DIAGNOSTIC TESTS

Test A Algebra

1. (a) $(-3)^4 = (-3)(-3)(-3)(-3) = 81$

   (b) $-3^4 = -(3)(3)(3)(3) = -81$

   (c) $3^{-4} = \dfrac{1}{3^4} = \dfrac{1}{81}$

   (d) $\dfrac{5^{23}}{5^{21}} = 5^{23-21} = 5^2 = 25$

   (e) $\left(\frac{2}{3}\right)^{-2} = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$

   (f) $16^{-3/4} = \dfrac{1}{16^{3/4}} = \dfrac{1}{\left(\sqrt[4]{16}\right)^3} = \dfrac{1}{2^3} = \dfrac{1}{8}$

2. (a) Note that $\sqrt{200} = \sqrt{100 \cdot 2} = 10\sqrt{2}$ and $\sqrt{32} = \sqrt{16 \cdot 2} = 4\sqrt{2}$. Thus $\sqrt{200} - \sqrt{32} = 10\sqrt{2} - 4\sqrt{2} = 6\sqrt{2}$.

   (b) $(3a^3b^3)(4ab^2)^2 = 3a^3b^3 16a^2b^4 = 48a^5b^7$

   (c) $\left(\dfrac{3x^{3/2}y^3}{x^2y^{-1/2}}\right)^{-2} = \left(\dfrac{x^2y^{-1/2}}{3x^{3/2}y^3}\right)^2 = \dfrac{(x^2y^{-1/2})^2}{(3x^{3/2}y^3)^2} = \dfrac{x^4y^{-1}}{9x^3y^6} = \dfrac{x^4}{9x^3y^6y} = \dfrac{x}{9y^7}$

3. (a) $3(x+6) + 4(2x-5) = 3x + 18 + 8x - 20 = 11x - 2$

   (b) $(x+3)(4x-5) = 4x^2 - 5x + 12x - 15 = 4x^2 + 7x - 15$

   (c) $\left(\sqrt{a} + \sqrt{b}\right)\left(\sqrt{a} - \sqrt{b}\right) = \left(\sqrt{a}\right)^2 - \sqrt{a}\sqrt{b} + \sqrt{a}\sqrt{b} - \left(\sqrt{b}\right)^2 = a - b$

   Or: Use the formula for the difference of two squares to see that $\left(\sqrt{a} + \sqrt{b}\right)\left(\sqrt{a} - \sqrt{b}\right) = \left(\sqrt{a}\right)^2 - \left(\sqrt{b}\right)^2 = a - b$.

   (d) $(2x+3)^2 = (2x+3)(2x+3) = 4x^2 + 6x + 6x + 9 = 4x^2 + 12x + 9$.

   Note: A quicker way to expand this binomial is to use the formula $(a+b)^2 = a^2 + 2ab + b^2$ with $a = 2x$ and $b = 3$: $(2x+3)^2 = (2x)^2 + 2(2x)(3) + 3^2 = 4x^2 + 12x + 9$

   (e) See Reference Page 1 for the binomial formula $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$. Using it, we get $(x+2)^3 = x^3 + 3x^2(2) + 3x(2^2) + 2^3 = x^3 + 6x^2 + 12x + 8$.

4. (a) Using the difference of two squares formula, $a^2 - b^2 = (a+b)(a-b)$, we have $4x^2 - 25 = (2x)^2 - 5^2 = (2x+5)(2x-5)$.

   (b) Factoring by trial and error, we get $2x^2 + 5x - 12 = (2x-3)(x+4)$.

   (c) Using factoring by grouping and the difference of two squares formula, we have $x^3 - 3x^2 - 4x + 12 = x^2(x-3) - 4(x-3) = (x^2-4)(x-3) = (x-2)(x+2)(x-3)$.

   (d) $x^4 + 27x = x(x^3 + 27) = x(x+3)(x^2 - 3x + 9)$

   This last expression was obtained using the sum of two cubes formula, $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$ with $a = x$ and $b = 3$. [See Reference Page 1 in the textbook.]

   (e) The smallest exponent on x is $-\frac{1}{2}$, so we will factor out $x^{-1/2}$.

   $3x^{3/2} - 9x^{1/2} + 6x^{-1/2} = 3x^{-1/2}(x^2 - 3x + 2) = 3x^{-1/2}(x-1)(x-2)$

   (f) $x^3y - 4xy = xy(x^2 - 4) = xy(x-2)(x+2)$

5. (a) $\dfrac{x^2+3x+2}{x^2-x-2} = \dfrac{(x+1)(x+2)}{(x+1)(x-2)} = \dfrac{x+2}{x-2}$

(b) $\dfrac{2x^2-x-1}{x^2-9} \cdot \dfrac{x+3}{2x+1} = \dfrac{(2x+1)(x-1)}{(x-3)(x+3)} \cdot \dfrac{x+3}{2x+1} = \dfrac{x-1}{x-3}$

(c) $\dfrac{x^2}{x^2-4} - \dfrac{x+1}{x+2} = \dfrac{x^2}{(x-2)(x+2)} - \dfrac{x+1}{x+2} = \dfrac{x^2}{(x-2)(x+2)} - \dfrac{x+1}{x+2} \cdot \dfrac{x-2}{x-2} = \dfrac{x^2-(x+1)(x-2)}{(x-2)(x+2)}$

$$= \frac{x^2-(x^2-x-2)}{(x+2)(x-2)} = \frac{x+2}{(x+2)(x-2)} = \frac{1}{x-2}$$

(d) $\dfrac{\dfrac{y}{x}-\dfrac{x}{y}}{\dfrac{1}{y}-\dfrac{1}{x}} = \dfrac{\dfrac{y}{x}-\dfrac{x}{y}}{\dfrac{1}{y}-\dfrac{1}{x}} \cdot \dfrac{xy}{xy} = \dfrac{y^2-x^2}{x-y} = \dfrac{(y-x)(y+x)}{-(y-x)} = \dfrac{y+x}{-1} = -(x+y)$

6. (a) $\dfrac{\sqrt{10}}{\sqrt{5}-2} = \dfrac{\sqrt{10}}{\sqrt{5}-2} \cdot \dfrac{\sqrt{5}+2}{\sqrt{5}+2} = \dfrac{\sqrt{50}+2\sqrt{10}}{\left(\sqrt{5}\right)^2-2^2} = \dfrac{5\sqrt{2}+2\sqrt{10}}{5-4} = 5\sqrt{2}+2\sqrt{10}$

(b) $\dfrac{\sqrt{4+h}-2}{h} = \dfrac{\sqrt{4+h}-2}{h} \cdot \dfrac{\sqrt{4+h}+2}{\sqrt{4+h}+2} = \dfrac{4+h-4}{h\left(\sqrt{4+h}+2\right)} = \dfrac{h}{h\left(\sqrt{4+h}+2\right)} = \dfrac{1}{\sqrt{4+h}+2}$

7. (a) $x^2+x+1 = \left(x^2+x+\frac{1}{4}\right)+1-\frac{1}{4} = \left(x+\frac{1}{2}\right)^2+\frac{3}{4}$

(b) $2x^2-12x+11 = 2(x^2-6x)+11 = 2(x^2-6x+9-9)+11 = 2(x^2-6x+9)-18+11 = 2(x-3)^2-7$

8. (a) $x+5 = 14-\frac{1}{2}x \quad \Leftrightarrow \quad x+\frac{1}{2}x = 14-5 \quad \Leftrightarrow \quad \frac{3}{2}x = 9 \quad \Leftrightarrow \quad x = \frac{2}{3}\cdot 9 \quad \Leftrightarrow \quad x = 6$

(b) $\dfrac{2x}{x+1} = \dfrac{2x-1}{x} \quad \Rightarrow \quad 2x^2 = (2x-1)(x+1) \quad \Leftrightarrow \quad 2x^2 = 2x^2+x-1 \quad \Leftrightarrow \quad x = 1$

(c) $x^2-x-12 = 0 \quad \Leftrightarrow \quad (x+3)(x-4) = 0 \quad \Leftrightarrow \quad x+3=0$ or $x-4=0 \quad \Leftrightarrow \quad x=-3$ or $x=4$

(d) By the quadratic formula, $2x^2+4x+1=0 \quad \Leftrightarrow$

$$x = \frac{-4\pm\sqrt{4^2-4(2)(1)}}{2(2)} = \frac{-4\pm\sqrt{8}}{4} = \frac{-4\pm2\sqrt{2}}{4} = \frac{2\left(-2\pm\sqrt{2}\right)}{4} = \frac{-2\pm\sqrt{2}}{2} = -1\pm\tfrac{1}{2}\sqrt{2}.$$

(e) $x^4-3x^2+2=0 \quad \Leftrightarrow \quad (x^2-1)(x^2-2)=0 \quad \Leftrightarrow \quad x^2-1=0$ or $x^2-2=0 \quad \Leftrightarrow \quad x^2=1$ or $x^2=2 \quad \Leftrightarrow$
$x=\pm1$ or $x=\pm\sqrt{2}$

(f) $3\,|x-4| = 10 \quad \Leftrightarrow \quad |x-4| = \frac{10}{3} \quad \Leftrightarrow \quad x-4 = -\frac{10}{3}$ or $x-4=\frac{10}{3} \quad \Leftrightarrow \quad x=\frac{2}{3}$ or $x=\frac{22}{3}$

(g) Multiplying through $2x(4-x)^{-1/2}-3\sqrt{4-x}=0$ by $(4-x)^{1/2}$ gives $2x-3(4-x)=0 \quad \Leftrightarrow$
$2x-12+3x=0 \quad \Leftrightarrow \quad 5x-12=0 \quad \Leftrightarrow \quad 5x=12 \quad \Leftrightarrow \quad x=\frac{12}{5}$.

9. (a) $-4 < 5-3x \le 17 \quad \Leftrightarrow \quad -9 < -3x \le 12 \quad \Leftrightarrow \quad 3 > x \ge -4$ or $-4 \le x < 3$.
In interval notation, the answer is $[-4, 3)$.

(b) $x^2 < 2x+8 \quad \Leftrightarrow \quad x^2-2x-8 < 0 \quad \Leftrightarrow \quad (x+2)(x-4) < 0$. Now, $(x+2)(x-4)$ will change sign at the critical values $x=-2$ and $x=4$. Thus the possible intervals of solution are $(-\infty,-2)$, $(-2,4)$, and $(4,\infty)$. By choosing a single test value from each interval, we see that $(-2,4)$ is the only interval that satisfies the inequality.

(c) The inequality $x(x-1)(x+2) > 0$ has critical values of $-2, 0$, and 1. The corresponding possible intervals of solution are $(-\infty, -2)$, $(-2, 0)$, $(0, 1)$ and $(1, \infty)$. By choosing a single test value from each interval, we see that both intervals $(-2, 0)$ and $(1, \infty)$ satisfy the inequality. Thus, the solution is the union of these two intervals: $(-2, 0) \cup (1, \infty)$.

(d) $|x-4| < 3 \quad \Leftrightarrow \quad -3 < x - 4 < 3 \quad \Leftrightarrow \quad 1 < x < 7$. In interval notation, the answer is $(1, 7)$.

(e) $\dfrac{2x-3}{x+1} \le 1 \quad \Leftrightarrow \quad \dfrac{2x-3}{x+1} - 1 \le 0 \quad \Leftrightarrow \quad \dfrac{2x-3}{x+1} - \dfrac{x+1}{x+1} \le 0 \quad \Leftrightarrow \quad \dfrac{2x-3-x-1}{x+1} \le 0 \quad \Leftrightarrow \quad \dfrac{x-4}{x+1} \le 0.$

Now, the expression $\dfrac{x-4}{x+1}$ may change signs at the critical values $x = -1$ and $x = 4$, so the possible intervals of solution are $(-\infty, -1)$, $(-1, 4]$, and $[4, \infty)$. By choosing a single test value from each interval, we see that $(-1, 4]$ is the only interval that satisfies the inequality.

10. (a) False. In order for the statement to be true, it must hold for all real numbers, so, to show that the statement is false, pick $p = 1$ and $q = 2$ and observe that $(1+2)^2 \neq 1^2 + 2^2$. In general, $(p+q)^2 = p^2 + 2pq + q^2$.

(b) True as long as a and b are nonnegative real numbers. To see this, think in terms of the laws of exponents: $\sqrt{ab} = (ab)^{1/2} = a^{1/2}b^{1/2} = \sqrt{a}\,\sqrt{b}$.

(c) False. To see this, let $p = 1$ and $q = 2$, then $\sqrt{1^2 + 2^2} \neq 1 + 2$.

(d) False. To see this, let $T = 1$ and $C = 2$, then $\dfrac{1 + 1(2)}{2} \neq 1 + 1$.

(e) False. To see this, let $x = 2$ and $y = 3$, then $\dfrac{1}{2-3} \neq \dfrac{1}{2} - \dfrac{1}{3}$.

(f) True since $\dfrac{1/x}{a/x - b/x} \cdot \dfrac{x}{x} = \dfrac{1}{a-b}$, as long as $x \neq 0$ and $a - b \neq 0$.

Test B Analytic Geometry

1. (a) Using the point $(2, -5)$ and $m = -3$ in the point-slope equation of a line, $y - y_1 = m(x - x_1)$, we get $y - (-5) = -3(x-2) \quad \Rightarrow \quad y + 5 = -3x + 6 \quad \Rightarrow \quad y = -3x + 1$.

(b) A line parallel to the x-axis must be horizontal and thus have a slope of 0. Since the line passes through the point $(2, -5)$, the y-coordinate of every point on the line is -5, so the equation is $y = -5$.

(c) A line parallel to the y-axis is vertical with undefined slope. So the x-coordinate of every point on the line is 2 and so the equation is $x = 2$.

(d) Note that $2x - 4y = 3 \quad \Rightarrow \quad -4y = -2x + 3 \quad \Rightarrow \quad y = \frac{1}{2}x - \frac{3}{4}$. Thus the slope of the given line is $m = \frac{1}{2}$. Hence, the slope of the line we're looking for is also $\frac{1}{2}$ (since the line we're looking for is required to be parallel to the given line). So the equation of the line is $y - (-5) = \frac{1}{2}(x-2) \quad \Rightarrow \quad y + 5 = \frac{1}{2}x - 1 \quad \Rightarrow \quad y = \frac{1}{2}x - 6$.

2. First we'll find the distance between the two given points in order to obtain the radius, r, of the circle: $r = \sqrt{[3-(-1)]^2 + (-2-4)^2} = \sqrt{4^2 + (-6)^2} = \sqrt{52}$. Next use the standard equation of a circle, $(x-h)^2 + (y-k)^2 = r^2$, where (h, k) is the center, to get $(x+1)^2 + (y-4)^2 = 52$.

3. We must rewrite the equation in standard form in order to identify the center and radius. Note that $x^2 + y^2 - 6x + 10y + 9 = 0 \quad\Rightarrow\quad x^2 - 6x + 9 + y^2 + 10y = 0$. For the left-hand side of the latter equation, we factor the first three terms and complete the square on the last two terms as follows: $x^2 - 6x + 9 + y^2 + 10y = 0 \quad\Rightarrow$ $(x-3)^2 + y^2 + 10y + 25 = 25 \quad\Rightarrow\quad (x-3)^2 + (y+5)^2 = 25$. Thus, the center of the circle is $(3, -5)$ and the radius is 5.

4. (a) $A(-7, 4)$ and $B(5, -12) \quad\Rightarrow\quad m_{AB} = \dfrac{-12-4}{5-(-7)} = \dfrac{-16}{12} = -\dfrac{4}{3}$

(b) $y - 4 = -\frac{4}{3}[x - (-7)] \quad\Rightarrow\quad y - 4 = -\frac{4}{3}x - \frac{28}{3} \quad\Rightarrow\quad 3y - 12 = -4x - 28 \quad\Rightarrow\quad 4x + 3y + 16 = 0$. Putting $y = 0$, we get $4x + 16 = 0$, so the x-intercept is -4, and substituting 0 for x results in a y-intercept of $-\frac{16}{3}$.

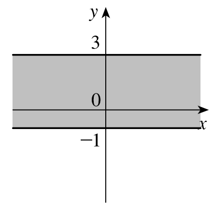
(c) The midpoint is obtained by averaging the corresponding coordinates of both points: $\left(\frac{-7+5}{2}, \frac{4+(-12)}{2}\right) = (-1, -4)$.

(d) $d = \sqrt{[5-(-7)]^2 + (-12-4)^2} = \sqrt{12^2 + (-16)^2} = \sqrt{144 + 256} = \sqrt{400} = 20$

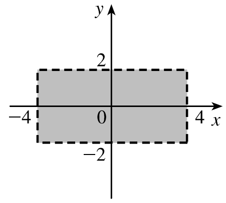
(e) The perpendicular bisector is the line that intersects the line segment $\overline{AB}$ at a right angle through its midpoint. Thus the perpendicular bisector passes through $(-1, -4)$ and has slope $\frac{3}{4}$ [the slope is obtained by taking the negative reciprocal of the answer from part (a)]. So the perpendicular bisector is given by $y + 4 = \frac{3}{4}[x - (-1)]$ or $3x - 4y = 13$.

(f) The center of the required circle is the midpoint of $\overline{AB}$, and the radius is half the length of $\overline{AB}$, which is 10. Thus, the equation is $(x+1)^2 + (y+4)^2 = 100$.

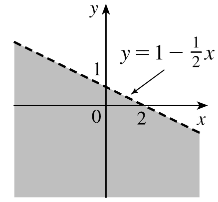
5. (a) Graph the corresponding horizontal lines (given by the equations $y = -1$ and $y = 3$) as solid lines. The inequality $y \geq -1$ describes the points (x, y) that lie on or *above* the line $y = -1$. The inequality $y \leq 3$ describes the points (x, y) that lie on or *below* the line $y = 3$. So the pair of inequalities $-1 \leq y \leq 3$ describes the points that lie on or *between* the lines $y = -1$ and $y = 3$.



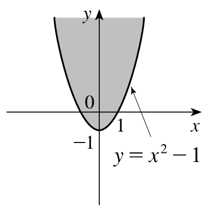
(b) Note that the given inequalities can be written as $-4 < x < 4$ and $-2 < y < 2$, respectively. So the region lies between the vertical lines $x = -4$ and $x = 4$ and between the horizontal lines $y = -2$ and $y = 2$. As shown in the graph, the region common to both graphs is a rectangle (minus its edges) centered at the origin.



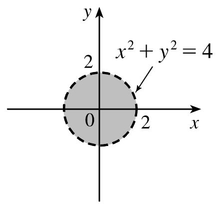
(c) We first graph $y = 1 - \frac{1}{2}x$ as a dotted line. Since $y < 1 - \frac{1}{2}x$, the points in the region lie *below* this line.



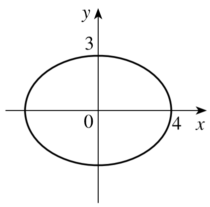
(d) We first graph the parabola $y = x^2 - 1$ using a solid curve. Since $y \geq x^2 - 1$, the points in the region lie on or *above* the parabola.



(e) We graph the circle $x^2 + y^2 = 4$ using a dotted curve. Since $\sqrt{x^2 + y^2} < 2$, the region consists of points whose distance from the origin is less than 2, that is, the points that lie *inside* the circle.



(f) The equation $9x^2 + 16y^2 = 144$ is an ellipse centered at $(0, 0)$. We put it in standard form by dividing by 144 and get $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$. The x-intercepts are located at a distance of $\sqrt{16} = 4$ from the center while the y-intercepts are a distance of $\sqrt{9} = 3$ from the center (see the graph).



Test C Functions

1. (a) Locate -1 on the x-axis and then go down to the point on the graph with an x-coordinate of -1. The corresponding y-coordinate is the value of the function at $x = -1$, which is -2. So, $f(-1) = -2$.

   (b) Using the same technique as in part (a), we get $f(2) \approx 2.8$.

   (c) Locate 2 on the y-axis and then go left and right to find all points on the graph with a y-coordinate of 2. The corresponding x-coordinates are the x-values we are searching for. So $x = -3$ and $x = 1$.

   (d) Using the same technique as in part (c), we get $x \approx -2.5$ and $x \approx 0.3$.

   (e) The domain is all the x-values for which the graph exists, and the range is all the y-values for which the graph exists. Thus, the domain is $[-3, 3]$, and the range is $[-2, 3]$.

2. Note that $f(2 + h) = (2 + h)^3$ and $f(2) = 2^3 = 8$. So the difference quotient becomes

$$\frac{f(2+h) - f(2)}{h} = \frac{(2+h)^3 - 8}{h} = \frac{8 + 12h + 6h^2 + h^3 - 8}{h} = \frac{12h + 6h^2 + h^3}{h} = \frac{h(12 + 6h + h^2)}{h} = 12 + 6h + h^2.$$

3. (a) Set the denominator equal to 0 and solve to find restrictions on the domain: $x^2 + x - 2 = 0 \quad \Rightarrow$ $(x - 1)(x + 2) = 0 \quad \Rightarrow \quad x = 1$ or $x = -2$. Thus, the domain is all real numbers except 1 or -2 or, in interval notation, $(-\infty, -2) \cup (-2, 1) \cup (1, \infty)$.

   (b) Note that the denominator is always greater than or equal to 1, and the numerator is defined for all real numbers. Thus, the domain is $(-\infty, \infty)$.

   (c) Note that the function h is the sum of two root functions. So h is defined on the intersection of the domains of these two root functions. The domain of a square root function is found by setting its radicand greater than or equal to 0. Now,

$4 - x \geq 0 \quad \Rightarrow \quad x \leq 4$ and $x^2 - 1 \geq 0 \quad \Rightarrow \quad (x-1)(x+1) \geq 0 \quad \Rightarrow \quad x \leq -1$ or $x \geq 1$. Thus, the domain of h is $(-\infty, -1] \cup [1, 4]$.

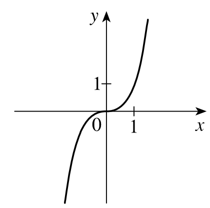
4. (a) Reflect the graph of f about the x-axis.

   (b) Stretch the graph of f vertically by a factor of 2, then shift 1 unit downward.

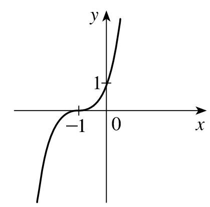
   (c) Shift the graph of f right 3 units, then up 2 units.

5. (a) Make a table and then connect the points with a smooth curve:

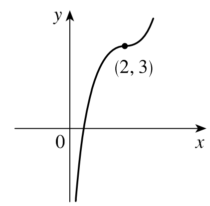
x	-2	-1	0	1	2
y	-8	-1	0	1	8



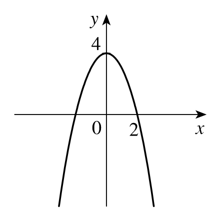
   (b) Shift the graph from part (a) left 1 unit.



   (c) Shift the graph from part (a) right 2 units and up 3 units.

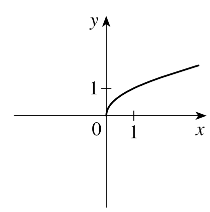


   (d) First plot $y = x^2$. Next, to get the graph of $f(x) = 4 - x^2$, reflect f about the x-axis and then shift it upward 4 units.

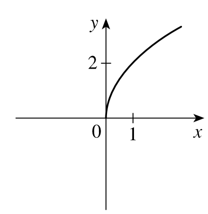


   (e) Make a table and then connect the points with a smooth curve:

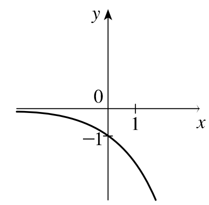
x	0	1	4	9
y	0	1	2	3



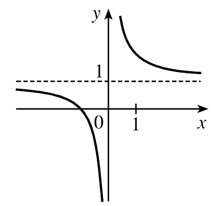
   (f) Stretch the graph from part (e) vertically by a factor of two.



(g) First plot $y = 2^x$. Next, get the graph of $y = -2^x$ by reflecting the graph of $y = 2^x$ about the x-axis.

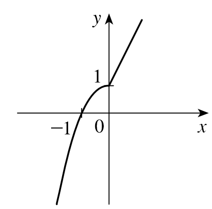


(h) Note that $y = 1 + x^{-1} = 1 + 1/x$. So first plot $y = 1/x$ and then shift it upward 1 unit.



6. (a) $f(-2) = 1 - (-2)^2 = -3$ and $f(1) = 2(1) + 1 = 3$

(b) For $x \leq 0$ plot $f(x) = 1 - x^2$ and, on the same plane, for $x > 0$ plot the graph of $f(x) = 2x + 1$.



7. (a) $(f \circ g)(x) = f(g(x)) = f(2x-3) = (2x-3)^2 + 2(2x-3) - 1 = 4x^2 - 12x + 9 + 4x - 6 - 1 = 4x^2 - 8x + 2$

(b) $(g \circ f)(x) = g(f(x)) = g(x^2 + 2x - 1) = 2(x^2 + 2x - 1) - 3 = 2x^2 + 4x - 2 - 3 = 2x^2 + 4x - 5$

(c) $(g \circ g \circ g)(x) = g(g(g(x))) = g(g(2x-3)) = g(2(2x-3)-3) = g(4x-9) = 2(4x-9) - 3$
$= 8x - 18 - 3 = 8x - 21$

Test D Trigonometry

1. (a) $300^\circ = 300^\circ \left(\dfrac{\pi}{180^\circ}\right) = \dfrac{300\pi}{180} = \dfrac{5\pi}{3}$

(b) $-18^\circ = -18^\circ \left(\dfrac{\pi}{180^\circ}\right) = -\dfrac{18\pi}{180} = -\dfrac{\pi}{10}$

2. (a) $\dfrac{5\pi}{6} = \dfrac{5\pi}{6}\left(\dfrac{180}{\pi}\right)^\circ = 150^\circ$

(b) $2 = 2\left(\dfrac{180}{\pi}\right)^\circ = \left(\dfrac{360}{\pi}\right)^\circ \approx 114.6^\circ$

3. We will use the arc length formula, $s = r\theta$, where s is arc length, r is the radius of the circle, and θ is the measure of the central angle in radians. First, note that $30^\circ = 30^\circ \left(\dfrac{\pi}{180^\circ}\right) = \dfrac{\pi}{6}$. So $s = (12)\left(\dfrac{\pi}{6}\right) = 2\pi$ cm.

4. (a) $\tan(\pi/3) = \sqrt{3}$ [You can read the value from a right triangle with sides 1, 2, and $\sqrt{3}$.]

(b) Note that $7\pi/6$ can be thought of as an angle in the third quadrant with reference angle $\pi/6$. Thus, $\sin(7\pi/6) = -\frac{1}{2}$, since the sine function is negative in the third quadrant.

(c) Note that $5\pi/3$ can be thought of as an angle in the fourth quadrant with reference angle $\pi/3$. Thus,
$\sec(5\pi/3) = \dfrac{1}{\cos(5\pi/3)} = \dfrac{1}{1/2} = 2$, since the cosine function is positive in the fourth quadrant.

5. $\sin\theta = a/24 \;\Rightarrow\; a = 24\sin\theta$ and $\cos\theta = b/24 \;\Rightarrow\; b = 24\cos\theta$

6. $\sin x = \frac{1}{3}$ and $\sin^2 x + \cos^2 x = 1 \;\Rightarrow\; \cos x = \sqrt{1-\frac{1}{9}} = \dfrac{2\sqrt{2}}{3}$. Also, $\cos y = \frac{4}{5} \;\Rightarrow\; \sin y = \sqrt{1-\frac{16}{25}} = \frac{3}{5}$.

So, using the sum identity for the sine, we have

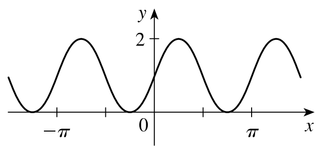
$$\sin(x+y) = \sin x\,\cos y + \cos x\,\sin y = \frac{1}{3}\cdot\frac{4}{5} + \frac{2\sqrt{2}}{3}\cdot\frac{3}{5} = \frac{4+6\sqrt{2}}{15} = \frac{1}{15}\left(4+6\sqrt{2}\right)$$

7. (a) $\tan\theta\,\sin\theta + \cos\theta = \dfrac{\sin\theta}{\cos\theta}\sin\theta + \cos\theta = \dfrac{\sin^2\theta}{\cos\theta} + \dfrac{\cos^2\theta}{\cos\theta} = \dfrac{1}{\cos\theta} = \sec\theta$

(b) $\dfrac{2\tan x}{1+\tan^2 x} = \dfrac{2\sin x/(\cos x)}{\sec^2 x} = 2\,\dfrac{\sin x}{\cos x}\,\cos^2 x = 2\sin x\,\cos x = \sin 2x$

8. $\sin 2x = \sin x \;\Leftrightarrow\; 2\sin x\,\cos x = \sin x \;\Leftrightarrow\; 2\sin x\,\cos x - \sin x = 0 \;\Leftrightarrow\; \sin x\,(2\cos x - 1) = 0 \;\Leftrightarrow\;$
$\sin x = 0$ or $\cos x = \frac{1}{2} \;\Rightarrow\; x = 0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi$.

9. We first graph $y = \sin 2x$ (by compressing the graph of $\sin x$ by a factor of 2) and then shift it upward 1 unit.



1 ☐ FUNCTIONS AND SEQUENCES

1.1 Four Ways to Represent a Function

1. The functions $f(x) = x + \sqrt{2 - x}$ and $g(u) = u + \sqrt{2 - u}$ give exactly the same output values for every input value, so f and g are equal.

2. $f(x) = \dfrac{x^2 - x}{x - 1} = \dfrac{x(x-1)}{x-1} = x$ for $x - 1 \neq 0$, so f and g [where $g(x) = x$] are not equal because $f(1)$ is undefined and $g(1) = 1$.

3. (a) The point $(1, 3)$ is on the graph of f, so $f(1) = 3$.
   (b) When $x = -1$, y is about -0.2, so $f(-1) \approx -0.2$.
   (c) $f(x) = 1$ is equivalent to $y = 1$. When $y = 1$, we have $x = 0$ and $x = 3$.
   (d) A reasonable estimate for x when $y = 0$ is $x = -0.8$.
   (e) The domain of f consists of all x-values on the graph of f. For this function, the domain is $-2 \leq x \leq 4$, or $[-2, 4]$. The range of f consists of all y-values on the graph of f. For this function, the range is $-1 \leq y \leq 3$, or $[-1, 3]$.
   (f) As x increases from -2 to 1, y increases from -1 to 3. Thus, f is increasing on the interval $[-2, 1]$.

4. (a) The point $(-4, -2)$ is on the graph of f, so $f(-4) = -2$. The point $(3, 4)$ is on the graph of g, so $g(3) = 4$.
   (b) We are looking for the values of x for which the y-values are equal. The y-values for f and g are equal at the points $(-2, 1)$ and $(2, 2)$, so the desired values of x are -2 and 2.
   (c) $f(x) = -1$ is equivalent to $y = -1$. When $y = -1$, we have $x = -3$ and $x = 4$.
   (d) As x increases from 0 to 4, y decreases from 3 to -1. Thus, f is decreasing on the interval $[0, 4]$.
   (e) The domain of f consists of all x-values on the graph of f. For this function, the domain is $-4 \leq x \leq 4$, or $[-4, 4]$. The range of f consists of all y-values on the graph of f. For this function, the range is $-2 \leq y \leq 3$, or $[-2, 3]$.
   (f) The domain of g is $[-4, 3]$ and the range is $[0.5, 4]$.

5. No, the curve is not the graph of a function because a vertical line intersects the curve more than once. Hence, the curve fails the Vertical Line Test.

6. Yes, the curve is the graph of a function because it passes the Vertical Line Test. The domain is $[-2, 2]$ and the range is $[-1, 2]$.

7. Yes, the curve is the graph of a function because it passes the Vertical Line Test. The domain is $[-3, 2]$ and the range is $[-3, -2) \cup [-1, 3]$.

8. No, the curve is not the graph of a function since for $x = 0$, ± 1, and ± 2, there are infinitely many points on the curve.

9. (a) The graph shows that the global average temperature in 1950 was $T(1950) \approx 13.8\,^{\circ}\mathrm{C}$.
   (b) By drawing the horizontal line $T = 14.2$ to the curve and then drawing the vertical line down to the horizontal axis, we see that $t \approx 1992$.
   (c) The temperature was smallest in 1910 and largest in 2006.
   (d) The range is $\{T \mid 13.5 \leq T \leq 14.5\} = [13.5, 14.5]$

10. (a) The range is $\{\text{Width} \mid 0 < \text{Width} \leq 1.6\} = (0, 1.6]$

(b) The graph shows an overall decline in global temperatures from 1500 to 1700, followed by an overall rise in temperatures. The fluctuations in temperature in the mid and late 19th century are reflective of the cooling effects caused by several large volcanic eruptions.

11. If we draw the horizontal line $\text{pH} = 4.0$, we can see that the pH curve is less than 4.0 between 12:23AM and 12:52AM. Therefore, a clinical acid reflux episode occurred approximately between 12:23AM and 12:52AM at which time the esophageal pH was less than 4.0.

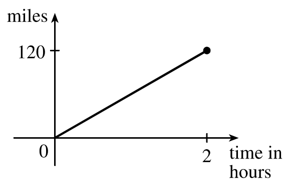
12. The graphs indicate that tadpoles raised in densely populated regions take longer to put on weight. This is sensible since more crowding leads to fewer resources available for each tadpole.

13. (a) At 30 °S and 20 °N, we expect approximately 100 and 134 ant species respectively.

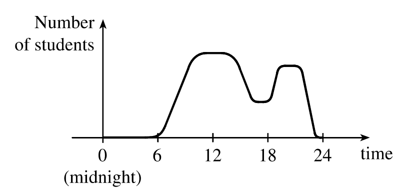
(b) By drawing the horizontal line at a species richness of 100, we see there are two points of intersection with the curve, each having latitude values of roughly 30 °N and 30 °S.

(c) The function is even since its graph is symmetric with respect to the y-axis.

