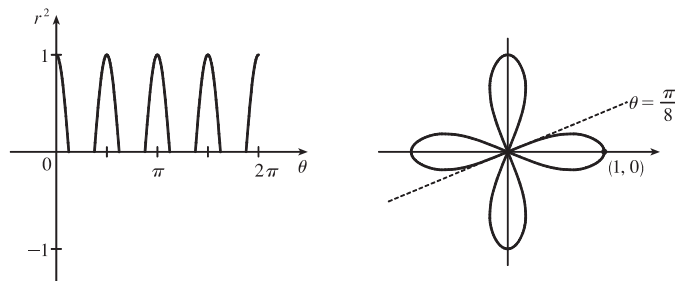
40.  $r = 2 + \sin \theta$



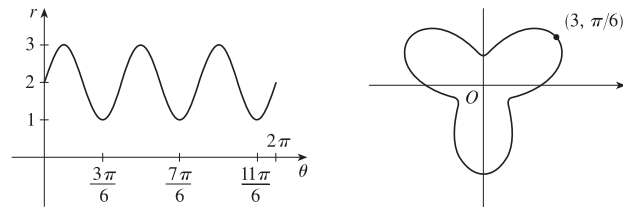
41.  $r^2 = 9 \sin 2\theta$



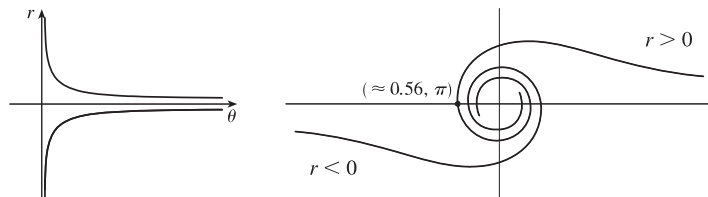
42.  $r^2 = \cos 4\theta$



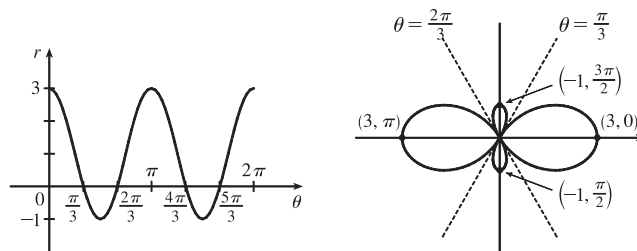
43.  $r = 2 + \sin 3\theta$



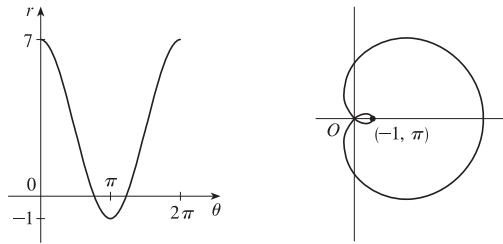
44.  $r^2\theta = 1 \Leftrightarrow r = \pm 1/\sqrt{\theta}$  for  $\theta > 0$



45.  $r = 1 + 2 \cos 2\theta$

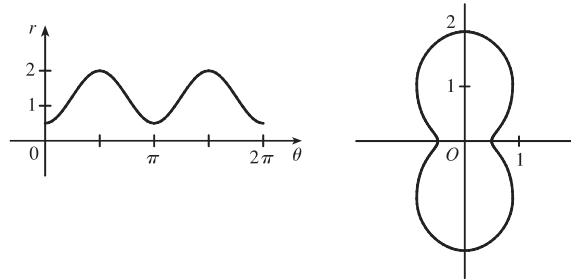


46.  $r = 3 + 4 \cos \theta$

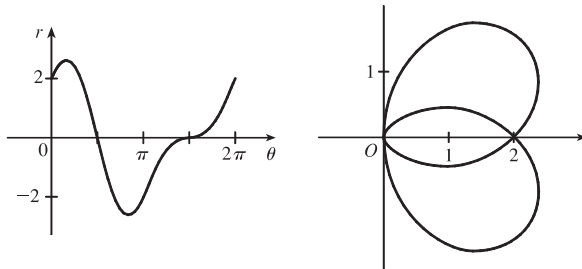


47. For  $\theta = 0, \pi$ , and  $2\pi$ ,  $r$  has its minimum value of about 0.5. For  $\theta = \frac{\pi}{2}$  and  $\frac{3\pi}{2}$ ,  $r$  attains its maximum value of 2.

We see that the graph has a similar shape for  $0 \leq \theta \leq \pi$  and  $\pi \leq \theta \leq 2\pi$ .



48.



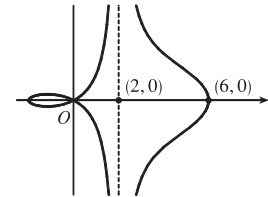
49.  $x = r \cos \theta = (4 + 2 \sec \theta) \cos \theta = 4 \cos \theta + 2$ . Now,  $r \rightarrow \infty \Rightarrow$

$(4 + 2 \sec \theta) \rightarrow \infty \Rightarrow \theta \rightarrow (\frac{\pi}{2})^-$  or  $\theta \rightarrow (\frac{3\pi}{2})^+$  [since we need only

consider  $0 \leq \theta < 2\pi$ ], so  $\lim_{r \rightarrow \infty} x = \lim_{\theta \rightarrow \pi/2^-} (4 \cos \theta + 2) = 2$ . Also,

$r \rightarrow -\infty \Rightarrow (4 + 2 \sec \theta) \rightarrow -\infty \Rightarrow \theta \rightarrow (\frac{\pi}{2})^+$  or  $\theta \rightarrow (\frac{3\pi}{2})^-$ , so

$\lim_{r \rightarrow -\infty} x = \lim_{\theta \rightarrow \pi/2^+} (4 \cos \theta + 2) = 2$ . Therefore,  $\lim_{r \rightarrow \pm\infty} x = 2 \Rightarrow x = 2$  is a vertical asymptote.



50.  $y = r \sin \theta = 2 \sin \theta - \csc \theta \sin \theta = 2 \sin \theta - 1$ .

$r \rightarrow \infty \Rightarrow (2 - \csc \theta) \rightarrow \infty \Rightarrow$

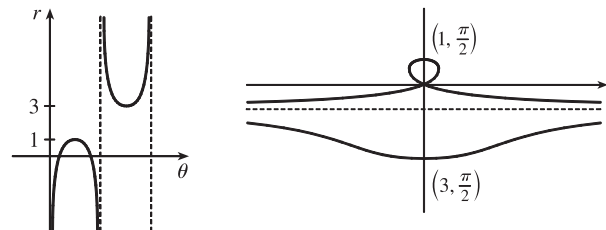
$\csc \theta \rightarrow -\infty \Rightarrow \theta \rightarrow \pi^+$  [since we need

only consider  $0 \leq \theta < 2\pi$ ] and so

$\lim_{r \rightarrow \infty} y = \lim_{\theta \rightarrow \pi^+} 2 \sin \theta - 1 = -1$ .

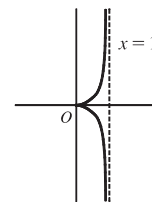
Also  $r \rightarrow -\infty \Rightarrow (2 - \csc \theta) \rightarrow -\infty \Rightarrow \csc \theta \rightarrow \infty \Rightarrow \theta \rightarrow \pi^-$  and so  $\lim_{r \rightarrow -\infty} y = \lim_{\theta \rightarrow \pi^-} 2 \sin \theta - 1 = -1$ .

Therefore  $\lim_{r \rightarrow \pm\infty} y = -1 \Rightarrow y = -1$  is a horizontal asymptote.



51. To show that  $x = 1$  is an asymptote we must prove  $\lim_{r \rightarrow \pm\infty} x = 1$ .

$$x = (r) \cos \theta = (\sin \theta \tan \theta) \cos \theta = \sin^2 \theta. \text{ Now, } r \rightarrow \infty \Rightarrow \sin \theta \tan \theta \rightarrow \infty \Rightarrow \theta \rightarrow \left(\frac{\pi}{2}\right)^-, \text{ so } \lim_{r \rightarrow \infty} x = \lim_{\theta \rightarrow \pi/2^-} \sin^2 \theta = 1. \text{ Also, } r \rightarrow -\infty \Rightarrow \sin \theta \tan \theta \rightarrow -\infty \Rightarrow \theta \rightarrow \left(\frac{\pi}{2}\right)^+, \text{ so } \lim_{r \rightarrow -\infty} x = \lim_{\theta \rightarrow \pi/2^+} \sin^2 \theta = 1. \text{ Therefore, } \lim_{r \rightarrow \pm\infty} x = 1 \Rightarrow x = 1 \text{ is}$$



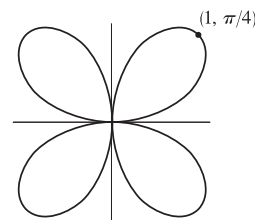
a vertical asymptote. Also notice that  $x = \sin^2 \theta \geq 0$  for all  $\theta$ , and  $x = \sin^2 \theta \leq 1$  for all  $\theta$ . And  $x \neq 1$ , since the curve is not defined at odd multiples of  $\frac{\pi}{2}$ . Therefore, the curve lies entirely within the vertical strip  $0 \leq x < 1$ .

52. The equation is  $(x^2 + y^2)^3 = 4x^2y^2$ , but using polar coordinates we know that

$$x^2 + y^2 = r^2 \text{ and } x = r \cos \theta \text{ and } y = r \sin \theta. \text{ Substituting into the given}$$

$$\text{equation: } r^6 = 4r^2 \cos^2 \theta r^2 \sin^2 \theta \Rightarrow r^2 = 4 \cos^2 \theta \sin^2 \theta \Rightarrow$$

$$r = \pm 2 \cos \theta \sin \theta = \pm \sin 2\theta. \text{ } r = \pm \sin 2\theta \text{ is sketched at right.}$$



53. (a) We see that the curve  $r = 1 + c \sin \theta$  crosses itself at the origin, where  $r = 0$  (in fact the inner loop corresponds to negative  $r$ -values,) so we solve the equation of the limaçon for  $r = 0 \Leftrightarrow c \sin \theta = -1 \Leftrightarrow \sin \theta = -1/c$ . Now if  $|c| < 1$ , then this equation has no solution and hence there is no inner loop. But if  $c < -1$ , then on the interval  $(0, 2\pi)$  the equation has the two solutions  $\theta = \sin^{-1}(-1/c)$  and  $\theta = \pi - \sin^{-1}(-1/c)$ , and if  $c > 1$ , the solutions are  $\theta = \pi + \sin^{-1}(1/c)$  and  $\theta = 2\pi - \sin^{-1}(1/c)$ . In each case,  $r < 0$  for  $\theta$  between the two solutions, indicating a loop.

(b) For  $0 < c < 1$ , the dimple (if it exists) is characterized by the fact that  $y$  has a local maximum at  $\theta = \frac{3\pi}{2}$ . So we

determine for what  $c$ -values  $\frac{d^2y}{d\theta^2}$  is negative at  $\theta = \frac{3\pi}{2}$ , since by the Second Derivative Test this indicates a maximum:

$$y = r \sin \theta = \sin \theta + c \sin^2 \theta \Rightarrow \frac{dy}{d\theta} = \cos \theta + 2c \sin \theta \cos \theta = \cos \theta + c \sin 2\theta \Rightarrow \frac{d^2y}{d\theta^2} = -\sin \theta + 2c \cos 2\theta.$$

At  $\theta = \frac{3\pi}{2}$ , this is equal to  $-(-1) + 2c(-1) = 1 - 2c$ , which is negative only for  $c > \frac{1}{2}$ . A similar argument shows that for  $-1 < c < 0$ ,  $y$  only has a local minimum at  $\theta = \frac{\pi}{2}$  (indicating a dimple) for  $c < -\frac{1}{2}$ .

54. (a)  $r = \sqrt{\theta}$ ,  $0 \leq \theta \leq 16\pi$ .  $r$  increases as  $\theta$  increases and there are eight full revolutions. The graph must be either II or V.

When  $\theta = 2\pi$ ,  $r = \sqrt{2\pi} \approx 2.5$  and when  $\theta = 16\pi$ ,  $r = \sqrt{16\pi} \approx 7$ , so the last revolution intersects the polar axis at approximately 3 times the distance that the first revolution intersects the polar axis, which is depicted in graph V.

(b)  $r = \theta^2$ ,  $0 \leq \theta \leq 16\pi$ . See part (a). This is graph II.

(c)  $r = \cos(\theta/3)$ .  $0 \leq \frac{\theta}{3} \leq 2\pi \Rightarrow 0 \leq \theta \leq 6\pi$ , so this curve will repeat itself every  $6\pi$  radians.

$$\cos\left(\frac{\theta}{3}\right) = 0 \Rightarrow \frac{\theta}{3} = \frac{\pi}{2} + \pi n \Rightarrow \theta = \frac{3\pi}{2} + 3\pi n, \text{ so there will be two "pole" values, } \frac{3\pi}{2} \text{ and } \frac{9\pi}{2}.$$

This is graph VI.

(d)  $r = 1 + 2 \cos \theta$  is a limaçon [see Exercise 53(a)] with  $c = 2$ . This is graph III.

(e) Since  $-1 \leq \sin 3\theta \leq 1$ ,  $1 \leq 2 + \sin 3\theta \leq 3$ , so  $r = 2 + \sin 3\theta$  is never 0; that is, the curve never intersects the pole.

This is graph I.

(f)  $r = 1 + 2 \sin 3\theta$ . Solving  $r = 0$  will give us many “pole” values, so this is graph IV.

$$55. r = 2 \sin \theta \Rightarrow x = r \cos \theta = 2 \sin \theta \cos \theta = \sin 2\theta, y = r \sin \theta = 2 \sin^2 \theta \Rightarrow$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2 \cdot 2 \sin \theta \cos \theta}{\cos 2\theta \cdot 2} = \frac{\sin 2\theta}{\cos 2\theta} = \tan 2\theta$$

$$\text{When } \theta = \frac{\pi}{6}, \frac{dy}{dx} = \tan\left(2 \cdot \frac{\pi}{6}\right) = \tan \frac{\pi}{3} = \sqrt{3}. \quad [\text{Another method: Use Equation 3.}]$$

$$56. r = 2 - \sin \theta \Rightarrow x = r \cos \theta = (2 - \sin \theta) \cos \theta, y = r \sin \theta = (2 - \sin \theta) \sin \theta \Rightarrow$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{(2 - \sin \theta) \cos \theta + \sin \theta(-\cos \theta)}{(2 - \sin \theta)(-\sin \theta) + \cos \theta(-\cos \theta)} = \frac{2 \cos \theta - 2 \sin \theta \cos \theta}{-2 \sin \theta + \sin^2 \theta - \cos^2 \theta} = \frac{2 \cos \theta - \sin 2\theta}{-2 \sin \theta - \cos 2\theta}$$

$$\text{When } \theta = \frac{\pi}{3}, \frac{dy}{dx} = \frac{2(1/2) - (\sqrt{3}/2)}{-2(\sqrt{3}/2) - (-1/2)} = \frac{1 - \sqrt{3}/2}{-\sqrt{3} + 1/2} \cdot \frac{2}{2} = \frac{2 - \sqrt{3}}{1 - 2\sqrt{3}}.$$

$$57. r = 1/\theta \Rightarrow x = r \cos \theta = (\cos \theta)/\theta, y = r \sin \theta = (\sin \theta)/\theta \Rightarrow$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sin \theta(-1/\theta^2) + (1/\theta) \cos \theta}{\cos \theta(-1/\theta^2) - (1/\theta) \sin \theta} \cdot \frac{\theta^2}{\theta^2} = \frac{-\sin \theta + \theta \cos \theta}{-\cos \theta - \theta \sin \theta}$$

$$\text{When } \theta = \pi, \frac{dy}{dx} = \frac{-0 + \pi(-1)}{-(-1) - \pi(0)} = \frac{-\pi}{1} = -\pi.$$

$$58. r = \cos(\theta/3) \Rightarrow x = r \cos \theta = \cos(\theta/3) \cos \theta, y = r \sin \theta = \cos(\theta/3) \sin \theta \Rightarrow$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos(\theta/3) \cos \theta + \sin \theta \left(-\frac{1}{3} \sin(\theta/3)\right)}{\cos(\theta/3) (-\sin \theta) + \cos \theta \left(-\frac{1}{3} \sin(\theta/3)\right)}$$

$$\text{When } \theta = \pi, \frac{dy}{dx} = \frac{\frac{1}{2}(-1) + (0)\left(-\frac{\sqrt{3}}{6}\right)}{\frac{1}{2}(0) + (-1)\left(-\frac{\sqrt{3}}{6}\right)} = \frac{-1/2}{\sqrt{3}/6} = -\frac{3}{\sqrt{3}} = -\sqrt{3}.$$

$$59. r = \cos 2\theta \Rightarrow x = r \cos \theta = \cos 2\theta \cos \theta, y = r \sin \theta = \cos 2\theta \sin \theta \Rightarrow$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos 2\theta \cos \theta + \sin \theta (-2 \sin 2\theta)}{\cos 2\theta (-\sin \theta) + \cos \theta (-2 \sin 2\theta)}$$

$$\text{When } \theta = \frac{\pi}{4}, \frac{dy}{dx} = \frac{0(\sqrt{2}/2) + (\sqrt{2}/2)(-2)}{0(-\sqrt{2}/2) + (\sqrt{2}/2)(-2)} = \frac{-\sqrt{2}}{-\sqrt{2}} = 1.$$

$$60. r = 1 + 2 \cos \theta \Rightarrow x = r \cos \theta = (1 + 2 \cos \theta) \cos \theta, y = r \sin \theta = (1 + 2 \cos \theta) \sin \theta \Rightarrow$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{(1 + 2 \cos \theta) \cos \theta + \sin \theta (-2 \sin \theta)}{(1 + 2 \cos \theta)(-\sin \theta) + \cos \theta (-2 \sin \theta)}$$

$$\text{When } \theta = \frac{\pi}{3}, \frac{dy}{dx} = \frac{2(\frac{1}{2}) + (\sqrt{3}/2)(-\sqrt{3})}{2(-\sqrt{3}/2) + \frac{1}{2}(-\sqrt{3})} \cdot \frac{2}{2} = \frac{2 - 3}{-2\sqrt{3} - \sqrt{3}} = \frac{-1}{-3\sqrt{3}} = \frac{\sqrt{3}}{9}.$$

61.  $r = 3 \cos \theta \Rightarrow x = r \cos \theta = 3 \cos \theta \cos \theta, y = r \sin \theta = 3 \cos \theta \sin \theta \Rightarrow$

$$\frac{dy}{d\theta} = -3 \sin^2 \theta + 3 \cos^2 \theta = 3 \cos 2\theta = 0 \Rightarrow 2\theta = \frac{\pi}{2} \text{ or } \frac{3\pi}{2} \Leftrightarrow \theta = \frac{\pi}{4} \text{ or } \frac{3\pi}{4}.$$

So the tangent is horizontal at  $\left(\frac{3}{\sqrt{2}}, \frac{\pi}{4}\right)$  and  $\left(-\frac{3}{\sqrt{2}}, \frac{3\pi}{4}\right)$  [same as  $\left(\frac{3}{\sqrt{2}}, -\frac{\pi}{4}\right)$ ].

$$\frac{dx}{d\theta} = -6 \sin \theta \cos \theta = -3 \sin 2\theta = 0 \Rightarrow 2\theta = 0 \text{ or } \pi \Leftrightarrow \theta = 0 \text{ or } \frac{\pi}{2}. \text{ So the tangent is vertical at } (3, 0) \text{ and } (0, \frac{\pi}{2}).$$

62.  $r = 1 - \sin \theta \Rightarrow x = r \cos \theta = \cos \theta (1 - \sin \theta), y = r \sin \theta = \sin \theta (1 - \sin \theta) \Rightarrow$

$$\frac{dy}{d\theta} = \sin \theta (-\cos \theta) + (1 - \sin \theta) \cos \theta = \cos \theta (1 - 2 \sin \theta) = 0 \Rightarrow \cos \theta = 0 \text{ or } \sin \theta = \frac{1}{2} \Rightarrow$$

$$\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \text{ or } \frac{3\pi}{2} \Rightarrow \text{horizontal tangent at } \left(\frac{1}{2}, \frac{\pi}{6}\right), \left(\frac{1}{2}, \frac{5\pi}{6}\right), \text{ and } \left(2, \frac{3\pi}{2}\right).$$

$$\begin{aligned} \frac{dx}{d\theta} &= \cos \theta (-\cos \theta) + (1 - \sin \theta)(-\sin \theta) = -\cos^2 \theta - \sin \theta + \sin^2 \theta = 2 \sin^2 \theta - \sin \theta - 1 \\ &= (2 \sin \theta + 1)(\sin \theta - 1) = 0 \Rightarrow \end{aligned}$$

$$\sin \theta = -\frac{1}{2} \text{ or } 1 \Rightarrow \theta = \frac{7\pi}{6}, \frac{11\pi}{6}, \text{ or } \frac{\pi}{2} \Rightarrow \text{vertical tangent at } \left(\frac{3}{2}, \frac{7\pi}{6}\right), \left(\frac{3}{2}, \frac{11\pi}{6}\right), \text{ and } (0, \frac{\pi}{2}).$$

Note that the tangent is vertical, not horizontal, when  $\theta = \frac{\pi}{2}$ , since

$$\lim_{\theta \rightarrow (\pi/2)^-} \frac{dy/d\theta}{dx/d\theta} = \lim_{\theta \rightarrow (\pi/2)^-} \frac{\cos \theta (1 - 2 \sin \theta)}{(2 \sin \theta + 1)(\sin \theta - 1)} = \infty \text{ and } \lim_{\theta \rightarrow (\pi/2)^+} \frac{dy/d\theta}{dx/d\theta} = -\infty.$$

63.  $r = 1 + \cos \theta \Rightarrow x = r \cos \theta = \cos \theta (1 + \cos \theta), y = r \sin \theta = \sin \theta (1 + \cos \theta) \Rightarrow$

$$\frac{dy}{d\theta} = (1 + \cos \theta) \cos \theta - \sin^2 \theta = 2 \cos^2 \theta + \cos \theta - 1 = (2 \cos \theta - 1)(\cos \theta + 1) = 0 \Rightarrow \cos \theta = \frac{1}{2} \text{ or } -1 \Rightarrow$$

$$\theta = \frac{\pi}{3}, \pi, \text{ or } \frac{5\pi}{3} \Rightarrow \text{horizontal tangent at } \left(\frac{3}{2}, \frac{\pi}{3}\right), (0, \pi), \text{ and } \left(\frac{3}{2}, \frac{5\pi}{3}\right).$$

$$\frac{dx}{d\theta} = -(1 + \cos \theta) \sin \theta - \cos \theta \sin \theta = -\sin \theta (1 + 2 \cos \theta) = 0 \Rightarrow \sin \theta = 0 \text{ or } \cos \theta = -\frac{1}{2} \Rightarrow$$

$$\theta = 0, \pi, \frac{2\pi}{3}, \text{ or } \frac{4\pi}{3} \Rightarrow \text{vertical tangent at } (2, 0), \left(\frac{1}{2}, \frac{2\pi}{3}\right), \text{ and } \left(\frac{1}{2}, \frac{4\pi}{3}\right).$$

Note that the tangent is horizontal, not vertical when  $\theta = \pi$ , since  $\lim_{\theta \rightarrow \pi} \frac{dy/d\theta}{dx/d\theta} = 0$ .

64.  $r = e^\theta \Rightarrow x = r \cos \theta = e^\theta \cos \theta, y = r \sin \theta = e^\theta \sin \theta \Rightarrow$

$$\frac{dy}{d\theta} = e^\theta \sin \theta + e^\theta \cos \theta = e^\theta (\sin \theta + \cos \theta) = 0 \Rightarrow \sin \theta = -\cos \theta \Rightarrow \tan \theta = -1 \Rightarrow$$

$$\theta = -\frac{1}{4}\pi + n\pi \text{ [n any integer]} \Rightarrow \text{horizontal tangents at } \left(e^{\pi(n-1/4)}, \pi(n - \frac{1}{4})\right).$$

$$\frac{dx}{d\theta} = e^\theta \cos \theta - e^\theta \sin \theta = e^\theta (\cos \theta - \sin \theta) = 0 \Rightarrow \sin \theta = \cos \theta \Rightarrow \tan \theta = 1 \Rightarrow$$

$$\theta = \frac{1}{4}\pi + n\pi \text{ [n any integer]} \Rightarrow \text{vertical tangents at } \left(e^{\pi(n+1/4)}, \pi(n + \frac{1}{4})\right).$$

65.  $r = a \sin \theta + b \cos \theta \Rightarrow r^2 = ar \sin \theta + br \cos \theta \Rightarrow x^2 + y^2 = ay + bx \Rightarrow$

$$x^2 - bx + \left(\frac{1}{2}b\right)^2 + y^2 - ay + \left(\frac{1}{2}a\right)^2 = \left(\frac{1}{2}b\right)^2 + \left(\frac{1}{2}a\right)^2 \Rightarrow \left(x - \frac{1}{2}b\right)^2 + \left(y - \frac{1}{2}a\right)^2 = \frac{1}{4}(a^2 + b^2), \text{ and this is a circle}$$

with center  $\left(\frac{1}{2}b, \frac{1}{2}a\right)$  and radius  $\frac{1}{2}\sqrt{a^2 + b^2}$ .

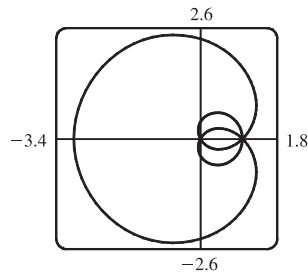
66. These curves are circles which intersect at the origin and at  $\left(\frac{1}{\sqrt{2}}a, \frac{\pi}{4}\right)$ . At the origin, the first circle has a horizontal

tangent and the second a vertical one, so the tangents are perpendicular here. For the first circle [ $r = a \sin \theta$ ],

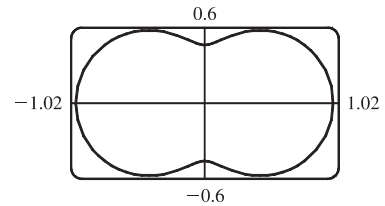
$$dy/d\theta = a \cos \theta \sin \theta + a \sin \theta \cos \theta = a \sin 2\theta = a \text{ at } \theta = \frac{\pi}{4} \text{ and } dx/d\theta = a \cos^2 \theta - a \sin^2 \theta = a \cos 2\theta = 0$$

at  $\theta = \frac{\pi}{4}$ , so the tangent here is vertical. Similarly, for the second circle  $[r = a \cos \theta]$ ,  $dy/d\theta = a \cos 2\theta = 0$  and  $dx/d\theta = -a \sin 2\theta = -a$  at  $\theta = \frac{\pi}{4}$ , so the tangent is horizontal, and again the tangents are perpendicular.

67.  $r = 1 + 2 \sin(\theta/2)$ . The parameter interval is  $[0, 4\pi]$ .

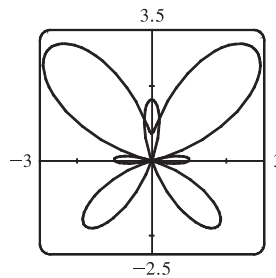


68.  $r = \sqrt{1 - 0.8 \sin^2 \theta}$ . The parameter interval is  $[0, 2\pi]$ .



69.  $r = e^{\sin \theta} - 2 \cos(4\theta)$ .

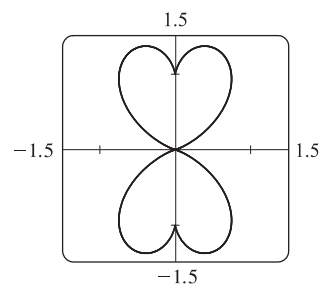
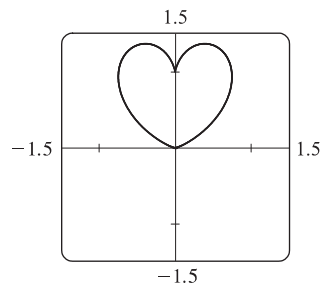
The parameter interval is  $[0, 2\pi]$ .



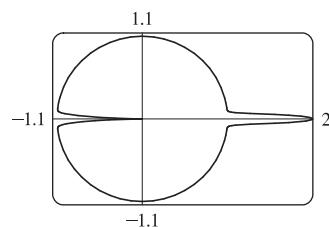
70.  $r = |\tan \theta|^{\cot \theta}$ .

The parameter interval  $[0, \pi]$  produces the heart-shaped valentine curve shown in the first window.

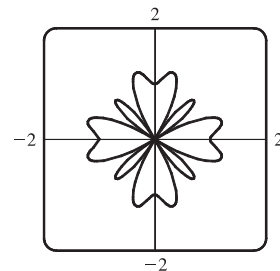
The complete curve, including the reflected heart, is produced by the parameter interval  $[0, 2\pi]$ , but perhaps you'll agree that the first curve is more appropriate.



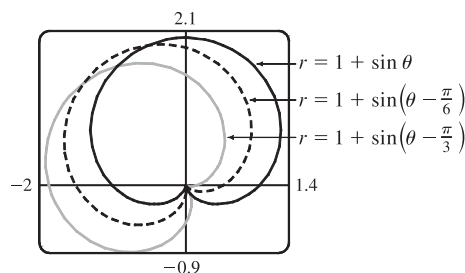
71.  $r = 1 + \cos^{999} \theta$ . The parameter interval is  $[0, 2\pi]$ .



72.  $r = \sin^2(4\theta) + \cos(4\theta)$ . The parameter interval is  $[0, 2\pi]$ .



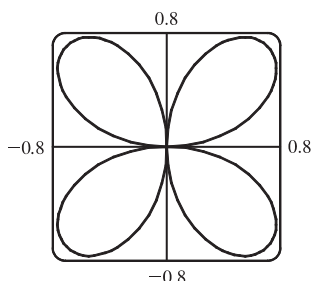
73. It appears that the graph of  $r = 1 + \sin(\theta - \frac{\pi}{6})$  is the same shape as the graph of  $r = 1 + \sin \theta$ , but rotated counterclockwise about the origin by  $\frac{\pi}{6}$ . Similarly, the graph of  $r = 1 + \sin(\theta - \frac{\pi}{3})$  is rotated by  $\frac{\pi}{3}$ . In general, the graph of  $r = f(\theta - \alpha)$  is the same shape as that of  $r = f(\theta)$ , but rotated counterclockwise through  $\alpha$  about the origin.



That is, for any point  $(r_0, \theta_0)$  on the curve  $r = f(\theta)$ , the point

$(r_0, \theta_0 + \alpha)$  is on the curve  $r = f(\theta - \alpha)$ , since  $r_0 = f(\theta_0) = f((\theta_0 + \alpha) - \alpha)$ .

74.



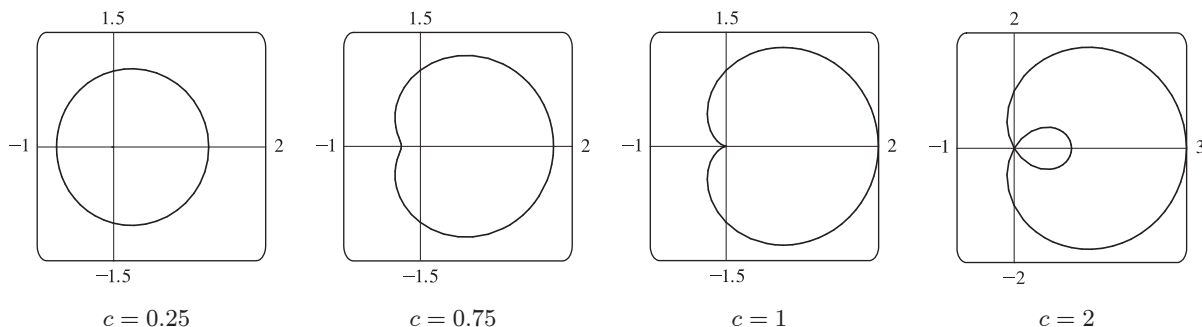
From the graph, the highest points seem to have  $y \approx 0.77$ . To find the exact value, we solve  $dy/d\theta = 0$ .  $y = r \sin \theta = \sin \theta \sin 2\theta \Rightarrow$

$$\begin{aligned} dy/d\theta &= 2 \sin \theta \cos 2\theta + \cos \theta \sin 2\theta \\ &= 2 \sin \theta (2 \cos^2 \theta - 1) + \cos \theta (2 \sin \theta \cos \theta) \\ &= 2 \sin \theta (3 \cos^2 \theta - 1) \end{aligned}$$

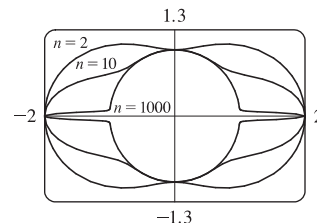
In the first quadrant, this is 0 when  $\cos \theta = \frac{1}{\sqrt{3}} \Leftrightarrow \sin \theta = \sqrt{\frac{2}{3}} \Leftrightarrow$

$$y = 2 \sin^2 \theta \cos \theta = 2 \cdot \frac{2}{3} \cdot \frac{1}{\sqrt{3}} = \frac{4}{9} \sqrt{3} \approx 0.77.$$

75. Consider curves with polar equation  $r = 1 + c \cos \theta$ , where  $c$  is a real number. If  $c = 0$ , we get a circle of radius 1 centered at the pole. For  $0 < c \leq 0.5$ , the curve gets slightly larger, moves right, and flattens out a bit on the left side. For  $0.5 < c < 1$ , the left side has a dimple shape. For  $c = 1$ , the dimple becomes a cusp. For  $c > 1$ , there is an internal loop. For  $c \geq 0$ , the rightmost point on the curve is  $(1 + c, 0)$ . For  $c < 0$ , the curves are reflections through the vertical axis of the curves with  $c > 0$ .

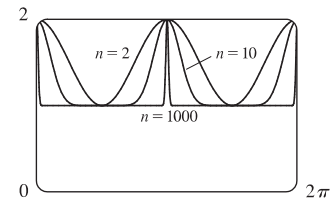


76. Consider the polar curves  $r = 1 + \cos^n \theta$ , where  $n$  is a positive integer. First, let  **$n$  be an even positive integer**. The first figure shows that the curve has a peanut shape for  $n = 2$ , but as  $n$  increases, the ends are squeezed. As  $n$  becomes large, the curves look more and more like the unit circle, but with spikes to the points  $(2, 0)$  and  $(2, \pi)$ .

