SOLUTIONS MANUAL

Applied Time Series Anlaysis with R, Second Edition

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CHAPTER 1 – Problem Solutions

Problem 1.1

$$\bar{X} = \frac{1}{n}\sum_{k=1}^{n} X_i = 71.25$$

$$\hat{\gamma}_0 = \frac{1}{n}\sum_{k=1}^{n}(X_k - \bar{X})^2 = \frac{1}{8}\sum_{k=1}^{8}(X_k - 71.25)^2 = \frac{279.5}{8} = 34.9375$$

$$\hat{\gamma}_1 = \frac{1}{8}\sum_{k=1}^{7}(X_k - 71.25)(X_{k+1} - 71.25) = \frac{117.9375}{8} = 14.74219$$

$$\hat{\rho}_0 = 1$$

$$\hat{\rho}_1 = \frac{14.74219}{34.9375} = .422$$

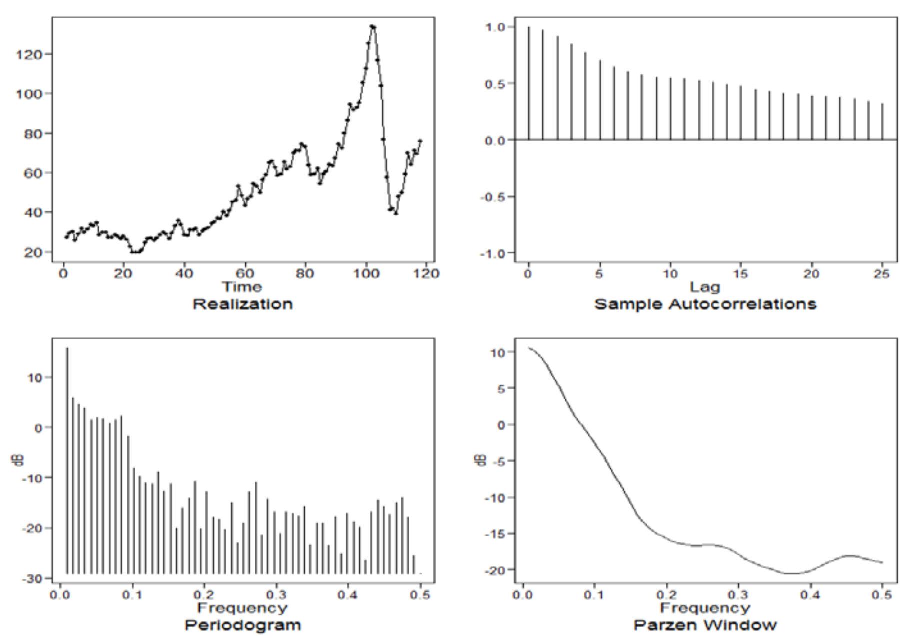
Problem 1.2

Using the tswge R code

```
data(wtcrude)
plotts.wge(wtcrude)
```

we obtain the following plots.

`wtcrude`

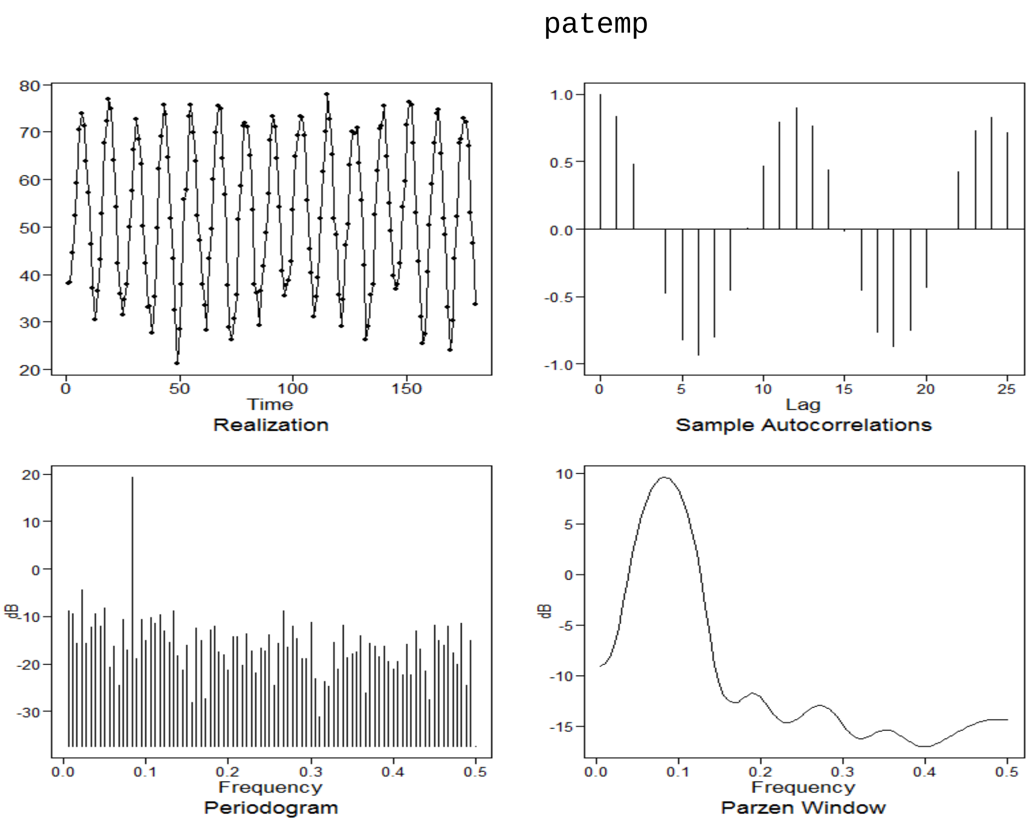


The data shows a non-cyclic wandering behavior with strong correlation between data values that are near each other in time. The sample autocorrelations show strong positive correlation (above 0.5 for $k \le 13$), and the periodogram and Parzen spectral estimator show peaks at zero with no indication of cyclic behavior. However, none of the diagnostic plots provide an indication of the precipitous drop in oil prices around t=100.

