1 PRECALCULUS REVIEW

1.1 Real Numbers, Functions, and Graphs

Preliminary Questions

1. Give an example of numbers a and b such that $a < b$ and $|a| > |b|$.

SOLUTION Take $a = -3$ and $b = 1$. Then $a < b$ but $|a| = 3 > 1 = |b|$.

2. Which numbers satisfy $|a| = a$? Which satisfy $|a| = -a$? What about $|-a| = a$?

SOLUTION The numbers $a \geq 0$ satisfy $|a| = a$ and $|-a| = a$. The numbers $a \leq 0$ satisfy $|a| = -a$.

3. Give an example of numbers a and b such that $|a + b| < |a| + |b|$.

SOLUTION Take $a = -3$ and $b = 1$. Then

$$|a + b| = |-3 + 1| = |-2| = 2, \qquad \text{but} \qquad |a| + |b| = |-3| + |1| = 3 + 1 = 4.$$

Thus, $|a + b| < |a| + |b|$.

4. Are there numbers a and b such that $|a + b| > |a| + |b|$?

SOLUTION No. By the Triangle inequality, $|a + b| \leq |a| + |b|$ for all real numbers a and b.

5. What are the coordinates of the point lying at the intersection of the lines $x = 9$ and $y = -4$?

SOLUTION The point $(9, -4)$ lies at the intersection of the lines $x = 9$ and $y = -4$.

6. In which quadrant do the following points lie?

(a) $(1, 4)$ **(b)** $(-3, 2)$ **(c)** $(4, -3)$ **(d)** $(-4, -1)$

SOLUTION

(a) Because both the x- and y-coordinates of the point $(1, 4)$ are positive, the point $(1, 4)$ lies in the first quadrant.

(b) Because the x-coordinate of the point $(-3, 2)$ is negative but the y-coordinate is positive, the point $(-3, 2)$ lies in the second quadrant.

(c) Because the x-coordinate of the point $(4, -3)$ is positive but the y-coordinate is negative, the point $(4, -3)$ lies in the fourth quadrant.

(d) Because both the x- and y-coordinates of the point $(-4, -1)$ are negative, the point $(-4, -1)$ lies in the third quadrant.

7. What is the radius of the circle with equation $(x - 7)^2 + (y - 8)^2 = 9$?

SOLUTION The circle with equation $(x - 7)^2 + (y - 8)^2 = 9$ has radius 3.

8. The equation $f(x) = 5$ has a solution if (choose one):

(a) 5 belongs to the domain of f.

(b) 5 belongs to the range of f.

SOLUTION The correct response is **(b)**: the equation $f(x) = 5$ has a solution if 5 belongs to the range of f.

9. What kind of symmetry does the graph have if $f(-x) = -f(x)$?

SOLUTION If $f(-x) = -f(x)$, then the graph of f is symmetric with respect to the origin.

10. Is there a function that is both even and odd?

SOLUTION Yes. The constant function $f(x) = 0$ for all real numbers x is both even and odd because

$$f(-x) = 0 = f(x)$$

and

$$f(-x) = 0 = -0 = -f(x)$$

for all real numbers x.

Exercises

1. Use a calculator to find a rational number r such that $|r - \pi^2| < 10^{-4}$.

SOLUTION r must satisfy $\pi^2 - 10^{-4} < r < \pi^2 + 10^{-4}$, or $9.869504 < r < 9.869705$. $r = 9.8696 = \frac{12337}{1250}$ would be one such number.

2. Which of (a)–(f) are true for $a = -3$ and $b = 2$?

(a) $a < b$
(b) $|a| < |b|$
(c) $ab > 0$
(d) $3a < 3b$
(e) $-4a < -4b$
(f) $\dfrac{1}{a} < \dfrac{1}{b}$

SOLUTION

(a) True.
(b) False, $|a| = 3 > 2 = |b|$.
(c) False, $(-3)(2) = -6 < 0$.
(d) True.
(e) False, $(-4)(-3) = 12 > -8 = (-4)(2)$.
(f) True.

In Exercises 3–8, express the interval in terms of an inequality involving absolute value.

3. $[-2, 2]$

SOLUTION $|x| \le 2$

4. $(-4, 4)$

SOLUTION $|x| < 4$

5. $(0, 4)$

SOLUTION The midpoint of the interval is $c = (0 + 4)/2 = 2$, and the radius is $r = (4 - 0)/2 = 2$; therefore, $(0, 4)$ can be expressed as $|x - 2| < 2$.

6. $[-4, 0]$

SOLUTION The midpoint of the interval is $c = (-4 + 0)/2 = -2$, and the radius is $r = (0 - (-4))/2 = 2$; therefore, the interval $[-4, 0]$ can be expressed as $|x + 2| \le 2$.

7. $[1, 5]$

SOLUTION The midpoint of the interval is $c = (1 + 5)/2 = 3$, and the radius is $r = (5 - 1)/2 = 2$; therefore, the interval $[1, 5]$ can be expressed as $|x - 3| \le 2$.

8. $(-2, 8)$

SOLUTION The midpoint of the interval is $c = (8 - 2)/2 = 3$, and the radius is $r = (8 - (-2))/2 = 5$; therefore, the interval $(-2, 8)$ can be expressed as $|x - 3| < 5$

In Exercises 9–12, write the inequality in the form $a < x < b$.

9. $|x| < 8$

SOLUTION $-8 < x < 8$

10. $|x - 12| < 8$

SOLUTION $-8 < x - 12 < 8$ so $4 < x < 20$

11. $|2x + 1| < 5$

SOLUTION $-5 < 2x + 1 < 5$ so $-6 < 2x < 4$ and $-3 < x < 2$

12. $|3x - 4| < 2$

SOLUTION $-2 < 3x - 4 < 2$ so $2 < 3x < 6$ and $\frac{2}{3} < x < 2$

In Exercises 13–18, express the set of numbers x satisfying the given condition as an interval.

13. $|x| < 4$

SOLUTION $(-4, 4)$

14. $|x| \le 9$

SOLUTION $[-9, 9]$

15. $|x - 4| < 2$

SOLUTION The expression $|x - 4| < 2$ is equivalent to $-2 < x - 4 < 2$. Therefore, $2 < x < 6$, which represents the interval $(2, 6)$.

16. $|x+7| < 2$

SOLUTION The expression $|x+7| < 2$ is equivalent to $-2 < x+7 < 2$. Therefore, $-9 < x < -5$, which represents the interval $(-9, -5)$.

17. $|4x-1| \le 8$

SOLUTION The expression $|4x-1| \le 8$ is equivalent to $-8 \le 4x-1 \le 8$ or $-7 \le 4x \le 9$. Therefore, $-\frac{7}{4} \le x \le \frac{9}{4}$, which represents the interval $[-\frac{7}{4}, \frac{9}{4}]$.

18. $|3x+5| < 1$

SOLUTION The expression $|3x+5| < 1$ is equivalent to $-1 < 3x+5 < 1$ or $-6 < 3x < -4$. Therefore, $-2 < x < -\frac{4}{3}$ which represents the interval $(-2, -\frac{4}{3})$

In Exercises 19–22, describe the set as a union of finite or infinite intervals.

19. $\{x : |x-4| > 2\}$

SOLUTION $x-4 > 2$ or $x-4 < -2 \Rightarrow x > 6$ or $x < 2 \Rightarrow (-\infty, 2) \cup (6, \infty)$

20. $\{x : |2x+4| > 3\}$

SOLUTION $2x+4 > 3$ or $2x+4 < -3 \Rightarrow 2x > -1$ or $2x < -7 \Rightarrow (-\infty, -\frac{7}{2}) \cup (-\frac{1}{2}, \infty)$

21. $\{x : |x^2-1| > 2\}$

SOLUTION $x^2-1 > 2$ or $x^2-1 < -2 \Rightarrow x^2 > 3$ or $x^2 < -1$ (this will never happen) $\Rightarrow x > \sqrt{3}$ or $x < -\sqrt{3} \Rightarrow (-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)$.

22. $\{x : |x^2+2x| > 2\}$

SOLUTION $x^2+2x > 2$ or $x^2+2x < -2 \Rightarrow x^2+2x-2 > 0$ or $x^2+2x+2 < 0$. For the first case, the zeroes are

$$x = -1 \pm \sqrt{3} \Rightarrow (-\infty, -1-\sqrt{3}) \cup (-1+\sqrt{3}, \infty).$$

For the second case, note there are no real zeros. Because the parabola opens upward and its vertex is located above the x-axis, there are no values of x for which $x^2+2x+2 < 0$. Hence, the solution set is $(-\infty, -1-\sqrt{3}) \cup (-1+\sqrt{3}, \infty)$.

23. Match (a)–(f) with (i)–(vi).

(a) $a > 3$

(b) $|a-5| < \dfrac{1}{3}$

(c) $\left|a - \dfrac{1}{3}\right| < 5$

(d) $|a| > 5$

(e) $|a-4| < 3$

(f) $1 \le a \le 5$

(i) a lies to the right of 3.
(ii) a lies between 1 and 7.
(iii) The distance from a to 5 is less than $\frac{1}{3}$.
(iv) The distance from a to 3 is at most 2.
(v) a is less than 5 units from $\frac{1}{3}$.
(vi) a lies either to the left of -5 or to the right of 5.

SOLUTION

(a) On the number line, numbers greater than 3 appear to the right; hence, $a > 3$ is equivalent to the numbers to the right of 3: **(i)**.

(b) $|a-5|$ measures the distance from a to 5; hence, $|a-5| < \frac{1}{3}$ is satisfied by those numbers less than $\frac{1}{3}$ of a unit from 5: **(iii)**.

(c) $|a-\frac{1}{3}|$ measures the distance from a to $\frac{1}{3}$; hence, $|a-\frac{1}{3}| < 5$ is satisfied by those numbers less than 5 units from $\frac{1}{3}$: **(v)**.

(d) The inequality $|a| > 5$ is equivalent to $a > 5$ or $a < -5$; that is, either a lies to the right of 5 or to the left of -5: **(vi)**.

(e) The interval described by the inequality $|a-4| < 3$ has a center at 4 and a radius of 3; that is, the interval consists of those numbers between 1 and 7: **(ii)**.

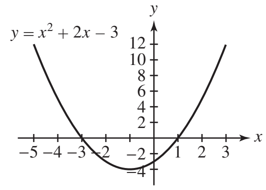
(f) The interval described by the inequality $1 < x < 5$ has a center at 3 and a radius of 2; that is, the interval consists of those numbers less than 2 units from 3: **(iv)**.

24. Describe $\left\{x : \dfrac{x}{x+1} < 0\right\}$ as an interval. *Hint:* Consider the sign of x and $x+1$ individually.

SOLUTION Case 1: $x < 0$ and $x+1 > 0$. This implies that $x < 0$ and $x > -1 \Rightarrow -1 < x < 0$.
Case 2: $x > 0$ and $x < -1$ for which there is no such x. Thus, solution set is therefore $(-1, 0)$.

25. Describe $\{x : x^2 + 2x < 3\}$ as an interval. *Hint:* Plot $y = x^2 + 2x - 3$.

SOLUTION The inequality $x^2 + 2x < 3$ is equivalent to $x^2 + 2x - 3 < 0$. The graph of $y = x^2 + 2x - 3$ is shown below. From this graph, it follows that $x^2 + 2x - 3 < 0$ for $-3 < x < 1$. Thus, the set $\{x : x^2 + 2x < 3\}$ is equivalent to the interval $(-3, 1)$.



26. Describe the set of real numbers satisfying $|x - 3| = |x - 2| + 1$ as a half-infinite interval.

SOLUTION Case 1: If $x \geq 3$, then $|x - 3| = x - 3$, $|x - 2| = x - 2$, and the equation $|x - 3| = |x - 2| + 1$ reduces to $x - 3 = x - 2 + 1$ or $-3 = -1$. As this is never true, the given equation has no solution for $x \geq 3$.

Case 2: If $2 \leq x < 3$, then $|x - 3| = -(x - 3) = 3 - x$, $|x - 2| = x - 2$, and the equation $|x - 3| = |x - 2| + 1$ reduces to $3 - x = x - 2 + 1$ or $x = 2$.

Case 3: If $x < 2$, then $|x - 3| = -(x - 3) = 3 - x$, $|x - 2| = -(x - 2) = 2 - x$, and the equation $|x - 3| = |x - 2| + 1$ reduces to $3 - x = 2 - x + 1$ or $1 = 1$. As this is always true, the given equation holds for all $x < 2$.

Combining the results from all three cases, it follows that the set of real numbers satisfying $|x - 3| = |x - 2| + 1$ is equivalent to the half-infinite interval $(-\infty, 2]$.

27. Show that if $a > b$, and $a, b \neq 0$, then $b^{-1} > a^{-1}$, provided that a and b have the same sign. What happens if $a > 0$ and $b < 0$?

SOLUTION Case 1a: If a and b are both positive, then $a > b \Rightarrow 1 > \frac{b}{a} \Rightarrow \frac{1}{b} > \frac{1}{a}$.

Case 1b: If a and b are both negative, then $a > b \Rightarrow 1 < \frac{b}{a}$ (since a is negative) $\Rightarrow \frac{1}{b} > \frac{1}{a}$ (again, since b is negative).

Case 2: If $a > 0$ and $b < 0$, then $\frac{1}{a} > 0$ and $\frac{1}{b} < 0$ so $\frac{1}{b} < \frac{1}{a}$. (See Exercise 2f for an example of this).

28. Which x satisfies both $|x - 3| < 2$ and $|x - 5| < 1$?

SOLUTION $|x - 3| < 2 \Rightarrow -2 < x - 3 < 2 \Rightarrow 1 < x < 5$. Also $|x - 5| < 1 \Rightarrow 4 < x < 6$. Since we want an x that satisfies both of these, we need the intersection of the two solution sets, that is, $4 < x < 5$.

29. Show that if $|a - 5| < \frac{1}{2}$ and $|b - 8| < \frac{1}{2}$, then $|(a + b) - 13| < 1$. *Hint:* Use the triangle inequality ($|a + b| \leq |a| + |b|$).

SOLUTION

$$\begin{aligned} |a + b - 13| &= |(a - 5) + (b - 8)| \\ &\leq |a - 5| + |b - 8| \quad \text{(by the triangle inequality)} \\ &< \frac{1}{2} + \frac{1}{2} = 1. \end{aligned}$$

30. Suppose that $|x - 4| \leq 1$.

(a) What is the maximum possible value of $|x + 4|$?

(b) Show that $|x^2 - 16| \leq 9$.

SOLUTION

(a) $|x - 4| \leq 1$ guarantees $3 \leq x \leq 5$. Thus, $7 \leq x + 4 \leq 9$, so $|x + 4| \leq 9$.

(b) $|x^2 - 16| = |x - 4| \cdot |x + 4| \leq 1 \cdot 9 = 9$.

31. Suppose that $|a - 6| \leq 2$ and $|b| \leq 3$.

(a) What is the largest possible value of $|a + b|$?

(b) What is the smallest possible value of $|a + b|$?

SOLUTION $|a - 6| \leq 2$ guarantees $4 \leq a \leq 8$, and $|b| \leq 3$ guarantees $-3 \leq b \leq 3$, so $1 \leq a + b \leq 11$. Based on this information,

(a) the largest possible value of $|a + b|$ is 11; and

(b) the smallest possible value of $|a + b|$ is 1.

32. Prove that $|x| - |y| \leq |x - y|$. *Hint:* Apply the triangle inequality to y and $x - y$.

SOLUTION First note

$$|x| = |x - y + y| \leq |x - y| + |y|$$

by the triangle inequality. Subtracting $|y|$ from both sides of this inequality yields

$$|x| - |y| \leq |x - y|.$$

33. Express $r_1 = 0.\overline{27}$ as a fraction. *Hint:* $100r_1 - r_1$ is an integer. Then express $r_2 = 0.2666\ldots$ as a fraction.

SOLUTION Let $r_1 = .\overline{27}$. We observe that $100r_1 = 27.\overline{27}$. Therefore, $100r_1 - r_1 = 27.\overline{27} - .\overline{27} = 27$ and

$$r_1 = \frac{27}{99} = \frac{3}{11}.$$

Now, let $r_2 = .2\overline{666}$. Then $10r_2 = 2.\overline{666}$ and $100r_2 = 26.\overline{666}$. Therefore, $100r_2 - 10r_2 = 26.\overline{666} - 2.\overline{666} = 24$ and

$$r_2 = \frac{24}{90} = \frac{4}{15}.$$

34. Represent $1/7$ and $4/27$ as repeating decimals.

SOLUTION $\frac{1}{7} = .\overline{142857}$; $\frac{4}{27} = .\overline{148}$

35. The text states: *If the decimal expansions of numbers a and b agree to k places, then* $|a - b| \le 10^{-k}$. Show that the converse is false: For all k there are numbers a and b whose decimal expansions *do not agree at all* but $|a - b| \le 10^{-k}$.

SOLUTION Let $a = 1$ and $b = .\overline{9}$ (see the discussion before Example 1). The decimal expansions of a and b do not agree, but $|1 - .\overline{9}| < 10^{-k}$ for all k.

36. Plot each pair of points and compute the distance between them:

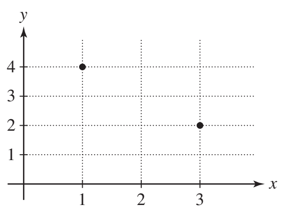
(a) $(1, 4)$ and $(3, 2)$ **(b)** $(2, 1)$ and $(2, 4)$

(c) $(0, 0)$ and $(-2, 3)$ **(d)** $(-3, -3)$ and $(-2, 3)$

SOLUTION

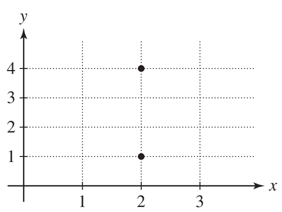
(a) The points $(1, 4)$ and $(3, 2)$ are plotted in the figure below. The distance between the points is

$$d = \sqrt{(3-1)^2 + (2-4)^2} = \sqrt{2^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}.$$



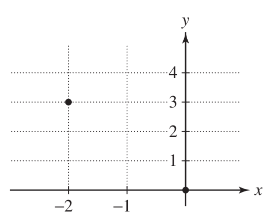
(b) The points $(2, 1)$ and $(2, 4)$ are plotted in the figure below. The distance between the points is

$$d = \sqrt{(2-2)^2 + (4-1)^2} = \sqrt{9} = 3.$$



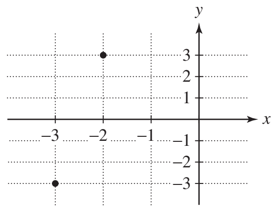
(c) The points $(0, 0)$ and $(-2, 3)$ are plotted in the figure below. The distance between the points is

$$d = \sqrt{(-2-0)^2 + (3-0)^2} = \sqrt{4+9} = \sqrt{13}.$$



(d) The points $(-3, -3)$ and $(-2, 3)$ are plotted in the figure below. The distance between the points is

$$d = \sqrt{(-3-(-2))^2 + (-3-3)^2} = \sqrt{1+36} = \sqrt{37}.$$



37. Find the equation of the circle with center $(2, 4)$:

(a) with radius $r = 3$.

(b) that passes through $(1, -1)$.

SOLUTION (a) The equation of the indicated circle is $(x-2)^2 + (y-4)^2 = 3^2 = 9$.

(b) First determine the radius as the distance from the center to the indicated point on the circle:

$$r = \sqrt{(2-1)^2 + (4-(-1))^2} = \sqrt{26}.$$

Thus, the equation of the circle is $(x-2)^2 + (y-4)^2 = 26$.

38. Find all points in the xy-plane with integer coordinates located at a distance 5 from the origin. Then find all points with integer coordinates located at a distance 5 from $(2, 3)$.

SOLUTION

- To be located a distance 5 from the origin, the points must lie on the circle $x^2 + y^2 = 25$. This leads to 12 points with integer coordinates:

$(5, 0)$	$(-5, 0)$	$(0, 5)$	$(0, -5)$
$(3, 4)$	$(-3, 4)$	$(3, -4)$	$(-3, -4)$
$(4, 3)$	$(-4, 3)$	$(4, -3)$	$(-4, -3)$

- To be located a distance 5 from the point $(2, 3)$, the points must lie on the circle $(x-2)^2 + (y-3)^2 = 25$, which implies that we must shift the points listed above two units to the right and three units up. This gives the 12 points:

$(7, 3)$	$(-3, 3)$	$(2, 8)$	$(2, -2)$
$(5, 7)$	$(-1, 7)$	$(5, -1)$	$(-1, -1)$
$(6, 6)$	$(-2, 6)$	$(6, 0)$	$(-2, 0)$

39. Determine the domain and range of the function

$$f : \{r, s, t, u\} \to \{A, B, C, D, E\}$$

defined by $f(r) = A,\ f(s) = B,\ f(t) = B,\ f(u) = E$.

SOLUTION The domain is the set $D = \{r, s, t, u\}$; the range is the set $R = \{A, B, E\}$.

40. Give an example of a function whose domain D has three elements and whose range R has two elements. Does a function exist whose domain D has two elements and whose range R has three elements?

SOLUTION Define f by $f : \{a, b, c\} \to \{1, 2\}$ where $f(a) = 1,\ f(b) = 1,\ f(c) = 2$.

There is no function whose domain has two elements and range has three elements. If that happened, one of the domain elements would get assigned to more than one element of the range, which would contradict the definition of a function.

In Exercises 41–48, find the domain and range of the function.

41. $f(x) = -x$

SOLUTION D : all reals; R : all reals

42. $g(t) = t^4$

SOLUTION D : all reals; $R : \{y : y \geq 0\}$

43. $f(x) = x^3$

SOLUTION D : all reals; R : all reals

44. $g(t) = \sqrt{2-t}$

SOLUTION $D: \{t : t \leq 2\}$; $R: \{y : y \geq 0\}$

45. $f(x) = |x|$

SOLUTION D : all reals; $R: \{y : y \geq 0\}$

46. $h(s) = \dfrac{1}{s}$

SOLUTION $D: \{s : s \neq 0\}$; $R: \{y : y \neq 0\}$

47. $f(x) = \dfrac{1}{x^2}$

SOLUTION $D: \{x : x \neq 0\}$; $R: \{y : y > 0\}$

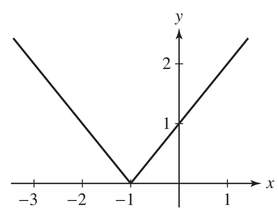
48. $g(t) = \cos \dfrac{1}{t}$

SOLUTION $D: \{t : t \neq 0\}$; $R: \{y : -1 \leq y \leq 1\}$

In Exercises 49–52, determine where f is increasing.

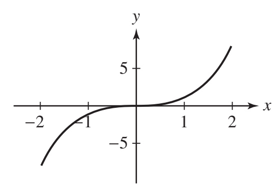
49. $f(x) = |x+1|$

SOLUTION A graph of the function $y = |x+1|$ is shown below. From the graph, we see that the function is increasing on the interval $(-1, \infty)$.



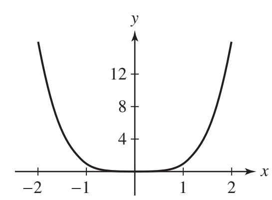
50. $f(x) = x^3$

SOLUTION A graph of the function $y = x^3$ is shown below. From the graph, we see that the function is increasing for all real numbers.



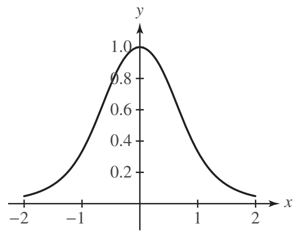
51. $f(x) = x^4$

SOLUTION A graph of the function $y = x^4$ is shown below. From the graph, we see that the function is increasing on the interval $(0, \infty)$.



52. $f(x) = \dfrac{1}{x^4 + x^2 + 1}$

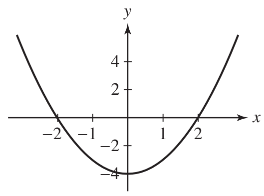
SOLUTION A graph of the function $y = \dfrac{1}{x^4 + x^2 + 1}$ is shown below. From the graph, we see that the function is increasing on the interval $(-\infty, 0)$.



In Exercises 53–58, find the zeros of f and sketch its graph by plotting points. Use symmetry and increase/decrease information where appropriate.

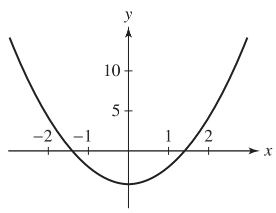
53. $f(x) = x^2 - 4$

SOLUTION Zeros: ± 2
Increasing: $x > 0$
Decreasing: $x < 0$
Symmetry: $f(-x) = f(x)$ (even function). So, y-axis symmetry.



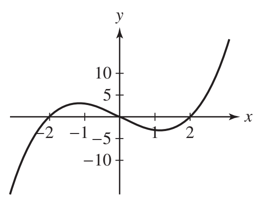
54. $f(x) = 2x^2 - 4$

SOLUTION Zeros: $\pm\sqrt{2}$
Increasing: $x > 0$
Decreasing: $x < 0$
Symmetry: $f(-x) = f(x)$ (even function). So, y-axis symmetry.