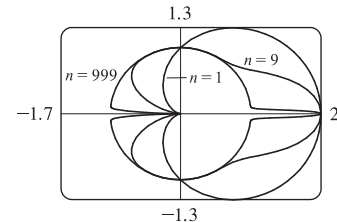




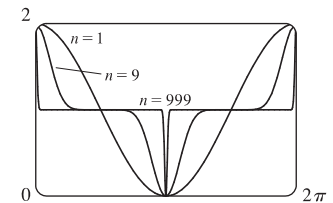
The second figure shows  $r$  as a function of  $\theta$  in Cartesian coordinates for the same values of  $n$ . We can see that for large  $n$ , the graph is similar to the graph of  $y = 1$ , but with spikes to  $y = 2$  for  $x = 0, \pi$ , and  $2\pi$ . (Note that when  $0 < \cos \theta < 1$ ,  $\cos^{1000} \theta$  is very small.)



Next, let  $n$  be an odd positive integer. The third figure shows that the curve is a cardioid for  $n = 1$ , but as  $n$  increases, the heart shape becomes more pronounced. As  $n$  becomes large, the curves again look more like the unit circle, but with an outward spike to  $(2, 0)$  and an inward spike to  $(0, \pi)$ .



The fourth figure shows  $r$  as a function of  $\theta$  in Cartesian coordinates for the same values of  $n$ . We can see that for large  $n$ , the graph is similar to the graph of  $y = 1$ , but spikes to  $y = 2$  for  $x = 0$  and  $\pi$ , and to  $y = 0$  for  $x = \pi$ .

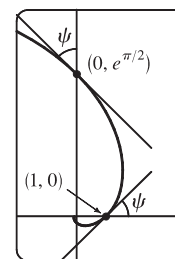


$$\begin{aligned}
 77. \tan \psi &= \tan(\phi - \theta) = \frac{\tan \phi - \tan \theta}{1 + \tan \phi \tan \theta} = \frac{\frac{dy}{dx} - \tan \theta}{1 + \frac{dy}{dx} \tan \theta} = \frac{\frac{dy/d\theta}{dx/d\theta} - \tan \theta}{1 + \frac{dy/d\theta}{dx/d\theta} \tan \theta} \\
 &= \frac{\frac{dy}{d\theta} - \frac{dx}{d\theta} \tan \theta}{\frac{dx}{d\theta} + \frac{dy}{d\theta} \tan \theta} = \frac{\left(\frac{dr}{d\theta} \sin \theta + r \cos \theta\right) - \tan \theta \left(\frac{dr}{d\theta} \cos \theta - r \sin \theta\right)}{\left(\frac{dr}{d\theta} \cos \theta - r \sin \theta\right) + \tan \theta \left(\frac{dr}{d\theta} \sin \theta + r \cos \theta\right)} = \frac{r \cos \theta + r \cdot \frac{\sin^2 \theta}{\cos \theta}}{\frac{dr}{d\theta} \cos \theta + \frac{dr}{d\theta} \cdot \frac{\sin^2 \theta}{\cos \theta}} \\
 &= \frac{r \cos^2 \theta + r \sin^2 \theta}{\frac{dr}{d\theta} \cos^2 \theta + \frac{dr}{d\theta} \sin^2 \theta} = \frac{r}{dr/d\theta}
 \end{aligned}$$

$$78. (a) r = e^\theta \Rightarrow dr/d\theta = e^\theta, \text{ so by Exercise 77, } \tan \psi = r/e^\theta = 1 \Rightarrow \psi = \arctan 1 = \frac{\pi}{4}.$$

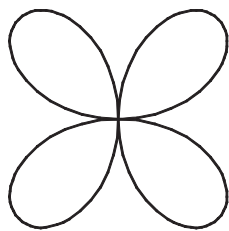
(b) The Cartesian equation of the tangent line at  $(1, 0)$  is  $y = x - 1$ , and that of the tangent line at  $(0, e^{\pi/2})$  is  $y = e^{\pi/2} - x$ .

(c) Let  $a$  be the tangent of the angle between the tangent and radial lines, that is,  $a = \tan \psi$ . Then, by Exercise 77,  $a = \frac{r}{dr/d\theta} \Rightarrow \frac{dr}{d\theta} = \frac{1}{a} r \Rightarrow r = Ce^{\theta/a}$  (by Theorem 9.4.2).

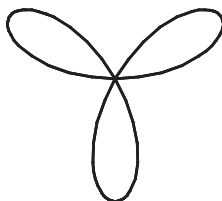


## LABORATORY PROJECT Families of Polar Curves

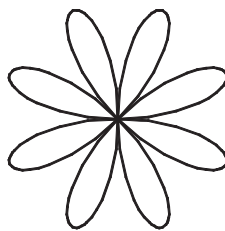
1. (a)  $r = \sin n\theta$ .



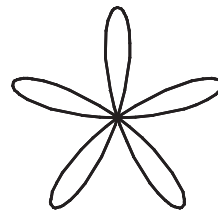
$n = 2$



$n = 3$



$n = 4$

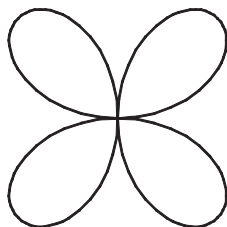


$n = 5$

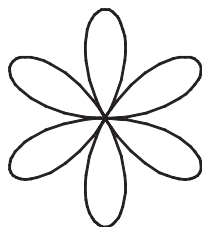
From the graphs, it seems that when  $n$  is even, the number of loops in the curve (called a rose) is  $2n$ , and when  $n$  is odd, the number of loops is simply  $n$ . This is because in the case of  $n$  odd, every point on the graph is traversed twice, due to the fact that

$$r(\theta + \pi) = \sin[n(\theta + \pi)] = \sin n\theta \cos n\pi + \cos n\theta \sin n\pi = \begin{cases} \sin n\theta & \text{if } n \text{ is even} \\ -\sin n\theta & \text{if } n \text{ is odd} \end{cases}$$

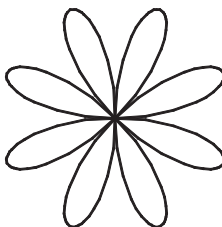
(b) The graph of  $r = |\sin n\theta|$  has  $2n$  loops whether  $n$  is odd or even, since  $r(\theta + \pi) = r(\theta)$ .



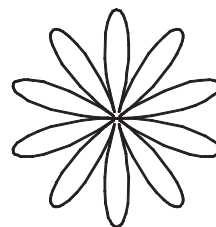
$n = 2$



$n = 3$



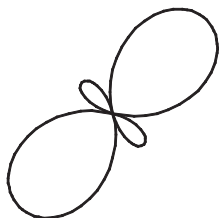
$n = 4$



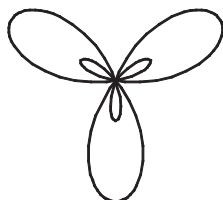
$n = 5$

2.  $r = 1 + c \sin n\theta$ . We vary  $n$  while keeping  $c$  constant at 2. As  $n$  changes, the curves change in the same way as those in Exercise 1: the number of loops increases. Note that if  $n$  is even, the smaller loops are outside the larger ones; if  $n$  is odd, they are inside.

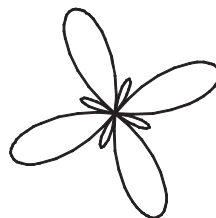
$c = 2$



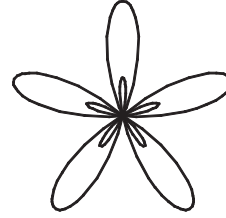
$n = 2$



$n = 3$

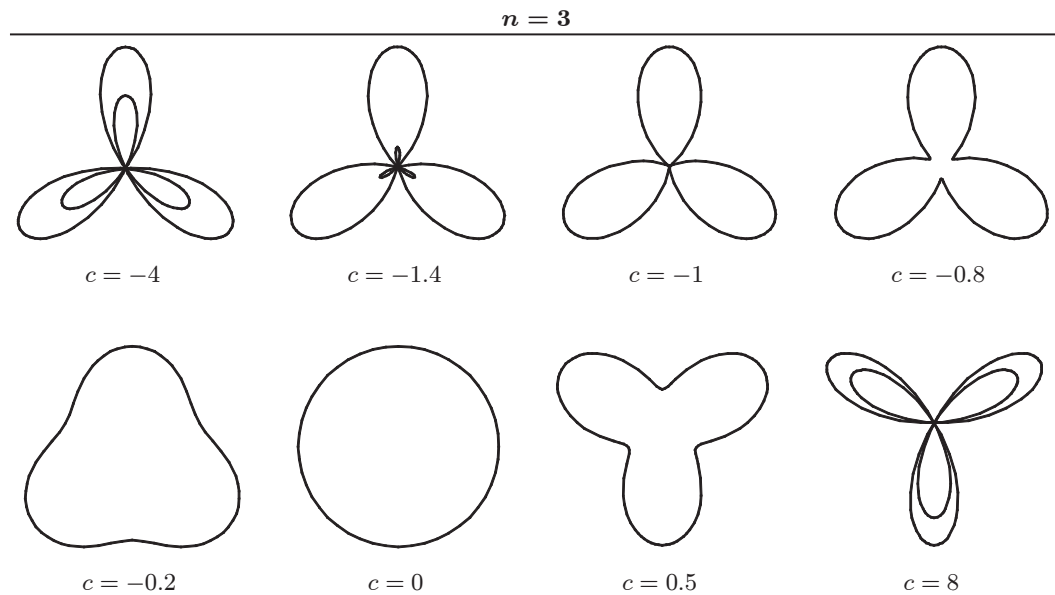


$n = 4$



$n = 5$

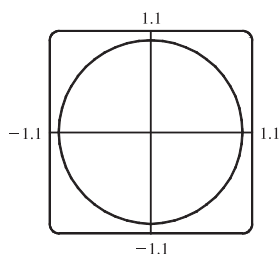
Now we vary  $c$  while keeping  $n = 3$ . As  $c$  increases toward 0, the entire graph gets smaller (the graphs below are not to scale) and the smaller loops shrink in relation to the large ones. At  $c = -1$ , the small loops disappear entirely, and for  $-1 < c < 1$ , the graph is a simple, closed curve (at  $c = 0$  it is a circle). As  $c$  continues to increase, the same changes are seen, but in reverse order, since  $1 + (-c) \sin n\theta = 1 + c \sin n(\theta + \pi)$ , so the graph for  $c = c_0$  is the same as that for  $c = -c_0$ , with a rotation through  $\pi$ . As  $c \rightarrow \infty$ , the smaller loops get relatively closer in size to the large ones. Note that the distance between the outermost points of corresponding inner and outer loops is always 2. Maple's `animate` command (or Mathematica's `Animate`) is very useful for seeing the changes that occur as  $c$  varies.



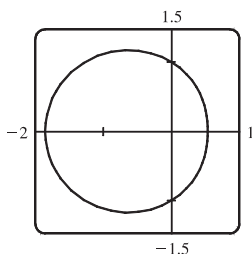
3.  $r = \frac{1 - a \cos \theta}{1 + a \cos \theta}$ . We start with  $a = 0$ , since in this case the curve is simply the circle  $r = 1$ .

As  $a$  increases, the graph moves to the left, and its right side becomes flattened. As  $a$  increases through about 0.4, the right side seems to grow a dimple, which upon closer investigation (with narrower  $\theta$ -ranges) seems to appear at  $a \approx 0.42$  [the actual value is  $\sqrt{2} - 1$ ]. As  $a \rightarrow 1$ , this dimple becomes more pronounced, and the curve begins to stretch out horizontally, until at  $a = 1$  the denominator vanishes at  $\theta = \pi$ , and the dimple becomes an actual cusp. For  $a > 1$  we must choose our parameter interval carefully, since  $r \rightarrow \infty$  as  $1 + a \cos \theta \rightarrow 0 \Leftrightarrow \theta \rightarrow \pm \cos^{-1}(-1/a)$ . As  $a$  increases from 1, the curve splits into two parts. The left part has a loop, which grows larger as  $a$  increases, and the right part grows broader vertically, and its left tip develops a dimple when  $a \approx 2.42$  [actually,  $\sqrt{2} + 1$ ]. As  $a$  increases, the dimple grows more and more pronounced. If  $a < 0$ , we get the same graph as we do for the corresponding positive  $a$ -value, but with a rotation through  $\pi$  about the pole, as happened when  $c$  was replaced with  $-c$  in Exercise 2.

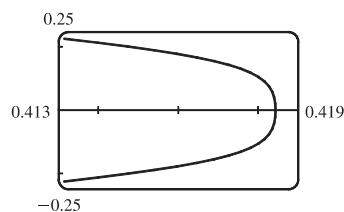
[continued]



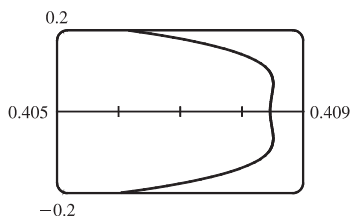
$$a = 0$$



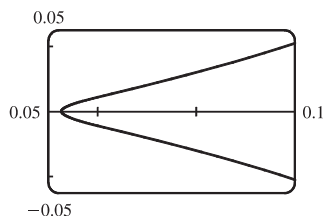
$$a = 0.3$$



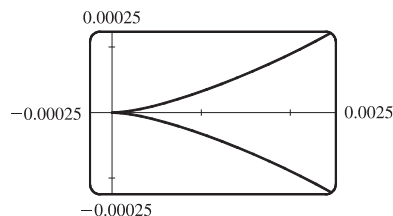
$$a = 0.41, |\theta| \leq 0.5$$



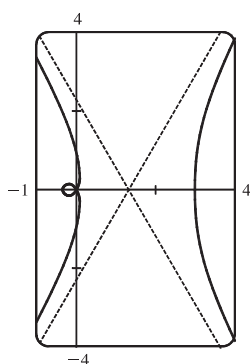
$$a = 0.42, |\theta| \leq 0.5$$



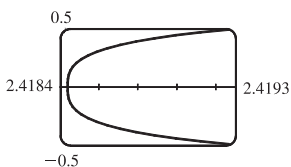
$$a = 0.9, |\theta| \leq 0.5$$



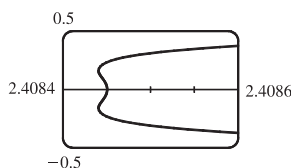
$$a = 1, |\theta| \leq 0.1$$



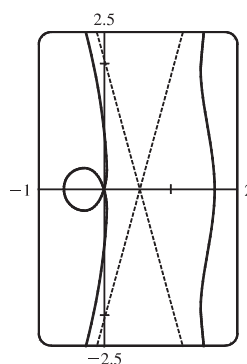
$$a = 2$$



$$a = 2.41, |\theta - \pi| \leq 0.2$$



$$a = 2.42, |\theta - \pi| \leq 0.2$$



$$a = 4$$

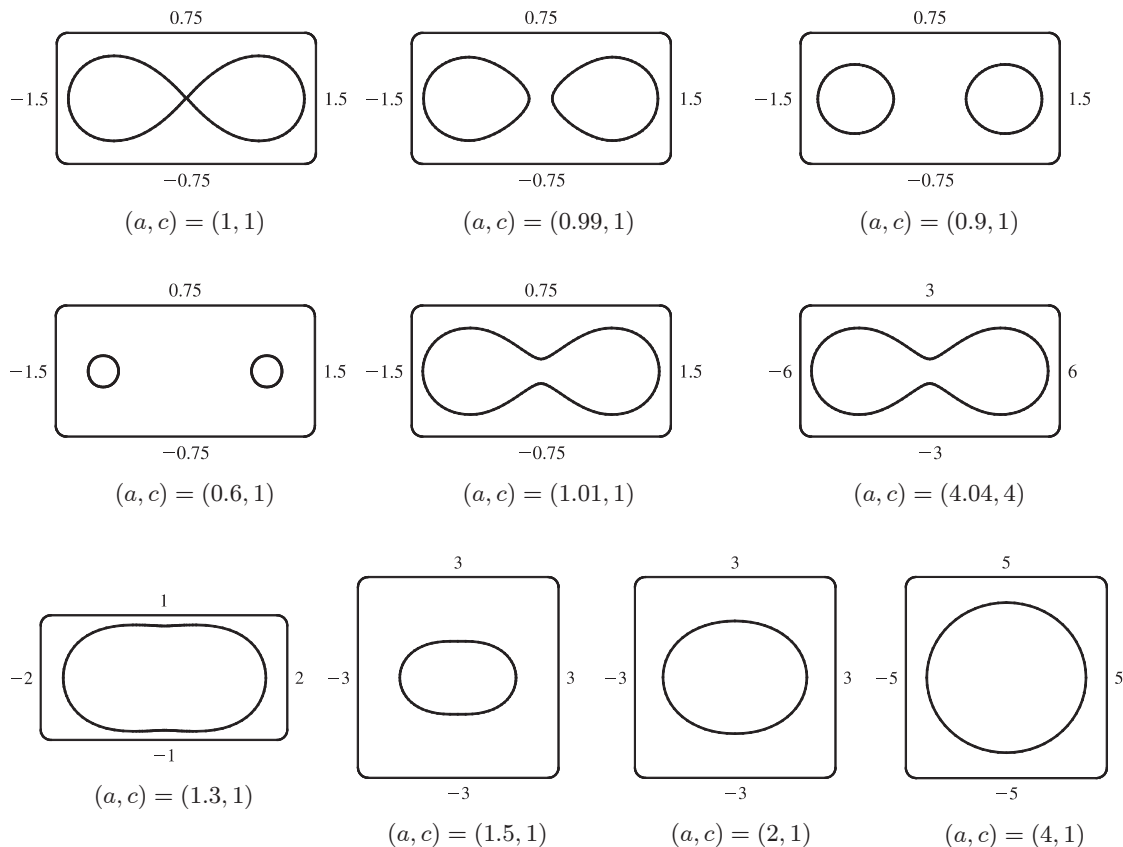
4. Most graphing devices cannot plot implicit polar equations, so we must first find an explicit expression (or expressions) for  $r$  in terms of  $\theta$ ,  $a$ , and  $c$ . We note that the given equation,  $r^4 - 2c^2r^2 \cos 2\theta + c^4 - a^4 = 0$ , is a quadratic in  $r^2$ , so we use the quadratic formula and find that

$$r^2 = \frac{2c^2 \cos 2\theta \pm \sqrt{4c^4 \cos^2 2\theta - 4(c^4 - a^4)}}{2} = c^2 \cos 2\theta \pm \sqrt{a^4 - c^4 \sin^2 2\theta}$$

so  $r = \pm \sqrt{c^2 \cos 2\theta \pm \sqrt{a^4 - c^4 \sin^2 2\theta}}$ . So for each graph, we must plot four curves to be sure of plotting all the points which satisfy the given equation. Note that all four functions have period  $\pi$ .

We start with the case  $a = c = 1$ , and the resulting curve resembles the symbol for infinity. If we let  $a$  decrease, the curve splits into two symmetric parts, and as  $a$  decreases further, the parts become smaller, further apart, and rounder. If instead we let  $a$  increase from 1, the two lobes of the curve join together, and as  $a$  increases further they continue to merge, until at

$a \approx 1.4$ , the graph no longer has dimples, and has an oval shape. As  $a \rightarrow \infty$ , the oval becomes larger and rounder, since the  $c^2$  and  $c^4$  terms lose their significance. Note that the shape of the graph seems to depend only on the ratio  $c/a$ , while the size of the graph varies as  $c$  and  $a$  jointly increase.



## 10.4 Areas and Lengths in Polar Coordinates

1.  $r = e^{-\theta/4}$ ,  $\pi/2 \leq \theta \leq \pi$ .

$$A = \int_{\pi/2}^{\pi} \frac{1}{2} r^2 d\theta = \int_{\pi/2}^{\pi} \frac{1}{2} (e^{-\theta/4})^2 d\theta = \int_{\pi/2}^{\pi} \frac{1}{2} e^{-\theta/2} d\theta = \frac{1}{2} \left[ -2e^{-\theta/2} \right]_{\pi/2}^{\pi} = -1(e^{-\pi/2} - e^{-\pi/4}) = e^{-\pi/4} - e^{-\pi/2}$$

2.  $r = \cos \theta$ ,  $0 \leq \theta \leq \pi/6$ .

$$A = \int_0^{\pi/6} \frac{1}{2} r^2 d\theta = \int_0^{\pi/6} \frac{1}{2} \cos^2 \theta d\theta = \frac{1}{2} \int_0^{\pi/6} \frac{1}{2} (1 + \cos 2\theta) d\theta = \frac{1}{4} \left[ \theta + \frac{1}{2} \sin 2\theta \right]_0^{\pi/6} \\ = \frac{1}{4} \left( \frac{\pi}{6} + \frac{1}{2} \cdot \frac{1}{2} \sqrt{3} \right) = \frac{\pi}{24} + \frac{1}{16} \sqrt{3}$$

3.  $r^2 = 9 \sin 2\theta$ ,  $r \geq 0$ ,  $0 \leq \theta \leq \pi/2$ .

$$A = \int_0^{\pi/2} \frac{1}{2} r^2 d\theta = \int_0^{\pi/2} \frac{1}{2} (9 \sin 2\theta) d\theta = \frac{9}{2} \left[ -\frac{1}{2} \cos 2\theta \right]_0^{\pi/2} = -\frac{9}{4} (-1 - 1) = \frac{9}{2}$$

4.  $r = \tan \theta$ ,  $\pi/6 \leq \theta \leq \pi/3$ .

$$\begin{aligned} A &= \int_{\pi/6}^{\pi/3} \frac{1}{2} r^2 d\theta = \int_{\pi/6}^{\pi/3} \frac{1}{2} \tan^2 \theta d\theta = \int_{\pi/6}^{\pi/3} \frac{1}{2} (\sec^2 \theta - 1) d\theta = \frac{1}{2} [\tan \theta - \theta]_{\pi/6}^{\pi/3} \\ &= \frac{1}{2} \left[ \left( \sqrt{3} - \frac{\pi}{3} \right) - \left( \frac{1}{3} \sqrt{3} - \frac{\pi}{6} \right) \right] = \frac{1}{2} \left[ \frac{2}{3} \sqrt{3} - \frac{\pi}{6} \right] = \frac{1}{3} \sqrt{3} - \frac{\pi}{12} \end{aligned}$$

5.  $r = \sqrt{\theta}$ ,  $0 \leq \theta \leq 2\pi$ .  $A = \int_0^{2\pi} \frac{1}{2} r^2 d\theta = \int_0^{2\pi} \frac{1}{2} (\sqrt{\theta})^2 d\theta = \int_0^{2\pi} \frac{1}{2} \theta d\theta = \left[ \frac{1}{4} \theta^2 \right]_0^{2\pi} = \pi^2$

6.  $r = 1 + \cos \theta$ ,  $0 \leq \theta \leq \pi$ .

$$\begin{aligned} A &= \int_0^\pi \frac{1}{2} (1 + \cos \theta)^2 d\theta = \frac{1}{2} \int_0^\pi (1 + 2 \cos \theta + \cos^2 \theta) d\theta = \frac{1}{2} \int_0^\pi \left[ 1 + 2 \cos \theta + \frac{1}{2} (1 + \cos 2\theta) \right] d\theta \\ &= \frac{1}{2} \int_0^\pi \left( \frac{3}{2} + 2 \cos \theta + \frac{1}{2} \cos 2\theta \right) d\theta = \frac{1}{2} \left[ \frac{3}{2} \theta + 2 \sin \theta + \frac{1}{4} \sin 2\theta \right]_0^\pi = \frac{1}{2} \left( \frac{3}{2} \pi + 0 + 0 \right) - \frac{1}{2} (0) = \frac{3\pi}{4} \end{aligned}$$

7.  $r = 4 + 3 \sin \theta$ ,  $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ .

$$\begin{aligned} A &= \int_{-\pi/2}^{\pi/2} \frac{1}{2} ((4 + 3 \sin \theta)^2) d\theta = \frac{1}{2} \int_{-\pi/2}^{\pi/2} (16 + 24 \sin \theta + 9 \sin^2 \theta) d\theta \\ &= \frac{1}{2} \int_{-\pi/2}^{\pi/2} (16 + 9 \sin^2 \theta) d\theta \quad [\text{by Theorem 4.5.6(b) [ET 5.5.7(b)]}] \\ &= \frac{1}{2} \cdot 2 \int_0^{\pi/2} [16 + 9 \cdot \frac{1}{2} (1 - \cos 2\theta)] d\theta \quad [\text{by Theorem 4.5.6(a) [ET 5.5.7(a)]}] \\ &= \int_0^{\pi/2} \left( \frac{41}{2} - \frac{9}{2} \cos 2\theta \right) d\theta = \left[ \frac{41}{2} \theta - \frac{9}{4} \sin 2\theta \right]_0^{\pi/2} = \left( \frac{41\pi}{4} - 0 \right) - (0 - 0) = \frac{41\pi}{4} \end{aligned}$$

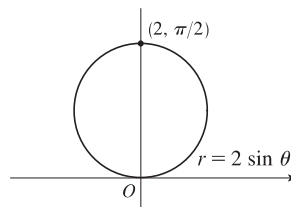
8.  $r = \sin 2\theta$ ,  $0 \leq \theta \leq \frac{\pi}{2}$ .

$$A = \int_0^{\pi/2} \frac{1}{2} \sin^2 2\theta d\theta = \frac{1}{2} \int_0^{\pi/2} \frac{1}{2} (1 - \cos 4\theta) d\theta = \frac{1}{4} \left[ \theta - \frac{1}{4} \sin 4\theta \right]_0^{\pi/2} = \frac{1}{4} \left( \frac{\pi}{2} \right) = \frac{\pi}{8}$$

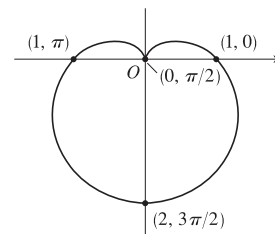
9. The area is bounded by  $r = 2 \sin \theta$  for  $\theta = 0$  to  $\theta = \pi$ .

$$\begin{aligned} A &= \int_0^\pi \frac{1}{2} r^2 d\theta = \frac{1}{2} \int_0^\pi (2 \sin \theta)^2 d\theta = \frac{1}{2} \int_0^\pi 4 \sin^2 \theta d\theta \\ &= 2 \int_0^\pi \frac{1}{2} (1 - \cos 2\theta) d\theta = \left[ \theta - \frac{1}{2} \sin 2\theta \right]_0^\pi = \pi \end{aligned}$$

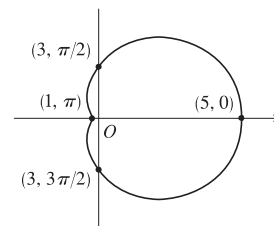
Also, note that this is a circle with radius 1, so its area is  $\pi(1)^2 = \pi$ .



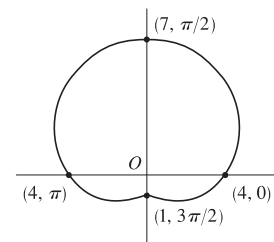
$$\begin{aligned} 10. \quad A &= \int_0^{2\pi} \frac{1}{2} r^2 d\theta = \int_0^{2\pi} \frac{1}{2} (1 - \sin \theta)^2 d\theta \\ &= \frac{1}{2} \int_0^{2\pi} (1 - 2 \sin \theta + \sin^2 \theta) d\theta = \frac{1}{2} \int_0^{2\pi} \left[ 1 - 2 \sin \theta + \frac{1}{2} (1 - \cos 2\theta) \right] d\theta \\ &= \frac{1}{2} \int_0^{2\pi} \left( \frac{3}{2} - 2 \sin \theta - \frac{1}{2} \cos 2\theta \right) d\theta = \frac{1}{2} \left[ \frac{3}{2} \theta + 2 \cos \theta - \frac{1}{4} \sin 2\theta \right]_0^{2\pi} \\ &= \frac{1}{2} [(3\pi + 2) - (2)] = \frac{3\pi}{2} \end{aligned}$$



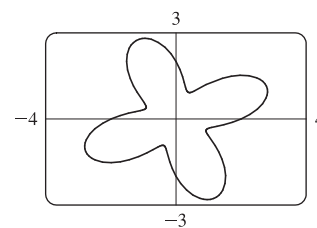
$$\begin{aligned}
 11. \quad A &= \int_0^{2\pi} \frac{1}{2} r^2 d\theta = \int_0^{2\pi} \frac{1}{2} (3 + 2 \cos \theta)^2 d\theta = \frac{1}{2} \int_0^{2\pi} (9 + 12 \cos \theta + 4 \cos^2 \theta) d\theta \\
 &= \frac{1}{2} \int_0^{2\pi} \left[ 9 + 12 \cos \theta + 4 \cdot \frac{1}{2} (1 + \cos 2\theta) \right] d\theta \\
 &= \frac{1}{2} \int_0^{2\pi} (11 + 12 \cos \theta + 2 \cos 2\theta) d\theta = \frac{1}{2} [11\theta + 12 \sin \theta + \sin 2\theta]_0^{2\pi} \\
 &= \frac{1}{2} (22\pi) = 11\pi
 \end{aligned}$$



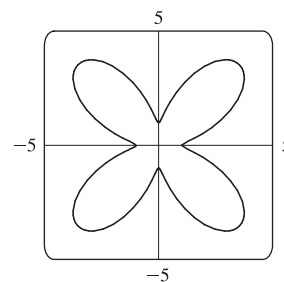
$$\begin{aligned}
 12. \quad A &= \int_0^{2\pi} \frac{1}{2} r^2 d\theta = \int_0^{2\pi} \frac{1}{2} (4 + 3 \sin \theta)^2 d\theta = \frac{1}{2} \int_0^{2\pi} (16 + 24 \sin \theta + 9 \sin^2 \theta) d\theta \\
 &= \frac{1}{2} \int_0^{2\pi} \left[ 16 + 24 \sin \theta + 9 \cdot \frac{1}{2} (1 - \cos 2\theta) \right] d\theta \\
 &= \frac{1}{2} \int_0^{2\pi} \left( \frac{41}{2} + 24 \sin \theta - \frac{9}{2} \cos 2\theta \right) d\theta = \frac{1}{2} \left[ \frac{41}{2} \theta - 24 \cos \theta - \frac{9}{4} \sin 2\theta \right]_0^{2\pi} \\
 &= \frac{1}{2} [(41\pi - 24) - (-24)] = \frac{41}{2} \pi
 \end{aligned}$$



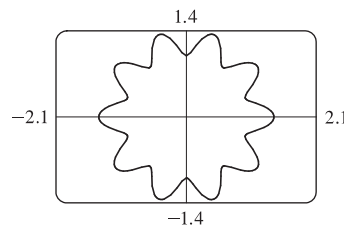
$$\begin{aligned}
 13. \quad A &= \int_0^{2\pi} \frac{1}{2} r^2 d\theta = \int_0^{2\pi} \frac{1}{2} (2 + \sin 4\theta)^2 d\theta = \frac{1}{2} \int_0^{2\pi} (4 + 4 \sin 4\theta + \sin^2 4\theta) d\theta \\
 &= \frac{1}{2} \int_0^{2\pi} \left[ 4 + 4 \sin 4\theta + \frac{1}{2} (1 - \cos 8\theta) \right] d\theta \\
 &= \frac{1}{2} \int_0^{2\pi} \left( \frac{9}{2} + 4 \sin 4\theta - \frac{1}{2} \cos 8\theta \right) d\theta = \frac{1}{2} \left[ \frac{9}{2} \theta - \cos 4\theta - \frac{1}{16} \sin 8\theta \right]_0^{2\pi} \\
 &= \frac{1}{2} [(9\pi - 1) - (-1)] = \frac{9}{2} \pi
 \end{aligned}$$



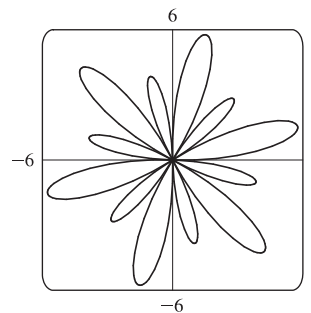
$$\begin{aligned}
 14. \quad A &= \int_0^{2\pi} \frac{1}{2} r^2 d\theta = \int_0^{2\pi} \frac{1}{2} (3 - 2 \cos 4\theta)^2 d\theta = \frac{1}{2} \int_0^{2\pi} (9 - 12 \cos 4\theta + 4 \cos^2 4\theta) d\theta \\
 &= \frac{1}{2} \int_0^{2\pi} \left[ 9 - 12 \cos 4\theta + 4 \cdot \frac{1}{2} (1 + \cos 8\theta) \right] d\theta \\
 &= \frac{1}{2} \int_0^{2\pi} (11 - 12 \cos 4\theta + 2 \cos 8\theta) d\theta = \frac{1}{2} [11\theta - 3 \sin 4\theta + \frac{1}{4} \sin 8\theta]_0^{2\pi} \\
 &= \frac{1}{2} (22\pi) = 11\pi
 \end{aligned}$$



$$\begin{aligned}
 15. \quad A &= \int_0^{2\pi} \frac{1}{2} r^2 d\theta = \int_0^{2\pi} \frac{1}{2} (\sqrt{1 + \cos^2 5\theta})^2 d\theta \\
 &= \frac{1}{2} \int_0^{2\pi} (1 + \cos^2 5\theta) d\theta = \frac{1}{2} \int_0^{2\pi} \left[ 1 + \frac{1}{2} (1 + \cos 10\theta) \right] d\theta \\
 &= \frac{1}{2} \left[ \frac{3}{2} \theta + \frac{1}{20} \sin 10\theta \right]_0^{2\pi} = \frac{1}{2} (3\pi) = \frac{3}{2} \pi
 \end{aligned}$$



$$\begin{aligned}
 16. \quad A &= \int_0^{2\pi} \frac{1}{2} r^2 d\theta = \int_0^{2\pi} \frac{1}{2} (1 + 5 \sin 6\theta)^2 d\theta \\
 &= \frac{1}{2} \int_0^{2\pi} (1 + 10 \sin 6\theta + 25 \sin^2 6\theta) d\theta \\
 &= \frac{1}{2} \int_0^{2\pi} \left[ 1 + 10 \sin 6\theta + 25 \cdot \frac{1}{2} (1 - \cos 12\theta) \right] d\theta \\
 &= \frac{1}{2} \int_0^{2\pi} \left[ \frac{27}{2} + 10 \sin 6\theta - \frac{25}{2} \cos 12\theta \right] d\theta = \frac{1}{2} \left[ \frac{27}{2} \theta - \frac{5}{3} \cos 6\theta - \frac{25}{24} \sin 12\theta \right]_0^{2\pi} \\
 &= \frac{1}{2} \left[ (27\pi - \frac{5}{3}) - (-\frac{5}{3}) \right] = \frac{27}{2} \pi
 \end{aligned}$$

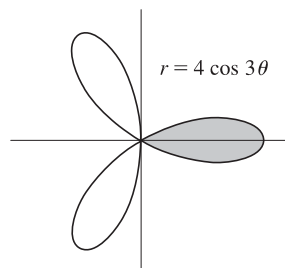


17. The curve passes through the pole when  $r = 0 \Rightarrow 4 \cos 3\theta = 0 \Rightarrow \cos 3\theta = 0 \Rightarrow 3\theta = \frac{\pi}{2} + \pi n \Rightarrow$

$\theta = \frac{\pi}{6} + \frac{\pi}{3}n$ . The part of the shaded loop above the polar axis is traced out for

$\theta = 0$  to  $\theta = \pi/6$ , so we'll use  $-\pi/6$  and  $\pi/6$  as our limits of integration.