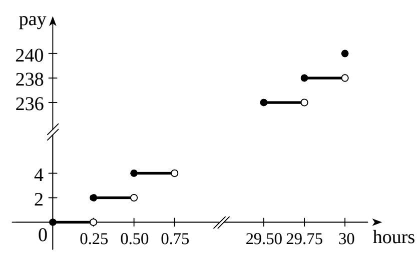
14. *Example 1:* A car is driven at 60 mi/h for 2 hours. The distance d traveled by the car is a function of the time t. The domain of the function is $\{t \mid 0 \leq t \leq 2\}$, where t is measured in hours. The range of the function is $\{d \mid 0 \leq d \leq 120\}$, where d is measured in miles.



Example 2: At a certain university, the number of students N on campus at any time on a particular day is a function of the time t after midnight. The domain of the function is $\{t \mid 0 \leq t \leq 24\}$, where t is measured in hours. The range of the function is $\{N \mid 0 \leq N \leq k\}$, where N is an integer and k is the largest number of students on campus at once.



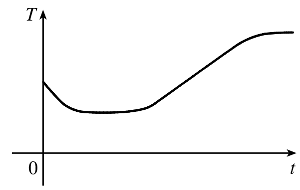
Example 3: A certain employee is paid \$8.00 per hour and works a maximum of 30 hours per week. The number of hours worked is rounded down to the nearest quarter of an hour. This employee's gross weekly pay P is a function of the number of hours worked h. The domain of the function is $[0, 30]$ and the range of the function is $\{0, 2.00, 4.00, \ldots, 238.00, 240.00\}$.



15. The person's weight increased to about 160 pounds at age 20 and stayed fairly steady for 10 years. The person's weight dropped to about 120 pounds for the next 5 years, then increased rapidly to about 170 pounds. The next 30 years saw a gradual increase to 190 pounds. Possible reasons for the drop in weight at 30 years of age: diet, exercise, health problems.

16. Initially, the person's forward moving heel contacts the ground resulting in a ground reaction force in the opposite or negative direction. In moving from heel-strike to toe-off, the foot transitions from a forward push to a backward push. Hence, the ground reaction force switches from a negative value to a positive value, becoming zero at some point in between.

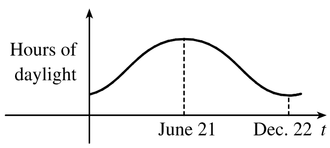
17. The water will cool down almost to freezing as the ice melts. Then, when the ice has melted, the water will slowly warm up to room temperature.



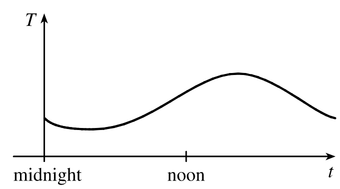
18. Runner A won the race, reaching the finish line at 100 meters in about 15 seconds, followed by runner B with a time of about 19 seconds, and then by runner C who finished in around 23 seconds. B initially led the race, followed by C, and then A. C then passed B to lead for a while. Then A passed first B, and then passed C to take the lead and finish first. Finally, B passed C to finish in second place. All three runners completed the race.

19. Initially, the bacteria population size remains constant during which nutrients are consumed in preparation for reproduction. In the second phase, the population size increases rapidly as the bacteria replicate. The population size plateaus in phase three at which point the "carrying capacity" has been reached and the available resources and space cannot support a larger population. Finally, the bacteria die due to starvation and waste toxicity and the population declines.

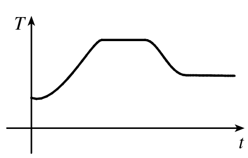
20. The summer solstice (the longest day of the year) is around June 21, and the winter solstice (the shortest day) is around December 22. (Exchange the dates for the southern hemisphere.)



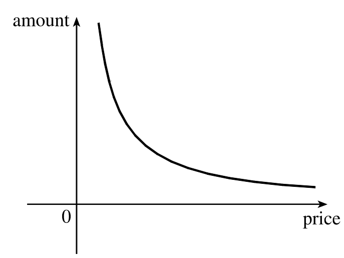
21. Of course, this graph depends strongly on the geographical location!



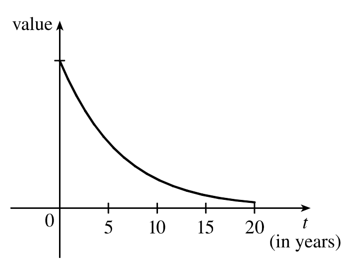
22. The temperature of the pie would increase rapidly, level off to oven temperature, decrease rapidly, and then level off to room temperature.



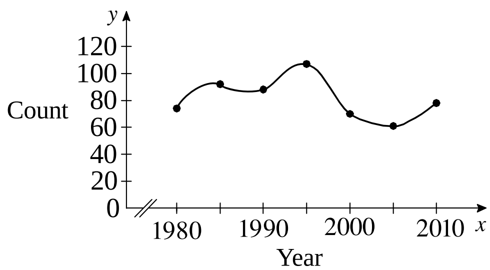
23. As the price increases, the amount sold decreases.



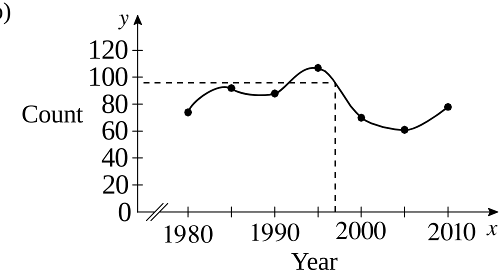
24. The value of the car decreases fairly rapidly initially, then somewhat less rapidly.



25. (a)

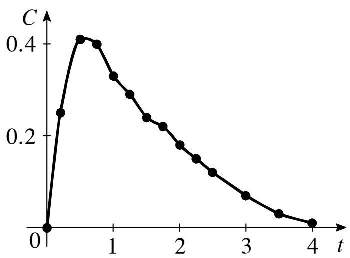


(b)



We see from the graph that there were approximately 92,000 birds in 1997.

26. (a)



(b) Alcohol concentration increases rapidly within the first hour of consumption and then slowly decreases over the following three hours.

27. $f(x) = 3x^2 - x + 2.$

$f(2) = 3(2)^2 - 2 + 2 = 12 - 2 + 2 = 12.$

$f(-2) = 3(-2)^2 - (-2) + 2 = 12 + 2 + 2 = 16.$

$f(a) = 3a^2 - a + 2.$

$f(-a) = 3(-a)^2 - (-a) + 2 = 3a^2 + a + 2.$

$f(a+1) = 3(a+1)^2 - (a+1) + 2 = 3(a^2 + 2a + 1) - a - 1 + 2 = 3a^2 + 6a + 3 - a + 1 = 3a^2 + 5a + 4.$

$2f(a) = 2 \cdot f(a) = 2(3a^2 - a + 2) = 6a^2 - 2a + 4.$

$f(2a) = 3(2a)^2 - (2a) + 2 = 3(4a^2) - 2a + 2 = 12a^2 - 2a + 2.$

$f(a^2) = 3(a^2)^2 - (a^2) + 2 = 3(a^4) - a^2 + 2 = 3a^4 - a^2 + 2.$

$$[f(a)]^2 = \left[3a^2 - a + 2\right]^2 = \left(3a^2 - a + 2\right)\left(3a^2 - a + 2\right)$$
$$= 9a^4 - 3a^3 + 6a^2 - 3a^3 + a^2 - 2a + 6a^2 - 2a + 4 = 9a^4 - 6a^3 + 13a^2 - 4a + 4.$$

$f(a+h) = 3(a+h)^2 - (a+h) + 2 = 3(a^2 + 2ah + h^2) - a - h + 2 = 3a^2 + 6ah + 3h^2 - a - h + 2.$

28. A spherical balloon with radius $r + 1$ has volume $V(r+1) = \frac{4}{3}\pi(r+1)^3 = \frac{4}{3}\pi(r^3 + 3r^2 + 3r + 1)$. We wish to find the amount of air needed to inflate the balloon from a radius of r to $r + 1$. Hence, we need to find the difference $V(r+1) - V(r) = \frac{4}{3}\pi(r^3 + 3r^2 + 3r + 1) - \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(3r^2 + 3r + 1).$

29. $f(x) = 4 + 3x - x^2$, so $f(3+h) = 4 + 3(3+h) - (3+h)^2 = 4 + 9 + 3h - (9 + 6h + h^2) = 4 - 3h - h^2$,

and $\dfrac{f(3+h) - f(3)}{h} = \dfrac{(4 - 3h - h^2) - 4}{h} = \dfrac{h(-3-h)}{h} = -3 - h.$

30. $f(x) = x^3$, so $f(a+h) = (a+h)^3 = a^3 + 3a^2h + 3ah^2 + h^3$,

and $\dfrac{f(a+h) - f(a)}{h} = \dfrac{(a^3 + 3a^2h + 3ah^2 + h^3) - a^3}{h} = \dfrac{h(3a^2 + 3ah + h^2)}{h} = 3a^2 + 3ah + h^2$.

31. $\dfrac{f(x) - f(a)}{x - a} = \dfrac{\dfrac{1}{x} - \dfrac{1}{a}}{x - a} = \dfrac{\dfrac{a - x}{xa}}{x - a} = \dfrac{a - x}{xa(x - a)} = \dfrac{-1(x - a)}{xa(x - a)} = -\dfrac{1}{ax}$

32. $\dfrac{f(x) - f(1)}{x - 1} = \dfrac{\dfrac{x+3}{x+1} - 2}{x - 1} = \dfrac{\dfrac{x + 3 - 2(x+1)}{x+1}}{x - 1} = \dfrac{x + 3 - 2x - 2}{(x+1)(x-1)}$

$= \dfrac{-x + 1}{(x+1)(x-1)} = \dfrac{-(x-1)}{(x+1)(x-1)} = -\dfrac{1}{x+1}$

33. $f(x) = (x+4)/(x^2 - 9)$ is defined for all x except when $0 = x^2 - 9 \iff 0 = (x+3)(x-3) \iff x = -3$ or 3, so the domain is $\{x \in \mathbb{R} \mid x \neq -3, 3\} = (-\infty, -3) \cup (-3, 3) \cup (3, \infty)$.

34. $f(x) = (2x^3 - 5)/(x^2 + x - 6)$ is defined for all x except when $0 = x^2 + x - 6 \iff 0 = (x+3)(x-2) \iff$ $x = -3$ or 2, so the domain is $\{x \in \mathbb{R} \mid x \neq -3, 2\} = (-\infty, -3) \cup (-3, 2) \cup (2, \infty)$.

35. $f(t) = \sqrt[3]{2t - 1}$ is defined for all real numbers. In fact $\sqrt[3]{p(t)}$, where $p(t)$ is a polynomial, is defined for all real numbers. Thus, the domain is $\mathbb{R}$, or $(-\infty, \infty)$.

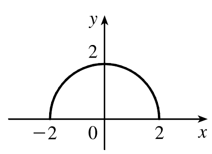
36. $g(t) = \sqrt{3 - t} - \sqrt{2 + t}$ is defined when $3 - t \geq 0 \iff t \leq 3$ and $2 + t \geq 0 \iff t \geq -2$. Thus, the domain is $-2 \leq t \leq 3$, or $[-2, 3]$.

37. $h(x) = 1 \big/ \sqrt[4]{x^2 - 5x}$ is defined when $x^2 - 5x > 0 \iff x(x-5) > 0$. Note that $x^2 - 5x \neq 0$ since that would result in division by zero. The expression $x(x-5)$ is positive if $x < 0$ or $x > 5$. (See Appendix A for methods for solving inequalities.) Thus, the domain is $(-\infty, 0) \cup (5, \infty)$.

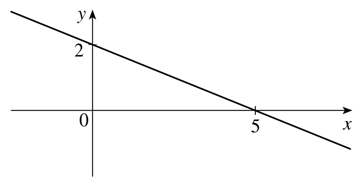
38. $f(u) = \dfrac{u+1}{1 + \dfrac{1}{u+1}}$ is defined when $u + 1 \neq 0$ $[u \neq -1]$ and $1 + \dfrac{1}{u+1} \neq 0$. Since $1 + \dfrac{1}{u+1} = 0 \Rightarrow$ $\dfrac{1}{u+1} = -1 \Rightarrow 1 = -u - 1 \Rightarrow u = -2$, the domain is $\{u \mid u \neq -2, u \neq -1\} = (-\infty, -2) \cup (-2, -1) \cup (-1, \infty)$.

39. $F(p) = \sqrt{2 - \sqrt{p}}$ is defined when $p \geq 0$ and $2 - \sqrt{p} \geq 0$. Since $2 - \sqrt{p} \geq 0 \Rightarrow 2 \geq \sqrt{p} \Rightarrow \sqrt{p} \leq 2 \Rightarrow$ $0 \leq p \leq 4$, the domain is $[0, 4]$.

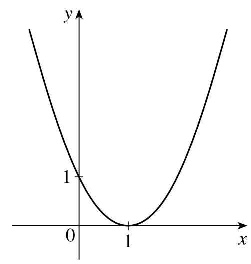
40. $h(x) = \sqrt{4 - x^2}$. Now $y = \sqrt{4 - x^2} \Rightarrow y^2 = 4 - x^2 \iff x^2 + y^2 = 4$, so the graph is the top half of a circle of radius 2 with center at the origin. The domain is $\{x \mid 4 - x^2 \geq 0\} = \{x \mid 4 \geq x^2\} = \{x \mid 2 \geq |x|\} = [-2, 2]$. From the graph, the range is $0 \leq y \leq 2$, or $[0, 2]$.



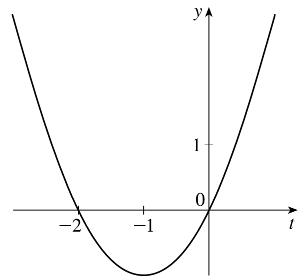
41. $f(x) = 2 - 0.4x$ is defined for all real numbers, so the domain is $\mathbb{R}$, or $(-\infty, \infty)$. The graph of f is a line with slope -0.4 and y-intercept 2.



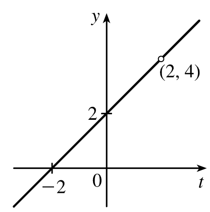
42. $F(x) = x^2 - 2x + 1 = (x-1)^2$ is defined for all real numbers, so the domain is $\mathbb{R}$, or $(-\infty, \infty)$. The graph of F is a parabola with vertex $(1, 0)$.



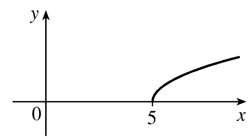
43. $f(t) = 2t + t^2$ is defined for all real numbers, so the domain is $\mathbb{R}$, or $(-\infty, \infty)$. The graph of f is a parabola opening upward since the coefficient of t^2 is positive. To find the t-intercepts, let $y = 0$ and solve for t. $0 = 2t + t^2 = t(2+t) \quad \Rightarrow \quad t = 0$ or $t = -2$. The t-coordinate of the vertex is halfway between the t-intercepts, that is, at $t = -1$. Since $f(-1) = 2(-1) + (-1)^2 = -2 + 1 = -1$, the vertex is $(-1, -1)$.



44. $H(t) = \dfrac{4 - t^2}{2 - t} = \dfrac{(2+t)(2-t)}{2-t}$, so for $t \neq 2$, $H(t) = 2 + t$. The domain is $\{t \mid t \neq 2\}$. So the graph of H is the same as the graph of the function $f(t) = t + 2$ (a line) except for the hole at $(2, 4)$.

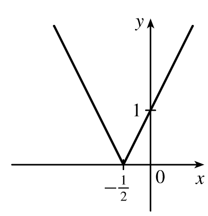


45. $g(x) = \sqrt{x-5}$ is defined when $x - 5 \geq 0$ or $x \geq 5$, so the domain is $[5, \infty)$. Since $y = \sqrt{x-5} \quad \Rightarrow \quad y^2 = x - 5 \quad \Rightarrow \quad x = y^2 + 5$, we see that g is the top half of a parabola.

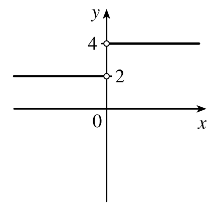


46. $$F(x) = |2x+1| = \begin{cases} 2x+1 & \text{if } 2x+1 \geq 0 \\ -(2x+1) & \text{if } 2x+1 < 1 \end{cases}$$
$$= \begin{cases} 2x+1 & \text{if } x \geq -\frac{1}{2} \\ -2x-1 & \text{if } x < -\frac{1}{2} \end{cases}$$

The domain is $\mathbb{R}$, or $(-\infty, \infty)$.



47. $G(x) = \dfrac{3x + |x|}{x}$. Since $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$, we have

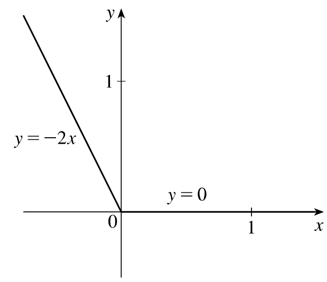


$$G(x) = \begin{cases} \dfrac{3x + x}{x} & \text{if } x > 0 \\ \dfrac{3x - x}{x} & \text{if } x < 0 \end{cases} = \begin{cases} \dfrac{4x}{x} & \text{if } x > 0 \\ \dfrac{2x}{x} & \text{if } x < 0 \end{cases} = \begin{cases} 4 & \text{if } x > 0 \\ 2 & \text{if } x < 0 \end{cases}$$

Note that G is not defined for $x = 0$. The domain is $(-\infty, 0) \cup (0, \infty)$.

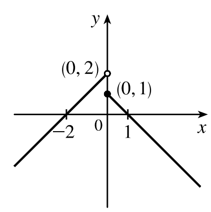
48. $g(x) = |x| - x = \begin{cases} x - x & \text{if } x \geq 0 \\ -x - x & \text{if } x < 0 \end{cases} = \begin{cases} 0 & \text{if } x \geq 0 \\ -2x & \text{if } x < 0 \end{cases}$.

The domain is $\mathbb{R}$, or $(-\infty, \infty)$.



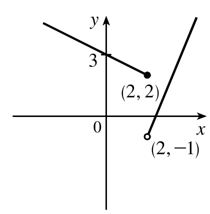
49. $f(x) = \begin{cases} x + 2 & \text{if } x < 0 \\ 1 - x & \text{if } x \geq 0 \end{cases}$

The domain is $\mathbb{R}$.



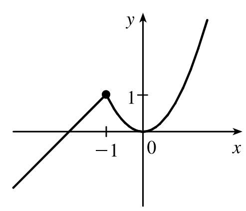
50. $f(x) = \begin{cases} 3 - \frac{1}{2}x & \text{if } x \leq 2 \\ 2x - 5 & \text{if } x > 2 \end{cases}$

The domain is $\mathbb{R}$.



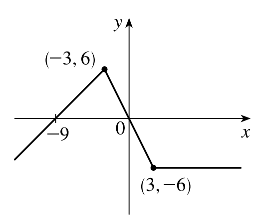
51. $f(x) = \begin{cases} x + 2 & \text{if } x \leq -1 \\ x^2 & \text{if } x > -1 \end{cases}$

Note that for $x = -1$, both $x + 2$ and x^2 are equal to 1. The domain is $\mathbb{R}$.



52. $f(x) = \begin{cases} x + 9 & \text{if } x < -3 \\ -2x & \text{if } |x| \leq 3 \\ -6 & \text{if } x > 3 \end{cases}$

Note that for $x = -3$, both $x + 9$ and $-2x$ are equal to 6; and for $x = 3$, both $-2x$ and -6 are equal to -6. The domain is $\mathbb{R}$.



53. Let the length and width of the rectangle be L and W. Then the perimeter is $2L + 2W = 20$ and the area is $A = LW$. Solving the first equation for W in terms of L gives $W = \dfrac{20 - 2L}{2} = 10 - L$. Thus, $A(L) = L(10 - L) = 10L - L^2$. Since lengths are positive, the domain of A is $0 < L < 10$. If we further restrict L to be larger than W, then $5 < L < 10$ would be the domain.

54. Let the length and width of the rectangle be L and W. Then the area is $LW = 16$, so that $W = 16/L$. The perimeter is $P = 2L + 2W$, so $P(L) = 2L + 2(16/L) = 2L + 32/L$, and the domain of P is $L > 0$, since lengths must be positive quantities. If we further restrict L to be larger than W, then $L > 4$ would be the domain.

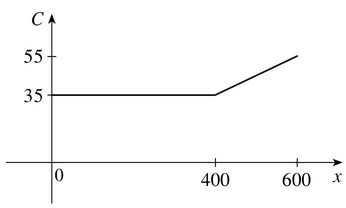
55. Let the length of a side of the equilateral triangle be x. Then by the Pythagorean Theorem, the height y of the triangle satisfies $y^2 + \left(\frac{1}{2}x\right)^2 = x^2$, so that $y^2 = x^2 - \frac{1}{4}x^2 = \frac{3}{4}x^2$ and $y = \frac{\sqrt{3}}{2}x$. Using the formula for the area A of a triangle, $A = \frac{1}{2}(\text{base})(\text{height})$, we obtain $A(x) = \frac{1}{2}(x)\left(\frac{\sqrt{3}}{2}x\right) = \frac{\sqrt{3}}{4}x^2$, with domain $x > 0$.

56. Let the volume of the cube be V and the length of an edge be L. Then $V = L^3$ so $L = \sqrt[3]{V}$, and the surface area is $S(V) = 6L^2 = 6\left(\sqrt[3]{V}\right)^2 = 6V^{2/3}$, with domain $V > 0$.

57. Let each side of the base of the box have length x, and let the height of the box be h. Since the volume is 2, we know that $2 = hx^2$, so that $h = 2/x^2$, and the surface area is $S = x^2 + 4xh$. Thus, $S(x) = x^2 + 4x(2/x^2) = x^2 + (8/x)$, with domain $x > 0$.

58. We can summarize the monthly cost with a piecewise defined function.

$$C(x) = \begin{cases} 35 & \text{if } 0 \le x \le 400 \\ 35 + 0.10(x - 400) & \text{if } x > 400 \end{cases}$$



59. We can summarize the total cost with a piecewise defined function.

$$T(x) = \begin{cases} 75x & \text{if } 0 < x \le 2 \\ 150 + 50(x - 2) & \text{if } x > 2 \end{cases}$$

60. One example is the amount paid for cable or telephone system repair in the home, usually measured to the nearest quarter hour. Another example is the amount paid by a student in tuition fees, if the fees vary according to the number of credits for which the student has registered.

61. The period can be estimated by measuring the peak-to-peak distance on the graph. This is approximately 77 hours. Note that the graph shown is for a single person's temperature. The period for this species of malaria is, on average, 72 hours.

62. The cycle of increased body temperature followed by a drop in temperature is indicative of a recurrent fever. This is typical of a *P. falciparum* infection. The period is approximately 48 hours, but the fever is also subsiding. This might be because the person is being treated for infection.

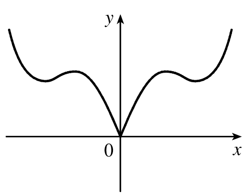
63. f is an odd function because its graph is symmetric about the origin. g is an even function because its graph is symmetric with respect to the y-axis.

64. f is not an even function since it is not symmetric with respect to the y-axis. f is not an odd function since it is not symmetric about the origin. Hence, f is *neither* even nor odd. g is an even function because its graph is symmetric with respect to the y-axis.

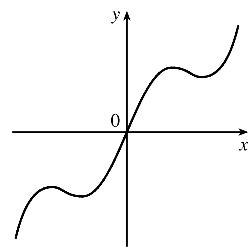
65. (a) Because an even function is symmetric with respect to the y-axis, and the point $(5, 3)$ is on the graph of this even function, the point $(-5, 3)$ must also be on its graph.

(b) Because an odd function is symmetric with respect to the origin, and the point $(5, 3)$ is on the graph of this odd function, the point $(-5, -3)$ must also be on its graph.

66. (a) If f is even, we get the rest of the graph by reflecting about the y-axis.



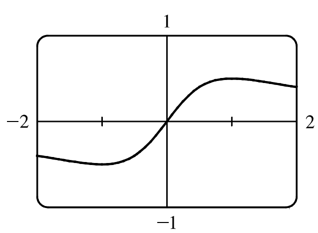
(b) If f is odd, we get the rest of the graph by rotating $180°$ about the origin.



67. $f(x) = \dfrac{x}{x^2+1}$.

$$f(-x) = \frac{-x}{(-x)^2+1} = \frac{-x}{x^2+1} = -\frac{x}{x^2+1} = -f(x).$$

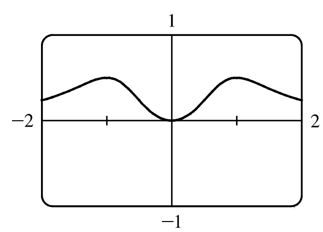
So f is an odd function.



68. $f(x) = \dfrac{x^2}{x^4+1}$.

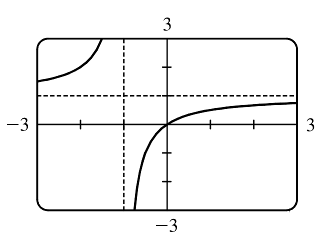
$$f(-x) = \frac{(-x)^2}{(-x)^4+1} = \frac{x^2}{x^4+1} = f(x).$$

So f is an even function.



69. $f(x) = \dfrac{x}{x+1}$, so $f(-x) = \dfrac{-x}{-x+1} = \dfrac{x}{x-1}$.

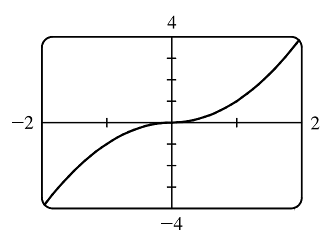
Since this is neither $f(x)$ nor $-f(x)$, the function f is neither even nor odd.



70. $f(x) = x\,|x|$.

$$\begin{aligned} f(-x) &= (-x)\,|-x| = (-x)\,|x| = -(x\,|x|) \\ &= -f(x) \end{aligned}$$

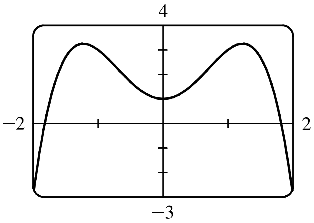
So f is an odd function.



71. $f(x) = 1 + 3x^2 - x^4$.

$f(-x) = 1+3(-x)^2-(-x)^4 = 1+3x^2-x^4 = f(x)$.

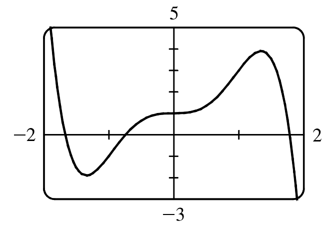
So f is an even function.



72. $f(x) = 1 + 3x^3 - x^5$, so

$$f(-x) = 1 + 3(-x)^3 - (-x)^5 = 1 + 3(-x^3) - (-x^5) = 1 - 3x^3 + x^5$$

Since this is neither $f(x)$ nor $-f(x)$, the function f is neither even nor odd.



73. (i) If f and g are both even functions, then $f(-x) = f(x)$ and $g(-x) = g(x)$. Now $(f+g)(-x) = f(-x) + g(-x) = f(x) + g(x) = (f+g)(x)$, so $f+g$ is an *even* function.

(ii) If f and g are both odd functions, then $f(-x) = -f(x)$ and $g(-x) = -g(x)$. Now $(f+g)(-x) = f(-x) + g(-x) = -f(x) + [-g(x)] = -[f(x) + g(x)] = -(f+g)(x)$, so $f+g$ is an *odd* function.

(iii) If f is an even function and g is an odd function, then $(f+g)(-x) = f(-x) + g(-x) = f(x) + [-g(x)] = f(x) - g(x)$, which is not $(f+g)(x)$ nor $-(f+g)(x)$, so $f+g$ is *neither* even nor odd. (Exception: if f is the zero function, then $f+g$ will be *odd*. If g is the zero function, then $f+g$ will be *even*.)

74. (i) If f and g are both even functions, then $f(-x) = f(x)$ and $g(-x) = g(x)$. Now $(fg)(-x) = f(-x)g(-x) = f(x)g(x) = (fg)(x)$, so fg is an *even* function.

(ii) If f and g are both odd functions, then $f(-x) = -f(x)$ and $g(-x) = -g(x)$. Now $(fg)(-x) = f(-x)g(-x) = [-f(x)][-g(x)] = f(x)g(x) = (fg)(x)$, so fg is an *even* function.

(iii) If f is an even function and g is an odd function, then $(fg)(-x) = f(-x)g(-x) = f(x)[-g(x)] = -[f(x)g(x)] = -(fg)(x)$, so fg is an *odd* function.

1.2 Mathematical Models: A Catalog of Essential Functions

1. (a) $f(x) = \log_2 x$ is a logarithmic function.

(b) $g(x) = \sqrt[4]{x}$ is a root function with $n = 4$.

(c) $h(x) = \dfrac{2x^3}{1 - x^2}$ is a rational function because it is a ratio of polynomials.

(d) $u(t) = 1 - 1.1t + 2.54t^2$ is a polynomial of degree 2 (also called a *quadratic function*).

(e) $v(t) = 5^t$ is an exponential function.

(f) $w(\theta) = \sin\theta\,\cos^2\theta$ is a trigonometric function.

2. (a) $y = \pi^x$ is an exponential function (notice that x is the *exponent*).

(b) $y = x^\pi$ is a power function (notice that x is the *base*).

(c) $y = x^2(2 - x^3) = 2x^2 - x^5$ is a polynomial of degree 5.

(d) $y = \tan t - \cos t$ is a trigonometric function.

(e) $y = s/(1 + s)$ is a rational function because it is a ratio of polynomials.

(f) $y = \sqrt{x^3 - 1}/(1 + \sqrt[3]{x})$ is an algebraic function because it involves polynomials and roots of polynomials.

3. We notice from the figure that g and h are even functions (symmetric with respect to the y-axis) and that f is an odd function (symmetric with respect to the origin). So (b) $\left[y = x^5\right]$ must be f. Since g is flatter than h near the origin, we must have (c) $\left[y = x^8\right]$ matched with g and (a) $\left[y = x^2\right]$ matched with h.

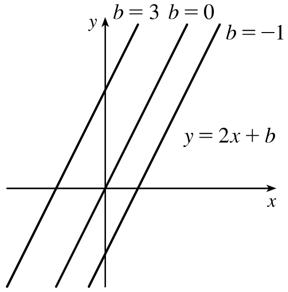
4. (a) The graph of $y = 3x$ is a line (choice G).

(b) $y = 3^x$ is an exponential function (choice f).

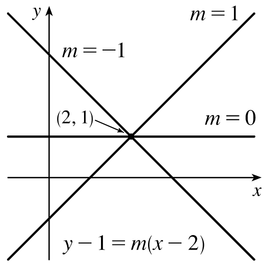
(c) $y = x^3$ is an odd polynomial function or power function (choice F).

(d) $y = \sqrt[3]{x} = x^{1/3}$ is a root function (choice g).

5. (a) An equation for the family of linear functions with slope 2 is $y = f(x) = 2x + b$, where b is the y-intercept.

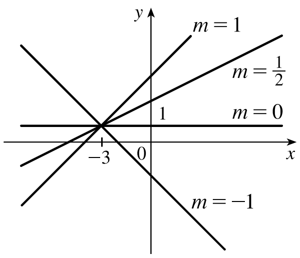


(b) $f(2) = 1$ means that the point $(2, 1)$ is on the graph of f. We can use the point-slope form of a line to obtain an equation for the family of linear functions through the point $(2, 1)$. $y - 1 = m(x - 2)$, which is equivalent to $y = mx + (1 - 2m)$ in slope-intercept form.

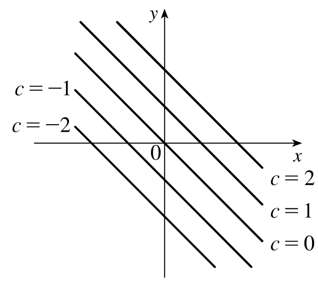


(c) To belong to both families, an equation must have slope $m = 2$, so the equation in part (b), $y = mx + (1 - 2m)$, becomes $y = 2x - 3$. It is the *only* function that belongs to both families.

6. All members of the family of linear functions $f(x) = 1 + m(x + 3)$ have graphs that are lines passing through the point $(-3, 1)$.



7. All members of the family of linear functions $f(x) = c - x$ have graphs that are lines with slope -1. The y-intercept is c.



8. The vertex of the parabola on the left is $(3, 0)$, so an equation is $y = a(x-3)^2 + 0$. Since the point $(4, 2)$ is on the parabola, we'll substitute 4 for x and 2 for y to find a. $2 = a(4-3)^2 \Rightarrow a = 2$, so an equation is $f(x) = 2(x-3)^2$.

   The y-intercept of the parabola on the right is $(0, 1)$, so an equation is $y = ax^2 + bx + 1$. Since the points $(-2, 2)$ and $(1, -2.5)$ are on the parabola, we'll substitute -2 for x and 2 for y as well as 1 for x and -2.5 for y to obtain two equations with the unknowns a and b.

   $(-2, 2)$: $\quad 2 = 4a - 2b + 1 \Rightarrow 4a - 2b = 1$ **(1)**

   $(1, -2.5)$: $\quad -2.5 = a + b + 1 \Rightarrow a + b = -3.5$ **(2)**

   $2 \cdot$ **(2)** $+$ **(1)** gives us $6a = -6 \Rightarrow a = -1$. From **(2)**, $-1 + b = -3.5 \Rightarrow b = -2.5$, so an equation is $g(x) = -x^2 - 2.5x + 1$.

9. Since $f(-1) = f(0) = f(2) = 0$, f has zeros of -1, 0, and 2, so an equation for f is $f(x) = a[x-(-1)](x-0)(x-2)$, or $f(x) = ax(x+1)(x-2)$. Because $f(1) = 6$, we'll substitute 1 for x and 6 for $f(x)$.
   $6 = a(1)(2)(-1) \Rightarrow -2a = 6 \Rightarrow a = -3$, so an equation for f is $f(x) = -3x(x+1)(x-2)$.

10. (a) For $T = 0.02t + 8.50$, the slope is 0.02, which means that the average surface temperature of the world is increasing at a rate of $0.02\,^\circ\mathrm{C}$ per year. The T-intercept is 8.50, which represents the average surface temperature in $^\circ\mathrm{C}$ in the year 1900.