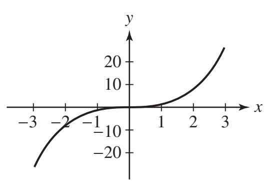
55. $f(x) = x^3 - 4x$

SOLUTION Zeros: $0, \pm 2$; Symmetry: $f(-x) = -f(x)$ (odd function). So origin symmetry.



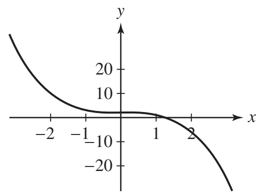
56. $f(x) = x^3$

SOLUTION Zeros: 0; Increasing for all x; Symmetry: $f(-x) = -f(x)$ (odd function). So origin symmetry.



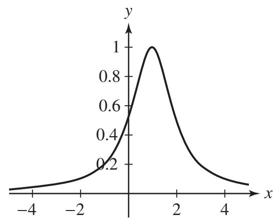
57. $f(x) = 2 - x^3$

SOLUTION This is an x-axis reflection of x^3 translated up 2 units. There is one zero at $x = \sqrt[3]{2}$.



58. $f(x) = \dfrac{1}{(x-1)^2 + 1}$

SOLUTION This is the graph of $\dfrac{1}{x^2+1}$ translated to the right 1 unit. The function has no zeros.



59. Which of the curves in Figure 26 is the graph of a function?

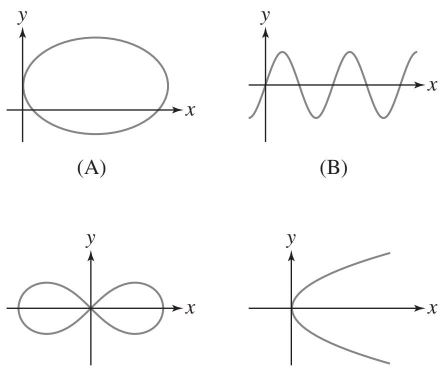


FIGURE 26

SOLUTION (B) is the graph of a function. (A), (C), and (D) all fail the vertical line test.

60. Determine whether the function is even, odd, or neither.

(a) $f(x) = x^5$ **(b)** $g(t) = t^3 - t^2$

(c) $F(t) = \dfrac{1}{t^4 + t^2}$

SOLUTION

(a) Because $f(-x) = (-x)^5 = -x^5 = -f(x)$, $f(x) = x^5$ is an odd function.

(b) Because $g(-t) = (-t)^3 - (-t)^2 = -t^3 - t^2$ equals neither $g(t)$ nor $-g(t)$, $g(t) = t^3 - t^2$ is neither an even function nor an odd function.

(c) Because $F(-t) = \dfrac{1}{(-t)^4 + (-t)^2} = \dfrac{1}{t^4 + t^2} = F(t)$, $F(t) = \dfrac{1}{t^4+t^2}$ is an even function.

61. Determine whether the function is even, odd, or neither.

(a) $f(t) = \dfrac{1}{t^4 + t + 1} - \dfrac{1}{t^4 - t + 1}$ **(b)** $g(t) = 2^t - 2^{-t}$

(c) $G(\theta) = \sin\theta + \cos\theta$ **(d)** $H(\theta) = \sin(\theta^2)$

SOLUTION

(a) Because

$$f(-t) = \frac{1}{(-t)^4 + (-t) + 1} - \frac{1}{(-t)^4 - (-t) + 1} = \frac{1}{t^4 - t + 1} - \frac{1}{t^4 + t + 1}$$
$$= -\left(\frac{1}{t^4 + t + 1} - \frac{1}{t^4 - t + 1}\right) = -f(t),$$

$f(t) = \dfrac{1}{t^4 + t + 1} - \dfrac{1}{t^4 - t + 1}$ is an odd function.

(b) Because $g(-t) = 2^{-t} - 2^{-(-t)} = 2^{-t} - 2^t = -(2^t - 2^{-t}) = -g(t)$, $g(t) = 2^t - 2^{-t}$ is an odd function.

(c) Because $G(-\theta) = \sin(-\theta) + \cos(-\theta) = -\sin\theta + \cos\theta$ equals neither $G(\theta)$ nor $-G(\theta)$, $G(\theta) = \sin\theta + \cos\theta$ is neither an even function nor an odd function.

(d) Because $H(-\theta) = \sin((-\theta)^2) = \sin(\theta^2) = H(\theta)$, $H(\theta) = \sin(\theta^2)$ is an even function.

62. Write $f(x) = 2x^4 - 5x^3 + 12x^2 - 3x + 4$ as the sum of an even and an odd function.

SOLUTION Let $g(x) = 2x^4 + 12x^2 + 4$ and $h(x) = -5x^3 - 3x$. Then

$$g(-x) = 2(-x)^4 + 12(-x)^2 + 4 = 2x^4 + 12x^2 + 4 = g(x)$$

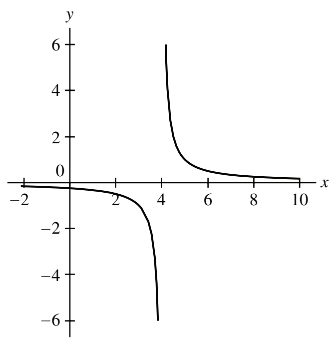
so that g is an even function,

$$h(-x) = -5(-x)^3 - 3(-x) = 5x^3 + 3x = -h(x)$$

so that h is an odd function, and $f(x) = g(x) + h(x)$.

63. Determine the values of x for which $f(x) = \dfrac{1}{x - 4}$ is increasing and for which decreasing.

SOLUTION A graph of the function is shown below. From this graph we can see that $f(x)$ is decreasing on $(-\infty, 4)$ and also decreasing on $(4, \infty)$.



64. State whether the function is increasing, decreasing, or neither.

(a) Surface area of a sphere as a function of its radius

(b) Temperature at a point on the equator as a function of time

(c) Price of an airline ticket as a function of the price of oil

(d) Pressure of the gas in a piston as a function of volume

SOLUTION

(a) Increasing **(b)** Neither **(c)** Increasing **(d)** Decreasing

In Exercises 65–70, let f be the function shown in Figure 27.

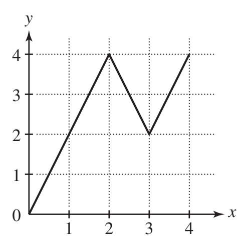


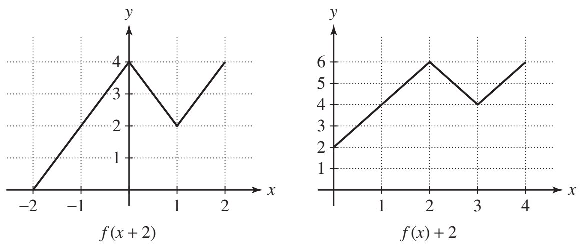
FIGURE 27

65. Find the domain and range of f.

SOLUTION $D : [0, 4]$; $R : [0, 4]$

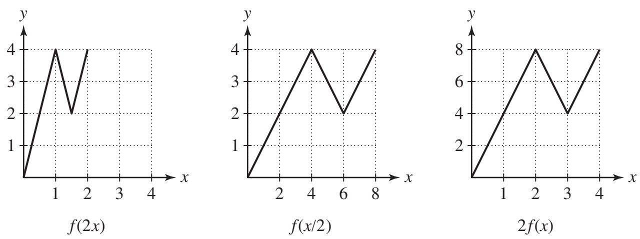
66. Sketch the graphs of $y = f(x + 2)$ and $y = f(x) + 2$.

SOLUTION The graph of $y = f(x + 2)$ is obtained by shifting the graph of $y = f(x)$ two units to the left (see the graph below on the left). The graph of $y = f(x) + 2$ is obtained by shifting the graph of $y = f(x)$ two units up (see the graph below on the right).



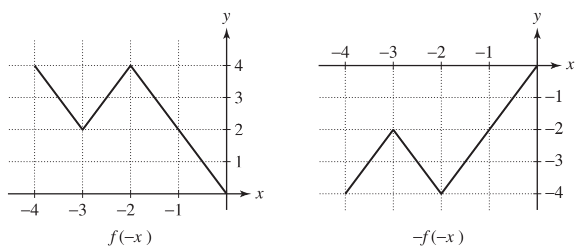
67. Sketch the graphs of $y = f(2x)$, $y = f\left(\frac{1}{2}x\right)$, and $y = 2f(x)$.

SOLUTION The graph of $y = f(2x)$ is obtained by compressing the graph of $y = f(x)$ horizontally by a factor of 2 (see the graph below on the left). The graph of $y = f(\frac{1}{2}x)$ is obtained by stretching the graph of $y = f(x)$ horizontally by a factor of 2 (see the graph below in the middle). The graph of $y = 2f(x)$ is obtained by stretching the graph of $y = f(x)$ vertically by a factor of 2 (see the graph below on the right).



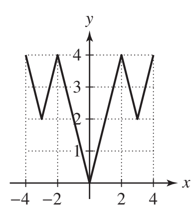
68. Sketch the graphs of $y = f(-x)$ and $y = -f(-x)$.

SOLUTION The graph of $y = f(-x)$ is obtained by reflecting the graph of $y = f(x)$ across the y-axis (see the graph below on the left). The graph of $y = -f(-x)$ is obtained by reflecting the graph of $y = f(x)$ across both the x- and y-axes, or equivalently, about the origin (see the graph below on the right).



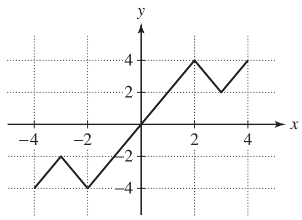
69. Extend the graph of f to $[-4, 4]$ so that it is an even function.

SOLUTION To continue the graph of $f(x)$ to the interval $[-4, 4]$ as an even function, reflect the graph of $f(x)$ across the y-axis (see the graph below).



70. Extend the graph of f to $[-4, 4]$ so that it is an odd function.

SOLUTION To continue the graph of $f(x)$ to the interval $[-4, 4]$ as an odd function, reflect the graph of $f(x)$ through the origin (see the graph below).



71. Suppose that f has domain [4, 8] and range [2, 6]. Find the domain and range of:

(a) $y = f(x) + 3$

(b) $y = f(x + 3)$

(c) $y = f(3x)$

(d) $y = 3f(x)$

SOLUTION

(a) $f(x) + 3$ is obtained by shifting $f(x)$ upward three units. Therefore, the domain remains [4, 8], while the range becomes [5, 9].

(b) $f(x + 3)$ is obtained by shifting $f(x)$ left three units. Therefore, the domain becomes [1, 5], while the range remains [2, 6].

(c) $f(3x)$ is obtained by compressing $f(x)$ horizontally by a factor of three. Therefore, the domain becomes $[\frac{4}{3}, \frac{8}{3}]$, while the range remains [2, 6].

(d) $3f(x)$ is obtained by stretching $f(x)$ vertically by a factor of three. Therefore, the domain remains [4, 8], while the range becomes [6, 18].

72. Let $f(x) = x^2$. Sketch the graph over $[-2, 2]$ of:

(a) $y = f(x + 1)$

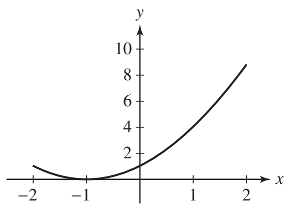
(b) $y = f(x) + 1$

(c) $y = f(5x)$

(d) $y = 5f(x)$

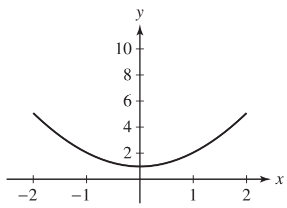
SOLUTION

(a) The graph of $y = f(x + 1)$ is obtained by shifting the graph of $y = f(x)$ one unit to the left.



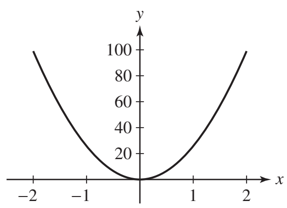
$f(x+1)$

(b) The graph of $y = f(x) + 1$ is obtained by shifting the graph of $y = f(x)$ one unit up.



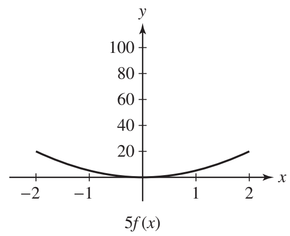
$f(x) + 1$

(c) The graph of $y = f(5x)$ is obtained by compressing the graph of $y = f(x)$ horizontally by a factor of 5.



$f(5x)$

(d) The graph of $y = 5f(x)$ is obtained by stretching the graph of $y = f(x)$ vertically by a factor of 5.



73. Suppose that the graph of $f(x) = \sin x$ is compressed horizontally by a factor of 2 and then shifted 5 units to the right.

(a) What is the equation for the new graph?

(b) What is the equation if you first shift by 5 and then compress by 2?

(c) GU Verify your answers by plotting your equations.

SOLUTION

(a) Let $f(x) = \sin x$. After compressing the graph of f horizontally by a factor of 2, we obtain the function $g(x) = f(2x) = \sin 2x$. Shifting the graph 5 units to the right then yields

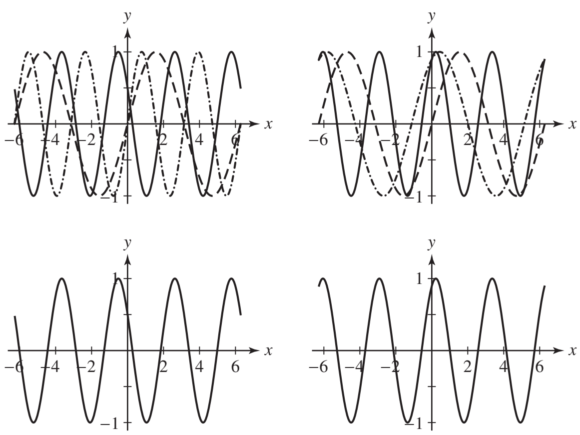
$$h(x) = g(x-5) = \sin 2(x-5) = \sin(2x-10).$$

(b) Let $f(x) = \sin x$. After shifting the graph 5 units to the right, we obtain the function $g(x) = f(x-5) = \sin(x-5)$. Compressing the graph horizontally by a factor of 2 then yields

$$h(x) = g(2x) = \sin(2x-5).$$

(c) The figure below at the top left shows the graphs of $y = \sin x$ (the dashed curve), the sine graph compressed horizontally by a factor of 2 (the dash, double dot curve) and then shifted right 5 units (the solid curve). Compare this last graph with the graph of $y = \sin(2x-10)$ shown at the bottom left.

The figure below at the top right shows the graphs of $y = \sin x$ (the dashed curve), the sine graph shifted to the right 5 units (the dash, double dot curve) and then compressed horizontally by a factor of 2 (the solid curve). Compare this last graph with the graph of $y = \sin(2x-5)$ shown at the bottom right.



74. Figure 28 shows the graph of $f(x) = |x| + 1$. Match the functions (a)–(e) with their graphs (i)–(v).

(a) $y = f(x-1)$ **(b)** $y = -f(x)$ **(c)** $y = -f(x) + 2$

(d) $y = f(x-1) - 2$ **(e)** $y = f(x+1)$

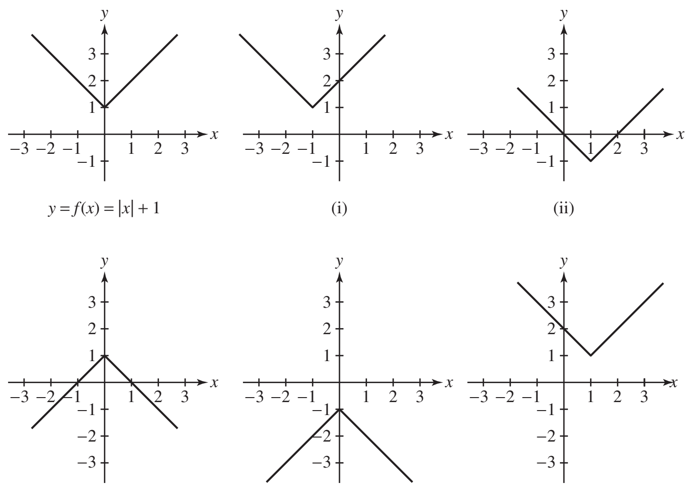


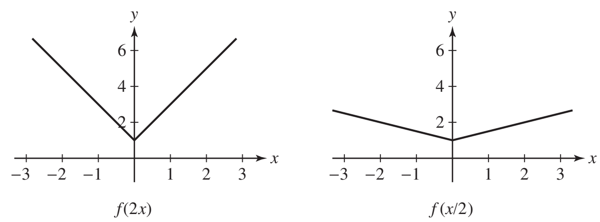
FIGURE 28

SOLUTION

(a) Shift graph to the right one unit: (v)
(b) Reflect graph across x-axis: (iv)
(c) Reflect graph across x-axis and then shift up two units: (iii)
(d) Shift graph to the right one unit and down two units: (ii)
(e) Shift graph to the left one unit: (i)

75. Sketch the graph of $y = f(2x)$ and $y = f\left(\frac{1}{2}x\right)$, where $f(x) = |x| + 1$ (Figure 28).

SOLUTION The graph of $y = f(2x)$ is obtained by compressing the graph of $y = f(x)$ horizontally by a factor of 2 (see the graph below on the left). The graph of $y = f(\frac{1}{2}x)$ is obtained by stretching the graph of $y = f(x)$ horizontally by a factor of 2 (see the graph below on the right).



76. Find the function f whose graph is obtained by shifting the parabola $y = x^2$ by 3 units to the right and 4 units down, as in Figure 29.

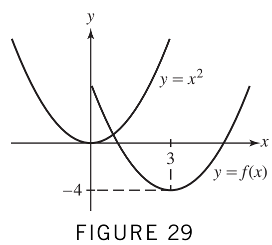
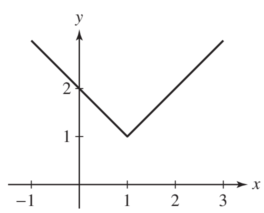


FIGURE 29

SOLUTION The new function is $f(x) = (x - 3)^2 - 4$

77. Define $f(x)$ to be the larger of x and $2 - x$. Sketch the graph of f. What are its domain and range? Express $f(x)$ in terms of the absolute value function.

SOLUTION



The graph of $y = f(x)$ is shown above. Clearly, the domain of f is the set of all real numbers while the range is $\{y \mid y \geq 1\}$. Notice the graph has the standard V-shape associated with the absolute value function, but the base of the V has been translated to the point $(1, 1)$. Thus, $f(x) = |x-1| + 1$.

78. For each curve in Figure 30, state whether it is symmetric with respect to the y-axis, the origin, both, or neither.

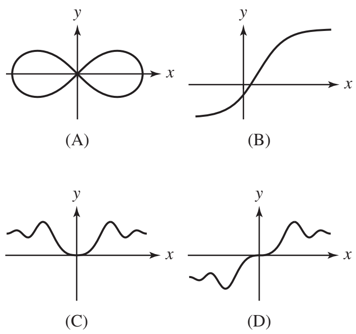


FIGURE 30

SOLUTION

(A) Both
(B) Neither
(C) y-axis
(D) Origin

79. Show that the sum of two even functions is even and the sum of two odd functions is odd.

SOLUTION Even: $(f+g)(-x) = f(-x) + g(-x) \overset{\text{even}}{=} f(x) + g(x) = (f+g)(x)$

Odd: $(f+g)(-x) = f(-x) + g(-x) \overset{\text{odd}}{=} -f(x) + -g(x) = -(f+g)(x)$

80. Suppose that f and g are both odd. Which of the following functions are even? Which are odd?

(a) $y = f(x)g(x)$

(b) $y = f(x)^3$

(c) $y = f(x) - g(x)$

(d) $y = \dfrac{f(x)}{g(x)}$

SOLUTION

(a) $f(-x)g(-x) = (-f(x))(-g(x)) = f(x)g(x) \Rightarrow$ Even

(b) $f(-x)^3 = [-f(x)]^3 = -f(x)^3 \Rightarrow$ Odd

(c) $f(-x) - g(-x) = -f(x) + g(x) = -(f(x) - g(x)) \Rightarrow$ Odd

(d) $\dfrac{f(-x)}{g(-x)} = \dfrac{-f(x)}{-g(x)} = \dfrac{f(x)}{g(x)} \Rightarrow$ Even

81. Prove that the only function whose graph is symmetric with respect to both the y-axis and the origin is the function $f(x) = 0$.

SOLUTION A circle of radius 1 with its center at the origin is symmetrical both with respect to the y-axis and the origin.

The only function having both symmetries is $f(x) = 0$. For if f is symmetric with respect to the y-axis, then $f(-x) = f(x)$. If f is also symmetric with respect to the origin, then $f(-x) = -f(x)$. Thus $f(x) = -f(x)$ or $2f(x) = 0$. Finally, $f(x) = 0$.

Further Insights and Challenges

82. Prove the triangle inequality ($|a+b| \leq |a| + |b|$) by adding the two inequalities

$$-|a| \leq a \leq |a|, \qquad -|b| \leq b \leq |b|$$

SOLUTION Adding the indicated inequalities gives

$$-(|a| + |b|) \leq a + b \leq |a| + |b|$$

and this is equivalent to $|a+b| \leq |a| + |b|$.

83. Show that a fraction $r = a/b$ in lowest terms has a *finite* decimal expansion if and only if

$$b = 2^n 5^m \quad \text{for some } n, m \geq 0.$$

Hint: Observe that r has a finite decimal expansion when $10^N r$ is an integer for some $N \geq 0$ (and hence b divides 10^N).

SOLUTION Suppose r has a finite decimal expansion. Then there exists an integer $N \geq 0$ such that $10^N r$ is an integer, call it k. Thus, $r = k/10^N$. Because the only prime factors of 10 are 2 and 5, it follows that when r is written in lowest terms, its denominator must be of the form $2^n 5^m$ for some integers $n, m \geq 0$.

Conversely, suppose $r = a/b$ in lowest with $b = 2^n 5^m$ for some integers $n, m \geq 0$. Then $r = \frac{a}{b} = \frac{a}{2^n 5^m}$ or $2^n 5^m r = a$. If $m \geq n$, then $2^m 5^m r = a 2^{m-n}$ or $r = \frac{a 2^{m-n}}{10^m}$ and thus r has a finite decimal expansion (less than or equal to m terms, to be precise). On the other hand, if $n > m$, then $2^n 5^n r = a 5^{n-m}$ or $r = \frac{a 5^{n-m}}{10^n}$ and once again r has a finite decimal expansion.

84. Let $p = p_1 \dots p_s$ be an integer with digits $p_1, \dots, p_s$. Show that

$$\frac{p}{10^s - 1} = 0.\overline{p_1 \dots p_s}$$

Use this to find the decimal expansion of $r = \frac{2}{11}$. Note that

$$r = \frac{2}{11} = \frac{18}{10^2 - 1}$$

SOLUTION Let $p = p_1 \dots p_s$ be an integer with digits $p_1, \dots, p_s$, and let $\overline{p} = .\overline{p_1 \dots p_s}$. Then

$$10^s \overline{p} - \overline{p} = p_1 \dots p_s.\overline{p_1 \dots p_s} - .\overline{p_1 \dots p_s} = p_1 \dots p_s = p.$$

Thus,

$$\frac{p}{10^s - 1} = \overline{p} = .\overline{p_1 \dots p_s}.$$

Consider the rational number $r = 2/11$. Because

$$r = \frac{2}{11} = \frac{18}{99} = \frac{18}{10^2 - 1},$$

it follows that the decimal expansion of r is $0.\overline{18}$.

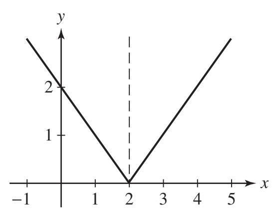
85. A function f is symmetric with respect to the vertical line $x = a$ if $f(a - x) = f(a + x)$.

(a) Draw the graph of a function that is symmetric with respect to $x = 2$.

(b) Show that if f is symmetric with respect to $x = a$, then $g(x) = f(x + a)$ is even.

SOLUTION

(a) There are many possibilities, one of which is:



$y = |x - 2|$

(b) Let $g(x) = f(x + a)$. Then

$$\begin{aligned} g(-x) = f(-x + a) &= f(a - x) \\ &= f(a + x) \qquad \text{symmetry with respect to } x = a \\ &= g(x) \end{aligned}$$

Thus, $g(x)$ is even.

86. Formulate a condition for f to be symmetric with respect to the point $(a, 0)$ on the x-axis.

SOLUTION In order for $f(x)$ to be symmetric with respect to the point $(a, 0)$, the value of f at a distance x units to the right of a must be opposite the value of f at a distance x units to the left of a. In other words, $f(x)$ is symmetrical with respect to $(a, 0)$ if $f(a + x) = -f(a - x)$.