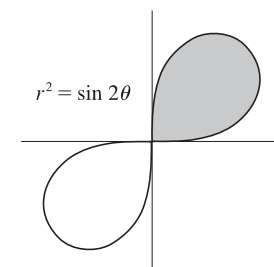
$$\begin{aligned}
 A &= \int_{-\pi/6}^{\pi/6} \frac{1}{2} (4 \cos 3\theta)^2 d\theta = 2 \int_{-\pi/6}^{\pi/6} \frac{1}{2} (16 \cos^2 3\theta) d\theta \\
 &= 16 \int_{-\pi/6}^{\pi/6} \frac{1}{2} (1 + \cos 6\theta) d\theta = 8 \left[ \theta + \frac{1}{6} \sin 6\theta \right]_{-\pi/6}^{\pi/6} = 8 \left( \frac{\pi}{6} \right) = \frac{4}{3} \pi
 \end{aligned}$$



18. For  $\theta = 0$  to  $\theta = \pi/2$ , the shaded loop is traced out by  $r = \sqrt{\sin 2\theta}$  and the

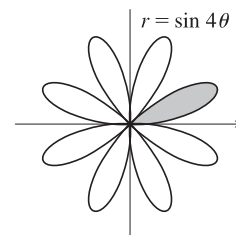
unshaded loop is traced out by  $r = -\sqrt{\sin 2\theta}$ .

$$\begin{aligned}
 A &= \int_0^{\pi/2} \frac{1}{2} r^2 d\theta = \int_0^{\pi/2} \frac{1}{2} \sin 2\theta d\theta \\
 &= \left[ -\frac{1}{4} \cos 2\theta \right]_0^{\pi/2} = \frac{1}{4} - \left( -\frac{1}{4} \right) = \frac{1}{2}
 \end{aligned}$$



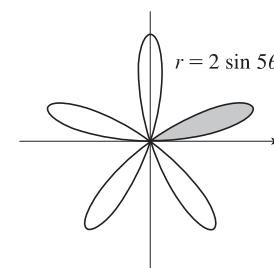
19.  $r = 0 \Rightarrow \sin 4\theta = 0 \Rightarrow 4\theta = \pi n \Rightarrow \theta = \frac{\pi}{4}n$ .

$$\begin{aligned}
 A &= \int_0^{\pi/4} \frac{1}{2} (\sin 4\theta)^2 d\theta = \frac{1}{2} \int_0^{\pi/4} \sin^2 4\theta d\theta = \frac{1}{2} \int_0^{\pi/4} \frac{1}{2} (1 - \cos 8\theta) d\theta \\
 &= \frac{1}{4} \left[ \theta - \frac{1}{8} \sin 8\theta \right]_0^{\pi/4} = \frac{1}{4} \left( \frac{\pi}{4} \right) = \frac{1}{16} \pi
 \end{aligned}$$



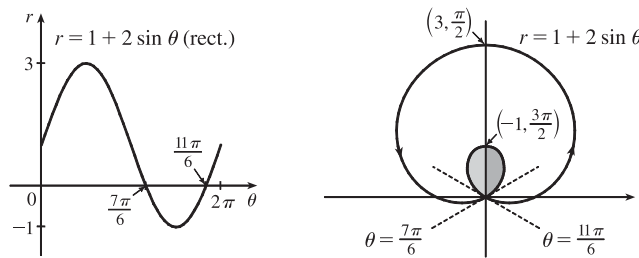
20.  $r = 0 \Rightarrow 2 \sin 5\theta = 0 \Rightarrow \sin 5\theta = 0 \Rightarrow 5\theta = \pi n \Rightarrow \theta = \frac{\pi}{5}n$ .

$$\begin{aligned}
 A &= \int_0^{\pi/5} \frac{1}{2} (2 \sin 5\theta)^2 d\theta = \frac{1}{2} \int_0^{\pi/5} 4 \sin^2 5\theta d\theta \\
 &= 2 \int_0^{\pi/5} \frac{1}{2} (1 - \cos 10\theta) d\theta = \left[ \theta - \frac{1}{10} \sin 10\theta \right]_0^{\pi/5} = \frac{\pi}{5}
 \end{aligned}$$





21.



This is a limaçon, with inner loop traced out between  $\theta = \frac{7\pi}{6}$  and  $\frac{11\pi}{6}$  [found by solving  $r = 0$ ].

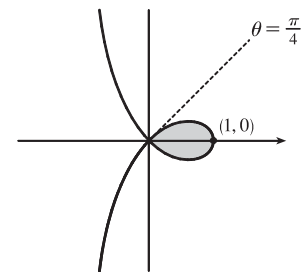
$$\begin{aligned} A &= 2 \int_{7\pi/6}^{3\pi/2} \frac{1}{2} (1 + 2 \sin \theta)^2 d\theta = \int_{7\pi/6}^{3\pi/2} (1 + 4 \sin \theta + 4 \sin^2 \theta) d\theta = \int_{7\pi/6}^{3\pi/2} [1 + 4 \sin \theta + 4 \cdot \frac{1}{2} (1 - \cos 2\theta)] d\theta \\ &= [\theta - 4 \cos \theta + 2\theta - \sin 2\theta]_{7\pi/6}^{3\pi/2} = \left(\frac{9\pi}{2}\right) - \left(\frac{7\pi}{2} + 2\sqrt{3} - \frac{\sqrt{3}}{2}\right) = \pi - \frac{3\sqrt{3}}{2} \end{aligned}$$

22. To determine when the strophoid  $r = 2 \cos \theta - \sec \theta$  passes through the pole, we solve

$$r = 0 \Rightarrow 2 \cos \theta - \frac{1}{\cos \theta} = 0 \Rightarrow 2 \cos^2 \theta - 1 = 0 \Rightarrow \cos^2 \theta = \frac{1}{2} \Rightarrow$$

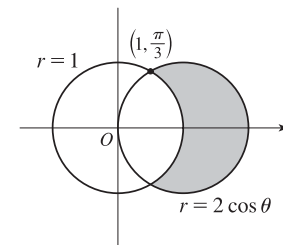
$$\cos \theta = \pm \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4} \text{ or } \theta = \frac{3\pi}{4} \text{ for } 0 \leq \theta \leq \pi \text{ with } \theta \neq \frac{\pi}{2}.$$

$$\begin{aligned} A &= 2 \int_0^{\pi/4} \frac{1}{2} (2 \cos \theta - \sec \theta)^2 d\theta = \int_0^{\pi/4} (4 \cos^2 \theta - 4 + \sec^2 \theta) d\theta \\ &= \int_0^{\pi/4} [4 \cdot \frac{1}{2} (1 + \cos 2\theta) - 4 + \sec^2 \theta] d\theta = \int_0^{\pi/4} (-2 + 2 \cos 2\theta + \sec^2 \theta) d\theta \\ &= [-2\theta + \sin 2\theta + \tan \theta]_0^{\pi/4} = \left(-\frac{\pi}{2} + 1 + 1\right) - 0 = 2 - \frac{\pi}{2} \end{aligned}$$



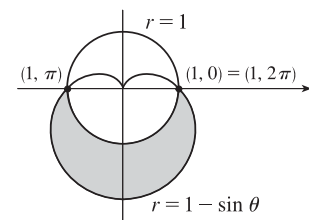
23.  $2 \cos \theta = 1 \Rightarrow \cos \theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3} \text{ or } \frac{5\pi}{3}.$

$$\begin{aligned} A &= 2 \int_0^{\pi/3} \frac{1}{2} [(2 \cos \theta)^2 - 1^2] d\theta = \int_0^{\pi/3} (4 \cos^2 \theta - 1) d\theta \\ &= \int_0^{\pi/3} \{4 [\frac{1}{2} (1 + \cos 2\theta)] - 1\} d\theta = \int_0^{\pi/3} (1 + 2 \cos 2\theta) d\theta \\ &= [\theta + \sin 2\theta]_0^{\pi/3} = \frac{\pi}{3} + \frac{\sqrt{3}}{2} \end{aligned}$$



24.  $1 - \sin \theta = 1 \Rightarrow \sin \theta = 0 \Rightarrow \theta = 0 \text{ or } \pi \Rightarrow$

$$\begin{aligned} A &= \int_{\pi}^{2\pi} \frac{1}{2} [(1 - \sin \theta)^2 - 1] d\theta = \frac{1}{2} \int_{\pi}^{2\pi} (\sin^2 \theta - 2 \sin \theta) d\theta \\ &= \frac{1}{4} \int_{\pi}^{2\pi} (1 - \cos 2\theta - 4 \sin \theta) d\theta = \frac{1}{4} [\theta - \frac{1}{2} \sin 2\theta + 4 \cos \theta]_{\pi}^{2\pi} \\ &= \frac{1}{4} \pi + 2 \end{aligned}$$



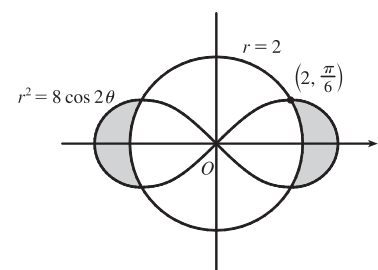
25. To find the area inside the lemniscate  $r^2 = 8 \cos 2\theta$  and outside the circle  $r = 2$ ,

we first note that the two curves intersect when  $r^2 = 8 \cos 2\theta$  and  $r = 2$ , that is, when  $\cos 2\theta = \frac{1}{2}$ . For  $-\pi < \theta \leq \pi$ ,  $\cos 2\theta = \frac{1}{2} \Leftrightarrow 2\theta = \pm\pi/3$

or  $\pm 5\pi/3 \Leftrightarrow \theta = \pm\pi/6$  or  $\pm 5\pi/6$ . The figure shows that the desired area is

4 times the area between the curves from 0 to  $\pi/6$ . Thus,

$$\begin{aligned} A &= 4 \int_0^{\pi/6} \left[ \frac{1}{2} (8 \cos 2\theta) - \frac{1}{2} (2)^2 \right] d\theta = 8 \int_0^{\pi/6} (2 \cos 2\theta - 1) d\theta \\ &= 8 \left[ \sin 2\theta - \theta \right]_0^{\pi/6} = 8 \left( \frac{\sqrt{3}}{2} - \frac{\pi}{6} \right) = 4\sqrt{3} - 4\pi/3 \end{aligned}$$

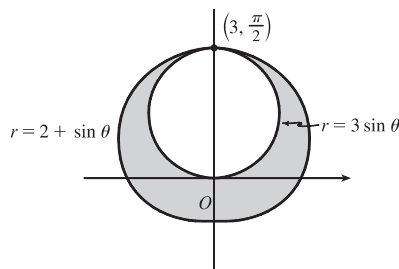


26. To find the shaded area  $A$ , we'll find the area  $A_1$  inside the curve  $r = 2 + \sin \theta$

and subtract  $\pi(\frac{3}{2})^2$  since  $r = 3 \sin \theta$  is a circle with radius  $\frac{3}{2}$ .

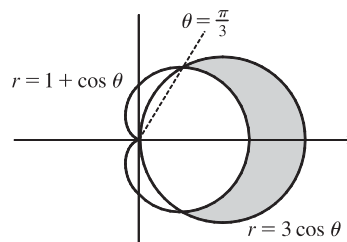
$$\begin{aligned} A_1 &= \int_0^{2\pi} \frac{1}{2} (2 + \sin \theta)^2 d\theta = \frac{1}{2} \int_0^{2\pi} (4 + 4 \sin \theta + \sin^2 \theta) d\theta \\ &= \frac{1}{2} \int_0^{2\pi} [4 + 4 \sin \theta + \frac{1}{2} (1 - \cos 2\theta)] d\theta \\ &= \frac{1}{2} \int_0^{2\pi} (\frac{9}{2} + 4 \sin \theta - \frac{1}{2} \cos 2\theta) d\theta \\ &= \frac{1}{2} [\frac{9}{2} \theta - 4 \cos \theta - \frac{1}{4} \sin 2\theta]_0^{2\pi} = \frac{1}{2} [(9\pi - 4) - (-4)] = \frac{9\pi}{2} \end{aligned}$$

So  $A = A_1 - \frac{9\pi}{4} = \frac{9\pi}{2} - \frac{9\pi}{4} = \frac{9\pi}{4}$ .



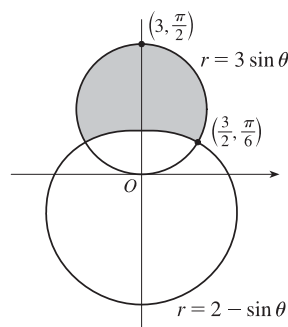
27.  $3 \cos \theta = 1 + \cos \theta \Leftrightarrow \cos \theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3} \text{ or } -\frac{\pi}{3}$ .

$$\begin{aligned} A &= 2 \int_0^{\pi/3} \frac{1}{2} [(3 \cos \theta)^2 - (1 + \cos \theta)^2] d\theta \\ &= \int_0^{\pi/3} (8 \cos^2 \theta - 2 \cos \theta - 1) d\theta = \int_0^{\pi/3} [4(1 + \cos 2\theta) - 2 \cos \theta - 1] d\theta \\ &= \int_0^{\pi/3} (3 + 4 \cos 2\theta - 2 \cos \theta) d\theta = [3\theta + 2 \sin 2\theta - 2 \sin \theta]_0^{\pi/3} \\ &= \pi + \sqrt{3} - \sqrt{3} = \pi \end{aligned}$$



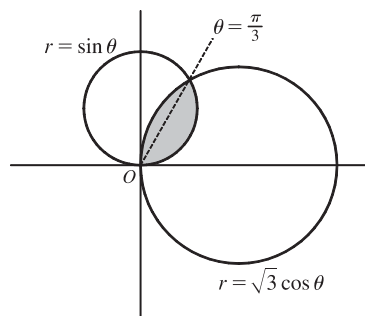
28.  $3 \sin \theta = 2 - \sin \theta \Rightarrow 4 \sin \theta = 2 \Rightarrow \sin \theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}$ .

$$\begin{aligned} A &= 2 \int_{\pi/6}^{\pi/2} \frac{1}{2} [(3 \sin \theta)^2 - (2 - \sin \theta)^2] d\theta \\ &= \int_{\pi/6}^{\pi/2} (9 \sin^2 \theta - 4 + 4 \sin \theta - \sin^2 \theta) d\theta \\ &= \int_{\pi/6}^{\pi/2} (8 \sin^2 \theta + 4 \sin \theta - 4) d\theta \\ &= 4 \int_{\pi/6}^{\pi/2} [2 \cdot \frac{1}{2} (1 - \cos 2\theta) + \sin \theta - 1] d\theta \\ &= 4 \int_{\pi/6}^{\pi/2} (\sin \theta - \cos 2\theta) d\theta = 4 [-\cos \theta - \frac{1}{2} \sin 2\theta]_{\pi/6}^{\pi/2} \\ &= 4 [(0 - 0) - (-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{4})] = 4 (\frac{3\sqrt{3}}{4}) = 3\sqrt{3} \end{aligned}$$

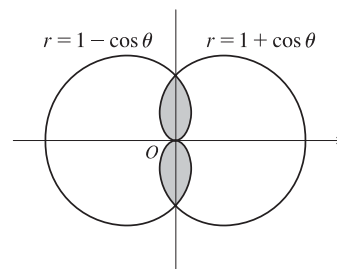


29.  $\sqrt{3} \cos \theta = \sin \theta \Rightarrow \sqrt{3} = \frac{\sin \theta}{\cos \theta} \Rightarrow \tan \theta = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3}$ .

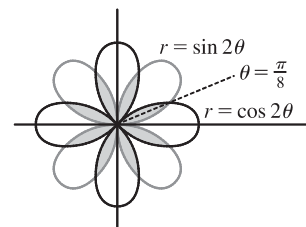
$$\begin{aligned} A &= \int_0^{\pi/3} \frac{1}{2} (\sin \theta)^2 d\theta + \int_{\pi/3}^{\pi/2} \frac{1}{2} (\sqrt{3} \cos \theta)^2 d\theta \\ &= \int_0^{\pi/3} \frac{1}{2} \cdot \frac{1}{2} (1 - \cos 2\theta) d\theta + \int_{\pi/3}^{\pi/2} \frac{1}{2} \cdot 3 \cdot \frac{1}{2} (1 + \cos 2\theta) d\theta \\ &= \frac{1}{4} [\theta - \frac{1}{2} \sin 2\theta]_0^{\pi/3} + \frac{3}{4} [\theta + \frac{1}{2} \sin 2\theta]_{\pi/3}^{\pi/2} \\ &= \frac{1}{4} \left[ \left( \frac{\pi}{3} - \frac{\sqrt{3}}{4} \right) - 0 \right] + \frac{3}{4} \left[ \left( \frac{\pi}{2} + 0 \right) - \left( \frac{\pi}{3} + \frac{\sqrt{3}}{4} \right) \right] \\ &= \frac{\pi}{12} - \frac{\sqrt{3}}{16} + \frac{\pi}{8} - \frac{3\sqrt{3}}{16} = \frac{5\pi}{24} - \frac{\sqrt{3}}{4} \end{aligned}$$



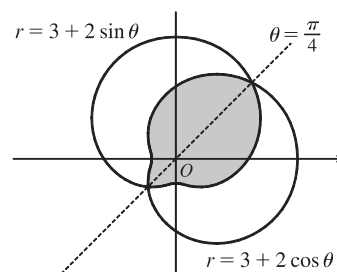
$$\begin{aligned}
 30. \quad A &= 4 \int_0^{\pi/2} \frac{1}{2} (1 - \cos \theta)^2 d\theta = 2 \int_0^{\pi/2} (1 - 2 \cos \theta + \cos^2 \theta) d\theta \\
 &= 2 \int_0^{\pi/2} \left[ 1 - 2 \cos \theta + \frac{1}{2} (1 + \cos 2\theta) \right] d\theta \\
 &= 2 \int_0^{\pi/2} \left( \frac{3}{2} - 2 \cos \theta + \frac{1}{2} \cos 2\theta \right) d\theta = \int_0^{\pi/2} (3 - 4 \cos \theta + \cos 2\theta) d\theta \\
 &= \left[ 3\theta - 4 \sin \theta + \frac{1}{2} \sin 2\theta \right]_0^{\pi/2} = \frac{3\pi}{2} - 4
 \end{aligned}$$



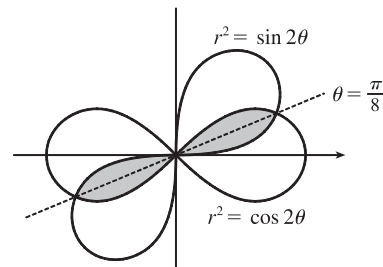
$$\begin{aligned}
 31. \quad \sin 2\theta &= \cos 2\theta \Rightarrow \frac{\sin 2\theta}{\cos 2\theta} = 1 \Rightarrow \tan 2\theta = 1 \Rightarrow 2\theta = \frac{\pi}{4} \Rightarrow \\
 \theta &= \frac{\pi}{8} \Rightarrow \\
 A &= 8 \cdot 2 \int_0^{\pi/8} \frac{1}{2} \sin^2 2\theta d\theta = 8 \int_0^{\pi/8} \frac{1}{2} (1 - \cos 4\theta) d\theta \\
 &= 4 \left[ \theta - \frac{1}{4} \sin 4\theta \right]_0^{\pi/8} = 4 \left( \frac{\pi}{8} - \frac{1}{4} \cdot 1 \right) = \frac{\pi}{2} - 1
 \end{aligned}$$



$$\begin{aligned}
 32. \quad 3 + 2 \cos \theta &= 3 + 2 \sin \theta \Rightarrow \cos \theta = \sin \theta \Rightarrow \theta = \frac{\pi}{4} \text{ or } \frac{5\pi}{4}. \\
 A &= 2 \int_{\pi/4}^{5\pi/4} \frac{1}{2} (3 + 2 \cos \theta)^2 d\theta = \int_{\pi/4}^{5\pi/4} (9 + 12 \cos \theta + 4 \cos^2 \theta) d\theta \\
 &= \int_{\pi/4}^{5\pi/4} \left[ 9 + 12 \cos \theta + 4 \cdot \frac{1}{2} (1 + \cos 2\theta) \right] d\theta \\
 &= \int_{\pi/4}^{5\pi/4} (11 + 12 \cos \theta + 2 \cos 2\theta) d\theta = \left[ 11\theta + 12 \sin \theta + \sin 2\theta \right]_{\pi/4}^{5\pi/4} \\
 &= \left( \frac{55\pi}{4} - 6\sqrt{2} + 1 \right) - \left( \frac{11\pi}{4} + 6\sqrt{2} + 1 \right) = 11\pi - 12\sqrt{2}
 \end{aligned}$$

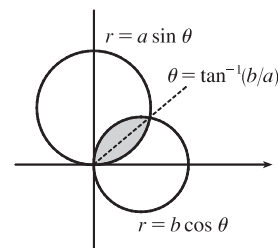


$$\begin{aligned}
 33. \quad \sin 2\theta &= \cos 2\theta \Rightarrow \tan 2\theta = 1 \Rightarrow 2\theta = \frac{\pi}{4} \Rightarrow \theta = \frac{\pi}{8} \\
 A &= 4 \int_0^{\pi/8} \frac{1}{2} \sin 2\theta d\theta \quad [\text{since } r^2 = \sin 2\theta] \\
 &= \int_0^{\pi/8} 2 \sin 2\theta d\theta = \left[ -\cos 2\theta \right]_0^{\pi/8} \\
 &= -\frac{1}{2} \sqrt{2} - (-1) = 1 - \frac{1}{2} \sqrt{2}
 \end{aligned}$$



34. Let  $\alpha = \tan^{-1}(b/a)$ . Then

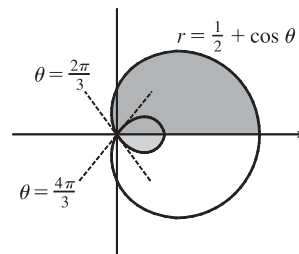
$$\begin{aligned}
 A &= \int_0^\alpha \frac{1}{2} (a \sin \theta)^2 d\theta + \int_\alpha^{\pi/2} \frac{1}{2} (b \cos \theta)^2 d\theta \\
 &= \frac{1}{4} a^2 \left[ \theta - \frac{1}{2} \sin 2\theta \right]_0^\alpha + \frac{1}{4} b^2 \left[ \theta + \frac{1}{2} \sin 2\theta \right]_\alpha^{\pi/2} \\
 &= \frac{1}{4} \alpha (a^2 - b^2) + \frac{1}{8} \pi b^2 - \frac{1}{4} (a^2 + b^2) (\sin \alpha \cos \alpha) \\
 &= \frac{1}{4} (a^2 - b^2) \tan^{-1}(b/a) + \frac{1}{8} \pi b^2 - \frac{1}{4} ab
 \end{aligned}$$



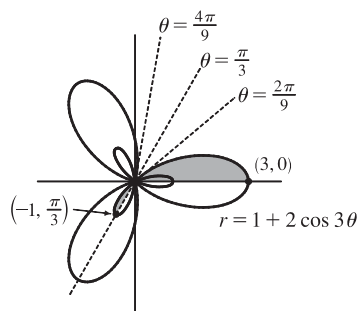
35. The darker shaded region (from  $\theta = 0$  to  $\theta = 2\pi/3$ ) represents  $\frac{1}{2}$  of the desired area plus  $\frac{1}{2}$  of the area of the inner loop.

From this area, we'll subtract  $\frac{1}{2}$  of the area of the inner loop (the lighter shaded region from  $\theta = 2\pi/3$  to  $\theta = \pi$ ), and then double that difference to obtain the desired area.

$$\begin{aligned} A &= 2 \left[ \int_0^{2\pi/3} \frac{1}{2} \left( \frac{1}{2} + \cos \theta \right)^2 d\theta - \int_{2\pi/3}^{\pi} \frac{1}{2} \left( \frac{1}{2} + \cos \theta \right)^2 d\theta \right] \\ &= \int_0^{2\pi/3} \left( \frac{1}{4} + \cos \theta + \cos^2 \theta \right) d\theta - \int_{2\pi/3}^{\pi} \left( \frac{1}{4} + \cos \theta + \cos^2 \theta \right) d\theta \\ &= \int_0^{2\pi/3} \left[ \frac{1}{4} + \cos \theta + \frac{1}{2}(1 + \cos 2\theta) \right] d\theta \\ &\quad - \int_{2\pi/3}^{\pi} \left[ \frac{1}{4} + \cos \theta + \frac{1}{2}(1 + \cos 2\theta) \right] d\theta \\ &= \left[ \frac{\theta}{4} + \sin \theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4} \right]_0^{2\pi/3} - \left[ \frac{\theta}{4} + \sin \theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4} \right]_{2\pi/3}^{\pi} \\ &= \left( \frac{\pi}{6} + \frac{\sqrt{3}}{2} + \frac{\pi}{3} - \frac{\sqrt{3}}{8} \right) - \left( \frac{\pi}{4} + \frac{\pi}{2} \right) + \left( \frac{\pi}{6} + \frac{\sqrt{3}}{2} + \frac{\pi}{3} - \frac{\sqrt{3}}{8} \right) \\ &= \frac{\pi}{4} + \frac{3}{4}\sqrt{3} = \frac{1}{4}(\pi + 3\sqrt{3}) \end{aligned}$$



36.  $r = 0 \Rightarrow 1 + 2\cos 3\theta = 0 \Rightarrow \cos 3\theta = -\frac{1}{2} \Rightarrow 3\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$  [for  $0 \leq 3\theta \leq 2\pi$ ]  $\Rightarrow \theta = \frac{2\pi}{9}, \frac{4\pi}{9}$ . The darker shaded region (from  $\theta = 0$  to  $\theta = 2\pi/9$ ) represents  $\frac{1}{2}$  of the desired area plus  $\frac{1}{2}$  of the area of the inner loop. From this area, we'll subtract  $\frac{1}{2}$  of the area of the inner loop (the lighter shaded region from  $\theta = 2\pi/9$  to  $\theta = \pi/3$ ), and then double that difference to obtain the desired area.



$$A = 2 \left[ \int_0^{2\pi/9} \frac{1}{2} (1 + 2\cos 3\theta)^2 d\theta - \int_{2\pi/9}^{\pi/3} \frac{1}{2} (1 + 2\cos 3\theta)^2 d\theta \right]$$

Now 
$$\begin{aligned} r^2 &= (1 + 2\cos 3\theta)^2 = 1 + 4\cos 3\theta + 4\cos^2 3\theta = 1 + 4\cos 3\theta + 4 \cdot \frac{1}{2}(1 + \cos 6\theta) \\ &= 1 + 4\cos 3\theta + 2 + 2\cos 6\theta = 3 + 4\cos 3\theta + 2\cos 6\theta \end{aligned}$$

and  $\int r^2 d\theta = 3\theta + \frac{4}{3}\sin 3\theta + \frac{1}{3}\sin 6\theta + C$ , so

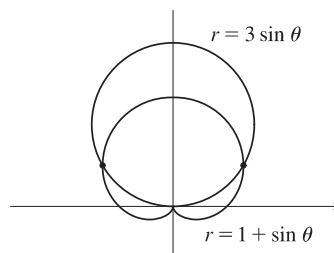
$$\begin{aligned} A &= \left[ 3\theta + \frac{4}{3}\sin 3\theta + \frac{1}{3}\sin 6\theta \right]_0^{2\pi/9} - \left[ 3\theta + \frac{4}{3}\sin 3\theta + \frac{1}{3}\sin 6\theta \right]_{2\pi/9}^{\pi/3} \\ &= \left[ \left( \frac{2\pi}{3} + \frac{4}{3} \cdot \frac{\sqrt{3}}{2} + \frac{1}{3} \cdot \frac{-\sqrt{3}}{2} \right) - 0 \right] - \left[ \left( \pi + 0 + 0 \right) - \left( \frac{2\pi}{3} + \frac{4}{3} \cdot \frac{\sqrt{3}}{2} + \frac{1}{3} \cdot \frac{-\sqrt{3}}{2} \right) \right] \\ &= \frac{4\pi}{3} + \frac{4}{3}\sqrt{3} - \frac{1}{3}\sqrt{3} - \pi = \frac{\pi}{3} + \sqrt{3} \end{aligned}$$

37. The pole is a point of intersection.

$$1 + \sin \theta = 3 \sin \theta \Rightarrow 1 = 2 \sin \theta \Rightarrow \sin \theta = \frac{1}{2} \Rightarrow$$

$$\theta = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}.$$

The other two points of intersection are  $(\frac{3}{2}, \frac{\pi}{6})$  and  $(\frac{3}{2}, \frac{5\pi}{6})$ .

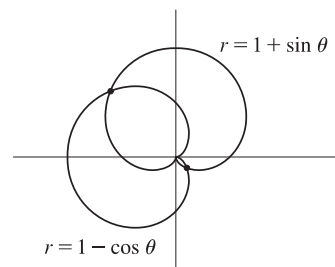


38. The pole is a point of intersection.

$$1 - \cos \theta = 1 + \sin \theta \Rightarrow -\cos \theta = \sin \theta \Rightarrow -1 = \tan \theta \Rightarrow$$

$$\theta = \frac{3\pi}{4} \text{ or } \frac{7\pi}{4}.$$

The other two points of intersection are  $\left(1 + \frac{\sqrt{2}}{2}, \frac{3\pi}{4}\right)$  and  $\left(1 - \frac{\sqrt{2}}{2}, \frac{7\pi}{4}\right)$ .



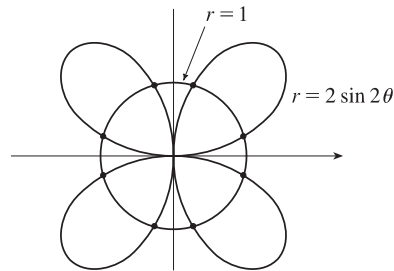
39.  $2 \sin 2\theta = 1 \Rightarrow \sin 2\theta = \frac{1}{2} \Rightarrow 2\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \text{ or } \frac{17\pi}{6}.$

By symmetry, the eight points of intersection are given by

$$(1, \theta), \text{ where } \theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \text{ and } \frac{17\pi}{12}, \text{ and}$$

$$(-1, \theta), \text{ where } \theta = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \text{ and } \frac{23\pi}{12}.$$

[There are many ways to describe these points.]

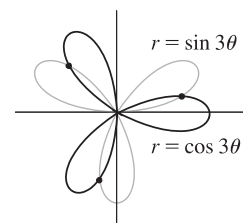


40. Clearly the pole lies on both curves.  $\sin 3\theta = \cos 3\theta \Rightarrow \tan 3\theta = 1 \Rightarrow$

$$3\theta = \frac{\pi}{4} + n\pi \text{ [n any integer]} \Rightarrow \theta = \frac{\pi}{12} + \frac{\pi}{3}n \Rightarrow$$

$$\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \text{ or } \frac{3\pi}{4}, \text{ so the three remaining intersection points are}$$

$$\left(\frac{1}{\sqrt{2}}, \frac{\pi}{12}\right), \left(-\frac{1}{\sqrt{2}}, \frac{5\pi}{12}\right), \text{ and } \left(\frac{1}{\sqrt{2}}, \frac{3\pi}{4}\right).$$

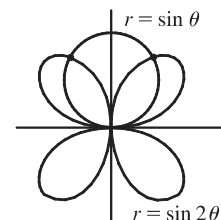


41. The pole is a point of intersection.  $\sin \theta = \sin 2\theta = 2 \sin \theta \cos \theta \Leftrightarrow$

$$\sin \theta (1 - 2 \cos \theta) = 0 \Leftrightarrow \sin \theta = 0 \text{ or } \cos \theta = \frac{1}{2} \Rightarrow$$

$$\theta = 0, \pi, \frac{\pi}{3}, \text{ or } -\frac{\pi}{3} \Rightarrow \text{the other intersection points are } \left(\frac{\sqrt{3}}{2}, \frac{\pi}{3}\right)$$

$$\text{and } \left(\frac{\sqrt{3}}{2}, \frac{2\pi}{3}\right) \text{ [by symmetry].}$$

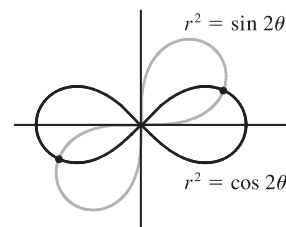


42. Clearly the pole is a point of intersection.  $\sin 2\theta = \cos 2\theta \Rightarrow$

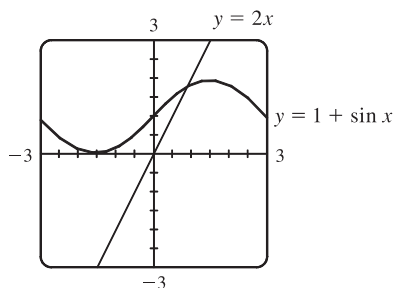
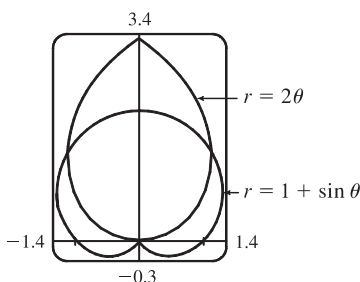
$$\tan 2\theta = 1 \Rightarrow 2\theta = \frac{\pi}{4} + 2n\pi \text{ [since } \sin 2\theta \text{ and } \cos 2\theta \text{ must be}$$

$$\text{positive in the equations]} \Rightarrow \theta = \frac{\pi}{8} + n\pi \Rightarrow \theta = \frac{\pi}{8} \text{ or } \frac{9\pi}{8}.$$

$$\text{So the curves also intersect at } \left(\frac{1}{\sqrt[4]{2}}, \frac{\pi}{8}\right) \text{ and } \left(\frac{1}{\sqrt[4]{2}}, \frac{9\pi}{8}\right).$$



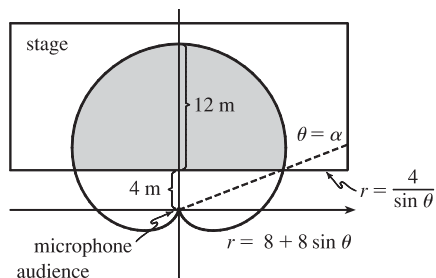
43.