    (b) $t = 2100 - 1900 = 200 \Rightarrow T = 0.02(200) + 8.50 = 12.50\,^\circ\mathrm{C}$

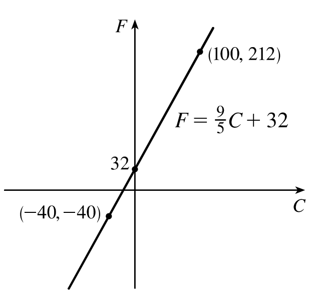
11. (a) $D = 200$, so $c = 0.0417D(a+1) = 0.0417(200)(a+1) = 8.34a + 8.34$. The slope is 8.34, which represents the change in mg of the dosage for a child for each change of 1 year in age.

    (b) For a newborn, $a = 0$, so $c = 8.34$ mg.

12. (a) We are given $\dfrac{\text{change in pressure}}{\text{10 feet change in depth}} = \dfrac{4.34}{10} = 0.434$. Using P for pressure and d for depth with the point $(d, P) = (0, 15)$, we have the slope-intercept form of the line, $P = 0.434d + 15$.

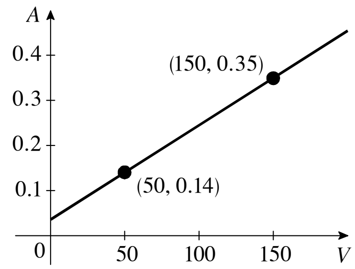
    (b) When $P = 100$, then $100 = 0.434d + 15 \Leftrightarrow 0.434d = 85 \Leftrightarrow d = \frac{85}{0.434} \approx 195.85$ feet. Thus, the pressure is $100\ \mathrm{lb/in}^2$ at a depth of approximately 196 feet.

13. (a)



    (b) The slope of $\frac{9}{5}$ means that F increases $\frac{9}{5}$ degrees for each increase of $1^\circ\mathrm{C}$. (Equivalently, F increases by 9 when C increases by 5 and F decreases by 9 when C decreases by 5.) The F-intercept of 32 is the Fahrenheit temperature corresponding to a Celsius temperature of 0.

14. (a) Assuming A is a linear function of V, we can sketch the graph of $A(V)$ by plotting the points $(150, 0.35)$ and $(50, 0.14)$ and drawing the straight line that passes through both these points.



(b) The slope is $m = \dfrac{A(150) - A(50)}{150 - 50} = \dfrac{0.35 - 0.14}{100} = 0.0021 \text{ min}^{-1}$. This represents the rate of change of absorption rate with respect to volume. The slope of 0.0021 means that A increases by 0.0021 mL/min for each 1 mL increase in V.

(c) The A-intercept of 0.035 mL/min is the absorption rate corresponding to a cerebrospinal fluid volume of 0 mL.

15. (a) Using N in place of x and T in place of y, we find the slope to be $\dfrac{T_2 - T_1}{N_2 - N_1} = \dfrac{80 - 70}{173 - 113} = \dfrac{10}{60} = \dfrac{1}{6}$. So a linear equation is $T - 80 = \frac{1}{6}(N - 173) \quad\Leftrightarrow\quad T - 80 = \frac{1}{6}N - \frac{173}{6} \quad\Leftrightarrow\quad T = \frac{1}{6}N + \frac{307}{6} \quad \left[\frac{307}{6} = 51.1\overline{6}\right]$.

(b) The slope of $\frac{1}{6}$ means that the temperature in Fahrenheit degrees increases one-sixth as rapidly as the number of cricket chirps per minute. Said differently, each increase of 6 cricket chirps per minute corresponds to an increase of 1°F.

(c) When $N = 150$, the temperature is given approximately by $T = \frac{1}{6}(150) + \frac{307}{6} = 76.1\overline{6}\,^\circ\text{F} \approx 76\,^\circ\text{F}$.

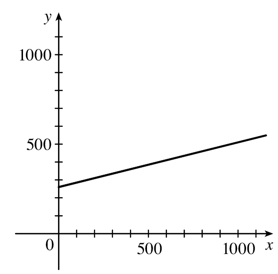
16. (a) Using d in place of x and C in place of y, we find the slope to be $\dfrac{C_2 - C_1}{d_2 - d_1} = \dfrac{460 - 380}{800 - 480} = \dfrac{80}{320} = \dfrac{1}{4}$.

So a linear equation is $C - 460 = \frac{1}{4}(d - 800) \quad\Leftrightarrow\quad C - 460 = \frac{1}{4}d - 200 \quad\Leftrightarrow\quad C = \frac{1}{4}d + 260$.

(b) Letting $d = 1500$ we get $C = \frac{1}{4}(1500) + 260 = 635$.

The cost of driving 1500 miles is \$635.

(c)



The slope of the line represents the cost per mile, \$0.25.

(d) The y-intercept represents the fixed cost, \$260.

(e) A linear function gives a suitable model in this situation because you have fixed monthly costs such as insurance and car payments, as well as costs that increase as you drive, such as gasoline, oil, and tires, and the cost of these for each additional mile driven is a constant.

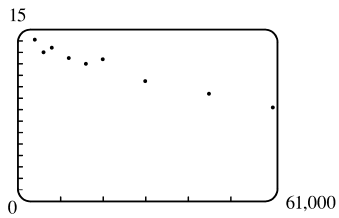
17. (a) The data appear to be periodic and a sine or cosine function would make the best model. A model of the form $f(x) = a\cos(bx) + c$ seems appropriate.

(b) The data appear to be decreasing in a linear fashion. A model of the form $f(x) = mx + b$ seems appropriate.

18. (a) The data appear to be increasing exponentially. A model of the form $f(x) = a \cdot b^x$ or $f(x) = a \cdot b^x + c$ seems appropriate.

(b) The data appear to be decreasing similarly to the values of the reciprocal function. A model of the form $f(x) = a/x$ seems appropriate.

19. (a)

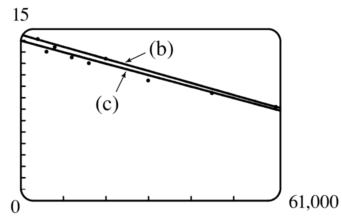


A linear model does seem appropriate.

(b) Using the points $(4000, 14.1)$ and $(60{,}000, 8.2)$, we obtain

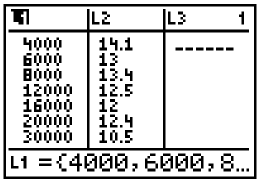
$y - 14.1 = \dfrac{8.2 - 14.1}{60{,}000 - 4000}(x - 4000)$ or, equivalently,

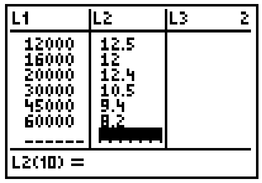
$y \approx -0.000105357x + 14.521429$.



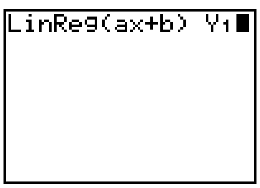
(c) Using a computing device, we obtain the least squares regression line $y = -0.0000997855x + 13.950764$. The following commands and screens illustrate how to find the least squares regression line on a TI-84 Plus.

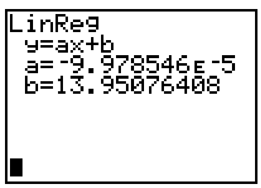
Enter the data into list one (L1) and list two (L2). Press [STAT] [1] to enter the editor.

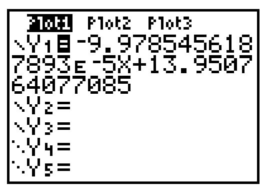




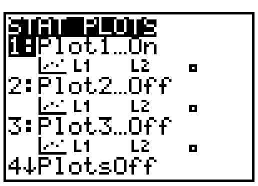
Find the regession line and store it in Y_1. Press [2nd] [QUIT] [STAT] [▶] [4] [VARS] [▶] [1] [1] [ENTER].

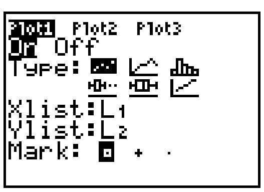




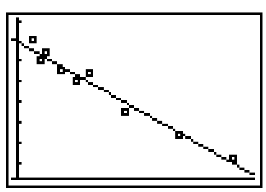


Note from the last figure that the regression line has been stored in Y_1 and that Plot1 has been turned on (Plot1 is highlighted). You can turn on Plot1 from the Y= menu by placing the cursor on Plot1 and pressing [ENTER] or by pressing [2nd] [STAT PLOT] [1] [ENTER].





Now press [ZOOM] [9] to produce a graph of the data and the regression line. Note that choice 9 of the ZOOM menu automatically selects a window that displays all of the data.

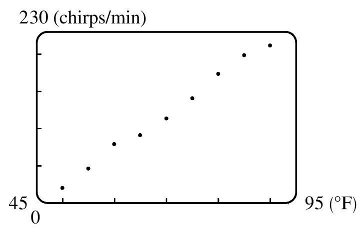


(d) When $x = 25{,}000$, $y \approx 11.456$; or about 11.5 per 100 population.

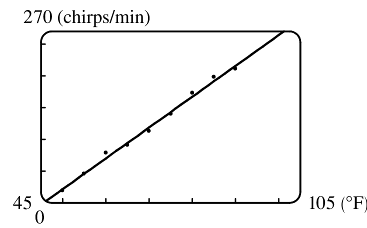
(e) When $x = 80{,}000$, $y \approx 5.968$; or about a 6% chance.

(f) When $x = 200{,}000$, y is negative, so the model does not apply.

20. (a)



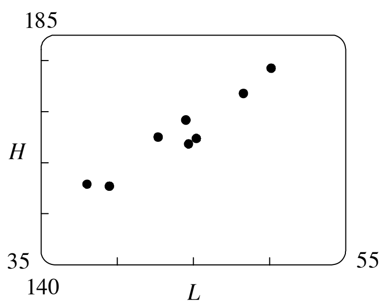
(b)



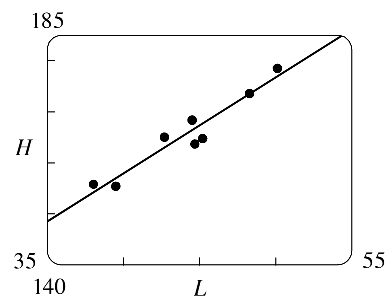
Using a computing device, we obtain the least squares regression line $y = 4.85\overline{6}x - 220.9\overline{6}$.

(c) When $x = 100^\circ\text{F}$, $y = 264.7 \approx 265$ chirps/min.

21. (a)



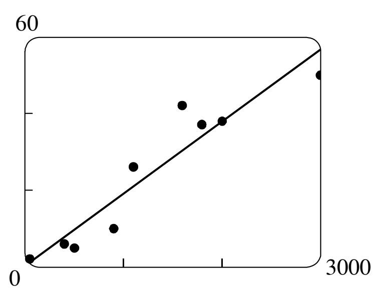
(b) Using a calculator to perform a linear regression gives $H = 1.8807L + 82.6497$ where H is the height in centimeters and L is the femur length in centimeters. This line, having slope 1.88 and H-intercept 82.65, is plotted below.



(c) The height of a person with $L = 53$ is $H(53) = (1.8807)(53) + 82.6497 \approx 182.3\,\text{cm}$.

22. (a) Using a calculator to perform a linear regression gives $y = 0.0188x + 0.3048$.

(b) The plot shows that the data is approximately linear. A higher degree polynomial fit, such as a cubic, may better model the data.



(c) The y-intercept represents the percentage of mice that developed tumors without any asbestos exposure.

23. If x is the original distance from the source, then the illumination is $f(x) = kx^{-2} = k/x^2$. Moving halfway to the lamp gives us an illumination of $f\left(\frac{1}{2}x\right) = k\left(\frac{1}{2}x\right)^{-2} = k(2/x)^2 = 4(k/x^2)$, so the light is 4 times as bright.

24. (a) Set $L = 90\,\text{in}$ and solve for W: $90 = 30.6W^{0.3952} \iff \frac{90}{30.6} = W^{0.3952} \iff W = \left(\frac{90}{30.6}\right)^{1/0.3952} \approx 15.33\,\text{lb}$

(b) Set $W = 300\,\text{lb}$ and calculate: $L = 30.6\,(300)^{0.3952} \approx 291.5\,\text{in}$

(c) According to the model, a 300 lb ostrich needs a wingspan of 292 in to fly. Therefore, an ostrich with a 72 in wingspan cannot generate enough lift for flight.

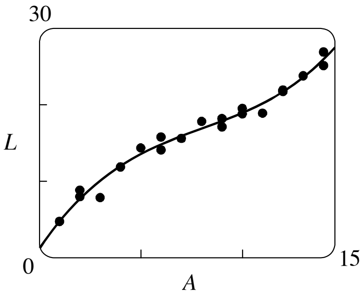
25. (a) Using a computing device, we obtain a power function $N = cA^b$, where $c \approx 3.1046$ and $b \approx 0.308$.

(b) If $A = 291$, then $N = cA^b \approx 17.8$, so you would expect to find 18 species of reptiles and amphibians on Dominica.

26. (a) $T = 1.000\,431\,227d^{\,1.499\,528\,750}$

(b) The power model in part (a) is approximately $T = d^{1.5}$. Squaring both sides gives us $T^2 = d^3$, so the model matches Kepler's Third Law, $T^2 = kd^3$.

27. (a) Using a calculator to perform a 3rd-degree polynomial regression gives $L = 0.0155A^3 - 0.3725A^2 + 3.9461A + 1.2108$ where A is age and L is length. This polynomial is plotted along with a scatterplot of the data.



(b) A 5-year old rock bass has a length of $L(5) = (0.0155)(5)^3 - (0.3725)(5)^2 + (3.9461)(5) + 1.2108 \approx 13.6$ in

(c) Using computer algebra software to solve for A in the equation $20 = 0.0155A^3 - 0.3725A^2 + 3.9461A + 1.2108$ gives $A \approx 10.88$ years. Alternatively, the graph from part (a) can be used to estimate the age when $L = 20$ by drawing a horizontal line at $L = 20$ to the curve and observing the age at this point.

1.3 New Functions from Old Functions

1. (a) If the graph of f is shifted 3 units upward, its equation becomes $y = f(x) + 3$.

(b) If the graph of f is shifted 3 units downward, its equation becomes $y = f(x) - 3$.

(c) If the graph of f is shifted 3 units to the right, its equation becomes $y = f(x - 3)$.

(d) If the graph of f is shifted 3 units to the left, its equation becomes $y = f(x + 3)$.

(e) If the graph of f is reflected about the x-axis, its equation becomes $y = -f(x)$.

(f) If the graph of f is reflected about the y-axis, its equation becomes $y = f(-x)$.

(g) If the graph of f is stretched vertically by a factor of 3, its equation becomes $y = 3f(x)$.

(h) If the graph of f is shrunk vertically by a factor of 3, its equation becomes $y = \frac{1}{3}f(x)$.

2. (a) To obtain the graph of $y = f(x) + 8$ from the graph of $y = f(x)$, shift the graph 8 units upward.

(b) To obtain the graph of $y = f(x + 8)$ from the graph of $y = f(x)$, shift the graph 8 units to the left.

(c) To obtain the graph of $y = 8f(x)$ from the graph of $y = f(x)$, stretch the graph vertically by a factor of 8.

(d) To obtain the graph of $y = f(8x)$ from the graph of $y = f(x)$, shrink the graph horizontally by a factor of 8.

(e) To obtain the graph of $y = -f(x) - 1$ from the graph of $y = f(x)$, first reflect the graph about the x-axis, and then shift it 1 unit downward.

(f) To obtain the graph of $y = 8f(\frac{1}{8}x)$ from the graph of $y = f(x)$, stretch the graph horizontally and vertically by a factor of 8.

3. (a) (graph 3) The graph of f is shifted 4 units to the right and has equation $y = f(x - 4)$.

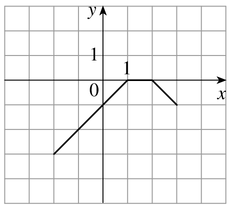
(b) (graph 1) The graph of f is shifted 3 units upward and has equation $y = f(x) + 3$.

(c) (graph 4) The graph of f is shrunk vertically by a factor of 3 and has equation $y = \frac{1}{3}f(x)$.

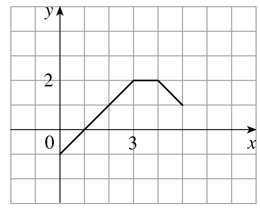
(d) (graph 5) The graph of f is shifted 4 units to the left and reflected about the x-axis. Its equation is $y = -f(x+4)$.

(e) (graph 2) The graph of f is shifted 6 units to the left and stretched vertically by a factor of 2. Its equation is $y = 2f(x+6)$.

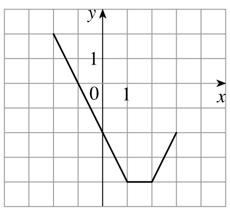
4. (a) To graph $y = f(x) - 2$, we shift the graph of f, 2 units downward.The point $(1, 2)$ on the graph of f corresponds to the point $(1, 2-2) = (1, 0)$.



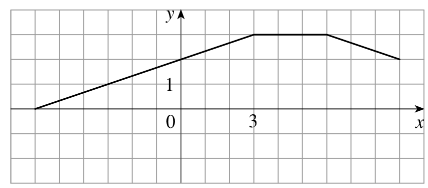
(b) To graph $y = f(x-2)$, we shift the graph of f, 2 units to the right.The point $(1, 2)$ on the graph of f corresponds to the point $(1+2, 2) = (3, 2)$.



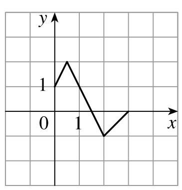
(c) To graph $y = -2f(x)$, we reflect the graph about the x-axis and stretch the graph vertically by a factor of 2. The point $(1, 2)$ on the graph of f corresponds to the point $(1, -2 \cdot 2) = (1, -4)$.



(d) To graph $y = f(\frac{1}{3}x) + 1$, we stretch the graph horizontally by a factor of 3 and shift it 1 unit upward. The point $(1, 2)$ on the graph of f corresponds to the point $(1 \cdot 3, 2+1) = (3, 3)$.

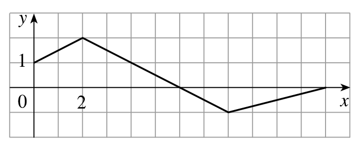


5. (a) To graph $y = f(2x)$ we shrink the graph of f horizontally by a factor of 2.



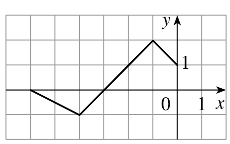
The point $(4, -1)$ on the graph of f corresponds to the point $\left(\frac{1}{2} \cdot 4, -1\right) = (2, -1)$.

(b) To graph $y = f\left(\frac{1}{2}x\right)$ we stretch the graph of f horizontally by a factor of 2.



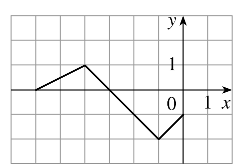
The point $(4, -1)$ on the graph of f corresponds to the point $(2 \cdot 4, -1) = (8, -1)$.

(c) To graph $y = f(-x)$ we reflect the graph of f about the y-axis.



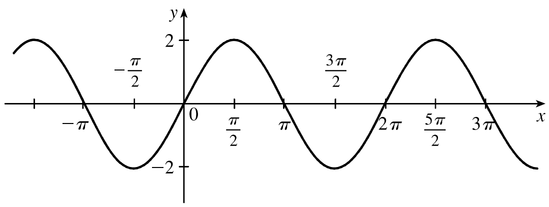
The point $(4, -1)$ on the graph of f corresponds to the point $(-1 \cdot 4, -1) = (-4, -1)$.

(d) To graph $y = -f(-x)$ we reflect the graph of f about the y-axis, then about the x-axis.

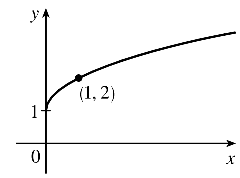


The point $(4, -1)$ on the graph of f corresponds to the point $(-1 \cdot 4, -1 \cdot -1) = (-4, 1)$.

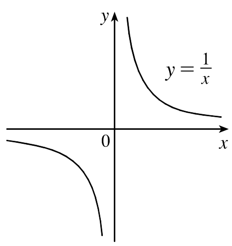
6. (a) The graph of $y = 2\sin x$ can be obtained from the graph of $y = \sin x$ by stretching it vertically by a factor of 2.

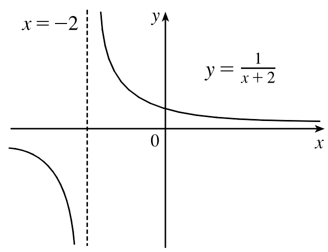


(b) The graph of $y = 1 + \sqrt{x}$ can be obtained from the graph of $y = \sqrt{x}$ by shifting it upward 1 unit.

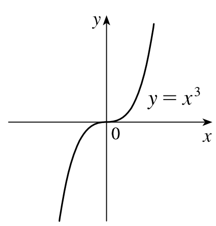


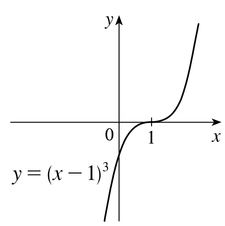
7. $y = \dfrac{1}{x+2}$: Start with the graph of the reciprocal function $y = 1/x$ and shift 2 units to the left.



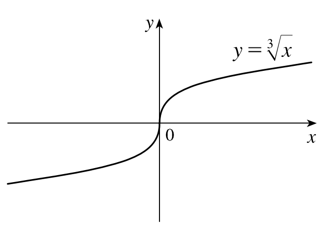


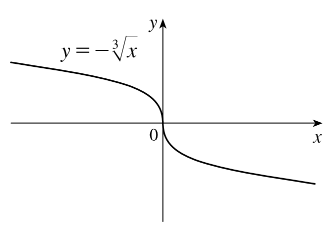
8. $y = (x-1)^3$: Start with the graph of $y = x^3$ and shift 1 unit to the right.



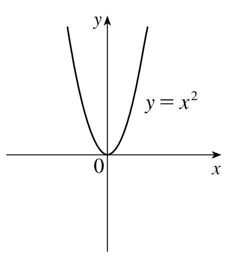


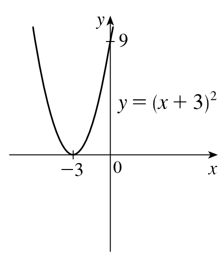
9. $y = -\sqrt[3]{x}$: Start with the graph of $y = \sqrt[3]{x}$ and reflect about the x-axis.

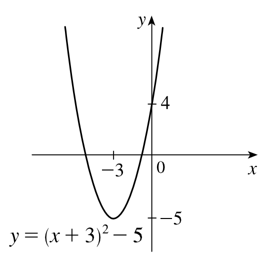




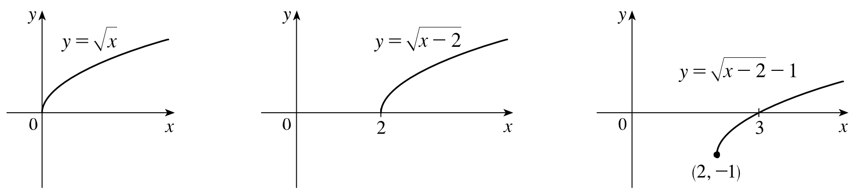
10. $y = x^2 + 6x + 4 = (x^2 + 6x + 9) - 5 = (x+3)^2 - 5$: Start with the graph of $y = x^2$, shift 3 units to the left, and then shift 5 units downward.



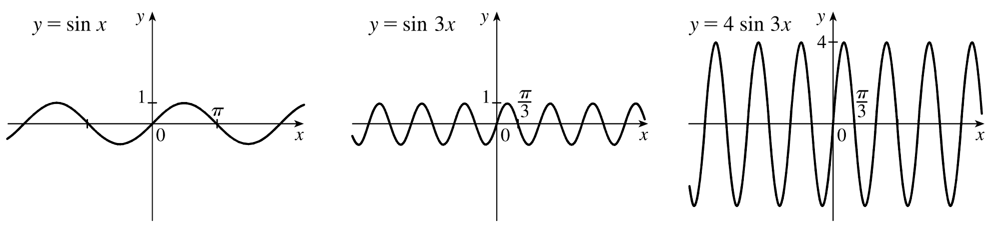




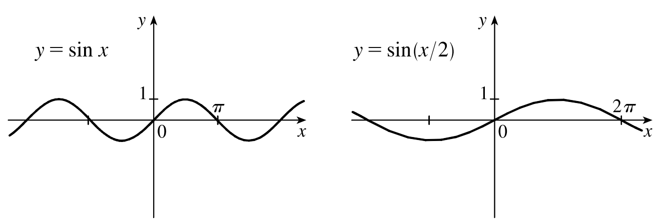
11. $y = \sqrt{x-2} - 1$: Start with the graph of $y = \sqrt{x}$, shift 2 units to the right, and then shift 1 unit downward.



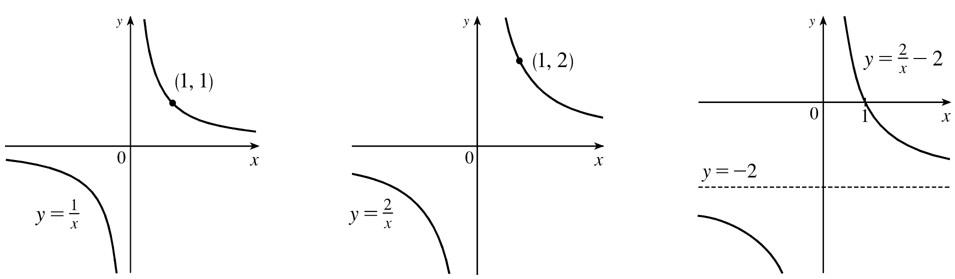
12. $y = 4\sin 3x$: Start with the graph of $y = \sin x$, compress horizontally by a factor of 3, and then stretch vertically by a factor of 4.



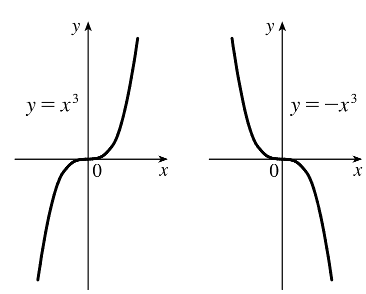
13. $y = \sin(x/2)$: Start with the graph of $y = \sin x$ and stretch horizontally by a factor of 2.



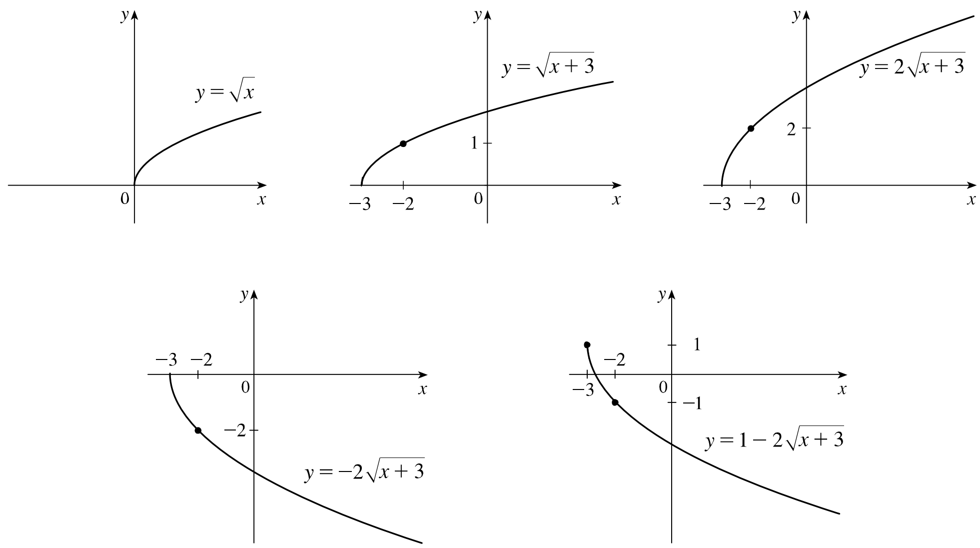
14. $y = \dfrac{2}{x} - 2$: Start with the graph of $y = \dfrac{1}{x}$, stretch vertically by a factor of 2, and then shift 2 units downward.



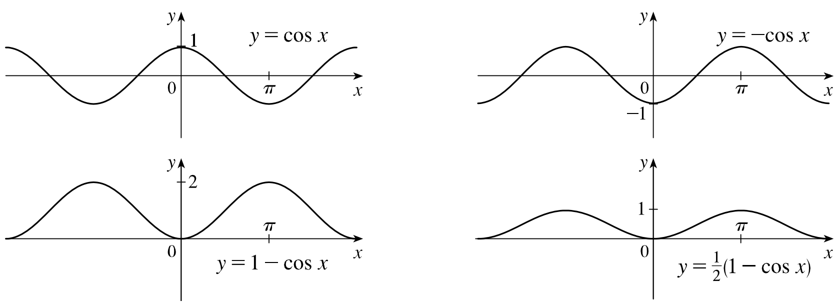
15. $y = -x^3$: Start with the graph of $y = x^3$ and reflect about the x-axis. Note: Reflecting about the y-axis gives the same result since substituting $-x$ for x gives us $y = (-x)^3 = -x^3$.



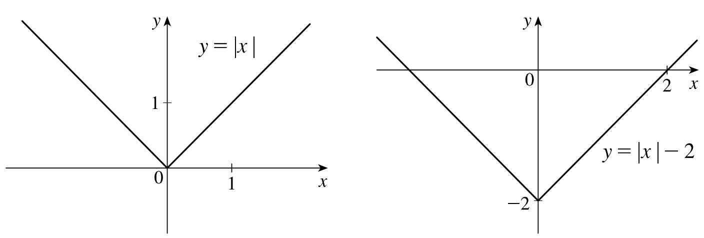
16. $y = 1 - 2\sqrt{x+3}$: Start with the graph of $y = \sqrt{x}$, shift 3 units to the left, stretch vertically by a factor of 2, reflect about the x-axis, and then shift 1 unit upward.



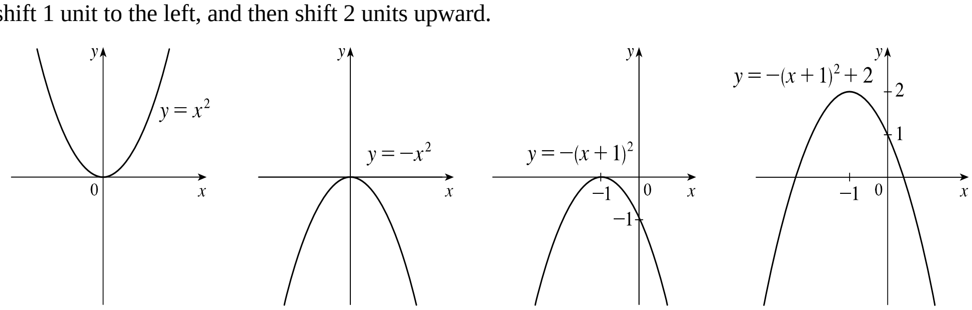
17. $y = \frac{1}{2}(1 - \cos x)$: Start with the graph of $y = \cos x$, reflect about the x-axis, shift 1 unit upward, and then shrink vertically by a factor of 2.



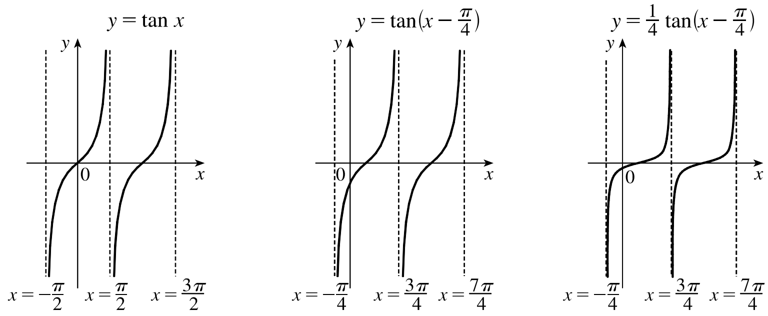
18. $y = |x| - 2$: Start with the graph of $y = |x|$ and shift 2 units downward.



19. $y = 1 - 2x - x^2 = -(x^2 + 2x) + 1 = -(x^2 + 2x + 1) + 2 = -(x+1)^2 + 2$: Start with the graph of $y = x^2$, reflect about the x-axis, shift 1 unit to the left, and then shift 2 units upward.



20. $y = \frac{1}{4}\tan(x - \frac{\pi}{4})$: Start with the graph of $y = \tan x$, shift $\frac{\pi}{4}$ units to the right, and then compress vertically by a factor of 4.



21. This is just like the solution to Example 4 except the amplitude of the curve (the 30°N curve in Figure 9 on June 21) is $14 - 12 = 2$. So the function is $L(t) = 12 + 2\sin\left[\frac{2\pi}{365}(t - 80)\right]$. March 31 is the 90th day of the year, so the model gives $L(90) \approx 12.34$ h. The daylight time (5:51 AM to 6:18 PM) is 12 hours and 27 minutes, or 12.45 h. The model value differs from the actual value by $\frac{12.45 - 12.34}{12.45} \approx 0.009$, less than 1%.

22. Using a sine function to model the brightness of Delta Cephei as a function of time, we take its period to be 5.4 days, its amplitude to be 0.35 (on the scale of magnitude), and its average magnitude to be 4.0. If we take $t = 0$ at a time of average brightness, then the magnitude (brightness) as a function of time t in days can be modeled by the formula $M(t) = 4.0 + 0.35\sin\left(\frac{2\pi}{5.4}t\right)$.

23. Let $D(t)$ be the water depth in meters at t hours after midnight. Apply the following transformations to the cosine function:
   - Vertical stretch by factor 5 since the amplitude needs to be $\frac{12-2}{2} = 5\,\text{m}$
   - Horizontal stretch by factor $\frac{12}{2\pi} = \frac{6}{\pi}$ since the period needs to be $12\,\text{h}$
   - Vertical shift 7 units upward since the function ranges between 2 and 12 which has a midpoint of $\frac{12+2}{2} = 7\,\text{m}$
   - Horizontal shift 6.75 units to right to position the maximum at $t = 6.75\,\text{h}$ (6:45AM)

   Combining these transformations gives the water depth function $D(t) = 5\cos\left(\frac{\pi}{6}(t - 6.75)\right) + 7$.