Using the tswge R code

```
data(patemp)
plotts.wge(patemp)
```

we obtain the following plots.



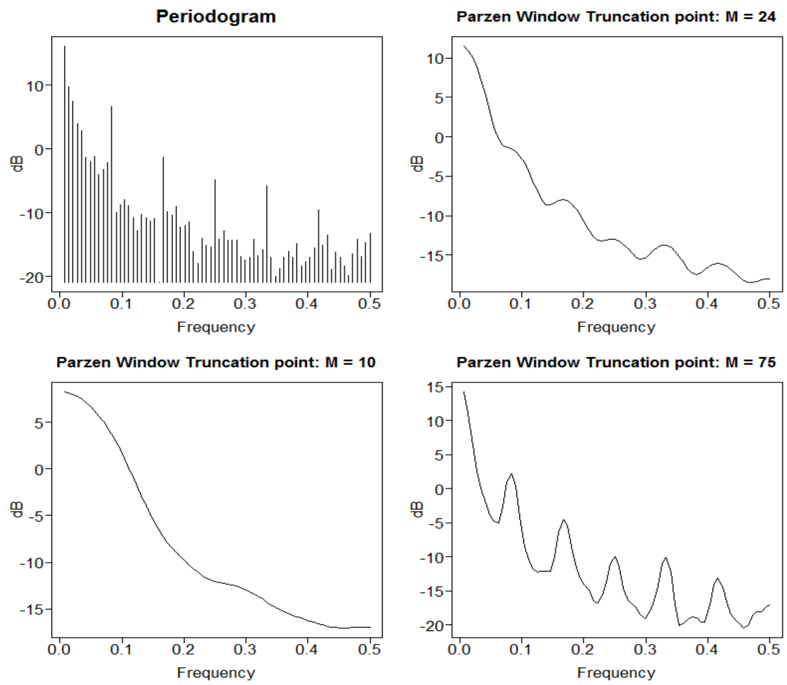
The data show a strong cyclic behavior (with a period of about 12 which makes sense (because this is monthly data). The sample autocorrelations show a very slowly damping cyclic behavior with cycle length 12 while the periodogram and Parzen spectral estimator show a strong peak at about f=1/12.

Problem 1.3

Using the tswge R code

```
data(airline)
airlog=log(airline)
plotts.parzen.wge(airlog,m2=c(10,75))
```

we obtain



All spectral estimator have a peak at zero, and all except M=10 show the peaks at the harmonic of the fundamental frequency 1/12. The spectral estimator that is closest to the AR spectral estimator in Figure 1.25 is the Parzen spectral estimator with M=75. The Parzen spectral estimator with M=10 is clearly too smooth.

Problem 1.4

Realization 1 has wandering behavior which is corresponds to fairly high positive autocorrelations for lags of modest length (d) and spectral density with a peak at zero (a): 1, d, a

Realization 2 has very little structure (random-line) which corresponds to small or zero autocorrelations (a) and flat spectral density (d): 2, a, d

Realization 3 is pseudo-cyclic with a period of about 10 which corresponds to the autocorrelations with damped sinusoidal behavior of about period 10 (b) and a peak in the spectral density at about f=0.1 (c): 3, b, c

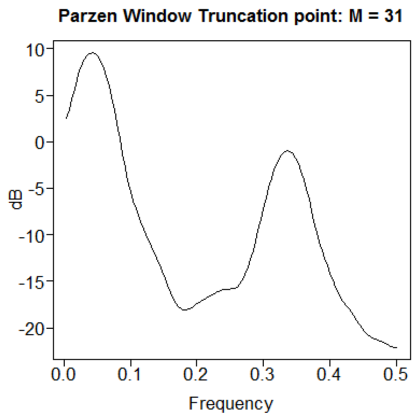
Realization 4 seems to have a pseudo-cyclic behavior with period a little less than 10 along with a higher frequency components. This corresponds to the autocorrelations in (c) that show a dames sinusoidal behavior of about 12 along with some high-frequency distortion. The spectral density in (b) shows a peak at about f=1/12 along with a higher frequency peak.: 4, c, b

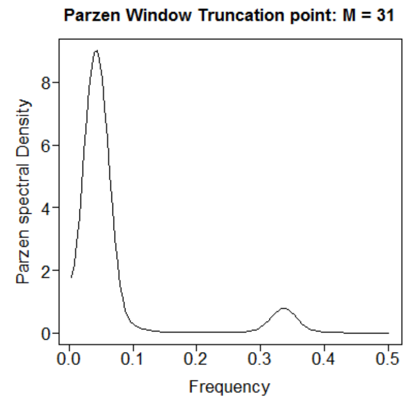
Problem 1.5

Using the tswge R code (M=31 is the default for *n*=100)

```
data(fig1.21a)
plotts.parzen.wge(fig1.21a)
plotts.parzen.wge(fuig1.21a,dbplot=FALSE)
```