1.2 Linear and Quadratic Functions

Preliminary Questions

1. What is the slope of the line $y = -4x - 9$?

SOLUTION The slope of the line $y = -4x - 9$ is -4, given by the coefficient of x.

2. Are the lines $y = 2x + 1$ and $y = -2x - 4$ perpendicular?

SOLUTION The slopes of perpendicular lines are negative reciprocals of one another. Because the slope of $y = 2x + 1$ is 2 and the slope of $y = -2x - 4$ is -2, these two lines are *not* perpendicular.

3. When is the line $ax + by = c$ parallel to the y-axis? To the x-axis?

SOLUTION The line $ax + by = c$ will be parallel to the y-axis when $b = 0$ and parallel to the x-axis when $a = 0$.

4. Suppose $y = 3x + 2$. What is Δy if x increases by 3?

SOLUTION Because $y = 3x + 2$ is a linear function with slope 3, increasing x by 3 will lead to $\Delta y = 3(3) = 9$.

5. What is the minimum of $f(x) = (x + 3)^2 - 4$?

SOLUTION Because $(x + 3)^2 \geq 0$, it follows that $(x + 3)^2 - 4 \geq -4$. Thus, the minimum value of $(x + 3)^2 - 4$ is -4.

6. What is the result of completing the square for $f(x) = x^2 + 1$?

SOLUTION Because there is no x term in $x^2 + 1$, completing the square on this expression leads to $(x - 0)^2 + 1$.

Exercises

In Exercises 1–4, find the slope, the y-intercept, and the x-intercept of the line with the given equation.

1. $y = 3x + 12$

SOLUTION Because the equation of the line is given in slope-intercept form, the slope is the coefficient of x and the y-intercept is the constant term: that is, $m = 3$ and the y-intercept is 12. To determine the x-intercept, substitute $y = 0$ and then solve for x: $0 = 3x + 12$ or $x = -4$.

2. $y = 4 - x$

SOLUTION Because the equation of the line is given in slope-intercept form, the slope is the coefficient of x and the y-intercept is the constant term: that is, $m = -1$ and the y-intercept is 4. To determine the x-intercept, substitute $y = 0$ and then solve for x: $0 = 4 - x$ or $x = 4$.

3. $4x + 9y = 3$

SOLUTION To determine the slope and y-intercept, we first solve the equation for y to obtain the slope-intercept form. This yields $y = -\frac{4}{9}x + \frac{1}{3}$. From here, we see that the slope is $m = -\frac{4}{9}$ and the y-intercept is $\frac{1}{3}$. To determine the x-intercept, substitute $y = 0$ and solve for x: $4x = 3$ or $x = \frac{3}{4}$.

4. $y - 3 = \frac{1}{2}(x - 6)$

SOLUTION The equation is in point-slope form, so we see that $m = \frac{1}{2}$. Substituting $x = 0$ yields $y - 3 = -3$ or $y = 0$. Thus, the x- and y-intercepts are both 0.

In Exercises 5–8, find the slope of the line.

5. $y = 3x + 2$

SOLUTION $m = 3$

6. $y = 3(x - 9) + 2$

SOLUTION $m = 3$

7. $3x + 4y = 12$

SOLUTION First solve the equation for y to obtain the slope-intercept form. This yields $y = -\frac{3}{4}x + 3$. The slope of the line is therefore $m = -\frac{3}{4}$.

8. $3x + 4y = -8$

SOLUTION First solve the equation for y to obtain the slope-intercept form. This yields $y = -\frac{3}{4}x - 2$. The slope of the line is therefore $m = -\frac{3}{4}$.

In Exercises 9–20, find the equation of the line with the given description.

9. Slope 3, y-intercept 8

SOLUTION Using the slope-intercept form for the equation of a line, we have $y = 3x + 8$.

10. Slope -2, y-intercept 3

SOLUTION Using the slope-intercept form for the equation of a line, we have $y = -2x + 3$.

11. Slope 3, passes through $(7, 9)$

SOLUTION Using the point-slope form for the equation of a line, we have $y - 9 = 3(x - 7)$ or $y = 3x - 12$.

12. Slope -5, passes through $(0, 0)$

SOLUTION Using the point-slope form for the equation of a line, we have $y - 0 = -5(x - 0)$ or $y = -5x$.

13. Horizontal, passes through $(0, -2)$

SOLUTION A horizontal line has a slope of 0. Using the point-slope form for the equation of a line, we have $y - (-2) = 0(x - 0)$ or $y = -2$.

14. Passes through $(-1, 4)$ and $(2, 7)$

SOLUTION The slope of the line that passes through $(-1, 4)$ and $(2, 7)$ is

$$m = \frac{7 - 4}{2 - (-1)} = 1.$$

Using the point-slope form for the equation of a line, we have $y - 7 = 1(x - 2)$ or $y = x + 5$.

15. Parallel to $y = 3x - 4$, passes through $(1, 1)$

SOLUTION Because the equation $y = 3x - 4$ is in slope-intercept form, we can readily identify that it has a slope of 3. Parallel lines have the same slope, so the slope of the requested line is also 3. Using the point-slope form for the equation of a line, we have $y - 1 = 3(x - 1)$ or $y = 3x - 2$.

16. Passes through $(1, 4)$ and $(12, -3)$

SOLUTION The slope of the line that passes through $(1, 4)$ and $(12, -3)$ is

$$m = \frac{-3 - 4}{12 - 1} = \frac{-7}{11}.$$

Using the point-slope form for the equation of a line, we have $y - 4 = -\frac{7}{11}(x - 1)$ or $y = -\frac{7}{11}x + \frac{51}{11}$.

17. Perpendicular to $3x + 5y = 9$, passes through $(2, 3)$

SOLUTION We start by solving the equation $3x + 5y = 9$ for y to obtain the slope-intercept form for the equation of a line. This yields

$$y = -\frac{3}{5}x + \frac{9}{5},$$

from which we identify the slope as $-\frac{3}{5}$. Perpendicular lines have slopes that are negative reciprocals of one another, so the slope of the desired line is $m_{\perp} = \frac{5}{3}$. Using the point-slope form for the equation of a line, we have $y - 3 = \frac{5}{3}(x - 2)$ or $y = \frac{5}{3}x - \frac{1}{3}$.

18. Vertical, passes through $(-4, 9)$

SOLUTION A vertical line has the equation $x = c$ for some constant c. Because the line needs to pass through the point $(-4, 9)$, we must have $c = -4$. The equation of the desired line is then $x = -4$.

19. Horizontal, passes through $(8, 4)$

SOLUTION A horizontal line has slope 0. Using the point slope form for the equation of a line, we have $y - 4 = 0(x - 8)$ or $y = 4$.

20. Slope 3, x-intercept 6

SOLUTION If the x-intercept is 6, then the line passes through the point $(6, 0)$. Using the point-slope form for the equation of a line, we have $y - 0 = 3(x - 6)$ or $y = 3x - 18$.

21. Find the equation of the perpendicular bisector of the segment joining (1, 2) and (5, 4) (Figure 12). *Hint:* The midpoint Q of the segment joining (a, b) and (c, d) is $\left(\frac{a+c}{2}, \frac{b+d}{2}\right)$.

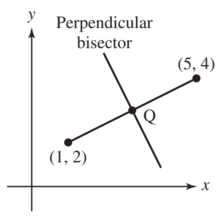


FIGURE 12

SOLUTION The slope of the segment joining (1, 2) and (5, 4) is

$$m = \frac{4-2}{5-1} = \frac{1}{2}$$

and the midpoint of the segment (Figure 12) is

$$\text{midpoint} = \left(\frac{1+5}{2}, \frac{2+4}{2}\right) = (3, 3)$$

The perpendicular bisector has slope $-1/m = -2$ and passes through (3, 3), so its equation is: $y - 3 = -2(x - 3)$ or $y = -2x + 9$.

22. Intercept-Intercept Form Show that if $a, b \neq 0$, then the line with x-intercept $x = a$ and y-intercept $y = b$ has equation (Figure 13)

$$\frac{x}{a} + \frac{y}{b} = 1$$

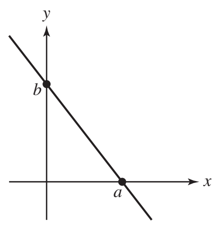


FIGURE 13

SOLUTION The line passes through the points $(a, 0)$ and $(0, b)$. Thus $m = -\frac{b}{a}$. Using the point-slope form for the equation of a line yields $y - 0 = -\frac{b}{a}(x - a) \Rightarrow y = -\frac{b}{a}x + b \Rightarrow \frac{b}{a}x + y = b \Rightarrow \frac{x}{a} + \frac{y}{b} = 1$.

23. Find an equation of the line with x-intercept $x = 4$ and y-intercept $y = 3$.

SOLUTION From Exercise 22, $\frac{x}{4} + \frac{y}{3} = 1$ or $3x + 4y = 12$.

24. Find y such that $(3, y)$ lies on the line of slope $m = 2$ through $(1, 4)$.

SOLUTION In order for the point $(3, y)$ to lie on the line through $(1, 4)$ of slope 2, the slope of the segment connecting $(1, 4)$ and $(3, y)$ must have slope 2. Therefore,

$$m = \frac{y-4}{3-1} = \frac{y-4}{2} = 2 \Rightarrow y - 4 = 4 \Rightarrow y = 8.$$

25. Determine whether there exists a constant c such that the line $x + cy = 1$:

(a) has slope 4.

(b) passes through (3, 1).

(c) is horizontal.

(d) is vertical.

SOLUTION

(a) Rewriting the equation of the line in slope-intercept form gives $y = -\frac{x}{c} + \frac{1}{c}$. To have slope 4 requires $-\frac{1}{c} = 4$ or $c = -\frac{1}{4}$.

(b) Substituting $x = 3$ and $y = 1$ into the equation of the line gives $3 + c = 1$ or $c = -2$.

(c) From (a), we know the slope of the line is $-\frac{1}{c}$. There is no value for c that will make this slope equal to 0.

(d) With $c = 0$, the equation becomes $x = 1$. This is the equation of a vertical line.

26. Assume that the number N of concert tickets that can be sold at a price of P dollars per ticket is a linear function $N(P)$ for $10 \leq P \leq 40$. Determine $N(P)$ (called the demand function) if $N(10) = 500$ and $N(40) = 0$. What is the decrease ΔN in the number of tickets sold if the price is increased by $\Delta P = 5$ dollars?

SOLUTION We first determine the slope of the line:

$$m = \frac{500 - 0}{10 - 40} = \frac{500}{-30} = -\frac{50}{3}.$$

Knowing that $N(40) = 0$, it follows that

$$N(P) = -\frac{50}{3}(P - 40) = -\frac{50}{3}P + \frac{2000}{3}.$$

Because the slope of the demand function is $-\frac{50}{3}$, a 5 dollar increase in price will lead to a decrease in the number of tickets sold of $\frac{50}{3}(5) = \frac{250}{3} = 83\frac{1}{3}$, or about 83 tickets.

27. Suppose that the number of a certain type of computer that can be sold when its price is P (in dollars) is given by a linear function $N(P)$. Determine $N(P)$ if $N(1000) = 10{,}000$ and $N(1500) = 7{,}500$. What is the change ΔN in the number of computers sold if the price is increased by $\Delta P = 100$ dollars?

SOLUTION We first determine the slope of the line:

$$m = \frac{10000 - 7500}{1000 - 1500} = \frac{2500}{-500} = -5.$$

Knowing that $N(1000) = 10000$, it follows that

$$N - 10000 = -5(P - 1000), \quad \text{or} \quad N(P) = -5P + 15000.$$

Because the slope of the demand function is -5, a 100 dollar increase in price will lead to a decrease in the number of computers sold of $5(100) = 500$ computers.

28. Suppose that the demand for Colin's kidney pies is linear in the price P. Determine the demand function N as a function of P giving the number of pies sold when the price is P if he can sell 100 pies when the price is \$5.00 and he can sell 40 pies when the price is \$10.00. Determine the revenue $(N \times P)$ for prices $P = 5, 6, 7, 8, 9, 10$ and then choose a price to maximize the revenue.

SOLUTION We first determine the slope of the line:

$$m = \frac{100 - 40}{5 - 10} = \frac{60}{-5} = -12.$$

Knowing that $N(5) = 100$, it follows that

$$N - 100 = -12(P - 5), \quad \text{or} \quad N(P) = -12P + 160.$$

The table below displays the revenue for prices $P = 5, 6, 7, 8, 9, 10$.

Price (P)	Demand (N)	Revenue ($N \times P$)
5	100	500
6	88	528
7	76	532
8	64	512
9	52	468
10	40	400

To determine the price that will maximize revenue, note that the revenue function, $R(P)$, is given by

$$R(P) = N(P) \times P = (160 - 12P)P = -12P^2 + 160P.$$

Completing the square on the revenue function yields

$$R(P) = -12\left(P^2 - \frac{40}{3}P + \frac{400}{9}\right) + \frac{1600}{9} = -12\left(P - \frac{20}{3}\right)^2 + \frac{1600}{6}.$$

From here, it follows that revenue is maximized when $P = \dfrac{20}{3}$, or roughly \$6.67.

29. Materials expand when heated. Consider a metal rod of length L_0 at temperature T_0. If the temperature is changed by an amount ΔT, then the rod's length approximately changes by $\Delta L = \alpha L_0 \Delta T$, where α is the thermal expansion coefficient and ΔT is not an extreme temperature change. For steel, $\alpha = 1.24 \times 10^{-5}\,^\circ\text{C}^{-1}$.

(a) A steel rod has length $L_0 = 40$ cm at $T_0 = 40^\circ$C. Find its length at $T = 90^\circ$C.

(b) Find its length at $T = 50^\circ$C if its length at $T_0 = 100^\circ$C is 65 cm.

(c) Express length L as a function of T if $L_0 = 65$ cm at $T_0 = 100^\circ$C.

SOLUTION

(a) With $T = 90°$C and $T_0 = 40°$C, $\Delta T = 50°$C. Therefore,

$$\Delta L = \alpha L_0 \Delta T = (1.24 \times 10^{-5})(40)(50) = .0248 \qquad \text{and} \qquad L = L_0 + \Delta L = 40.0248 \text{ cm}.$$

(b) With $T = 50°$C and $T_0 = 100°$C, $\Delta T = -50°$C. Therefore,

$$\Delta L = \alpha L_0 \Delta T = (1.24 \times 10^{-5})(65)(-50) = -.0403 \qquad \text{and} \qquad L = L_0 + \Delta L = 64.9597 \text{ in}.$$

(c) $L = L_0 + \Delta L = L_0 + \alpha L_0 \Delta T = L_0(1 + \alpha \Delta T) = 65(1 + \alpha(T - 100))$

30. Do the points (0.5, 1), (1, 1.2), (2, 2) lie on a line?

SOLUTION Examine the slope between consecutive data points. The first pair of data points yields a slope of

$$\frac{1.2 - 1}{1 - .5} = \frac{.2}{.5} = .4,$$

while the second pair of data points yields a slope of

$$\frac{2 - 1.2}{2 - 1} = \frac{.8}{1} = .8.$$

Because the slopes are not equal, the three points do not lie on a line.

31. Find b such that $(2, -1)$, $(3, 2)$, and $(b, 5)$ lie on a line.

SOLUTION The slope of the line determined by the points $(2, -1)$ and $(3, 2)$ is

$$\frac{2 - (-1)}{3 - 2} = 3.$$

To lie on the same line, the slope between $(3, 2)$ and $(b, 5)$ must also be 3. Thus, we require

$$\frac{5 - 2}{b - 3} = \frac{3}{b - 3} = 3,$$

or $b = 4$.

32. Find an expression for the velocity v as a linear function of t that matches the following data:

t (s)	0	2	4	6
v (m/s)	39.2	58.6	78	97.4

SOLUTION Examine the slope between consecutive data points. The first pair of data points yields a slope of

$$\frac{58.6 - 39.2}{2 - 0} = 9.7,$$

while the second pair of data points yields a slope of

$$\frac{78 - 58.6}{4 - 2} = 9.7,$$

and the last pair of data points yields a slope of

$$\frac{97.4 - 78}{6 - 4} = 9.7$$

Thus, the data suggests a linear function with slope 9.7. Finally,

$$v - 39.2 = 9.7(t - 0) \Rightarrow v = 9.7t + 39.2$$

33. The period T of a pendulum is measured for pendulums of several different lengths L. Based on the following data, does T appear to be a linear function of L?

L (cm)	20	30	40	50
T (s)	0.9	1.1	1.27	1.42

SOLUTION Examine the slope between consecutive data points. The first pair of data points yields a slope of

$$\frac{1.1 - 0.9}{30 - 20} = 0.02,$$

while the second pair of data points yields a slope of

$$\frac{1.27 - 1.1}{40 - 30} = 0.017,$$

and the last pair of data points yields a slope of

$$\frac{1.42 - 1.27}{50 - 40} = 0.015$$

Because the three slopes are not equal, T does not appear to be a linear function of L.

34. Show that f is linear of slope m if and only if

$$f(x + h) - f(x) = mh \quad \text{(for all } x \text{ and } h\text{)}$$

That is to say, prove the following two statements:

(a) f is linear of slope m implies that $f(x + h) - f(x) = mh$ (for all x and h).

(b) $f(x + h) - f(x) = mh$ (for all x and h) implies that f is linear of slope m.

SOLUTION

(a) First, suppose $f(x)$ is linear. Then the slope between $(x, f(x))$ and $(x + h, f(x + h))$ is

$$m = \frac{f(x + h) - f(x)}{h} \Rightarrow mh = f(x + h) - f(x).$$

(b) Conversely, suppose $f(x + h) - f(x) = mh$ for all x and for all h. Then

$$m = \frac{f(x + h) - f(x)}{h} = \frac{f(x + h) - f(x)}{x + h - x},$$

which is the slope between $(x, f(x))$ and $(x + h, f(x + h))$. Since this is true for all x and h, f must be linear (it has constant slope).

35. Find the roots of the quadratic polynomials:

(a) $f(x) = 4x^2 - 3x - 1$

(b) $f(x) = x^2 - 2x - 1$

SOLUTION

(a) $x = \dfrac{3 \pm \sqrt{9 - 4(4)(-1)}}{2(4)} = \dfrac{3 \pm \sqrt{25}}{8} = 1 \text{ or } -\dfrac{1}{4}$

(b) $x = \dfrac{2 \pm \sqrt{4 - (4)(1)(-1)}}{2} = \dfrac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}$

In Exercises 36–43, complete the square and find the minimum or maximum value of the quadratic function.

36. $y = x^2 + 2x + 5$

SOLUTION $y = x^2 + 2x + 1 - 1 + 5 = (x + 1)^2 + 4$; therefore, the minimum value of the quadratic polynomial is 4, and this occurs at $x = -1$.

37. $y = x^2 - 6x + 9$

SOLUTION $y = (x - 3)^2$; therefore, the minimum value of the quadratic polynomial is 0, and this occurs at $x = 3$.

38. $y = -9x^2 + x$

SOLUTION $y = -9(x^2 - x/9) = -9(x^2 - \frac{x}{9} + \frac{1}{324}) + \frac{9}{324} = -9(x - \frac{1}{18})^2 + \frac{1}{36}$; therefore, the maximum value of the quadratic polynomial is $\frac{1}{36}$, and this occurs at $x = \frac{1}{18}$.

39. $y = x^2 + 6x + 2$

SOLUTION $y = x^2 + 6x + 9 - 9 + 2 = (x + 3)^2 - 7$; therefore, the minimum value of the quadratic polynomial is -7, and this occurs at $x = -3$.

40. $y = 2x^2 - 4x - 7$

SOLUTION $y = 2(x^2 - 2x + 1 - 1) - 7 = 2(x^2 - 2x + 1) - 7 - 2 = 2(x - 1)^2 - 9$; therefore, the minimum value of the quadratic polynomial is -9, and this occurs at $x = 1$.

41. $y = -4x^2 + 3x + 8$

SOLUTION $y = -4x^2 + 3x + 8 = -4(x^2 - \frac{3}{4}x + \frac{9}{64}) + 8 + \frac{9}{16} = -4(x - \frac{3}{8})^2 + \frac{137}{16}$; therefore, the maximum value of the quadratic polynomial is $\frac{137}{16}$, and this occurs at $x = \frac{3}{8}$.

42. $y = 3x^2 + 12x - 5$

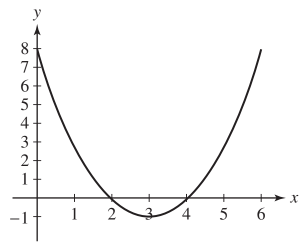
SOLUTION $y = 3(x^2 + 4x + 4) - 5 - 12 = 3(x + 2)^2 - 17$; therefore, the minimum value of the quadratic polynomial is -17, and this occurs at $x = -2$.

43. $y = 4x - 12x^2$

SOLUTION $y = -12(x^2 - \frac{x}{3}) = -12(x^2 - \frac{x}{3} + \frac{1}{36}) + \frac{1}{3} = -12(x - \frac{1}{6})^2 + \frac{1}{3}$; therefore, the maximum value of the quadratic polynomial is $\frac{1}{3}$, and this occurs at $x = \frac{1}{6}$.

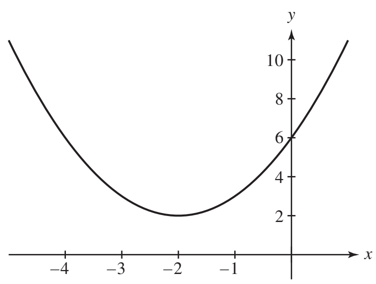
44. Sketch the graph of $y = x^2 - 6x + 8$ by plotting the roots and the minimum point.

SOLUTION $y = x^2 - 6x + 9 - 9 + 8 = (x - 3)^2 - 1$ so the vertex is located at $(3, -1)$ and the roots are $x = 2$ and $x = 4$. This is the graph of x^2 moved right 3 units and down 1 unit.



45. Sketch the graph of $y = x^2 + 4x + 6$ by plotting the minimum point, the y-intercept, and one other point.

SOLUTION $y = x^2 + 4x + 4 - 4 + 6 = (x + 2)^2 + 2$ so the minimum occurs at $(-2, 2)$. If $x = 0$, then $y = 6$ and if $x = -4$, $y = 6$. This is the graph of x^2 moved left 2 units and up 2 units.



46. If the alleles A and B of the cystic fibrosis gene occur in a population with frequencies p and $1 - p$ (where p is a fraction between 0 and 1), then the frequency of heterozygous carriers (carriers with both alleles) is $2p(1 - p)$. Which value of p gives the largest frequency of heterozygous carriers?

SOLUTION Let

$$f = 2p - 2p^2 = -2\left(p^2 - p + \frac{1}{4}\right) + \frac{1}{2} = -2\left(p - \frac{1}{2}\right)^2 + \frac{1}{2}.$$

Then $p = \frac{1}{2}$ yields a maximum.

47. For which values of c does $f(x) = x^2 + cx + 1$ have a double root? No real roots?

SOLUTION A double root occurs when $c^2 - 4(1)(1) = 0$ or $c^2 = 4$. Thus, $c = \pm 2$.

There are no real roots when $c^2 - 4(1)(1) < 0$ or $c^2 < 4$. Thus, $-2 < c < 2$.

48. Let f be a quadratic function and c a constant. Which of the following statements is correct? Explain graphically.

(a) There is a unique value of c such that $y = f(x) - c$ has a double root.

(b) There is a unique value of c such that $y = f(x - c)$ has a double root.

SOLUTION First note that because $f(x)$ is a quadratic function, its graph is a parabola.

(a) This is true. Because $f(x) - c$ is a vertical translation of the graph of $f(x)$, there is one and only one value of c that will move the vertex of the parabola to the x-axis.

(b) This is false. Observe that $f(x - c)$ is a horizontal translation of the graph of $f(x)$. If $f(x)$ has a double root, then $f(x - c)$ will have a double root for any value of c; on the other hand, if $f(x)$ does not have a double root, then there is no value of c for which $f(x - c)$ will have a double root.

49. Prove that $x + \frac{1}{x} \ge 2$ for all $x > 0$. *Hint:* Consider $(x^{1/2} - x^{-1/2})^2$.

SOLUTION Let $x > 0$. Then

$$\left(x^{1/2} - x^{-1/2}\right)^2 = x - 2 + \frac{1}{x}.$$

Because $(x^{1/2} - x^{-1/2})^2 \ge 0$, it follows that

$$x - 2 + \frac{1}{x} \ge 0 \qquad \text{or} \qquad x + \frac{1}{x} \ge 2.$$

50. Let $a, b > 0$. Show that the *geometric mean* $\sqrt{ab}$ is not larger than the *arithmetic mean* $(a + b)/2$. *Hint:* Use a variation of the hint given in Exercise 49.