24. Let $V(t)$ be the total volume of air in mL after t seconds. Because the respiratory cycle is periodic, a sine function can be used as a model by applying the following transformations:
   - Vertical stretch by factor 250 since the amplitude needs to be $\frac{500}{2} = 250\,\text{mL}$
   - Horizontal stretch by factor $\frac{4}{2\pi} = \frac{2}{\pi}$ since the period needs to be $4\,\text{s}$
   - Vertical shift 2250 units upward since the function ranges between 2000 and 2500 which has a midpoint of $\frac{2000+2500}{2} = 2250\,\text{mL}$

   Combining these transformations gives the volume function $V(t) = 250\sin\left(\frac{\pi}{2}t\right) + 2250$.

25. Let $f(t)$ be the gene frequency after t years. The gene frequency dynamics can be modeled using a sine function with the following transformations:
   - Vertical stretch by factor 30 since the amplitude needs to be $\frac{80-20}{2} = 30\%$
   - Horizontal stretch by factor $\frac{3}{2\pi}$ since the period needs to be $3\,\text{years}$
   - Vertical shift 50 units upward since the function ranges between 80 and 20 which has a midpoint of $\frac{80+20}{2} = 50$

   Combining these transformations gives the gene frequency function $f(t) = 30\sin\left(\frac{2\pi}{3}t\right) + 50$.

26. Let $D(t)$ be the density of neutrophils in cells/μL after t days. The density is periodic and can be modeled using a cosine function with the following transformations:
   - Vertical stretch by factor 1000 since the amplitude needs to be $\frac{2000-0}{2} = 1000$
   - Horizontal stretch by factor $\frac{21}{2\pi}$ since the period needs to be 21 days (or 3 weeks)
   - Vertical shift 1000 units upward since the function ranges between 0 and 2000 which has a midpoint of $\frac{2000+0}{2} = 1000$

   Combining these transformations gives the density function $D(t) = 1000\cos\left(\frac{2\pi}{21}t\right) + 1000$.

27. $f(x) = x^3 + 2x^2$; $g(x) = 3x^2 - 1$. $D = \mathbb{R}$ for both f and g.

(a) $(f + g)(x) = (x^3 + 2x^2) + (3x^2 - 1) = x^3 + 5x^2 - 1,\ D = \mathbb{R}$.

(b) $(f - g)(x) = (x^3 + 2x^2) - (3x^2 - 1) = x^3 - x^2 + 1,\ D = \mathbb{R}$.

(c) $(fg)(x) = (x^3 + 2x^2)(3x^2 - 1) = 3x^5 + 6x^4 - x^3 - 2x^2,\ D = \mathbb{R}$.

(d) $\left(\dfrac{f}{g}\right)(x) = \dfrac{x^3 + 2x^2}{3x^2 - 1},\ D = \left\{x \mid x \neq \pm\dfrac{1}{\sqrt{3}}\right\}$ since $3x^2 - 1 \neq 0$.

28. $f(x) = \sqrt{3 - x},\ D = (-\infty, 3]$; $g(x) = \sqrt{x^2 - 1},\ D = (-\infty, -1] \cup [1, \infty)$.

(a) $(f + g)(x) = \sqrt{3 - x} + \sqrt{x^2 - 1},\ D = (-\infty, -1] \cup [1, 3]$, which is the intersection of the domains of f and g.

(b) $(f - g)(x) = \sqrt{3 - x} - \sqrt{x^2 - 1},\ D = (-\infty, -1] \cup [1, 3]$.

(c) $(fg)(x) = \sqrt{3 - x} \cdot \sqrt{x^2 - 1},\ D = (-\infty, -1] \cup [1, 3]$.

(d) $\left(\dfrac{f}{g}\right)(x) = \dfrac{\sqrt{3 - x}}{\sqrt{x^2 - 1}},\ D = (-\infty, -1) \cup (1, 3]$. We must exclude $x = \pm 1$ since these values would make $\dfrac{f}{g}$ undefined.

29. $f(x) = x^2 - 1,\ D = \mathbb{R}$; $g(x) = 2x + 1,\ D = \mathbb{R}$.

(a) $(f \circ g)(x) = f(g(x)) = f(2x + 1) = (2x + 1)^2 - 1 = (4x^2 + 4x + 1) - 1 = 4x^2 + 4x,\ D = \mathbb{R}$.

(b) $(g \circ f)(x) = g(f(x)) = g(x^2 - 1) = 2(x^2 - 1) + 1 = (2x^2 - 2) + 1 = 2x^2 - 1,\ D = \mathbb{R}$.

(c) $(f \circ f)(x) = f(f(x)) = f(x^2 - 1) = (x^2 - 1)^2 - 1 = (x^4 - 2x^2 + 1) - 1 = x^4 - 2x^2,\ D = \mathbb{R}$.

(d) $(g \circ g)(x) = g(g(x)) = g(2x + 1) = 2(2x + 1) + 1 = (4x + 2) + 1 = 4x + 3,\ D = \mathbb{R}$.

30. $f(x) = x - 2$; $g(x) = x^2 + 3x + 4$. $D = \mathbb{R}$ for both f and g, and hence for their composites.

(a) $(f \circ g)(x) = f(g(x)) = f(x^2 + 3x + 4) = (x^2 + 3x + 4) - 2 = x^2 + 3x + 2$.

(b) $(g \circ f)(x) = g(f(x)) = g(x - 2) = (x - 2)^2 + 3(x - 2) + 4 = x^2 - 4x + 4 + 3x - 6 + 4 = x^2 - x + 2$.

(c) $(f \circ f)(x) = f(f(x)) = f(x - 2) = (x - 2) - 2 = x - 4$.

(d)
$$
\begin{aligned}
(g \circ g)(x) = g(g(x)) = g(x^2 + 3x + 4) &= (x^2 + 3x + 4)^2 + 3(x^2 + 3x + 4) + 4 \\
&= (x^4 + 9x^2 + 16 + 6x^3 + 8x^2 + 24x) + 3x^2 + 9x + 12 + 4 \\
&= x^4 + 6x^3 + 20x^2 + 33x + 32
\end{aligned}
$$

31. $f(x) = 1 - 3x$; $g(x) = \cos x$. $D = \mathbb{R}$ for both f and g, and hence for their composites.

(a) $(f \circ g)(x) = f(g(x)) = f(\cos x) = 1 - 3\cos x$.

(b) $(g \circ f)(x) = g(f(x)) = g(1 - 3x) = \cos(1 - 3x)$.

(c) $(f \circ f)(x) = f(f(x)) = f(1 - 3x) = 1 - 3(1 - 3x) = 1 - 3 + 9x = 9x - 2$.

(d) $(g \circ g)(x) = g(g(x)) = g(\cos x) = \cos(\cos x)$ [Note that this is *not* $\cos x \cdot \cos x$.]

32. $f(x) = \sqrt{x},\ D = [0, \infty)$; $g(x) = \sqrt[3]{1 - x},\ D = \mathbb{R}$.

(a) $(f \circ g)(x) = f(g(x)) = f\left(\sqrt[3]{1 - x}\right) = \sqrt{\sqrt[3]{1 - x}} = \sqrt[6]{1 - x}$.

The domain of $f \circ g$ is $\{x \mid \sqrt[3]{1 - x} \geq 0\} = \{x \mid 1 - x \geq 0\} = \{x \mid x \leq 1\} = (-\infty, 1]$.

(b) $(g \circ f)(x) = g(f(x)) = g(\sqrt{x}) = \sqrt[3]{1 - \sqrt{x}}$.
The domain of $g \circ f$ is $\{x \mid x$ is in the domain of f and $f(x)$ is in the domain of $g\}$. This is the domain of f, that is, $[0, \infty)$.

(c) $(f \circ f)(x) = f(f(x)) = f(\sqrt{x}) = \sqrt{\sqrt{x}} = \sqrt[4]{x}$. The domain of $f \circ f$ is $\{x \mid x \geq 0$ and $\sqrt{x} \geq 0\} = [0, \infty)$.

(d) $(g \circ g)(x) = g(g(x)) = g\left(\sqrt[3]{1-x}\right) = \sqrt[3]{1-\sqrt[3]{1-x}}$, and the domain is $(-\infty, \infty)$.

33. $f(x) = x + \dfrac{1}{x},\ D = \{x \mid x \neq 0\};\quad g(x) = \dfrac{x+1}{x+2},\ D = \{x \mid x \neq -2\}$

(a) $(f \circ g)(x) = f(g(x)) = f\left(\dfrac{x+1}{x+2}\right) = \dfrac{x+1}{x+2} + \dfrac{1}{\dfrac{x+1}{x+2}} = \dfrac{x+1}{x+2} + \dfrac{x+2}{x+1}$

$$= \frac{(x+1)(x+1)+(x+2)(x+2)}{(x+2)(x+1)} = \frac{\left(x^2+2x+1\right)+\left(x^2+4x+4\right)}{(x+2)(x+1)} = \frac{2x^2+6x+5}{(x+2)(x+1)}$$

Since $g(x)$ is not defined for $x = -2$ and $f(g(x))$ is not defined for $x = -2$ and $x = -1$,

the domain of $(f \circ g)(x)$ is $D = \{x \mid x \neq -2, -1\}$.

(b) $(g \circ f)(x) = g(f(x)) = g\left(x + \dfrac{1}{x}\right) = \dfrac{\left(x+\dfrac{1}{x}\right)+1}{\left(x+\dfrac{1}{x}\right)+2} = \dfrac{\dfrac{x^2+1+x}{x}}{\dfrac{x^2+1+2x}{x}} = \dfrac{x^2+x+1}{x^2+2x+1} = \dfrac{x^2+x+1}{(x+1)^2}$

Since $f(x)$ is not defined for $x = 0$ and $g(f(x))$ is not defined for $x = -1$,
the domain of $(g \circ f)(x)$ is $D = \{x \mid x \neq -1, 0\}$.

(c) $(f \circ f)(x) = f(f(x)) = f\left(x + \dfrac{1}{x}\right) = \left(x + \dfrac{1}{x}\right) + \dfrac{1}{x + \frac{1}{x}} = x + \dfrac{1}{x} + \dfrac{1}{\frac{x^2+1}{x}} = x + \dfrac{1}{x} + \dfrac{x}{x^2+1}$

$$= \frac{x(x)\left(x^2+1\right)+1\left(x^2+1\right)+x(x)}{x(x^2+1)} = \frac{x^4+x^2+x^2+1+x^2}{x(x^2+1)}$$

$$= \frac{x^4+3x^2+1}{x(x^2+1)}, \quad D = \{x \mid x \neq 0\}$$

(d) $(g \circ g)(x) = g(g(x)) = g\left(\dfrac{x+1}{x+2}\right) = \dfrac{\dfrac{x+1}{x+2}+1}{\dfrac{x+1}{x+2}+2} = \dfrac{\dfrac{x+1+1(x+2)}{x+2}}{\dfrac{x+1+2(x+2)}{x+2}} = \dfrac{x+1+x+2}{x+1+2x+4} = \dfrac{2x+3}{3x+5}$

Since $g(x)$ is not defined for $x = -2$ and $g(g(x))$ is not defined for $x = -\frac{5}{3}$,

the domain of $(g \circ g)(x)$ is $D = \left\{x \mid x \neq -2, -\frac{5}{3}\right\}$.

34. $f(x) = \dfrac{x}{1+x},\ D = \{x \mid x \neq -1\};\quad g(x) = \sin 2x,\ D = \mathbb{R}$.

(a) $(f \circ g)(x) = f(g(x)) = f(\sin 2x) = \dfrac{\sin 2x}{1 + \sin 2x}$

Domain: $1 + \sin 2x \neq 0 \quad\Rightarrow\quad \sin 2x \neq -1 \quad\Rightarrow\quad 2x \neq \dfrac{3\pi}{2} + 2\pi n \quad\Rightarrow\quad x \neq \dfrac{3\pi}{4} + \pi n$ [n an integer].

(b) $(g \circ f)(x) = g(f(x)) = g\left(\dfrac{x}{1+x}\right) = \sin\left(\dfrac{2x}{1+x}\right)$.

Domain: $\{x \mid x \neq -1\}$

(c) $(f \circ f)(x) = f(f(x)) = f\left(\dfrac{x}{1+x}\right) = \dfrac{\dfrac{x}{1+x}}{1+\dfrac{x}{1+x}} = \dfrac{\left(\dfrac{x}{1+x}\right)\cdot(1+x)}{\left(1+\dfrac{x}{1+x}\right)\cdot(1+x)} = \dfrac{x}{1+x+x} = \dfrac{x}{2x+1}$

Since $f(x)$ is not defined for $x = -1$, and $f(f(x))$ is not defined for $x = -\frac{1}{2}$,

the domain of $(f \circ f)(x)$ is $D = \{x \mid x \neq -1, -\frac{1}{2}\}$.

(d) $(g \circ g)(g) = g(g(x)) = g(\sin 2x) = \sin(2 \sin 2x)$.
Domain: $\mathbb{R}$

35. $(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x^2)) = f(\sin(x^2)) = 3\sin(x^2) - 2$

36. $(f \circ g \circ h)(x) = f(g(h(x))) = f(g(\sqrt{x})) = f(2^{\sqrt{x}}) = \left|2^{\sqrt{x}} - 4\right|$

37. $(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x^3 + 2)) = f[(x^3 + 2)^2]$
$= f(x^6 + 4x^3 + 4) = \sqrt{(x^6 + 4x^3 + 4) - 3} = \sqrt{x^6 + 4x^3 + 1}$

38. $(f \circ g \circ h)(x) = f(g(h(x))) = f(g(\sqrt[3]{x})) = f\left(\dfrac{\sqrt[3]{x}}{\sqrt[3]{x} - 1}\right) = \tan\left(\dfrac{\sqrt[3]{x}}{\sqrt[3]{x} - 1}\right)$

39. Let $g(x) = 2x + x^2$ and $f(x) = x^4$. Then $(f \circ g)(x) = f(g(x)) = f(2x + x^2) = (2x + x^2)^4 = F(x)$.

40. Let $g(x) = \cos x$ and $f(x) = x^2$. Then $(f \circ g)(x) = f(g(x)) = f(\cos x) = (\cos x)^2 = \cos^2 x = F(x)$.

41. Let $g(x) = \sqrt[3]{x}$ and $f(x) = \dfrac{x}{1 + x}$. Then $(f \circ g)(x) = f(g(x)) = f(\sqrt[3]{x}) = \dfrac{\sqrt[3]{x}}{1 + \sqrt[3]{x}} = F(x)$.

42. Let $g(x) = \dfrac{x}{1 + x}$ and $f(x) = \sqrt[3]{x}$. Then $(f \circ g)(x) = f(g(x)) = f\left(\dfrac{x}{1 + x}\right) = \sqrt[3]{\dfrac{x}{1 + x}} = G(x)$.

43. Let $g(t) = t^2$ and $f(t) = \sec t \tan t$. Then $(f \circ g)(t) = f(g(t)) = f(t^2) = \sec(t^2)\tan(t^2) = v(t)$.

44. Let $g(t) = \tan t$ and $f(t) = \dfrac{t}{1 + t}$. Then $(f \circ g)(t) = f(g(t)) = f(\tan t) = \dfrac{\tan t}{1 + \tan t} = u(t)$.

45. Let $h(x) = \sqrt{x}$, $g(x) = x - 1$, and $f(x) = \sqrt{x}$. Then
$(f \circ g \circ h)(x) = f(g(h(x))) = f(g(\sqrt{x})) = f(\sqrt{x} - 1) = \sqrt{\sqrt{x} - 1} = R(x)$.

46. Let $h(x) = |x|$, $g(x) = 2 + x$, and $f(x) = \sqrt[8]{x}$. Then
$(f \circ g \circ h)(x) = f(g(h(x))) = f(g(|x|)) = f(2 + |x|) = \sqrt[8]{2 + |x|} = H(x)$.

47. Let $h(x) = \sqrt{x}$, $g(x) = \sec x$, and $f(x) = x^4$. Then
$(f \circ g \circ h)(x) = f(g(h(x))) = f(g(\sqrt{x})) = f(\sec\sqrt{x}) = (\sec\sqrt{x})^4 = \sec^4(\sqrt{x}) = H(x)$.

48. (a) $f(g(1)) = f(6) = 5$
(b) $g(f(1)) = g(3) = 2$
(c) $f(f(1)) = f(3) = 4$
(d) $g(g(1)) = g(6) = 3$
(e) $(g \circ f)(3) = g(f(3)) = g(4) = 1$
(f) $(f \circ g)(6) = f(g(6)) = f(3) = 4$

49. (a) $g(2) = 5$, because the point $(2, 5)$ is on the graph of g. Thus, $f(g(2)) = f(5) = 4$, because the point $(5, 4)$ is on the graph of f.

(b) $g(f(0)) = g(0) = 3$

(c) $(f \circ g)(0) = f(g(0)) = f(3) = 0$

(d) $(g \circ f)(6) = g(f(6)) = g(6)$. This value is not defined, because there is no point on the graph of g that has x-coordinate 6.

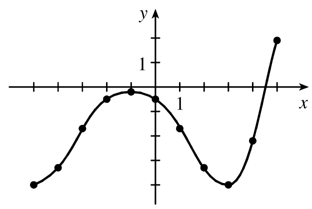
(e) $(g \circ g)(-2) = g(g(-2)) = g(1) = 4$

(f) $(f \circ f)(4) = f(f(4)) = f(2) = -2$

50. To find a particular value of $f(g(x))$, say for $x = 0$, we note from the graph that $g(0) \approx 2.8$ and $f(2.8) \approx -0.5$. Thus, $f(g(0)) \approx f(2.8) \approx -0.5$. The other values listed in the table were obtained in a similar fashion.

x	$g(x)$	$f(g(x))$
-5	-0.2	-4
-4	1.2	-3.3
-3	2.2	-1.7
-2	2.8	-0.5
-1	3	-0.2

x	$g(x)$	$f(g(x))$
0	2.8	-0.5
1	2.2	-1.7
2	1.2	-3.3
3	-0.2	-4
4	-1.9	-2.2
5	-4.1	1.9



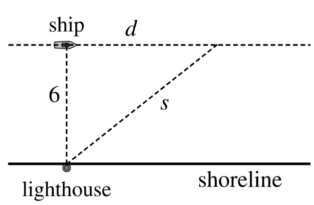
51. (a) Using the relationship *distance* = *rate* · *time* with the radius r as the distance, we have $r(t) = 60t$.

(b) $A = \pi r^2 \quad \Rightarrow \quad (A \circ r)(t) = A(r(t)) = \pi(60t)^2 = 3600\pi t^2$. This formula gives us the extent of the rippled area (in cm^2) at any time t.

52. (a) The radius r of the balloon is increasing at a rate of 2 cm/s, so $r(t) = (2\text{ cm/s})(t\text{ s}) = 2t$ (in cm).

(b) Using $V = \frac{4}{3}\pi r^3$, we get $(V \circ r)(t) = V(r(t)) = V(2t) = \frac{4}{3}\pi(2t)^3 = \frac{32}{3}\pi t^3$.

The result, $V = \frac{32}{3}\pi t^3$, gives the volume of the balloon (in cm^3) as a function of time (in s).



53. (a) From the figure, we have a right triangle with legs 6 and d, and hypotenuse s. By the Pythagorean Theorem, $d^2 + 6^2 = s^2 \quad \Rightarrow \quad s = f(d) = \sqrt{d^2 + 36}$.

(b) Using $d = rt$, we get $d = (30\text{ km/h})(t\text{ hours}) = 30t$ (in km). Thus, $d = g(t) = 30t$.

(c) $(f \circ g)(t) = f(g(t)) = f(30t) = \sqrt{(30t)^2 + 36} = \sqrt{900t^2 + 36}$. This function represents the distance between the lighthouse and the ship as a function of the time elapsed since noon.

54. (a) The passage of the drug through the body can be represented as inputs into the defined functions as follows:

$$x \xrightarrow[\text{dose}]{\text{oral}} h \xrightarrow[\text{stream}]{\text{blood}} g \xrightarrow[\text{infection}]{\text{site of}} f \longrightarrow \text{\# surviving bacteria}$$

Therefore, the amount of the drug that reaches the site of infection is

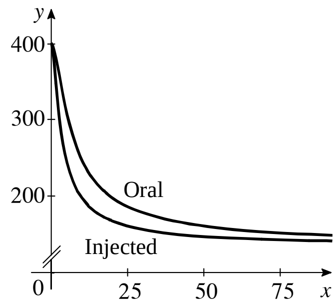
$g \circ h = g(h(x)) = \dfrac{4h(x)}{h(x)+4} = \dfrac{4\left(\frac{1}{2}x\right)}{\left(\frac{1}{2}x\right)+4} = \dfrac{4x}{x+8}$, and the number of surviving bacteria is given by

$$f \circ g \circ h = f(g(h(x))) = f\left(\frac{4x}{x+8}\right) = \frac{3200}{8 + \left(\dfrac{4x}{x+8}\right)^2} = \frac{400}{1 + \dfrac{2x^2}{(x+8)^2}} = \frac{400\,(x+8)^2}{3x^2 + 16x + 64}.$$

(b) With direct injections, the bioavailability function is no longer required since the entire antibiotic dosage is administered directly into the bloodstream. In this case, the number of surviving bacteria is given by

$$f \circ g = f(g(x)) = f\left(\frac{4x}{x+4}\right) = \frac{3200}{8 + \left(\dfrac{4x}{x+4}\right)^2} = \frac{3200}{8 + \dfrac{16x^2}{(x+4)^2}} = \frac{400}{1 + \dfrac{2x^2}{(x+4)^2}} = \frac{400\,(x+4)^2}{3x^2 + 8x + 16}$$

(c)



55. (a) The diameter d of the tumor is increasing at a rate of g mm/year, so $d(t) = (g \text{ mm/year})(t \text{ year}) = gt$ (in mm).

(b) Using $S = 4\pi r^2 = \pi d^2$ for the surface area of a sphere, we get $(S \circ d)(t) = S(d(t)) = S(gt) = \pi(gt)^2 = \pi g^2 t^2$. Now, since P is proportional to the surface area, we have $P(S) = kS$ where k is a proportionality constant. Thus, $(P \circ S \circ d)(t) = P(S(d(t))) = P(\pi g^2 t^2) = k\pi g^2 t^2$.

The result, $P = k\pi g^2 t^2$, gives the rate of enzyme production as a function of time.

Alternative Solution: If we assume the initial tumor size is nonzero so that $d(0) = d_0$, then $d(t) = d_0 + gt$. This gives $(P \circ S \circ d)(t) = k\pi\,(d_0 + gt)^2$.

56. If $A(x) = 1.04x$, then

$(A \circ A)(x) = A(A(x)) = A(1.04x) = 1.04(1.04x) = (1.04)^2 x$,

$(A \circ A \circ A)(x) = A((A \circ A)(x)) = A((1.04)^2 x) = 1.04(1.04)^2 x = (1.04)^3 x$, and

$(A \circ A \circ A \circ A)(x) = A((A \circ A \circ A)(x)) = A((1.04)^3 x) = 1.04(1.04)^3 x, = (1.04)^4 x$.

These compositions represent the amount of the investment after 2, 3, and 4 years.

Based on this pattern, when we compose n copies of A, we get the formula $\underbrace{(A \circ A \circ \cdots \circ A)}_{n\ A's}(x) = (1.04)^n x$.

57. If $f(x) = m_1 x + b_1$ and $g(x) = m_2 x + b_2$, then

$(f \circ g)(x) = f(g(x)) = f(m_2 x + b_2) = m_1(m_2 x + b_2) + b_1 = m_1 m_2 x + m_1 b_2 + b_1$.

So $f \circ g$ is a linear function with slope $m_1 m_2$.

58. We need to examine $h(-x)$.

$$h(-x) = (f \circ g)(-x) = f(g(-x)) = f(g(x)) \quad \text{[because } g \text{ is even]} \quad = h(x)$$

Because $h(-x) = h(x)$, h is an even function.

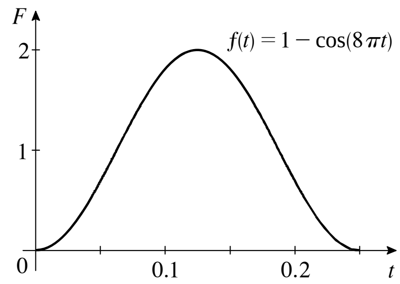
59. $h(-x) = f(g(-x)) = f(-g(x))$. At this point, we can't simplify the expression, so we might try to find a counterexample to show that h is not an odd function. Let $g(x) = x$, an odd function, and $f(x) = x^2 + x$. Then $h(x) = x^2 + x$, which is neither even nor odd.

Now suppose f is an odd function. Then $f(-g(x)) = -f(g(x)) = -h(x)$. Hence, $h(-x) = -h(x)$, and so h is odd if both f and g are odd.

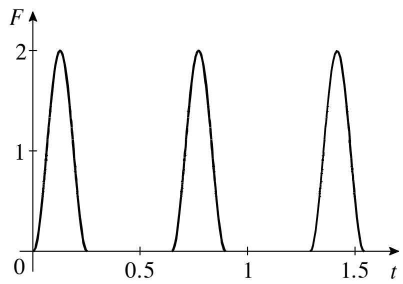
Now suppose f is an even function. Then $f(-g(x)) = f(g(x)) = h(x)$. Hence, $h(-x) = h(x)$, and so h is even if g is odd and f is even.

PROJECT The Biomechanics of Human Movement

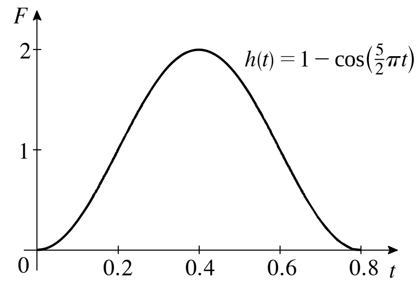
1. (a) The function $f(t) = 1 - \cos(8\pi t)$ has a value of zero at $t = 0\,\mathrm{s}$ (foot-strike), smoothly increases to a peak value of 2 at $t = 0.125\,\mathrm{s}$, and smoothly decreases to zero at $t = 0.25\,\mathrm{s}$ (toe-off). This effectively models the foot-strike cycle described.



(b)

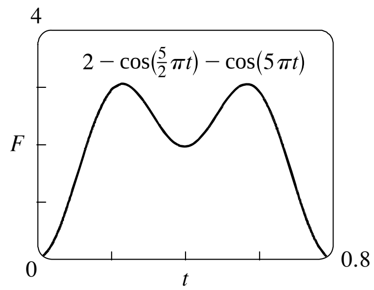


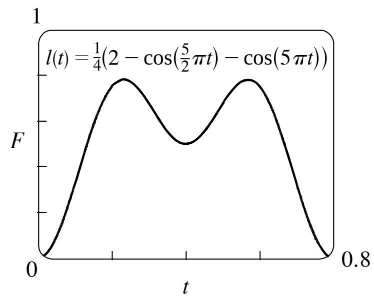
2. (a) Applying a horizontal stretch by a factor of $\frac{0.8\,\mathrm{s}}{0.25\,\mathrm{s}} = \frac{16}{5}$ to $f(t)$ gives the new function $h(t) = f(\frac{5}{16}t) = 1 - \cos(\frac{5}{2}\pi t)$ which has a value of zero at $t = 0\,\mathrm{s}$ and $t = 0.8\,\mathrm{s}$.



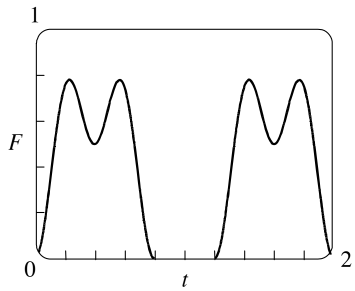
(b) Applying a horizontal compression by a factor of 2 to $h(t)$ gives a new function that oscillates twice as fast: $h(2t) = 1 - \cos(5\pi t)$. Adding this to $h(t)$ from part (a) gives $2 - \cos(\frac{5}{2}\pi t) - \cos(5\pi t)$. This function has the correct shape but the peak force is too high ($3.125\,\mathrm{kN}$). Scaling this new function by a factor of 4 gives a function that closely approximates the stride-cycle in Figure 1 (b):

$l(t) = \left(2 - \cos(\frac{5}{2}\pi t) - \cos(5\pi t)\right)/4$

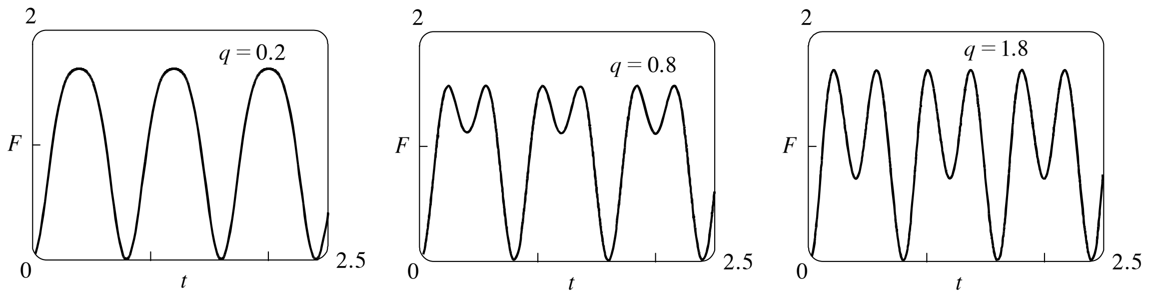




(c)



3. (a)



The constant q acts as a transition weight between the functions $f_1(t) = 1 - \cos\left(\frac{5}{2}\pi t\right)$ and $f_2(t) = 1 - \cos(5\pi t)$. When $q = 0 \Rightarrow g(t) = f_1(t)$. As q increases, $g(t)$ smoothly transitions from $f_1(t)$ to $f_2(t)$.

(b) The graphs of $g(t)$ are similar in shape to the graph of $l(t)$ from Problem 2(b) for constant values $q = 0.8$ and $q = 1.8$.

Note, however, that the peak values of $g(t)$ are higher than those of $l(t)$.

1.4 Exponential Functions

1. (a) $\dfrac{4^{-3}}{2^{-8}} = \dfrac{2^8}{4^3} = \dfrac{2^8}{(2^2)^3} = \dfrac{2^8}{2^6} = 2^{8-6} = 2^2 = 4$ (b) $\dfrac{1}{\sqrt[3]{x^4}} = \dfrac{1}{x^{4/3}} = x^{-4/3}$

2. (a) $8^{4/3} = (8^{1/3})^4 = 2^4 = 16$ (b) $x(3x^2)^3 = x \cdot 3^3(x^2)^3 = 27x \cdot x^6 = 27x^7$

3. (a) $b^8(2b)^4 = b^8 \cdot 2^4 b^4 = 16b^{12}$ (b) $\dfrac{(6y^3)^4}{2y^5} = \dfrac{6^4(y^3)^4}{2y^5} = \dfrac{1296y^{12}}{2y^5} = 648y^7$

4. (a) $\dfrac{x^{2n} \cdot x^{3n-1}}{x^{n+2}} = \dfrac{x^{2n+3n-1}}{x^{n+2}} = \dfrac{x^{5n-1}}{x^{n+2}} = x^{4n-3}$

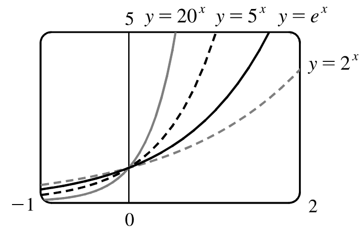
(b) $\dfrac{\sqrt{a\sqrt{b}}}{\sqrt[3]{ab}} = \dfrac{\sqrt{a}\sqrt{\sqrt{b}}}{\sqrt[3]{a}\sqrt[3]{b}} = \dfrac{a^{1/2}b^{1/4}}{a^{1/3}b^{1/3}} = a^{(1/2-1/3)}b^{(1/4-1/3)} = a^{1/6}b^{-1/12}$

5. (a) $f(x) = b^x,\ \ b > 0$ (b) $\mathbb{R}$ (c) $(0, \infty)$ (d) See Figures 5(c), 5(b), and 5(a), respectively.

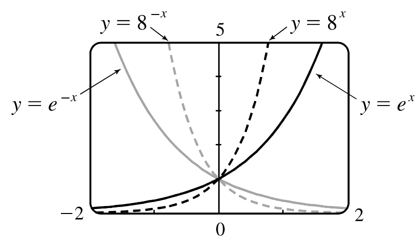
6. (a) The number e is the value of b such that the slope of the tangent line at $x = 0$ on the graph of $y = b^x$ is exactly 1.

(b) $e \approx 2.71828$ (c) $f(x) = e^x$

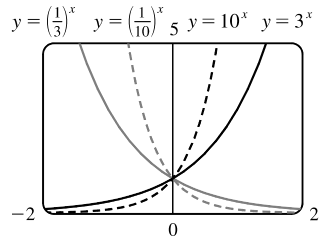
7. All of these graphs approach 0 as $x \to -\infty$, all of them pass through the point $(0, 1)$, and all of them are increasing and approach ∞ as $x \to \infty$. The larger the base, the faster the function increases for $x > 0$, and the faster it approaches 0 as $x \to -\infty$.



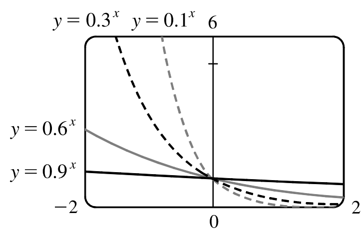
8. The graph of e^{-x} is the reflection of the graph of e^x about the y-axis, and the graph of 8^{-x} is the reflection of that of 8^x about the y-axis. The graph of 8^x increases more quickly than that of e^x for $x > 0$, and approaches 0 faster as $x \to -\infty$.