We get the following plots (not showing the periodogram that is also plotted):





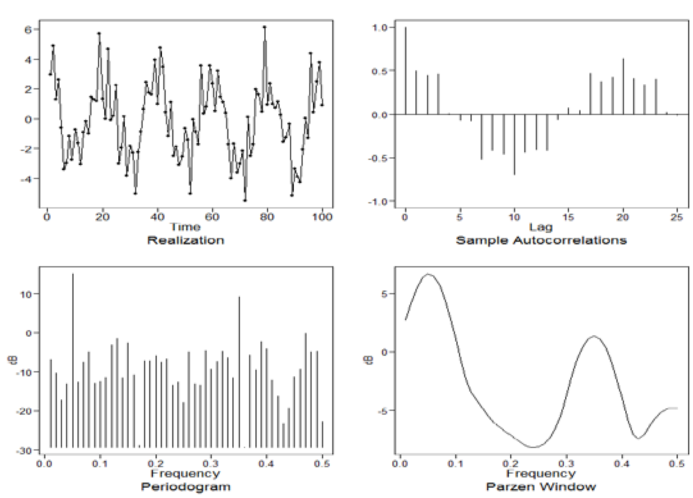
Both show strong indication of a peak at abouit *f*=0.05. However, the plot in dB shows the secondary peak at about *f*=0.33 much more clearly.

Problem 1.6

(a-d) The tswge R code

```
x=gen.sigplusnoise.wge(n=100,coef=c(3,1.5),freq=c(.05,.35),psi=c(0,2))
plotts.sample.wge(x)
```

produces the following plots.



The realizations shows a dominant frequency with period about 20 along with higher-frequency behavior. The autocorrelations show the periodic behavior of associated with the period 20 (f=0.05) with some indication of a higher frequency component. These two plots provide very little evidence regarding the nature of the higher frequency behavior. The two spectral plots clearly suggest frequency behavior at both f=0.05 and f=0.35.

Problem 1.7
Answers will vary

Theoretical Problems

Problem 1.8

$$\begin{aligned}\text{Cov}(Z_t,\ Z_{t+k}) &= E[(Z_t-\mu)(Z_{t+k}-\mu)] \\ &= E[(Z_{t-k}-\mu)(Z_t-\mu)] \\ &= E[(Z_t-\mu)(Z_{t-k}-\mu)] \\ &= \text{Cov}(Z_t,\ Z_{t-k}).\end{aligned}$$

Problem 1.9 $Z_t = a_t$ where a_t is drawn independently for any t and where the distribution of a_t is given by

$$a_t = \begin{cases} c & \text{with prob. } \frac{1}{2} \\ -c & \text{with prob. } \frac{1}{2} \end{cases} \qquad \text{for } c > 0.$$

(a)

$$\begin{aligned}E[Z_t] &= E[a_t] \\ &= \frac{1}{2}c + \frac{1}{2}(-c) = 0\end{aligned}$$

$E[Z_t] = 0$ is a constant .

$$\begin{aligned}\text{Var}(Z_t) &= \text{Var}(a_t) \\ &= E[a_t^2] \text{ since } E[a_t] = 0 \\ &= \frac{1}{2}c^2 + \frac{1}{2}c^2 = c^2 < \infty\end{aligned}$$

$\text{Var}(Z_t) = c^2$ is a finite a constant.

$$
\begin{aligned}
\gamma_k &= Cov(Z_t, Z_{t+k}) \\
&= E[Z_t Z_{t+k}] - 0 \\
&= E[a_t a_{t+k}]
\end{aligned}
$$

$$\gamma_0 = c^2$$

$$\gamma_1 = 0$$

$$\gamma_k = 0, \quad k \geq 1$$

$$\gamma_k = \begin{cases} c^2, & k = 0 \\ 0, & k \neq 0 \end{cases}$$

So, γ_k depends only on lag *k*, and the process is covariance stationary.

(b) This part was answered in the solution to part (a).

Problem 1.10

Assume $X_t^{(1)}, \ldots, X_t^{(k)}$ are k uncorrelated covariance stationary time series and

$$E(X_t^{(i)}) = \mu_i$$

$$\text{Var}(X_t^{(i)}) = \sigma_i^2 < \infty \qquad i = 1, \ldots, k$$

Now let's consider the sum of the k series

$$E\left(\sum_{i=1}^{k} X_t^{(i)}\right) = \sum_{i=1}^{k} \mu_i \quad \text{constant}$$

$$\text{Var}(\sum_{i=1}^{k} X_t^{(i)}) = \sum_{i=1}^{k} \sigma_i^2 < \infty \text{ constant (since the } X_t^{(i)}\text{'s are uncorrelated)}$$

$$
\begin{aligned}
\gamma(h) &= E\left[\left(\sum_{i=1}^{k} X_t^{(i)} - \sum_{i=1}^{k} \mu_i\right)\left(\sum_{i=1}^{k} X_{t+h}^{(i)} - \sum_{i=1}^{k} \mu_i\right)\right] \\
&= E\left[\sum_{i=1}^{k}\left(X_t^{(i)} - \mu_i\right)\sum_{j=1}^{k}\left(X_{t+h}^{(j)} - \mu_j\right)\right] \\
&= E\left[\sum_{i=1}^{k}\sum_{j=1}^{k}\left(X_t^{(i)} - \mu_i\right)\left(X_{t+h}^{(j)} - \mu_j\right)\right] \\
&= E\left[\sum_{i=1}^{k}\left(X_t^{(i)} - \mu_i\right)\left(X_{t+h}^{(i)} - \mu_j\right)\right] \text{ (since } E\left[\left(X_t^{(i)} - \mu_i\right)\left(X_{t+h}^{(j)} - \mu_j\right)\right] = 0
\end{aligned}
$$

(unless $j = i$)

$$= \sum_{i=1}^{k} \gamma_h^{(i)} \text{ which depends only on lag } h.$$

Problem 1.11
Based on Example 1.3.