From the first graph, we see that the pole is one point of intersection. By zooming in or using the cursor, we find the  $\theta$ -values of the intersection points to be  $\alpha \approx 0.88786 \approx 0.89$  and  $\pi - \alpha \approx 2.25$ . (The first of these values may be more easily estimated by plotting  $y = 1 + \sin x$  and  $y = 2x$  in rectangular coordinates; see the second graph.) By symmetry, the total area contained is twice the area contained in the first quadrant, that is,

$$\begin{aligned} A &= 2 \int_0^\alpha \frac{1}{2} (2\theta)^2 d\theta + 2 \int_\alpha^{\pi/2} \frac{1}{2} (1 + \sin \theta)^2 d\theta = \int_0^\alpha 4\theta^2 d\theta + \int_\alpha^{\pi/2} [1 + 2\sin \theta + \frac{1}{2}(1 - \cos 2\theta)] d\theta \\ &= \left[ \frac{4}{3}\theta^3 \right]_0^\alpha + \left[ \theta - 2\cos \theta + \left( \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta \right) \right]_\alpha^{\pi/2} = \frac{4}{3}\alpha^3 + \left[ \left( \frac{\pi}{2} + \frac{\pi}{4} \right) - \left( \alpha - 2\cos \alpha + \frac{1}{2}\alpha - \frac{1}{4}\sin 2\alpha \right) \right] \approx 3.4645 \end{aligned}$$

44.



We need to find the shaded area  $A$  in the figure. The horizontal line representing the front of the stage has equation  $y = 4 \Leftrightarrow$

$$r \sin \theta = 4 \Rightarrow r = 4 / \sin \theta. \text{ This line intersects the curve}$$

$$r = 8 + 8 \sin \theta \text{ when } 8 + 8 \sin \theta = \frac{4}{\sin \theta} \Rightarrow$$

$$8 \sin \theta + 8 \sin^2 \theta = 4 \Rightarrow 2 \sin^2 \theta + 2 \sin \theta - 1 = 0 \Rightarrow$$

$$\sin \theta = \frac{-2 \pm \sqrt{4+8}}{4} = \frac{-2 \pm 2\sqrt{3}}{4} = \frac{-1 \pm \sqrt{3}}{2} \quad [\text{the other value is less than } -1] \Rightarrow \theta = \sin^{-1} \left( \frac{\sqrt{3}-1}{2} \right).$$

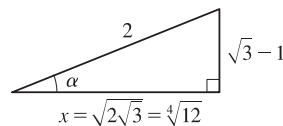
This angle is about  $21.5^\circ$  and is denoted by  $\alpha$  in the figure.

$$\begin{aligned} A &= 2 \int_\alpha^{\pi/2} \frac{1}{2} (8 + 8 \sin \theta)^2 d\theta - 2 \int_\alpha^{\pi/2} \frac{1}{2} (4 \csc \theta)^2 d\theta = 64 \int_\alpha^{\pi/2} (1 + 2 \sin \theta + \sin^2 \theta) d\theta - 16 \int_\alpha^{\pi/2} \csc^2 \theta d\theta \\ &= 64 \int_\alpha^{\pi/2} \left( 1 + 2 \sin \theta + \frac{1}{2} - \frac{1}{2} \cos 2\theta \right) d\theta + 16 \int_\alpha^{\pi/2} (-\csc^2 \theta) d\theta = 64 \left[ \frac{3}{2}\theta - 2 \cos \theta - \frac{1}{4} \sin 2\theta \right]_\alpha^{\pi/2} + 16 [\cot \theta]_\alpha^{\pi/2} \\ &= 16 [6\theta - 8 \cos \theta - \sin 2\theta + \cot \theta]_\alpha^{\pi/2} = 16 [(3\pi - 0 - 0 + 0) - (6\alpha - 8 \cos \alpha - \sin 2\alpha + \cot \alpha)] \\ &= 48\pi - 96\alpha + 128 \cos \alpha + 16 \sin 2\alpha - 16 \cot \alpha \end{aligned}$$

$$\text{From the figure, } x^2 + (\sqrt{3}-1)^2 = 2^2 \Rightarrow x^2 = 4 - (3 - 2\sqrt{3} + 1) \Rightarrow$$

$$x^2 = 2\sqrt{3} = \sqrt{12}, \text{ so } x = \sqrt{2\sqrt{3}} = \sqrt[4]{12}. \text{ Using the trigonometric relationships}$$

for a right triangle and the identity  $\sin 2\alpha = 2 \sin \alpha \cos \alpha$ , we continue:



$$\begin{aligned} A &= 48\pi - 96\alpha + 128 \cdot \frac{\sqrt[4]{12}}{2} + 16 \cdot 2 \cdot \frac{\sqrt{3}-1}{2} \cdot \frac{\sqrt[4]{12}}{2} - 16 \cdot \frac{\sqrt[4]{12}}{\sqrt{3}-1} \cdot \frac{\sqrt{3}+1}{\sqrt{3}+1} \\ &= 48\pi - 96\alpha + 64 \sqrt[4]{12} + 8 \sqrt[4]{12} (\sqrt{3}-1) - 8 \sqrt[4]{12} (\sqrt{3}+1) = 48\pi + 48 \sqrt[4]{12} - 96 \sin^{-1} \left( \frac{\sqrt{3}-1}{2} \right) \\ &\approx 204.16 \text{ m}^2 \end{aligned}$$

$$\begin{aligned}
 45. \quad L &= \int_a^b \sqrt{r^2 + (dr/d\theta)^2} d\theta = \int_0^\pi \sqrt{(2\cos\theta)^2 + (-2\sin\theta)^2} d\theta \\
 &= \int_0^\pi \sqrt{4(\cos^2\theta + \sin^2\theta)} d\theta = \int_0^\pi \sqrt{4} d\theta = [2\theta]_0^\pi = 2\pi
 \end{aligned}$$

As a check, note that the curve is a circle of radius 1, so its circumference is  $2\pi(1) = 2\pi$ .

$$\begin{aligned}
 46. \quad L &= \int_a^b \sqrt{r^2 + (dr/d\theta)^2} d\theta = \int_0^{2\pi} \sqrt{(5^\theta)^2 + (5^\theta \ln 5)^2} d\theta = \int_0^{2\pi} \sqrt{5^{2\theta}[1 + (\ln 5)^2]} d\theta \\
 &= \sqrt{1 + (\ln 5)^2} \int_0^{2\pi} \sqrt{5^{2\theta}} d\theta = \sqrt{1 + (\ln 5)^2} \int_0^{2\pi} 5^\theta d\theta = \sqrt{1 + (\ln 5)^2} \left[ \frac{5^\theta}{\ln 5} \right]_0^{2\pi} \\
 &= \sqrt{1 + (\ln 5)^2} \left( \frac{5^{2\pi}}{\ln 5} - \frac{1}{\ln 5} \right) = \frac{\sqrt{1 + (\ln 5)^2}}{\ln 5} (5^{2\pi} - 1)
 \end{aligned}$$

$$\begin{aligned}
 47. \quad L &= \int_a^b \sqrt{r^2 + (dr/d\theta)^2} d\theta = \int_0^{2\pi} \sqrt{(\theta^2)^2 + (2\theta)^2} d\theta = \int_0^{2\pi} \sqrt{\theta^4 + 4\theta^2} d\theta \\
 &= \int_0^{2\pi} \theta \sqrt{\theta^2 + 4} d\theta = \int_0^{2\pi} \theta \sqrt{\theta^2 + 4} d\theta
 \end{aligned}$$

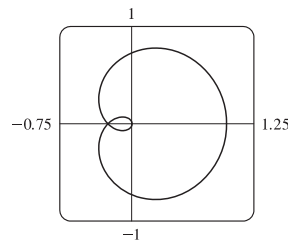
Now let  $u = \theta^2 + 4$ , so that  $du = 2\theta d\theta$  [ $\theta d\theta = \frac{1}{2} du$ ] and

$$\int_0^{2\pi} \theta \sqrt{\theta^2 + 4} d\theta = \int_4^{4\pi^2 + 4} \frac{1}{2} \sqrt{u} du = \frac{1}{2} \cdot \frac{2}{3} \left[ u^{3/2} \right]_4^{4\pi^2 + 4} = \frac{1}{3} [4^{3/2}(\pi^2 + 1)^{3/2} - 4^{3/2}] = \frac{8}{3} [(\pi^2 + 1)^{3/2} - 1]$$

$$\begin{aligned}
 48. \quad L &= \int_a^b \sqrt{r^2 + (dr/d\theta)^2} d\theta = \int_0^{2\pi} \sqrt{[2(1 + \cos\theta)]^2 + (-2\sin\theta)^2} d\theta = \int_0^{2\pi} \sqrt{4 + 8\cos\theta + 4\cos^2\theta + 4\sin^2\theta} d\theta \\
 &= \int_0^{2\pi} \sqrt{8 + 8\cos\theta} d\theta = \sqrt{8} \int_0^{2\pi} \sqrt{1 + \cos\theta} d\theta = \sqrt{8} \int_0^{2\pi} \sqrt{2 \cdot \frac{1}{2}(1 + \cos\theta)} d\theta \\
 &= \sqrt{8} \int_0^{2\pi} \sqrt{2 \cos^2 \frac{\theta}{2}} d\theta = \sqrt{8} \sqrt{2} \int_0^{2\pi} \left| \cos \frac{\theta}{2} \right| d\theta = 4 \cdot 2 \int_0^\pi \cos \frac{\theta}{2} d\theta \quad [\text{by symmetry}] \\
 &= 8 \left[ 2 \sin \frac{\theta}{2} \right]_0^\pi = 8(2) = 16
 \end{aligned}$$

49. The curve  $r = \cos^4(\theta/4)$  is completely traced with  $0 \leq \theta \leq 4\pi$ .

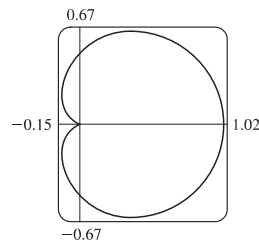
$$\begin{aligned}
 r^2 + (dr/d\theta)^2 &= [\cos^4(\theta/4)]^2 + [4\cos^3(\theta/4) \cdot (-\sin(\theta/4)) \cdot \frac{1}{4}]^2 \\
 &= \cos^8(\theta/4) + \cos^6(\theta/4) \sin^2(\theta/4) \\
 &= \cos^6(\theta/4) [\cos^2(\theta/4) + \sin^2(\theta/4)] = \cos^6(\theta/4)
 \end{aligned}$$



$$\begin{aligned}
 L &= \int_0^{4\pi} \sqrt{\cos^6(\theta/4)} d\theta = \int_0^{4\pi} |\cos^3(\theta/4)| d\theta \\
 &= 2 \int_0^{2\pi} \cos^3(\theta/4) d\theta \quad [\text{since } \cos^3(\theta/4) \geq 0 \text{ for } 0 \leq \theta \leq 2\pi] = 8 \int_0^{\pi/2} \cos^3 u du \quad [u = \frac{1}{4}\theta] \\
 &= 8 \int_0^{\pi/2} (1 - \sin^2 u) \cos u du = 8 \int_0^1 (1 - x^2) dx \quad \left[ \begin{array}{l} x = \sin u, \\ dx = \cos u du \end{array} \right] \\
 &= 8 \left[ x - \frac{1}{3}x^3 \right]_0^1 = 8 \left( 1 - \frac{1}{3} \right) = \frac{16}{3}
 \end{aligned}$$

50. The curve  $r = \cos^2(\theta/2)$  is completely traced with  $0 \leq \theta \leq 2\pi$ .

$$\begin{aligned} r^2 + (dr/d\theta)^2 &= [\cos^2(\theta/2)]^2 + [2\cos(\theta/2) \cdot (-\sin(\theta/2)) \cdot \frac{1}{2}]^2 \\ &= \cos^4(\theta/2) + \cos^2(\theta/2) \sin^2(\theta/2) \\ &= \cos^2(\theta/2) [\cos^2(\theta/2) + \sin^2(\theta/2)] \\ &= \cos^2(\theta/2) \end{aligned}$$



$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{\cos^2(\theta/2)} d\theta = \int_0^{2\pi} |\cos(\theta/2)| d\theta = 2 \int_0^{\pi} \cos(\theta/2) d\theta \quad [\text{since } \cos(\theta/2) \geq 0 \text{ for } 0 \leq \theta \leq \pi] \\ &= 4 \int_0^{\pi/2} \cos u du \quad [u = \frac{1}{2}\theta] = 4[\sin u]_0^{\pi/2} = 4(1 - 0) = 4 \end{aligned}$$

51. One loop of the curve  $r = \cos 2\theta$  is traced with  $-\pi/4 \leq \theta \leq \pi/4$ .

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = \cos^2 2\theta + (-2\sin 2\theta)^2 = \cos^2 2\theta + 4\sin^2 2\theta = 1 + 3\sin^2 2\theta \Rightarrow L \int_{-\pi/4}^{\pi/4} \sqrt{1 + 3\sin^2 2\theta} d\theta \approx 2.4221.$$

$$52. r^2 + \left(\frac{dr}{d\theta}\right)^2 = \tan^2 \theta + (\sec^2 \theta)^2 \Rightarrow L \int_{\pi/6}^{\pi/3} \sqrt{\tan^2 \theta + \sec^4 \theta} d\theta \approx 1.2789$$

53. The curve  $r = \sin(6\sin \theta)$  is completely traced with  $0 \leq \theta \leq \pi$ .  $r = \sin(6\sin \theta) \Rightarrow \frac{dr}{d\theta} = \cos(6\sin \theta) \cdot 6\cos \theta$ , so

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = \sin^2(6\sin \theta) + 36\cos^2 \theta \cos^2(6\sin \theta) \Rightarrow L \int_0^{\pi} \sqrt{\sin^2(6\sin \theta) + 36\cos^2 \theta \cos^2(6\sin \theta)} d\theta \approx 8.0091.$$

54. The curve  $r = \sin(\theta/4)$  is completely traced with  $0 \leq \theta \leq 8\pi$ .  $r = \sin(\theta/4) \Rightarrow \frac{dr}{d\theta} = \frac{1}{4}\cos(\theta/4)$ , so

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = \sin^2(\theta/4) + \frac{1}{16}\cos^2(\theta/4) \Rightarrow L \int_0^{8\pi} \sqrt{\sin^2(\theta/4) + \frac{1}{16}\cos^2(\theta/4)} d\theta \approx 17.1568.$$

55. (a) From (10.2.6),

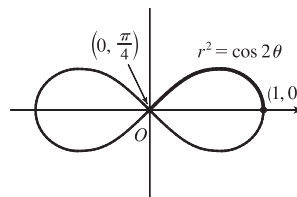
$$\begin{aligned} S &= \int_a^b 2\pi y \sqrt{(dx/d\theta)^2 + (dy/d\theta)^2} d\theta \\ &= \int_a^b 2\pi y \sqrt{r^2 + (dr/d\theta)^2} d\theta \quad [\text{from the derivation of Equation 10.4.5}] \\ &= \int_a^b 2\pi r \sin \theta \sqrt{r^2 + (dr/d\theta)^2} d\theta \end{aligned}$$

(b) The curve  $r^2 = \cos 2\theta$  goes through the pole when  $\cos 2\theta = 0 \Rightarrow$

$2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$ . We'll rotate the curve from  $\theta = 0$  to  $\theta = \frac{\pi}{4}$  and double this value to obtain the total surface area generated.

$$r^2 = \cos 2\theta \Rightarrow 2r \frac{dr}{d\theta} = -2\sin 2\theta \Rightarrow \left(\frac{dr}{d\theta}\right)^2 = \frac{\sin^2 2\theta}{r^2} = \frac{\sin^2 2\theta}{\cos 2\theta}.$$

$$\begin{aligned} S &= 2 \int_0^{\pi/4} 2\pi \sqrt{\cos 2\theta} \sin \theta \sqrt{\cos 2\theta + (\sin^2 2\theta)/\cos 2\theta} d\theta = 4\pi \int_0^{\pi/4} \sqrt{\cos 2\theta} \sin \theta \sqrt{\frac{\cos^2 2\theta + \sin^2 2\theta}{\cos 2\theta}} d\theta \\ &= 4\pi \int_0^{\pi/4} \sqrt{\cos 2\theta} \sin \theta \frac{1}{\sqrt{\cos 2\theta}} d\theta = 4\pi \int_0^{\pi/4} \sin \theta d\theta = 4\pi [-\cos \theta]_0^{\pi/4} = -4\pi \left(\frac{\sqrt{2}}{2} - 1\right) = 2\pi(2 - \sqrt{2}) \end{aligned}$$





56. (a) Rotation around  $\theta = \frac{\pi}{2}$  is the same as rotation around the  $y$ -axis, that is,  $S = \int_a^b 2\pi x \, ds$  where

$ds = \sqrt{(dx/dt)^2 + (dy/dt)^2} \, dt$  for a parametric equation, and for the special case of a polar equation,  $x = r \cos \theta$  and

$ds = \sqrt{(dx/d\theta)^2 + (dy/d\theta)^2} \, d\theta = \sqrt{r^2 + (dr/d\theta)^2} \, d\theta$  [see the derivation of Equation 10.4.5]. Therefore, for a polar

equation rotated around  $\theta = \frac{\pi}{2}$ ,  $S = \int_a^b 2\pi r \cos \theta \sqrt{r^2 + (dr/d\theta)^2} \, d\theta$ .

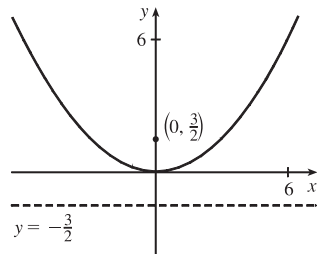
(b) As in the solution for Exercise 55(b), we can double the surface area generated by rotating the curve from  $\theta = 0$  to  $\theta = \frac{\pi}{4}$  to obtain the total surface area.

$$\begin{aligned} S &= 2 \int_0^{\pi/4} 2\pi \sqrt{\cos 2\theta} \cos \theta \sqrt{\cos 2\theta + (\sin^2 2\theta)/\cos 2\theta} \, d\theta = 4\pi \int_0^{\pi/4} \sqrt{\cos 2\theta} \cos \theta \sqrt{\frac{\cos^2 2\theta + \sin^2 2\theta}{\cos 2\theta}} \, d\theta \\ &= 4\pi \int_0^{\pi/4} \sqrt{\cos 2\theta} \cos \theta \frac{1}{\sqrt{\cos 2\theta}} \, d\theta = 4\pi \int_0^{\pi/4} \cos \theta \, d\theta = 4\pi [\sin \theta]_0^{\pi/4} = 4\pi \left( \frac{\sqrt{2}}{2} - 0 \right) = 2\sqrt{2}\pi \end{aligned}$$

## 10.5 Conic Sections

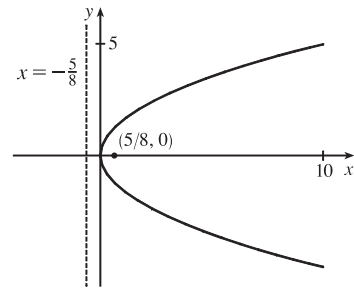
1.  $x^2 = 6y$  and  $x^2 = 4py \Rightarrow 4p = 6 \Rightarrow p = \frac{3}{2}$ .

The vertex is  $(0, 0)$ , the focus is  $(0, \frac{3}{2})$ , and the directrix is  $y = -\frac{3}{2}$ .



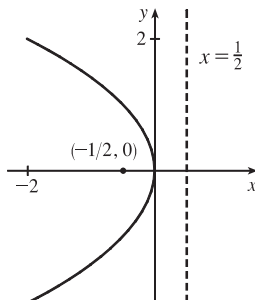
2.  $2y^2 = 5x \Rightarrow y^2 = \frac{5}{2}x$ .  $4p = \frac{5}{2} \Rightarrow p = \frac{5}{8}$ .

The vertex is  $(0, 0)$ , the focus is  $(\frac{5}{8}, 0)$ , and the directrix is  $x = -\frac{5}{8}$ .



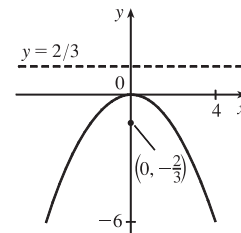
3.  $2x = -y^2 \Rightarrow y^2 = -2x$ .  $4p = -2 \Rightarrow p = -\frac{1}{2}$ .

The vertex is  $(0, 0)$ , the focus is  $(-\frac{1}{2}, 0)$ , and the directrix is  $x = \frac{1}{2}$ .

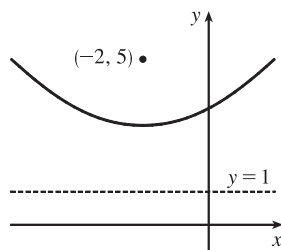


4.  $3x^2 + 8y = 0 \Rightarrow 3x^2 = -8y \Rightarrow x^2 = -\frac{8}{3}y$ .

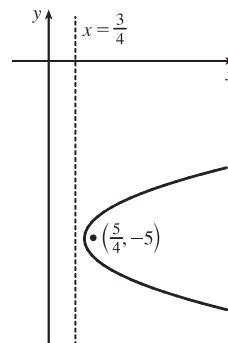
$4p = -\frac{8}{3} \Rightarrow p = -\frac{2}{3}$ . The vertex is  $(0, 0)$ , the focus is  $(0, -\frac{2}{3})$ , and the directrix is  $y = \frac{2}{3}$ .



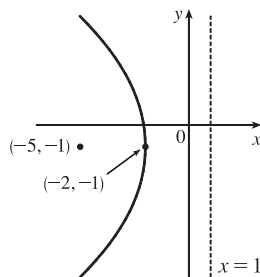
5.  $(x+2)^2 = 8(y-3)$ .  $4p = 8$ , so  $p = 2$ . The vertex is  $(-2, 3)$ , the focus is  $(-2, 5)$ , and the directrix is  $y = 1$ .



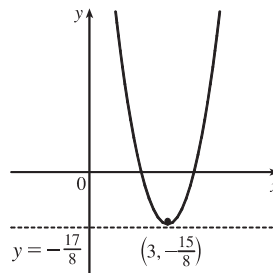
6.  $x-1 = (y+5)^2$ .  $4p = 1$ , so  $p = \frac{1}{4}$ . The vertex is  $(1, -5)$ , the focus is  $(\frac{5}{4}, -5)$ , and the directrix is  $x = \frac{3}{4}$ .



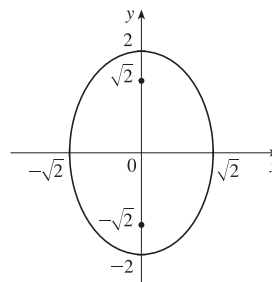
7.  $y^2 + 2y + 12x + 25 = 0 \Rightarrow y^2 + 2y + 1 = -12x - 24 \Rightarrow (y+1)^2 = -12(x+2)$ .  $4p = -12$ , so  $p = -3$ . The vertex is  $(-2, -1)$ , the focus is  $(-5, -1)$ , and the directrix is  $x = 1$ .



8.  $y + 12x - 2x^2 = 16 \Rightarrow 2x^2 - 12x = y - 16 \Rightarrow 2(x^2 - 6x + 9) = y - 16 + 18 \Rightarrow 2(x-3)^2 = y+2 \Rightarrow (x-3)^2 = \frac{1}{2}(y+2)$ .  $4p = \frac{1}{2}$ , so  $p = \frac{1}{8}$ . The vertex is  $(3, -2)$ , the focus is  $(3, -\frac{15}{8})$ , and the directrix is  $y = -\frac{17}{8}$ .

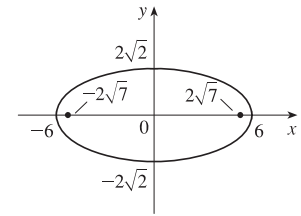


9. The equation has the form  $y^2 = 4px$ , where  $p < 0$ . Since the parabola passes through  $(-1, 1)$ , we have  $1^2 = 4p(-1)$ , so  $4p = -1$  and an equation is  $y^2 = -x$  or  $x = -y^2$ .  $4p = -1$ , so  $p = -\frac{1}{4}$  and the focus is  $(-\frac{1}{4}, 0)$  while the directrix is  $x = \frac{1}{4}$ .
10. The vertex is  $(2, -2)$ , so the equation is of the form  $(x-2)^2 = 4p(y+2)$ , where  $p > 0$ . The point  $(0, 0)$  is on the parabola, so  $4 = 4p(2)$  and  $4p = 2$ . Thus, an equation is  $(x-2)^2 = 2(y+2)$ .  $4p = 2$ , so  $p = \frac{1}{2}$  and the focus is  $(2, -\frac{3}{2})$  while the directrix is  $y = -\frac{5}{2}$ .
11.  $\frac{x^2}{2} + \frac{y^2}{4} = 1 \Rightarrow a = \sqrt{4} = 2, b = \sqrt{2}, c = \sqrt{a^2 - b^2} = \sqrt{4 - 2} = \sqrt{2}$ . The ellipse is centered at  $(0, 0)$ , with vertices at  $(0, \pm 2)$ . The foci are  $(0, \pm\sqrt{2})$ .



12.  $\frac{x^2}{36} + \frac{y^2}{8} = 1 \Rightarrow a = \sqrt{36} = 6, b = \sqrt{8},$

$c = \sqrt{a^2 - b^2} = \sqrt{36 - 8} = \sqrt{28} = 2\sqrt{7}.$  The ellipse is centered at  $(0, 0)$ , with vertices at  $(\pm 6, 0)$ . The foci are  $(\pm 2\sqrt{7}, 0)$ .

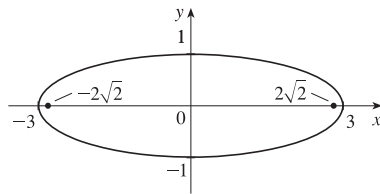


13.  $x^2 + 9y^2 = 9 \Leftrightarrow \frac{x^2}{9} + \frac{y^2}{1} = 1 \Rightarrow a = \sqrt{9} = 3,$

$b = \sqrt{1} = 1, c = \sqrt{a^2 - b^2} = \sqrt{9 - 1} = \sqrt{8} = 2\sqrt{2}.$

The ellipse is centered at  $(0, 0)$ , with vertices  $(\pm 3, 0)$ .

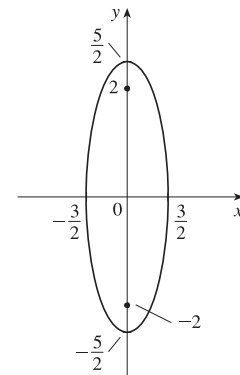
The foci are  $(\pm 2\sqrt{2}, 0)$ .



14.  $100x^2 + 36y^2 = 225 \Leftrightarrow \frac{x^2}{\frac{225}{100}} + \frac{y^2}{\frac{225}{36}} = 1 \Leftrightarrow$

$\frac{x^2}{\frac{9}{4}} + \frac{y^2}{\frac{25}{4}} = 1 \Rightarrow a = \sqrt{\frac{25}{4}} = \frac{5}{2}, b = \sqrt{\frac{9}{4}} = \frac{3}{2},$

$c = \sqrt{a^2 - b^2} = \sqrt{\frac{25}{4} - \frac{9}{4}} = 2.$  The ellipse is centered at  $(0, 0)$ , with vertices  $(0, \pm \frac{5}{2})$ . The foci are  $(0, \pm 2)$ .



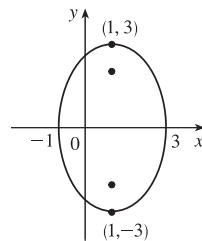
15.  $9x^2 - 18x + 4y^2 = 27 \Leftrightarrow$

$9(x^2 - 2x + 1) + 4y^2 = 27 + 9 \Leftrightarrow$

$9(x - 1)^2 + 4y^2 = 36 \Leftrightarrow \frac{(x - 1)^2}{4} + \frac{y^2}{9} = 1 \Rightarrow$

$a = 3, b = 2, c = \sqrt{5} \Rightarrow \text{center } (1, 0),$

vertices  $(1, \pm 3)$ , foci  $(1, \pm \sqrt{5})$



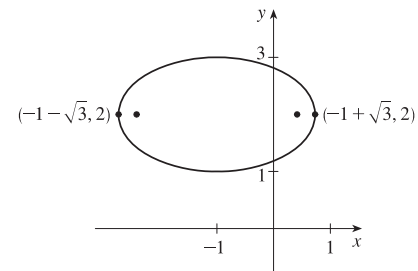
16.  $x^2 + 3y^2 + 2x - 12y + 10 = 0 \Leftrightarrow$

$x^2 + 2x + 1 + 3(y^2 - 4y + 4) = -10 + 1 + 12 \Leftrightarrow$

$(x + 1)^2 + 3(y - 2)^2 = 3 \Leftrightarrow$

$\frac{(x + 1)^2}{3} + \frac{(y - 2)^2}{1} = 1 \Rightarrow a = \sqrt{3}, b = 1,$

$c = \sqrt{2} \Rightarrow \text{center } (-1, 2), \text{ vertices } (-1 \pm \sqrt{3}, 2),$   
foci  $(-1 \pm \sqrt{2}, 2)$

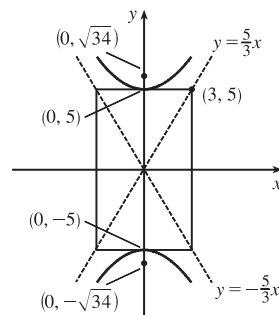


17. The center is  $(0, 0)$ ,  $a = 3$ , and  $b = 2$ , so an equation is  $\frac{x^2}{4} + \frac{y^2}{9} = 1.$   $c = \sqrt{a^2 - b^2} = \sqrt{5}$ , so the foci are  $(0, \pm \sqrt{5})$ .

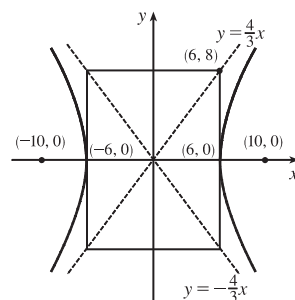
18. The ellipse is centered at  $(2, 1)$ , with  $a = 3$  and  $b = 2$ . An equation is  $\frac{(x-2)^2}{9} + \frac{(y-1)^2}{4} = 1$ .  $c = \sqrt{a^2 - b^2} = \sqrt{5}$ , so the foci are  $(2 \pm \sqrt{5}, 1)$ .

19.  $\frac{y^2}{25} - \frac{x^2}{9} = 1 \Rightarrow a = 5, b = 3, c = \sqrt{25 + 9} = \sqrt{34} \Rightarrow$   
center  $(0, 0)$ , vertices  $(0, \pm 5)$ , foci  $(0, \pm \sqrt{34})$ , asymptotes  $y = \pm \frac{5}{3}x$ .

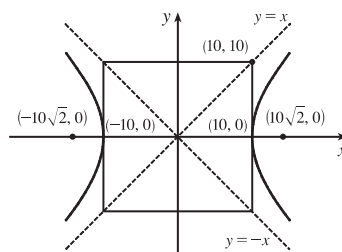
*Note:* It is helpful to draw a  $2a$ -by- $2b$  rectangle whose center is the center of the hyperbola. The asymptotes are the extended diagonals of the rectangle.