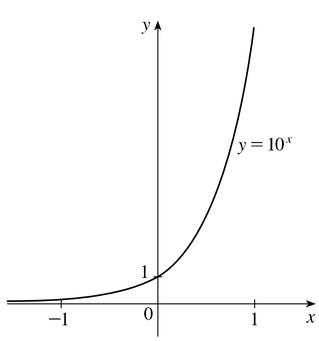
9. The functions with bases greater than 1 (3^x and 10^x) are increasing, while those with bases less than 1 $\left[\left(\frac{1}{3}\right)^x \text{ and } \left(\frac{1}{10}\right)^x\right]$ are decreasing. The graph of $\left(\frac{1}{3}\right)^x$ is the reflection of that of 3^x about the y-axis, and the graph of $\left(\frac{1}{10}\right)^x$ is the reflection of that of 10^x about the y-axis. The graph of 10^x increases more quickly than that of 3^x for $x > 0$, and approaches 0 faster as $x \to -\infty$.

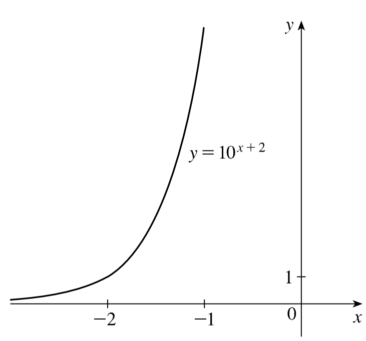


10. Each of the graphs approaches ∞ as $x \to -\infty$, and each approaches 0 as $x \to \infty$. The smaller the base, the faster the function grows as $x \to -\infty$, and the faster it approaches 0 as $x \to \infty$.

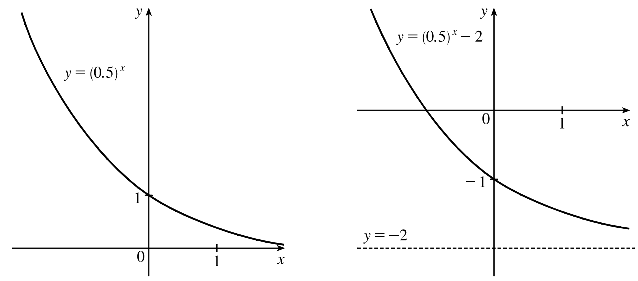


11. We start with the graph of $y = 10^x$ (Figure 4) and shift it 2 units to the left to obtain the graph of $y = 10^{x+2}$.

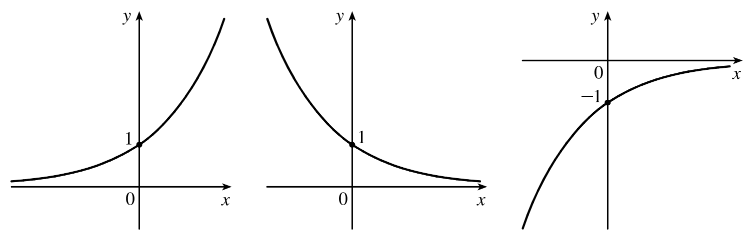




12. We start with the graph of $y = (0.5)^x$ (Figure 4) and shift it 2 units downward to obtain the graph of $y = (0.5)^x - 2$. The horizontal asymptote of the final graph is $y = -2$.

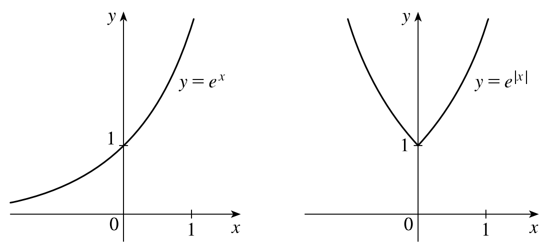


13. We start with the graph of $y = 2^x$ (Figure 4), reflect it about the y-axis, and then about the x-axis (or just rotate $180°$ to handle both reflections) to obtain the graph of $y = -2^{-x}$. In each graph, $y = 0$ is the horizontal asymptote.

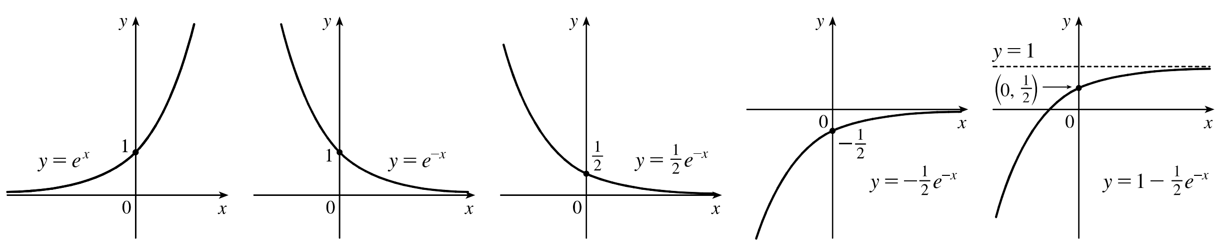


$y = 2^x$ $\qquad$ $y = 2^{-x}$ $\qquad$ $y = -2^{-x}$

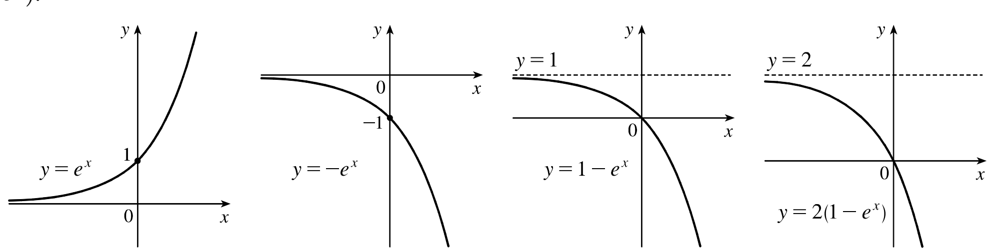
14. We start with the graph of $y = e^x$ (Figure 17) and reflect the portion of the graph in the first quadrant about the y-axis to obtain the graph of $y = e^{|x|}$.



15. We start with the graph of $y = e^x$ (Figure 17) and reflect about the y-axis to get the graph of $y = e^{-x}$. Then we compress the graph vertically by a factor of 2 to obtain the graph of $y = \frac{1}{2}e^{-x}$ and then reflect about the x-axis to get the graph of $y = -\frac{1}{2}e^{-x}$. Finally, we shift the graph upward one unit to get the graph of $y = 1 - \frac{1}{2}e^{-x}$.



16. We start with the graph of $y = e^x$ (Figure 17) and reflect about the x-axis to get the graph of $y = -e^x$. Then shift the graph upward one unit to get the graph of $y = 1 - e^x$. Finally, we stretch the graph vertically by a factor of 2 to obtain the graph of $y = 2(1 - e^x)$.



17. (a) To find the equation of the graph that results from shifting the graph of $y = e^x$ 2 units downward, we subtract 2 from the original function to get $y = e^x - 2$.

 (b) To find the equation of the graph that results from shifting the graph of $y = e^x$ 2 units to the right, we replace x with $x - 2$ in the original function to get $y = e^{(x-2)}$.

 (c) To find the equation of the graph that results from reflecting the graph of $y = e^x$ about the x-axis, we multiply the original function by -1 to get $y = -e^x$.

 (d) To find the equation of the graph that results from reflecting the graph of $y = e^x$ about the y-axis, we replace x with $-x$ in the original function to get $y = e^{-x}$.

 (e) To find the equation of the graph that results from reflecting the graph of $y = e^x$ about the x-axis and then about the y-axis, we first multiply the original function by -1 (to get $y = -e^x$) and then replace x with $-x$ in this equation to get $y = -e^{-x}$.

18. (a) This reflection consists of first reflecting the graph about the x-axis (giving the graph with equation $y = -e^x$) and then shifting this graph $2 \cdot 4 = 8$ units upward. So the equation is $y = -e^x + 8$.

 (b) This reflection consists of first reflecting the graph about the y-axis (giving the graph with equation $y = e^{-x}$) and then shifting this graph $2 \cdot 2 = 4$ units to the right. So the equation is $y = e^{-(x-4)}$.

19. (a) The denominator is zero when $1 - e^{1-x^2} = 0 \quad \Leftrightarrow \quad e^{1-x^2} = 1 \quad \Leftrightarrow \quad 1 - x^2 = 0 \quad \Leftrightarrow \quad x = \pm 1$. Thus, the function $f(x) = \dfrac{1 - e^{x^2}}{1 - e^{1-x^2}}$ has domain $\{x \mid x \neq \pm 1\} = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

 (b) The denominator is never equal to zero, so the function $f(x) = \dfrac{1 + x}{e^{\cos x}}$ has domain $\mathbb{R}$, or $(-\infty, \infty)$.

20. (a) The sine and exponential functions have domain $\mathbb{R}$, so $g(t) = \sin(e^{-t})$ also has domain $\mathbb{R}$.

 (b) The function $g(t) = \sqrt{1 - 2^t}$ has domain $\{t \mid 1 - 2^t \geq 0\} = \{t \mid 2^t \leq 1\} = \{t \mid t \leq 0\} = (-\infty, 0]$.

21. Use $y = Cb^x$ with the points $(1, 6)$ and $(3, 24)$. $6 = Cb^1 \quad \left[C = \frac{6}{b}\right]$ and $24 = Cb^3 \quad \Rightarrow \quad 24 = \left(\dfrac{6}{b}\right)b^3 \quad \Rightarrow$ $4 = b^2 \quad \Rightarrow \quad b = 2 \quad$ [since $b > 0$] and $C = \frac{6}{2} = 3$. The function is $f(x) = 3 \cdot 2^x$.

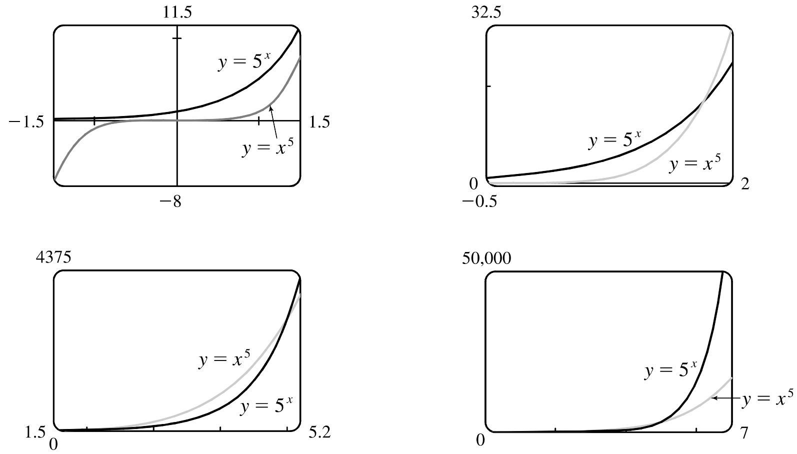
22. Use $y = Cb^x$ with the points $(-1, 3)$ and $\left(1, \frac{4}{3}\right)$. From the point $(-1, 3)$, we have $3 = Cb^{-1}$, hence $C = 3b$. Using this and the point $\left(1, \frac{4}{3}\right)$, we get $\frac{4}{3} = Cb^1 \quad \Rightarrow \quad \frac{4}{3} = (3b)b \quad \Rightarrow \quad \frac{4}{9} = b^2 \quad \Rightarrow \quad b = \frac{2}{3}$ [since $b > 0$] and $C = 3(\frac{2}{3}) = 2$. The function is $f(x) = 2(\frac{2}{3})^x$.

23. If $f(x) = 5^x$, then $\dfrac{f(x+h) - f(x)}{h} = \dfrac{5^{x+h} - 5^x}{h} = \dfrac{5^x 5^h - 5^x}{h} = \dfrac{5^x\left(5^h - 1\right)}{h} = 5^x\left(\dfrac{5^h - 1}{h}\right)$.

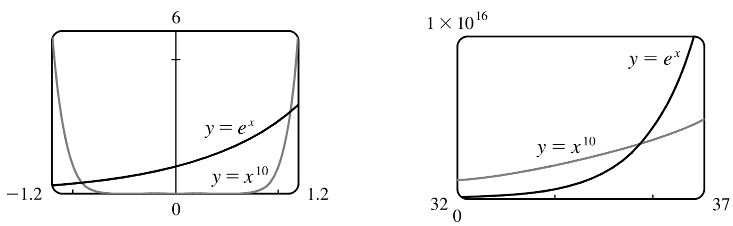
24. Suppose the month is February. Your payment on the 28th day would be $2^{28-1} = 2^{27} = 134{,}217{,}728$ cents, or \$1,342,177.28. Clearly, the second method of payment results in a larger amount for any month.

25. 2 ft $= 24$ in, $f(24) = 24^2$ in $= 576$ in $= 48$ ft. $\quad g(24) = 2^{24}$ in $= 2^{24}/(12 \cdot 5280)$ mi ≈ 265 mi

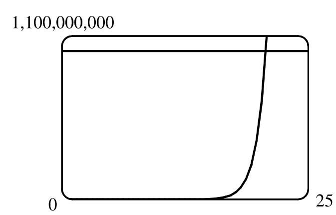
26. We see from the graphs that for x less than about 1.8, $g(x) = 5^x > f(x) = x^5$, and then near the point $(1.8, 17.1)$ the curves intersect. Then $f(x) > g(x)$ from $x \approx 1.8$ until $x = 5$. At $(5, 3125)$ there is another point of intersection, and for $x > 5$ we see that $g(x) > f(x)$. In fact, g increases much more rapidly than f beyond that point.



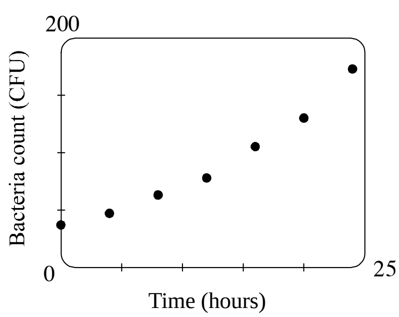
27. The graph of g finally surpasses that of f at $x \approx 35.8$.



28. We graph $y = e^x$ and $y = 1{,}000{,}000{,}000$ and determine where $e^x = 1 \times 10^9$. This seems to be true at $x \approx 20.723$, so $e^x > 1 \times 10^9$ for $x > 20.723$.

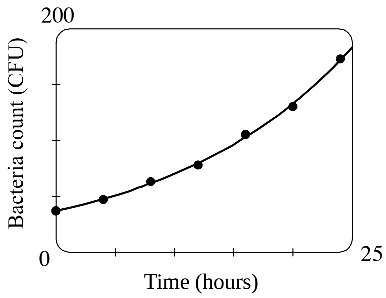


29. (a)



(b) Using a calculator to fit an exponential curve to the data gives $f(t) = (36.78) \cdot (1.07)^t$.

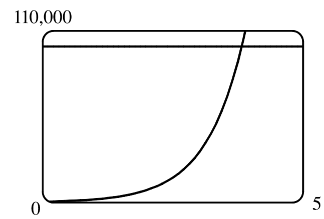
(c)



Using the TRACE feature of a calculator, we find that the bacteria count increases from 50 CFU to 100 CFU in about 10.8 hours. Therefore it takes approximately 10.8 hours for the bacteria count to double.

30. (a) Three hours represents 6 doubling periods (one doubling period is 30 minutes). $500 \cdot 2^6 = 32{,}000$

(b) In t hours, there will be $2t$ doubling periods. The initial population is 500, so the population y at time t is $y = 500 \cdot 2^{2t}$.

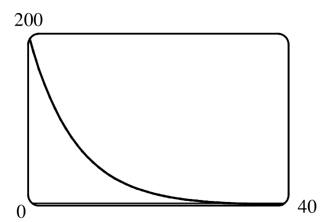


(c) $t = \frac{40}{60} = \frac{2}{3} \quad \Rightarrow \quad y = 500 \cdot 2^{2(2/3)} \approx 1260$

(d) We graph $y_1 = 500 \cdot 2^{2t}$ and $y_2 = 100{,}000$. The two curves intersect at $t \approx 3.82$, so the population reaches 100,000 in about 3.82 hours.

31. (a) Fifteen days represents 3 half-life periods (one half-life period is 5 days). $200\left(\frac{1}{2}\right)^3 = 25\text{ mg}$

(b) In t hours, there will be $t/5$ half-life periods. The initial amount is 200 mg, so the amount remaining after t days is $y = 200\left(\frac{1}{2}\right)^{t/5}$, or equivalently, $y = 200 \cdot 2^{-t/5}$.

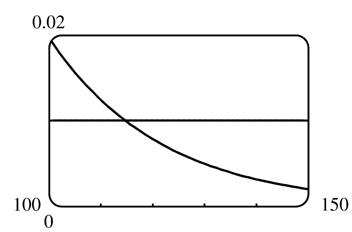


(c) $t = 3\text{ weeks} = 21\text{ days} \quad \Rightarrow \quad y = 200 \cdot 2^{-21/5} \approx 10.9\text{ mg}$

(d) We graph $y_1 = 200 \cdot 2^{-t/5}$ and $y_2 = 1$. The two curves intersect at $t \approx 38.2$, so the mass will be reduced to 1 mg in about 38.2 days.

32. (a) Sixty hours represents 4 half-life periods. $2 \cdot \left(\frac{1}{2}\right)^4 = \frac{1}{8}$ g

(b) In t hours, there will be $t/15$ half-life periods. The initial mass is 2 g, so the mass y at time t is $y = 2 \cdot \left(\frac{1}{2}\right)^{t/15}$.



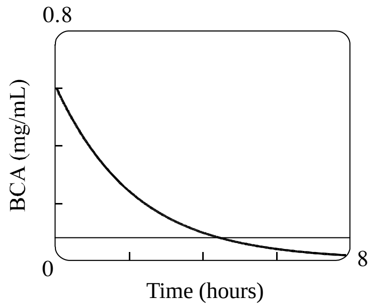
(c) 4 days $= 4 \cdot 24 = 96$ hours. $t = 96 \quad \Rightarrow \quad y = 2 \cdot \left(\frac{1}{2}\right)^{96/15} \approx 0.024$ g

(d) $y = 0.01 \quad \Rightarrow \quad t \approx 114.7$ hours

33. The half-life is approximately 3.5 days since the RNA load drops from 40 to 20 in that time.

34. (a) Let $C(t) = a \cdot b^t$ represent the blood alcohol concentration t hours after midnight. The initial concentration at midnight is $C(0) = a \cdot b^0 = a = 0.6$. This will drop by half after 1.5 hours implying $\frac{a}{2} = a \cdot b^{1.5} \quad \Leftrightarrow \quad \frac{1}{2} = b^{1.5} \quad \Leftrightarrow$ $b = \left(\frac{1}{2}\right)^{1/1.5} \approx 0.62996$. So the exponential decay model is $C(t) = (0.6) \cdot (0.62996)^t$.

(b)

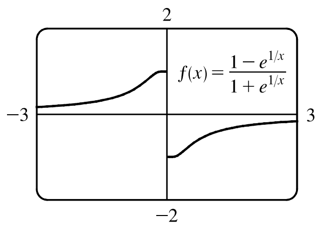


By graphing the exponential model $C(t) = (0.6) \cdot (0.62996)^t$ along with the line $C = 0.08$, we observe that one can legally drive home after about 4.4 hours.

35. Let $t = 0$ correspond to 1950 to get the model $P = ab^t$, where $a \approx 2614.086$ and $b \approx 1.01693$. To estimate the population in 1993, let $t = 43$ to obtain $P \approx 5381$ million. To predict the population in 2020, let $t = 70$ to obtain $P \approx 8466$ million.

36. Let $t = 0$ correspond to 1900 to get the model $P = ab^t$, where $a \approx 80.8498$ and $b \approx 1.01269$. To estimate the population in 1925, let $t = 25$ to obtain $P \approx 111$ million. To predict the population in 2020, let $t = 120$ to obtain $P \approx 367$ million.

37.



From the graph, it appears that f is an odd function (f is undefined for $x = 0$). To prove this, we must show that $f(-x) = -f(x)$.

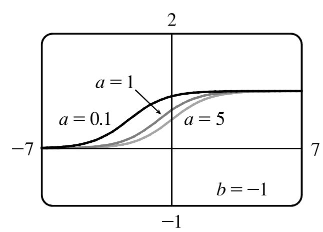
$$f(-x) = \frac{1 - e^{1/(-x)}}{1 + e^{1/(-x)}} = \frac{1 - e^{(-1/x)}}{1 + e^{(-1/x)}} = \frac{1 - \dfrac{1}{e^{1/x}}}{1 + \dfrac{1}{e^{1/x}}} \cdot \frac{e^{1/x}}{e^{1/x}} = \frac{e^{1/x} - 1}{e^{1/x} + 1}$$

$$= -\frac{1 - e^{1/x}}{1 + e^{1/x}} = -f(x)$$

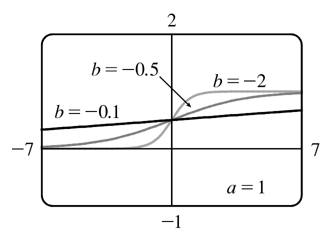
so f is an odd function.

38. We'll start with $b = -1$ and graph $f(x) = \dfrac{1}{1 + ae^{bx}}$ for $a = 0.1$, 1, and 5.

From the graph, we see that there is a horizontal asymptote $y = 0$ as $x \to -\infty$ and a horizontal asymptote $y = 1$ as $x \to \infty$. If $a = 1$, the y-intercept is $\left(0, \frac{1}{2}\right)$. As a gets smaller (close to 0), the graph of f moves left. As a gets larger, the graph of f moves right.

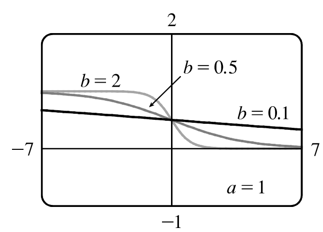


As b changes from -1 to 0, the graph of f is stretched horizontally. As b changes through large negative values, the graph of f is compressed horizontally. (This takes care of negatives values of b.)



If b is positive, the graph of f is reflected through the y-axis.

Last, if $b = 0$, the graph of f is the horizontal line $y = 1/(1 + a)$.



1.5 Logarithms; Semi-log and Log-log Plots

1. (a) See Definition 1.

   (b) It must pass the Horizontal Line Test.

2. (a) $f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y$ for any y in B. The domain of f^{-1} is B and the range of f^{-1} is A.

   (b) See the steps in (5).

   (c) Reflect the graph of f about the line $y = x$.

3. f is not one-to-one because $2 \neq 6$, but $f(2) = 2.0 = f(6)$.

4. f is one-to-one because it never takes on the same value twice.

5. We could draw a horizontal line that intersects the graph in more than one point. Thus, by the Horizontal Line Test, the function is not one-to-one.

6. No horizontal line intersects the graph more than once. Thus, by the Horizontal Line Test, the function is one-to-one.

7. No horizontal line intersects the graph more than once. Thus, by the Horizontal Line Test, the function is one-to-one.

8. We could draw a horizontal line that intersects the graph in more than one point. Thus, by the Horizontal Line Test, the function is not one-to-one.

9. The graph of $f(x) = x^2 - 2x$ is a parabola with axis of symmetry $x = -\dfrac{b}{2a} = -\dfrac{-2}{2(1)} = 1$. Pick any x-values equidistant from 1 to find two equal function values. For example, $f(0) = 0$ and $f(2) = 0$, so f is not one-to-one.

10. The graph of $f(x) = 10 - 3x$ is a line with slope -3. It passes the Horizontal Line Test, so f is one-to-one.

    Algebraic solution: If $x_1 \neq x_2$, then $-3x_1 \neq -3x_2 \quad \Rightarrow \quad 10 - 3x_1 \neq 10 - 3x_2 \quad \Rightarrow \quad f(x_1) \neq f(x_2)$, so f is one-to-one.

11. $g(x) = 1/x. \quad x_1 \neq x_2 \quad \Rightarrow \quad 1/x_1 \neq 1/x_2 \quad \Rightarrow \quad g(x_1) \neq g(x_2)$, so g is one-to-one.

    Geometric solution: The graph of g is the hyperbola shown in Figure 14 in Section 1.2. It passes the Horizontal Line Test, so g is one-to-one.

12. $g(x) = \cos x. \quad g(0) = 1 = g(2\pi)$, so g is not one-to-one.

13. A football will attain every height h up to its maximum height twice: once on the way up, and again on the way down. Thus, even if t_1 does not equal t_2, $f(t_1)$ may equal $f(t_2)$, so f is not 1-1.

14. f is not 1-1 because eventually we all stop growing and therefore, there are two times at which we have the same height.

15. (a) Since f is 1-1, $f(6) = 17 \quad \Leftrightarrow \quad f^{-1}(17) = 6$.

    (b) Since f is 1-1, $f^{-1}(3) = 2 \quad \Leftrightarrow \quad f(2) = 3$.

16. First, we must determine x such that $f(x) = 3$. By inspection, we see that if $x = 1$, then $f(1) = 3$. Since f is 1-1 (f is an increasing function), it has an inverse, and $f^{-1}(3) = 1$. If f is a 1-1 function, then $f(f^{-1}(a)) = a$, so $f(f^{-1}(2)) = 2$.

17. First, we must determine x such that $g(x) = 4$. By inspection, we see that if $x = 0$, then $g(x) = 4$. Since g is 1-1 (g is an increasing function), it has an inverse, and $g^{-1}(4) = 0$.

18. (a) f is 1-1 because it passes the Horizontal Line Test.

(b) Domain of $f = [-3, 3] =$ Range of f^{-1}. Range of $f = [-1, 3] =$ Domain of f^{-1}.

(c) Since $f(0) = 2$, $f^{-1}(2) = 0$.

(d) Since $f(-1.7) \approx 0$, $f^{-1}(0) \approx -1.7$.

19. We solve $C = \frac{5}{9}(F - 32)$ for F: $\frac{9}{5}C = F - 32 \quad \Rightarrow \quad F = \frac{9}{5}C + 32$. This gives us a formula for the inverse function, that is, the Fahrenheit temperature F as a function of the Celsius temperature C. $F \geq -459.67 \quad \Rightarrow \quad \frac{9}{5}C + 32 \geq -459.67 \quad \Rightarrow \quad \frac{9}{5}C \geq -491.67 \quad \Rightarrow \quad C \geq -273.15$, the domain of the inverse function.

20. $m = \dfrac{m_0}{\sqrt{1 - v^2/c^2}} \quad \Rightarrow \quad 1 - \dfrac{v^2}{c^2} = \dfrac{m_0^2}{m^2} \quad \Rightarrow \quad \dfrac{v^2}{c^2} = 1 - \dfrac{m_0^2}{m^2} \quad \Rightarrow \quad v^2 = c^2\left(1 - \dfrac{m_0^2}{m^2}\right) \quad \Rightarrow \quad v = c\sqrt{1 - \dfrac{m_0^2}{m^2}}$.

This formula gives us the speed v of the particle in terms of its mass m, that is, $v = f^{-1}(m)$.

21. $y = f(x) = 1 + \sqrt{2 + 3x} \quad (y \geq 1) \quad \Rightarrow \quad y - 1 = \sqrt{2 + 3x} \quad \Rightarrow \quad (y - 1)^2 = 2 + 3x \quad \Rightarrow \quad (y - 1)^2 - 2 = 3x \quad \Rightarrow$ $x = \frac{1}{3}(y - 1)^2 - \frac{2}{3}$. Interchange x and y: $y = \frac{1}{3}(x - 1)^2 - \frac{2}{3}$. So $f^{-1}(x) = \frac{1}{3}(x - 1)^2 - \frac{2}{3}$. Note that the domain of f^{-1} is $x \geq 1$.

22. $y = f(x) = \dfrac{4x - 1}{2x + 3} \quad \Rightarrow \quad y(2x + 3) = 4x - 1 \quad \Rightarrow \quad 2xy + 3y = 4x - 1 \quad \Rightarrow \quad 3y + 1 = 4x - 2xy \quad \Rightarrow$

$3y + 1 = (4 - 2y)x \quad \Rightarrow \quad x = \dfrac{3y + 1}{4 - 2y}$. Interchange x and y: $y = \dfrac{3x + 1}{4 - 2x}$. So $f^{-1}(x) = \dfrac{3x + 1}{4 - 2x}$.

23. $y = f(x) = e^{2x-1} \quad \Rightarrow \quad \ln y = 2x - 1 \quad \Rightarrow \quad 1 + \ln y = 2x \quad \Rightarrow \quad x = \frac{1}{2}(1 + \ln y)$.

Interchange x and y: $y = \frac{1}{2}(1 + \ln x)$. So $f^{-1}(x) = \frac{1}{2}(1 + \ln x)$.

24. $y = f(x) = x^2 - x \quad (x \geq \frac{1}{2}) \quad \Rightarrow \quad y = x^2 - x + \frac{1}{4} - \frac{1}{4} \quad \Rightarrow \quad y = (x - \frac{1}{2})^2 - \frac{1}{4} \quad \Rightarrow$

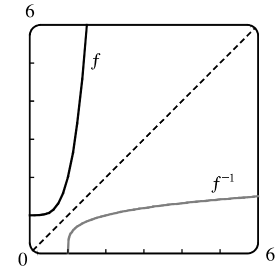
$y + \frac{1}{4} = (x - \frac{1}{2})^2 \quad \Rightarrow \quad x - \frac{1}{2} = \sqrt{y + \frac{1}{4}} \quad \Rightarrow \quad x = \frac{1}{2} + \sqrt{y + \frac{1}{4}}$. Interchange x and y: $y = \frac{1}{2} + \sqrt{x + \frac{1}{4}}$. So $f^{-1}(x) = \frac{1}{2} + \sqrt{x + \frac{1}{4}}$.

25. $y = f(x) = \ln(x + 3) \quad \Rightarrow \quad x + 3 = e^y \quad \Rightarrow \quad x = e^y - 3$. Interchange x and y: $y = e^x - 3$. So $f^{-1}(x) = e^x - 3$.

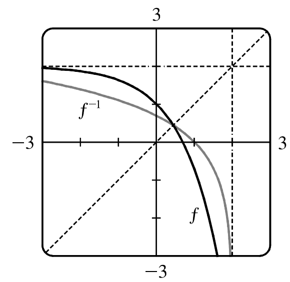
26. $y = f(x) = \dfrac{e^x}{1 + 2e^x} \quad \Rightarrow \quad y + 2ye^x = e^x \quad \Rightarrow \quad y = e^x - 2ye^x \quad \Rightarrow \quad y = e^x(1 - 2y) \quad \Rightarrow \quad e^x = \dfrac{y}{1 - 2y} \quad \Rightarrow$

$x = \ln\left(\dfrac{y}{1 - 2y}\right)$. Interchange x and y: $y = \ln\left(\dfrac{x}{1 - 2x}\right)$. So $f^{-1}(x) = \ln\left(\dfrac{x}{1 - 2x}\right)$. Note that the range of f and the domain of f^{-1} is $(0, \frac{1}{2})$.

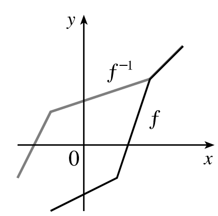
27. $y = f(x) = x^4 + 1 \quad \Rightarrow \quad y - 1 = x^4 \quad \Rightarrow \quad x = \sqrt[4]{y-1}$ [not $\pm$ since $x \geq 0$]. Interchange x and y: $y = \sqrt[4]{x-1}$. So $f^{-1}(x) = \sqrt[4]{x-1}$. The graph of $y = \sqrt[4]{x-1}$ is just the graph of $y = \sqrt[4]{x}$ shifted right one unit. From the graph, we see that f and f^{-1} are reflections about the line $y = x$.



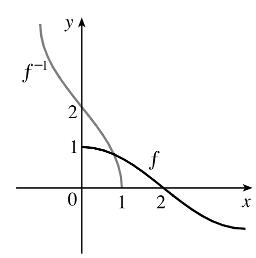
28. $y = f(x) = 2 - e^x \quad \Rightarrow \quad e^x = 2 - y \quad \Rightarrow \quad x = \ln(2 - y)$. Interchange x and y: $y = \ln(2 - x)$. So $f^{-1}(x) = \ln(2 - x)$. From the graph, we see that f and f^{-1} are reflections about the line $y = x$.



29. Reflect the graph of f about the line $y = x$. The points $(-1, -2)$, $(1, -1)$, $(2, 2)$, and $(3, 3)$ on f are reflected to $(-2, -1)$, $(-1, 1)$, $(2, 2)$, and $(3, 3)$ on f^{-1}.

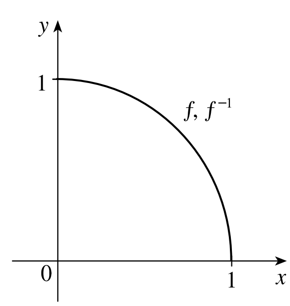


30. Reflect the graph of f about the line $y = x$.



31. (a) $y = f(x) = \sqrt{1 - x^2} \quad (0 \leq x \leq 1$ and note that $y \geq 0) \quad \Rightarrow \quad y^2 = 1 - x^2 \quad \Rightarrow \quad x^2 = 1 - y^2 \quad \Rightarrow \quad x = \sqrt{1 - y^2}$. So $f^{-1}(x) = \sqrt{1 - x^2}$, $0 \leq x \leq 1$. We see that $f^{-1}$and f are the same function.

(b) The graph of f is the portion of the circle $x^2 + y^2 = 1$ with $0 \leq x \leq 1$ and $0 \leq y \leq 1$ (quarter-circle in the first quadrant). The graph of f is symmetric with respect to the line $y = x$, so its reflection about $y = x$ is itself, that is, $f^{-1} = f$.



32. Find the inverse function by solving for v in the equation $P = 1 - \dfrac{1}{R_0\,(1-v)} \quad\Leftrightarrow\quad \dfrac{1}{R_0\,(1-v)} = 1 - P \quad\Leftrightarrow$

$R_0\,(1-v) = \dfrac{1}{1-P} \quad\Leftrightarrow\quad 1 - v = \dfrac{1}{R_0\,(1-P)} \quad\Leftrightarrow\quad v = 1 - \dfrac{1}{R_0\,(1-P)}$. Observe that the inverse function is identical to the original function if P and v are interchanged. This implies the original function is symmetric about the line $P = v$.

33. (a) It is defined as the inverse of the exponential function with base b, that is, $\log_b x = y \quad\Leftrightarrow\quad b^y = x$.