SOLUTION Let $a, b > 0$ and note

$$0 \le \left(\sqrt{a} - \sqrt{b}\right)^2 = a - 2\sqrt{ab} + b.$$

Therefore,

$$\sqrt{ab} \le \frac{a+b}{2}.$$

51. If objects of weights x and w_1 are suspended from the balance in Figure 14(A), the cross-beam is horizontal if $bx = aw_1$. If the lengths a and b are known, we may use this equation to determine an unknown weight x by selecting w_1 such that the cross-beam is horizontal. If a and b are not known precisely, we might proceed as follows. First balance x by w_1 on the left as in (A). Then switch places and balance x by w_2 on the right as in (B). The average $\bar{x} = \frac{1}{2}(w_1 + w_2)$ gives an estimate for x. Show that $\bar{x}$ is greater than or equal to the true weight x.

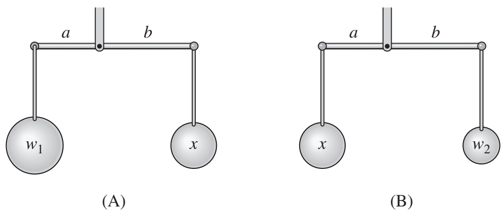


FIGURE 14

SOLUTION First note $bx = aw_1$ and $ax = bw_2$. Thus,

$$\begin{aligned}\bar{x} &= \frac{1}{2}(w_1 + w_2)\\ &= \frac{1}{2}\left(\frac{bx}{a} + \frac{ax}{b}\right)\\ &= \frac{x}{2}\left(\frac{b}{a} + \frac{a}{b}\right)\\ &\ge \frac{x}{2}(2) \qquad \text{by Exercise 49}\\ &= x\end{aligned}$$

52. Find numbers x and y with sum 10 and product 24. *Hint:* Find a quadratic polynomial satisfied by x.

SOLUTION Let x and y be numbers whose sum is 10 and product is 24. Then $x + y = 10$ and $xy = 24$. From the second equation, $y = \frac{24}{x}$. Substituting this expression for y in the first equation gives $x + \frac{24}{x} = 10$ or $x^2 - 10x + 24 = (x - 4)(x - 6) = 0$, whence $x = 4$ or $x = 6$. If $x = 4$, then $y = \frac{24}{4} = 6$. On the other hand, if $x = 6$, then $y = \frac{24}{6} = 4$. Thus, the two numbers are 4 and 6.

53. Find a pair of numbers whose sum and product are both equal to 8.

SOLUTION Let x and y be numbers whose sum and product are both equal to 8. Then $x + y = 8$ and $xy = 8$. From the second equation, $y = \frac{8}{x}$. Substituting this expression for y in the first equation gives $x + \frac{8}{x} = 8$ or $x^2 - 8x + 8 = 0$. By the quadratic formula,

$$x = \frac{8 \pm \sqrt{64 - 32}}{2} = 4 \pm 2\sqrt{2}.$$

If $x = 4 + 2\sqrt{2}$, then

$$y = \frac{8}{4+2\sqrt{2}} = \frac{8}{4+2\sqrt{2}} \cdot \frac{4-2\sqrt{2}}{4-2\sqrt{2}} = 4 - 2\sqrt{2}.$$

On the other hand, if $x = 4 - 2\sqrt{2}$, then

$$y = \frac{8}{4-2\sqrt{2}} = \frac{8}{4-2\sqrt{2}} \cdot \frac{4+2\sqrt{2}}{4+2\sqrt{2}} = 4 + 2\sqrt{2}.$$

Thus, the two numbers are $4 + 2\sqrt{2}$ and $4 - 2\sqrt{2}$.

54. Show that the parabola $y = x^2$ consists of all points P such that $d_1 = d_2$, where d_1 is the distance from P to $\left(0, \frac{1}{4}\right)$ and d_2 is the distance from P to the line $y = -\frac{1}{4}$ (Figure 15).

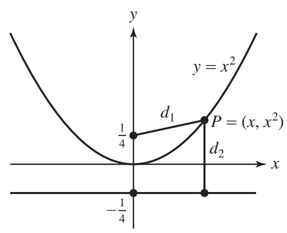


FIGURE 15

SOLUTION Let P be a point on the graph of the parabola $y = x^2$. Then P has coordinates (x, x^2) for some real number x. Now $d_2 = x^2 + \frac{1}{4}$ and

$$d_1 = \sqrt{(x-0)^2 + \left(x^2 - \frac{1}{4}\right)^2} = \sqrt{x^2 + x^4 - \frac{1}{2}x^2 + \frac{1}{16}} = \sqrt{\left(x^2 + \frac{1}{4}\right)^2} = x^2 + \frac{1}{4} = d_2.$$

Further Insights and Challenges

55. Show that if f and g are linear, then so is $f + g$. Is the same true of fg?

SOLUTION If $f(x) = mx + b$ and $g(x) = nx + d$, then

$$f(x) + g(x) = mx + b + nx + d = (m+n)x + (b+d),$$

which is linear. fg is not generally linear. Take, for example, $f(x) = g(x) = x$. Then $f(x)g(x) = x^2$.

56. Show that if f and g are linear functions such that $f(0) = g(0)$ and $f(1) = g(1)$, then $f = g$.

SOLUTION Suppose $f(x) = mx + b$ and $g(x) = nx + d$. Then $f(0) = b$ and $g(0) = d$, which implies $b = d$. Thus $f(x) = mx + b$ and $g(x) = nx + b$. Now, $f(1) = m + b$ and $g(1) = n + b$ so $m + b = n + b$ and $m = n$. Thus $f = g$.

57. Show that $\Delta y/\Delta x$ for the function $f(x) = x^2$ over the interval $[x_1, x_2]$ is not a constant, but depends on the interval. Determine the exact dependence of $\Delta y/\Delta x$ on x_1 and x_2.

SOLUTION For x^2, $\dfrac{\Delta y}{\Delta x} = \dfrac{x_2^2 - x_1^2}{x_2 - x_1} = x_2 + x_1$.

58. Complete the square and use the result to derive the quadratic formula for the roots of $ax^2 + bx + c = 0$.

SOLUTION Consider the equation $ax^2 + bx + c = 0$. First, complete the square to obtain

$$a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} = 0.$$

Then

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \qquad \text{and} \qquad \left|x + \frac{b}{2a}\right| = \sqrt{\frac{b^2 - 4ac}{4a^2}} = \frac{\sqrt{b^2 - 4ac}}{2a}.$$

Dropping the absolute values yields

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a} \qquad \text{or} \qquad x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

59. Let $a, c \neq 0$. Show that the roots of

$$ax^2 + bx + c = 0 \qquad \text{and} \qquad cx^2 + bx + a = 0$$

are reciprocals of each other.

SOLUTION Let r_1 and r_2 be the roots of $ax^2 + bx + c$ and r_3 and r_4 be the roots of $cx^2 + bx + a$. Without loss of generality, let

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \Rightarrow \frac{1}{r_1} = \frac{2a}{-b + \sqrt{b^2 - 4ac}} \cdot \frac{-b - \sqrt{b^2 - 4ac}}{-b - \sqrt{b^2 - 4ac}}$$

$$= \frac{2a(-b - \sqrt{b^2 - 4ac})}{b^2 - b^2 + 4ac} = \frac{-b - \sqrt{b^2 - 4ac}}{2c} = r_4.$$

Similarly, you can show $\dfrac{1}{r_2} = r_3$.

60. Show, by completing the square, that the parabola

$$y = ax^2 + bx + c$$

is congruent to $y = ax^2$ by a vertical and horizontal translation.

SOLUTION

$$y = a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + c - \frac{b^2}{4a} = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}.$$

Thus, the first parabola is just the second translated horizontally by $-\frac{b}{2a}$ and vertically by $\frac{4ac-b^2}{4a}$.

61. Prove **Viète's Formulas**: The quadratic polynomial with α and β as roots is $x^2 + bx + c$, where $b = -\alpha - \beta$ and $c = \alpha\beta$.

SOLUTION If a quadratic polynomial has roots α and β, then the polynomial is

$$(x - \alpha)(x - \beta) = x^2 - \alpha x - \beta x + \alpha\beta = x^2 + (-\alpha - \beta)x + \alpha\beta.$$

Thus, $b = -\alpha - \beta$ and $c = \alpha\beta$.

1.3 The Basic Classes of Functions

Preliminary Questions

1. Give an example of a rational function.

SOLUTION One example is $\dfrac{3x^2 - 2}{7x^3 + x - 1}$.

2. Is $y = |x|$ a polynomial function? What about $y = |x^2 + 1|$?

SOLUTION $|x|$ is not a polynomial; however, because $x^2 + 1 > 0$ for all x, it follows that $|x^2 + 1| = x^2 + 1$, which is a polynomial.

3. What is unusual about the domain of the composite function $f \circ g$ for the functions $f(x) = x^{1/2}$ and $g(x) = -1 - |x|$?

SOLUTION Recall that $(f \circ g)(x) = f(g(x))$. Now, for any real number x, $g(x) = -1 - |x| \leq -1 < 0$. Because we cannot take the square root of a negative number, it follows that $f(g(x))$ is not defined for any real number. In other words, the domain of $f(g(x))$ is the empty set.

4. Is $f(x) = \left(\frac{1}{2}\right)^x$ increasing or decreasing?

SOLUTION The function $f(x) = (\frac{1}{2})^x$ is an exponential function with base $b = \frac{1}{2} < 1$. Therefore, f is a decreasing function.

5. Give an example of a transcendental function.

SOLUTION One possibility is $f(x) = e^x - \sin x$.

Exercises

In Exercises 1–12, determine the domain of the function.

1. $f(x) = x^{1/4}$

SOLUTION $x \geq 0$

2. $g(t) = t^{2/3}$

SOLUTION All reals

3. $f(x) = x^3 + 3x - 4$

SOLUTION All reals

4. $h(z) = z^3 + z^{-3}$

SOLUTION $z \neq 0$

5. $g(t) = \dfrac{1}{t+2}$

SOLUTION $t \neq -2$

6. $f(x) = \dfrac{1}{x^2+4}$

SOLUTION All reals

7. $G(u) = \dfrac{1}{u^2-4}$

SOLUTION $u \neq \pm 2$

8. $f(x) = \dfrac{\sqrt{x}}{x^2-9}$

SOLUTION $x \geq 0, x \neq 3$

9. $f(x) = x^{-4} + (x-1)^{-3}$

SOLUTION $x \neq 0, 1$

10. $F(s) = \sin\left(\dfrac{s}{s+1}\right)$

SOLUTION $s \neq -1$

11. $g(y) = 10^{\sqrt{y}+y^{-1}}$

SOLUTION $y > 0$

12. $f(x) = \dfrac{x + x^{-1}}{(x-3)(x+4)}$

SOLUTION $x \neq 0, 3, -4$

In Exercises 13–24, identify each of the following functions as polynomial, rational, algebraic, or transcendental.

13. $f(x) = 4x^3 + 9x^2 - 8$

SOLUTION Polynomial

14. $f(x) = x^{-4}$

SOLUTION Rational

15. $f(x) = \sqrt{x}$

SOLUTION Algebraic

16. $f(x) = \sqrt{1-x^2}$

SOLUTION Algebraic

17. $f(x) = \dfrac{x^2}{x + \sin x}$

SOLUTION Transcendental

18. $f(x) = 2^x$

SOLUTION Transcendental

19. $f(x) = \dfrac{2x^3 + 3x}{9 - 7x^2}$

SOLUTION Rational

20. $f(x) = \dfrac{3x - 9x^{-1/2}}{9 - 7x^2}$

SOLUTION Algebraic

21. $f(x) = \sin(x^2)$

SOLUTION Transcendental

22. $f(x) = \dfrac{x}{\sqrt{x} + 1}$

SOLUTION Algebraic

23. $f(x) = x^2 + 3x^{-1}$

SOLUTION Rational

24. $f(x) = \sin(3^x)$

SOLUTION Transcendental

25. Is $f(x) = 2^{x^2}$ a transcendental function?

SOLUTION Yes.

26. Show that $f(x) = x^2 + 3x^{-1}$ and $g(x) = 3x^3 - 9x + x^{-2}$ are rational functions—that is, quotients of polynomials.

SOLUTION $f(x) = x^2 + 3x^{-1} = x^2 + \dfrac{3}{x} = \dfrac{x^3 + 3}{x}$

$$g(x) = 3x^3 - 9x + x^{-2} = \frac{3x^5 - 9x^3 + 1}{x^2}$$

In Exercises 27–34, calculate the composite functions $f \circ g$ and $g \circ f$, and determine their domains.

27. $f(x) = \sqrt{x}, \quad g(x) = x + 1$

SOLUTION $f(g(x)) = \sqrt{x + 1}$; D: $x \geq -1$, $\quad g(f(x)) = \sqrt{x} + 1$; D: $x \geq 0$

28. $f(x) = \dfrac{1}{x}, \quad g(x) = x^{-4}$

SOLUTION $f(g(x)) = x^4$; D: $x \neq 0$, $\quad g(f(x)) = x^4$: D: $x \neq 0$

29. $f(x) = 2^x, \quad g(x) = x^2$

SOLUTION $f(g(x)) = 2^{x^2}$; D: $\mathbf{R}$, $\quad g(f(x)) = (2^x)^2 = 2^{2x}$; D: $\mathbf{R}$

30. $f(x) = |x|, \quad g(\theta) = \sin\theta$

SOLUTION $f(g(\theta)) = |\sin\theta|$; D: $\mathbf{R}$, $\quad g(f(x)) = \sin|x|$; D: $\mathbf{R}$

31. $f(\theta) = \cos\theta, \quad g(x) = x^3 + x^2$

SOLUTION $f(g(x)) = \cos(x^3 + x^2)$; D: $\mathbf{R}$, $\quad g(f(\theta)) = \cos^3\theta + \cos^2\theta$; D: $\mathbf{R}$

32. $f(x) = \dfrac{1}{x^2 + 1}, \quad g(x) = x^{-2}$

SOLUTION $f(g(x)) = \dfrac{1}{(x^{-2})^2 + 1} = \dfrac{1}{x^{-4} + 1}$; D: $x \neq 0$, $\quad g(f(x)) = \left(\dfrac{1}{x^2 + 1}\right)^{-2} = (x^2 + 1)^2$; D: $\mathbf{R}$

33. $f(t) = \dfrac{1}{\sqrt{t}}, \quad g(t) = -t^2$

SOLUTION $f(g(t)) = \dfrac{1}{\sqrt{-t^2}}$; D: Not valid for any t, $\quad g(f(t)) = -\left(\frac{1}{\sqrt{t}}\right)^2 = -\frac{1}{t}$; D: $t > 0$

34. $f(t) = \sqrt{t}, \quad g(t) = 1 - t^3$

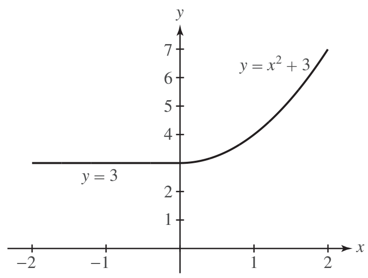
SOLUTION $f(g(t)) = \sqrt{1 - t^3}$; D: $t \leq 1$, $\quad g(f(t)) = 1 - t^{3/2}$; D: $t \geq 0$

In Exercises 35–38, draw the graphs of each of the piecewise-defined functions.

35.

$$f(x) = \begin{cases} 3 & \text{when } x < 0 \\ x^2 + 3 & \text{when } x \geq 0 \end{cases}$$

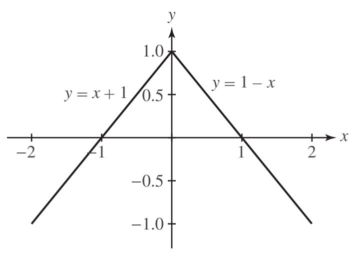
SOLUTION



36.

$$f(x) = \begin{cases} x + 1 & \text{when } x < 0 \\ 1 - x & \text{when } x \geq 0 \end{cases}$$

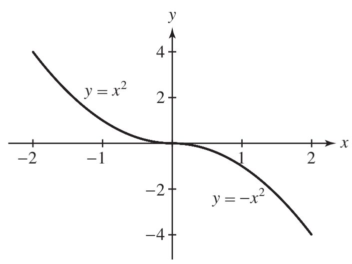
SOLUTION



37.

$$f(x) = \begin{cases} x^2 & \text{when } x < 0 \\ -x^2 & \text{when } x \geq 0 \end{cases}$$

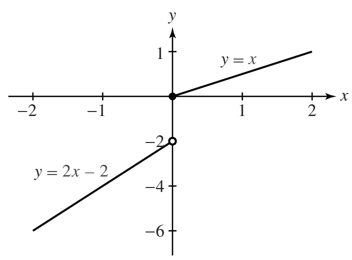
SOLUTION



38.

$$f(x) = \begin{cases} 2x - 2 & \text{when } x < 0 \\ x & \text{when } x \geq 0 \end{cases}$$

SOLUTION



39. The population (in millions) of a country as a function of time t (years) is $P(t) = 30 \cdot 2^{0.1t}$. Show that the population doubles every 10 years. Show more generally that for any positive constants a and k, the function $g(t) = a2^{kt}$ doubles after $1/k$ years.

SOLUTION Let $P(t) = 30 \cdot 2^{0.1t}$. Then

$$P(t+10) = 30 \cdot 2^{0.1(t+10)} = 30 \cdot 2^{0.1t+1} = 2(30 \cdot 2^{0.1t}) = 2P(t).$$

Hence, the population doubles in size every 10 years. In the more general case, let $g(t) = a2^{kt}$. Then

$$g\left(t + \frac{1}{k}\right) = a2^{k(t+1/k)} = a2^{kt+1} = 2a2^{kt} = 2g(t).$$

Hence, the function g doubles after $1/k$ years.

40. Find all values of c such that $f(x) = \dfrac{x+1}{x^2+2cx+4}$ has domain $\mathbf{R}$.

SOLUTION The domain of f will consist of all real numbers provided the denominator has no real roots. The roots of $x^2 + 2cx + 4 = 0$ are

$$x = \frac{-2c \pm \sqrt{4c^2 - 16}}{2} = -c \pm \sqrt{c^2 - 4}.$$

There will be no real roots when $c^2 < 4$ or when $-2 < c < 2$.

Further Insights and Challenges

In Exercises 41–47, we define the first difference δf *of a function* f *by* $\delta f(x) = f(x+1) - f(x)$.

41. Show that if $f(x) = x^2$, then $\delta f(x) = 2x + 1$. Calculate δf for $f(x) = x$ and $f(x) = x^3$.

SOLUTION $f(x) = x^2$: $\delta f(x) = f(x+1) - f(x) = (x+1)^2 - x^2 = 2x + 1$
$f(x) = x$: $\delta f(x) = x + 1 - x = 1$
$f(x) = x^3$: $\delta f(x) = (x+1)^3 - x^3 = 3x^2 + 3x + 1$

42. Show that $\delta(10^x) = 9 \cdot 10^x$ and, more generally, that $\delta(b^x) = (b-1)b^x$.

SOLUTION $\delta(10^x) = 10^{x+1} - 10^x = 10 \cdot 10^x - 10^x = 10^x(10-1) = 9 \cdot 10^x$
$\delta(b^x) = b^{x+1} - b^x = b^x(b-1)$

43. Show that for any two functions f and g, $\delta(f+g) = \delta f + \delta g$ and $\delta(cf) = c\delta(f)$, where c is any constant.

SOLUTION $\delta(f+g) = (f(x+1) + g(x+1)) - (f(x) - g(x))$

$$= (f(x+1) - f(x)) + (g(x+1) - g(x)) = \delta f(x) + \delta g(x)$$

$$\delta(cf) = cf(x+1) - cf(x) = c(f(x+1) - f(x)) = c\delta f(x).$$

44. Suppose we can find a function P such that $\delta P(x) = (x+1)^k$ and $P(0) = 0$. Prove that $P(1) = 1^k$, $P(2) = 1^k + 2^k$, and, more generally, for every whole number n,

$$P(n) = 1^k + 2^k + \cdots + n^k \qquad \boxed{1}$$

SOLUTION Suppose we have found a function $P(x)$ such that $\delta P(x) = (x+1)^k$ and $P(0) = 0$. Taking $x = 0$, we have $\delta P(0) = P(1) - P(0) = (0+1)^k = 1^k$. Therefore, $P(1) = P(0) + 1^k = 1^k$. Next, take $x = 1$. Then $\delta P(1) = P(2) - P(1) = (1+1)^k = 2^k$, and $P(2) = P(1) + 2^k = 1^k + 2^k$.

To prove the general result, we will proceed by induction. The basis step, proving that $P(1) = 1^k$ is given above, so we move on to the induction step. Assume that, for some integer j, $P(j) = 1^k + 2^k + \cdots + j^k$. Then $\delta P(j) = P(j+1) - P(j) = (j+1)^k$ and

$$P(j+1) = P(j) + (j+1)^k = 1^k + 2^k + \cdots + j^k + (j+1)^k.$$

Therefore, by mathematical induction, for every whole number n, $P(n) = 1^k + 2^k + \cdots + n^k$.

45. First show that

$$P(x) = \frac{x(x+1)}{2}$$

satisfies $\delta P = (x+1)$. Then apply Exercise 44 to conclude that

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

SOLUTION Let $P(x) = x(x+1)/2$. Then

$$\delta P(x) = P(x+1) - P(x) = \frac{(x+1)(x+2)}{2} - \frac{x(x+1)}{2} = \frac{(x+1)(x+2-x)}{2} = x+1.$$

Also, note that $P(0) = 0$. Thus, by Exercise 44, with $k = 1$, it follows that

$$P(n) = \frac{n(n+1)}{2} = 1 + 2 + 3 + \cdots + n.$$

46. Calculate $\delta(x^3)$, $\delta(x^2)$, and $\delta(x)$. Then find a polynomial P of degree 3 such that $\delta P = (x+1)^2$ and $P(0) = 0$. Conclude that $P(n) = 1^2 + 2^2 + \cdots + n^2$.

SOLUTION From Exercise 41, we know

$$\delta x = 1, \qquad \delta x^2 = 2x + 1, \qquad \text{and} \qquad \delta x^3 = 3x^2 + 3x + 1.$$

Therefore,

$$\frac{1}{3}\delta x^3 + \frac{1}{2}\delta x^2 + \frac{1}{6}\delta x = x^2 + 2x + 1 = (x+1)^2.$$

Now, using the properties of the first difference from Exercise 43, it follows that

$$\frac{1}{3}\delta x^3 + \frac{1}{2}\delta x^2 + \frac{1}{6}\delta x = \delta\left(\frac{1}{3}x^3\right) + \delta\left(\frac{1}{2}x^2\right) + \delta\left(\frac{1}{6}x\right) = \delta\left(\frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{6}x\right) = \delta\left(\frac{2x^3 + 3x^2 + x}{6}\right).$$

Finally, let

$$P(x) = \frac{2x^3 + 3x^2 + x}{6}.$$

Then $\delta P(x) = (x+1)^2$ and $P(0) = 0$, so by Exercise 44, with $k = 2$, it follows that

$$P(n) = \frac{2n^3 + 3n^2 + n}{6} = 1^2 + 2^2 + 3^2 + \cdots + n^2.$$

47. This exercise combined with Exercise 44 shows that for all whole numbers k, there exists a polynomial P satisfying Eq. (1). The solution requires the Binomial Theorem and proof by induction (see Appendix C).

(a) Show that $\delta(x^{k+1}) = (k+1)\,x^k + \cdots$, where the dots indicate terms involving smaller powers of x.

(b) Show by induction that there exists a polynomial of degree $k+1$ with leading coefficient $1/(k+1)$:

$$P(x) = \frac{1}{k+1}x^{k+1} + \cdots$$

such that $\delta P = (x+1)^k$ and $P(0) = 0$.

SOLUTION

(a) By the Binomial Theorem:

$$\delta(x^{n+1}) = (x+1)^{n+1} - x^{n+1} = \left(x^{n+1} + \binom{n+1}{1}x^n + \binom{n+1}{2}x^{n-1} + \cdots + 1\right) - x^{n+1}$$
$$= \binom{n+1}{1}x^n + \binom{n+1}{2}x^{n-1} + \cdots + 1$$

Thus,

$$\delta(x^{n+1}) = (n+1)\,x^n + \cdots$$

where the dots indicate terms involving smaller powers of x.

(b) For $k = 0$, note that $P(x) = x$ satisfies $\delta P = (x+1)^0 = 1$ and $P(0) = 0$.

Now suppose the polynomial

$$P(x) = \frac{1}{k}x^k + p_{k-1}x^{k-1} + \cdots + p_1 x$$

which clearly satisfies $P(0) = 0$ also satisfies $\delta P = (x+1)^{k-1}$. We try to prove the existence of

$$Q(x) = \frac{1}{k+1}x^{k+1} + q_k x^k + \cdots + q_1 x$$

such that $\delta Q = (x+1)^k$. Observe that $Q(0) = 0$.