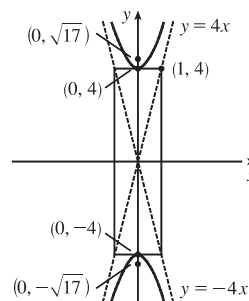
20.  $\frac{x^2}{36} - \frac{y^2}{64} = 1 \Rightarrow a = 6, b = 8, c = \sqrt{36 + 64} = 10 \Rightarrow$   
center  $(0, 0)$ , vertices  $(\pm 6, 0)$ , foci  $(\pm 10, 0)$ , asymptotes  $y = \pm \frac{8}{6}x = \pm \frac{4}{3}x$



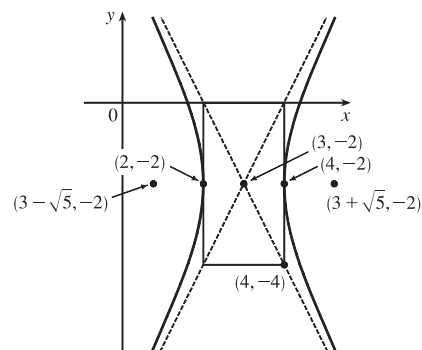
21.  $x^2 - y^2 = 100 \Leftrightarrow \frac{x^2}{100} - \frac{y^2}{100} = 1 \Rightarrow a = b = 10,$   
 $c = \sqrt{100 + 100} = 10\sqrt{2} \Rightarrow$  center  $(0, 0)$ , vertices  $(\pm 10, 0)$ ,  
foci  $(\pm 10\sqrt{2}, 0)$ , asymptotes  $y = \pm \frac{10}{10}x = \pm x$



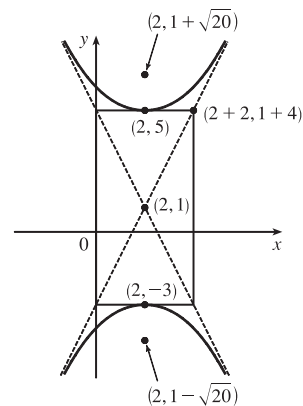
22.  $y^2 - 16x^2 = 16 \Leftrightarrow \frac{y^2}{16} - \frac{x^2}{1} = 1 \Rightarrow a = 4, b = 1,$   
 $c = \sqrt{16 + 1} = \sqrt{17} \Rightarrow$  center  $(0, 0)$ , vertices  $(0, \pm 4)$ ,  
foci  $(0, \pm \sqrt{17})$ , asymptotes  $y = \pm \frac{4}{1}x = \pm 4x$



$$\begin{aligned}
 23. \quad & 4x^2 - y^2 - 24x - 4y + 28 = 0 \Leftrightarrow \\
 & 4(x^2 - 6x + 9) - (y^2 + 4y + 4) = -28 + 36 - 4 \Leftrightarrow \\
 & 4(x - 3)^2 - (y + 2)^2 = 4 \Leftrightarrow \frac{(x - 3)^2}{1} - \frac{(y + 2)^2}{4} = 1 \Rightarrow \\
 & a = \sqrt{1} = 1, b = \sqrt{4} = 2, c = \sqrt{1 + 4} = \sqrt{5} \Rightarrow \\
 & \text{center } (3, -2), \text{ vertices } (4, -2) \text{ and } (2, -2), \text{ foci } (3 \pm \sqrt{5}, -2), \\
 & \text{asymptotes } y + 2 = \pm 2(x - 3).
 \end{aligned}$$



$$\begin{aligned}
 24. \quad & y^2 - 4x^2 - 2y + 16x = 31 \Leftrightarrow \\
 & (y^2 - 2y + 1) - 4(x^2 - 4x + 4) = 31 + 1 - 16 \Leftrightarrow \\
 & (y - 1)^2 - 4(x - 2)^2 = 16 \Leftrightarrow \\
 & \frac{(y - 1)^2}{16} - \frac{(x - 2)^2}{4} = 1 \Rightarrow a = \sqrt{16} = 4, b = \sqrt{4} = 2, \\
 & c = \sqrt{16 + 4} = \sqrt{20} \Rightarrow \text{center } (2, 1), \text{ vertices } (2, 1 \pm 4), \\
 & \text{foci } (2, 1 \pm \sqrt{20}), \text{ asymptotes } y - 1 = \pm 2(x - 2).
 \end{aligned}$$



$$25. \quad x^2 = y + 1 \Leftrightarrow x^2 = 1(y + 1). \text{ This is an equation of a } \textit{parabola} \text{ with } 4p = 1, \text{ so } p = \frac{1}{4}. \text{ The vertex is } (0, -1) \text{ and the focus is } (0, -\frac{3}{4}).$$

$$26. \quad x^2 = y^2 + 1 \Leftrightarrow x^2 - y^2 = 1. \text{ This is an equation of a } \textit{hyperbola} \text{ with vertices } (\pm 1, 0). \text{ The foci are at } (\pm\sqrt{1+1}, 0) = (\pm\sqrt{2}, 0).$$

$$27. \quad x^2 = 4y - 2y^2 \Leftrightarrow x^2 + 2y^2 - 4y = 0 \Leftrightarrow x^2 + 2(y^2 - 2y + 1) = 2 \Leftrightarrow x^2 + 2(y - 1)^2 = 2 \Leftrightarrow \frac{x^2}{2} + \frac{(y - 1)^2}{1} = 1. \text{ This is an equation of an } \textit{ellipse} \text{ with vertices at } (\pm\sqrt{2}, 1). \text{ The foci are at } (\pm\sqrt{2-1}, 1) = (\pm 1, 1).$$

$$28. \quad y^2 - 8y = 6x - 16 \Leftrightarrow y^2 - 8y + 16 = 6x \Leftrightarrow (y - 4)^2 = 6x. \text{ This is an equation of a } \textit{parabola} \text{ with } 4p = 6, \text{ so } p = \frac{3}{2}. \text{ The vertex is } (0, 4) \text{ and the focus is } (\frac{3}{2}, 4).$$

$$29. \quad y^2 + 2y = 4x^2 + 3 \Leftrightarrow y^2 + 2y + 1 = 4x^2 + 4 \Leftrightarrow (y + 1)^2 - 4x^2 = 4 \Leftrightarrow \frac{(y + 1)^2}{4} - x^2 = 1. \text{ This is an equation of a } \textit{hyperbola} \text{ with vertices } (0, -1 \pm 2) = (0, 1) \text{ and } (0, -3). \text{ The foci are at } (0, -1 \pm \sqrt{4+1}) = (0, -1 \pm \sqrt{5}).$$

$$30. \quad 4x^2 + 4x + y^2 = 0 \Leftrightarrow 4(x^2 + x + \frac{1}{4}) + y^2 = 1 \Leftrightarrow 4(x + \frac{1}{2})^2 + y^2 = 1 \Leftrightarrow \frac{(x + \frac{1}{2})^2}{1/4} + y^2 = 1. \text{ This is an equation of an } \textit{ellipse} \text{ with vertices } (-\frac{1}{2}, 0 \pm 1) = (-\frac{1}{2}, \pm 1). \text{ The foci are at } (-\frac{1}{2}, 0 \pm \sqrt{1 - \frac{1}{4}}) = (-\frac{1}{2}, \pm \sqrt{3}/2).$$

31. The parabola with vertex  $(0, 0)$  and focus  $(1, 0)$  opens to the right and has  $p = 1$ , so its equation is  $y^2 = 4px$ , or  $y^2 = 4x$ .
32. The parabola with focus  $(0, 0)$  and directrix  $y = 6$  has vertex  $(0, 3)$  and opens downward, so  $p = -3$  and its equation is  $(x - 0)^2 = 4p(y - 3)$ , or  $x^2 = -12(y - 3)$ .
33. The distance from the focus  $(-4, 0)$  to the directrix  $x = 2$  is  $2 - (-4) = 6$ , so the distance from the focus to the vertex is  $\frac{1}{2}(6) = 3$  and the vertex is  $(-1, 0)$ . Since the focus is to the left of the vertex,  $p = -3$ . An equation is  $y^2 = 4p(x + 1) \Rightarrow y^2 = -12(x + 1)$ .
34. The distance from the focus  $(3, 6)$  to the vertex  $(3, 2)$  is  $6 - 2 = 4$ . Since the focus is above the vertex,  $p = 4$ .  
An equation is  $(x - 3)^2 = 4p(y - 2) \Rightarrow (x - 3)^2 = 16(y - 2)$ .
35. A parabola with vertical axis and vertex  $(2, 3)$  has equation  $y - 3 = a(x - 2)^2$ . Since it passes through  $(1, 5)$ , we have  $5 - 3 = a(1 - 2)^2 \Rightarrow a = 2$ , so an equation is  $y - 3 = 2(x - 2)^2$ .
36. A parabola with horizontal axis has equation  $x = ay^2 + by + c$ . Since the parabola passes through the point  $(-1, 0)$ , substitute  $-1$  for  $x$  and  $0$  for  $y$ :  $-1 = 0 + 0 + c$ . Now with  $c = -1$ , substitute  $1$  for  $x$  and  $-1$  for  $y$ :  $1 = a - b - 1$  **(1)**; and then  $3$  for  $x$  and  $1$  for  $y$ :  $3 = a + b - 1$  **(2)**. Add **(1)** and **(2)** to get  $4 = 2a - 2 \Rightarrow a = 3$  and then  $b = 1$ .  
Thus, the equation is  $x = 3y^2 + y - 1$ .
37. The ellipse with foci  $(\pm 2, 0)$  and vertices  $(\pm 5, 0)$  has center  $(0, 0)$  and a horizontal major axis, with  $a = 5$  and  $c = 2$ ,  
so  $b^2 = a^2 - c^2 = 25 - 4 = 21$ . An equation is  $\frac{x^2}{25} + \frac{y^2}{21} = 1$ .
38. The ellipse with foci  $(0, \pm 5)$  and vertices  $(0, \pm 13)$  has center  $(0, 0)$  and a vertical major axis, with  $c = 5$  and  $a = 13$ ,  
so  $b = \sqrt{a^2 - c^2} = 12$ . An equation is  $\frac{x^2}{144} + \frac{y^2}{169} = 1$ .
39. Since the vertices are  $(0, 0)$  and  $(0, 8)$ , the ellipse has center  $(0, 4)$  with a vertical axis and  $a = 4$ . The foci at  $(0, 2)$  and  $(0, 6)$  are 2 units from the center, so  $c = 2$  and  $b = \sqrt{a^2 - c^2} = \sqrt{4^2 - 2^2} = \sqrt{12}$ . An equation is  $\frac{(x - 0)^2}{b^2} + \frac{(y - 4)^2}{a^2} = 1 \Rightarrow \frac{x^2}{12} + \frac{(y - 4)^2}{16} = 1$ .
40. Since the foci are  $(0, -1)$  and  $(8, -1)$ , the ellipse has center  $(4, -1)$  with a horizontal axis and  $c = 4$ .  
The vertex  $(9, -1)$  is 5 units from the center, so  $a = 5$  and  $b = \sqrt{a^2 - c^2} = \sqrt{5^2 - 4^2} = \sqrt{9}$ . An equation is  $\frac{(x - 4)^2}{a^2} + \frac{(y + 1)^2}{b^2} = 1 \Rightarrow \frac{(x - 4)^2}{25} + \frac{(y + 1)^2}{9} = 1$ .
41. An equation of an ellipse with center  $(-1, 4)$  and vertex  $(-1, 0)$  is  $\frac{(x + 1)^2}{b^2} + \frac{(y - 4)^2}{4^2} = 1$ . The focus  $(-1, 6)$  is 2 units from the center, so  $c = 2$ . Thus,  $b^2 + 2^2 = 4^2 \Rightarrow b^2 = 12$ , and the equation is  $\frac{(x + 1)^2}{12} + \frac{(y - 4)^2}{16} = 1$ .

42. Foci  $F_1(-4, 0)$  and  $F_2(4, 0) \Rightarrow c = 4$  and an equation is  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ . The ellipse passes through  $P(-4, 1.8)$ , so

$$2a = |PF_1| + |PF_2| \Rightarrow 2a = 1.8 + \sqrt{8^2 + (1.8)^2} \Rightarrow 2a = 1.8 + 8.2 \Rightarrow a = 5.$$

$$b^2 = a^2 - c^2 = 25 - 16 = 9 \text{ and the equation is } \frac{x^2}{25} + \frac{y^2}{9} = 1.$$

43. An equation of a hyperbola with vertices  $(\pm 3, 0)$  is  $\frac{x^2}{3^2} - \frac{y^2}{b^2} = 1$ . Foci  $(\pm 5, 0) \Rightarrow c = 5$  and  $3^2 + b^2 = 5^2 \Rightarrow$

$$b^2 = 25 - 9 = 16, \text{ so the equation is } \frac{x^2}{9} - \frac{y^2}{16} = 1.$$

44. An equation of a hyperbola with vertices  $(0, \pm 2)$  is  $\frac{y^2}{2^2} - \frac{x^2}{b^2} = 1$ . Foci  $(0, \pm 5) \Rightarrow c = 5$  and  $2^2 + b^2 = 5^2 \Rightarrow$

$$b^2 = 25 - 4 = 21, \text{ so the equation is } \frac{y^2}{4} - \frac{x^2}{21} = 1.$$

45. The center of a hyperbola with vertices  $(-3, -4)$  and  $(-3, 6)$  is  $(-3, 1)$ , so  $a = 5$  and an equation is

$$\frac{(y-1)^2}{5^2} - \frac{(x+3)^2}{b^2} = 1. \text{ Foci } (-3, -7) \text{ and } (-3, 9) \Rightarrow c = 8, \text{ so } 5^2 + b^2 = 8^2 \Rightarrow b^2 = 64 - 25 = 39 \text{ and the}$$

$$\text{equation is } \frac{(y-1)^2}{25} - \frac{(x+3)^2}{39} = 1.$$

46. The center of a hyperbola with vertices  $(-1, 2)$  and  $(7, 2)$  is  $(3, 2)$ , so  $a = 4$  and an equation is  $\frac{(x-3)^2}{4^2} - \frac{(y-2)^2}{b^2} = 1$ .

$$\text{Foci } (-2, 2) \text{ and } (8, 2) \Rightarrow c = 5, \text{ so } 4^2 + b^2 = 5^2 \Rightarrow b^2 = 25 - 16 = 9 \text{ and the equation is}$$

$$\frac{(x-3)^2}{16} - \frac{(y-2)^2}{9} = 1.$$

47. The center of a hyperbola with vertices  $(\pm 3, 0)$  is  $(0, 0)$ , so  $a = 3$  and an equation is  $\frac{x^2}{3^2} - \frac{y^2}{b^2} = 1$ .

$$\text{Asymptotes } y = \pm 2x \Rightarrow \frac{b}{a} = 2 \Rightarrow b = 2(3) = 6 \text{ and the equation is } \frac{x^2}{9} - \frac{y^2}{36} = 1.$$

48. The center of a hyperbola with foci  $(2, 0)$  and  $(2, 8)$  is  $(2, 4)$ , so  $c = 4$  and an equation is  $\frac{(y-4)^2}{a^2} - \frac{(x-2)^2}{b^2} = 1$ .

$$\text{The asymptote } y = 3 + \frac{1}{2}x \text{ has slope } \frac{1}{2}, \text{ so } \frac{a}{b} = \frac{1}{2} \Rightarrow b = 2a \text{ and } a^2 + b^2 = c^2 \Rightarrow a^2 + (2a)^2 = 4^2 \Rightarrow$$

$$5a^2 = 16 \Rightarrow a^2 = \frac{16}{5} \text{ and so } b^2 = 16 - \frac{16}{5} = \frac{64}{5}. \text{ Thus, an equation is } \frac{(y-4)^2}{16/5} - \frac{(x-2)^2}{64/5} = 1.$$

49. In Figure 8, we see that the point on the ellipse closest to a focus is the closer vertex (which is a distance

$a - c$  from it) while the farthest point is the other vertex (at a distance of  $a + c$ ). So for this lunar orbit,

$$(a - c) + (a + c) = 2a = (1728 + 110) + (1728 + 314), \text{ or } a = 1940; \text{ and } (a + c) - (a - c) = 2c = 314 - 110,$$

$$\text{or } c = 102. \text{ Thus, } b^2 = a^2 - c^2 = 3,753,196, \text{ and the equation is } \frac{x^2}{3,763,600} + \frac{y^2}{3,753,196} = 1.$$

50. (a) Choose  $V$  to be the origin, with  $x$ -axis through  $V$  and  $F$ . Then  $F$  is  $(p, 0)$ ,  $A$  is  $(p, 5)$ , so substituting  $A$  into the equation  $y^2 = 4px$  gives  $25 = 4p^2$  so  $p = \frac{5}{2}$  and  $y^2 = 10x$ .

(b)  $x = 11 \Rightarrow y = \sqrt{110} \Rightarrow |CD| = 2\sqrt{110}$

51. (a) Set up the coordinate system so that  $A$  is  $(-200, 0)$  and  $B$  is  $(200, 0)$ .

$$|PA| - |PB| = (1200)(980) = 1,176,000 \text{ ft} = \frac{2450}{11} \text{ mi} = 2a \Rightarrow a = \frac{1225}{11}, \text{ and } c = 200 \text{ so}$$

$$b^2 = c^2 - a^2 = \frac{3,339,375}{121} \Rightarrow \frac{121x^2}{1,500,625} - \frac{121y^2}{3,339,375} = 1.$$

(b) Due north of  $B \Rightarrow x = 200 \Rightarrow \frac{(121)(200)^2}{1,500,625} - \frac{121y^2}{3,339,375} = 1 \Rightarrow y = \frac{133,575}{539} \approx 248 \text{ mi}$

52.  $|PF_1| - |PF_2| = \pm 2a \Leftrightarrow \sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = \pm 2a \Leftrightarrow$

$$\sqrt{(x+c)^2 + y^2} = \sqrt{(x-c)^2 + y^2} \pm 2a \Leftrightarrow (x+c)^2 + y^2 = (x-c)^2 + y^2 + 4a^2 \pm 4a\sqrt{(x-c)^2 + y^2} \Leftrightarrow$$

$$4cx - 4a^2 = \pm 4a\sqrt{(x-c)^2 + y^2} \Leftrightarrow c^2x^2 - 2a^2cx + a^4 = a^2(x^2 - 2cx + c^2 + y^2) \Leftrightarrow$$

$$(c^2 - a^2)x^2 - a^2y^2 = a^2(c^2 - a^2) \Leftrightarrow b^2x^2 - a^2y^2 = a^2b^2 \text{ [where } b^2 = c^2 - a^2] \Leftrightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

53. The function whose graph is the upper branch of this hyperbola is concave upward. The function is

$$y = f(x) = a\sqrt{1 + \frac{x^2}{b^2}} = \frac{a}{b}\sqrt{b^2 + x^2}, \text{ so } y' = \frac{a}{b}x(b^2 + x^2)^{-1/2} \text{ and}$$

$$y'' = \frac{a}{b}[(b^2 + x^2)^{-1/2} - x^2(b^2 + x^2)^{-3/2}] = ab(b^2 + x^2)^{-3/2} > 0 \text{ for all } x, \text{ and so } f \text{ is concave upward.}$$

54. We can follow exactly the same sequence of steps as in the derivation of Formula 4, except we use the points  $(1, 1)$  and

$$(-1, -1) \text{ in the distance formula (first equation of that derivation) so } \sqrt{(x-1)^2 + (y-1)^2} + \sqrt{(x+1)^2 + (y+1)^2} = 4$$

$$\text{will lead (after moving the second term to the right, squaring, and simplifying) to } 2\sqrt{(x+1)^2 + (y+1)^2} = x + y + 4,$$

$$\text{which, after squaring and simplifying again, leads to } 3x^2 - 2xy + 3y^2 = 8.$$

55. (a) If  $k > 16$ , then  $k - 16 > 0$ , and  $\frac{x^2}{k} + \frac{y^2}{k-16} = 1$  is an *ellipse* since it is the sum of two squares on the left side.

(b) If  $0 < k < 16$ , then  $k - 16 < 0$ , and  $\frac{x^2}{k} + \frac{y^2}{k-16} = 1$  is a *hyperbola* since it is the difference of two squares on the left side.

(c) If  $k < 0$ , then  $k - 16 < 0$ , and there is *no curve* since the left side is the sum of two negative terms, which cannot equal 1.

(d) In case (a),  $a^2 = k$ ,  $b^2 = k - 16$ , and  $c^2 = a^2 - b^2 = 16$ , so the foci are at  $(\pm 4, 0)$ . In case (b),  $k - 16 < 0$ , so  $a^2 = k$ ,  $b^2 = 16 - k$ , and  $c^2 = a^2 + b^2 = 16$ , and so again the foci are at  $(\pm 4, 0)$ .

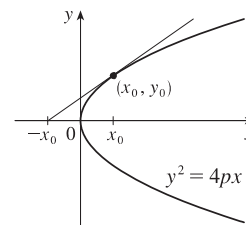


56. (a)  $y^2 = 4px \Rightarrow 2yy' = 4p \Rightarrow y' = \frac{2p}{y}$ , so the tangent line is

$$y - y_0 = \frac{2p}{y_0}(x - x_0) \Rightarrow yy_0 - y_0^2 = 2p(x - x_0) \Leftrightarrow$$

$$yy_0 - 4px_0 = 2px - 2px_0 \Rightarrow yy_0 = 2p(x + x_0).$$

- (b) The  $x$ -intercept is  $-x_0$ .



57.  $x^2 = 4py \Rightarrow 2x = 4py' \Rightarrow y' = \frac{x}{2p}$ , so the tangent line at  $(x_0, y_0)$  is

$$y - \frac{x_0^2}{4p} = \frac{x_0}{2p}(x - x_0). \text{ This line passes through the point } (a, -p) \text{ on the}$$

$$\text{directrix, so } -p - \frac{x_0^2}{4p} = \frac{x_0}{2p}(a - x_0) \Rightarrow -4p^2 - x_0^2 = 2ax_0 - 2x_0^2 \Leftrightarrow$$

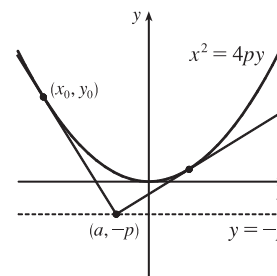
$$x_0^2 - 2ax_0 - 4p^2 = 0 \Leftrightarrow x_0^2 - 2ax_0 + a^2 = a^2 + 4p^2 \Leftrightarrow$$

$$(x_0 - a)^2 = a^2 + 4p^2 \Leftrightarrow x_0 = a \pm \sqrt{a^2 + 4p^2}. \text{ The slopes of the tangent lines at } x = a \pm \sqrt{a^2 + 4p^2}$$

are  $\frac{a \pm \sqrt{a^2 + 4p^2}}{2p}$ , so the product of the two slopes is

$$\frac{a + \sqrt{a^2 + 4p^2}}{2p} \cdot \frac{a - \sqrt{a^2 + 4p^2}}{2p} = \frac{a^2 - (a^2 + 4p^2)}{4p^2} = \frac{-4p^2}{4p^2} = -1,$$

showing that the tangent lines are perpendicular.



58. Without a loss of generality, let the ellipse, hyperbola, and foci be as shown in the figure.

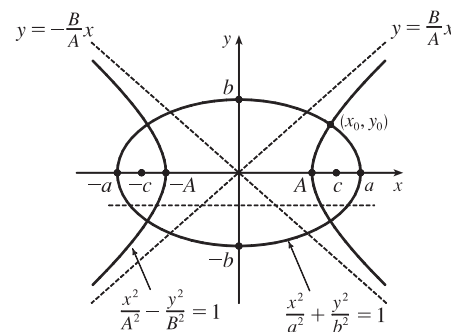
The curves intersect (eliminate  $y^2$ )  $\Rightarrow$

$$B^2 \left( \frac{x^2}{A^2} - \frac{y^2}{B^2} \right) + b^2 \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} \right) = B^2 + b^2 \Rightarrow$$

$$\frac{B^2 x^2}{A^2} + \frac{b^2 x^2}{a^2} = B^2 + b^2 \Rightarrow x^2 \left( \frac{B^2}{A^2} + \frac{b^2}{a^2} \right) = B^2 + b^2 \Rightarrow$$

$$x^2 = \frac{B^2 + b^2}{\frac{B^2}{A^2} + \frac{b^2}{a^2}} = \frac{A^2 a^2 (B^2 + b^2)}{a^2 B^2 + b^2 A^2}.$$

$$\text{Similarly, } y^2 = \frac{B^2 b^2 (a^2 - A^2)}{b^2 A^2 + a^2 B^2}.$$



$$\text{Next we find the slopes of the tangent lines of the curves: } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{2x}{a^2} + \frac{2yy'}{b^2} = 0 \Rightarrow \frac{yy'}{b^2} = -\frac{x}{a^2} \Rightarrow$$

$$y'_E = -\frac{b^2}{a^2} \frac{x}{y} \text{ and } \frac{x^2}{A^2} - \frac{y^2}{B^2} = 1 \Rightarrow \frac{2x}{A^2} - \frac{2yy'}{B^2} = 0 \Rightarrow \frac{yy'}{B^2} = \frac{x}{A^2} \Rightarrow y'_H = \frac{B^2}{A^2} \frac{x}{y}. \text{ The product of the slopes}$$

$$\text{at } (x_0, y_0) \text{ is } y'_E y'_H = -\frac{b^2 B^2 x_0^2}{a^2 A^2 y_0^2} = -\frac{b^2 B^2 \left[ \frac{A^2 a^2 (B^2 + b^2)}{a^2 B^2 + b^2 A^2} \right]}{a^2 A^2 \left[ \frac{B^2 b^2 (a^2 - A^2)}{b^2 A^2 + a^2 B^2} \right]} = -\frac{B^2 + b^2}{a^2 - A^2}. \text{ Since } a^2 - b^2 = c^2 \text{ and } A^2 + B^2 = c^2,$$

we have  $a^2 - b^2 = A^2 + B^2 \Rightarrow a^2 - A^2 = b^2 + B^2$ , so the product of the slopes is  $-1$ , and hence, the tangent lines at each point of intersection are perpendicular.

59.  $9x^2 + 4y^2 = 36 \Leftrightarrow \frac{x^2}{4} + \frac{y^2}{9} = 1$ . We use the parametrization  $x = 2 \cos t$ ,  $y = 3 \sin t$ ,  $0 \leq t \leq 2\pi$ . The circumference is given by

$$L = \int_0^{2\pi} \sqrt{(dx/dt)^2 + (dy/dt)^2} dt = \int_0^{2\pi} \sqrt{(-2 \sin t)^2 + (3 \cos t)^2} dt = \int_0^{2\pi} \sqrt{4 \sin^2 t + 9 \cos^2 t} dt \\ = \int_0^{2\pi} \sqrt{4 + 5 \cos^2 t} dt$$

Now use Simpson's Rule with  $n = 8$ ,  $\Delta t = \frac{2\pi - 0}{8} = \frac{\pi}{4}$ , and  $f(t) = \sqrt{4 + 5 \cos^2 t}$  to get

$$L \approx S_8 = \frac{\pi/4}{3} [f(0) + 4f(\frac{\pi}{4}) + 2f(\frac{\pi}{2}) + 4f(\frac{3\pi}{4}) + 2f(\pi) + 4f(\frac{5\pi}{4}) + 2f(\frac{3\pi}{2}) + 4f(\frac{7\pi}{4}) + f(2\pi)] \approx 15.9.$$

60. The length of the major axis is  $2a$ , so  $a = \frac{1}{2}(1.18 \times 10^{10}) = 5.9 \times 10^9$ . The length of the minor axis is  $2b$ , so

$b = \frac{1}{2}(1.14 \times 10^{10}) = 5.7 \times 10^9$ . An equation of the ellipse is  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ , or converting into parametric equations,  $x = a \cos \theta$  and  $y = b \sin \theta$ . So

$$L = 4 \int_0^{\pi/2} \sqrt{(dx/d\theta)^2 + (dy/d\theta)^2} d\theta = 4 \int_0^{\pi/2} \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta} d\theta$$

Using Simpson's Rule with  $n = 10$ ,  $\Delta \theta = \frac{\pi/2 - 0}{10} = \frac{\pi}{20}$ , and  $f(\theta) = \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta}$ , we get

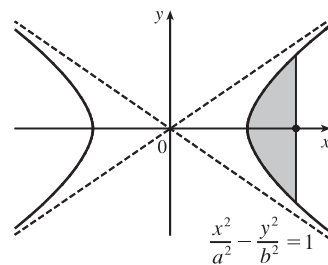
$$L \approx 4 \cdot S_{10} = 4 \cdot \frac{\pi}{20 \cdot 3} [f(0) + 4f(\frac{\pi}{20}) + 2f(\frac{2\pi}{20}) + \cdots + 2f(\frac{8\pi}{20}) + 4f(\frac{9\pi}{20}) + f(\frac{\pi}{2})] \approx 3.64 \times 10^{10} \text{ km}$$

61.  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow \frac{y^2}{b^2} = \frac{x^2 - a^2}{a^2} \Rightarrow y = \pm \frac{b}{a} \sqrt{x^2 - a^2}$ .

$$A = 2 \int_a^c \frac{b}{a} \sqrt{x^2 - a^2} dx \stackrel{39}{=} \frac{2b}{a} \left[ \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \ln |x + \sqrt{x^2 - a^2}| \right]_a^c \\ = \frac{b}{a} [c \sqrt{c^2 - a^2} - a^2 \ln |c + \sqrt{c^2 - a^2}| + a^2 \ln |a|]$$

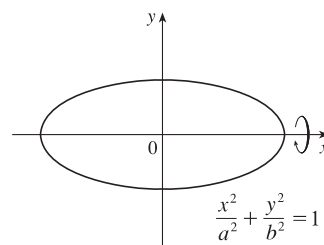
Since  $a^2 + b^2 = c^2$ ,  $c^2 - a^2 = b^2$ , and  $\sqrt{c^2 - a^2} = b$ .

$$= \frac{b}{a} [cb - a^2 \ln(c + b) + a^2 \ln a] = \frac{b}{a} [cb + a^2 (\ln a - \ln(b + c))] \\ = b^2 c/a + ab \ln[a/(b + c)], \text{ where } c^2 = a^2 + b^2.$$



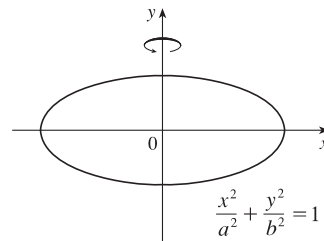
62. (a)  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{y^2}{b^2} = \frac{a^2 - x^2}{a^2} \Rightarrow y = \pm \frac{b}{a} \sqrt{a^2 - x^2}$ .

$$V = \int_{-a}^a \pi \left( \frac{b}{a} \sqrt{a^2 - x^2} \right)^2 dx = 2\pi \frac{b^2}{a^2} \int_0^a (a^2 - x^2) dx \\ = \frac{2\pi b^2}{a^2} \left[ a^2 x - \frac{1}{3} x^3 \right]_0^a = \frac{2\pi b^2}{a^2} \left( \frac{2a^3}{3} \right) = \frac{4}{3} \pi b^2 a$$



- (b)  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{x^2}{a^2} = \frac{b^2 - y^2}{b^2} \Rightarrow x = \pm \frac{a}{b} \sqrt{b^2 - y^2}$ .

$$V = \int_{-b}^b \pi \left( \frac{a}{b} \sqrt{b^2 - y^2} \right)^2 dy = 2\pi \frac{a^2}{b^2} \int_0^b (b^2 - y^2) dy \\ = \frac{2\pi a^2}{b^2} \left[ b^2 y - \frac{1}{3} y^3 \right]_0^b = \frac{2\pi a^2}{b^2} \left( \frac{2b^3}{3} \right) = \frac{4}{3} \pi a^2 b$$

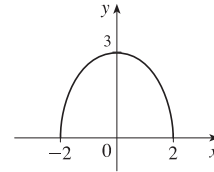


63.  $9x^2 + 4y^2 = 36 \Leftrightarrow \frac{x^2}{4} + \frac{y^2}{9} = 1 \Rightarrow a = 3, b = 2$ . By symmetry,  $\bar{x} = 0$ . By Example 2 in Section 7.3, the area of the top half of the ellipse is  $\frac{1}{2}(\pi ab) = 3\pi$ . Solve  $9x^2 + 4y^2 = 36$  for  $y$  to get an equation for the top half of the ellipse:

$$9x^2 + 4y^2 = 36 \Leftrightarrow 4y^2 = 36 - 9x^2 \Leftrightarrow y^2 = \frac{9}{4}(4 - x^2) \Rightarrow y = \frac{3}{2}\sqrt{4 - x^2}.$$
 Now

$$\begin{aligned} \bar{y} &= \frac{1}{A} \int_a^b \frac{1}{2}[f(x)]^2 dx = \frac{1}{3\pi} \int_{-2}^2 \frac{1}{2} \left( \frac{3}{2} \sqrt{4 - x^2} \right)^2 dx = \frac{3}{8\pi} \int_{-2}^2 (4 - x^2) dx \\ &= \frac{3}{8\pi} \cdot 2 \int_0^2 (4 - x^2) dx = \frac{3}{4\pi} \left[ 4x - \frac{1}{3}x^3 \right]_0^2 = \frac{3}{4\pi} \left( \frac{16}{3} \right) = \frac{4}{\pi} \end{aligned}$$

so the centroid is  $(0, 4/\pi)$ .