(b) $(0, \infty)$ (c) $\mathbb{R}$ (d) See Figure 11.

34. (a) The natural logarithm is the logarithm with base e, denoted $\ln x$.

(b) The common logarithm is the logarithm with base 10, denoted $\log x$.

(c) See Figure 13.

35. (a) $\log_5 125 = 3$ since $5^3 = 125$. (b) $\log_3 \dfrac{1}{27} = -3$ since $3^{-3} = \dfrac{1}{3^3} = \dfrac{1}{27}$.

36. (a) $\ln(1/e) = \ln 1 - \ln e = 0 - 1 = -1$ (b) $\log_{10} \sqrt{10} = \log_{10} 10^{1/2} = \frac{1}{2}$ by (2).

37. (a)

$$\begin{aligned} \log_2 6 - \log_2 15 + \log_2 20 &= \log_2\!\left(\tfrac{6}{15}\right) + \log_2 20 && \text{[by Law 2]} \\ &= \log_2\!\left(\tfrac{6}{15} \cdot 20\right) && \text{[by Law 1]} \\ &= \log_2 8, \text{ and } \log_2 8 = 3 \text{ since } 2^3 = 8. \end{aligned}$$

(b)

$$\begin{aligned} \log_3 100 - \log_3 18 - \log_3 50 &= \log_3\!\left(\tfrac{100}{18}\right) - \log_3 50 = \log_3\!\left(\tfrac{100}{18 \cdot 50}\right) \\ &= \log_3\!\left(\tfrac{1}{9}\right), \text{ and } \log_3\!\left(\tfrac{1}{9}\right) = -2 \text{ since } 3^{-2} = \tfrac{1}{9}. \end{aligned}$$

38. (a) $e^{-2\ln 5} = \left(e^{\ln 5}\right)^{-2} \overset{(6)}{=} 5^{-2} = \dfrac{1}{5^2} = \dfrac{1}{25}$ (b) $\ln\!\left(\ln e^{e^{10}}\right) \overset{(6)}{=} \ln(e^{10}) \overset{(6)}{=} 10$

39.

$$\begin{aligned} \ln 5 + 5\ln 3 &= \ln 5 + \ln 3^5 && \text{[by Law 3]} \\ &= \ln(5 \cdot 3^5) && \text{[by Law 1]} \\ &= \ln 1215 \end{aligned}$$

40.

$$\begin{aligned} \ln(a+b) + \ln(a-b) - 2\ln c &= \ln[(a+b)(a-b)] - \ln c^2 && \text{[by Laws 1, 3]} \\ &= \ln \frac{(a+b)(a-b)}{c^2} && \text{[by Law 2]} \\ &\text{or } \ln \frac{a^2 - b^2}{c^2} \end{aligned}$$

41.

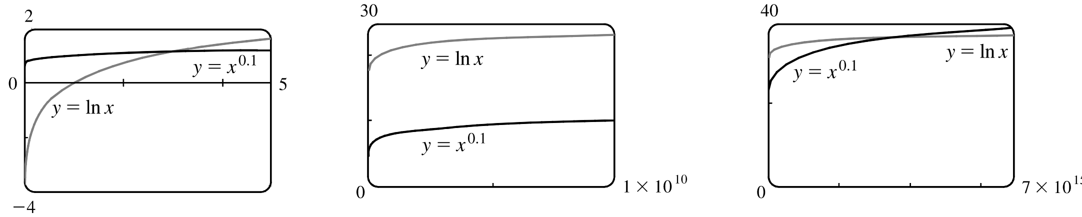
$$\begin{aligned} \tfrac{1}{3}\ln(x+2)^3 + \tfrac{1}{2}\left[\ln x - \ln(x^2+3x+2)^2\right] &= \ln[(x+2)^3]^{1/3} + \tfrac{1}{2}\ln \frac{x}{(x^2+3x+2)^2} && \text{[by Laws 3, 2]} \\ &= \ln(x+2) + \ln \frac{\sqrt{x}}{x^2+3x+2} && \text{[by Law 3]} \\ &= \ln \frac{(x+2)\sqrt{x}}{(x+1)(x+2)} && \text{[by Law 1]} \\ &= \ln \frac{\sqrt{x}}{x+1} \end{aligned}$$

Note that since $\ln x$ is defined for $x > 0$, we have $x+1$, $x+2$, and $x^2 + 3x + 2$ all positive, and hence their logarithms are defined.

42. (a) $\log_{12} 10 = \dfrac{\ln 10}{\ln 12} \approx 0.926628$ (b) $\log_2 8.4 = \dfrac{\ln 8.4}{\ln 2} \approx 3.070389$

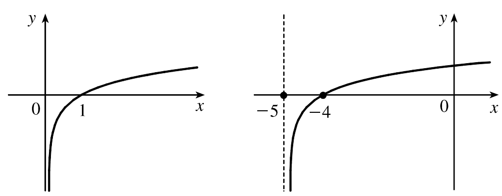
43. 3 ft $= 36$ in, so we need x such that $\log_2 x = 36 \quad \Leftrightarrow \quad x = 2^{36} = 68{,}719{,}476{,}736$. In miles, this is $68{,}719{,}476{,}736 \text{ in} \cdot \dfrac{1 \text{ ft}}{12 \text{ in}} \cdot \dfrac{1 \text{ mi}}{5280 \text{ ft}} \approx 1{,}084{,}587.7$ mi.

44.



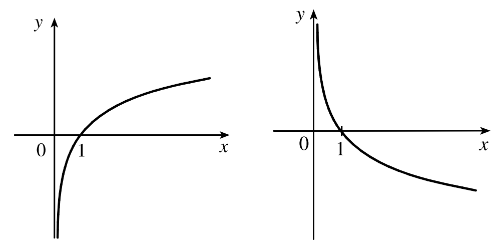
From the graphs, we see that $f(x) = x^{0.1} > g(x) = \ln x$ for approximately $0 < x < 3.06$, and then $g(x) > f(x)$ for $3.06 < x < 3.43 \times 10^{15}$ (approximately). At that point, the graph of f finally surpasses the graph of g for good.

45. (a) Shift the graph of $y = \log_{10} x$ five units to the left to obtain the graph of $y = \log_{10}(x + 5)$. Note the vertical asymptote of $x = -5$.



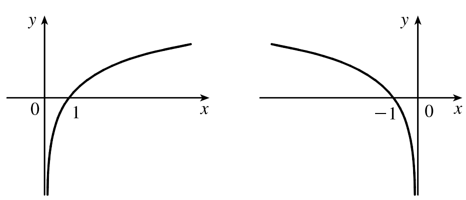
$y = \log_{10} x$ $\quad$ $y = \log_{10}(x + 5)$

(b) Reflect the graph of $y = \ln x$ about the x-axis to obtain the graph of $y = -\ln x$.



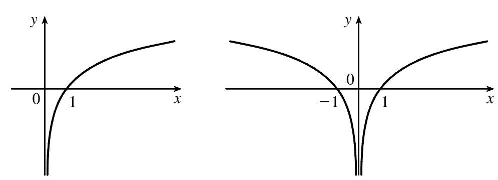
$y = \ln x$ $\quad$ $y = -\ln x$

46. (a) Reflect the graph of $y = \ln x$ about the y-axis to obtain the graph of $y = \ln(-x)$.



$y = \ln x$ $\quad$ $y = \ln(-x)$

(b) Reflect the portion of the graph of $y = \ln x$ to the right of the y-axis about the y-axis. The graph of $y = \ln|x|$ is that reflection in addition to the original portion.



$y = \ln x$ $\quad$ $y = \ln|x|$

47. (a) $e^{7-4x} = 6 \quad \Leftrightarrow \quad 7 - 4x = \ln 6 \quad \Leftrightarrow \quad 7 - \ln 6 = 4x \quad \Leftrightarrow \quad x = \frac{1}{4}(7 - \ln 6)$

(b) $\ln(3x - 10) = 2 \quad \Leftrightarrow \quad 3x - 10 = e^2 \quad \Leftrightarrow \quad 3x = e^2 + 10 \quad \Leftrightarrow \quad x = \frac{1}{3}(e^2 + 10)$

48. (a) $\ln(x^2 - 1) = 3 \quad \Leftrightarrow \quad x^2 - 1 = e^3 \quad \Leftrightarrow \quad x^2 = 1 + e^3 \quad \Leftrightarrow \quad x = \pm\sqrt{1 + e^3}$.

(b) $e^{2x} - 3e^x + 2 = 0 \;\Leftrightarrow\; (e^x - 1)(e^x - 2) = 0 \;\Leftrightarrow\; e^x = 1$ or $e^x = 2 \;\Leftrightarrow\; x = \ln 1$ or $x = \ln 2$, so $x = 0$ or $\ln 2$.

49. (a) $2^{x-5} = 3 \quad \Leftrightarrow \quad \log_2 3 = x - 5 \quad \Leftrightarrow \quad x = 5 + \log_2 3$.

Or: $2^{x-5} = 3 \quad \Leftrightarrow \quad \ln\left(2^{x-5}\right) = \ln 3 \quad \Leftrightarrow \quad (x - 5)\ln 2 = \ln 3 \quad \Leftrightarrow \quad x - 5 = \dfrac{\ln 3}{\ln 2} \quad \Leftrightarrow \quad x = 5 + \dfrac{\ln 3}{\ln 2}$

(b) $\ln x + \ln(x - 1) = \ln(x(x - 1)) = 1 \quad \Leftrightarrow \quad x(x - 1) = e^1 \quad \Leftrightarrow \quad x^2 - x - e = 0$. The quadratic formula (with $a = 1$, $b = -1$, and $c = -e$) gives $x = \frac{1}{2}\left(1 \pm \sqrt{1 + 4e}\right)$, but we reject the negative root since the natural logarithm is not defined for $x < 0$. So $x = \frac{1}{2}\left(1 + \sqrt{1 + 4e}\right)$.

50. (a) $\ln(\ln x) = 1 \quad \Leftrightarrow \quad e^{\ln(\ln x)} = e^1 \quad \Leftrightarrow \quad \ln x = e^1 = e \quad \Leftrightarrow \quad e^{\ln x} = e^e \quad \Leftrightarrow \quad x = e^e$

(b) $e^{ax} = Ce^{bx} \quad \Leftrightarrow \quad \ln e^{ax} = \ln[C(e^{bx})] \quad \Leftrightarrow \quad ax = \ln C + \ln e^{bx} \quad \Leftrightarrow \quad ax = \ln C + bx \quad \Leftrightarrow$

$ax - bx = \ln C \quad \Leftrightarrow \quad (a-b)x = \ln C \quad \Leftrightarrow \quad x = \dfrac{\ln C}{a-b}$

51. (a) $\ln x < 0 \quad \Rightarrow \quad x < e^0 \quad \Rightarrow \quad x < 1$. Since the domain of $f(x) = \ln x$ is $x > 0$, the solution of the original inequality is $0 < x < 1$.

(b) $e^x > 5 \quad \Rightarrow \quad \ln e^x > \ln 5 \quad \Rightarrow \quad x > \ln 5$

52. (a) $1 < e^{3x-1} < 2 \quad \Rightarrow \quad \ln 1 < 3x - 1 < \ln 2 \quad \Rightarrow \quad 0 < 3x - 1 < \ln 2 \quad \Rightarrow \quad 1 < 3x < 1 + \ln 2 \quad \Rightarrow$
$\frac{1}{3} < x < \frac{1}{3}(1 + \ln 2)$

(b) $1 - 2\ln x < 3 \quad \Rightarrow \quad -2\ln x < 2 \quad \Rightarrow \quad \ln x > -1 \quad \Rightarrow \quad x > e^{-1}$

53. (a) Solve for t in the equation: $c(t) = c_0 e^{-Kt/V} \quad \Rightarrow \quad 0.60 = 1.65 e^{-340t/32941} \quad \Leftrightarrow \quad \ln\left(\frac{0.60}{1.65}\right) = \ln\left(e^{-340t/32941}\right) \quad \Leftrightarrow$
$t = -\frac{32941}{340} \ln\left(\frac{0.60}{1.65}\right) \approx 98.0\,\text{minutes}$

(b) Solve for T in the equation: $c(T) = c_0 e^{-KT/V} \quad \Leftrightarrow \quad \dfrac{c(T)}{c_0} = e^{-KT/V} \quad \Leftrightarrow \quad \ln\left(\dfrac{c(T)}{c_0}\right) = -KT/V \quad \Leftrightarrow$

$T = -\dfrac{V}{K} \ln\left(\dfrac{c(T)}{c_0}\right)$

54. (a) Since Kt represents the volume of blood processed in t hours, the quantity Kt/V is the amount of blood processed relative to total blood volume. Kt has units $\left[\frac{\text{mL}}{\text{minute}}\right] \cdot [\text{minute}] = [\text{mL}]$ and V has units $[\text{mL}]$, so Kt/V is unitless.

(b) The fractional reduction in urea is

$$U = \frac{\text{Initial Concentration} - \text{Final Concentration}}{\text{Initial Concentration}} = \frac{c_0 - c(t)}{c_0} = 1 - \frac{c(t)}{c_0} = 1 - e^{-Kt/V}$$

As Kt/V increases, the term $e^{-Kt/V}$ decreases so that U increases. Similarly, as Kt/V decreases, U decreases.

55. (a) We must have $e^x - 3 > 0 \quad \Rightarrow \quad e^x > 3 \quad \Rightarrow \quad x > \ln 3$. Thus, the domain of $f(x) = \ln(e^x - 3)$ is $(\ln 3, \infty)$.

(b) $y = \ln(e^x - 3) \quad \Rightarrow \quad e^y = e^x - 3 \quad \Rightarrow \quad e^x = e^y + 3 \quad \Rightarrow \quad x = \ln(e^y + 3)$, so $f^{-1}(x) = \ln(e^x + 3)$.

Now $e^x + 3 > 0 \quad \Rightarrow \quad e^x > -3$, which is true for any real x, so the domain of f^{-1} is $\mathbb{R}$.

56. (a) By (6), $e^{\ln 300} = 300$ and $\ln(e^{300}) = 300$.

(b) A calculator gives $e^{\ln 300} = 300$ and an error message for $\ln(e^{300})$ since e^{300} is larger than most calculators can evaluate.

57. (a) Find the inverse by solving for t: $\quad n = 500 \cdot 4^t \quad \Leftrightarrow \quad \ln\left(\dfrac{n}{500}\right) = \ln\left(4^t\right) \quad \Leftrightarrow \quad \ln\left(\dfrac{n}{500}\right) = t\ln(4) \quad \Leftrightarrow$

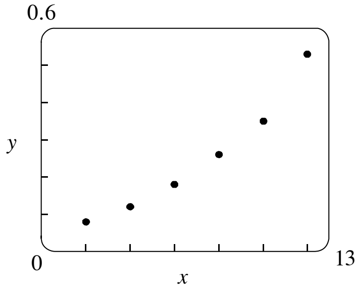
$t = \dfrac{\log(n/500)}{\log(4)}$. The inverse function gives the number of hours that have passed when the population size reaches n.

(b) Substituting $n = 10,000$ into the inverse function gives $t = \dfrac{\ln(10,000/500)}{\ln(4)} = \dfrac{\ln(20)}{\ln(4)} \approx 2.16\,\text{hours}$.

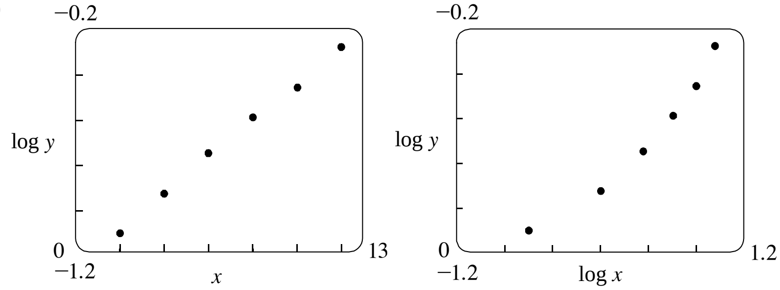
58. (a) $Q = Q_0(1 - e^{-t/a}) \quad \Rightarrow \quad \dfrac{Q}{Q_0} = 1 - e^{-t/a} \quad \Rightarrow \quad e^{-t/a} = 1 - \dfrac{Q}{Q_0} \quad \Rightarrow \quad -\dfrac{t}{a} = \ln\left(1 - \dfrac{Q}{Q_0}\right) \quad \Rightarrow$
$t = -a\ln(1 - Q/Q_0)$. This gives us the time t necessary to obtain a given charge Q.

(b) $Q = 0.9Q_0$ and $a = 2 \quad \Rightarrow \quad t = -2\ln(1 - 0.9Q_0/Q_0) = -2\ln 0.1 \approx 4.6$ seconds.

59. (a)

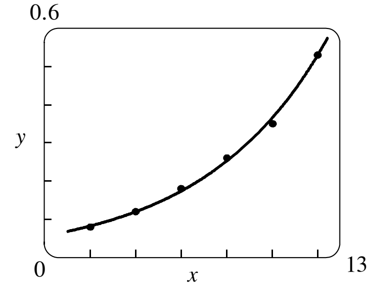


(b)

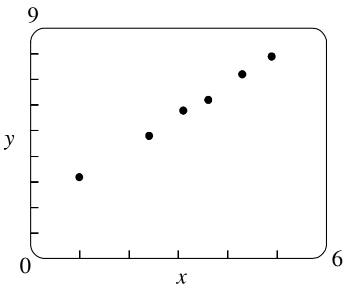


(c) Since the semi log plot is approximately linear, an exponential model is appropriate.

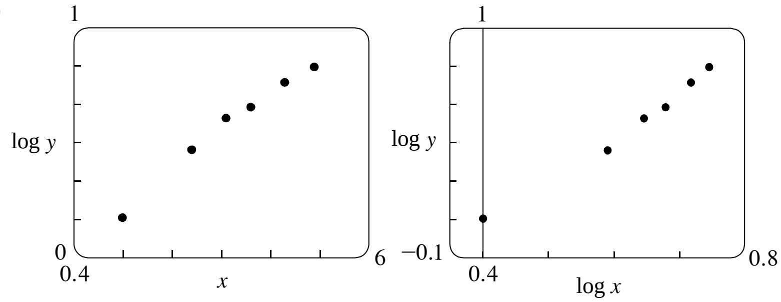
(d) Using a calculator to fit an exponential curve to the data gives $y = (0.056769)(1.204651)^x$.



60. (a)

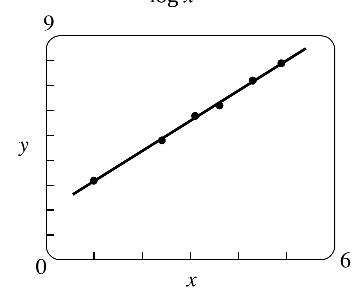


(b)

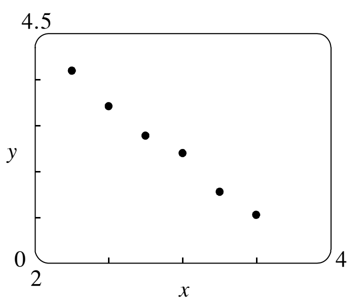


(c) Since the scatter plot is approximately linear, a linear model is appropriate.

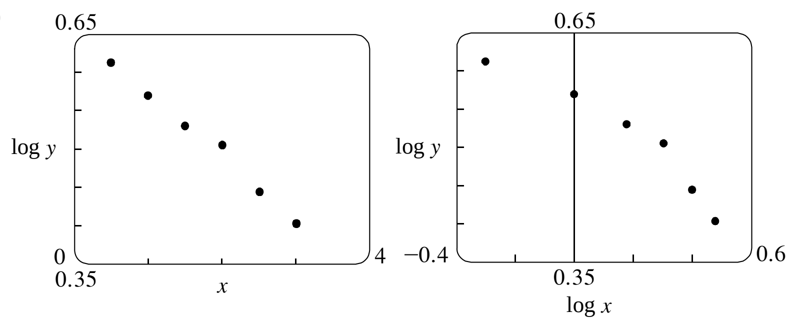
(d) Using a calculator to fit a line to the data gives $y = (1.208925)x + 1.961293$.



61. (a)

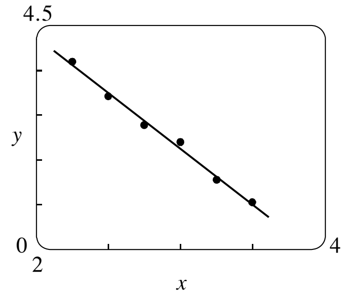


(b)

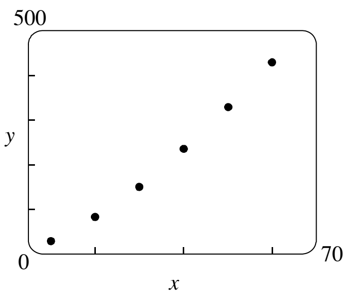


(c) Since the scatter plot is approximately linear, a linear model is appropriate.

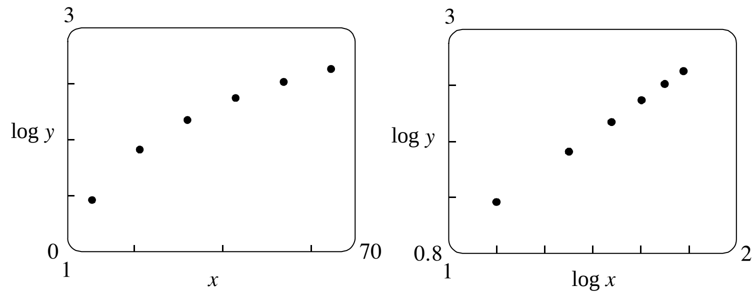
(d) Using a calculator to fit a line to the data gives $y = (-0.618857)\,x + 4.368000$.



62. (a)

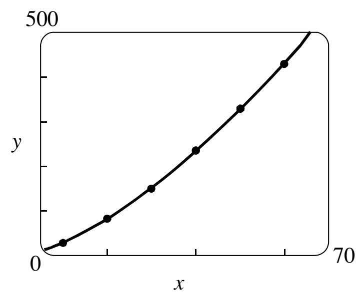


(b)

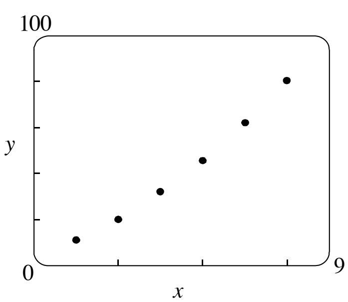


(c) Since the log-log plot is approximately linear, a power model is appropriate.

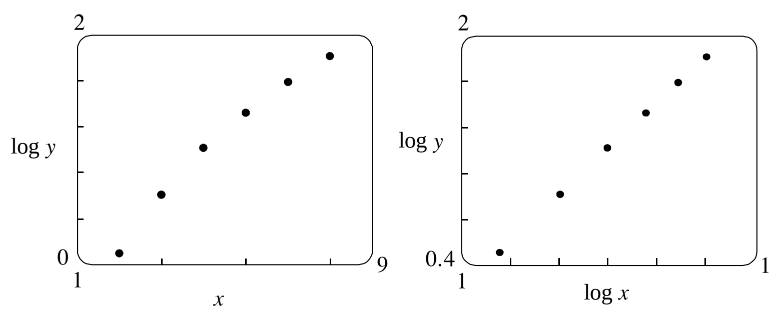
(d) Using a calculator to fit power curve to the data gives $y = (0.894488) \cdot x^{1.509230}$.



63. (a)

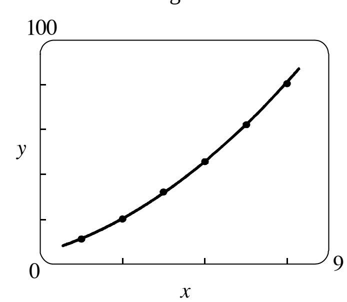


(b)

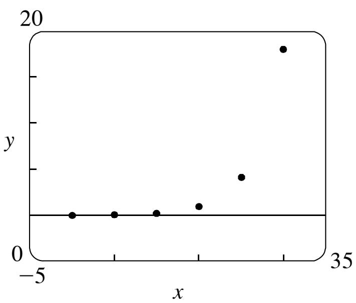


(c) Since the log-log plot is approximately linear, a power model is appropriate.

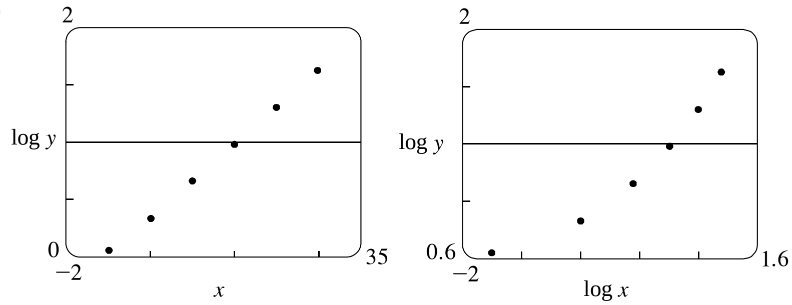
(d) Using a calculator to fit a power curve to the data gives $y = (1.260294) \cdot x^{2.002959}$



64. (a)

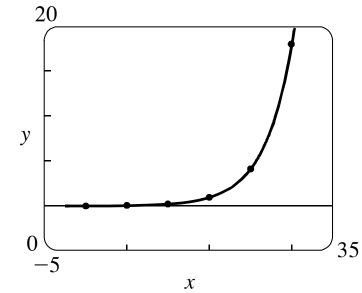


(b)

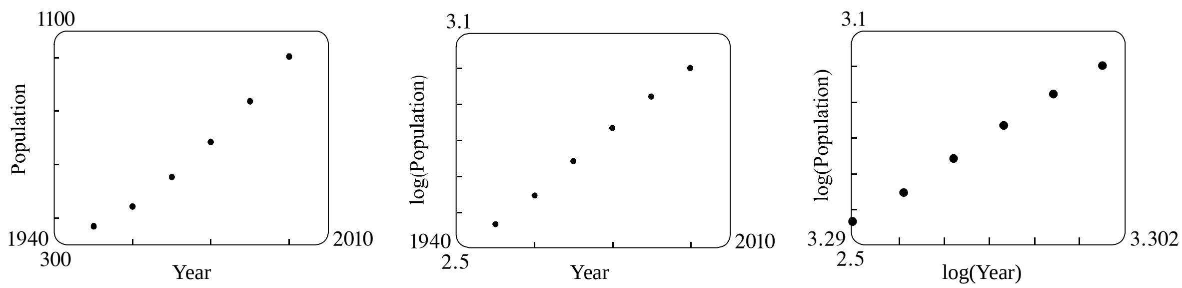


(c) Since the semi log plot is approximately linear, an exponential model is appropriate.

(d) Using a calculator to fit an exponential curve to the data gives $y = (0.002717) \cdot (1.339539)^x$.



65. (a)

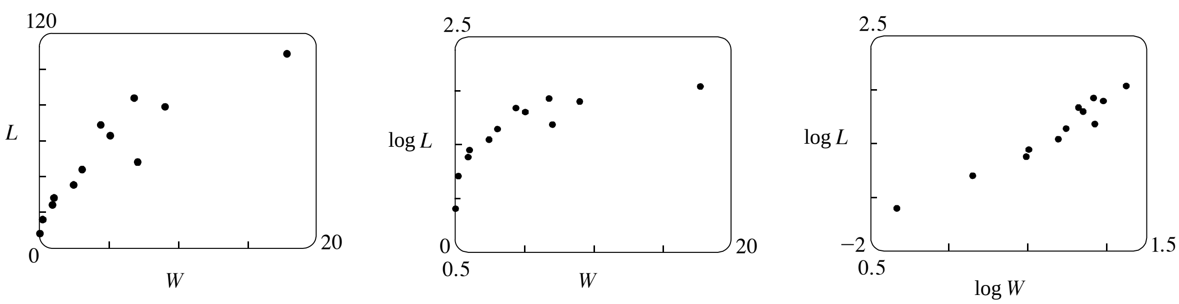


Since the semi log plot is approximately linear, an exponential model is appropriate.

(b) Using a calculator to fit an exponential curve to the data gives $P = \left(2.276131 \cdot 10^{-15}\right) \cdot (1.020529)^Y$ where P is the population in millions and Y is the year. Alternatively, we could have defined Y to be the number of years since 1950.

(c) In 2010, the model predicts a population of $P = \left(2.276131 \cdot 10^{-15}\right) \cdot (1.020529)^{2010} \approx 1247\,\text{million}$. The model overestimates the true population by $1247 - 1173 = 74\,\text{million}$. Therefore, this exponential model does not generalize well to the future population growth in India.

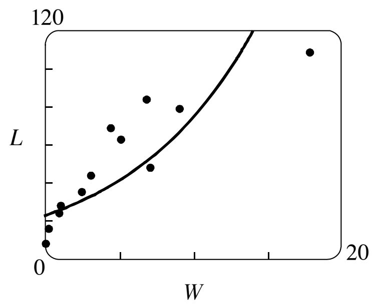
66. (a)



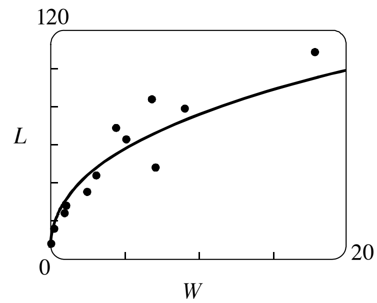
Since the log-log plot is approximately linear, a power model is appropriate.

(b) Using a calculator to fit an exponential curve to the data gives $L = (22.874763) \cdot (1.126290)^W$. Fitting a power curve to the data gives $L = (30.562377) \cdot W^{0.395199}$

(c)



Exponential Model

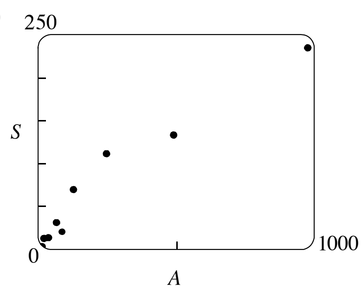


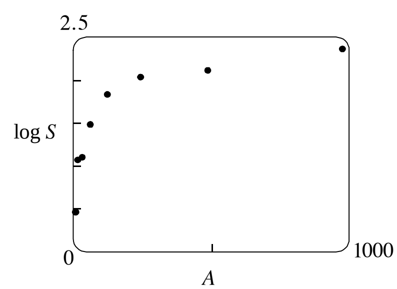
Power Model

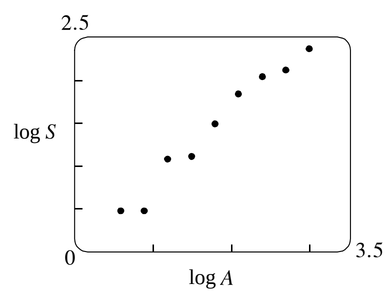
The power model is a more suitable fit to the data.

(d) As predicted by the power model in (b), the wingspan of a 45-lb bird is $L(45) = (30.562377) \cdot (45)^{0.395199} \approx 137.57\,\text{inches}$. This suggests the dodo bird would require a wingspan close to $138\,$inches in order to fly. The actual wingspan of the dodo was much shorter and therefore the bird could not create enough lift for flight.

67. (a)

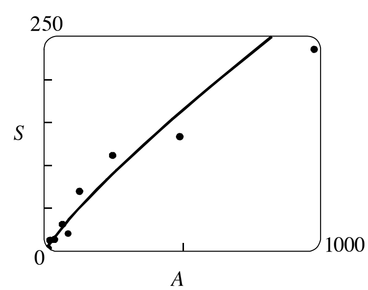




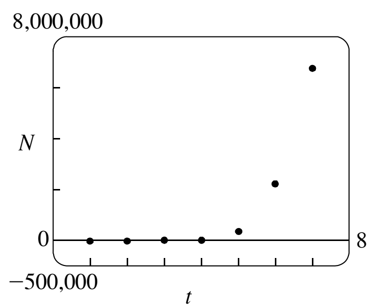


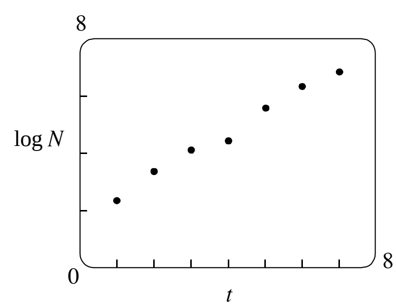
Since the log-log plot is approximately linear, a power model is appropriate.

(b) Using a calculator to fit a power curve to the data gives $S = (0.881518) \cdot A^{0.841701}$.

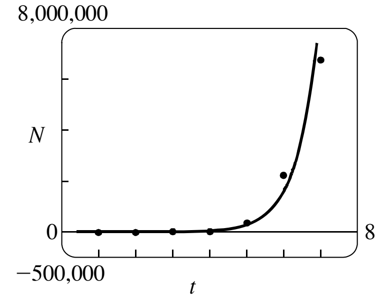


68. (a)

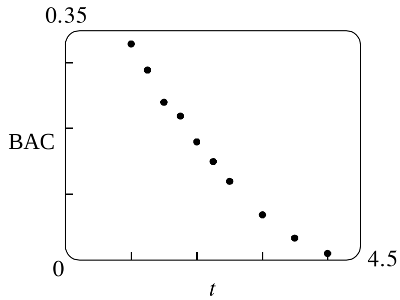


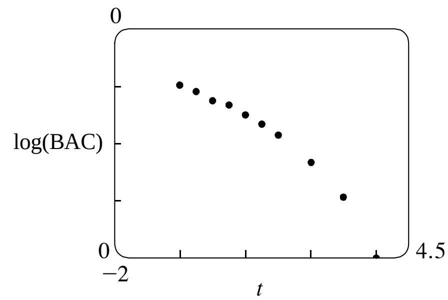


(b) Using a calculator to fit an exponential curve to the data gives $N = (54.980508) \cdot (5.543613)^t$. The exponential function fits the curve well.