If $\delta Q = (x+1)^k$ and $\delta P = (x+1)^{k-1}$, then

$$\delta Q = (x+1)^k = (x+1)\delta P = x\delta P(x) + \delta P$$

By the linearity of δ (Exercise 43), we find $\delta Q - \delta P = x\delta P$ or $\delta(Q-P) = x\delta P$. By definition,

$$Q - P = \frac{1}{k+1}x^{k+1} + \left(q_k - \frac{1}{k}\right)x^k + \cdots + (q_1 - p_1)x,$$

so, by the linearity of δ,

$$\delta(Q-P) = \frac{1}{k+1}\delta(x^{k+1}) + \left(q_k - \frac{1}{k}\right)\delta(x^k) + \cdots + (q_1 - p_1) = x(x+1)^{k-1} \tag{1}$$

By part (a),

$$\begin{aligned}
\delta(x^{k+1}) &= (k+1)x^k + L_{k-1,k-1}x^{k-1} + \ldots + L_{k-1,1}x + 1\\
\delta(x^k) &= kx^{k-1} + L_{k-2,k-2}x^{k-2} + \ldots + L_{k-2,1}x + 1\\
&\vdots\\
\delta(x^2) &= 2x + 1
\end{aligned}$$

where the $L_{i,j}$ are real numbers for each i, j.

To construct Q, we have to group like powers of x on both sides of (1). This yields the system of equations

$$\begin{aligned}
\frac{1}{k+1}\left((k+1)x^k\right) &= x^k\\
\frac{1}{k+1}L_{k-1,k-1}x^{k-1} + \left(q_k - \frac{1}{k}\right)kx^{k-1} &= (k-1)x^{k-1}\\
&\vdots\\
\frac{1}{k+1} + \left(q_k - \frac{1}{k}\right) + (q_{k-1} - p_{k-1}) + \cdots + (q_1 - p_1) &= 0.
\end{aligned}$$

The first equation is identically true, and the second equation can be solved immediately for q_k. Substituting the value of q_k into the third equation of the system, we can then solve for q_{k-1}. We continue this process until we substitute the values of $q_k, q_{k-1}, \ldots q_2$ into the last equation, and then solve for q_1.

1.4 Trigonometric Functions

Preliminary Questions

1. How is it possible for two different rotations to define the same angle?

SOLUTION Working from the same initial radius, two rotations that differ by a whole number of full revolutions will have the same ending radius; consequently, the two rotations will define the same angle even though the measures of the rotations will be different.

2. Give two different positive rotations that define the angle $\pi/4$.

SOLUTION The angle $\pi/4$ is defined by any rotation of the form $\frac{\pi}{4} + 2\pi k$ where k is an integer. Thus, two different positive rotations that define the angle $\pi/4$ are

$$\frac{\pi}{4} + 2\pi(1) = \frac{9\pi}{4} \qquad \text{and} \qquad \frac{\pi}{4} + 2\pi(5) = \frac{41\pi}{4}.$$

3. Give a negative rotation that defines the angle $\pi/3$.

SOLUTION The angle $\pi/3$ is defined by any rotation of the form $\frac{\pi}{3} + 2\pi k$ where k is an integer. Thus, a negative rotation that defines the angle $\pi/3$ is

$$\frac{\pi}{3} + 2\pi(-1) = -\frac{5\pi}{3}.$$

4. The definition of $\cos\theta$ using right triangles applies when (choose the correct answer):

(a) $0 < \theta < \frac{\pi}{2}$ **(b)** $0 < \theta < \pi$ **(c)** $0 < \theta < 2\pi$

SOLUTION The correct response is **(a)**: $0 < \theta < \frac{\pi}{2}$.

5. What is the unit circle definition of $\sin\theta$?

SOLUTION Let O denote the center of the unit circle, and let P be a point on the unit circle such that the radius $\overline{OP}$ makes an angle θ with the positive x-axis. Then, $\sin\theta$ is the y-coordinate of the point P.

6. How does the periodicity of $f(x) = \sin\theta$ and $f(x) = \cos\theta$ follow from the unit circle definition?

SOLUTION Let O denote the center of the unit circle, and let P be a point on the unit circle such that the radius $\overline{OP}$ makes an angle θ with the positive x-axis. Then, $\cos\theta$ and $\sin\theta$ are the x- and y-coordinates, respectively, of the point P. The angle $\theta + 2\pi$ is obtained from the angle θ by making one full revolution around the circle. The angle $\theta + 2\pi$ will therefore have the radius $\overline{OP}$ as its terminal side. Thus

$$\cos(\theta + 2\pi) = \cos\theta \qquad \text{and} \qquad \sin(\theta + 2\pi) = \sin\theta.$$

In other words, $\sin\theta$ and $\cos\theta$ are periodic functions.

Exercises

1. Find the angle between 0 and 2π equivalent to $13\pi/4$.

SOLUTION Because $13\pi/4 > 2\pi$, we repeatedly subtract 2π until we arrive at a radian measure that is between 0 and 2π. After one subtraction, we have $13\pi/4 - 2\pi = 5\pi/4$. Because $0 < 5\pi/4 < 2\pi$, $5\pi/4$ is the angle measure between 0 and 2π that is equivalent to $13\pi/4$.

2. Describe $\theta = \pi/6$ by an angle of negative radian measure.

SOLUTION If we subtract 2π from $\pi/6$, we obtain $\theta = -11\pi/6$. Thus, the angle $\theta = \pi/6$ is equivalent to the angle $\theta = -11\pi/6$.

3. Convert from radians to degrees:

(a) 1 **(b)** $\frac{\pi}{3}$ **(c)** $\frac{5}{12}$ **(d)** $-\frac{3\pi}{4}$

SOLUTION

(a) $1\left(\frac{180°}{\pi}\right) = \frac{180°}{\pi} \approx 57.1°$ **(b)** $\frac{\pi}{3}\left(\frac{180°}{\pi}\right) = 60°$

(c) $\frac{5}{12}\left(\frac{180°}{\pi}\right) = \frac{75°}{\pi} \approx 23.87°$ **(d)** $-\frac{3\pi}{4}\left(\frac{180°}{\pi}\right) = -135°$

4. Convert from degrees to radians:

(a) 1° **(b)** 30° **(c)** 25° **(d)** 120°

SOLUTION

(a) $1°\left(\frac{\pi}{180°}\right) = \frac{\pi}{180}$ **(b)** $30°\left(\frac{\pi}{180°}\right) = \frac{\pi}{6}$ **(c)** $25°\left(\frac{\pi}{180°}\right) = \frac{5\pi}{36}$ **(d)** $120°\left(\frac{\pi}{180°}\right) = \frac{2\pi}{3}$

5. Find the lengths of the arcs subtended by the angles θ and ϕ radians in Figure 20.

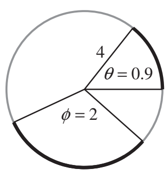


FIGURE 20 Circle of radius 4.

SOLUTION $s = r\theta = 4(.9) = 3.6$; $s = r\phi = 4(2) = 8$

6. Calculate the values of the six standard trigonometric functions for the angle θ in Figure 21.

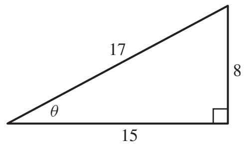


FIGURE 21

SOLUTION Using the definition of the six trigonometric functions in terms of the ratio of sides of a right triangle, we find $\sin\theta = 8/17$; $\cos\theta = 15/17$; $\tan\theta = 8/15$; $\csc\theta = 17/8$; $\sec\theta = 17/15$; $\cot\theta = 15/8$.

7. Fill in the remaining values of $(\cos\theta, \sin\theta)$ for the points in Figure 22.

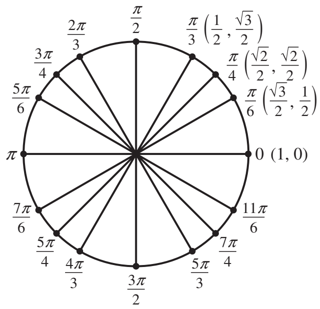


FIGURE 22

SOLUTION

θ	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$
$(\cos\theta, \sin\theta)$	$(0, 1)$	$\left(\frac{-1}{2}, \frac{\sqrt{3}}{2}\right)$	$\left(\frac{-\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$	$\left(\frac{-\sqrt{3}}{2}, \frac{1}{2}\right)$	$(-1, 0)$	$\left(\frac{-\sqrt{3}}{2}, \frac{-1}{2}\right)$
θ	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$
$(\cos\theta, \sin\theta)$	$\left(\frac{-\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}\right)$	$\left(\frac{-1}{2}, \frac{-\sqrt{3}}{2}\right)$	$(0, -1)$	$\left(\frac{1}{2}, \frac{-\sqrt{3}}{2}\right)$	$\left(\frac{\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}\right)$	$\left(\frac{\sqrt{3}}{2}, \frac{-1}{2}\right)$

8. Find the values of the six standard trigonometric functions at $\theta = 11\pi/6$.

SOLUTION From Figure 22, we see that

$$\sin\frac{11\pi}{6} = -\frac{1}{2} \qquad \text{and} \qquad \cos\frac{11\pi}{6} = \frac{\sqrt{3}}{2}.$$

Then,

$$\tan\frac{11\pi}{6} = \frac{\sin\frac{11\pi}{6}}{\cos\frac{11\pi}{6}} = -\frac{\sqrt{3}}{3};$$

$$\cot\frac{11\pi}{6} = \frac{\cos\frac{11\pi}{6}}{\sin\frac{11\pi}{6}} = -\sqrt{3};$$

$$\csc\frac{11\pi}{6} = \frac{1}{\sin\frac{11\pi}{6}} = -2;$$

$$\sec\frac{11\pi}{6} = \frac{1}{\cos\frac{11\pi}{6}} = \frac{2\sqrt{3}}{3}.$$

In Exercises 9–14, use Figure 22 to find all angles between 0 *and* 2π *satisfying the given condition.*

9. $\cos\theta = \frac{1}{2}$

SOLUTION $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$

10. $\tan\theta = 1$

SOLUTION $\theta = \frac{\pi}{4}, \frac{5\pi}{4}$

11. $\tan\theta = -1$

SOLUTION $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$

12. $\csc\theta = 2$

SOLUTION $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

13. $\sin x = \frac{\sqrt{3}}{2}$

SOLUTION $x = \frac{\pi}{3}, \frac{2\pi}{3}$

14. $\sec t = 2$

SOLUTION $t = \frac{\pi}{3}, \frac{5\pi}{3}$

15. Fill in the following table of values:

θ	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$
$\tan\theta$							
$\sec\theta$							

SOLUTION

θ	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$
$\tan\theta$	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	und	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$
$\sec\theta$	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	und	-2	$-\sqrt{2}$	$-\frac{2}{\sqrt{3}}$

16. Complete the following table of signs:

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
$0 < \theta < \frac{\pi}{2}$	+	+				
$\frac{\pi}{2} < \theta < \pi$						
$\pi < \theta < \frac{3\pi}{2}$						
$\frac{3\pi}{2} < \theta < 2\pi$						

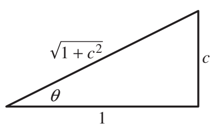
SOLUTION

θ	sin	cos	tan	cot	sec	csc
$0 < \theta < \frac{\pi}{2}$	+	+	+	+	+	+
$\frac{\pi}{2} < \theta < \pi$	+	−	−	−	−	+
$\pi < \theta < \frac{3\pi}{2}$	−	−	+	+	−	−
$\frac{3\pi}{2} < \theta < 2\pi$	−	+	−	−	+	−

17. Show that if $\tan\theta = c$ and $0 \le \theta < \pi/2$, then $\cos\theta = 1/\sqrt{1+c^2}$. *Hint:* Draw a right triangle whose opposite and adjacent sides have lengths c and 1.

SOLUTION Because $0 \le \theta < \pi/2$, we can use the definition of the trigonometric functions in terms of right triangles. $\tan\theta$ is the ratio of the length of the side opposite the angle θ to the length of the adjacent side. With $c = \frac{c}{1}$, we label the length of the opposite side as c and the length of the adjacent side as 1 (see the diagram below). By the Pythagorean theorem, the length of the hypotenuse is $\sqrt{1+c^2}$. Finally, we use the fact that $\cos\theta$ is the ratio of the length of the adjacent side to the length of the hypotenuse to obtain

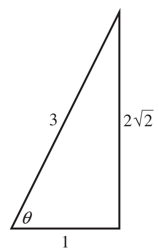
$$\cos\theta = \frac{1}{\sqrt{1+c^2}}.$$



18. Suppose that $\cos\theta = \frac{1}{3}$.
(a) Show that if $0 \le \theta < \pi/2$, then $\sin\theta = 2\sqrt{2}/3$ and $\tan\theta = 2\sqrt{2}$.
(b) Find $\sin\theta$ and $\tan\theta$ if $3\pi/2 \le \theta < 2\pi$.

SOLUTION
(a) Because $0 \le \theta < \pi/2$, we can use the definition of the trigonometric functions in terms of right triangles. $\cos\theta$ is the ratio of the length of the side adjacent to the angle θ to the length of the hypotenuse, so we label the length of the adjacent side as 1 and the length of the hypotenuse as 3 (see the diagram below). By the Pythagorean theorem, the length of the side opposite the angle θ is $\sqrt{3^2 - 1^2} = 2\sqrt{2}$. Finally, we use the definitions of $\sin\theta$ as the ratio of the length of the opposite side to the length of the hypotenuse and of $\tan\theta$ as the ratio of the length of the opposite side to the length of the adjacent side to obtain

$$\sin\theta = \frac{2\sqrt{2}}{3} \qquad \text{and} \qquad \tan\theta = \frac{2\sqrt{2}}{1} = 2\sqrt{2}.$$



(b) If $3\pi/2 \le \theta < 2\pi$, then θ is in the fourth quadrant and $\sin\theta$ and $\tan\theta$ are negative but have the same magnitude as found in part (a). Thus,

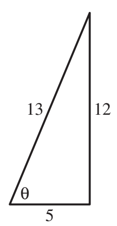
$$\sin\theta = -\frac{2\sqrt{2}}{3} \qquad \text{and} \qquad \tan\theta = -2\sqrt{2}.$$

In Exercises 19–24, assume that $0 \le \theta < \pi/2$.

19. Find $\sin\theta$ and $\tan\theta$ if $\cos\theta = \frac{5}{13}$.

SOLUTION Consider the triangle below. The lengths of the side adjacent to the angle θ and the hypotenuse have been labeled so that $\cos\theta = \frac{5}{13}$. The length of the side opposite the angle θ has been calculated using the Pythagorean theorem: $\sqrt{13^2 - 5^2} = 12$. From the triangle, we see that

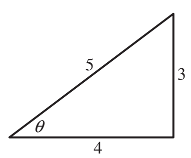
$$\sin\theta = \frac{12}{13} \qquad \text{and} \qquad \tan\theta = \frac{12}{5}.$$



20. Find $\cos\theta$ and $\tan\theta$ if $\sin\theta = \frac{3}{5}$.

SOLUTION Consider the triangle below. The lengths of the side opposite the angle θ and the hypotenuse have been labeled so that $\sin\theta = \frac{3}{5}$. The length of the side adjacent to the angle θ has been calculated using the Pythagorean theorem: $\sqrt{5^2 - 3^2} = 4$. From the triangle, we see that

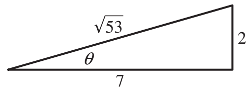
$$\cos\theta = \frac{4}{5} \qquad \text{and} \qquad \tan\theta = \frac{3}{4}.$$



21. Find $\sin\theta$, $\sec\theta$, and $\cot\theta$ if $\tan\theta = \frac{2}{7}$.

SOLUTION If $\tan\theta = \frac{2}{7}$, then $\cot\theta = \frac{7}{2}$. For the remaining trigonometric functions, consider the triangle below. The lengths of the sides opposite and adjacent to the angle θ have been labeled so that $\tan\theta = \frac{2}{7}$. The length of the hypotenuse has been calculated using the Pythagorean theorem: $\sqrt{2^2 + 7^2} = \sqrt{53}$. From the triangle, we see that

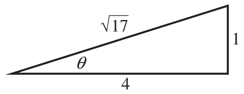
$$\sin\theta = \frac{2}{\sqrt{53}} = \frac{2\sqrt{53}}{53} \qquad \text{and} \qquad \sec\theta = \frac{\sqrt{53}}{7}.$$



22. Find $\sin\theta$, $\cos\theta$, and $\sec\theta$ if $\cot\theta = 4$.

SOLUTION Consider the triangle below. The lengths of the sides opposite and adjacent to the angle θ have been labeled so that $\cot\theta = 4 = \frac{4}{1}$. The length of the hypotenuse has been calculated using the Pythagorean theorem: $\sqrt{4^2 + 1^2} = \sqrt{17}$. From the triangle, we see that

$$\sin\theta = \frac{1}{\sqrt{17}} = \frac{\sqrt{17}}{17}, \quad \cos\theta = \frac{4}{\sqrt{17}} = \frac{4\sqrt{17}}{17} \quad \text{and} \quad \sec\theta = \frac{\sqrt{17}}{4}.$$



23. Find $\cos 2\theta$ if $\sin\theta = \frac{1}{5}$.

SOLUTION Using the double angle formula $\cos 2\theta = \cos^2\theta - \sin^2\theta$ and the fundamental identity $\sin^2\theta + \cos^2\theta = 1$, we find that $\cos 2\theta = 1 - 2\sin^2\theta$. Thus, $\cos 2\theta = 1 - 2(1/25) = 23/25$.

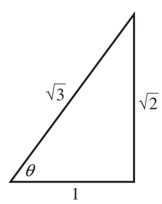
24. Find $\sin 2\theta$ and $\cos 2\theta$ if $\tan\theta = \sqrt{2}$.

SOLUTION By the double angle formulas, $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = \cos^2\theta - \sin^2\theta$. We can determine $\sin\theta$ and $\cos\theta$ using the triangle shown below. The lengths of the sides opposite and adjacent to the angle θ have been labeled so that $\tan\theta = \sqrt{2}$. The hypotenuse was calculated using the Pythagorean theorem: $\sqrt{1^2 + (\sqrt{2})^2} = \sqrt{3}$. Thus,

$$\sin\theta = \frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{6}}{3} \qquad \text{and} \qquad \cos\theta = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}.$$

Finally,

$$\sin 2\theta = 2\frac{\sqrt{6}}{3} \cdot \frac{\sqrt{3}}{3} = \frac{2\sqrt{2}}{3}$$
$$\cos 2\theta = \frac{1}{3} - \frac{2}{3} = -\frac{1}{3}.$$



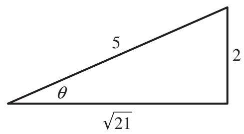
25. Find $\cos\theta$ and $\tan\theta$ if $\sin\theta = 0.4$ and $\pi/2 \le \theta < \pi$.

SOLUTION We can determine the "magnitude" of $\cos\theta$ and $\tan\theta$ using the triangle shown below. The lengths of the side opposite the angle θ and the hypotenuse have been labeled so that $\sin\theta = 0.4 = \frac{2}{5}$. The length of the side adjacent to the angle θ was calculated using the Pythagorean theorem: $\sqrt{5^2 - 2^2} = \sqrt{21}$. From the triangle, we see that

$$|\cos\theta| = \frac{\sqrt{21}}{5} \qquad \text{and} \qquad |\tan\theta| = \frac{2}{\sqrt{21}} = \frac{2\sqrt{21}}{21}.$$

Because $\pi/2 \le \theta < \pi$, both $\cos\theta$ and $\tan\theta$ are negative; consequently,

$$\cos\theta = -\frac{\sqrt{21}}{5} \qquad \text{and} \qquad \tan\theta = -\frac{2\sqrt{21}}{21}.$$



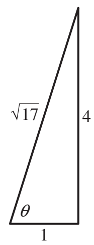
26. Find $\cos\theta$ and $\sin\theta$ if $\tan\theta = 4$ and $\pi \le \theta < 3\pi/2$.

SOLUTION We can determine the "magnitude" of $\cos\theta$ and $\sin\theta$ using the triangle shown below. The lengths of the sides opposite and adjacent to the angle θ have been labeled so that $\tan\theta = 4 = \frac{4}{1}$. The length of the hypotenuse was calculated using the Pythagorean theorem: $\sqrt{1^2 + 4^2} = \sqrt{17}$. From the triangle, we see that

$$|\cos\theta| = \frac{1}{\sqrt{17}} = \frac{\sqrt{17}}{17} \qquad \text{and} \qquad |\sin\theta| = \frac{4}{\sqrt{14}} = \frac{4\sqrt{17}}{17}.$$

Because $\pi \le \theta < 3\pi/2$, both $\cos\theta$ and $\sin\theta$ are negative; consequently,

$$\cos\theta = -\frac{\sqrt{17}}{17} \qquad \text{and} \qquad \sin\theta = -\frac{4\sqrt{17}}{17}.$$



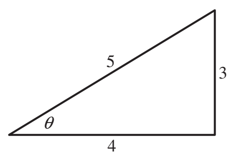
27. Find $\cos\theta$ if $\cot\theta = \frac{4}{3}$ and $\sin\theta < 0$.

SOLUTION We can determine the "magnitude" of $\cos\theta$ using the triangle shown below. The lengths of the sides opposite and adjacent to the angle θ have been labeled so that $\cot\theta = \frac{4}{3}$. The length of the hypotenuse was calculated using the Pythagorean theorem: $\sqrt{3^2 + 4^2} = 5$. From the triangle, we see that

$$|\cos\theta| = \frac{4}{5}.$$

Because $\cot\theta > 0$ and $\sin\theta < 0$, the angle θ must be in the third quadrant so that $\cos\theta$ must be negative; consequently,

$$\cos\theta = -\frac{4}{5}.$$



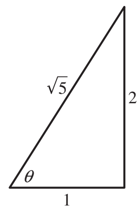
28. Find $\tan\theta$ if $\sec\theta = \sqrt{5}$ and $\sin\theta < 0$.

SOLUTION We can determine the "magnitude" of $\tan\theta$ using the triangle shown below. The lengths of the side adjacent to the angle θ and the hypotenuse have been labeled so that $\sec\theta = \sqrt{5} = \frac{\sqrt{5}}{1}$. The length of the side opposite the angle θ was calculated using the Pythagorean theorem: $\sqrt{(\sqrt{5})^2 - 1^2} = 2$. From the triangle, we see that

$$|\tan\theta| = \frac{2}{1} = 2.$$

Because $\sec\theta > 0$ and $\sin\theta < 0$, the angle θ must be in the fourth quadrant so that $\tan\theta$ must be negative; consequently,

$$\tan\theta = -2.$$



29. Find the values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ for the angles corresponding to the eight points on the unit circles in Figure 23(A) and (B).

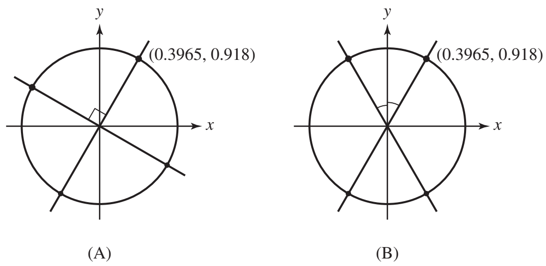


FIGURE 23

SOLUTION Let's start with the four points in Figure 23(A).

- The point in the first quadrant has coordinates (0.3965, 0.918). Therefore,

$$\sin\theta = 0.918, \quad \cos\theta = 0.3965, \quad \text{and} \quad \tan\theta = \frac{0.918}{0.3965} = 2.3153.$$

- The coordinates of the point in the second quadrant are (−0.918, 0.3965). Therefore,

$$\sin\theta = 0.3965, \quad \cos\theta = -0.918, \quad \text{and} \quad \tan\theta = \frac{0.3965}{-0.918} = -0.4319.$$

- Because the point in the third quadrant is symmetric to the point in the first quadrant with respect to the origin, its coordinates are (−0.3965, −0.918). Therefore,

$$\sin\theta = -0.918, \quad \cos\theta = -0.3965, \quad \text{and} \quad \tan\theta = \frac{-0.918}{-0.3965} = 2.3153.$$

- Because the point in the fourth quadrant is symmetric to the point in the second quadrant with respect to the origin, its coordinates are (0.918, −0.3965). Therefore,

$$\sin\theta = -0.3965, \quad \cos\theta = 0.918, \quad \text{and} \quad \tan\theta = \frac{-0.3965}{0.918} = -0.4319.$$

Now consider the four points in Figure 23(B).