If $Y \sim U(0, 2\pi)$ then $\cos(\lambda t + Y)$ is covariance stationary. Since
$\psi_1, \psi_2, \psi_3 \sim U(0, 2\pi)$, then $\cos(2\pi(0.025)t + \psi_2)$, $1.5\cos(2\pi(0.1)t + \psi_2)$, and $2\cos(2\pi(0.15)t + \psi_0$

are all covariance stationary.

ψ_1, ψ_2, ψ_3 uncorrelated $\Rightarrow \cos(2\pi(0.025)t + \psi_1)$, $1.5\cos(2\pi(0.1)t + \psi_2)$, and $2\cos(2\pi(0.15)t + \psi_2$ are all covariance stationary.

By Exercise 1.10 $X_t = \cos(2\pi(0.015)t\psi_1) + 1.5\cos(2\pi(0.1)t + \psi_2) + 2\cos(2\pi(0.15)t + \psi_3)$

is covariance stationary.

(a) By Example 1.3 $E(X_t) = 0$

$X^{(1)}(t) = \cos(2\pi(0.025)t + \psi_1)$, $X^{(2)}(t) = 1.5\cos(2\pi(0.1)t + \psi_2)$, $X^{(3)}(t) = 2\cos(2\pi(0.15)t + \psi_3)$

$$\begin{aligned}\gamma(h) &= \gamma^{(1)}(h) + \gamma^{(2)}(h) + \gamma^{(3)}(h) \\ &= \frac{1}{2}\cos(2\pi(.025)h + \frac{1.5^2}{2}\cos(2\pi(.1)h) + \frac{2^2}{2}\cos(2\pi(.15)h) \\ &= .5\cos(2\pi(.025)h + 1.125\cos(2\pi(.1)h) + 2\cos(2\pi(.15)h)\end{aligned}$$

(b) $\sigma_X^2 = \gamma(0) = .5 + 1.125 + 2 = 3.625$.

(c) $$\begin{aligned}\rho(h) = \frac{\gamma(h)}{\gamma(0)} &= \frac{.5\cos(2\pi(.025)h + 1.125\cos(2\pi(.1)h) + 2\cos(2\pi(.15)h)}{3.625} . \\ &= .138\cos(2\pi(.025)h + .310\cos(2\pi(.1)h) + .552\cos(2\pi(.15)h)\end{aligned}$$

Problem 1.12
$Y_t = C_1\cos(2\pi f_1 t + \psi_1) + N_t, \qquad t = 0, \pm 1, \ldots$.
From Example 1.3

$H_t = C_1 \cos(2\pi f_1 t + \psi_1)$ is covariance stationary.

i. $E[H_t] = 0$

ii. $\text{Var}(H_t) = \frac{C_1^2}{2} < \infty$

iii. $\gamma_k = \frac{C_1^2}{2}\cos(2\pi f_1 k)$

Also, N_t is discrete iswhite noise, $\Rightarrow$ the N_t's are identically distributed.

i. $E[N_t] = 0$

ii. $\text{Var}(N_t) = \sigma_N^2 < \infty$ (fixed and finite)

iii. $\gamma_k^{(N)} = 0$ if $k \neq 0$ and $\gamma_k^{(N)} = \sigma_N^2 < \infty$

So,

i. $E[Y_t] = E[H_t] + E[N_t] = 0$ (constant).

ii. $\text{Var}(Y_t) = \text{Var}(H_t) + \text{Var}(N_t) = \frac{C_1^2}{2} + \sigma_N^2 < \infty$ (constant)

iii.
$$\begin{aligned}\gamma_k &= \text{Cov}(Y_t, Y_{t+k}) \\ &= E[Y_t Y_{t+k}] \\ &= E[(H_t + N_t)(H_{t+k} + N_{t+k})] \\ &= E[(H_t H_{t+k})] + E[H_t N_{t+k}] + E[N_t H_{t+k}] + E[N_t N_{t+k}] \\ &= E[H_t H_{t+k}] + E[N_t N_{t+k}]\end{aligned}$$

So,
$$\begin{aligned}\gamma_k &= \frac{C_1^2}{2} + \sigma_N^2, \quad \text{if } k = 0 \\ &= \frac{C_1^2}{2}\cos(2\pi f_1 k), \quad \text{if } k \neq 0\end{aligned}$$

which only depends on k. Thus, Y_t is also a covariance stationary process.