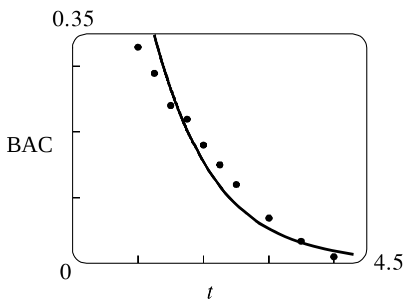


69. (a)





(b)



Using a calculator to fit an exponential curve to the data gives $C(t) = (1.343328) \cdot (0.338676)^t$ where $C(t)$ is the blood alcohol concentration after t hours. The exponential function overestimates BAC for small values of t.

(c) Solve for t in the equation: $C(t) \leq 0.08 \quad \Leftrightarrow \quad (1.343328) \cdot (0.338676)^t \leq 0.08 \quad \Leftrightarrow \quad \ln\left(0.338676^t\right) \leq \ln\left(\frac{0.08}{1.343328}\right)$

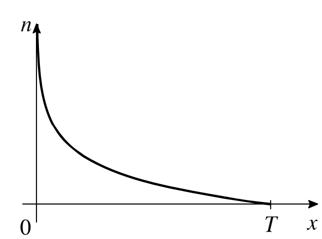
$\Leftrightarrow \quad t \ln(0.338676) \leq \ln\left(\frac{0.08}{1.343328}\right) \quad \Leftrightarrow \quad t \geq \dfrac{\ln(0.08/1.343328)}{\ln(0.338676)} \quad \left[\begin{array}{l}\text{inequality switched direction} \\ \text{because } \ln(0.338676) < 0\end{array}\right] \approx 2.61\,\text{hr}$. Therefore, the driver's blood alcohol concentration will be under the legal limit after approximately 2.6 hours.

70. (a) Let $C(n)$ be the number of DNA molecules after n cycles. The number of molecules doubles every cycle so that $C(1) = 2x, \quad C(2) = 2C(1) = 2^2 x, \quad C(3) = 2C(2) = 2^3 x$ and in general $C(n) = x \cdot 2^n$.

(b) The threshold is reached when $C(n) = T \quad \Leftrightarrow \quad x \cdot 2^n = T \quad \Leftrightarrow \quad \ln(2^n) = \ln\left(\dfrac{T}{x}\right) \quad \Leftrightarrow \quad n = \dfrac{\ln(T/x)}{\ln(2)}$

(c) Observe that we can rewrite

$n(x) = \dfrac{\ln(T/x)}{\ln(2)} = -\dfrac{\ln(x/T)}{\ln(2)} = -\left(\dfrac{1}{\ln 2}\right) \ln\left(\dfrac{1}{T} \cdot x\right)$. Therefore, the graph of $n(x)$ can be obtained from the graph of $y = \ln x$ by horizontally stretching by factor T, reflecting about the x-axis, and vertically compressing by factor $\ln 2$. These transformations lead to the sketch shown. We see that having a larger initial number of DNA molecules leads to shorter times to reach the detection threshold.



PROJECT The Coding Function of DNA

1. The domain is $\{A, T, C, G\}$. Since the coding function maps an input (codon) to exactly one output (amino acid), the biggest possible range is 4 amino acids.

2. With two-base codons, there are $4^2 = 16$ possibilities which can be found by combining the bases in all possible arrangements. Thus, the domain is $\{AA, AT, AC, AG, TA, TT, TC, TG, CA, CT, CC, CG, GA, GT, GC, GG\}$. The biggest possible range is 16 output amino acids, which would be achieved if every codon mapped to a distinct amino acid.

3. With three-base codons, there are $4^3 = 64$ possibilities. The domain of 64 codons can be found by iterating through all possible three-base codons using the 4 bases. Formally, the domain is $\{(i, j, k) \mid i$ and j and $k \in \{A, T, C, G\}\}$. The biggest possible range is 64 amino acids.

4. As observed in Problems 1 and 2, codons with 1 or 2 bases have too small a range to generate the 20 different amino acids required to build proteins. Three-base codons have 64 possible "words" which is more than enough to code for 20 amino acids. Codons with 4 or more bases would have excess redundancy in the genetic code, thus making three-bases the optimal number for coding amino acids.

5. Three-base codons can code up to 64 amino acids, however, there are only 20 distinct amino acids that are coded for by the DNA of living organisms. Therefore, there are multiple codons that produce the same amino acid. E.g. the codons AAG and AAA both code for the amino acid Lysine. This implies the coding function is not one-to-one.

1.6 Sequences and Difference Equations

1. $a_n = \dfrac{2n}{n^2+1}$, so the sequence is $\left\{\dfrac{2}{1+1}, \dfrac{4}{4+1}, \dfrac{6}{9+1}, \dfrac{8}{16+1}, \dfrac{10}{25+1}, \ldots\right\} = \left\{1, \dfrac{4}{5}, \dfrac{3}{5}, \dfrac{8}{17}, \dfrac{5}{13}, \ldots\right\}$.

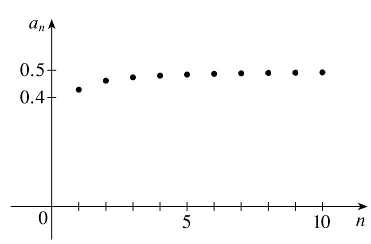
2. $a_n = \dfrac{3^n}{1+2^n}$, so the sequence is $\left\{\dfrac{3}{1+2}, \dfrac{9}{1+4}, \dfrac{27}{1+8}, \dfrac{81}{1+16}, \dfrac{243}{1+32}, \ldots\right\} = \left\{1, \dfrac{9}{5}, 3, \dfrac{81}{17}, \dfrac{81}{11}, \ldots\right\}$.

3. $a_n = \dfrac{(-1)^{n-1}}{5^n}$, so the sequence is $\left\{\dfrac{1}{5^1}, \dfrac{-1}{5^2}, \dfrac{1}{5^3}, \dfrac{-1}{5^4}, \dfrac{1}{5^5}, \ldots\right\} = \left\{\dfrac{1}{5}, -\dfrac{1}{25}, \dfrac{1}{125}, -\dfrac{1}{625}, \dfrac{1}{3125}, \ldots\right\}$.

4. $a_n = \cos\dfrac{n\pi}{2}$, so the sequence is $\left\{\cos\dfrac{\pi}{2}, \cos\pi, \cos\dfrac{3\pi}{2}, \cos 2\pi, \cos\dfrac{5\pi}{2}, \ldots\right\} = \{0, -1, 0, 1, 0, \ldots\}$.

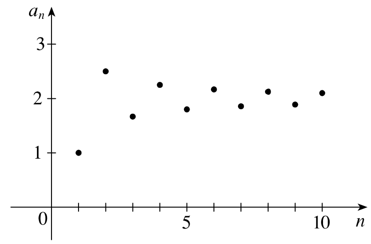
5.

n	$a_n = \dfrac{3n}{1+6n}$
1	0.4286
2	0.4615
3	0.4737
4	0.4800
5	0.4839
6	0.4865
7	0.4884
8	0.4898
9	0.4909
10	0.4918



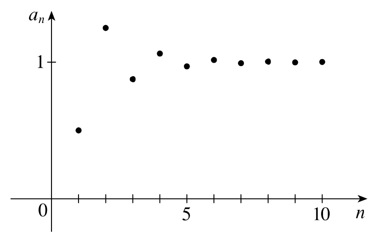
6.

n	$a_n = 2 + \dfrac{(-1)^n}{n}$
1	1.0000
2	2.5000
3	1.6667
4	2.2500
5	1.8000
6	2.1667
7	1.8571
8	2.1250
9	1.8889
10	2.1000



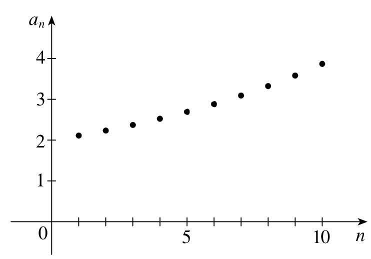
7.

n	$a_n = 1 + \left(-\frac{1}{2}\right)^n$
1	0.5000
2	1.2500
3	0.8750
4	1.0625
5	0.9688
6	1.0156
7	0.9922
8	1.0039
9	0.9980
10	1.0010



8.

n	$a_n = 1 + \dfrac{10^n}{9^n}$
1	2.1111
2	2.2346
3	2.3717
4	2.5242
5	2.6935
6	2.8817
7	3.0908
8	3.3231
9	3.5812
10	3.8680



9. $\left\{1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \ldots\right\}$. The denominator of the *n*th term is the *n*th positive odd integer, so $a_n = \dfrac{1}{2n-1}$.

10. $\left\{1, -\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \frac{1}{81}, \ldots\right\}$. Each term is $-\frac{1}{3}$ times the preceding term, so $a_n = \left(-\frac{1}{3}\right)^{n-1}$.

11. $\left\{-3, 2, -\frac{4}{3}, \frac{8}{9}, -\frac{16}{27}, \ldots\right\}$. The first term is -3 and each term is $-\frac{2}{3}$ times the preceding one, so $a_n = -3\left(-\frac{2}{3}\right)^{n-1}$.

12. $\{5, 8, 11, 14, 17, \ldots\}$. Each term is larger than the preceding term by 3, so $a_n = a_1 + d(n-1) = 5 + 3(n-1) = 3n + 2$.

13. $\left\{\frac{1}{2}, -\frac{4}{3}, \frac{9}{4}, -\frac{16}{5}, \frac{25}{6}, \ldots\right\}$. The numerator of the *n*th term is n^2 and its denominator is $n+1$. Including the alternating signs, we get $a_n = (-1)^{n+1}\dfrac{n^2}{n+1}$.

14. $\{1, 0, -1, 0, 1, 0, -1, 0, \ldots\}$. Two possibilities are $a_n = \sin\dfrac{n\pi}{2}$ and $a_n = \cos\dfrac{(n-1)\pi}{2}$.

15. $a_1 = 1$
$a_2 = 5a_1 - 3 = 5(1) - 3 = 2$
$a_3 = 5a_2 - 3 = 5(2) - 3 = 7$
$a_4 = 5a_3 - 3 = 5(7) - 3 = 32$
$a_5 = 5a_4 - 3 = 5(32) - 3 = 157$
$a_6 = 5a_5 - 3 = 5(157) - 3 = 782$

16. $a_1 = 6$
$a_2 = a_1/1 = 6/6 = 6$
$a_3 = a_2/2 = 6/2 = 3$
$a_4 = a_3/3 = 3/3 = 1$
$a_5 = a_4/4 = 1/4 = 1/4$
$a_6 = a_5/5 = (1/4)/5 = 1/20$

17. $a_1 = 2$
$a_2 = a_1/(1+a_1) = 2/(1+2) = 2/3$
$a_3 = a_2/(1+a_2) = (2/3)/(1+2/3) = 2/5$
$a_4 = a_3/(1+a_3) = (2/5)/(1+2/5) = 2/7$
$a_5 = a_4/(1+a_4) = (2/7)/(1+2/7) = 2/9$
$a_6 = a_5/(1+a_5) = (2/9)/(1+2/9) = 2/11$

18. $a_1 = 1$
$a_2 = 4 - a_1 = 4 - 1 = 3$
$a_3 = 4 - a_2 = 4 - 3 = 1$
$a_4 = 4 - a_3 = 4 - 1 = 3$
$a_5 = 4 - a_4 = 4 - 3 = 1$
$a_6 = 4 - a_5 = 4 - 1 = 3$

19. $a_1 = 1$
$a_2 = \sqrt{3a_1} = (3 \cdot 1)^{1/2} = 3^{1/2}$
$a_3 = \sqrt{3a_2} = (3 \cdot 3^{1/2})^{1/2} = 3^{3/4}$
$a_4 = \sqrt{3a_3} = (3 \cdot 3^{3/4})^{1/2} = 3^{7/8}$
$a_5 = \sqrt{3a_4} = (3 \cdot 3^{7/8})^{1/2} = 3^{15/16}$
$a_6 = \sqrt{3a_5} = (3 \cdot 3^{15/16})^{1/2} = 3^{31/32}$

20. $a_1 = 3$
$a_2 = \sqrt{3a_1} = (3 \cdot 3)^{1/2} = 3$
$a_3 = \sqrt{3a_2} = (3 \cdot 3)^{1/2} = 3$
$a_4 = \sqrt{3a_3} = (3 \cdot 3)^{1/2} = 3$
$a_5 = \sqrt{3a_4} = (3 \cdot 3)^{1/2} = 3$
$a_6 = \sqrt{3a_5} = (3 \cdot 3)^{1/2} = 3$

21. $a_1 = 2$
$a_2 = 1$
$a_3 = a_2 - a_1 = 1 - 2 = -1$
$a_4 = a_3 - a_2 = -1 - 1 = -2$
$a_5 = a_4 - a_3 = -2 - (-1) = -1$
$a_6 = a_5 - a_4 = -1 - (-2) = 1$

22. $a_1 = 1$
$a_2 = 2$
$a_3 = a_2 + 2a_1 = 2 + 2(1) = 4$
$a_4 = a_3 + 2a_2 = 4 + 2(2) = 8$
$a_5 = a_4 + 2a_3 = 8 + 2(4) = 16$
$a_6 = a_5 + 2a_4 = 16 + 2(8) = 32$

23. Let a_n be the number of rabbit pairs in the *n*th month. Clearly $a_1 = 1 = a_2$. In the *n*th month, each pair that is 2 or more months old (that is, a_{n-2} pairs) will produce a new pair to add to the a_{n-1} pairs already present. Thus, $a_n = a_{n-1} + a_{n-2}$, so that $\{a_n\} = \{f_n\}$, the Fibonacci sequence.

24. (a) We are given that the initial population is 5000, so $P_0 = 5000$. The number of catfish increases by 8% per month and is decreased by 300 per month, so $P_1 = P_0 + 8\%P_0 - 300 = 1.08P_0 - 300$, $P_2 = 1.08P_1 - 300$, and so on. Thus, $P_n = 1.08P_{n-1} - 300$.

(b) Using the recursive formula with $P_0 = 5000$, we get $P_1 = 5100$, $P_2 = 5208$, $P_3 = 5325$ (rounding any portion of a catfish), $P_4 = 5451$, $P_5 = 5587$, and $P_6 = 5734$, which is the number of catfish in the pond after six months.

25. The solution to the difference equation $N_{t+1} = RN_t$ as given in equation (2) is $N_t = N_0R^t$. When $N_0 = 1$, the solution is $N_t = R^t$.

(a) The solution $N_t = R^t$ says that the tth term is found by multiplying R by itself t times. If $R < 1$, N_t will decrease as t increases. For example, consider the case when $R = \frac{1}{2}$, so that $N_1 = \frac{1}{2}, N_2 = \frac{1}{4}, N_3 = \frac{1}{8}$, and if t is very large, say 100, then $N_{100} = \frac{1}{2^{100}} \approx 8 \cdot 10^{-31} \approx 0$. Therefore, we infer that when $R < 1$, the value of N_t approaches zero as t becomes large.

(b) When $R = 1$, the general solution is $N_t = (1)^t = 1$. That is all terms in the sequence have a value of one.

(c) When $R > 1$, the solution $N_t = R^t$ will increase as t increases. For example, consider the case when $R = 2$, so that $N_1 = 2, N_2 = 4, N_3 = 8$, and if t is very large, say 100, then $N_{100} = 2^{100} \approx 10^{30}$. Therefore, we infer that when $R > 1$, the sequence grows indefinitely as t increases.

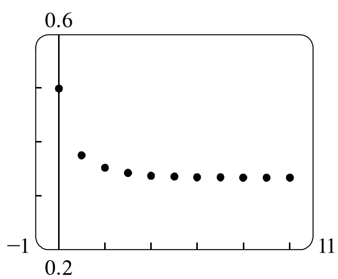
26. (a) If $N_{t+1} = f(N_t)$, then $(f \circ f)(N_t) = f\left(f(N_t)\right) = f(N_{t+1}) = N_{t+2}$. Therefore, $f \circ f$ represents the population size two time steps ahead of the current time.

(b) $f^{-1}(N_{t+1}) = f^{-1}(f(N_t)) = N_t$ since f is a one-to-one function. Therefore, f^{-1} takes as input the population size at a given time and outputs the population one time step earlier.

27-31 A calculator was used to compute the first 10 terms of each sequence and these (t, x_t) data points were then graphed.

27.

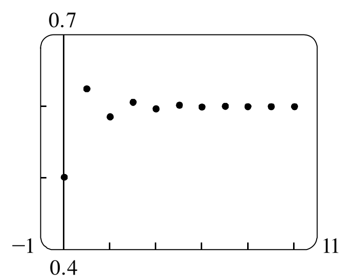
t	x_t
0	0.5000
1	0.3750
2	0.3516
3	0.3419
4	0.3375
5	0.3354
6	0.3344
7	0.3338
8	0.3336
9	0.3335
10	0.3334



The sequence decreases and approaches $1/3$.

28.

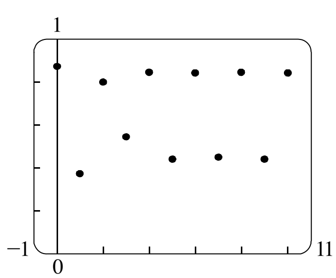
t	x_t
0	0.5000
1	0.6250
2	0.5859
3	0.6065
4	0.5966
5	0.6017
6	0.5992
7	0.6004
8	0.5998
9	0.6001
10	0.5999



The sequence oscillates above and below 0.6, approaching 0.6 as t increases.

29.

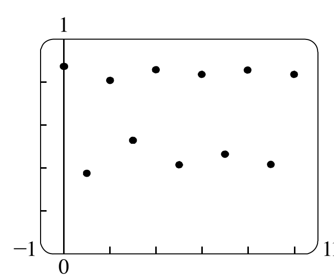
t	x_t
0	0.8750
1	0.3741
2	0.8008
3	0.5456
4	0.8479
5	0.4411
6	0.8431
7	0.4523
8	0.8472
9	0.4427
10	0.8438



As t increases, the sequence cycles near two values (0.44 and 0.84).

30.

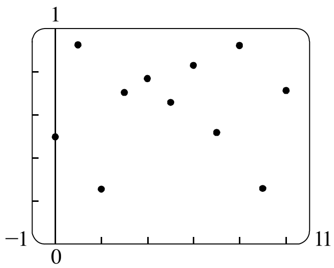
t	x_t
0	0.8750
1	0.3773
2	0.8106
3	0.5297
4	0.8595
5	0.4167
6	0.8386
7	0.4670
8	0.8587
9	0.4185
10	0.8396



As t increases, the sequence cycles near two values (0.42 and 0.84).

31.

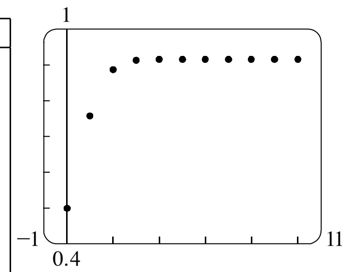
t	x_t
0	0.5000
1	0.9250
2	0.2567
3	0.7060
4	0.7681
5	0.6591
6	0.8313
7	0.5189
8	0.9237
9	0.2608
10	0.7134



As t increases, the sequence cycles irregularly among a range of values between 0.25 and 0.93.

32. (a)

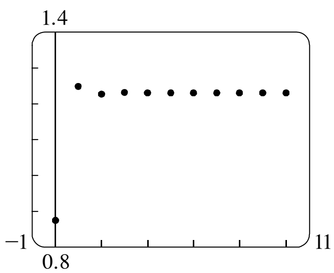
t	x_t
0	0.5000
1	0.7582
2	0.8880
3	0.9135
4	0.9161
5	0.9163
6	0.9163
7	0.9163
8	0.9163
9	0.9163
10	0.9163



(b) The Ricker model increases toward a value of 0.9163 while the logistic model converges in an oscillatory fashion toward 0.6.

33.

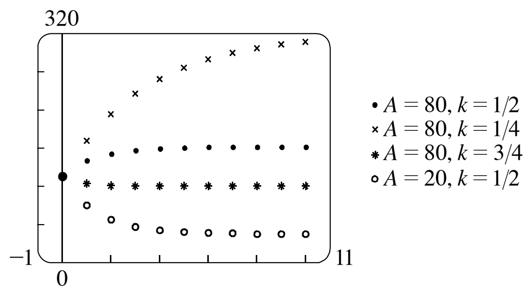
t	x_t
0	0.8750
1	1.2475
2	1.2254
3	1.2306
4	1.2294
5	1.2297
6	1.2296
7	1.2296
8	1.2296
9	1.2296
10	1.2296



The Ricker model approaches 1.2296 in an oscillatory fashion. This is very different compared to the cycling behavior observed from the logistic model in Exercise 29.

34. (a) At each time step, there is an increase in concentration of A and a decrease of kC_t. The recursion is $c_{t+1} = c_t + \text{inflow} - \text{outflow} = c_t + A - kc_t$

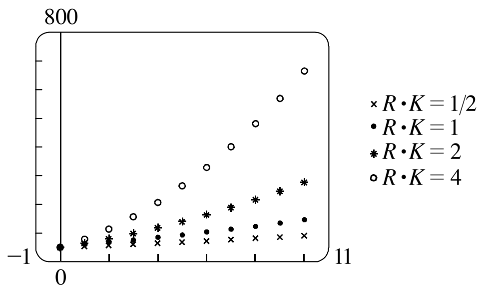
(b) A calculator was used to calculate and graph the first 10 terms of the sequence using $c_0 = 120$ for several different values of A and k. In every case, the concentration converges toward a constant value as t increases. Larger values of A result in higher long-run concentrations while larger values of k give lower long-term concentrations. Note that each sequence converges toward the value A/k.



35. (a) Since area is proportional to number of bacteria, the relationship between colony radius, r, and the population size can be found as follows: $A_{\text{circle}} = kN \quad \Leftrightarrow \quad \pi r^2 = kN \quad \Leftrightarrow \quad r = \sqrt{\frac{k}{\pi} N}$ where k is a proportionality constant.

Since I is proportional to the colony circumference, C, the input of new individuals is
$I = RC = R(2\pi r) = 2\pi R\sqrt{\frac{k}{\pi}N} = RK\sqrt{N}$ where $K = 2\pi\sqrt{\frac{k}{\pi}}$ is a constant. This gives the recursion equation
$N_{t+1} = N_t + RK\sqrt{N_t}$.

(b) A calculator was used to calculate and graph the first 10 terms of the sequence using $N_0 = 40$ for several different values of $R \cdot K$.



36. Let N_t be the number of individuals in the colony at time t. Since volume is proportional to N_t, the relationship between colony radius, r, and the population size can be found as follows: $V_{\text{sphere}} = kN_t \quad \Leftrightarrow \quad \frac{4}{3}\pi r^3 = kN_t \quad \Leftrightarrow \quad r = \sqrt[3]{\frac{3k}{4\pi}N_t}$ where k is a proportionality constant. Because growth occurs only at the surface-resource interface, the input of new individuals, I, is proportional to the surface area, A, of the spherical colony. With proportionality constant R, this gives $I = R \cdot A = R\left(4\pi r^2\right) = 4\pi R\left(\frac{3k}{4\pi}N_t\right)^{2/3} = C \cdot (N_t)^{2/3}$ where $C = 4\pi R\left(\frac{3k}{4\pi}\right)^{2/3}$ is a constant. Therefore, the difference equation is $N_{t+1} = N_t + I = N_t + C \cdot (N_t)^{2/3}$.

37. (a) Let n_t represent the number of fish at time t. First, the fish face predation reducing the population to $n^* = n_t - dn_t$. The n^* fish then produce offspring and die resulting in a population of $n^{**} = n^* + bn^* - n^* = bn^*$. Finally, m additional fish are added to the population and all swim to sea giving the recursion $n_{t+1} = n^{**} + m = bn^* + m = b\,(1-d)\,n_t + m$.

(b) In this case, the fish first reproduce and die resulting in a population of $n^* = bn_t$. Then, m fish are added increasing the population to $n^{**} = n^* + m$. Lastly, all the fish face predation while swimming downstream giving a final population of $n_{t+1} = n^{**} - dn^{**} = (1-d)\,(n^* + m) = (1-d)\,(bn_t + m)$.

(c) The difference in recursions from parts (a) and (b) is
$[b\,(1-d)\,n_t + m] - [(1-d)\,(bn_t + m)] = m\,(1-(1-d)) = md > 0$ since $m > 0$ and $d > 0$. Hence, the recursion from part (a) gives the largest increase in population from one year to the next. This seems sensible since all the offspring and additional fish face predation in part (b), whereas all the additional fish survive in part (a).

38. Let f_t represent the fraction of methylated DNA at time t. The fraction of unmethylated DNA is then given by $1 - f_t$. First, m unmethylated locations become methylated giving a new fraction of $f^* = f_t + m\,(1-f_t)$. Then u methylated locations become unmethylated giving a final fraction of $f_{t+1} = f^* - uf^* = (1-u)\,[f_t + m\,(1-f_t)] = (1-m)\,(1-u)\,f_t + m\,(1-u)$.

39. $$p_{t+1} = \frac{a_{t+1}}{a_{t+1}+b_{t+1}} = \frac{R_a a_t}{R_a a_t + R_b b_t} = \frac{\frac{R_a}{R_b}a_t}{\frac{R_a}{R_b}a_t + b_t} = \frac{\alpha a_t}{\alpha a_t + b_t} \quad \text{where } \alpha = R_a/R_b$$

$$= \frac{\alpha\dfrac{a_t}{a_t+b_t}}{\alpha\dfrac{a_t}{a_t+b_t} + \dfrac{b_t}{a_t+b_t}} = \frac{\alpha p_t}{\alpha p_t + \dfrac{a_t+b_t-a_t}{a_t+b_t}} = \frac{\alpha p_t}{\alpha p_t + 1 - \dfrac{a_t}{a_t+b_t}} = \frac{\alpha p_t}{\alpha p_t + 1 - p_t}$$

40. $a_{n+1} = \begin{cases} \frac{1}{2}a_n & \text{if } a_n \text{ is an even number} \\ 3a_n + 1 & \text{if } a_n \text{ is an odd number} \end{cases}$ When $a_1 = 11$, the first 40 terms are 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4. When $a_1 = 25$, the first 40 terms are 25, 76, 38, 19, 58, 29, 88, 44, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4.

The famous Collatz conjecture is that this sequence always reaches 1, regardless of the starting point a_1.

PROJECT Drug Resistance in Malaria

1. Let g_{ij} represent the frequency of ij individuals. For random and independent unions, the frequency of RR individuals will be given by the probability that any two R individuals unite, that is, $g_{\text{RR}} = p_t \cdot p_t = p_t^2$. Similarly, the probability that any two S individuals unite is $g_{\text{SS}} = (1 - p_t) \cdot (1 - p_t) = (1 - p_t)^2$. The probability that an RS diploid individual forms is $g_{\text{RS}} = 2p_t (1 - p_t)$. We include a factor of 2 since there are two ways of selecting RS: first an R individual, then an S individual, and vice versa. Observe that the sum of the frequencies $g_{\text{RR}} + g_{\text{RS}} + g_{\text{SS}} = 1$ as is required.

2. Scaling each frequency by the respective probabilities of survival gives $W_{\text{RR}}p_t^2$, $W_{\text{SS}} (1 - p_t)^2$, and $2W_{\text{RS}}p_t (1 - p_t)$. These new quantities no longer sum to one, but instead sum to $\overline{W} = W_{\text{RR}}p_t^2 + W_{\text{SS}} (1 - p_t)^2 + 2W_{\text{RS}}p_t (1 - p_t)$. To ensure the new diploid frequencies (after survival) g^* sum to one, we normalize or divide by $\overline{W}$ giving $g_{\text{RR}}^* = \frac{W_{\text{RR}}}{\overline{W}}p_t^2$, $g_{\text{SS}}^* = \frac{W_{\text{SS}}}{\overline{W}} (1 - p_t)^2$, and $g_{\text{RS}}^* = 2\frac{W_{\text{RS}}}{\overline{W}}p_t (1 - p_t)$.

3. If there are a total of N individuals in the population, then after reproduction, the number of R-type haploids is $bN \left(g_{\text{RR}}^* + \frac{1}{2}g_{\text{RS}}^*\right) = bN \left(\frac{W_{\text{RR}}}{\overline{W}}p_t^2 + \frac{W_{\text{RS}}}{\overline{W}}p_t (1 - p_t)\right)$ and the total number of haploid individuals is $bN \left(g_{\text{RR}}^* + g_{\text{RS}}^* + g_{\text{SS}}^*\right) = bN (1) = bN$. Therefore, the frequency of R-type haploids is

$$p_{t+1} = \frac{bN \left(\frac{W_{\text{RR}}}{\overline{W}}p_t^2 + \frac{W_{\text{RS}}}{\overline{W}}p_t (1 - p_t)\right)}{bN} = \frac{W_{\text{RR}}}{\overline{W}}p_t^2 + \frac{W_{\text{RS}}}{\overline{W}}p_t (1 - p_t) = \frac{W_{\text{RR}}p_t^2 + W_{\text{RS}}p_t (1 - p_t)}{W_{\text{RR}}p_t^2 + 2W_{\text{RS}}p_t (1 - p_t) + W_{\text{SS}} (1 - p_t)^2}.$$

4. Substituting $W_{\text{RR}} = \frac{1}{4}$ and $W_{\text{RS}} = W_{\text{SS}} = \frac{1}{2}$ into p_{t+1} gives

$$\begin{aligned} p_{t+1} &= \frac{\frac{1}{4}p_t^2 + \frac{1}{2}p_t (1 - p_t)}{\frac{1}{4}p_t^2 + 2 \cdot \frac{1}{2}p_t (1 - p_t) + \frac{1}{2} (1 - p_t)^2} \\ &= \frac{p_t^2 + 2p_t (1 - p_t)}{p_t^2 + 4p_t (1 - p_t) + 2 (1 - p_t)^2} \\ &= \frac{-p_t^2 + 2p_t}{-3p_t^2 + 4p_t + 2 - 4p_t + 2p_t^2} \\ &= \frac{p_t^2 - 2p_t}{p_t^2 - 2} \end{aligned}$$

This is the same rational function discussed in Section 1.2.

1 Review

TRUE-FALSE QUIZ

1. False. Let $f(x) = x^2$, $s = -1$, and $t = 1$. Then $f(s+t) = (-1+1)^2 = 0^2 = 0$, but $f(s) + f(t) = (-1)^2 + 1^2 = 2 \neq 0 = f(s+t)$.

2. False. Let $f(x) = x^2$. Then $f(-2) = 4 = f(2)$, but $-2 \neq 2$.

3. False. Let $f(x) = x^2$. Then $f(3x) = (3x)^2 = 9x^2$ and $3f(x) = 3x^2$. So $f(3x) \neq 3f(x)$.

4. True. If $x_1 < x_2$ and f is a decreasing function, then the y-values get smaller as we move from left to right. Thus, $f(x_1) > f(x_2)$.

5. True. See the Vertical Line Test.

6. False. Let $f(x) = x^2$ and $g(x) = 2x$. Then $(f \circ g)(x) = f(g(x)) = f(2x) = (2x)^2 = 4x^2$ and $(g \circ f)(x) = g(f(x)) = g(x^2) = 2x^2$. So $f \circ g \neq g \circ f$.

7. False. Let $f(x) = x^3$. Then f is one-to-one and $f^{-1}(x) = \sqrt[3]{x}$. But $1/f(x) = 1/x^3$, which is not equal to $f^{-1}(x)$.

8. True. We can divide by e^x since $e^x \neq 0$ for every x.

9. True. The function $\ln x$ is an increasing function on $(0, \infty)$.

10. False. Let $x = e$. Then $(\ln x)^6 = (\ln e)^6 = 1^6 = 1$, but $6 \ln x = 6 \ln e = 6 \cdot 1 = 6 \neq 1 = (\ln x)^6$. What *is* true, however, is that $\ln(x^6) = 6 \ln x$ for $x > 0$.

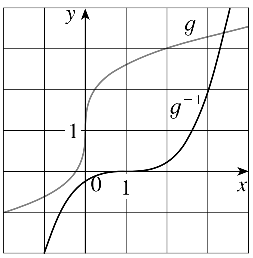
11. False. Let $x = e^2$ and $a = e$. Then $\dfrac{\ln x}{\ln a} = \dfrac{\ln e^2}{\ln e} = \dfrac{2 \ln e}{\ln e} = 2$ and $\ln \dfrac{x}{a} = \ln \dfrac{e^2}{e} = \ln e = 1$, so in general the statement is false. What *is* true, however, is that $\ln \dfrac{x}{a} = \ln x - \ln a$.

12. False. For example, if $x = -3$, then $\sqrt{(-3)^2} = \sqrt{9} = 3$, not -3.

EXERCISES

1. (a) When $x = 2$, $y \approx 2.7$. Thus, $f(2) \approx 2.7$.
   (b) $f(x) = 3 \quad \Rightarrow \quad x \approx 2.3, 5.6$
   (c) The domain of f is $-6 \leq x \leq 6$, or $[-6, 6]$.
   (d) The range of f is $-4 \leq y \leq 4$, or $[-4, 4]$.
   (e) f is increasing on $[-4, 4]$, that is, on $-4 \leq x \leq 4$.
   (f) f is not one-to-one since it fails the Horizontal Line Test.
   (g) f is odd since its graph is symmetric about the origin.

2. (a) When $x = 2$, $y = 3$. Thus, $g(2) = 3$.
   (b) g is one-to-one because it passes the Horizontal Line Test.
   (c) When $y = 2$, $x \approx 0.2$. So $g^{-1}(2) \approx 0.2$.
   (d) The range of g is $[-1, 3.5]$, which is the same as the domain of g^{-1}.
   (e) We reflect the graph of g through the line $y = x$ to obtain the graph of g^{-1}.



3. (a) $S(1000) \approx -36\,\text{m}$

   (b) The sea level was lowest about $18,000$ years ago and highest about $121,000$ ago present.

   (c) $\{S \,|\, -114 \le S \le 8\} = [-114, 8]$

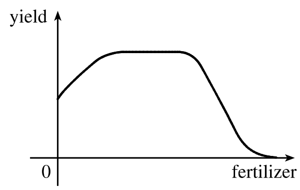
   (d) The drops in sea level around 150,000 and 18,000 years ago correspond to periods of glaciation during which large amounts of Earth's water was frozen in ice sheets.