- The point in the first quadrant has coordinates (0.3965, 0.918). Therefore,

$$\sin\theta = 0.918, \quad \cos\theta = 0.3965, \quad \text{and} \quad \tan\theta = \frac{0.918}{0.3965} = 2.3153.$$

- The point in the second quadrant is a reflection through the y-axis of the point in the first quadrant. Its coordinates are therefore (−0.3965, 0.918) and

$$\sin\theta = 0.918, \quad \cos\theta = -0.3965, \quad \text{and} \quad \tan\theta = \frac{0.918}{0.3965} = -2.3153.$$

- Because the point in the third quadrant is symmetric to the point in the first quadrant with respect to the origin, its coordinates are (−0.3965, −0.918). Therefore,

$$\sin\theta = -0.918, \quad \cos\theta = -0.3965, \quad \text{and} \quad \tan\theta = \frac{-0.918}{-0.3965} = 2.3153.$$

- Because the point in the fourth quadrant is symmetric to the point in the second quadrant with respect to the origin, its coordinates are (0.3965, −0.918). Therefore,

$$\sin\theta = -0.918, \quad \cos\theta = 0.3965, \quad \text{and} \quad \tan\theta = \frac{-0.918}{0.3965} = -2.3153.$$

30. Refer to Figure 24(A). Express the functions $\sin\theta$, $\tan\theta$, and $\csc\theta$ in terms of c.

SOLUTION By the Pythagorean theorem, the length of the side adjacent to the angle θ in Figure 24 (A) is $\sqrt{1-c^2}$. Consequently,

$$\sin\theta = \frac{c}{1} = c, \quad \cos\theta = \frac{\sqrt{1-c^2}}{1} = \sqrt{1-c^2}, \quad \text{and} \quad \tan\theta = \frac{c}{\sqrt{1-c^2}}.$$

31. Refer to Figure 24(B). Compute $\cos\psi$, $\sin\psi$, $\cot\psi$, and $\csc\psi$.

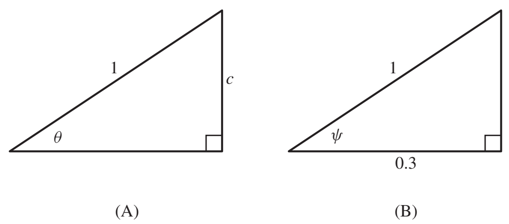


FIGURE 24

SOLUTION By the Pythagorean theorem, the length of the side opposite the angle ψ in Figure 24 (B) is $\sqrt{1-0.3^2} = \sqrt{0.91}$. Consequently,

$$\cos\psi = \frac{0.3}{1} = 0.3, \quad \sin\psi = \frac{\sqrt{0.91}}{1} = \sqrt{0.91}, \quad \cot\psi = \frac{0.3}{\sqrt{0.91}} \quad \text{and} \quad \csc\psi = \frac{1}{\sqrt{0.91}}.$$

32. Express $\cos\left(\theta + \frac{\pi}{2}\right)$ and $\sin\left(\theta + \frac{\pi}{2}\right)$ in terms of $\cos\theta$ and $\sin\theta$. *Hint:* Find the relation between the coordinates (a, b) and (c, d) in Figure 25.

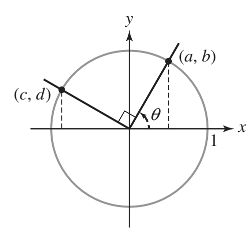


FIGURE 25

SOLUTION Note the triangle in the second quadrant in Figure 25 is congruent to the triangle in the first quadrant rotated 90° clockwise. Thus, $c = -b$ and $d = a$. But $a = \cos\theta$, $b = \sin\theta$, $c = \cos\left(\theta + \frac{\pi}{2}\right)$ and $d = \sin\left(\theta + \frac{\pi}{2}\right)$; therefore,

$$\cos\left(\theta + \frac{\pi}{2}\right) = -\sin\theta \qquad \text{and} \qquad \sin\left(\theta + \frac{\pi}{2}\right) = \cos\theta.$$

33. Use the addition formula to compute $\cos\left(\frac{\pi}{3} + \frac{\pi}{4}\right)$ exactly.

SOLUTION Using the addition formula for the cosine function, we find

$$\cos\left(\frac{\pi}{3} + \frac{\pi}{4}\right) = \cos\frac{\pi}{3}\cos\frac{\pi}{4} - \sin\frac{\pi}{3}\sin\frac{\pi}{4} = \frac{1}{2}\cdot\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{2}-\sqrt{6}}{4}.$$

34. Use the addition formula to compute $\sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$ exactly.

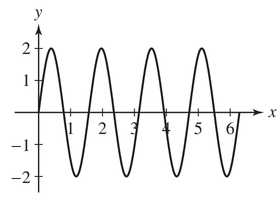
SOLUTION Using the addition formula for the sine function, we find

$$\begin{aligned}\sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right) &= \sin\left[\frac{\pi}{3} + \left(-\frac{\pi}{4}\right)\right] \\ &= \sin\frac{\pi}{3}\cos\left(-\frac{\pi}{4}\right) + \cos\frac{\pi}{3}\sin\left(-\frac{\pi}{4}\right) = \sin\frac{\pi}{3}\cos\frac{\pi}{4} - \cos\frac{\pi}{3}\sin\frac{\pi}{4} \\ &= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} - \frac{1}{2}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}.\end{aligned}$$

In Exercises 35–38, sketch the graph over $[0, 2\pi]$.

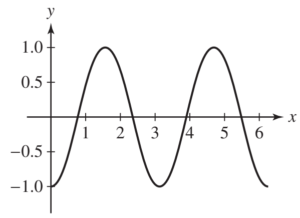
35. $f(\theta) = 2\sin 4\theta$

SOLUTION



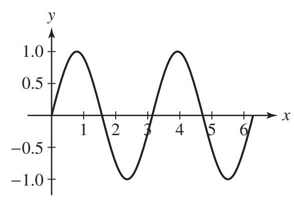
36. $f(\theta) = \cos\left(2\left(\theta - \frac{\pi}{2}\right)\right)$

SOLUTION



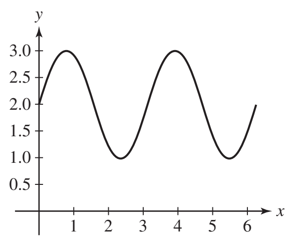
37. $f(\theta) = \cos\left(2\theta - \frac{\pi}{2}\right)$

SOLUTION



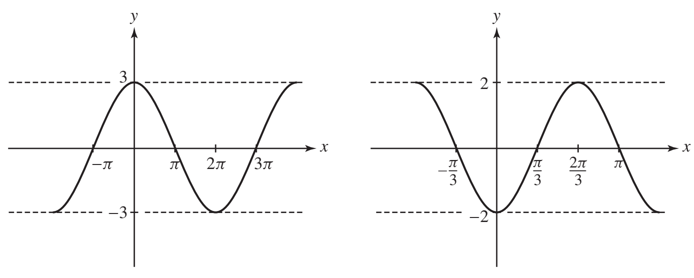
38. $f(\theta) = \sin\left(2\left(\theta - \frac{\pi}{2}\right) + \pi\right) + 2$

SOLUTION



39. Determine a function that would have a graph as in Figure 26(A), stating the period and amplitude.

SOLUTION The graph in Figure 26 (A) appears to be a cosine function with an amplitude of 3 and a period of 4π. Thus, one possible function is $y = 3\cos\frac{x}{2}$.



(A) (B)

FIGURE 26

40. Determine a function that would have a graph as in Figure 26(B), stating the period and amplitude.

SOLUTION The graph in Figure 26 (B) appears to be a cosine function with an amplitude of 2, a period of $\frac{4\pi}{3}$, and that has been reflected across the x-axis. Thus, one possible function is $y = -2\cos\frac{3x}{2}$.

41. How many points lie on the intersection of the horizontal line $y = c$ and the graph of $y = \sin x$ for $0 \le x < 2\pi$? *Hint:* The answer depends on c.

SOLUTION Recall that for any x, $-1 \le \sin x \le 1$. Thus, if $|c| > 1$, the horizontal line $y = c$ and the graph of $y = \sin x$ never intersect. If $c = +1$, then $y = c$ and $y = \sin x$ intersect at the peak of the sine curve; that is, they intersect at $x = \frac{\pi}{2}$. On the other hand, if $c = -1$, then $y = c$ and $y = \sin x$ intersect at the bottom of the sine curve; that is, they intersect at $x = \frac{3\pi}{2}$. Finally, if $|c| < 1$, the graphs of $y = c$ and $y = \sin x$ intersect twice.

42. How many points lie on the intersection of the horizontal line $y = c$ and the graph of $y = \tan x$ for $0 \le x < 2\pi$?

SOLUTION Recall that the graph of $y = \tan x$ consists of an infinite collection of "branches," each between two consecutive vertical asymptotes. Because each branch is increasing and has a range of all real numbers, the graph of the horizontal line $y = c$ will intersect each branch of the graph of $y = \tan x$ once, regardless of the value of c. The interval $0 \le x < 2\pi$ covers the equivalent of two branches of the tangent function, so over this interval there are two points of intersection for each value of c.

In Exercises 43–46, solve for $0 \le \theta < 2\pi$ *(see Example 4).*

43. $\sin 2\theta + \sin 3\theta = 0$

SOLUTION $\sin\alpha = -\sin\beta$ when $\alpha = -\beta + 2\pi k$ or $\alpha = \pi + \beta + 2\pi k$. Substituting $\alpha = 2\theta$ and $\beta = 3\theta$, we have either $2\theta = -3\theta + 2\pi k$ or $2\theta = \pi + 3\theta + 2\pi k$. Solving each of these equations for θ yields $\theta = \frac{2}{5}\pi k$ or $\theta = -\pi - 2\pi k$. The solutions on the interval $0 \le \theta < 2\pi$ are then

$$\theta = 0, \frac{2\pi}{5}, \frac{4\pi}{5}, \pi, \frac{6\pi}{5}, \frac{8\pi}{5}.$$

44. $\sin\theta = \sin 2\theta$

SOLUTION Using the double angle formula for the sine function, we rewrite the equation as $\sin\theta = 2\sin\theta\cos\theta$ or $\sin\theta(1 - 2\cos\theta) = 0$. Thus, either $\sin\theta = 0$ or $\cos\theta = \frac{1}{2}$. The solutions on the interval $0 \le \theta < 2\pi$ are then

$$\theta = 0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}.$$

45. $\cos 4\theta + \cos 2\theta = 0$

SOLUTION $\cos\alpha = -\cos\beta$ when $\alpha + \beta = \pi + 2\pi k$ or $\alpha = \beta + \pi + 2\pi k$. Substituting $\alpha = 4\theta$ and $\beta = 2\theta$, we have either $6\theta = \pi + 2\pi k$ or $4\theta = 2\theta + \pi + 2\pi k$. Solving each of these equations for θ yields $\theta = \frac{\pi}{6} + \frac{\pi}{3}k$ or $\theta = \frac{\pi}{2} + \pi k$. The solutions on the interval $0 \le \theta < 2\pi$ are then

$$\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}.$$

46. $\sin\theta = \cos 2\theta$

SOLUTION Solving the double angle formula $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$ for $\cos 2\theta$ yields $\cos 2\theta = 1 - 2\sin^2\theta$. We can therefore rewrite the original equation as $\sin\theta = 1 - 2\sin^2\theta$ or $2\sin^2\theta + \sin\theta - 1 = 0$. The left-hand side of this latter equation factors as $(2\sin\theta - 1)(\sin\theta + 1)$, so we have either $\sin\theta = \frac{1}{2}$ or $\sin\theta = -1$. The solutions on the interval $0 \le \theta < 2\pi$ are

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}.$$

In Exercises 47–56, derive the identity using the identities listed in this section.

47. $\cos 2\theta = 2\cos^2\theta - 1$

SOLUTION Starting from the double angle formula for cosine, $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$, we solve for $\cos 2\theta$. This gives $2\cos^2\theta = 1 + \cos 2\theta$ and then $\cos 2\theta = 2\cos^2\theta - 1$.

48. $\cos^2\frac{\theta}{2} = \frac{1 + \cos\theta}{2}$

SOLUTION Substitute $x = \theta/2$ into the double angle formula for cosine, $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$ to obtain $\cos^2\left(\frac{\theta}{2}\right) = \frac{1 + \cos\theta}{2}$.

49. $\sin\frac{\theta}{2} = \sqrt{\frac{1 - \cos\theta}{2}}$

SOLUTION Substitute $x = \theta/2$ into the double angle formula for sine, $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ to obtain $\sin^2\left(\frac{\theta}{2}\right) = \frac{1 - \cos\theta}{2}$. Taking the square root of both sides yields $\sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{1 - \cos\theta}{2}}$.

50. $\sin(\theta + \pi) = -\sin\theta$

SOLUTION From the addition formula for the sine function, we have

$$\sin(\theta + \pi) = \sin\theta\cos\pi + \cos\theta\sin\pi = -\sin\theta$$

51. $\cos(\theta + \pi) = -\cos\theta$

SOLUTION From the addition formula for the cosine function, we have

$$\cos(\theta + \pi) = \cos\theta\cos\pi - \sin\theta\sin\pi = \cos\theta(-1) = -\cos\theta$$

52. $\tan x = \cot\left(\frac{\pi}{2} - x\right)$

SOLUTION Using the Complementary Angle Identity,

$$\cot\left(\frac{\pi}{2} - x\right) = \frac{\cos(\pi/2 - x)}{\sin(\pi/2 - x)} = \frac{\sin x}{\cos x} = \tan x.$$

53. $\tan(\pi - \theta) = -\tan\theta$

SOLUTION Using Exercises 50 and 51,

$$\tan(\pi - \theta) = \frac{\sin(\pi - \theta)}{\cos(\pi - \theta)} = \frac{\sin(\pi + (-\theta))}{\cos(\pi + (-\theta))} = \frac{-\sin(-\theta)}{-\cos(-\theta)} = \frac{\sin\theta}{-\cos\theta} = -\tan\theta.$$

The second to last equality occurs because $\sin x$ is an odd function and $\cos x$ is an even function.

54. $\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$

SOLUTION Using the definition of the tangent function and the double angle formulas for sine and cosine, we find

$$\tan 2x = \frac{\sin 2x}{\cos 2x} = \frac{2\sin x\cos x}{\cos^2 x - \sin^2 x} \cdot \frac{1/\cos^2 x}{1/\cos^2 x} = \frac{2\tan x}{1 - \tan^2 x}.$$

55. $\tan x = \frac{\sin 2x}{1 + \cos 2x}$

SOLUTION Using the addition formula for the sine function, we find

$$\sin 2x = \sin(x + x) = \sin x\cos x + \cos x\sin x = 2\sin x\cos x.$$

By Exercise 47, we know that $\cos 2x = 2\cos^2 x - 1$. Therefore,

$$\frac{\sin 2x}{1 + \cos 2x} = \frac{2\sin x\cos x}{1 + 2\cos^2 x - 1} = \frac{2\sin x\cos x}{2\cos^2 x} = \frac{\sin x}{\cos x} = \tan x.$$

56. $\sin^2 x\cos^2 x = \frac{1 - \cos 4x}{8}$

SOLUTION Using the double angle formulas for sine and cosine, we find

$$\sin^2 x\cos^2 x = \frac{1}{2}(1 - \cos 2x) \cdot \frac{1}{2}(1 + \cos 2x) = \frac{1}{4}(1 - \cos^2 2x)$$
$$= \frac{1}{4}\left(1 - \frac{1}{2} - \frac{1}{2}\cos 4x\right) = \frac{1}{8}(1 - \cos 4x).$$

57. Use Exercises 50 and 51 to show that $\tan\theta$ and $\cot\theta$ are periodic with period π.

SOLUTION By Exercises 50 and 51,

$$\tan(\theta + \pi) = \frac{\sin(\theta + \pi)}{\cos(\theta + \pi)} = \frac{-\sin\theta}{-\cos\theta} = \tan\theta,$$

and

$$\cot(\theta + \pi) = \frac{\cos(\theta + \pi)}{\sin(\theta + \pi)} = \frac{-\cos\theta}{-\sin\theta} = \cot\theta.$$

Thus, both $\tan\theta$ and $\cot\theta$ are periodic with period π.

58. Use the identity of Exercise 48 to show that $\cos\frac{\pi}{8}$ is equal to $\sqrt{\frac{1}{2}+\frac{\sqrt{2}}{4}}$.

SOLUTION Using the identity of Exercise 48 with $\theta = \frac{\pi}{4}$ yields

$$\cos^2\frac{\pi}{8} = \frac{1+\cos\frac{\pi}{4}}{2} = \frac{1}{2}+\frac{\sqrt{2}}{4}.$$

Because $\frac{\pi}{8}$ is a first quadrant angle, $\cos\frac{\pi}{8} > 0$; therefore,

$$\cos\frac{\pi}{8} = \sqrt{\frac{1}{2}+\frac{\sqrt{2}}{4}}.$$

59. Use the Law of Cosines to find the distance from P to Q in Figure 27.

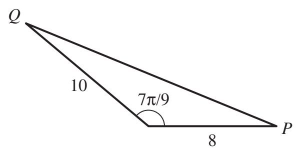


FIGURE 27

SOLUTION By the Law of Cosines, the distance from P to Q is

$$\sqrt{10^2+8^2-2(10)(8)\cos\frac{7\pi}{9}} = 16.928.$$

Further Insights and Challenges

60. Use Figure 28 to derive the Law of Cosines from the Pythagorean Theorem.

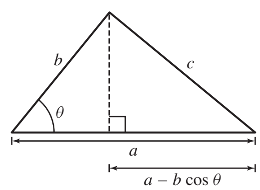


FIGURE 28

SOLUTION Applying the Pythagorean Theorem to the right triangle on the right-hand side of Figure 28 and noting that the length of the vertical leg of this triangle is $b\sin\theta$ yields

$$\begin{aligned} c^2 &= (b\sin\theta)^2 + (a-b\cos\theta)^2 \\ &= b^2\sin^2\theta + a^2 - 2ab\cos\theta + b^2\cos^2\theta \\ &= a^2 + b^2(\sin^2\theta+\cos^2\theta) - 2ab\cos\theta \\ &= a^2+b^2-2ab\cos\theta. \end{aligned}$$

61. Use the addition formula to prove

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

SOLUTION

$$\begin{aligned} \cos 3\theta &= \cos(2\theta+\theta) = \cos 2\theta\cos\theta - \sin 2\theta\sin\theta = (2\cos^2\theta - 1)\cos\theta - (2\sin\theta\cos\theta)\sin\theta \\ &= \cos\theta(2\cos^2\theta - 1 - 2\sin^2\theta) = \cos\theta(2\cos^2\theta - 1 - 2(1-\cos^2\theta)) \\ &= \cos\theta(2\cos^2\theta - 1 - 2 + 2\cos^2\theta) = 4\cos^3\theta - 3\cos\theta \end{aligned}$$

62. Use the addition formulas for sine and cosine to prove

$$\tan(a+b) = \frac{\tan a + \tan b}{1-\tan a\tan b}$$

$$\cot(a-b) = \frac{\cot a\cot b + 1}{\cot b - \cot a}$$

SOLUTION

$$\tan(a+b)=\frac{\sin(a+b)}{\cos(a+b)}=\frac{\sin a\cos b+\cos a\sin b}{\cos a\cos b-\sin a\sin b}=\frac{\frac{\sin a\cos b}{\cos a\cos b}+\frac{\cos a\sin b}{\cos a\cos b}}{\frac{\cos a\cos b}{\cos a\cos b}-\frac{\sin a\sin b}{\cos a\cos b}}=\frac{\tan a+\tan b}{1-\tan a\tan b}$$

$$\cot(a-b)=\frac{\cos(a-b)}{\sin(a-b)}=\frac{\cos a\cos b+\sin a\sin b}{\sin a\cos b-\cos a\sin b}=\frac{\frac{\cos a\cos b}{\sin a\sin b}+\frac{\sin a\sin b}{\sin a\sin b}}{\frac{\sin a\cos b}{\sin a\sin b}-\frac{\cos a\sin b}{\sin a\sin b}}=\frac{\cot a\cot b+1}{\cot b-\cot a}$$

63. Let θ be the angle between the line $y=mx+b$ and the x-axis [Figure 29(A)]. Prove that $m=\tan\theta$.

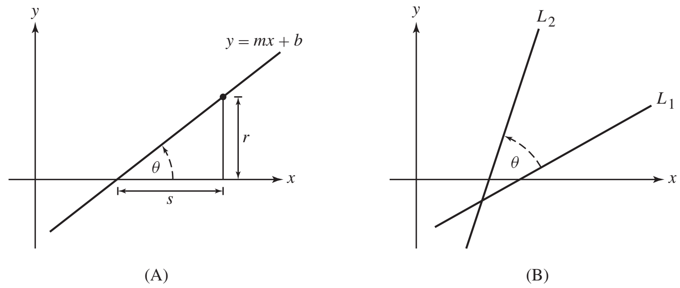


FIGURE 29

SOLUTION Using the distances labeled in Figure 29(A), we see that the slope of the line is given by the ratio r/s. The tangent of the angle θ is given by the same ratio. Therefore, $m=\tan\theta$.

64. Let L_1 and L_2 be the lines of slope m_1 and m_2 [Figure 29(B)]. Show that the angle θ between L_1 and L_2 satisfies $\cot\theta=\dfrac{m_2m_1+1}{m_2-m_1}$.

SOLUTION Measured from the positive x-axis, let α and β satisfy $\tan\alpha=m_1$ and $\tan\beta=m_2$. Without loss of generality, let $\beta\geq\alpha$. Then the angle between the two lines will be $\theta=\beta-\alpha$. Then from Exercise 62,

$$\cot\theta=\cot(\beta-\alpha)=\frac{\cot\beta\cot\alpha+1}{\cot\alpha-\cot\beta}=\frac{(\frac{1}{m_1})(\frac{1}{m_2})+1}{\frac{1}{m_1}-\frac{1}{m_2}}=\frac{1+m_1m_2}{m_2-m_1}$$

65. Perpendicular Lines Use Exercise 64 to prove that two lines with nonzero slopes m_1 and m_2 are perpendicular if and only if $m_2=-1/m_1$.

SOLUTION If lines are perpendicular, then the angle between them is $\theta=\pi/2\Rightarrow$

$$\cot(\pi/2)=\frac{1+m_1m_2}{m_1-m_2}$$
$$0=\frac{1+m_1m_2}{m_1-m_2}$$

$\Rightarrow m_1m_2=-1\Rightarrow m_1=-\dfrac{1}{m_2}$

66. Apply the double-angle formula to prove:

(a) $\cos\dfrac{\pi}{8}=\dfrac{1}{2}\sqrt{2+\sqrt{2}}$

(b) $\cos\dfrac{\pi}{16}=\dfrac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}}$

Guess the values of $\cos\dfrac{\pi}{32}$ and of $\cos\dfrac{\pi}{2^n}$ for all n.

SOLUTION

(a) $\cos\dfrac{\pi}{8}=\cos\dfrac{\pi/4}{2}=\sqrt{\dfrac{1+\cos\frac{\pi}{4}}{2}}=\sqrt{\dfrac{1+\frac{\sqrt{2}}{2}}{2}}=\dfrac{1}{2}\sqrt{2+\sqrt{2}}.$

(b) $\cos\dfrac{\pi}{16}=\sqrt{\dfrac{1+\cos\frac{\pi}{8}}{2}}=\sqrt{\dfrac{1+\frac{1}{2}\sqrt{2+\sqrt{2}}}{2}}=\dfrac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}}.$

(c) Observe that $8 = 2^3$ and $\cos\frac{\pi}{8}$ involves two nested square roots of 2; further, $16 = 2^4$ and $\cos\frac{\pi}{16}$ involves three nested square roots of 2. Since $32 = 2^5$, it seems plausible that

$$\cos\frac{\pi}{32} = \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}},$$

and that $\cos\frac{\pi}{2^n}$ involves $n-1$ nested square roots of 2. Note that the general case can be proven by induction.

1.5 Technology: Calculators and Computers

Preliminary Questions

1. Is there a definite way of choosing the optimal viewing rectangle, or is it best to experiment until you find a viewing rectangle appropriate to the problem at hand?

SOLUTION It is best to experiment with the window size until one is found that is appropriate for the problem at hand.

2. Describe the calculator screen produced when the function $y = 3 + x^2$ is plotted with a viewing rectangle:

(a) $[-1, 1] \times [0, 2]$ **(b)** $[0, 1] \times [0, 4]$

SOLUTION

(a) Using the viewing rectangle $[-1, 1]$ by $[0, 2]$, the screen will display nothing as the minimum value of $y = 3 + x^2$ is $y = 3$.

(b) Using the viewing rectangle $[0, 1]$ by $[0, 4]$, the screen will display the portion of the parabola between the points $(0, 3)$ and $(1, 4)$.

3. According to the evidence in Example 4, it appears that $f(n) = (1 + 1/n)^n$ never takes on a value greater than 3 for $n > 0$. Does this evidence *prove* that $f(n) \le 3$ for $n > 0$?

SOLUTION No, this evidence does not constitute a proof that $f(n) \le 3$ for $n \ge 0$.

4. How can a graphing calculator be used to find the minimum value of a function?

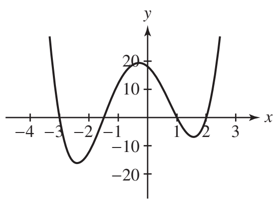
SOLUTION Experiment with the viewing window to zoom in on the lowest point on the graph of the function. The y-coordinate of the lowest point on the graph is the minimum value of the function.

Exercises

The exercises in this section should be done using a graphing calculator or computer algebra system.