Problem 1.13

$Z_t \sim N(0, \sigma^2)$

(a) Stationary

i. $E(X_t) = E(a + bZ_t) + cZ_{t-1}) = a$ (constant)

ii. $\text{Var}(X_t) = \text{Var}\left[a + bZ_t + cZ_{t-1}\right] = \text{Var}(a) + \text{Var}(bZ_t) + \text{Var}(cZ_{t-1})$ (since uncorrelated)

$= b^2\sigma^2 + c^2\sigma^2 = (b^2 + c^2)\sigma^2$ (finite constant)

iii. $\gamma(t_1,t_2)=\text{cov}(X_{t_1},Z_{t_2})=\text{cov}(a+bZ_{t_1}+cZ_{t_1-1},a+bZ_{t_2}+cZ_{t_2-1})$

$$=b^2\text{cov}(Z_{t_1},Z_{t_2})+bc(\text{cov}(Z_{t_1},Z_{t_2-1})+\text{cov}(Z_{t_1},Z_{t_2}))+c^2\text{cov}(Z_{t_1-1},Z_{t_2-1})$$

$$=\begin{cases}(b^2+c^2)\sigma^2 & t_1=t_2\\ bc\sigma^2 & |t_1-t_2|=1.\\ 0 & \text{otherwise}\end{cases}$$

(b) Stationary

i. $E(X_t)=E(a+bZ_0)=a+bE(Z_0)=a$

ii. $\text{Var}(X_t)=\text{Var}(a+bZ_0)=b^2\sigma^2$

iii. $\gamma(t_1,t_2)=\text{cov}(a+bZ_0,a+bZ_0)=b^2\text{Var}(Z_0)=b^2\sigma^2$

(c) Not stationary

i. $E(X_t)=E(Z_t\cos(ct)+Z_{t-1}\sin(ct))=0$

ii. $\text{Var}(X_t)=\text{Var}(Z_t\cos(ct)+Z_{t-1}\sin(ct))=\cos^2(ct)\text{Var}(Z_t)+\sin^2(ct)\text{Var}(Z_{t-1})$

$$=(\cos^2(ct)+\sin^2(ct))\sigma^2=\sigma^2\quad\text{(finite constant)}$$

iii. $\gamma(t_1,t_2)=\text{cov}[Z_{t_1}\cos(ct_1)+Z_{t_1}\sin(ct_1),Z_{t_2}\cos(ct_1)+Z_{t_2-1}\sin(ct_2)]$

$$=\text{cov}[Z_{t_1}\cos(ct_1),Z_{t_2}\cos(ct_2)]+\text{cov}[Z_{t_1}\cos(ct_1),Z_{t_2-1}\sin(ct_2)]$$

$$+\text{cov}[Z_{t_1-1}\sin(ct_1),Z_{t_2}\cos(ct_2)]+\text{cov}[Z_{t_1-1}\sin(ct_1),Z_{t_2-1}\sin(ct_2)]$$

$$=\begin{cases}[\cos^2(ct_1)+\sin^2(ct_1)]\sigma^2=\sigma^2 & t_1-t_2=0\\ \cos(ct_1)\sin(ct_2)\sigma^2 & t_2-t_1=1\\ \cos(ct_2)\sin(ct_1)\sigma^2 & t_2-t_1=-1\\ 0 & \text{otherwise}\end{cases}$$

Depends on t_1,t_2, no only depends on t_2-t_1 .

(d) Not stationary

$\text{Var}(X_t)=\cos^2(ct)\sigma^2$ which is a function of t

Problem 1.14

(1) $X_t^{(1)}=X_t-0.5X_{t-1}$.

Let $\mu=E(X_t)$ and $\sigma^2=\text{Var}(X_t)$.

i. $E(X_t^{(1)}) = E(X_t - 0.5X_{t-1}) = \mu - 0.5\mu = 0.5\mu$ (constant for all t) .

ii. $\text{Var}(X_t^{(1)}) = \text{Var}(X_t - 0.5X_{t-1}) = \text{Var}(X_t) + 0.25\text{Var}(X_{t-1}) - \text{cov}(X_t, X_{t-1})$.

$= 1.25\sigma^2 - \gamma_1$ (which is a finite constant)

iii. Let $k \neq 0$ be an integer.

$$
\begin{aligned}
\gamma_k^{(1)} &= \text{Cov}[X_t - .5X_{t-1}, X_{t+k} - .5X_{t+k1}] \\
&= E[\{(X_t - \mu) - .5(X_{t-1} - \mu)\}\{(X_{t+k} - \mu) - .5(X_{t+k-1} - \mu)\}] \\
&= E[(X_t - \mu)(X_{t+k} - \mu)] - .5E[(X_{t-1} - \mu)(X_{t+k} - \mu)] \\
&\quad - .5E[(X_t - \mu)(X_{t+k-1} - \mu)] + .25E[(X_{t-1} - \mu)(X_{t+k-1} - \mu)] \\
&= \gamma_k - .5\gamma_{k+1} - .5\gamma_{k-1} + .25\gamma_k \\
&= 1.25\gamma_k - .5\gamma_{k+1} - .5\gamma_{k-1}, \quad \text{if } k \neq 0 \\
&\qquad = 1.25\sigma^2 - \gamma_1, \quad \text{if } k = 0
\end{aligned}
$$

which depends only on lag k. So, $X_t^{(1)}$ is covariance stationary.