4. (a) When $F = 70$, $t \approx 1982$.

   (b) The lowest fish catch was about 18 million and the largest fish catch was about 86 million. So the range of F is approximately $\{F \,|\, 18 \le F \le 86\} = [18, 86]$.

5. $f(x) = x^2 - 2x + 3$, so $f(a+h) = (a+h)^2 - 2(a+h) + 3 = a^2 + 2ah + h^2 - 2a - 2h + 3$, and
$$\frac{f(a+h) - f(a)}{h} = \frac{(a^2 + 2ah + h^2 - 2a - 2h + 3) - (a^2 - 2a + 3)}{h} = \frac{h(2a + h - 2)}{h} = 2a + h - 2.$$

6. There will be some yield with no fertilizer, increasing yields with increasing fertilizer use, a leveling-off of yields at some point, and disaster with too much fertilizer use.



7. $f(x) = 2/(3x-1)$. Domain: $3x - 1 \ne 0 \;\Rightarrow\; 3x \ne 1 \;\Rightarrow\; x \ne \frac{1}{3}$. $D = \left(-\infty, \frac{1}{3}\right) \cup \left(\frac{1}{3}, \infty\right)$
   Range: all reals except 0 ($y = 0$ is the horizontal asymptote for f.) $R = (-\infty, 0) \cup (0, \infty)$

8. $g(x) = \sqrt{16 - x^4}$. Domain: $16 - x^4 \ge 0 \;\Rightarrow\; x^4 \le 16 \;\Rightarrow\; |x| \le \sqrt[4]{16} \;\Rightarrow\; |x| \le 2$. $D = [-2, 2]$
   Range: $y \ge 0$ and $y \le \sqrt{16} \;\Rightarrow\; 0 \le y \le 4$. $R = [0, 4]$

9. $h(x) = \ln(x + 6)$. Domain: $x + 6 > 0 \;\Rightarrow\; x > -6$. $D = (-6, \infty)$
   Range: $x + 6 > 0$, so $\ln(x + 6)$ takes on all real numbers and, hence, the range is $\mathbb{R}$.
   $R = (-\infty, \infty)$

10. $y = F(t) = 3 + \cos 2t$. Domain: $\mathbb{R}$. $D = (-\infty, \infty)$
   Range: $-1 \le \cos 2t \le 1 \;\Rightarrow\; 2 \le 3 + \cos 2t \le 4 \;\Rightarrow\; 2 \le y \le 4$. $R = [2, 4]$

11. (a) To obtain the graph of $y = f(x) + 8$, we shift the graph of $y = f(x)$ up 8 units.

   (b) To obtain the graph of $y = f(x + 8)$, we shift the graph of $y = f(x)$ left 8 units.

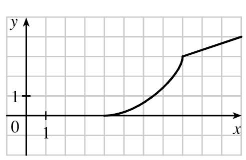
   (c) To obtain the graph of $y = 1 + 2f(x)$, we stretch the graph of $y = f(x)$ vertically by a factor of 2, and then shift the resulting graph 1 unit upward.

   (d) To obtain the graph of $y = f(x - 2) - 2$, we shift the graph of $y = f(x)$ right 2 units (for the "-2" inside the parentheses), and then shift the resulting graph 2 units downward.

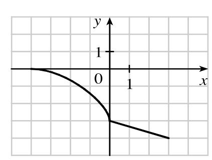
   (e) To obtain the graph of $y = -f(x)$, we reflect the graph of $y = f(x)$ about the x-axis.

   (f) To obtain the graph of $y = f^{-1}(x)$, we reflect the graph of $y = f(x)$ about the line $y = x$ (assuming f is one–to-one).

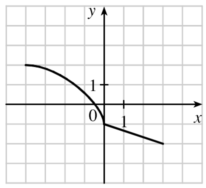
12. (a) To obtain the graph of $y = f(x - 8)$, we shift the graph of $y = f(x)$ right 8 units.



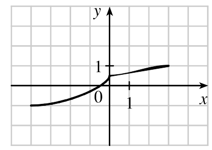
   (b) To obtain the graph of $y = -f(x)$, we reflect the graph of $y = f(x)$ about the x-axis.



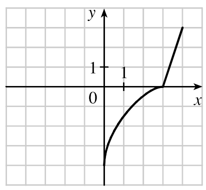
(c) To obtain the graph of $y = 2 - f(x)$, we reflect the graph of $y = f(x)$ about the x-axis, and then shift the resulting graph 2 units upward.



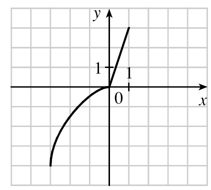
(d) To obtain the graph of $y = \frac{1}{2}f(x) - 1$, we shrink the graph of $y = f(x)$ by a factor of 2, and then shift the resulting graph 1 unit downward.



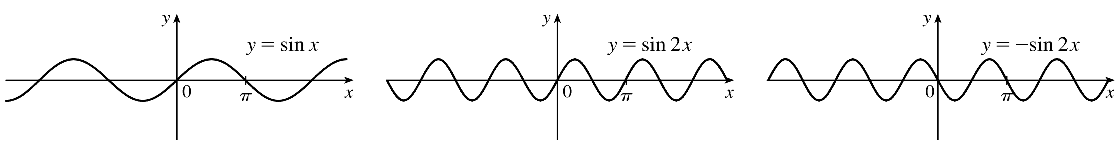
(e) To obtain the graph of $y = f^{-1}(x)$, we reflect the graph of $y = f(x)$ about the line $y = x$.



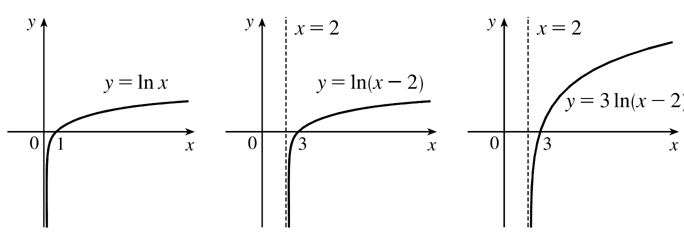
(f) To obtain the graph of $y = f^{-1}(x+3)$, we reflect the graph of $y = f(x)$ about the line $y = x$ [see part (e)], and then shift the resulting graph left 3 units.



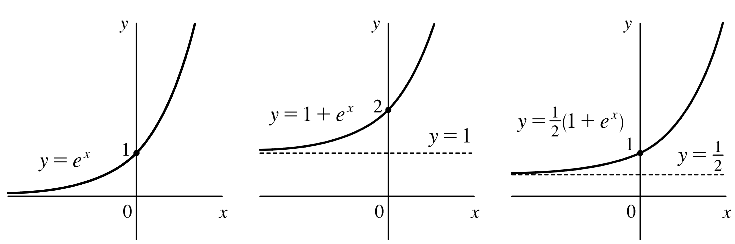
13. $y = -\sin 2x$: Start with the graph of $y = \sin x$, compress horizontally by a factor of 2, and reflect about the x-axis.



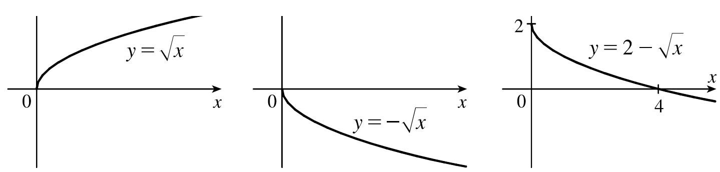
14. $y = 3\ln(x-2)$:
Start with the graph of $y = \ln x$, shift 2 units to the right, and stretch vertically by a factor of 3.



15. $y = \frac{1}{2}(1 + e^x)$:
Start with the graph of $y = e^x$, shift 1 unit upward, and compress vertically by a factor of 2.

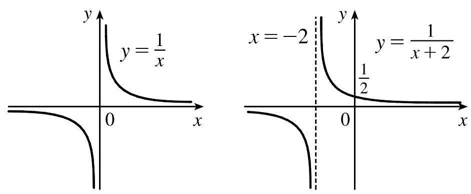


16. $y = 2 - \sqrt{x}$:
Start with the graph of $y = \sqrt{x}$, reflect about the x-axis, and shift 2 units upward.



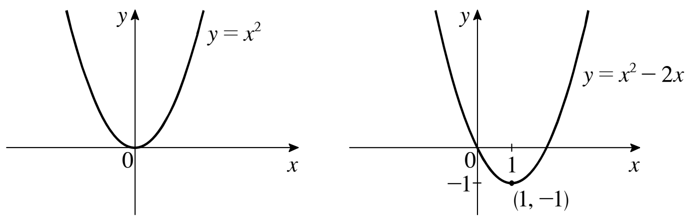
17. $f(x) = \dfrac{1}{x+2}$:

Start with the graph of $f(x) = 1/x$ and shift 2 units to the left.



18. $f(x) = x^2 - 2x$
$= (x^2 - 2x + 1) - 1$
$= (x-1)^2 - 1$

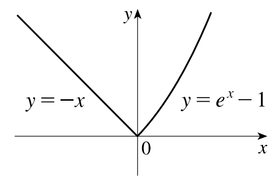
Start with the graph of $y = x^2$, shift 1 unit down, and shift 1 unit right.



19. $f(x) = \begin{cases} -x & \text{if } x < 0 \\ e^x - 1 & \text{if } x \geq 0 \end{cases}$

On $(-\infty, 0)$, graph $y = -x$ (the line with slope -1 and y-intercept 0) with open endpoint $(0, 0)$.

On $[0, \infty)$, graph $y = e^x - 1$ (the graph of $y = e^x$ shifted 1 unit downward) with closed endpoint $(0, 0)$.



20. (a) The terms of f are a mixture of odd and even powers of x, so f is neither even nor odd.

(b) The terms of f are all odd powers of x, so f is odd.

(c) $f(-x) = e^{-(-x)^2} = e^{-x^2} = f(x)$, so f is even.

(d) $f(-x) = 1 + \sin(-x) = 1 - \sin x$. Now $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$, so f is neither even nor odd.

21. $f(x) = \ln x$, $\quad D = (0, \infty)$; $\quad g(x) = x^2 - 9$, $\quad D = \mathbb{R}$.

(a) $(f \circ g)(x) = f(g(x)) = f(x^2 - 9) = \ln(x^2 - 9)$.

Domain: $x^2 - 9 > 0 \quad \Rightarrow \quad x^2 > 9 \quad \Rightarrow \quad |x| > 3 \quad \Rightarrow \quad x \in (-\infty, -3) \cup (3, \infty)$

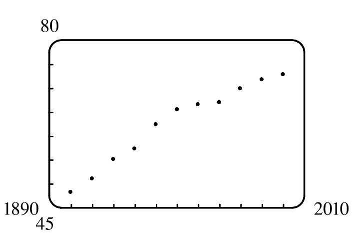
(b) $(g \circ f)(x) = g(f(x)) = g(\ln x) = (\ln x)^2 - 9$. Domain: $x > 0$, or $(0, \infty)$

(c) $(f \circ f)(x) = f(f(x)) = f(\ln x) = \ln(\ln x)$. Domain: $\ln x > 0 \quad \Rightarrow \quad x > e^0 = 1$, or $(1, \infty)$

(d) $(g \circ g)(x) = g(g(x)) = g(x^2 - 9) = (x^2 - 9)^2 - 9$. Domain: $x \in \mathbb{R}$, or $(-\infty, \infty)$

22. Let $h(x) = x + \sqrt{x}$, $g(x) = \sqrt{x}$, and $f(x) = 1/x$. Then $(f \circ g \circ h)(x) = \dfrac{1}{\sqrt{x + \sqrt{x}}} = F(x)$.

23.

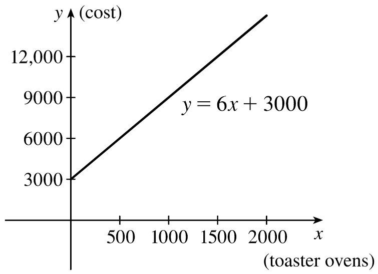


Many models appear to be plausible. Your choice depends on whether you think medical advances will keep increasing life expectancy, or if there is bound to be a natural leveling-off of life expectancy. A linear model, $y = 0.2493x - 423.4818$, gives us an estimate of 77.6 years for the year 2010.

24. (a) Let x denote the number of toaster ovens produced in one week and y the associated cost. Using the points $(1000, 9000)$ and $(1500, 12{,}000)$, we get an equation of a line:

$$y - 9000 = \frac{12{,}000 - 9000}{1500 - 1000}(x - 1000) \quad \Rightarrow$$

$$y = 6(x - 1000) + 9000 \quad \Rightarrow \quad y = 6x + 3000.$$



(b) The slope of 6 means that each additional toaster oven produced adds \$6 to the weekly production cost.

(c) The y-intercept of 3000 represents the overhead cost—the cost incurred without producing anything.

25. We need to know the value of x such that $f(x) = 2x + \ln x = 2$. Since $x = 1$ gives us $y = 2$, $f^{-1}(2) = 1$.

26. $y = \dfrac{x+1}{2x+1}$. Interchanging x and y gives us $x = \dfrac{y+1}{2y+1} \;\Rightarrow\; 2xy + x = y + 1 \;\Rightarrow\; 2xy - y = 1 - x \;\Rightarrow$

$y(2x - 1) = 1 - x \;\Rightarrow\; y = \dfrac{1-x}{2x-1} = f^{-1}(x)$.

27. (a) $e^{2\ln 3} = (e^{\ln 3})^2 = 3^2 = 9$

(b) $\log_{10} 25 + \log_{10} 4 = \log_{10}(25 \cdot 4) = \log_{10} 100 = \log_{10} 10^2 = 2$

(c) $\tan\left(\arcsin \frac{1}{2}\right) = \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}$

(d) Let $\theta = \cos^{-1} \frac{4}{5}$, so $\cos\theta = \frac{4}{5}$. Then $\sin\left(\cos^{-1} \frac{4}{5}\right) = \sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1 - \left(\frac{4}{5}\right)^2} = \sqrt{\frac{9}{25}} = \frac{3}{5}$.

28. (a) $e^x = 5 \;\Rightarrow\; x = \ln 5$ (b) $\ln x = 2 \;\Rightarrow\; x = e^2$ (c) $e^{e^x} = 2 \;\Rightarrow\; e^x = \ln 2 \;\Rightarrow\; x = \ln(\ln 2)$

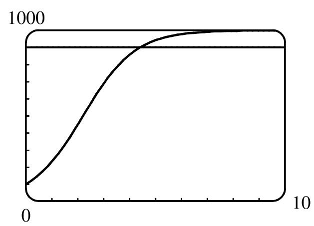
29. (a) After 4 days, $\frac{1}{2}$ gram remains; after 8 days, $\frac{1}{4}$ g; after 12 days, $\frac{1}{8}$ g; after 16 days, $\frac{1}{16}$ g.

(b) $m(4) = \dfrac{1}{2}$, $m(8) = \dfrac{1}{2^2}$, $m(12) = \dfrac{1}{2^3}$, $m(16) = \dfrac{1}{2^4}$. From the pattern, we see that $m(t) = \dfrac{1}{2^{t/4}}$, or $2^{-t/4}$.

(c) $m = 2^{-t/4} \;\Rightarrow\; \log_2 m = -t/4 \;\Rightarrow\; t = -4\log_2 m$; this is the time elapsed when there are m grams of ^{100}Pd.

(d) $m = 0.01 \;\Rightarrow\; t = -4\log_2 0.01 = -4\left(\dfrac{\ln 0.01}{\ln 2}\right) \approx 26.6$ days

30. (a)



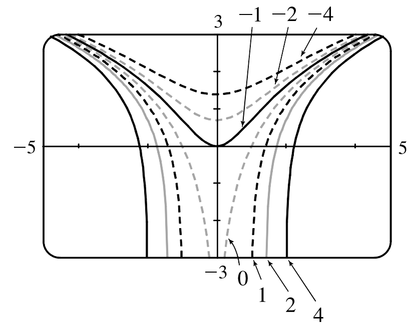
The population would reach 900 in about 4.4 years.

(b) $P = \dfrac{100{,}000}{100 + 900e^{-t}} \;\Rightarrow\; 100P + 900Pe^{-t} = 100{,}000 \;\Rightarrow\; 900Pe^{-t} = 100{,}000 - 100P \;\Rightarrow$

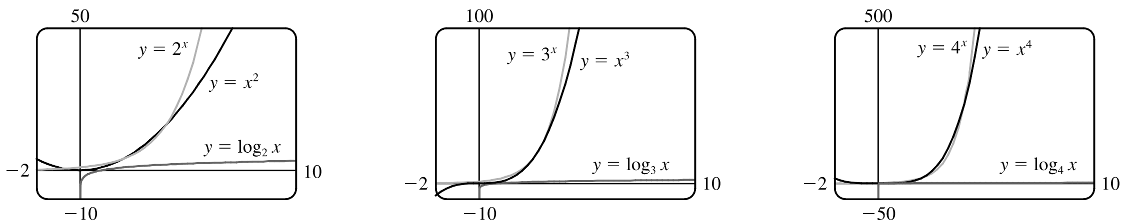
$e^{-t} = \dfrac{100{,}000 - 100P}{900P} \;\Rightarrow\; -t = \ln\left(\dfrac{1000 - P}{9P}\right) \;\Rightarrow\; t = -\ln\left(\dfrac{1000 - P}{9P}\right)$, or $\ln\left(\dfrac{9P}{1000 - P}\right)$; this is the time required for the population to reach a given number P.

(c) $P = 900 \;\Rightarrow\; t = \ln\left(\dfrac{9 \cdot 900}{1000 - 900}\right) = \ln 81 \approx 4.4$ years, as in part (a).

31. $f(x) = \ln(x^2 - c)$. If $c < 0$, the domain of f is $\mathbb{R}$. If $c = 0$, the domain of f is $(-\infty, 0) \cup (0, \infty)$. If $c > 0$, the domain of f is $(-\infty, -\sqrt{c}) \cup (\sqrt{c}, \infty)$. As c increases, the dip at $x = 0$ becomes deeper. For $c \geq 0$, the graph has asymptotes at $x = \pm\sqrt{c}$.

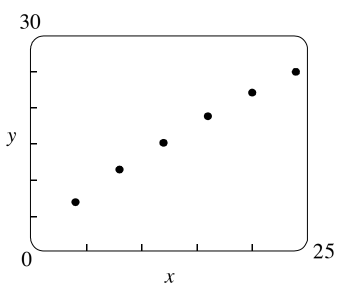


32.

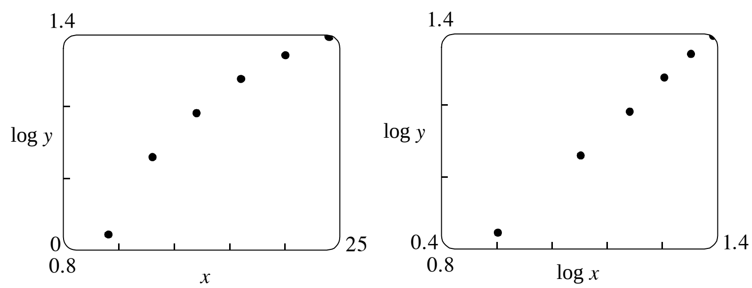


For large values of x, $y = a^x$ has the largest y-values and $y = \log_a x$ has the smallest y-values. This makes sense because they are inverses of each other.

33. (a)



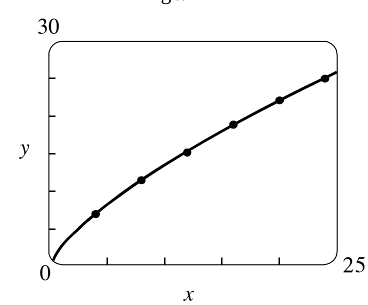
(b)



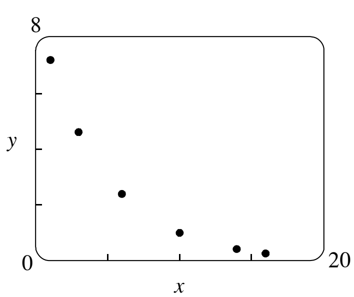
(c) Since the log-log plot is approximately linear, a power model is appropriate.

(d) Using computer software to fit a power curve to the data gives

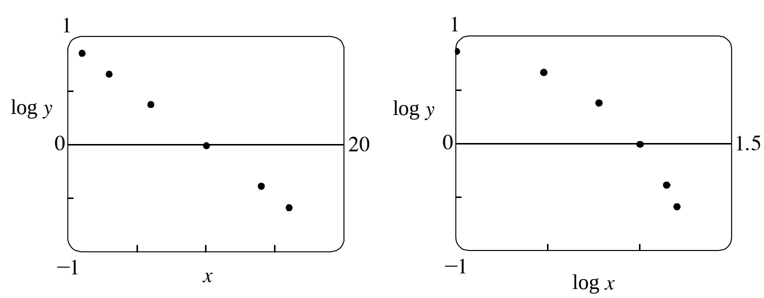
$y = (2.608377) \cdot x^{0.712277}$.



34. (a)

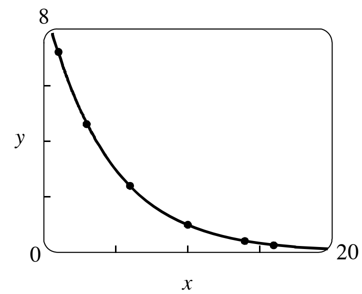


(b)

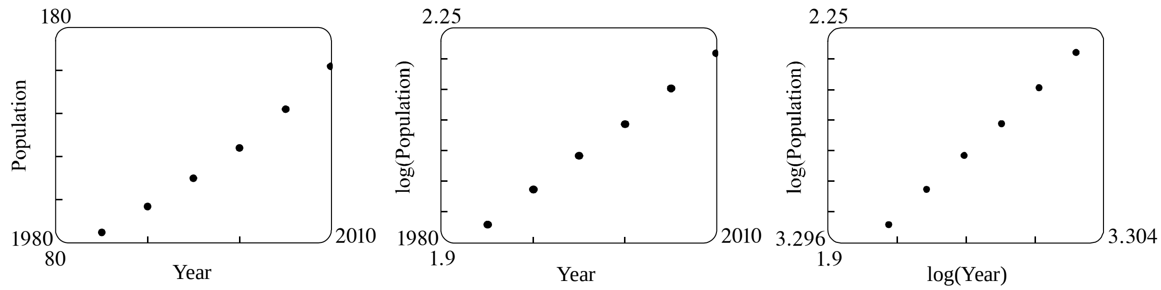


(c) Since the semi-log plot is approximately linear, an exponential model is appropriate.

(d) Using computer software to fit an exponential curve to the data gives $y = (8.982193) \cdot (0.801749)^x$.



35. (a)



Both the semi-log and log-log plots are approximately linear, so an exponential or power model is appropriate.

(b) Using computer software to fit an exponential curve to the data gives $P = \left(6.6326 \cdot 10^{-21}\right) \cdot (1.025977)^Y$ where P is the population in millions and Y is the year. Alternatively, we could have defined Y to be the number of years since 1985.

(c) $P(2008) = \left(6.6326 \cdot 10^{-21}\right) \cdot (1.025977)^{2008} \approx 153\,\text{million}$

$P(2020) = \left(6.6326 \cdot 10^{-21}\right) \cdot (1.025977)^{2020} \approx 209\,\text{million}$

36. $a_1 = \sin(1 \cdot \pi/3) = \sqrt{3}/2$

$a_2 = \sin(2 \cdot \pi/3) = \sqrt{3}/2$

$a_3 = \sin(3 \cdot \pi/3) = 0$

$a_4 = \sin(4 \cdot \pi/3) = -\sqrt{3}/2$

$a_5 = \sin(5 \cdot \pi/3) = -\sqrt{3}/2$

$a_6 = \sin(6 \cdot \pi/3) = 0$

37. $a_1 = 3$

$a_2 = 1 + 2a_1 - 1 = 1 + 2(3) - 1 = 6$

$a_3 = 2 + 2a_2 - 1 = 2 + 2(6) - 1 = 13$

$a_4 = 3 + 2a_3 - 1 = 3 + 2(13) - 1 = 28$

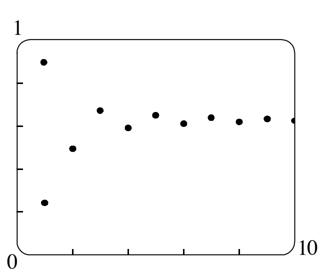
$a_5 = 4 + 2a_4 - 1 = 4 + 2(28) - 1 = 59$

$a_6 = 5 + 2a_5 - 1 = 5 + 2(59) - 1 = 122$

38. Writing the first term in the sequence as a fraction gives $-\frac{3}{1}, \frac{5}{4}, -\frac{7}{9}, \frac{9}{16}, -\frac{11}{25}$. Observe the numerator of the fractions start at 3 and increases by 2 in succeeding terms. Hence, the nth term will have numerator $2n+1$. The denominator of the nth term is n^2. The signs of each term alternate from positive to negative so we multiply by $(-1)^n$. Therefore $a_n = (-1)^n \dfrac{2n+1}{n^2}$.

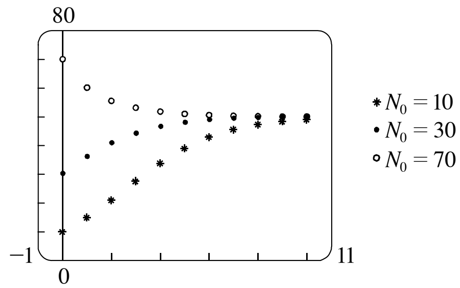
39.

t	x_t	t	x_t
0	0.9000	6	0.6126
1	0.2430	7	0.6407
2	0.4967	8	0.6215
3	0.6750	9	0.6351
4	0.5923	10	0.6257
5	0.6520		

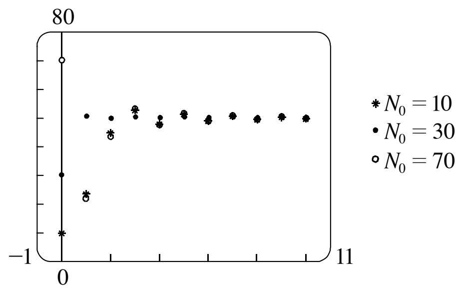


The sequence approaches 0.63 in an oscillatory fashion.

40. (a) A calculator was used to calculate and plot the first 10 terms of the Beverton-Holt recursion model with $c = 1.7$.

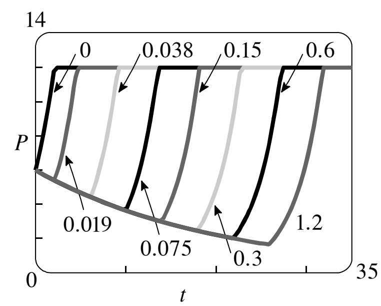


(b) A calculator was used to calculate and plot the first 10 terms of the discrete logistic equation with $r = 1.7$ and $K = 50$. Observe that both the discrete logistic model and the Beverton-Holt model from part (a) converge toward the carrying capacity. The BH model has populations either increase or decrease toward K while the logistic model has populations converge in an oscillatory fashion toward K.



CASE STUDY 1a Kill Curves and Antibiotic Effectiveness

1. When c_0 has values $\{0.019, 0.038, 0.075, 0.15, 0.3, 0.6, 1.2\}$, the respective values of a are approximately $\{2.17, 6.13, 10.01, 13.98, 17.94, 21.9, 25.86\}$. These are the times at which the bacteria population changes from exponential decay to exponential growth. We plot $P(t)$ using the piecewise function (2a) when $c_0 = 0$ ($c_0 < MIC$) and we use equation (2b) for all other values of c_0 ($c_0 \geq MIC$). This gives the graph at right.



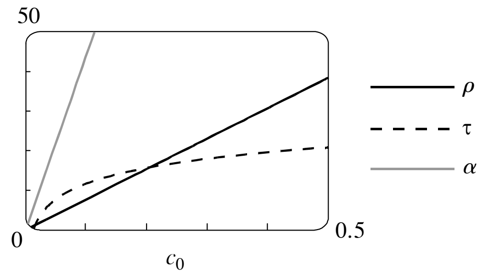
The kill curves from the data and the model show an initial decrease in bacteria population and then an increase to a maximum value of 12 CFU/mL (when $c_0 \geq 0.013$). The larger the initial concentration c_0 the longer it takes the population to reach its maximum value. When $c_0 = 0$ the bacteria population increases immediately in both the model and data. The kill curves obtained from the data are more jagged and follow an irregular path up to the maximum value of 12 as compared to the model curves. The data kill curves also appear to reach a minimum value earlier than the model kill curves.

2. The antibiotic concentration $c(t) = c_0 e^{-kt}$ is an exponential decay function that has an initial value at $c(0) = c_0$ and decreases as time passes. Thus, the peak antibiotic concentration is c_0. This gives $\rho = \dfrac{c_{\text{max}}}{MIC} = \dfrac{c_0}{MIC}$. The other measure τ must satisfy $c(\tau) = MIC \;\Rightarrow\; c_0 e^{-k\tau} = MIC \;\Rightarrow\; -k\tau = \ln\left(\dfrac{MIC}{c_0}\right) \;\Rightarrow\; \tau = \dfrac{1}{k}\ln\left(\dfrac{c_0}{MIC}\right)$.

3. $\rho = \dfrac{c_0}{MIC} = \dfrac{c_0\ \mu\text{g/mL}}{0.013\ \mu\text{g/mL}} \approx 76.92c_0$ (unitless)

$$\tau = \frac{1}{k}\ln\left(\frac{c_0}{MIC}\right) = \frac{1}{0.175\ (1/\text{hours})}\ln\left(\frac{c_0\ \mu\text{g/mL}}{0.013\ \mu\text{g/mL}}\right)$$
$$\approx 5.71\ln\left(76.92c_0\right)\ \text{hours}$$

$$\alpha = \frac{1}{0.175\ (1/\text{hours})}\left(\frac{c_0\ \mu\text{g/mL}}{0.013\ \mu\text{g/mL}}\right) \approx 439.56c_0\ \text{hours}$$



4. As seen in Problem 1 and Equation (2b), the bacteria population starts to rebound when $t = a$. So the drop in population size is

$$\Delta = P(0) - P(a) = 6 - 6Ae^{a/3} = 6\left[1 - (77co)^{-2.2}\, e^{5.7\ln(77c_0)/3}\right] = 6\left[1 - (77co)^{-2.2}\,(77c_0)^{5.7/3}\right]$$
$$= 6\left[1 - (77co)^{-0.3}\right]$$

Observe from the population functions $P(t)$ in equation (2a) and (2b) that $P(0) = 6$. When $c_0 < 0.013$, there is no drop in the bacteria count since $6e^{t/3}$ is an increasing function. Focusing on the other case $c_0 \geq 0.013$, the piecewise function decreases in the interval $t < a$. Thus, the measure T must satisfy $P(T) = 0.9P(0) \quad\Rightarrow\quad 6e^{-T/20} = 0.9\,(6) \quad\Rightarrow$ $e^{-T/20} = 0.9 \quad\Rightarrow\quad T = -20\ln 0.9$.

5. $T = f(c_0) = -20\ln 0.9 \approx 2.11$ seconds (a constant function)

$\Delta = g\,(c_0) = 6\left[1 - (77co)^{-0.3}\right]$

$\alpha = h(c_0) = \dfrac{1}{0.175\ (1/\text{hours})}\left(\dfrac{c_0\ \mu\text{g/mL}}{0.013\ \mu\text{g/mL}}\right) \approx 439.56c_0$ hours (as in question 3)

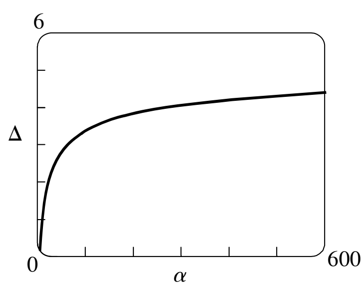
6. If the inverse function of $\alpha = h(c_0)$ exists, we can find it by solving the equation for c_0 to give $c_0 = h^{-1}(\alpha)$. Substituting this into $\Delta = g\,(c_0)$ gives the composite function $\Delta = g\left(h^{-1}(\alpha)\right) = \left(g \circ h^{-1}\right)(\alpha)$. Thus, we write Δ as a function of α by first finding $h^{-1}(\alpha)$ and then determining $\Delta = \left(g \circ h^{-1}\right)(\alpha)$ as follows:

$$\alpha = \frac{c_0}{(0.175)\,(0.013)} \quad\Rightarrow\quad c_0 = h^{-1}(\alpha) = (0.175)\,(0.013)\,\alpha \quad\Rightarrow$$

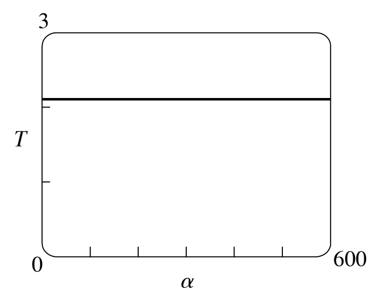
$$\Delta = g\left(h^{-1}(\alpha)\right) = g\left((0.175)\,(0.013)\,\alpha\right) = 6\left[1 - (77\,(0.175)\,(0.013)\,\alpha)^{-0.3}\right] \approx 6\left[1 - (0.175\alpha)^{-0.3}\right]$$

7. In Problem 5, we found that T is a constant function, that is, it is independent of c_0 and hence, also independent of α. So $T = f(\alpha) = -20\ln 0.9 \approx 2.11$ seconds.

8.



$\Delta = 6\left(1 - (0.175\alpha)^{-0.3}\right)$



$T = -20\ln 0.9 \approx 2.11$

9. In the bacteria population model $P(t)$ in equation (2b), the population decreases at the same rate for all initial drug concentrations governed by $P(t) = 6e^{-t/20}$ when $t < a$. Thus, the time taken to reach 90% of the initial population size (T) is the same for all initial drug concentrations (c_0) as seen in Figure 9. However, the duration of the population decline phase increases as the initial drug concentration increases because the concentration remains above the minimum inhibitory concentration (MIC) for a longer period of time. So larger initial concentrations lead to longer time periods in the population decline phase that, in turn, lead to larger drops in population size before rebound (Δ) as seen in Figure 8.