1. Plot $f(x) = 2x^4 + 3x^3 - 14x^2 - 9x + 18$ in the appropriate viewing rectangles and determine its roots.

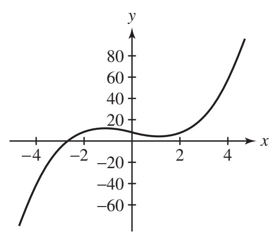
SOLUTION Using a viewing rectangle of $[-4, 3]$ by $[-20, 20]$, we obtain the plot below.



Now, the roots of $f(x)$ are the x-intercepts of the graph of $y = f(x)$. From the plot, we can identify the x-intercepts as $-3, -1.5, 1$, and 2. The roots of $f(x)$ are therefore $x = -3$, $x = -1.5$, $x = 1$, and $x = 2$.

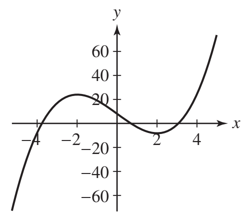
2. How many solutions does $x^3 - 4x + 8 = 0$ have?

SOLUTION Solutions to the equation $x^3 - 4x + 8 = 0$ are the x-intercepts of the graph of $y = x^3 - 4x + 8$. From the figure below, we see that the graph has one x-intercept (between $x = -4$ and $x = -2$), so the equation has one solution.



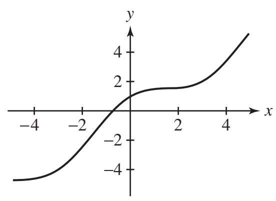
3. How many *positive* solutions does $x^3 - 12x + 8 = 0$ have?

SOLUTION The graph of $y = x^3 - 12x + 8$ shown below has two x-intercepts to the right of the origin; therefore the equation $x^3 - 12x + 8 = 0$ has two positive solutions.



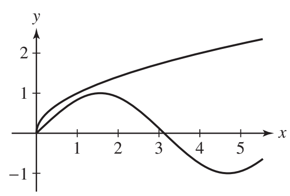
4. Does $\cos x + x = 0$ have a solution? A positive solution?

SOLUTION The graph of $y = \cos x + x$ shown below has one x-intercept; therefore, the equation $\cos x + x = 0$ has one solution. The lone x-intercept is to the left of the origin, so the equation has no positive solutions.



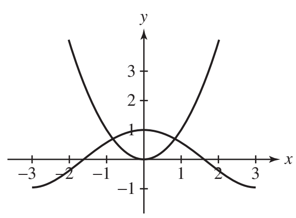
5. Find all the solutions of $\sin x = \sqrt{x}$ for $x > 0$.

SOLUTION Solutions to the equation $\sin x = \sqrt{x}$ correspond to points of intersection between the graphs of $y = \sin x$ and $y = \sqrt{x}$. The two graphs are shown below; the only point of intersection is at $x = 0$. Therefore, there are no solutions of $\sin x = \sqrt{x}$ for $x > 0$.



6. How many solutions does $\cos x = x^2$ have?

SOLUTION Solutions to the equation $\cos x = x^2$ correspond to points of intersection between the graphs of $y = \cos x$ and $y = x^2$. The two graphs are shown below; there are two points of intersection. Thus, the equation $\cos x = x^2$ has two solutions.

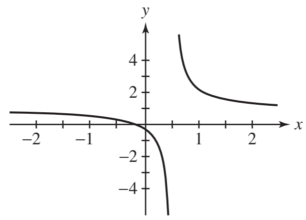


7. Let $f(x) = (x - 100)^2 + 1000$. What will the display show if you graph f in the viewing rectangle $[-10, 10]$ by $[-10, 10]$? Find an appropriate viewing rectangle.

SOLUTION Because $(x - 100)^2 \geq 0$ for all x, it follows that $f(x) = (x - 100)^2 + 1000 \geq 1000$ for all x. Thus, using a viewing rectangle of $[-10, 10]$ by $[-10, 10]$ will display nothing. The minimum value of the function occurs when $x = 100$, so an appropriate viewing rectangle would be $[50, 150]$ by $[1000, 2000]$.

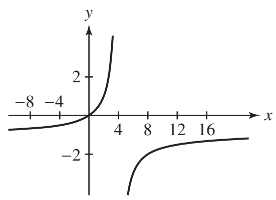
8. Plot $f(x) = \dfrac{8x+1}{8x-4}$ in an appropriate viewing rectangle. What are the vertical and horizontal asymptotes?

SOLUTION From the graph of $y = \dfrac{8x+1}{8x-4}$ shown below, we see that the vertical asymptote is $x = \frac{1}{2}$ and the horizontal asymptote is $y = 1$.



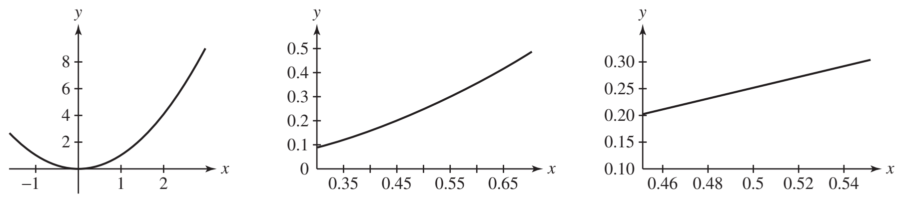
9. Plot the graph of $f(x) = x/(4-x)$ in a viewing rectangle that clearly displays the vertical and horizontal asymptotes.

SOLUTION From the graph of $y = \dfrac{x}{4-x}$ shown below, we see that the vertical asymptote is $x = 4$ and the horizontal asymptote is $y = -1$.



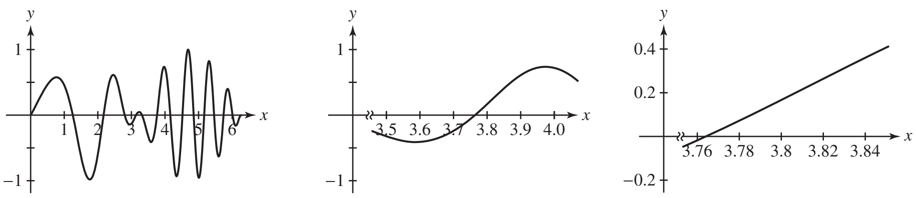
10. Illustrate local linearity for $f(x) = x^2$ by zooming in on the graph at $x = 0.5$ (see Example 6).

SOLUTION The following three graphs display $f(x) = x^2$ over the intervals $[-1, 3]$, $[0.3, 0.7]$ and $[0.45, 0.55]$. The final graph looks like a straight line.



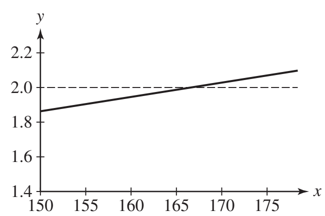
11. Plot $f(x) = \cos(x^2)\sin x$ for $0 \le x \le 2\pi$. Then illustrate local linearity at $x = 3.8$ by choosing appropriate viewing rectangles.

SOLUTION The following three graphs display $f(x) = \cos(x^2)\sin x$ over the intervals $[0, 2\pi]$, $[3.5, 4.1]$ and $[3.75, 3.85]$. The final graph looks like a straight line.



12. If P_0 dollars are deposited in a bank account paying 5% interest compounded monthly, then the account has value $P_0\left(1 + \frac{0.05}{12}\right)^N$ after N months. Find, to the nearest integer N, the number of months after which the account value doubles.

SOLUTION $P(N) = P_0(1 + \frac{0.05}{12})^N$. This doubles when $P(N) = 2P_0$, or when $2 = (1 + \frac{0.05}{12})^N$. The graphs of $y = 2$ and $y = (1 + \frac{0.05}{12})^N$ are shown below; they appear to intersect at $N = 167$. Thus, it will take approximately 167 months for money earning $r = 5\%$ interest compounded monthly to double in value.

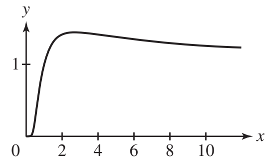


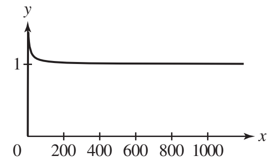
In Exercises 13–18, investigate the behavior of the function as n or x grows large by making a table of function values and plotting a graph (see Example 4). Describe the behavior in words.

13. $f(n) = n^{1/n}$

SOLUTION The table and graphs below suggest that as n gets large, $n^{1/n}$ approaches 1.

n	$n^{1/n}$
10	1.258925412
10^2	1.047128548
10^3	1.006931669
10^4	1.000921458
10^5	1.000115136
10^6	1.000013816

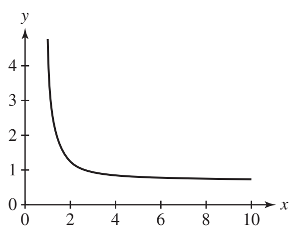


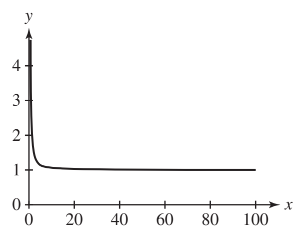


14. $f(n) = \dfrac{4n+1}{6n-5}$

SOLUTION The table and graphs below suggest that as n gets large, $\dfrac{4n+1}{6n-5}$ approaches $\dfrac{2}{3}$.

n	$\dfrac{4n+1}{6n-5}$
10	.7454545455
10^2	.6739495798
10^3	.6673894912
10^4	.6667388949
10^5	.6666738889
10^6	.6666673889

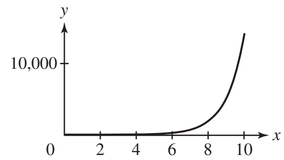


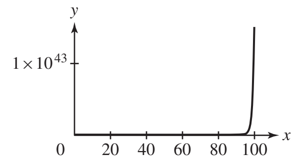


15. $f(n) = \left(1 + \dfrac{1}{n}\right)^{n^2}$

SOLUTION The table and graphs below suggest that as n gets large, $f(n)$ tends toward ∞.

n	$\left(1 + \dfrac{1}{n}\right)^{n^2}$
10	13780.61234
10^2	$1.635828711 \times 10^{43}$
10^3	$1.195306603 \times 10^{434}$
10^4	$5.341783312 \times 10^{4342}$
10^5	$1.702333054 \times 10^{43429}$
10^6	$1.839738749 \times 10^{434294}$

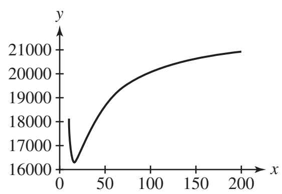


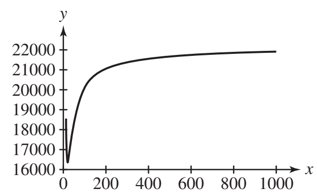


16. $f(x) = \left(\frac{x+6}{x-4}\right)^x$

SOLUTION The table and graphs below suggest that as x gets large, $f(x)$ roughly tends toward 22026.

x	$\left(\frac{x+6}{x-4}\right)^x$
10	18183.91210
10^2	20112.36934
10^3	21809.33633
10^4	22004.43568
10^5	22024.26311
10^6	22025.36451
10^7	22026.35566

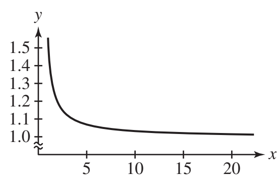


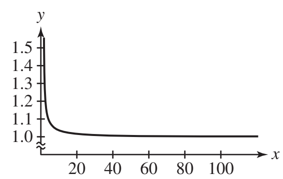


17. $f(x) = \left(x \tan \frac{1}{x}\right)^x$

SOLUTION The table and graphs below suggest that as x gets large, $f(x)$ approaches 1.

x	$\left(x \tan \frac{1}{x}\right)^x$
10	1.033975759
10^2	1.003338973
10^3	1.000333389
10^4	1.000033334
10^5	1.000003333
10^6	1.000000333

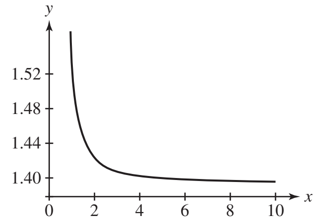


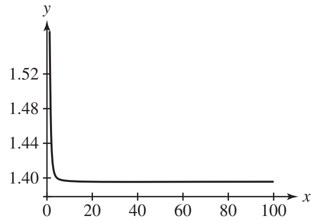


18. $f(x) = \left(x \tan \frac{1}{x}\right)^{x^2}$

SOLUTION The table and graphs below suggest that as x gets large, $f(x)$ approaches 1.39561.

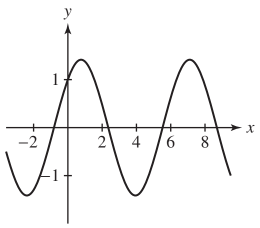
x	$\left(x \tan \frac{1}{x}\right)^{x^2}$
10	1.396701388
10^2	1.395623280
10^3	1.395612534
10^4	1.395612426
10^5	1.395612425
10^6	1.395612425



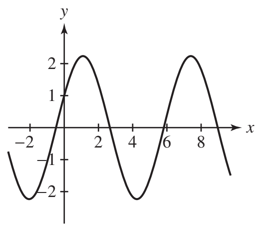


19. The graph of $f(\theta) = A\cos\theta + B\sin\theta$ is a sinusoidal wave for any constants A and B. Confirm this for $(A, B) = (1, 1)$, $(1, 2)$, and $(3, 4)$ by plotting f.

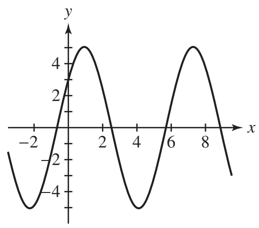
SOLUTION The graphs of $f(\theta) = \cos\theta + \sin\theta$, $f(\theta) = \cos\theta + 2\sin\theta$ and $f(\theta) = 3\cos\theta + 4\sin\theta$ are shown below.



$(A, B) = (1, 1)$



$(A, B) = (1, 2)$



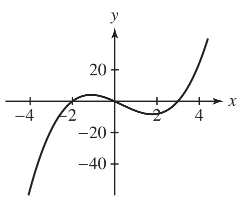
$(A, B) = (3, 4)$

20. Find the maximum value of f for the graphs produced in Exercise 19. Can you guess the formula for the maximum value in terms of A and B?

SOLUTION For $A = 1$ and $B = 1$, max $\approx 1.4 \approx \sqrt{2}$
For $A = 1$ and $B = 2$, max $\approx 2.25 \approx \sqrt{5}$
For $A = 3$ and $B = 4$, max $\approx 5 = \sqrt{3^2 + 4^2}$
Max $= \sqrt{A^2 + B^2}$

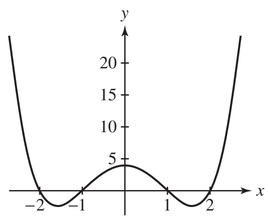
21. Find the intervals on which $f(x) = x(x + 2)(x - 3)$ is positive by plotting a graph.

SOLUTION The function $f(x) = x(x + 2)(x - 3)$ is positive when the graph of $y = x(x + 2)(x - 3)$ lies above the x-axis. The graph of $y = x(x + 2)(x - 3)$ is shown below. Clearly, the graph lies above the x-axis and the function is positive for $x \in (-2, 0) \cup (3, \infty)$.



22. Find the set of solutions to the inequality $(x^2 - 4)(x^2 - 1) < 0$ by plotting a graph.

SOLUTION To solve the inequality $(x^2 - 4)(x^2 - 1) < 0$, we can plot the graph of $y = (x^2 - 4)(x^2 - 1)$ and identify when the graph lies below the x-axis. The graph of $y = (x^2 - 4)(x^2 - 1)$ is shown below. The solution set for the inequality $(x^2 - 4)(x^2 - 1) < 0$ is clearly $x \in (-2, -1) \cup (1, 2)$.



Further Insights and Challenges

23. CAS Let $f_1(x) = x$ and define a sequence of functions by $f_{n+1}(x) = \frac{1}{2}(f_n(x) + x/f_n(x))$. For example, $f_2(x) = \frac{1}{2}(x+1)$. Use a computer algebra system to compute $f_n(x)$ for $n = 3, 4, 5$ and plot $y = f_n(x)$ together with $y = \sqrt{x}$ for $x \geq 0$. What do you notice?

SOLUTION With $f_1(x) = x$ and $f_2(x) = \frac{1}{2}(x+1)$, we calculate

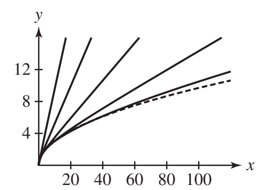
$$f_3(x) = \frac{1}{2}\left(\frac{1}{2}(x+1) + \frac{x}{\frac{1}{2}(x+1)}\right) = \frac{x^2+6x+1}{4(x+1)}$$

$$f_4(x) = \frac{1}{2}\left(\frac{x^2+6x+1}{4(x+1)} + \frac{x}{\frac{x^2+6x+1}{4(x+1)}}\right) = \frac{x^4+28x^3+70x^2+28x+1}{8(1+x)(1+6x+x^2)}$$

and

$$f_5(x) = \frac{1+120x+1820x^2+8008x^3+12870x^4+8008x^5+1820x^6+120x^7+x^8}{16(1+x)(1+6x+x^2)(1+28x+70x^2+28x^3+x^4)}.$$

A plot of $f_1(x)$, $f_2(x)$, $f_3(x)$, $f_4(x)$, $f_5(x)$ and $\sqrt{x}$ is shown below, with the graph of $\sqrt{x}$ shown as a dashed curve. It seems as if the f_n are asymptotic to $\sqrt{x}$.



24. Set $P_0(x) = 1$ and $P_1(x) = x$. The **Chebyshev polynomials** (useful in approximation theory) are defined inductively by the formula $P_{n+1}(x) = 2xP_n(x) - P_{n-1}(x)$.

(a) Show that $P_2(x) = 2x^2 - 1$.

(b) Compute $P_n(x)$ for $3 \leq n \leq 6$ using a computer algebra system or by hand, and plot $y = P_n(x)$ over $[-1, 1]$.

(c) Check that your plots confirm two interesting properties: (a) $y = P_n(x)$ has n real roots in $[-1, 1]$ and (b) for $x \in [-1, 1]$, $P_n(x)$ lies between -1 and 1.

SOLUTION

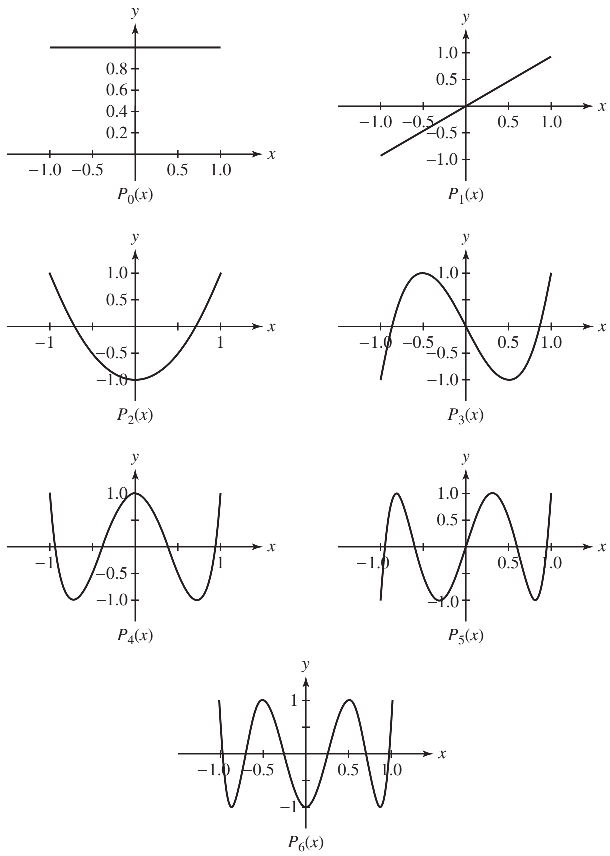
(a) With $P_0(x) = 1$ and $P_1(x) = x$, we calculate

$$P_2(x) = 2x(P_1(x)) - P_0(x) = 2x(x) - 1 = 2x^2 - 1.$$

(b) Using the formula $P_{n+1}(x) = 2xP_n(x) - P_{n-1}(x)$ with $n = 2, 3, 4$ and 5, we find

$$P_3(x) = 2x(2x^2-1) - x = 4x^3 - 3x$$
$$P_4(x) = 2x(4x^3-3x) - (2x^2-1) = 8x^4 - 8x^2 + 1$$
$$P_5(x) = 16x^5 - 20x^3 + 5x$$
$$P_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$$

The graphs of the functions $P_n(x)$ for $0 \leq n \leq 6$ are shown below.



(c) From the graphs shown above, it is clear that for each n, the polynomial $P_n(x)$ has precisely n roots on the interval $[-1, 1]$ and that $-1 \le P_n(x) \le 1$ for $x \in [-1, 1]$.

CHAPTER REVIEW EXERCISES

1. Express $(4, 10)$ as a set $\{x : |x - a| < c\}$ for suitable a and c.

SOLUTION The center of the interval $(4, 10)$ is $\frac{4+10}{2} = 7$ and the radius is $\frac{10-4}{2} = 3$. Therefore, the interval $(4, 10)$ is equivalent to the set $\{x : |x - 7| < 3\}$.

2. Express as an interval:

(a) $\{x : |x - 5| < 4\}$ **(b)** $\{x : |5x + 3| \le 2\}$

SOLUTION

(a) Upon dropping the absolute value, the inequality $|x - 5| < 4$ becomes $-4 < x - 5 < 4$ or $1 < x < 9$. The set $\{x : |x - 5| < 4\}$ can therefore be expressed as the interval $(1, 9)$.

(b) Upon dropping the absolute value, the inequality $|5x + 3| \le 2$ becomes $-2 \le 5x + 3 \le 2$ or $-1 \le x \le -\frac{1}{5}$. The set $\{x : |5x + 3| \le 2\}$ can therefore be expressed as the interval $[-1, -\frac{1}{5}]$.

3. Express $\{x : 2 \le |x - 1| \le 6\}$ as a union of two intervals.

SOLUTION The set $\{x : 2 \le |x - 1| \le 6\}$ consists of those numbers that are at least 2 but at most 6 units from 1. The numbers larger than 1 that satisfy these conditions are $3 \le x \le 7$, while the numbers smaller than 1 that satisfy these conditions are $-5 \le x \le -1$. Therefore $\{x : 2 \le |x - 1| \le 6\} = [-5, -1] \cup [3, 7]$.

4. Give an example of numbers x, y such that $|x| + |y| = x - y$.

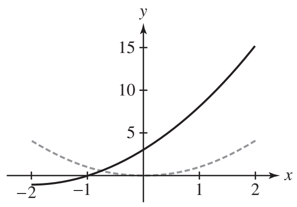
SOLUTION Let $x = 3$ and $y = -1$. Then $|x| + |y| = 3 + 1 = 4$ and $x - y = 3 - (-1) = 4$.

5. Describe the pairs of numbers x, y such that $|x + y| = x - y$.

SOLUTION First consider the case when $x + y \ge 0$. Then $|x + y| = x + y$ and we obtain the equation $x + y = x - y$. The solution of this equation is $y = 0$. Thus, the pairs $(x, 0)$ with $x \ge 0$ satisfy $|x + y| = x - y$. Next, consider the case when $x + y < 0$. Then $|x + y| = -(x + y) = -x - y$ and we obtain the equation $-x - y = x - y$. The solution of this equation is $x = 0$. Thus, the pairs $(0, y)$ with $y < 0$ also satisfy $|x + y| = x - y$.

6. Sketch the graph of $y = f(x+2) - 1$, where $f(x) = x^2$ for $-2 \le x \le 2$.

SOLUTION The graph of $y = f(x+2) - 1$ is obtained by shifting the graph of $y = f(x)$ two units to the left and one unit down. In the figure below, the graph of $y = f(x)$ is shown as the dashed curve, and the graph of $y = f(x+2) - 1$ is shown as the solid curve.



In Exercises 7–10, let $f(x)$ be the function shown in Figure 1.

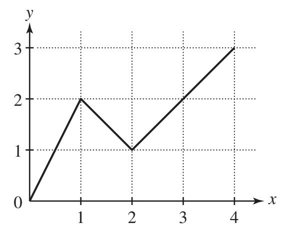
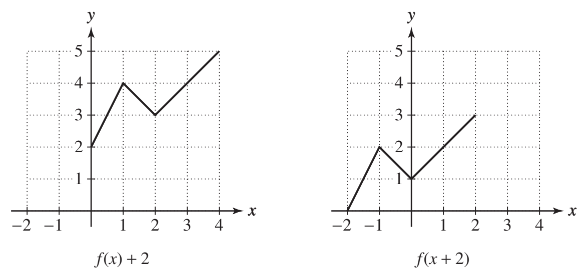


FIGURE 1

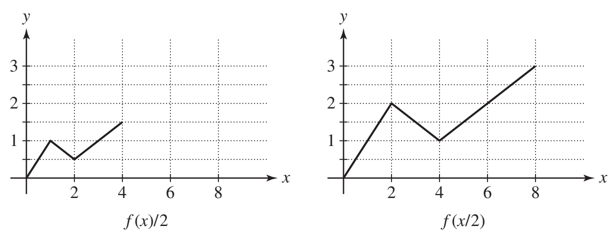
7. Sketch the graphs of $y = f(x) + 2$ and $y = f(x+2)$.