(2) $X_t^{(2)} = X_t^{(1)} - X_{t-1}^{(1)}$

i. $E(X_t^{(2)}) = E(X_t^{(1)} - X_{t-1}^{(1)}) = E(X_{t-1}^{(1)}) = 0.5\mu - 0.5\mu = 0$ (constant)

ii. $\text{Var}(X_t^{(2)}) = \text{Var}(X_t^{(1)} - X_{t-1}^{(1)}) = \text{Var}(X_t^{(1)}) + \text{Var}(X_{t-1}^{(1)}) - 2\,\text{cov}(X_t^{(1)}, X_{t-1}^{(1)})$

$$
\begin{aligned}
&= 1.25\sigma^2 - \gamma_1 + 1.25\sigma^2 - \gamma_1 - 2\gamma_1^{(1)} \\
&= 2.5\sigma^2 - 2\gamma_1 - 2(1.25\gamma_1 - .5\gamma_2 - .5\gamma_0) \\
&= 3.5\sigma^2 - 4.5\gamma_1 + \gamma_2
\end{aligned}
$$

iii. $\gamma^{(2)}(t_1, t_2) = \text{C ov}(X_{t_1}^{(1)} - X_{t_1-1}^{(1)}, X_{t_2}^{(1)} - X_{t_2-1}^{(1)})$

$$
\begin{aligned}
&= E[\{(X_{t_1}^{(1)} - .5\mu) - (X_{t_1-1}^{(1)} - .5\mu)\}\{(X_{t_2}^{(1)} - .5\mu) - (X_{t_2-1}^{(1)} - .5\mu)\}] \\
&= E[(X_{t_1}^{(1)} - .5\mu)(X_{t_2}^{(1)} - .5\mu) - E[(X_{t_1-1}^{(1)} - .5\mu)(X_{t_2}^{(1)} - .5\mu)] \\
&\quad - E[(X_{t_2-1}^{(1)} - .5\mu)(X_{t_1}^{(1)} - .5\mu)] + E[(X_{t_1-1}^{(1)} - .5\mu)(X_{t_2-1}^{(1)} - .5\mu)] \\
&= \gamma^{(1)}(t_2 - t_1) - \gamma^{(1)}(t_2 - t_1 - 1) - \gamma^{(1)}(t_1 - t_2 - 1) + \gamma^{(1)}(t_2 - t_1) \\
&= \gamma_k^{(1)} - \gamma_{k-1}^{(1)} - \gamma_{k+1}^{(1)} + \gamma_k^{(1)} \\
&= 1.25\gamma_k - .5\gamma_{k+1} - .5\gamma_{k-1} - (1.25\gamma_{k-1} - .5\gamma_k - .5\gamma_{k-2}) \\
&\quad - (1.25\gamma_{k+1} - .5\gamma_{k+2} - .5\gamma_k) + 1.25\gamma_k - .5\gamma_{k+1} - .5\gamma_{k-1} \\
&= 3.5\gamma_k - 2.25\gamma_{k+1} - 2.25\gamma_{k-1} + .5\gamma_{k+2} + .5\gamma_{k-2}
\end{aligned}
$$

which only depends on lag k. So, $X_t^{(2)}$ is covariance stationary.

Problem 1.15

(a) Joint distribution of $(Z_{t_1}, ..., Z_{t_n})$ is that of $(X, X, ..., X)$

Joint distribution of $(Z_{t_1+h}, ..., Z_{t_n+h})$ is that of $(X, X, ..., X)$

Hence it is strictly stationary.

For weak stationarity:

i. $E(Z_t) = \mu$ constant

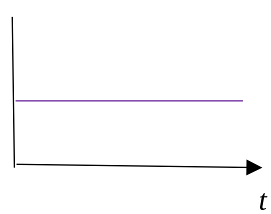
ii. $\text{Var}(Z_t) = \sigma^2 < \infty$ constant

iii. $\gamma(k) = E(Z_t - \mu)(Z_{t+k} - \mu)$
$= \sigma^2$ which does not depend on t.

Therefore, Z_t is weakly stationary.

(b) $\gamma(k) = \sigma^2$ as shown in (a) and so $\rho(k) = \dfrac{\gamma(k)}{\gamma(0)} = 1$ for all k .

(c) A typical realization is



(d) $\bar{Z} = \dfrac{1}{n}\sum_{t=1}^{n} Z_t = X$

and $\lim_{n\to\infty} E(\bar{Z} - \mu)^2 = \lim_{n\to\infty} E(X - \mu)^2 = \lim_{n\to\infty} \sigma^2 = \sigma^2 \neq 0$

$\therefore$ Z_t is not ergodic for μ.

Problem 1.16

(a) $E[Y_t] = a + bt + \mu$ (where $E[X_t] = \mu$, $\text{Var}[X_t] = \sigma^2 < \infty$)

which depends on t if $b \neq 0$ so Y_t is not covariance stationary.

(b) $Z_t = a + bt + X_t - (a + b(t-a) + X_{t-1})$

$= b + X_t - X_{t-1}$

i. $E[Z_t] = b$

ii. $\text{Var}[Z_t] = \text{Var}[X_t] + \text{Var}[X_{t-1}] - 2\text{Cov}[X_t, X_{t-1}]$

$= 2\sigma^2 - 2\gamma_1$

$= 2(\sigma^2 - \gamma_1)$ which is a finite, positive constant

iii. $\gamma(k) = \text{Cov}(X_t - X_{t-1}, X_{t+k} - X_{t+k-1})$

$$= E[\{(X_t - \mu) - (X_{t-1} - \mu)\}\{(X_{t+k} - \mu) - (X_{t+k-1} - \mu)\}]$$

$$= E[(X_t - \mu)(X_{t+k} - \mu)] - E[(X_{t-1} - \mu)(X_{t+k} - \mu)]$$

$$- E[(X_t - \mu)(X_{t+k-1} - \mu)] + E[(X_{t-1} - \mu)(X_{t+k-1} - \mu)]$$

$$= \gamma_k - \gamma_{k+1} - \gamma_{k-1} + \gamma_k$$

$$= 2\gamma_k - \gamma_{k+1} - \gamma_{k-1}, \quad \text{if } k \neq 0$$

$$= 2(\sigma^2 - \gamma_1), \quad \text{if } k = 0$$

which only depends on lag k. So, Y_t is covariance stationary.