64. (a) Consider the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  with  $a > b$ , so that the major axis is the  $x$ -axis. Let the ellipse be parametrized by

$$x = a \cos t, y = b \sin t, 0 \leq t \leq 2\pi. \text{ Then}$$

$$\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 = a^2 \sin^2 t + b^2 \cos^2 t = a^2(1 - \cos^2 t) + b^2 \cos^2 t = a^2 + (b^2 - a^2) \cos^2 t = a^2 - c^2 \cos^2 t,$$

where  $c^2 = a^2 - b^2$ . Using symmetry and rotating the ellipse about the major axis gives us surface area

$$\begin{aligned} S &= \int 2\pi y ds = 2 \int_0^{\pi/2} 2\pi(b \sin t) \sqrt{a^2 - c^2 \cos^2 t} dt = 4\pi b \int_c^0 \sqrt{a^2 - u^2} \left( -\frac{1}{c} du \right) \quad \left[ \begin{array}{l} u = c \cos t \\ du = -c \sin t dt \end{array} \right] \\ &= \frac{4\pi b}{c} \int_0^c \sqrt{a^2 - u^2} du \stackrel{30}{=} \frac{4\pi b}{c} \left[ \frac{u}{2} \sqrt{a^2 - u^2} + \frac{a^2}{2} \sin^{-1} \left( \frac{u}{a} \right) \right]_0^c = \frac{2\pi b}{c} \left[ c \sqrt{a^2 - c^2} + a^2 \sin^{-1} \left( \frac{c}{a} \right) \right] \\ &= \frac{2\pi b}{c} \left[ bc + a^2 \sin^{-1} \left( \frac{c}{a} \right) \right] \end{aligned}$$

- (b) As in part (a),

$$\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 = a^2 \sin^2 t + b^2 \cos^2 t = a^2 \sin^2 t + b^2(1 - \sin^2 t) = b^2 + (a^2 - b^2) \sin^2 t = b^2 + c^2 \sin^2 t.$$

Rotating about the minor axis gives us

$$\begin{aligned} S &= \int 2\pi x ds = 2 \int_0^{\pi/2} 2\pi(a \cos t) \sqrt{b^2 + c^2 \sin^2 t} dt = 4\pi a \int_0^c \sqrt{b^2 + u^2} \left( \frac{1}{c} du \right) \quad \left[ \begin{array}{l} u = c \sin t \\ du = c \cos t dt \end{array} \right] \\ &\stackrel{21}{=} \frac{4\pi a}{c} \left[ \frac{u}{2} \sqrt{b^2 + u^2} + \frac{b^2}{2} \ln(u + \sqrt{b^2 + u^2}) \right]_0^c = \frac{2\pi a}{c} [c \sqrt{b^2 + c^2} + b^2 \ln(c + \sqrt{b^2 + c^2}) - b^2 \ln b] \\ &= \frac{2\pi a}{c} \left[ ac + b^2 \ln \left( \frac{a + c}{b} \right) \right] \end{aligned}$$

65. Differentiating implicitly,  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{2x}{a^2} + \frac{2yy'}{b^2} = 0 \Rightarrow y' = -\frac{b^2 x}{a^2 y}$  [ $y \neq 0$ ]. Thus, the slope of the tangent

line at  $P$  is  $-\frac{b^2 x_1}{a^2 y_1}$ . The slope of  $F_1 P$  is  $\frac{y_1}{x_1 + c}$  and of  $F_2 P$  is  $\frac{y_1}{x_1 - c}$ . By the formula from Problems Plus, we have

$$\begin{aligned} \tan \alpha &= \frac{\frac{y_1}{x_1 + c} + \frac{b^2 x_1}{a^2 y_1}}{1 - \frac{b^2 x_1 y_1}{a^2 y_1 (x_1 + c)}} = \frac{\frac{a^2 y_1^2 + b^2 x_1 (x_1 + c)}{a^2 y_1 (x_1 + c)}}{\frac{a^2 y_1^2 + b^2 x_1 y_1}{a^2 y_1 (x_1 + c)}} = \frac{a^2 b^2 + b^2 c x_1}{c^2 x_1 y_1 + a^2 c y_1} \quad \left[ \begin{array}{l} \text{using } b^2 x_1^2 + a^2 y_1^2 = a^2 b^2, \\ \text{and } a^2 - b^2 = c^2 \end{array} \right] \\ &= \frac{b^2 (c x_1 + a^2)}{c y_1 (c x_1 + a^2)} = \frac{b^2}{c y_1} \end{aligned}$$

[continued]

and

$$\tan \beta = \frac{-\frac{b^2 x_1}{a^2 y_1} - \frac{y_1}{x_1 - c}}{1 - \frac{b^2 x_1 y_1}{a^2 y_1 (x_1 - c)}} = \frac{-a^2 y_1^2 - b^2 x_1 (x_1 - c)}{a^2 y_1 (x_1 - c) - b^2 x_1 y_1} = \frac{-a^2 b^2 + b^2 c x_1}{c^2 x_1 y_1 - a^2 c y_1} = \frac{b^2 (c x_1 - a^2)}{c y_1 (c x_1 - a^2)} = \frac{b^2}{c y_1}$$

Thus,  $\alpha = \beta$ .

66. The slopes of the line segments  $F_1 P$  and  $F_2 P$  are  $\frac{y_1}{x_1 + c}$  and  $\frac{y_1}{x_1 - c}$ , where  $P$  is  $(x_1, y_1)$ . Differentiating implicitly,

$$\frac{2x}{a^2} - \frac{2yy'}{b^2} = 0 \Rightarrow y' = \frac{b^2 x}{a^2 y} \Rightarrow \text{the slope of the tangent at } P \text{ is } \frac{b^2 x_1}{a^2 y_1}, \text{ so by the formula in Problem 19 on text page 271,}$$

$$\tan \alpha = \frac{\frac{b^2 x_1}{a^2 y_1} - \frac{y_1}{x_1 + c}}{1 + \frac{b^2 x_1 y_1}{a^2 y_1 (x_1 + c)}} = \frac{b^2 x_1 (x_1 + c) - a^2 y_1^2}{a^2 y_1 (x_1 + c) + b^2 x_1 y_1} = \frac{b^2 (c x_1 + a^2)}{c y_1 (c x_1 + a^2)} \left[ \begin{array}{l} \text{using } x_1^2/a^2 - y_1^2/b^2 = 1, \\ \text{and } a^2 + b^2 = c^2 \end{array} \right] = \frac{b^2}{c y_1}$$

and

$$\tan \beta = \frac{-\frac{b^2 x_1}{a^2 y_1} + \frac{y_1}{x_1 - c}}{1 + \frac{b^2 x_1 y_1}{a^2 y_1 (x_1 - c)}} = \frac{-b^2 x_1 (x_1 - c) + a^2 y_1^2}{a^2 y_1 (x_1 - c) + b^2 x_1 y_1} = \frac{b^2 (c x_1 - a^2)}{c y_1 (c x_1 - a^2)} = \frac{b^2}{c y_1}$$

So  $\alpha = \beta$ .

## 10.6 Conic Sections in Polar Coordinates

1. The directrix  $x = 4$  is to the right of the focus at the origin, so we use the form with “+  $e \cos \theta$ ” in the denominator.

(See Theorem 6 and Figure 2.) An equation is  $r = \frac{ed}{1 + e \cos \theta} = \frac{\frac{1}{2} \cdot 4}{1 + \frac{1}{2} \cos \theta} = \frac{4}{2 + \cos \theta}$ .

2. The directrix  $x = -3$  is to the left of the focus at the origin, so we use the form with “−  $e \cos \theta$ ” in the denominator.

$e = 1$  for a parabola, so an equation is  $r = \frac{ed}{1 - e \cos \theta} = \frac{1 \cdot 3}{1 - 1 \cos \theta} = \frac{3}{1 - \cos \theta}$ .

3. The directrix  $y = 2$  is above the focus at the origin, so we use the form with “+  $e \sin \theta$ ” in the denominator. An equation is

$$r = \frac{ed}{1 + e \sin \theta} = \frac{1.5(2)}{1 + 1.5 \sin \theta} = \frac{6}{2 + 3 \sin \theta}.$$

4. The directrix  $x = 3$  is to the right of the focus at the origin, so we use the form with “+  $e \cos \theta$ ” in the denominator. An

equation is  $r = \frac{ed}{1 + e \cos \theta} = \frac{3 \cdot 3}{1 + 3 \cos \theta} = \frac{9}{1 + 3 \cos \theta}$ .

5. The vertex  $(4, 3\pi/2)$  is 4 units below the focus at the origin, so the directrix is 8 units below the focus ( $d = 8$ ), and we use the form with “−  $e \sin \theta$ ” in the denominator.  $e = 1$  for a parabola, so an equation is

$$r = \frac{ed}{1 - e \sin \theta} = \frac{1(8)}{1 - 1 \sin \theta} = \frac{8}{1 - \sin \theta}.$$

6. The vertex  $P(1, \pi/2)$  is 1 unit above the focus  $F$  at the origin, so  $|PF| = 1$  and we use the form with “+  $e \sin \theta$ ” in the denominator. The distance from the focus to the directrix  $l$  is  $d$ , so

$$e = \frac{|PF|}{|Pl|} \Rightarrow 0.8 = \frac{1}{d-1} \Rightarrow 0.8d - 0.8 = 1 \Rightarrow 0.8d = 1.8 \Rightarrow d = 2.25.$$

$$\text{An equation is } r = \frac{ed}{1 + e \sin \theta} = \frac{0.8(2.25)}{1 + 0.8 \sin \theta} \cdot \frac{5}{5} = \frac{9}{5 + 4 \sin \theta}.$$

7. The directrix  $r = 4 \sec \theta$  (equivalent to  $r \cos \theta = 4$  or  $x = 4$ ) is to the right of the focus at the origin, so we will use the form with “+  $e \cos \theta$ ” in the denominator. The distance from the focus to the directrix is  $d = 4$ , so an equation is

$$r = \frac{ed}{1 + e \cos \theta} = \frac{\frac{1}{2}(4)}{1 + \frac{1}{2} \cos \theta} \cdot \frac{2}{2} = \frac{4}{2 + \cos \theta}.$$

8. The directrix  $r = -6 \csc \theta$  (equivalent to  $r \sin \theta = -6$  or  $y = -6$ ) is below the focus at the origin, so we will use the form with “-  $e \sin \theta$ ” in the denominator. The distance from the focus to the directrix is  $d = 6$ , so an equation is

$$r = \frac{ed}{1 - e \sin \theta} = \frac{3(6)}{1 - 3 \sin \theta} = \frac{18}{1 - 3 \sin \theta}.$$

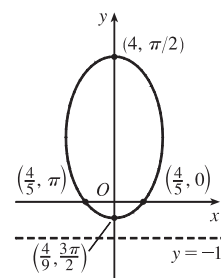
9.  $r = \frac{4}{5 - 4 \sin \theta} \cdot \frac{1/5}{1/5} = \frac{4/5}{1 - \frac{4}{5} \sin \theta}$ , where  $e = \frac{4}{5}$  and  $ed = \frac{4}{5} \Rightarrow d = 1$ .

(a) Eccentricity =  $e = \frac{4}{5}$

(b) Since  $e = \frac{4}{5} < 1$ , the conic is an ellipse.

(c) Since “-  $e \sin \theta$ ” appears in the denominator, the directrix is below the focus at the origin,  $d = |Fl| = 1$ , so an equation of the directrix is  $y = -1$ .

(d) The vertices are  $(4, \frac{\pi}{2})$  and  $(\frac{4}{9}, \frac{3\pi}{2})$ .



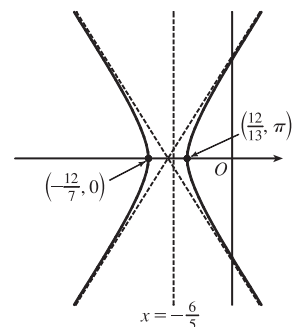
10.  $r = \frac{12}{3 - 10 \cos \theta} \cdot \frac{1/3}{1/3} = \frac{4}{1 - \frac{10}{3} \cos \theta}$ , where  $e = \frac{10}{3}$  and  $ed = 4 \Rightarrow d = 4(\frac{3}{10}) = \frac{6}{5}$ .

(a) Eccentricity =  $e = \frac{10}{3}$

(b) Since  $e = \frac{10}{3} > 1$ , the conic is a hyperbola.

(c) Since “-  $e \cos \theta$ ” appears in the denominator, the directrix is to the left of the focus at the origin.  $d = |Fl| = \frac{6}{5}$ , so an equation of the directrix is  $x = -\frac{6}{5}$ .

(d) The vertices are  $(-\frac{12}{7}, 0)$  and  $(\frac{12}{13}, \pi)$ , so the center is midway between them, that is,  $(\frac{120}{91}, \pi)$ .



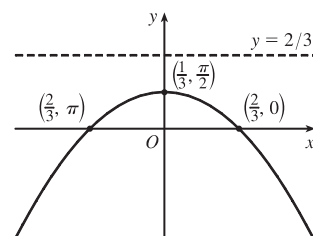
11.  $r = \frac{2}{3 + 3 \sin \theta} \cdot \frac{1/3}{1/3} = \frac{2/3}{1 + \sin \theta}$ , where  $e = 1$  and  $ed = \frac{2}{3} \Rightarrow d = \frac{2}{3}$ .

(a) Eccentricity =  $e = 1$

(b) Since  $e = 1$ , the conic is a parabola.

(c) Since “+  $e \sin \theta$ ” appears in the denominator, the directrix is above the focus at the origin.  $d = |Fl| = \frac{2}{3}$ , so an equation of the directrix is  $y = \frac{2}{3}$ .

(d) The vertex is at  $(\frac{1}{3}, \frac{\pi}{2})$ , midway between the focus and directrix.



$$12. r = \frac{3}{2 + 2\cos\theta} \cdot \frac{1/2}{1/2} = \frac{3/2}{1 + \cos\theta}, \text{ where } e = 1 \text{ and } ed = \frac{3}{2} \Rightarrow d = \frac{3}{2}.$$

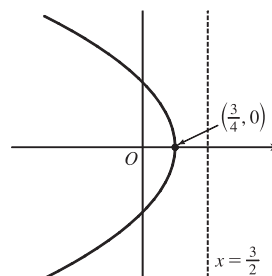
(a) Eccentricity =  $e = 1$

(b) Since  $e = 1$ , the conic is a parabola.

(c) Since “ $+e\cos\theta$ ” appears in the denominator, the directrix is to the right of the focus at the origin.  $d = |Fl| = \frac{3}{2}$ , so an equation of the directrix is

$$x = \frac{3}{2}.$$

(d) The vertex is at  $(\frac{3}{4}, 0)$ , midway between the focus and directrix.



$$13. r = \frac{9}{6 + 2\cos\theta} \cdot \frac{1/6}{1/6} = \frac{3/2}{1 + \frac{1}{3}\cos\theta}, \text{ where } e = \frac{1}{3} \text{ and } ed = \frac{3}{2} \Rightarrow d = \frac{9}{2}.$$

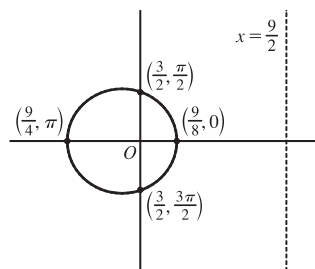
(a) Eccentricity =  $e = \frac{1}{3}$

(b) Since  $e = \frac{1}{3} < 1$ , the conic is an ellipse.

(c) Since “ $+e\cos\theta$ ” appears in the denominator, the directrix is to the right of the focus at the origin.  $d = |Fl| = \frac{9}{2}$ , so an equation of the directrix is

$$x = \frac{9}{2}.$$

(d) The vertices are  $(\frac{9}{8}, 0)$  and  $(\frac{9}{4}, \pi)$ , so the center is midway between them, that is,  $(\frac{9}{16}, \pi)$ .



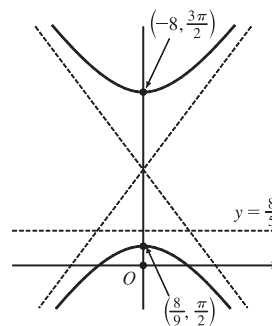
$$14. r = \frac{8}{4 + 5\sin\theta} \cdot \frac{1/4}{1/4} = \frac{2}{1 + \frac{5}{4}\sin\theta}, \text{ where } e = \frac{5}{4} \text{ and } ed = 2 \Rightarrow d = 2(\frac{4}{5}) = \frac{8}{5}.$$

(a) Eccentricity =  $e = \frac{5}{4}$

(b) Since  $e = \frac{5}{4} > 1$ , the conic is a hyperbola.

(c) Since “ $+e\sin\theta$ ” appears in the denominator, the directrix is above the focus at the origin.  $d = |Fl| = \frac{8}{5}$ , so an equation of the directrix is  $y = \frac{8}{5}$ .

(d) The vertices are  $(\frac{8}{9}, \frac{\pi}{2})$  and  $(-8, \frac{3\pi}{2})$ , so the center is midway between them, that is,  $(\frac{40}{9}, \frac{\pi}{2})$ .



$$15. r = \frac{3}{4 - 8\cos\theta} \cdot \frac{1/4}{1/4} = \frac{3/4}{1 - 2\cos\theta}, \text{ where } e = 2 \text{ and } ed = \frac{3}{4} \Rightarrow d = \frac{3}{8}.$$

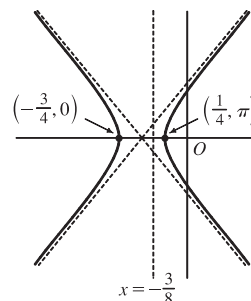
(a) Eccentricity =  $e = 2$

(b) Since  $e = 2 > 1$ , the conic is a hyperbola.

(c) Since “ $-e\cos\theta$ ” appears in the denominator, the directrix is to the left of the focus at the origin.  $d = |Fl| = \frac{3}{8}$ , so an equation of the directrix is

$$x = -\frac{3}{8}.$$

(d) The vertices are  $(-\frac{3}{4}, 0)$  and  $(\frac{1}{4}, \pi)$ , so the center is midway between them, that is,  $(\frac{1}{2}, \pi)$ .



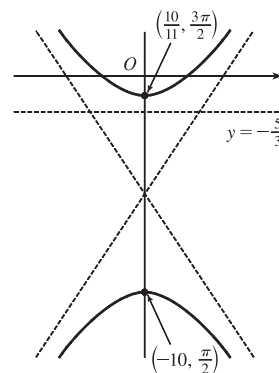
16.  $r = \frac{10}{5 - 6 \sin \theta} \cdot \frac{1/5}{1/5} = \frac{2}{1 - \frac{6}{5} \sin \theta}$ , where  $e = \frac{6}{5}$  and  $ed = 2 \Rightarrow d = 2\left(\frac{5}{6}\right) = \frac{5}{3}$ .

(a) Eccentricity  $= e = \frac{6}{5}$

(b) Since  $e = \frac{6}{5} > 1$ , the conic is a hyperbola.

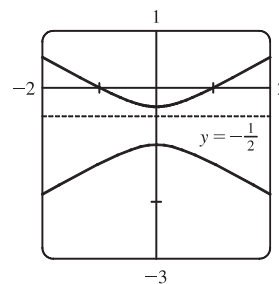
(c) Since “ $-e \sin \theta$ ” appears in the denominator, the directrix is below the focus at the origin.  $d = |Fl| = \frac{5}{3}$ , so an equation of the directrix is  $y = -\frac{5}{3}$ .

(d) The vertices are  $(-10, \frac{\pi}{2})$  and  $(\frac{10}{11}, \frac{3\pi}{2})$ , so the center is midway between them, that is,  $(\frac{60}{11}, \frac{3\pi}{2})$ .



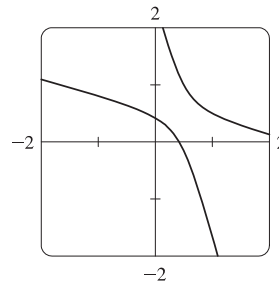
17. (a)  $r = \frac{1}{1 - 2 \sin \theta}$ , where  $e = 2$  and  $ed = 1 \Rightarrow d = \frac{1}{2}$ . The eccentricity

$e = 2 > 1$ , so the conic is a hyperbola. Since “ $-e \sin \theta$ ” appears in the denominator, the directrix is below the focus at the origin.  $d = |Fl| = \frac{1}{2}$ , so an equation of the directrix is  $y = -\frac{1}{2}$ . The vertices are  $(-1, \frac{\pi}{2})$  and  $(\frac{1}{3}, \frac{3\pi}{2})$ , so the center is midway between them, that is,  $(\frac{2}{3}, \frac{3\pi}{2})$ .



(b) By the discussion that precedes Example 4, the equation

$$\text{is } r = \frac{1}{1 - 2 \sin(\theta - \frac{3\pi}{4})}.$$

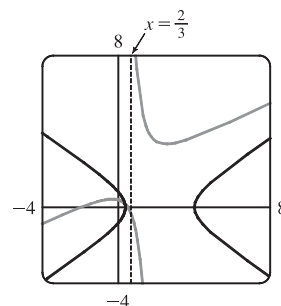


18.  $r = \frac{4}{5 + 6 \cos \theta} = \frac{4/5}{1 + \frac{6}{5} \cos \theta}$ , so  $e = \frac{6}{5}$  and  $ed = \frac{4}{5} \Rightarrow d = \frac{2}{3}$ .

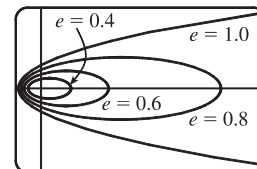
An equation of the directrix is  $x = \frac{2}{3} \Rightarrow r \cos \theta = \frac{2}{3} \Rightarrow r = \frac{2}{3 \cos \theta}$ .

If the hyperbola is rotated about its focus (the origin) through an angle  $\pi/3$ , its equation is the same as that of the original, with  $\theta$  replaced by  $\theta - \frac{\pi}{3}$

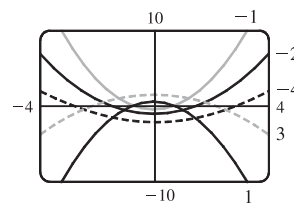
(see Example 4), so  $r = \frac{4}{5 + 6 \cos(\theta - \frac{\pi}{3})}$ .



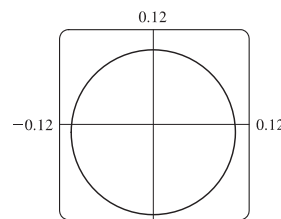
19. For  $e < 1$  the curve is an ellipse. It is nearly circular when  $e$  is close to 0. As  $e$  increases, the graph is stretched out to the right, and grows larger (that is, its right-hand focus moves to the right while its left-hand focus remains at the origin.) At  $e = 1$ , the curve becomes a parabola with focus at the origin.



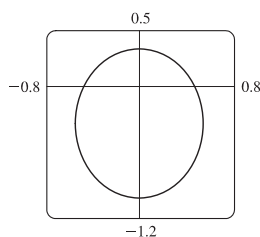
20. (a) The value of  $d$  does not seem to affect the shape of the conic (a parabola) at all, just its size, position, and orientation (for  $d < 0$  it opens upward, for  $d > 0$  it opens downward).



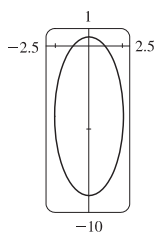
- (b) We consider only positive values of  $e$ . When  $0 < e < 1$ , the conic is an ellipse. As  $e \rightarrow 0^+$ , the graph approaches perfect roundness and zero size. As  $e$  increases, the ellipse becomes more elongated, until at  $e = 1$  it turns into a parabola. For  $e > 1$ , the conic is a hyperbola, which moves downward and gets broader as  $e$  continues to increase.



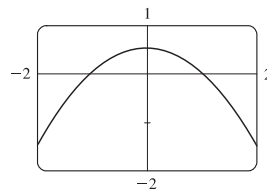
$e = 0.1$



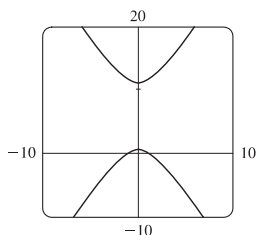
$e = 0.5$



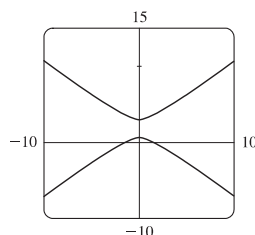
$e = 0.9$



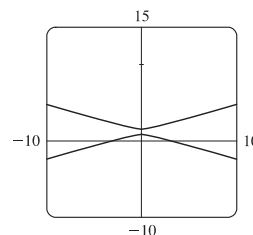
$e = 1$



$e = 1.1$

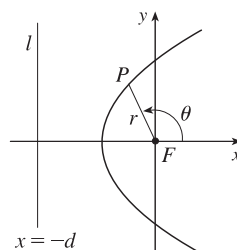


$e = 1.5$

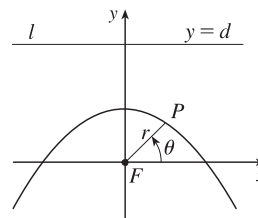


$e = 10$

$$21. |PF| = e|Pl| \Rightarrow r = e[d - r \cos(\pi - \theta)] = e(d + r \cos \theta) \Rightarrow r(1 - e \cos \theta) = ed \Rightarrow r = \frac{ed}{1 - e \cos \theta}$$



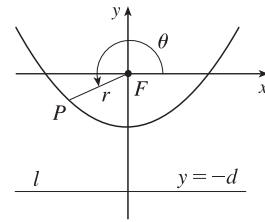
$$22. |PF| = e|Pl| \Rightarrow r = e[d - r \sin \theta] \Rightarrow r(1 + e \sin \theta) = ed \Rightarrow r = \frac{ed}{1 + e \sin \theta}$$





$$23. |PF| = e|Pl| \Rightarrow r = e[d - r \sin(\theta - \pi)] = e(d + r \sin \theta) \Rightarrow$$

$$r(1 - e \sin \theta) = ed \Rightarrow r = \frac{ed}{1 - e \sin \theta}$$



$$24. \text{ The parabolas intersect at the two points where } \frac{c}{1 + \cos \theta} = \frac{d}{1 - \cos \theta} \Rightarrow \cos \theta = \frac{c-d}{c+d} \Rightarrow r = \frac{c+d}{2}.$$

For the first parabola,  $\frac{dr}{d\theta} = \frac{c \sin \theta}{(1 + \cos \theta)^2}$ , so

$$\frac{dy}{dx} = \frac{(dr/d\theta) \sin \theta + r \cos \theta}{(dr/d\theta) \cos \theta - r \sin \theta} = \frac{c \sin^2 \theta + c \cos \theta (1 + \cos \theta)}{c \sin \theta \cos \theta - c \sin \theta (1 + \cos \theta)} = \frac{1 + \cos \theta}{-\sin \theta}$$

and similarly for the second,  $\frac{dy}{dx} = \frac{1 - \cos \theta}{\sin \theta} = \frac{\sin \theta}{1 + \cos \theta}$ . Since the product of these slopes is  $-1$ , the parabolas intersect at right angles.

25. We are given  $e = 0.093$  and  $a = 2.28 \times 10^8$ . By (7), we have

$$r = \frac{a(1 - e^2)}{1 + e \cos \theta} = \frac{2.28 \times 10^8 [1 - (0.093)^2]}{1 + 0.093 \cos \theta} \approx \frac{2.26 \times 10^8}{1 + 0.093 \cos \theta}$$

26. We are given  $e = 0.048$  and  $2a = 1.56 \times 10^9 \Rightarrow a = 7.8 \times 10^8$ . By (7), we have

$$r = \frac{a(1 - e^2)}{1 + e \cos \theta} = \frac{7.8 \times 10^8 [1 - (0.048)^2]}{1 + 0.048 \cos \theta} \approx \frac{7.78 \times 10^8}{1 + 0.048 \cos \theta}$$

27. Here  $2a = \text{length of major axis} = 36.18 \text{ AU} \Rightarrow a = 18.09 \text{ AU}$  and  $e = 0.97$ . By (7), the equation of the orbit is

$$r = \frac{18.09[1 - (0.97)^2]}{1 + 0.97 \cos \theta} \approx \frac{1.07}{1 + 0.97 \cos \theta}. \text{ By (8), the maximum distance from the comet to the sun is}$$

$$18.09(1 + 0.97) \approx 35.64 \text{ AU or about 3.314 billion miles.}$$

28. Here  $2a = \text{length of major axis} = 356.5 \text{ AU} \Rightarrow a = 178.25 \text{ AU}$  and  $e = 0.9951$ . By (7), the equation of the orbit

$$\text{is } r = \frac{178.25[1 - (0.9951)^2]}{1 + 0.9951 \cos \theta} \approx \frac{1.7426}{1 + 0.9951 \cos \theta}. \text{ By (8), the minimum distance from the comet to the sun is}$$

$$178.25(1 - 0.9951) \approx 0.8734 \text{ AU or about 81 million miles.}$$

29. The minimum distance is at perihelion, where  $4.6 \times 10^7 = r = a(1 - e) = a(1 - 0.206) = a(0.794) \Rightarrow$

$$a = 4.6 \times 10^7 / 0.794. \text{ So the maximum distance, which is at aphelion, is}$$

$$r = a(1 + e) = (4.6 \times 10^7 / 0.794)(1.206) \approx 7.0 \times 10^7 \text{ km.}$$

30. At perihelion,  $r = a(1 - e) = 4.43 \times 10^9$ , and at aphelion,  $r = a(1 + e) = 7.37 \times 10^9$ . Adding, we get  $2a = 11.80 \times 10^9$ ,

$$\text{so } a = 5.90 \times 10^9 \text{ km. Therefore } 1 + e = a(1 + e)/a = \frac{7.37}{5.90} \approx 1.249 \text{ and } e \approx 0.249.$$

31. From Exercise 29, we have  $e = 0.206$  and  $a(1 - e) = 4.6 \times 10^7$  km. Thus,  $a = 4.6 \times 10^7 / 0.794$ . From (7), we can write the

equation of Mercury's orbit as  $r = a \frac{1 - e^2}{1 + e \cos \theta}$ . So since

$$\frac{dr}{d\theta} = \frac{a(1 - e^2)e \sin \theta}{(1 + e \cos \theta)^2} \Rightarrow$$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = \frac{a^2(1 - e^2)^2}{(1 + e \cos \theta)^2} + \frac{a^2(1 - e^2)^2 e^2 \sin^2 \theta}{(1 + e \cos \theta)^4} = \frac{a^2(1 - e^2)^2}{(1 + e \cos \theta)^4} (1 + 2e \cos \theta + e^2)$$

the length of the orbit is

$$L = \int_0^{2\pi} \sqrt{r^2 + (dr/d\theta)^2} d\theta = a(1 - e^2) \int_0^{2\pi} \frac{\sqrt{1 + e^2 + 2e \cos \theta}}{(1 + e \cos \theta)^2} d\theta \approx 3.6 \times 10^8 \text{ km}$$

This seems reasonable, since Mercury's orbit is nearly circular, and the circumference of a circle of radius  $a$  is  $2\pi a \approx 3.6 \times 10^8$  km.

## 10 Review

### CONCEPT CHECK

1. (a) A parametric curve is a set of points of the form  $(x, y) = (f(t), g(t))$ , where  $f$  and  $g$  are continuous functions of a variable  $t$ .  
 (b) Sketching a parametric curve, like sketching the graph of a function, is difficult to do in general. We can plot points on the curve by finding  $f(t)$  and  $g(t)$  for various values of  $t$ , either by hand or with a calculator or computer. Sometimes, when  $f$  and  $g$  are given by formulas, we can eliminate  $t$  from the equations  $x = f(t)$  and  $y = g(t)$  to get a Cartesian equation relating  $x$  and  $y$ . It may be easier to graph that equation than to work with the original formulas for  $x$  and  $y$  in terms of  $t$ .
2. (a) You can find  $\frac{dy}{dx}$  as a function of  $t$  by calculating  $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$  [if  $dx/dt \neq 0$ ].  
 (b) Calculate the area as  $\int_a^b y dx = \int_a^\beta g(t) f'(t) dt$  [or  $\int_\beta^\alpha g(t) f'(t) dt$  if the leftmost point is  $(f(\beta), g(\beta))$  rather than  $(f(\alpha), g(\alpha))$ ].
3. (a)  $L = \int_a^\beta \sqrt{(dx/dt)^2 + (dy/dt)^2} dt = \int_a^\beta \sqrt{[f'(t)]^2 + [g'(t)]^2} dt$   
 (b)  $S = \int_a^\beta 2\pi y \sqrt{(dx/dt)^2 + (dy/dt)^2} dt = \int_a^\beta 2\pi g(t) \sqrt{[f'(t)]^2 + [g'(t)]^2} dt$
4. (a) See Figure 5 in Section 10.3.  
 (b)  $x = r \cos \theta$ ,  $y = r \sin \theta$   
 (c) To find a polar representation  $(r, \theta)$  with  $r \geq 0$  and  $0 \leq \theta < 2\pi$ , first calculate  $r = \sqrt{x^2 + y^2}$ . Then  $\theta$  is specified by  $\cos \theta = x/r$  and  $\sin \theta = y/r$ .
5. (a) Calculate  $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{d}{d\theta}(y)}{\frac{d}{d\theta}(x)} = \frac{\frac{d}{d\theta}(r \sin \theta)}{\frac{d}{d\theta}(r \cos \theta)} = \frac{\left(\frac{dr}{d\theta}\right) \sin \theta + r \cos \theta}{\left(\frac{dr}{d\theta}\right) \cos \theta - r \sin \theta}$ , where  $r = f(\theta)$ .  
 (b) Calculate  $A = \int_a^b \frac{1}{2} r^2 d\theta = \int_a^b \frac{1}{2} [f(\theta)]^2 d\theta$   
 (c)  $L = \int_a^b \sqrt{(dx/d\theta)^2 + (dy/d\theta)^2} d\theta = \int_a^b \sqrt{r^2 + (dr/d\theta)^2} d\theta = \int_a^b \sqrt{[f(\theta)]^2 + [f'(\theta)]^2} d\theta$

6. (a) A parabola is a set of points in a plane whose distances from a fixed point  $F$  (the focus) and a fixed line  $l$  (the directrix) are equal.
- (b)  $x^2 = 4py$ ;  $y^2 = 4px$
7. (a) An ellipse is a set of points in a plane the sum of whose distances from two fixed points (the foci) is a constant.
- (b)  $\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1$ .
8. (a) A hyperbola is a set of points in a plane the difference of whose distances from two fixed points (the foci) is a constant. This difference should be interpreted as the larger distance minus the smaller distance.
- (b)  $\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1$
- (c)  $y = \pm \frac{\sqrt{c^2 - a^2}}{a}x$
9. (a) If a conic section has focus  $F$  and corresponding directrix  $l$ , then the eccentricity  $e$  is the fixed ratio  $|PF| / |Pl|$  for points  $P$  of the conic section.
- (b)  $e < 1$  for an ellipse;  $e > 1$  for a hyperbola;  $e = 1$  for a parabola.
- (c)  $x = d$ :  $r = \frac{ed}{1 + e \cos \theta}$ .  $x = -d$ :  $r = \frac{ed}{1 - e \cos \theta}$ .  $y = d$ :  $r = \frac{ed}{1 + e \sin \theta}$ .  $y = -d$ :  $r = \frac{ed}{1 - e \sin \theta}$ .

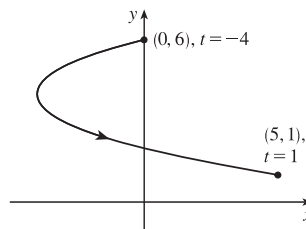
## TRUE-FALSE QUIZ

1. False. Consider the curve defined by  $x = f(t) = (t - 1)^3$  and  $y = g(t) = (t - 1)^2$ . Then  $g'(t) = 2(t - 1)$ , so  $g'(1) = 0$ , but its graph has a vertical tangent when  $t = 1$ . Note: The statement is true if  $f'(1) \neq 0$  when  $g'(1) = 0$ .
2. False. If  $x = f(t)$  and  $y = g(t)$  are twice differentiable, then  $\frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{\frac{d}{dt} \left( \frac{dy}{dx} \right)}{\frac{dx}{dt}}$ .
3. False. For example, if  $f(t) = \cos t$  and  $g(t) = \sin t$  for  $0 \leq t \leq 4\pi$ , then the curve is a circle of radius 1, hence its length is  $2\pi$ , but  $\int_0^{4\pi} \sqrt{[f'(t)]^2 + [g'(t)]^2} dt = \int_0^{4\pi} \sqrt{(-\sin t)^2 + (\cos t)^2} dt = \int_0^{4\pi} 1 dt = 4\pi$ , since as  $t$  increases from 0 to  $4\pi$ , the circle is traversed twice.
4. False. If  $(r, \theta) = (1, \pi)$ , then  $(x, y) = (-1, 0)$ , so  $\tan^{-1}(y/x) = \tan^{-1} 0 = 0 \neq \theta$ . The statement is true for points in quadrants I and IV.
5. True. The curve  $r = 1 - \sin 2\theta$  is unchanged if we rotate it through  $180^\circ$  about  $O$  because  $1 - \sin 2(\theta + \pi) = 1 - \sin(2\theta + 2\pi) = 1 - \sin 2\theta$ . So it's unchanged if we replace  $r$  by  $-r$ . (See the discussion after Example 8 in Section 10.3.) In other words, it's the same curve as  $r = -(1 - \sin 2\theta) = \sin 2\theta - 1$ .
6. True. The polar equation  $r = 2$ , the Cartesian equation  $x^2 + y^2 = 4$ , and the parametric equations  $x = 2 \sin 3t$ ,  $y = 2 \cos 3t$  [ $0 \leq t \leq 2\pi$ ] all describe the circle of radius 2 centered at the origin.