SOLUTION The graph of $y = f(x) + 2$ is obtained by shifting the graph of $y = f(x)$ up 2 units (see the graph below at the left). The graph of $y = f(x+2)$ is obtained by shifting the graph of $y = f(x)$ to the left 2 units (see the graph below at the right).



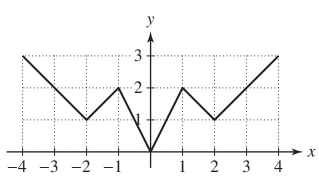
8. Sketch the graphs of $y = \frac{1}{2}f(x)$ and $y = f\left(\frac{1}{2}x\right)$.

SOLUTION The graph of $y = \frac{1}{2}f(x)$ is obtained by compressing the graph of $y = f(x)$ vertically by a factor of 2 (see the graph below at the left). The graph of $y = f(\frac{1}{2}x)$ is obtained by stretching the graph of $y = f(x)$ horizontally be a factor of 2 (see the graph below at the right).



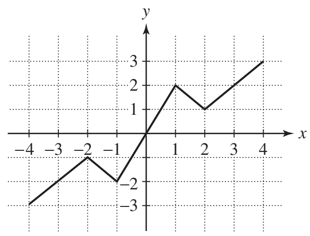
9. Continue the graph of f to the interval $[-4, 4]$ as an even function.

SOLUTION To continue the graph of $f(x)$ to the interval $[-4, 4]$ as an even function, reflect the graph of $f(x)$ across the y-axis (see the graph below).



10. Continue the graph of f to the interval $[-4, 4]$ as an odd function.

SOLUTION To continue the graph of $f(x)$ to the interval $[-4, 4]$ as an odd function, reflect the graph of $f(x)$ through the origin (see the graph below).



In Exercises 11–14, find the domain and range of the function.

11. $f(x) = \sqrt{x+1}$

SOLUTION The domain of the function $f(x) = \sqrt{x+1}$ is $\{x : x \geq -1\}$ and the range is $\{y : y \geq 0\}$.

12. $f(x) = \dfrac{4}{x^4+1}$

SOLUTION The domain of the function $f(x) = \dfrac{4}{x^4+1}$ is the set of all real numbers and the range is $\{y : 0 < y \leq 4\}$.

13. $f(x) = \dfrac{2}{3-x}$

SOLUTION The domain of the function $f(x) = \dfrac{2}{3-x}$ is $\{x : x \neq 3\}$ and the range is $\{y : y \neq 0\}$.

14. $f(x) = \sqrt{x^2 - x + 5}$

SOLUTION Because

$$x^2 - x + 5 = \left(x^2 - x + \frac{1}{4}\right) + 5 - \frac{1}{4} = \left(x - \frac{1}{2}\right)^2 + \frac{19}{4},$$

$x^2 - x + 5 \geq \frac{19}{4}$ for all x. It follows that the domain of the function $f(x) = \sqrt{x^2 - x + 5}$ is all real numbers and the range is $\{y : y \geq \sqrt{19}/2\}$.

15. Determine whether the function is increasing, decreasing, or neither:

(a) $f(x) = 3^{-x}$

(b) $f(x) = \dfrac{1}{x^2+1}$

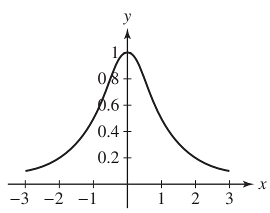
(c) $g(t) = t^2 + t$

(d) $g(t) = t^3 + t$

SOLUTION

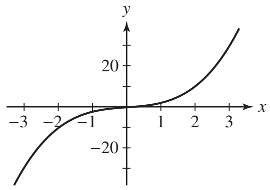
(a) The function $f(x) = 3^{-x}$ can be rewritten as $f(x) = (\frac{1}{3})^x$. This is an exponential function with a base less than 1; therefore, this is a decreasing function.

(b) From the graph of $y = 1/(x^2+1)$ shown below, we see that this function is neither increasing nor decreasing for all x (though it is increasing for $x < 0$ and decreasing for $x > 0$).



(c) The graph of $y = t^2 + t$ is an upward opening parabola; therefore, this function is neither increasing nor decreasing for all t. By completing the square we find $y = (t + \frac{1}{2})^2 - \frac{1}{4}$. The vertex of this parabola is then at $t = -\frac{1}{2}$, so the function is decreasing for $t < -\frac{1}{2}$ and increasing for $t > -\frac{1}{2}$.

(d) From the graph of $y = t^3 + t$ shown below, we see that this is an increasing function.



16. Determine whether the function is even, odd, or neither:

(a) $f(x) = x^4 - 3x^2$

(b) $g(x) = \sin(x + 1)$

(c) $f(x) = 2^{-x^2}$

SOLUTION

(a) $f(-x) = (-x)^4 - 3(-x)^2 = x^4 - 3x^2 = f(x)$, so this function is even.

(b) $g(-x) = \sin(-x + 1)$, which is neither equal to $g(x)$ nor to $-g(x)$, so this function is neither even nor odd.

(c) $f(-x) = 2^{-(-x)^2} = 2^{-x^2} = f(x)$, so this function is even.

In Exercises 17–24, find the equation of the line.

17. Line passing through $(-1, 4)$ and $(2, 6)$

SOLUTION The slope of the line passing through $(-1, 4)$ and $(2, 6)$ is

$$m = \frac{6 - 4}{2 - (-1)} = \frac{2}{3}.$$

The equation of the line passing through $(-1, 4)$ and $(2, 6)$ is therefore $y - 4 = \frac{2}{3}(x + 1)$ or $2x - 3y = -14$.

18. Line passing through $(-1, 4)$ and $(-1, 6)$

SOLUTION The line passing through $(-1, 4)$ and $(-1, 6)$ is vertical with an x-coordinate of -1. Therefore, the equation of the line is $x = -1$.

19. Line of slope 6 through $(9, 1)$

SOLUTION Using the point-slope form for the equation of a line, the equation of the line of slope 6 and passing through $(9, 1)$ is $y - 1 = 6(x - 9)$ or $6x - y = 53$.

20. Line of slope $-\frac{3}{2}$ through $(4, -12)$

SOLUTION Using the point-slope form for the equation of a line, the equation of the line of slope $-\frac{3}{2}$ and passing through $(4, -12)$ is $y + 12 = -\frac{3}{2}(x - 4)$ or $3x + 2y = -12$.

21. Line through $(2, 1)$ perpendicular to the line given by $y = 3x + 7$

SOLUTION The equation $y = 3x + 7$ is in slope-intercept form; it follows that the slope of this line is 3. Perpendicular lines have slopes that are negative reciprocals of one another, so we are looking for the equation of the line of slope $-\frac{1}{3}$ and passing through $(2, 1)$. The equation of this line is $y - 1 = -\frac{1}{3}(x - 2)$ or $x + 3y = 5$.

22. Line through $(3, 4)$ perpendicular to the line given by $y = 4x - 2$

SOLUTION The equation $y = 4x - 2$ is in slope-intercept form; it follows that the slope of this line is 4. Perpendicular lines have slopes that are negative reciprocals of one another, so we are looking for the equation of the line of slope $-\frac{1}{4}$ and passing through $(3, 4)$. The equation of this line is $y - 4 = -\frac{1}{4}(x - 3)$ or $x + 4y = 19$.

23. Line through $(2, 3)$ parallel to $y = 4 - x$

SOLUTION The equation $y = 4 - x$ is in slope-intercept form; it follows that the slope of this line is -1. Any line parallel to $y = 4 - x$ will have the same slope, so we are looking for the equation of the line of slope -1 and passing through $(2, 3)$. The equation of this line is $y - 3 = -(x - 2)$ or $x + y = 5$.

24. Horizontal line through $(-3, 5)$

SOLUTION A horizontal line has a slope of 0; the equation of the specified line is therefore $y - 5 = 0(x + 3)$ or $y = 5$.

25. Does the following table of market data suggest a linear relationship between price and number of homes sold during a one-year period? Explain.

Price (thousands of $)	180	195	220	240
No. of homes sold	127	118	103	91

SOLUTION Examine the slope between consecutive data points. The first pair of data points yields a slope of

$$\frac{118 - 127}{195 - 180} = -\frac{9}{15} = -\frac{3}{5},$$

while the second pair of data points yields a slope of

$$\frac{103 - 118}{220 - 195} = -\frac{15}{25} = -\frac{3}{5}$$

and the last pair of data points yields a slope of

$$\frac{91 - 103}{240 - 220} = -\frac{12}{20} = -\frac{3}{5}.$$

Because all three slopes are equal, the data does suggest a linear relationship between price and the number of homes sold.

26. Does the following table of revenue data for a computer manufacturer suggest a linear relation between revenue and time? Explain.

Year	2005	2009	2011	2014
Revenue (billions of $)	13	18	15	11

SOLUTION Examine the slope between consecutive data points. The first pair of data points yields a slope of

$$\frac{18 - 13}{2009 - 2005} = \frac{5}{4},$$

while the second pair of data points yields a slope of

$$\frac{15 - 18}{2011 - 2009} = -\frac{3}{2}$$

and the last pair of data points yields a slope of

$$\frac{11 - 15}{2014 - 2011} = -\frac{4}{3}.$$

Because the three slopes are not equal, the data does not suggest a linear relationship between revenue and time.

27. Suppose that a cell phone plan that is offered at a price of P dollars per month attracts C customers, where $C(P)$ is a linear demand function for $\$100 \le P \le \$500$. If $C(100) = 1{,}000{,}000$ and $C(500) = 100{,}000$, determine the demand function C. What is the decrease in the number of customers for each increase of \$100 in the price?

SOLUTION We first determine the slope of the line:

$$m = \frac{1000000 - 100000}{100 - 500} = \frac{900000}{-400} = -2250.$$

Knowing that $C(100) = 1000000$, it follows that

$$C - 1000000 = -2250(P - 100), \quad \text{or} \quad C(P) = -2250P + 1225000.$$

Because the slope of the demand function is -2250, a 100 dollar increase in price will lead to a decrease in the number of customers of $2250(100) = 225000$ customers.

28. Suppose that Internet domain names are sold at a price of $\$P$ per month for $\$2 \le P \le \$100$. The number of customers C who buy the domain names is a linear function of the price. If 10,000 customers buy a domain name when the price is \$2 per month and 1000 customers buy when the price is \$100 per month, determine the demand function C. What is the decrease in the number of customers for every \$1 increase in the cost of the domain names?

SOLUTION We first determine the slope of the line:

$$m = \frac{10000 - 1000}{2 - 100} = \frac{9000}{-98} = -\frac{4500}{49}.$$

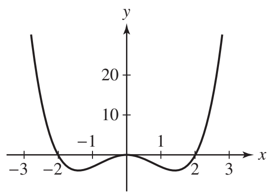
Knowing that $C(2) = 10000$, it follows that

$$C - 10000 = -\frac{4500}{49}(P - 2), \quad \text{or} \quad C(P) = -\frac{4500}{49}P + \frac{499000}{49}.$$

Because the slope of the demand function is $-\frac{4500}{49}$, a 1 dollar increase in the cost of domain names will lead to a decrease in the number of customers of $\frac{4500}{49} \approx 91.84$, or about 92 customers.

29. Find the roots of $f(x) = x^4 - 4x^2$ and sketch its graph. On which intervals is f decreasing?

SOLUTION The roots of $f(x) = x^4 - 4x^2$ are obtained by solving the equation $x^4 - 4x^2 = x^2(x-2)(x+2) = 0$, which yields $x = -2$, $x = 0$ and $x = 2$. The graph of $y = f(x)$ is shown below. From this graph we see that $f(x)$ is decreasing for x less than approximately -1.4 and for x between 0 and approximately 1.4.



30. Let $h(z) = -2z^2 + 12z + 3$. Complete the square and find the maximum value of h.

SOLUTION Let $h(z) = -2z^2 + 12z + 3$. Completing the square yields

$$h(z) = -2(z^2 - 6z) + 3 = -2(z^2 - 6z + 9) + 3 + 18 = -2(z-3)^2 + 21.$$

Because $(z-3)^2 \geq 0$ for all z, it follows that $h(z) = -2(z-3)^2 + 21 \leq 21$ for all z. Thus, the maximum value of h is 21.

31. Let $f(x)$ be the square of the distance from the point $(2, 1)$ to a point $(x, 3x+2)$ on the line $y = 3x + 2$. Show that f is a quadratic function, and find its minimum value by completing the square.

SOLUTION Let $f(x)$ denote the square of the distance from the point $(2, 1)$ to a point $(x, 3x+2)$ on the line $y = 3x + 2$. Then

$$f(x) = (x-2)^2 + (3x+2-1)^2 = x^2 - 4x + 4 + 9x^2 + 6x + 1 = 10x^2 + 2x + 5,$$

which is a quadratic function. Completing the square, we find

$$f(x) = 10\left(x^2 + \frac{1}{5}x + \frac{1}{100}\right) + 5 - \frac{1}{10} = 10\left(x + \frac{1}{10}\right)^2 + \frac{49}{10}.$$

Because $\left(x + \frac{1}{10}\right)^2 \geq 0$ for all x, it follows that $f(x) \geq \frac{49}{10}$ for all x. Hence, the minimum value of $f(x)$ is $\frac{49}{10}$.

32. Prove that $x^2 + 3x + 3 \geq 0$ for all x.

SOLUTION Observe that

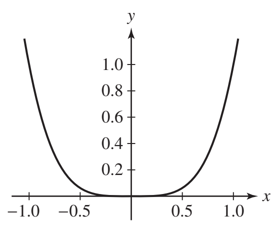
$$x^2 + 3x + 3 = \left(x^2 + 3x + \frac{9}{4}\right) + 3 - \frac{9}{4} = \left(x + \frac{3}{2}\right)^2 + \frac{3}{4}.$$

Thus, $x^2 + 3x + 3 \geq \frac{3}{4} > 0$ for all x.

In Exercises 33–38, sketch the graph by hand.

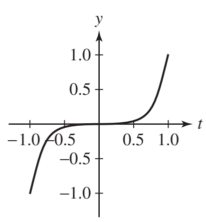
33. $y = t^4$

SOLUTION



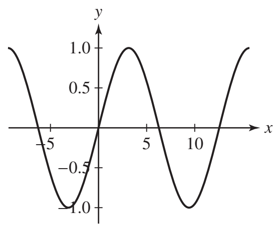
34. $y = t^5$

SOLUTION



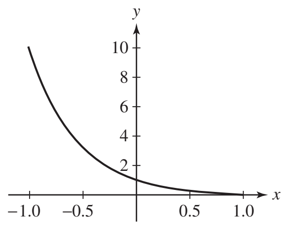
35. $y = \sin \dfrac{\theta}{2}$

SOLUTION



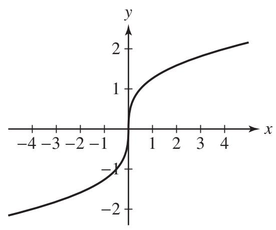
36. $y = 10^{-x}$

SOLUTION



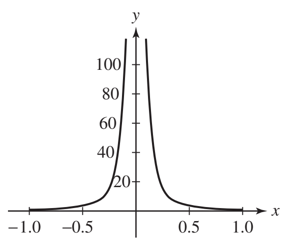
37. $y = x^{1/3}$

SOLUTION



38. $y = \dfrac{1}{x^2}$

SOLUTION



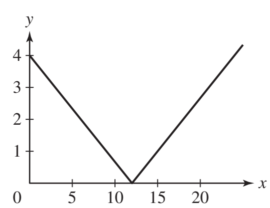
39. Show that the graph of $y = f\left(\frac{1}{3}x - b\right)$ is obtained by shifting the graph of $y = f\left(\frac{1}{3}x\right)$ to the right $3b$ units. Use this observation to sketch the graph of $y = \left|\frac{1}{3}x - 4\right|$.

SOLUTION Let $g(x) = f(\frac{1}{3}x)$. Then

$$g(x - 3b) = f\left(\frac{1}{3}(x - 3b)\right) = f\left(\frac{1}{3}x - b\right).$$

Thus, the graph of $y = f(\frac{1}{3}x - b)$ is obtained by shifting the graph of $y = f(\frac{1}{3}x)$ to the right $3b$ units.

The graph of $y = |\frac{1}{3}x - 4|$ is the graph of $y = |\frac{1}{3}x|$ shifted right 12 units (see the graph below).



40. Let $h(x) = \cos x$ and $g(x) = x^{-1}$. Compute the composite functions $h(g)$ and $g(h)$, and find their domains.

SOLUTION Let $h(x) = \cos x$ and $g(x) = x^{-1}$. Then

$$h(g(x)) = h(x^{-1}) = \cos x^{-1}.$$

The domain of this function is $x \neq 0$. On the other hand,

$$g(h(x)) = g(\cos x) = \frac{1}{\cos x} = \sec x.$$

The domain of this function is

$$x \neq \frac{(2n+1)\pi}{2} \quad \text{for any integer } n.$$

41. Find functions f and g such that the function

$$f(g(t)) = (12t + 9)^4$$

SOLUTION One possible choice is $f(t) = t^4$ and $g(t) = 12t + 9$. Then

$$f(g(t)) = f(12t + 9) = (12t + 9)^4$$

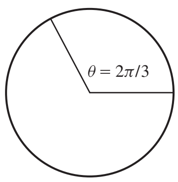
as desired.

42. Sketch the points on the unit circle corresponding to the following three angles, and find the values of the six standard trigonometric functions at each angle:

(a) $\frac{2\pi}{3}$ **(b)** $\frac{7\pi}{4}$ **(c)** $\frac{7\pi}{6}$

SOLUTION

(a)

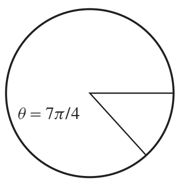


$$\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2} \qquad \cos\frac{2\pi}{3} = -\frac{1}{2}$$

$$\tan\frac{2\pi}{3} = -\sqrt{3} \qquad \cot\frac{2\pi}{3} = -\frac{\sqrt{3}}{3}$$

$$\sec\frac{2\pi}{3} = -2 \qquad \csc\frac{2\pi}{3} = \frac{2\sqrt{3}}{3}$$

(b)

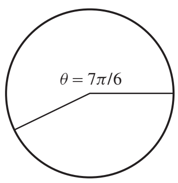


$$\sin\frac{7\pi}{4} = -\frac{\sqrt{2}}{2} \qquad \cos\frac{7\pi}{4} = \frac{\sqrt{2}}{2}$$

$$\tan\frac{7\pi}{4} = -1 \qquad \cot\frac{7\pi}{4} = -1$$

$$\sec\frac{7\pi}{4} = \sqrt{2} \qquad \csc\frac{7\pi}{4} = -\sqrt{2}$$

(c)



$$\sin\frac{7\pi}{6} = -\frac{1}{2} \qquad \cos\frac{7\pi}{6} = -\frac{\sqrt{3}}{2}$$

$$\tan\frac{7\pi}{6} = \frac{\sqrt{3}}{3} \qquad \cot\frac{7\pi}{6} = \sqrt{3}$$

$$\sec\frac{7\pi}{6} = -\frac{2\sqrt{3}}{2} \qquad \csc\frac{7\pi}{6} = -2$$

43. What are the periods of these functions?

(a) $y = \sin 2\theta$ **(b)** $y = \sin\frac{\theta}{2}$

(c) $y = \sin 2\theta + \sin\frac{\theta}{2}$

SOLUTION

(a) π

(b) 4π

(c) The function $\sin 2\theta$ has a period of π, and the function $\sin(\theta/2)$ has a period of 4π. Because 4π is a multiple of π, the period of the function $g(\theta) = \sin 2\theta + \sin\theta/2$ is 4π.

44. Assume that $\sin\theta = \frac{4}{5}$, where $\pi/2 < \theta < \pi$. Find:

(a) $\tan\theta$ **(b)** $\sin 2\theta$ **(c)** $\csc\frac{\theta}{2}$

SOLUTION If $\sin\theta = 4/5$, then by the fundamental trigonometric identity,

$$\cos^2\theta = 1 - \sin^2\theta = 1 - \left(\frac{4}{5}\right)^2 = \frac{9}{25}.$$

Because $\pi/2 < \theta < \pi$, it follows that $\cos\theta$ must be negative. Hence, $\cos\theta = -3/5$.

(a) $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{4/5}{-3/5} = -\dfrac{4}{3}.$

(b) $\sin(2\theta) = 2\sin\theta\cos\theta = 2\cdot\dfrac{4}{5}\cdot -\dfrac{3}{5} = -\dfrac{24}{25}.$

(c) We first note that

$$\sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{1-(-3/5)}{2}} = 2\frac{\sqrt{5}}{5}.$$

Thus,

$$\csc\left(\frac{\theta}{2}\right) = \frac{\sqrt{5}}{2}.$$

45. Give an example of values a, b such that

(a) $\cos(a+b) \neq \cos a + \cos b$

(b) $\cos\dfrac{a}{2} \neq \dfrac{\cos a}{2}$

SOLUTION

(a) Take $a = b = \pi/2$. Then $\cos(a+b) = \cos\pi = -1$ but

$$\cos a + \cos b = \cos\frac{\pi}{2} + \cos\frac{\pi}{2} = 0 + 0 = 0.$$

(b) Take $a = \pi$. Then

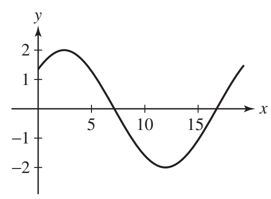
$$\cos\left(\frac{a}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$$

but

$$\frac{\cos a}{2} = \frac{\cos\pi}{2} = \frac{-1}{2} = -\frac{1}{2}.$$

46. Let $f(x) = \cos x$. Sketch the graph of $y = 2f\left(\frac{1}{3}x - \frac{\pi}{4}\right)$ for $0 \le x \le 6\pi$.

SOLUTION



47. Solve $\sin 2x + \cos x = 0$ for $0 \le x < 2\pi$.

SOLUTION Using the double angle formula for the sine function, we rewrite the equation as $2\sin x\cos x + \cos x = \cos x(2\sin x + 1) = 0$. Thus, either $\cos x = 0$ or $\sin x = -1/2$. From here we see that the solutions are $x = \pi/2$, $x = 7\pi/6$, $x = 3\pi/2$ and $x = 11\pi/6$.

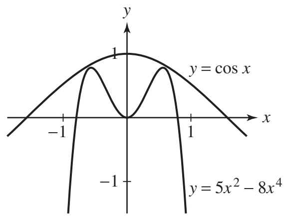
48. How does $h(n) = n^2/2^n$ behave for large whole-number values of n? Does $h(n)$ tend to infinity?

SOLUTION The table below suggests that for large whole number values of n, $h(n) = \dfrac{n^2}{2^n}$ tends toward 0.

n	$h(n) = n^2/2^n$
10	0.09765625000
10^2	$7.888609052 \times 10^{-27}$
10^3	$9.332636185 \times 10^{-296}$
10^4	$5.012372749 \times 10^{-3003}$
10^5	$1.000998904 \times 10^{-30093}$
10^6	$1.010034059 \times 10^{-301018}$

49. GU Use a graphing calculator to determine whether the equation $\cos x = 5x^2 - 8x^4$ has any solutions.

SOLUTION The graphs of $y = \cos x$ and $y = 5x^2 - 8x^4$ are shown below. Because the graphs do not intersect, there are no solutions to the equation $\cos x = 5x^2 - 8x^4$.



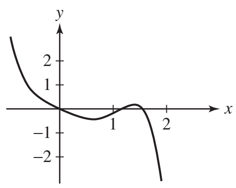
50. GU Using a graphing calculator, find the number of real roots and estimate the largest root to two decimal places:

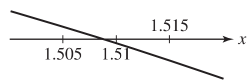
(a) $f(x) = 1.8x^4 - x^5 - x$

(b) $g(x) = 1.7x^4 - x^5 - x$

SOLUTION

(a) The graph of $y = 1.8x^4 - x^5 - x$ is shown below at the left. Because the graph has three x-intercepts, the function $f(x) = 1.8x^4 - x^5 - x$ has three real roots. From the graph shown below at the right, we see that the largest root of $f(x) = 1.8x^4 - x^5 - x$ is approximately $x = 1.51$.





(b) The graph of $y = 1.7x^4 - x^5 - x$ is shown below. Because the graph has only one x-intercept, the function $f(x) = 1.7x^4 - x^5 - x$ has only one real root. From the graph, we see that the largest root of $f(x) = 1.7x^4 - x^5 - x$ is approximately $x = 0$.