Problem 1.17

$E[X_t^{(1)}] = E[X_t^{(2)}] = 0$

$\text{Var}[X_t^{(1)}] = \sigma_a^2 + \theta^2\sigma_a^2 = \sigma_a^2(1+\theta^2)$

$\text{Var}[X_t^{(2)}] = \sigma_a^2 + \dfrac{1}{\theta^2}\sigma_a^2 = \sigma_a^2(1+\dfrac{1}{\theta^2})$

$$\gamma_k^{(1)} = \text{Cov}[X_t^{(1)}, X_{t+1}^{(1)}] = E[(a_t - \theta a_{t-1})(a_{t+1} - \theta a_t)]$$

$$= E[a_t a_{t+1}] - \theta E[a_{t-1}a_{t+1}] - \theta E[a_t^2] + \theta^2 E[a_{t-1}a_t]$$

$$= -\theta\sigma_a^2 \quad \text{(since } a_t \text{ is white noise)}$$

$$\gamma_k^{(2)} = \text{Cov}[X_t^{(2)}, X_{t+1}^{(2)}] = E[(a_t - \frac{1}{\theta}a_{t-1})(a_{t+1} - \frac{1}{\theta}a_t)]$$

$$= E[a_t a_{t+1}] - \frac{1}{\theta}E[a_{t-1}a_{t+1}] - \frac{1}{\theta}E[a_t^2] + \frac{1}{\theta^2}E[a_{t-1}a_t]$$

$$= -\frac{1}{\theta}\sigma_a^2 \quad \text{(since } a_t \text{ is white noise)}$$

Note that if k > 1 then the associated covariances are zero (there are no subscripts that "match").

So, $\rho_k^{(1)} = \frac{\gamma_k^{(1)}}{\gamma_0^{(1)}}$. For $k = 1, \rho_1^{(1)} = \frac{\gamma_1^{(1)}}{\gamma_0^{(1)}} = \frac{-\theta}{1+\theta^2}$. For $k \geq 2$, $\rho_k^{(1)} = 0$.

and $\rho_k^{(2)} = \frac{\gamma_k^{(2)}}{\gamma_0^{(2)}}$. For $k = 1, \rho_1^{(2)} = \frac{\gamma_1^{(2)}}{\gamma_0^{(2)}} = \frac{-\frac{1}{\theta}}{1+\frac{1}{\theta^2}} = \frac{-\theta}{\theta^2+1}$. For $k \geq 2$, $\rho_k^{(2)} = 0$.

Problem 1.18

Using the trigonometric identity for the cosine of a sum, the result follows by noting that

$$\begin{aligned}
\cos[2\pi(1.3)k] &= \cos[2\pi k + 2\pi(.3)k] \\
&= \cos[2\pi k]\cos[2\pi(.3)k] - \sin[2\pi k]\sin[2\pi(.3)k] \\
&= \cos[2\pi(.3)k] \text{ because for integer } k,\ \cos[2\pi k] = 1 \text{ and } \sin[2\pi k] = 0
\end{aligned}$$

Problem 1.19

(a)
$$\begin{aligned}
P_x(-f) &= \sum_{k=-\infty}^{-\infty} \exp(-2\pi ik(-f))\gamma_k \\
&= \sum_{k=\infty}^{-\infty} \exp(-2\pi i(-k)(-f))\gamma_{-k} \\
&= \sum_{k=-\infty}^{\infty} \exp(-2\pi ikf)\gamma_k \\
&= P_x(f)
\end{aligned}$$

$$S_x(-f) = \frac{P_x(-f)}{\sigma_x^2} = \frac{P_x(f)}{\sigma_x^2} = S_x(f)$$

(b) $P_X(f) = \sum_{k=-\infty}^{\infty} \gamma_k e^{-2\pi ifk}$

$$= \sum_{k=-\infty}^{\infty} \gamma_k [\cos(2\pi fk) - i\sin(2\pi fk)] \quad \text{(Euler's formula)}$$

$$= \sum_{k=-\infty}^{-1} \gamma_k [\cos(2\pi fk)] + \sum_{k=1}^{\infty} \gamma_k [\cos(2\pi fk)] + \gamma_0$$

$$- i\left[\sum_{k=-\infty}^{-1} \gamma_k [\sin(2\pi fk)] + \sum_{k=1}^{\infty} \gamma_k [\sin(2\pi fk)] \right]$$

$$= 2\sum_{k=1}^{\infty} \gamma_k [\cos(2\pi fk)] + \gamma_0 - i\left[\sum_{k=1}^{\infty} \gamma_k [\sin(2\pi fk)] + \gamma_{-k}[\sin(2\pi fk)] \right]$$

$$= 2\sum_{k=1}^{\infty} \gamma_k [\cos(2\pi fk)] + \sigma_X^2$$

$$S_x(f) = \frac{P_x(f)}{\sigma_x^2} = 1 + 2\sum_{k=1}^{\infty} \rho_k \cos[2\pi fk]$$