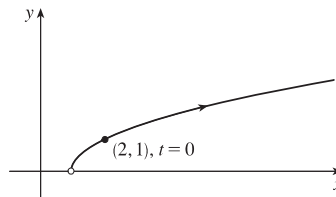
7. False. The first pair of equations gives the portion of the parabola  $y = x^2$  with  $x \geq 0$ , whereas the second pair of equations traces out the whole parabola  $y = x^2$ .
8. True.  $y^2 = 2y + 3x \Leftrightarrow (y - 1)^2 = 3x + 1 = 3(x + \frac{1}{3}) = 4(\frac{3}{4})(x + \frac{1}{3})$ , which is the equation of a parabola with vertex  $(-\frac{1}{3}, 1)$  and focus  $(-\frac{1}{3} + \frac{3}{4}, 1)$ , opening to the right.
9. True. By rotating and translating the parabola, we can assume it has an equation of the form  $y = cx^2$ , where  $c > 0$ . The tangent at the point  $(a, ca^2)$  is the line  $y - ca^2 = 2ca(x - a)$ ; i.e.,  $y = 2cax - ca^2$ . This tangent meets the parabola at the points  $(x, cx^2)$  where  $cx^2 = 2cax - ca^2$ . This equation is equivalent to  $x^2 = 2ax - a^2$  [since  $c > 0$ ]. But  $x^2 = 2ax - a^2 \Leftrightarrow x^2 - 2ax + a^2 = 0 \Leftrightarrow (x - a)^2 = 0 \Leftrightarrow x = a \Leftrightarrow (x, cx^2) = (a, ca^2)$ . This shows that each tangent meets the parabola at exactly one point.
10. True. Consider a hyperbola with focus at the origin, oriented so that its polar equation is  $r = \frac{ed}{1 + e \cos \theta}$ , where  $e > 1$ . The directrix is  $x = d$ , but along the hyperbola we have  $x = r \cos \theta = \frac{ed \cos \theta}{1 + e \cos \theta} = d \left( \frac{e \cos \theta}{1 + e \cos \theta} \right) \neq d$ .

## EXERCISES

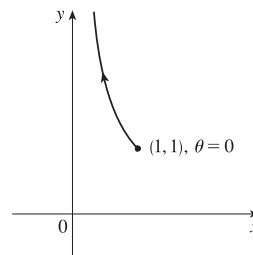
1.  $x = t^2 + 4t$ ,  $y = 2 - t$ ,  $-4 \leq t \leq 1$ .  $t = 2 - y$ , so  
 $x = (2 - y)^2 + 4(2 - y) = 4 - 4y + y^2 + 8 - 4y = y^2 - 8y + 12 \Leftrightarrow$   
 $x + 4 = y^2 - 8y + 16 = (y - 4)^2$ . This is part of a parabola with vertex  $(-4, 4)$ , opening to the right.



2.  $x = 1 + e^{2t}$ ,  $y = e^t$ .  
 $x = 1 + e^{2t} = 1 + (e^t)^2 = 1 + y^2$ ,  $y > 0$ .

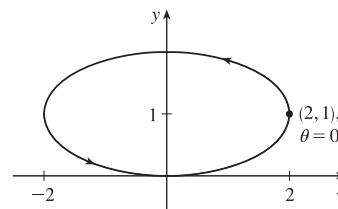


3.  $y = \sec \theta = \frac{1}{\cos \theta} = \frac{1}{x}$ . Since  $0 \leq \theta \leq \pi/2$ ,  $0 < x \leq 1$  and  $y \geq 1$ .  
 This is part of the hyperbola  $y = 1/x$ .



4.  $x = 2 \cos \theta, y = 1 + \sin \theta, \cos^2 \theta + \sin^2 \theta = 1 \Rightarrow$

$\left(\frac{x}{2}\right)^2 + (y - 1)^2 = 1 \Rightarrow \frac{x^2}{4} + (y - 1)^2 = 1$ . This is an ellipse, centered at  $(0, 1)$ , with semimajor axis of length 2 and semiminor axis of length 1.



5. Three different sets of parametric equations for the curve  $y = \sqrt{x}$  are

(i)  $x = t, y = \sqrt{t}$

(ii)  $x = t^4, y = t^2$

(iii)  $x = \tan^2 t, y = \tan t, 0 \leq t < \pi/2$

There are many other sets of equations that also give this curve.

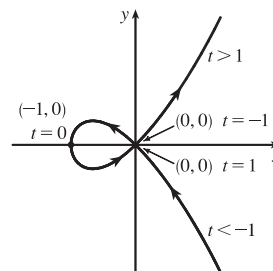
6. For  $t < -1, x > 0$  and  $y < 0$  with  $x$  decreasing and  $y$  increasing. When

$t = -1, (x, y) = (0, 0)$ . When  $-1 < t < 0$ , we have  $-1 < x < 0$  and

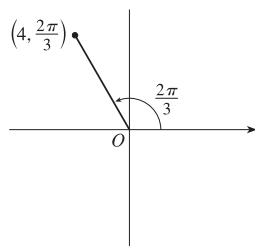
$0 < y < 1/2$ . When  $t = 0, (x, y) = (-1, 0)$ . When  $0 < t < 1$ ,

$-1 < x < 0$  and  $-\frac{1}{2} < y < 0$ . When  $t = 1, (x, y) = (0, 0)$  again.

When  $t > 1$ , both  $x$  and  $y$  are positive and increasing.



7. (a)



The Cartesian coordinates are  $x = 4 \cos \frac{2\pi}{3} = 4(-\frac{1}{2}) = -2$  and

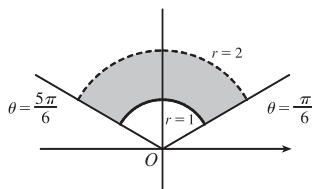
$y = 4 \sin \frac{2\pi}{3} = 4(\frac{\sqrt{3}}{2}) = 2\sqrt{3}$ , that is, the point  $(-2, 2\sqrt{3})$ .

(b) Given  $x = -3$  and  $y = 3$ , we have  $r = \sqrt{(-3)^2 + 3^2} = \sqrt{18} = 3\sqrt{2}$ . Also,  $\tan \theta = \frac{y}{x} \Rightarrow \tan \theta = \frac{3}{-3}$ , and since

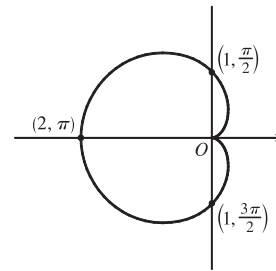
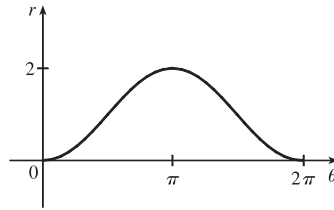
$(-3, 3)$  is in the second quadrant,  $\theta = \frac{3\pi}{4}$ . Thus, one set of polar coordinates for  $(-3, 3)$  is  $(3\sqrt{2}, \frac{3\pi}{4})$ , and two others are

$(3\sqrt{2}, \frac{11\pi}{4})$  and  $(-3\sqrt{2}, \frac{7\pi}{4})$ .

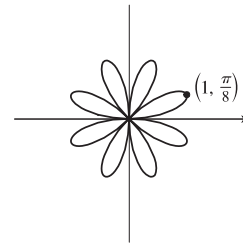
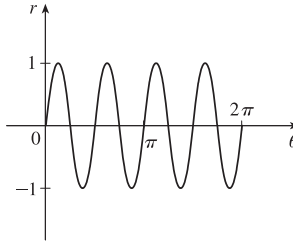
8.  $1 \leq r < 2, \frac{\pi}{6} \leq \theta \leq \frac{5\pi}{6}$



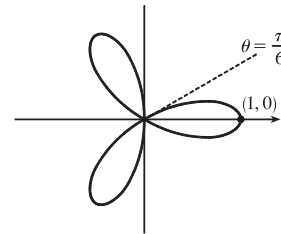
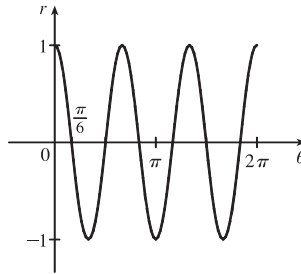
9.  $r = 1 - \cos \theta$ . This cardioid is symmetric about the polar axis.



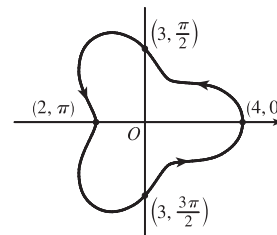
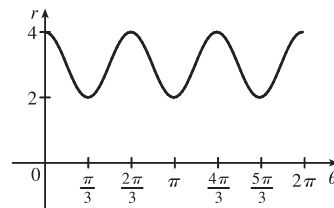
10.  $r = \sin 4\theta$ . This is an eight-leaved rose.



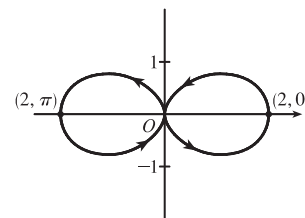
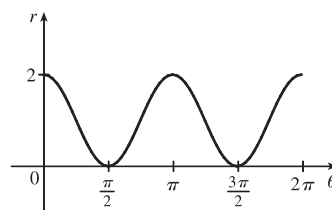
11.  $r = \cos 3\theta$ . This is a three-leaved rose. The curve is traced twice.



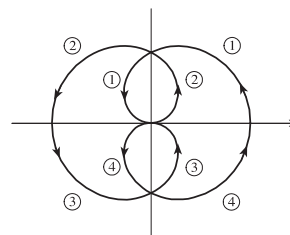
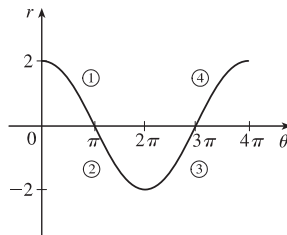
12.  $r = 3 + \cos 3\theta$ . The curve is symmetric about the horizontal axis.



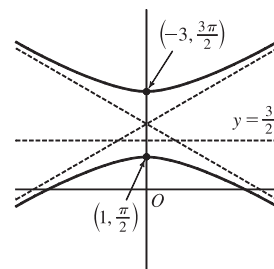
13.  $r = 1 + \cos 2\theta$ . The curve is symmetric about the pole and both the horizontal and vertical axes.



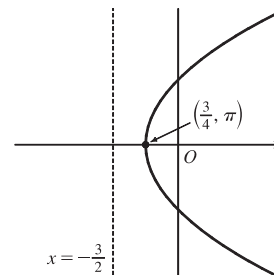
14.  $r = 2 \cos (\theta/2)$ . The curve is symmetric about the pole and both the horizontal and vertical axes.



15.  $r = \frac{3}{1 + 2 \sin \theta} \Rightarrow e = 2 > 1$ , so the conic is a hyperbola.  $de = 3 \Rightarrow d = \frac{3}{2}$  and the form “ $+2 \sin \theta$ ” imply that the directrix is above the focus at the origin and has equation  $y = \frac{3}{2}$ . The vertices are  $(1, \frac{\pi}{2})$  and  $(-3, \frac{3\pi}{2})$ .



16.  $r = \frac{3}{2 - 2 \cos \theta} \cdot \frac{1/2}{1/2} = \frac{3/2}{1 - 1 \cos \theta} \Rightarrow e = 1$ , so the conic is a parabola.  $de = \frac{3}{2} \Rightarrow d = \frac{3}{2}$  and the form “ $-2 \cos \theta$ ” imply that the directrix is to the left of the focus at the origin and has equation  $x = -\frac{3}{2}$ . The vertex is  $(\frac{3}{4}, \pi)$ .

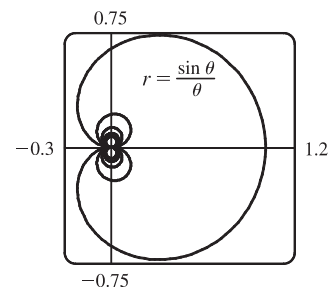
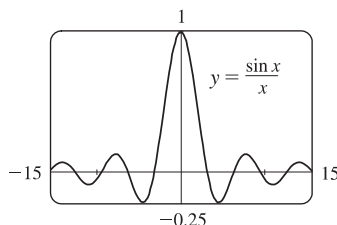


17.  $x + y = 2 \Leftrightarrow r \cos \theta + r \sin \theta = 2 \Leftrightarrow r(\cos \theta + \sin \theta) = 2 \Leftrightarrow r = \frac{2}{\cos \theta + \sin \theta}$

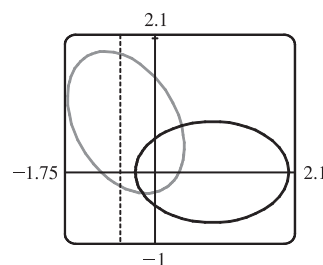
18.  $x^2 + y^2 = 2 \Rightarrow r^2 = 2 \Rightarrow r = \sqrt{2}$ . [ $r = -\sqrt{2}$  gives the same curve.]

19.  $r = (\sin \theta)/\theta$ . As  $\theta \rightarrow \pm\infty$ ,  $r \rightarrow 0$ .

As  $\theta \rightarrow 0$ ,  $r \rightarrow 1$ . In the first figure, there are an infinite number of  $x$ -intercepts at  $x = \pi n$ ,  $n$  a nonzero integer. These correspond to pole points in the second figure.



20.  $r = \frac{2}{4 - 3 \cos \theta} = \frac{1/2}{1 - \frac{3}{4} \cos \theta} \Rightarrow e = \frac{3}{4}$  and  $d = \frac{2}{3}$ . The equation of the directrix is  $x = -\frac{2}{3} \Rightarrow r = -2/(3 \cos \theta)$ . To obtain the equation of the rotated ellipse, we replace  $\theta$  in the original equation with  $\theta - \frac{2\pi}{3}$ , and get  $r = \frac{2}{4 - 3 \cos(\theta - \frac{2\pi}{3})}$ .



21.  $x = \ln t$ ,  $y = 1 + t^2$ ;  $t = 1$ .  $\frac{dy}{dt} = 2t$  and  $\frac{dx}{dt} = \frac{1}{t}$ , so  $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{1/t} = 2t^2$ .

When  $t = 1$ ,  $(x, y) = (0, 2)$  and  $dy/dx = 2$ .

22.  $x = t^3 + 6t + 1$ ,  $y = 2t - t^2$ ;  $t = -1$ .  $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2 - 2t}{3t^2 + 6}$ . When  $t = -1$ ,  $(x, y) = (-6, -3)$  and  $\frac{dy}{dx} = \frac{4}{9}$ .

$$23. r = e^{-\theta} \Rightarrow y = r \sin \theta = e^{-\theta} \sin \theta \text{ and } x = r \cos \theta = e^{-\theta} \cos \theta \Rightarrow$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{dr}{d\theta} \sin \theta + r \cos \theta}{\frac{dr}{d\theta} \cos \theta - r \sin \theta} = \frac{-e^{-\theta} \sin \theta + e^{-\theta} \cos \theta}{-e^{-\theta} \cos \theta - e^{-\theta} \sin \theta} \cdot \frac{-e^{\theta}}{-e^{\theta}} = \frac{\sin \theta - \cos \theta}{\cos \theta + \sin \theta}.$$

$$\text{When } \theta = \pi, \frac{dy}{dx} = \frac{0 - (-1)}{-1 + 0} = \frac{1}{-1} = -1.$$

$$24. r = 3 + \cos 3\theta \Rightarrow \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{dr}{d\theta} \sin \theta + r \cos \theta}{\frac{dr}{d\theta} \cos \theta - r \sin \theta} = \frac{-3 \sin 3\theta \sin \theta + (3 + \cos 3\theta) \cos \theta}{-3 \sin 3\theta \cos \theta - (3 + \cos 3\theta) \sin \theta}.$$

$$\text{When } \theta = \pi/2, \frac{dy}{dx} = \frac{(-3)(-1)(1) + (3 + 0) \cdot 0}{(-3)(-1)(0) - (3 + 0) \cdot 1} = \frac{3}{-3} = -1.$$

$$25. x = t + \sin t, y = t - \cos t \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1 + \sin t}{1 + \cos t} \Rightarrow$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt} \left( \frac{dy}{dx} \right)}{dx/dt} = \frac{\frac{(1 + \cos t) \cos t - (1 + \sin t)(-\sin t)}{(1 + \cos t)^2}}{1 + \cos t} = \frac{\cos t + \cos^2 t + \sin t + \sin^2 t}{(1 + \cos t)^3} = \frac{1 + \cos t + \sin t}{(1 + \cos t)^3}$$

$$26. x = 1 + t^2, y = t - t^3. \frac{dy}{dt} = 1 - 3t^2 \text{ and } \frac{dx}{dt} = 2t, \text{ so } \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1 - 3t^2}{2t} = \frac{1}{2}t^{-1} - \frac{3}{2}t.$$

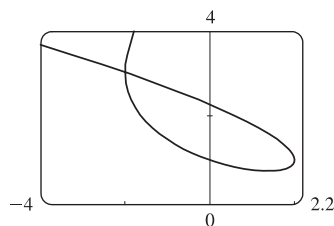
$$\frac{d^2y}{dx^2} = \frac{d(dy/dx)/dt}{dx/dt} = \frac{-\frac{1}{2}t^{-2} - \frac{3}{2}}{2t} = -\frac{1}{4}t^{-3} - \frac{3}{4}t^{-1} = -\frac{1}{4t^3}(1 + 3t^2) = -\frac{3t^2 + 1}{4t^3}.$$

$$27. \text{ We graph the curve } x = t^3 - 3t, y = t^2 + t + 1 \text{ for } -2.2 \leq t \leq 1.2.$$

By zooming in or using a cursor, we find that the lowest point is about

(1.4, 0.75). To find the exact values, we find the  $t$ -value at which

$$dy/dt = 2t + 1 = 0 \Leftrightarrow t = -\frac{1}{2} \Leftrightarrow (x, y) = \left(\frac{11}{8}, \frac{3}{4}\right).$$



28. We estimate the coordinates of the point of intersection to be  $(-2, 3)$ . In fact this is exact, since both  $t = -2$  and  $t = 1$  give the point  $(-2, 3)$ . So the area enclosed by the loop is

$$\begin{aligned} \int_{t=-2}^{t=1} y dx &= \int_{-2}^1 (t^2 + t + 1)(3t^2 - 3) dt = \int_{-2}^1 (3t^4 + 3t^3 - 3t - 3) dt \\ &= \left[ \frac{3}{5}t^5 + \frac{3}{4}t^4 - \frac{3}{2}t^2 - 3t \right]_{-2}^1 = \left( \frac{3}{5} + \frac{3}{4} - \frac{3}{2} - 3 \right) - \left[ -\frac{96}{5} + 12 - 6 - (-6) \right] = \frac{81}{20} \end{aligned}$$

$$29. x = 2a \cos t - a \cos 2t \Rightarrow \frac{dx}{dt} = -2a \sin t + 2a \sin 2t = 2a \sin t(2 \cos t - 1) = 0 \Leftrightarrow$$

$$\sin t = 0 \text{ or } \cos t = \frac{1}{2} \Rightarrow t = 0, \frac{\pi}{3}, \pi, \text{ or } \frac{5\pi}{3}.$$

$$y = 2a \sin t - a \sin 2t \Rightarrow \frac{dy}{dt} = 2a \cos t - 2a \cos 2t = 2a(1 + \cos t - 2 \cos^2 t) = 2a(1 - \cos t)(1 + 2 \cos t) = 0 \Rightarrow$$

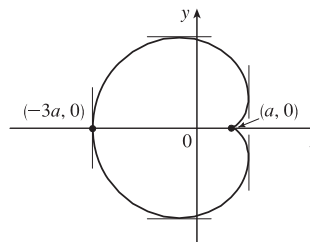
$$t = 0, \frac{2\pi}{3}, \text{ or } \frac{4\pi}{3}.$$

Thus the graph has vertical tangents where  $t = \frac{\pi}{3}, \pi$  and  $\frac{5\pi}{3}$ , and horizontal tangents where  $t = \frac{2\pi}{3}$  and  $\frac{4\pi}{3}$ . To determine



what the slope is where  $t = 0$ , we use l'Hospital's Rule to evaluate  $\lim_{t \rightarrow 0} \frac{dy/dt}{dx/dt} = 0$ , so there is a horizontal tangent there.

$t$	$x$	$y$
0	$a$	0
$\frac{\pi}{3}$	$\frac{3}{2}a$	$\frac{\sqrt{3}}{2}a$
$\frac{2\pi}{3}$	$-\frac{1}{2}a$	$\frac{3\sqrt{3}}{2}a$
$\pi$	$-3a$	0
$\frac{4\pi}{3}$	$-\frac{1}{2}a$	$-\frac{3\sqrt{3}}{2}a$
$\frac{5\pi}{3}$	$\frac{3}{2}a$	$-\frac{\sqrt{3}}{2}a$



30. From Exercise 29,  $x = 2a \cos t - a \cos 2t$ ,  $y = 2a \sin t - a \sin 2t \Rightarrow$

$$\begin{aligned} A &= 2 \int_{\pi}^0 (2a \sin t - a \sin 2t)(-2a \sin t + 2a \sin 2t) dt = 4a^2 \int_0^{\pi} (2 \sin^2 t + \sin^2 2t - 3 \sin t \sin 2t) dt \\ &= 4a^2 \int_0^{\pi} \left[ (1 - \cos 2t) + \frac{1}{2}(1 - \cos 4t) - 6 \sin^2 t \cos t \right] dt = 4a^2 \left[ t - \frac{1}{2} \sin 2t + \frac{1}{2}t - \frac{1}{8} \sin 4t - 2 \sin^3 t \right]_0^{\pi} \\ &= 4a^2 \left( \frac{3}{2} \right) \pi = 6\pi a^2 \end{aligned}$$

31. The curve  $r^2 = 9 \cos 5\theta$  has 10 “petals.” For instance, for  $-\frac{\pi}{10} \leq \theta \leq \frac{\pi}{10}$ , there are two petals, one with  $r > 0$  and one with  $r < 0$ .

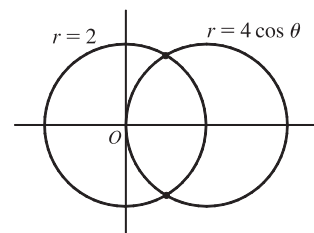
$$A = 10 \int_{-\pi/10}^{\pi/10} \frac{1}{2} r^2 d\theta = 5 \int_{-\pi/10}^{\pi/10} 9 \cos 5\theta d\theta = 5 \cdot 9 \cdot 2 \int_0^{\pi/10} \cos 5\theta d\theta = 18 [\sin 5\theta]_0^{\pi/10} = 18$$

32.  $r = 1 - 3 \sin \theta$ . The inner loop is traced out as  $\theta$  goes from  $\alpha = \sin^{-1}(\frac{1}{3})$  to  $\pi - \alpha$ , so

$$\begin{aligned} A &= \int_{\alpha}^{\pi-\alpha} \frac{1}{2} r^2 d\theta = \int_{\alpha}^{\pi-\alpha} (1 - 3 \sin \theta)^2 d\theta = \int_{\alpha}^{\pi-\alpha} [1 - 6 \sin \theta + \frac{9}{2}(1 - \cos 2\theta)] d\theta \\ &= \left[ \frac{11}{2}\theta + 6 \cos \theta - \frac{9}{4} \sin 2\theta \right]_{\alpha}^{\pi-\alpha} = \frac{11}{4}\pi - \frac{11}{2} \sin^{-1}\left(\frac{1}{3}\right) - 3\sqrt{2} \end{aligned}$$

33. The curves intersect when  $4 \cos \theta = 2 \Rightarrow \cos \theta = \frac{1}{2} \Rightarrow \theta = \pm \frac{\pi}{3}$

for  $-\pi \leq \theta \leq \pi$ . The points of intersection are  $(2, \frac{\pi}{3})$  and  $(2, -\frac{\pi}{3})$ .

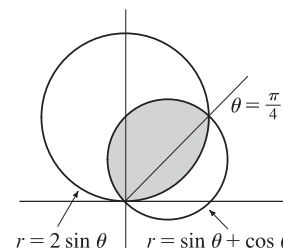


34. The two curves clearly both contain the pole. For other points of intersection,  $\cot \theta = 2 \cos(\theta + 2n\pi)$  or  $-2 \cos(\theta + \pi + 2n\pi)$ , both of which reduce to  $\cot \theta = 2 \cos \theta \Leftrightarrow \cos \theta = 2 \sin \theta \cos \theta \Leftrightarrow \cos \theta(1 - 2 \sin \theta) = 0 \Rightarrow \cos \theta = 0$  or  $\sin \theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$  or  $\frac{3\pi}{2} \Rightarrow$  intersection points are  $(0, \frac{\pi}{2})$ ,  $(\sqrt{3}, \frac{\pi}{6})$ , and  $(\sqrt{3}, \frac{11\pi}{6})$ .

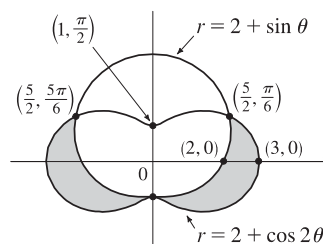
35. The curves intersect where  $2 \sin \theta = \sin \theta + \cos \theta \Rightarrow$

$\sin \theta = \cos \theta \Rightarrow \theta = \frac{\pi}{4}$ , and also at the origin (at which  $\theta = \frac{3\pi}{4}$  on the second curve).

$$\begin{aligned} A &= \int_0^{\pi/4} \frac{1}{2} (2 \sin \theta)^2 d\theta + \int_{\pi/4}^{3\pi/4} \frac{1}{2} (\sin \theta + \cos \theta)^2 d\theta \\ &= \int_0^{\pi/4} (1 - \cos 2\theta) d\theta + \frac{1}{2} \int_{\pi/4}^{3\pi/4} (1 + \sin 2\theta) d\theta \\ &= \left[ \theta - \frac{1}{2} \sin 2\theta \right]_0^{\pi/4} + \left[ \frac{1}{2}\theta - \frac{1}{4} \cos 2\theta \right]_{\pi/4}^{3\pi/4} = \frac{1}{2}(\pi - 1) \end{aligned}$$



$$\begin{aligned}
 36. A &= 2 \int_{-\pi/2}^{\pi/6} \frac{1}{2} [(2 + \cos 2\theta)^2 - (2 + \sin \theta)^2] d\theta \\
 &= \int_{-\pi/2}^{\pi/6} [4 \cos 2\theta + \cos^2 2\theta - 4 \sin \theta - \sin^2 \theta] d\theta \\
 &= \left[ 2 \sin 2\theta + \frac{1}{2} \theta + \frac{1}{8} \sin 4\theta + 4 \cos \theta - \frac{1}{2} \theta + \frac{1}{4} \sin 2\theta \right]_{-\pi/2}^{\pi/6} \\
 &= \frac{51}{16} \sqrt{3}
 \end{aligned}$$



$$37. x = 3t^2, y = 2t^3.$$

$$\begin{aligned}
 L &= \int_0^2 \sqrt{(dx/dt)^2 + (dy/dt)^2} dt = \int_0^2 \sqrt{(6t)^2 + (6t^2)^2} dt = \int_0^2 \sqrt{36t^2 + 36t^4} dt = \int_0^2 \sqrt{36t^2} \sqrt{1 + t^2} dt \\
 &= \int_0^2 6|t| \sqrt{1 + t^2} dt = 6 \int_0^2 t \sqrt{1 + t^2} dt = 6 \int_1^5 u^{1/2} \left(\frac{1}{2} du\right) \quad [u = 1 + t^2, du = 2t dt] \\
 &= 6 \cdot \frac{1}{2} \cdot \frac{2}{3} \left[ u^{3/2} \right]_1^5 = 2(5^{3/2} - 1) = 2(5\sqrt{5} - 1)
 \end{aligned}$$

$$38. x = 2 + 3t, y = \cosh 3t \Rightarrow (dx/dt)^2 + (dy/dt)^2 = 3^2 + (3 \sinh 3t)^2 = 9(1 + \sinh^2 3t) = 9 \cosh^2 3t, \text{ so}$$

$$L = \int_0^1 \sqrt{9 \cosh^2 3t} dt = \int_0^1 |3 \cosh 3t| dt = \int_0^1 3 \cosh 3t dt = [\sinh 3t]_0^1 = \sinh 3 - \sinh 0 = \sinh 3.$$

$$\begin{aligned}
 39. L &= \int_{\pi}^{2\pi} \sqrt{r^2 + (dr/d\theta)^2} d\theta = \int_{\pi}^{2\pi} \sqrt{(1/\theta)^2 + (-1/\theta^2)^2} d\theta = \int_{\pi}^{2\pi} \frac{\sqrt{\theta^2 + 1}}{\theta^2} d\theta \\
 &\stackrel{24}{=} \left[ -\frac{\sqrt{\theta^2 + 1}}{\theta} + \ln(\theta + \sqrt{\theta^2 + 1}) \right]_{\pi}^{2\pi} = \frac{\sqrt{\pi^2 + 1}}{\pi} - \frac{\sqrt{4\pi^2 + 1}}{2\pi} + \ln\left(\frac{2\pi + \sqrt{4\pi^2 + 1}}{\pi + \sqrt{\pi^2 + 1}}\right) \\
 &= \frac{2\sqrt{\pi^2 + 1} - \sqrt{4\pi^2 + 1}}{2\pi} + \ln\left(\frac{2\pi + \sqrt{4\pi^2 + 1}}{\pi + \sqrt{\pi^2 + 1}}\right)
 \end{aligned}$$

$$\begin{aligned}
 40. L &= \int_0^{\pi} \sqrt{r^2 + (dr/d\theta)^2} d\theta = \int_0^{\pi} \sqrt{\sin^6(\frac{1}{3}\theta) + \sin^4(\frac{1}{3}\theta) \cos^2(\frac{1}{3}\theta)} d\theta \\
 &= \int_0^{\pi} \sin^2(\frac{1}{3}\theta) d\theta = \left[ \frac{1}{2} \left( \theta - \frac{3}{2} \sin(\frac{2}{3}\theta) \right) \right]_0^{\pi} = \frac{1}{2} \pi - \frac{3}{8} \sqrt{3}
 \end{aligned}$$

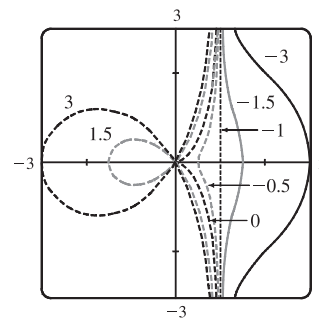
$$41. x = 4\sqrt{t}, y = \frac{t^3}{3} + \frac{1}{2t^2}, 1 \leq t \leq 4 \Rightarrow$$

$$\begin{aligned}
 S &= \int_1^4 2\pi y \sqrt{(dx/dt)^2 + (dy/dt)^2} dt = \int_1^4 2\pi \left( \frac{1}{3}t^3 + \frac{1}{2}t^{-2} \right) \sqrt{(2/\sqrt{t})^2 + (t^2 - t^{-3})^2} dt \\
 &= 2\pi \int_1^4 \left( \frac{1}{3}t^3 + \frac{1}{2}t^{-2} \right) \sqrt{(t^2 + t^{-3})^2} dt = 2\pi \int_1^4 \left( \frac{1}{3}t^5 + \frac{5}{6} + \frac{1}{2}t^{-5} \right) dt = 2\pi \left[ \frac{1}{18}t^6 + \frac{5}{6}t - \frac{1}{8}t^{-4} \right]_1^4 = \frac{471,295}{1024} \pi
 \end{aligned}$$

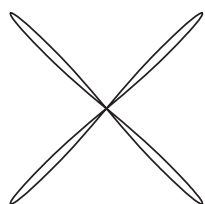
$$42. x = 2 + 3t, y = \cosh 3t \Rightarrow (dx/dt)^2 + (dy/dt)^2 = 3^2 + (3 \sinh 3t)^2 = 9(1 + \sinh^2 3t) = 9 \cosh^2 3t, \text{ so}$$

$$\begin{aligned}
 S &= \int_0^1 2\pi y ds = \int_0^1 2\pi \cosh 3t \sqrt{9 \cosh^2 3t} dt = \int_0^1 2\pi \cosh 3t |3 \cosh 3t| dt = \int_0^1 2\pi \cosh 3t \cdot 3 \cosh 3t dt \\
 &= 6\pi \int_0^1 \cosh^2 3t dt = 6\pi \int_0^1 \frac{1}{2}(1 + \cosh 6t) dt = 3\pi \left[ t + \frac{1}{6} \sinh 6t \right]_0^1 = 3\pi \left( 1 + \frac{1}{6} \sinh 6 \right) = 3\pi + \frac{\pi}{2} \sinh 6
 \end{aligned}$$

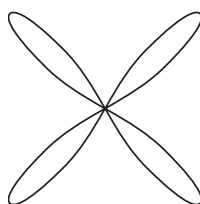
43. For all  $c$  except  $-1$ , the curve is asymptotic to the line  $x = 1$ . For  $c < -1$ , the curve bulges to the right near  $y = 0$ . As  $c$  increases, the bulge becomes smaller, until at  $c = -1$  the curve is the straight line  $x = 1$ . As  $c$  continues to increase, the curve bulges to the left, until at  $c = 0$  there is a cusp at the origin. For  $c > 0$ , there is a loop to the left of the origin, whose size and roundness increase as  $c$  increases. Note that the  $x$ -intercept of the curve is always  $-c$ .



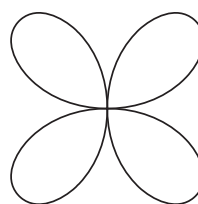
44. For  $a$  close to 0, the graph consists of four thin petals. As  $a$  increases, the petals get wider, until as  $a \rightarrow \infty$ , each petal occupies almost its entire quarter-circle.



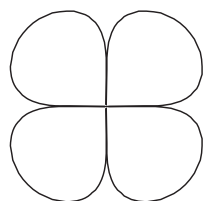
$a = 0.01$



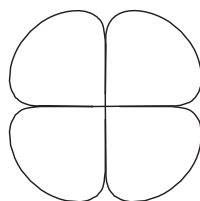
$a = 0.1$



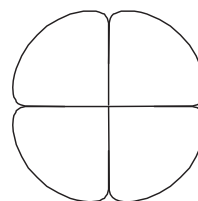
$a = 1$



$a = 5$



$a = 10$

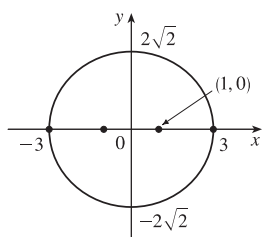


$a = 25$

45.  $\frac{x^2}{9} + \frac{y^2}{8} = 1$  is an ellipse with center  $(0, 0)$ .

$$a = 3, b = 2\sqrt{2}, c = 1 \Rightarrow$$

foci  $(\pm 1, 0)$ , vertices  $(\pm 3, 0)$ .

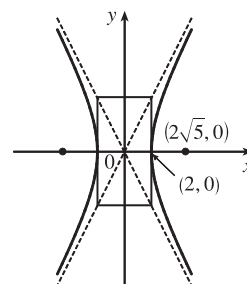


46.  $4x^2 - y^2 = 16 \Leftrightarrow \frac{x^2}{4} - \frac{y^2}{16} = 1$  is a hyperbola

with center  $(0, 0)$ , vertices  $(\pm 2, 0)$ ,  $a = 2$ ,  $b = 4$ ,

$c = \sqrt{16 + 4} = 2\sqrt{5}$ , foci  $(\pm 2\sqrt{5}, 0)$  and

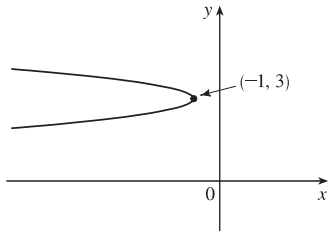
asymptotes  $y = \pm 2x$ .



47.  $6y^2 + x - 36y + 55 = 0 \Leftrightarrow$

$$6(y^2 - 6y + 9) = -(x + 1) \Leftrightarrow$$

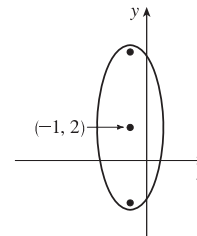
$(y - 3)^2 = -\frac{1}{6}(x + 1)$ , a parabola with vertex  $(-1, 3)$ , opening to the left,  $p = -\frac{1}{24} \Rightarrow$  focus  $(-\frac{25}{24}, 3)$  and directrix  $x = -\frac{23}{24}$ .



48.  $25x^2 + 4y^2 + 50x - 16y = 59 \Leftrightarrow$

$$25(x + 1)^2 + 4(y - 2)^2 = 100 \Leftrightarrow$$

$\frac{1}{4}(x + 1)^2 + \frac{1}{25}(y - 2)^2 = 1$  is an ellipse centered at  $(-1, 2)$  with foci on the line  $x = -1$ , vertices  $(-1, 7)$  and  $(-1, -3)$ ;  $a = 5$ ,  $b = 2 \Rightarrow c = \sqrt{21} \Rightarrow$  foci  $(-1, 2 \pm \sqrt{21})$ .



49. The ellipse with foci  $(\pm 4, 0)$  and vertices  $(\pm 5, 0)$  has center  $(0, 0)$  and a horizontal major axis, with  $a = 5$  and  $c = 4$ ,

so  $b^2 = a^2 - c^2 = 5^2 - 4^2 = 9$ . An equation is  $\frac{x^2}{25} + \frac{y^2}{9} = 1$ .

50. The distance from the focus  $(2, 1)$  to the directrix  $x = -4$  is  $2 - (-4) = 6$ , so the distance from the focus to the vertex is  $\frac{1}{2}(6) = 3$  and the vertex is  $(-1, 1)$ . Since the focus is to the right of the vertex,  $p = 3$ . An equation is

$$(y - 1)^2 = 4 \cdot 3[x - (-1)], \text{ or } (y - 1)^2 = 12(x + 1).$$

51. The center of a hyperbola with foci  $(0, \pm 4)$  is  $(0, 0)$ , so  $c = 4$  and an equation is  $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ .

The asymptote  $y = 3x$  has slope 3, so  $\frac{a}{b} = \frac{3}{1} \Rightarrow a = 3b$  and  $a^2 + b^2 = c^2 \Rightarrow (3b)^2 + b^2 = 4^2 \Rightarrow$

$10b^2 = 16 \Rightarrow b^2 = \frac{8}{5}$  and so  $a^2 = 16 - \frac{8}{5} = \frac{72}{5}$ . Thus, an equation is  $\frac{y^2}{72/5} - \frac{x^2}{8/5} = 1$ , or  $\frac{5y^2}{72} - \frac{5x^2}{8} = 1$ .

52. Center is  $(3, 0)$ , and  $a = \frac{8}{2} = 4$ ,  $c = 2 \Leftrightarrow b = \sqrt{4^2 - 2^2} = \sqrt{12} \Rightarrow$

an equation of the ellipse is  $\frac{(x - 3)^2}{12} + \frac{y^2}{16} = 1$ .

53.  $x^2 = -(y - 100)$  has its vertex at  $(0, 100)$ , so one of the vertices of the ellipse is  $(0, 100)$ . Another form of the equation of a parabola is  $x^2 = 4p(y - 100)$  so  $4p(y - 100) = -(y - 100) \Rightarrow 4p = -1 \Rightarrow p = -\frac{1}{4}$ . Therefore the shared focus is

found at  $(0, \frac{399}{4})$  so  $2c = \frac{399}{4} - 0 \Rightarrow c = \frac{399}{8}$  and the center of the ellipse is  $(0, \frac{399}{8})$ . So  $a = 100 - \frac{399}{8} = \frac{401}{8}$  and

$b^2 = a^2 - c^2 = \frac{401^2 - 399^2}{8^2} = 25$ . So the equation of the ellipse is  $\frac{x^2}{b^2} + \frac{(y - \frac{399}{8})^2}{a^2} = 1 \Rightarrow \frac{x^2}{25} + \frac{(y - \frac{399}{8})^2}{(\frac{401}{8})^2} = 1$ ,

or  $\frac{x^2}{25} + \frac{(8y - 399)^2}{160,801} = 1$ .

54.  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{2x}{a^2} + \frac{2y}{b^2} \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{b^2}{a^2} \frac{x}{y}$ . Therefore  $\frac{dy}{dx} = m \Leftrightarrow y = -\frac{b^2}{a^2} \frac{x}{m}$ . Combining this

condition with  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ , we find that  $x = \pm \frac{a^2 m}{\sqrt{a^2 m^2 + b^2}}$ . In other words, the two points on the ellipse where the

tangent has slope  $m$  are  $\left(\pm \frac{a^2 m}{\sqrt{a^2 m^2 + b^2}}, \mp \frac{b^2}{\sqrt{a^2 m^2 + b^2}}\right)$ . The tangent lines at these points have the equations

$$y \pm \frac{b^2}{\sqrt{a^2 m^2 + b^2}} = m \left( x \mp \frac{a^2 m}{\sqrt{a^2 m^2 + b^2}} \right) \text{ or } y = mx \mp \frac{a^2 m^2}{\sqrt{a^2 m^2 + b^2}} \mp \frac{b^2}{\sqrt{a^2 m^2 + b^2}} = mx \mp \sqrt{a^2 m^2 + b^2}.$$

55. Directrix  $x = 4 \Rightarrow d = 4$ , so  $e = \frac{1}{3} \Rightarrow r = \frac{ed}{1 + e \cos \theta} = \frac{4}{3 + \cos \theta}$ .

56. See the end of the proof of Theorem 10.6.1. If  $e > 1$ , then  $1 - e^2 < 0$  and Equations 10.6.4 become  $a^2 = \frac{e^2 d^2}{(e^2 - 1)^2}$  and

$b^2 = \frac{e^2 d^2}{e^2 - 1}$ , so  $\frac{b^2}{a^2} = e^2 - 1$ . The asymptotes  $y = \pm \frac{b}{a}x$  have slopes  $\pm \frac{b}{a} = \pm \sqrt{e^2 - 1}$ , so the angles they make with the polar axis are  $\pm \tan^{-1}[\sqrt{e^2 - 1}] = \cos^{-1}(\pm 1/e)$ .

57. (a) If  $(a, b)$  lies on the curve, then there is some parameter value  $t_1$  such that  $\frac{3t_1}{1 + t_1^3} = a$  and  $\frac{3t_1^2}{1 + t_1^3} = b$ . If  $t_1 = 0$ ,

the point is  $(0, 0)$ , which lies on the line  $y = x$ . If  $t_1 \neq 0$ , then the point corresponding to  $t = \frac{1}{t_1}$  is given by

$$x = \frac{3(1/t_1)}{1 + (1/t_1)^3} = \frac{3t_1^2}{t_1^3 + 1} = b, y = \frac{3(1/t_1)^2}{1 + (1/t_1)^3} = \frac{3t_1}{t_1^3 + 1} = a. \text{ So } (b, a) \text{ also lies on the curve. [Another way to see}$$

this is to do part (e) first; the result is immediate.] The curve intersects the line  $y = x$  when  $\frac{3t}{1 + t^3} = \frac{3t^2}{1 + t^3} \Rightarrow$

$$t = t^2 \Rightarrow t = 0 \text{ or } 1, \text{ so the points are } (0, 0) \text{ and } \left(\frac{3}{2}, \frac{3}{2}\right).$$

(b)  $\frac{dy}{dt} = \frac{(1 + t^3)(6t) - 3t^2(3t^2)}{(1 + t^3)^2} = \frac{6t - 3t^4}{(1 + t^3)^2} = 0$  when  $6t - 3t^4 = 3t(2 - t^3) = 0 \Rightarrow t = 0$  or  $t = \sqrt[3]{2}$ , so there are horizontal tangents at  $(0, 0)$  and  $(\sqrt[3]{2}, \sqrt[3]{4})$ . Using the symmetry from part (a), we see that there are vertical tangents at  $(0, 0)$  and  $(\sqrt[3]{4}, \sqrt[3]{2})$ .

(c) Notice that as  $t \rightarrow -1^+$ , we have  $x \rightarrow -\infty$  and  $y \rightarrow \infty$ . As  $t \rightarrow -1^-$ , we have  $x \rightarrow \infty$  and  $y \rightarrow -\infty$ . Also

$y - (-x - 1) = y + x + 1 = \frac{3t + 3t^2 + (1 + t^3)}{1 + t^3} = \frac{(t + 1)^3}{1 + t^3} = \frac{(t + 1)^2}{t^2 - t + 1} \rightarrow 0$  as  $t \rightarrow -1$ . So  $y = -x - 1$  is a slant asymptote.

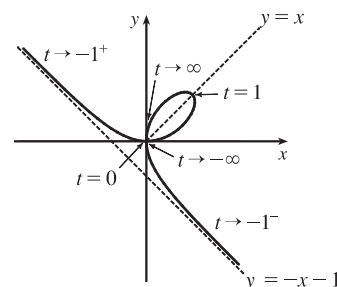
(d)  $\frac{dx}{dt} = \frac{(1 + t^3)(3) - 3t(3t^2)}{(1 + t^3)^2} = \frac{3 - 6t^3}{(1 + t^3)^2}$  and from part (b) we have  $\frac{dy}{dt} = \frac{6t - 3t^4}{(1 + t^3)^2}$ . So  $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{t(2 - t^3)}{1 - 2t^3}$ .

$$\text{Also } \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \frac{2(1 + t^3)^4}{3(1 - 2t^3)^3} > 0 \Leftrightarrow t < \frac{1}{\sqrt[3]{2}}.$$

So the curve is concave upward there and has a minimum point at  $(0, 0)$

and a maximum point at  $(\sqrt[3]{2}, \sqrt[3]{4})$ . Using this together with the

information from parts (a), (b), and (c), we sketch the curve.



$$(e) \ x^3 + y^3 = \left(\frac{3t}{1+t^3}\right)^3 + \left(\frac{3t^2}{1+t^3}\right)^3 = \frac{27t^3 + 27t^6}{(1+t^3)^3} = \frac{27t^3(1+t^3)}{(1+t^3)^3} = \frac{27t^3}{(1+t^3)^2} \text{ and}$$

$$3xy = 3\left(\frac{3t}{1+t^3}\right)\left(\frac{3t^2}{1+t^3}\right) = \frac{27t^3}{(1+t^3)^2}, \text{ so } x^3 + y^3 = 3xy.$$

(f) We start with the equation from part (e) and substitute  $x = r \cos \theta$ ,  $y = r \sin \theta$ . Then  $x^3 + y^3 = 3xy \Rightarrow$

$$r^3 \cos^3 \theta + r^3 \sin^3 \theta = 3r^2 \cos \theta \sin \theta. \text{ For } r \neq 0, \text{ this gives } r = \frac{3 \cos \theta \sin \theta}{\cos^3 \theta + \sin^3 \theta}. \text{ Dividing numerator and denominator}$$

$$\text{by } \cos^3 \theta, \text{ we obtain } r = \frac{3\left(\frac{1}{\cos \theta}\right) \frac{\sin \theta}{\cos \theta}}{1 + \frac{\sin^3 \theta}{\cos^3 \theta}} = \frac{3 \sec \theta \tan \theta}{1 + \tan^3 \theta}.$$

(g) The loop corresponds to  $\theta \in (0, \frac{\pi}{2})$ , so its area is

$$\begin{aligned} A &= \int_0^{\pi/2} \frac{r^2}{2} d\theta = \frac{1}{2} \int_0^{\pi/2} \left(\frac{3 \sec \theta \tan \theta}{1 + \tan^3 \theta}\right)^2 d\theta = \frac{9}{2} \int_0^{\pi/2} \frac{\sec^2 \theta \tan^2 \theta}{(1 + \tan^3 \theta)^2} d\theta = \frac{9}{2} \int_0^\infty \frac{u^2 du}{(1 + u^3)^2} \quad [\text{let } u = \tan \theta] \\ &= \lim_{b \rightarrow \infty} \frac{9}{2} \left[-\frac{1}{3}(1 + u^3)^{-1}\right]_0^b = \frac{3}{2} \end{aligned}$$

(h) By symmetry, the area between the folium and the line  $y = -x - 1$  is equal to the enclosed area in the third quadrant,

plus twice the enclosed area in the fourth quadrant. The area in the third quadrant is  $\frac{1}{2}$ , and since  $y = -x - 1 \Rightarrow$

$$r \sin \theta = -r \cos \theta - 1 \Rightarrow r = -\frac{1}{\sin \theta + \cos \theta}, \text{ the area in the fourth quadrant is}$$

$$\frac{1}{2} \int_{-\pi/2}^{-\pi/4} \left[ \left(-\frac{1}{\sin \theta + \cos \theta}\right)^2 - \left(\frac{3 \sec \theta \tan \theta}{1 + \tan^3 \theta}\right)^2 \right] d\theta \stackrel{\text{CAS}}{=} \frac{1}{2}. \text{ Therefore, the total area is } \frac{1}{2} + 2\left(\frac{1}{2}\right) = \frac{3}{2}.$$

## □ PROBLEMS PLUS

1.  $x = \int_1^t \frac{\cos u}{u} du$ ,  $y = \int_1^t \frac{\sin u}{u} du$ , so by FTC1, we have  $\frac{dx}{dt} = \frac{\cos t}{t}$  and  $\frac{dy}{dt} = \frac{\sin t}{t}$ . Vertical tangent lines occur when  $\frac{dx}{dt} = 0 \Leftrightarrow \cos t = 0$ . The parameter value corresponding to  $(x, y) = (0, 0)$  is  $t = 1$ , so the nearest vertical tangent occurs when  $t = \frac{\pi}{2}$ . Therefore, the arc length between these points is

$$L = \int_1^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_1^{\pi/2} \sqrt{\frac{\cos^2 t}{t^2} + \frac{\sin^2 t}{t^2}} dt = \int_1^{\pi/2} \frac{dt}{t} = [\ln t]_1^{\pi/2} = \ln \frac{\pi}{2}$$

2. (a) The curve  $x^4 + y^4 = x^2 + y^2$  is symmetric about both axes and about the line  $y = x$  (since interchanging  $x$  and  $y$  does not change the equation) so we need only consider  $y \geq x \geq 0$  to begin with. Implicit differentiation gives

$$4x^3 + 4y^3 y' = 2x + 2yy' \Rightarrow y' = \frac{x(1 - 2x^2)}{y(2y^2 - 1)} \Rightarrow y' = 0 \text{ when } x = 0 \text{ and when } x = \pm \frac{1}{\sqrt{2}}. \text{ If } x = 0, \text{ then}$$

$$y^4 = y^2 \Rightarrow y^2(y^2 - 1) = 0 \Rightarrow y = 0 \text{ or } \pm 1. \text{ The point } (0, 0) \text{ can't be a highest or lowest point because it is isolated. [If } -1 < x < 1 \text{ and } -1 < y < 1, \text{ then } x^4 < x^2 \text{ and } y^4 < y^2 \Rightarrow x^4 + y^4 < x^2 + y^2, \text{ except for } (0, 0).]$$

$$\text{If } x = \frac{1}{\sqrt{2}}, \text{ then } x^2 = \frac{1}{2}, x^4 = \frac{1}{4}, \text{ so } \frac{1}{4} + y^4 = \frac{1}{2} + y^2 \Rightarrow 4y^4 - 4y^2 - 1 = 0 \Rightarrow y^2 = \frac{4 \pm \sqrt{16 + 16}}{8} = \frac{1 \pm \sqrt{2}}{2}.$$

But  $y^2 > 0$ , so  $y^2 = \frac{1 + \sqrt{2}}{2} \Rightarrow y = \pm \sqrt{\frac{1 + \sqrt{2}}{2}}$ . Near the point  $(0, 1)$ , the denominator of  $y'$  is positive and the numerator changes from negative to positive as  $x$  increases through 0, so  $(0, 1)$  is a local minimum point. At

$\left(\frac{1}{\sqrt{2}}, \sqrt{\frac{1 + \sqrt{2}}{2}}\right)$ ,  $y'$  changes from positive to negative, so that point gives a maximum. By symmetry, the highest points

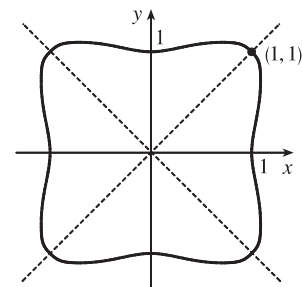
on the curve are  $\left(\pm \frac{1}{\sqrt{2}}, \sqrt{\frac{1 + \sqrt{2}}{2}}\right)$  and the lowest points are  $\left(\pm \frac{1}{\sqrt{2}}, -\sqrt{\frac{1 + \sqrt{2}}{2}}\right)$ .

- (b) We use the information from part (a), together with symmetry with respect to the axes and the lines  $y = \pm x$ , to sketch the curve.

- (c) In polar coordinates,  $x^4 + y^4 = x^2 + y^2$  becomes  $r^4 \cos^4 \theta + r^4 \sin^4 \theta = r^2$  or

$$r^2 = \frac{1}{\cos^4 \theta + \sin^4 \theta}. \text{ By the symmetry shown in part (b), the area enclosed by}$$

$$\text{the curve is } A = 8 \int_0^{\pi/4} \frac{1}{2} r^2 d\theta = 4 \int_0^{\pi/4} \frac{d\theta}{\cos^4 \theta + \sin^4 \theta} \stackrel{\text{CAS}}{=} \sqrt{2} \pi.$$



3. In terms of  $x$  and  $y$ , we have  $x = r \cos \theta = (1 + c \sin \theta) \cos \theta = \cos \theta + c \sin \theta \cos \theta = \cos \theta + \frac{1}{2} c \sin 2\theta$  and  $y = r \sin \theta = (1 + c \sin \theta) \sin \theta = \sin \theta + c \sin^2 \theta$ . Now  $-1 \leq \sin \theta \leq 1 \Rightarrow -1 \leq \sin \theta + c \sin^2 \theta \leq 1 + c \leq 2$ , so  $-1 \leq y \leq 2$ . Furthermore,  $y = 2$  when  $c = 1$  and  $\theta = \frac{\pi}{2}$ , while  $y = -1$  for  $c = 0$  and  $\theta = \frac{3\pi}{2}$ . Therefore, we need a viewing rectangle with  $-1 \leq y \leq 2$ .

To find the  $x$ -values, look at the equation  $x = \cos \theta + \frac{1}{2} c \sin 2\theta$  and use the fact that  $\sin 2\theta \geq 0$  for  $0 \leq \theta \leq \frac{\pi}{2}$  and  $\sin 2\theta \leq 0$  for  $-\frac{\pi}{2} \leq \theta \leq 0$ . [Because  $r = 1 + c \sin \theta$  is symmetric about the  $y$ -axis, we only need to consider

$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ .] So for  $-\frac{\pi}{2} \leq \theta \leq 0$ ,  $x$  has a maximum value when  $c = 0$  and then  $x = \cos \theta$  has a maximum value

of 1 at  $\theta = 0$ . Thus, the maximum value of  $x$  must occur on  $[0, \frac{\pi}{2}]$  with  $c = 1$ . Then  $x = \cos \theta + \frac{1}{2} \sin 2\theta \Rightarrow$

$$\frac{dx}{d\theta} = -\sin \theta + \cos 2\theta = -\sin \theta + 1 - 2\sin^2 \theta \Rightarrow \frac{dx}{d\theta} = -(2\sin \theta - 1)(\sin \theta + 1) = 0 \text{ when } \sin \theta = -1 \text{ or } \frac{1}{2}$$

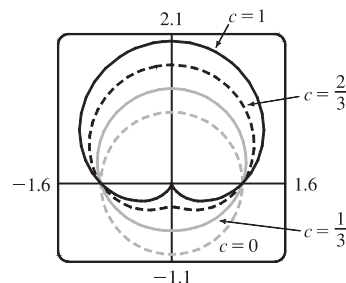
[but  $\sin \theta \neq -1$  for  $0 \leq \theta \leq \frac{\pi}{2}$ ]. If  $\sin \theta = \frac{1}{2}$ , then  $\theta = \frac{\pi}{6}$  and

$$x = \cos \frac{\pi}{6} + \frac{1}{2} \sin \frac{\pi}{3} = \frac{3}{4}\sqrt{3}. \text{ Thus, the maximum value of } x \text{ is } \frac{3}{4}\sqrt{3}, \text{ and,}$$

by symmetry, the minimum value is  $-\frac{3}{4}\sqrt{3}$ . Therefore, the smallest

viewing rectangle that contains every member of the family of polar curves

$$r = 1 + c \sin \theta, \text{ where } 0 \leq c \leq 1, \text{ is } [-\frac{3}{4}\sqrt{3}, \frac{3}{4}\sqrt{3}] \times [-1, 2].$$



4. (a) Let us find the polar equation of the path of the bug that starts in the upper right corner of the square. If the polar coordinates of this bug, at a particular moment, are  $(r, \theta)$ , then the polar coordinates of the bug that it is crawling toward must be  $(r, \theta + \frac{\pi}{2})$ . (The next bug must be the same distance from the origin and the angle between the lines joining the bugs to the pole must be  $\frac{\pi}{2}$ .) The Cartesian coordinates of the first bug are

$(r \cos \theta, r \sin \theta)$  and for the second bug we have

$$x = r \cos (\theta + \frac{\pi}{2}) = -r \sin \theta, y = r \sin (\theta + \frac{\pi}{2}) = r \cos \theta. \text{ So the slope of the line joining the bugs is}$$

$$\frac{r \cos \theta - r \sin \theta}{-r \sin \theta - r \cos \theta} = \frac{\sin \theta - \cos \theta}{\sin \theta + \cos \theta}. \text{ This must be equal to the slope of the tangent line at } (r, \theta), \text{ so by}$$

$$\text{Equation 10.3.3 we have } \frac{(dr/d\theta) \sin \theta + r \cos \theta}{(dr/d\theta) \cos \theta - r \sin \theta} = \frac{\sin \theta - \cos \theta}{\sin \theta + \cos \theta}. \text{ Solving for } \frac{dr}{d\theta}, \text{ we get}$$

$$\frac{dr}{d\theta} \sin^2 \theta + \frac{dr}{d\theta} \sin \theta \cos \theta + r \sin \theta \cos \theta + r \cos^2 \theta = \frac{dr}{d\theta} \sin \theta \cos \theta - \frac{dr}{d\theta} \cos^2 \theta - r \sin^2 \theta + r \sin \theta \cos \theta \Rightarrow$$

$$\frac{dr}{d\theta} (\sin^2 \theta + \cos^2 \theta) + r (\cos^2 \theta + \sin^2 \theta) = 0 \Rightarrow \frac{dr}{d\theta} = -r. \text{ Solving this differential equation as a separable}$$

equation (as in Section 9.3), or using Theorem 9.4.2 with  $k = -1$ , we get  $r = Ce^{-\theta}$ . To determine  $C$  we use the fact that,

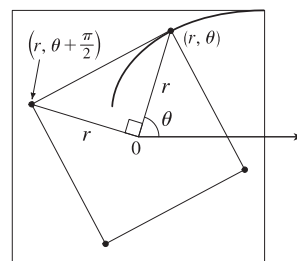
at its starting position,  $\theta = \frac{\pi}{4}$  and  $r = \frac{1}{\sqrt{2}}a$ , so  $\frac{1}{\sqrt{2}}a = Ce^{-\pi/4} \Rightarrow C = \frac{1}{\sqrt{2}}ae^{\pi/4}$ . Therefore, a polar equation of the

bug's path is  $r = \frac{1}{\sqrt{2}}ae^{\pi/4}e^{-\theta}$  or  $r = \frac{1}{\sqrt{2}}ae^{(\pi/4)-\theta}$ .

- (b) The distance traveled by this bug is  $L = \int_{\pi/4}^{\infty} \sqrt{r^2 + (dr/d\theta)^2} d\theta$ , where  $\frac{dr}{d\theta} = \frac{a}{\sqrt{2}}e^{\pi/4}(-e^{-\theta})$  and so

$$r^2 + (dr/d\theta)^2 = \frac{1}{2}a^2e^{\pi/2}e^{-2\theta} + \frac{1}{2}a^2e^{\pi/2}e^{-2\theta} = a^2e^{\pi/2}e^{-2\theta}. \text{ Thus}$$

$$\begin{aligned} L &= \int_{\pi/4}^{\infty} ae^{\pi/4}e^{-\theta} d\theta = ae^{\pi/4} \lim_{t \rightarrow \infty} \int_{\pi/4}^t e^{-\theta} d\theta = ae^{\pi/4} \lim_{t \rightarrow \infty} [-e^{-\theta}]_{\pi/4}^t \\ &= ae^{\pi/4} \lim_{t \rightarrow \infty} [e^{-\pi/4} - e^{-t}] = ae^{\pi/4}e^{-\pi/4} = a \end{aligned}$$





5. Without loss of generality, assume the hyperbola has equation  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ . Use implicit differentiation to get

$$\frac{2x}{a^2} - \frac{2y y'}{b^2} = 0, \text{ so } y' = \frac{b^2 x}{a^2 y}. \text{ The tangent line at the point } (c, d) \text{ on the hyperbola has equation } y - d = \frac{b^2 c}{a^2 d}(x - c).$$

The tangent line intersects the asymptote  $y = \frac{b}{a}x$  when  $\frac{b}{a}x - d = \frac{b^2 c}{a^2 d}(x - c) \Rightarrow abdx - a^2 d^2 = b^2 cx - b^2 c^2 \Rightarrow$

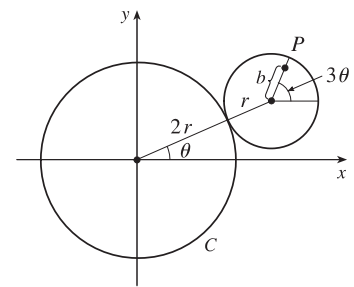
$$abdx - b^2 cx = a^2 d^2 - b^2 c^2 \Rightarrow x = \frac{a^2 d^2 - b^2 c^2}{b(ad - bc)} = \frac{ad + bc}{b} \text{ and the } y\text{-value is } \frac{b}{a} \frac{ad + bc}{b} = \frac{ad + bc}{a}.$$

Similarly, the tangent line intersects  $y = -\frac{b}{a}x$  at  $\left(\frac{bc - ad}{b}, \frac{ad - bc}{a}\right)$ . The midpoint of these intersection points is

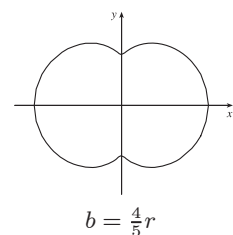
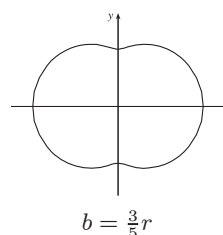
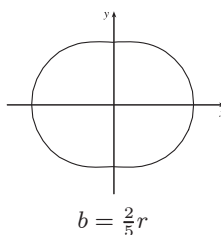
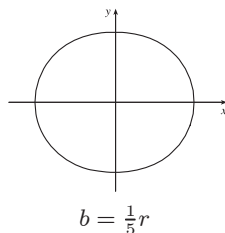
$$\left(\frac{1}{2}\left(\frac{ad + bc}{b} + \frac{bc - ad}{b}\right), \frac{1}{2}\left(\frac{ad + bc}{a} + \frac{ad - bc}{a}\right)\right) = \left(\frac{1}{2}\frac{2bc}{b}, \frac{1}{2}\frac{2ad}{a}\right) = (c, d), \text{ the point of tangency.}$$

*Note:* If  $y = 0$ , then at  $(\pm a, 0)$ , the tangent line is  $x = \pm a$ , and the points of intersection are clearly equidistant from the point of tangency.

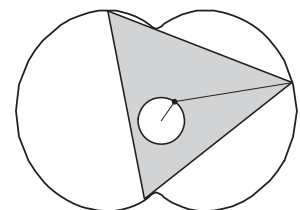
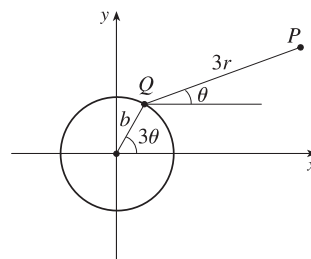
6. (a) Since the smaller circle rolls without slipping around  $C$ , the amount of arc traversed on  $C$  ( $2r\theta$  in the figure) must equal the amount of arc of the smaller circle that has been in contact with  $C$ . Since the smaller circle has radius  $r$ , it must have turned through an angle of  $2r\theta/r = 2\theta$ . In addition to turning through an angle  $2\theta$ , the little circle has rolled through an angle  $\theta$  against  $C$ . Thus,  $P$  has turned through an angle of  $3\theta$  as shown in the figure. (If the little circle had turned through an angle of  $2\theta$  with its center pinned to the  $x$ -axis, then  $P$  would have turned only  $2\theta$  instead of  $3\theta$ . The movement of the little circle around  $C$  adds  $\theta$  to the angle.) From the figure, we see that the center of the small circle has coordinates  $(3r \cos \theta, 3r \sin \theta)$ . Thus,  $P$  has coordinates  $(x, y)$ , where  $x = b \cos 3\theta + 3r \cos \theta$  and  $y = b \sin 3\theta + 3r \sin \theta$ .



(b)



- (c) The diagram gives an alternate description of point  $P$  on the epitrochoid.  $Q$  moves around a circle of radius  $b$ , and  $P$  rotates one-third as fast with respect to  $Q$  at a distance of  $3r$ . Place an equilateral triangle with sides of length  $3\sqrt{3}r$  so that its centroid is at  $Q$  and



one vertex is at  $P$ . (The distance from the centroid to a vertex is  $\frac{1}{\sqrt{3}}$  times the length of a side of the equilateral triangle.)

As  $\theta$  increases by  $\frac{2\pi}{3}$ , the point  $Q$  travels once around the circle of radius  $b$ , returning to its original position. At the same time,  $P$  (and the rest of the triangle) rotate through an angle of  $\frac{2\pi}{3}$  about  $Q$ , so  $P$ 's position is occupied by another vertex. In this way, we see that the epitrochoid traced out by  $P$  is simultaneously traced out by the other two vertices as well. The whole equilateral triangle sits inside the epitrochoid (touching it only with its vertices) and each vertex traces out the curve once while the centroid moves around the circle three times.

- (d) We view the epitrochoid as being traced out in the same way as in part (c), by a rotor for which the distance from its center to each vertex is  $3r$ , so it has radius  $6r$ . To show that the rotor fits inside the epitrochoid, it suffices to show that for any position of the tracing point  $P$ , there are no points on the opposite side of the rotor which are outside the epitrochoid. But the most likely case of intersection is when  $P$  is on the  $y$ -axis, so as long as the diameter of the rotor (which is  $3\sqrt{3}r$ ) is less than the distance between the  $y$ -intercepts, the rotor will fit. The  $y$ -intercepts occur when  $\theta = \frac{\pi}{2}$  or  $\theta = \frac{3\pi}{2} \Rightarrow y = -b + 3r$  or  $y = b - 3r$ , so the distance between the intercepts is  $(-b + 3r) - (b - 3r) = 6r - 2b$ , and the rotor will fit if  $3\sqrt{3}r \leq 6r - 2b \Leftrightarrow 2b \leq 6r - 3\sqrt{3}r \Leftrightarrow b \leq \frac{3}{2}(2 - \sqrt{3})r$ .