(c) $\int_{-.5}^{.5} P_X(f) e^{2\pi ifk} df = \int_{-.5}^{.5} \left[\sum_{\ell=-\infty}^{\infty} \gamma_\ell e^{-2\pi if\ell} \right] e^{2\pi ifk} df$

$$= \sum_{\ell=-\infty}^{\infty} \gamma_\ell \int_{-.5}^{.5} e^{2\pi if(k-\ell)} df$$

$$= \gamma_k + \sum_{\ell \neq k} \frac{\gamma_\ell}{2\pi if(k-\ell)} \left[e^{\pi i(k-\ell)} - e^{-\pi i(k-\ell)} \right]$$

$$= \gamma_k + \sum_{\ell \neq k} \frac{\gamma_\ell}{2\pi if(k-\ell)} \left[2i\sin(\pi(k-\ell)) \right]$$

$$= \gamma_k \quad \text{since } \sin(\pi k) = 0 \text{ for integer } k$$

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \int_{-.5}^{.5} \frac{P_X}{\sigma_X^2}(f) e^{2\pi ifk} df = \int_{-.5}^{.5} S_X(f) e^{2\pi ifk} df$$

(d) Let $k = 0$ in (c). We immediately get

$$\sigma_X^2 = \int_{-.5}^{.5} P_X(f) e^{2\pi i f(0)} df = \int_{-.5}^{.5} P_X(f) df$$

$$\rho_0 = 1 = \int_{-.5}^{.5} S_X(f) e^{2\pi i f(0)} df = \int_{-.5}^{.5} S_X(f) df$$

(e) $$\gamma_k = \int_{-.5}^{.5} P_X(f) e^{2\pi i f k} df$$

$$= \int_{-.5}^{.5} P_X(f)\left[\cos(2\pi fk\right] df + i\int_{-.5}^{.5} P_X(f)\left[\sin(2\pi fk\right] df$$

$$= 2\int_{0}^{.5} P_X(f)\cos(2\pi fk) df$$

since $P_X(f)\cos(2\pi fk)$ is an odd function

$P_X(f)\sin(2\pi fk)$ is an even function

$$\rho_k = \frac{\gamma_k}{\sigma_X^2} = 2\int_0^{.5} \frac{P_X(f)}{\sigma_X^2}\cos(2\pi fk) df = 2\int_0^{.5} S_X(f)\cos(2\pi fk) df$$

Problem 1.20

(II) $$\frac{1}{n}\left|\sum_{t=1}^{n} X_t e^{-2\pi ikt/n}\right|^2 = \frac{1}{n}\left|\sum_{t=1}^{n} X_t \cos(2\pi kt/n) - i\sum_{t=1}^{n} X_t \sin(2\pi kt/n)\right|^2$$

$$= \frac{1}{n}\left[\left(\sum_{t=1}^{n} X_t \cos(2\pi kt/n)\right)^2 + \left(\sum_{t=1}^{n} X_t \sin(2\pi kt/n)\right)^2\right]$$

$$= \frac{1}{n}\frac{n^2}{4}\left[a_k^2 + b_k^2\right]$$

(III) $$= \frac{n}{4}\left[a_k^2 + b_k^2\right]$$

$$= \frac{n}{4}\frac{4}{n^2}\left[\left(\sum_{t=1}^{n}(X_t-\bar{X})\cos\frac{2\pi k}{n}t\right)^2+\left(\sum_{t=1}^{n}(X_t-\bar{X})\sin\frac{2\pi k}{n}t\right)^2\right]$$

$$= \frac{1}{n}\sum_{t=1}^{n}\sum_{s=1}^{n}(X_t-\bar{X})(X_s-\bar{X})\left(\cos(\frac{2\pi k}{n}t)\cos(\frac{2\pi k}{n}s)+\sin(\frac{2\pi k}{n}t)\sin(\frac{2\pi k}{n}s)\right)$$

$$= \frac{1}{n}\sum_{t=1}^{n}\sum_{s=1}^{n}(X_t-\bar{X})(X_s-\bar{X})\cos\left(\frac{2\pi k}{n}(t-s)\right)$$

$$= \frac{1}{n}\left[\sum_{t=1}^{n}(X_t-\bar{X})^2+2\sum_{1\le t<s\le n}(X_t-\bar{X})(X_s-\bar{X})\cos\left(\frac{2\pi k}{n}(t-s)\right)\right]$$

$$= \frac{1}{n}\left[\sum_{t=1}^{n}(X_t-\bar{X})^2+2\sum_{v=1}^{n-1}\left(\sum_{t=1}^{n-v}(X_t-\bar{X})(X_{t+v}-\bar{X})\cos\left(\frac{2\pi k}{n}v\right)\right)\right]$$

$$= \frac{1}{n}\sum_{t=1}^{n}(X_t-\bar{X})^2+2\sum_{v=1}^{n-1}\left(\frac{1}{n}\sum_{t=1}^{n-v}(X_t-\bar{X})(X_{t+v}-\bar{X})\cos\left(\frac{2\pi k}{n}v\right)\right)$$

(I) $$= \hat{\gamma}_0+2\sum_{v=1}^{n-1}\hat{\gamma}_v\cos\left(\frac{2\pi k}{n}v\right)$$

That is, forms I, II, and III are equivalent.

Problem 1.21 Without loss of generality we assume $\mu = 0$.

$$\mathrm{Var}(\bar{X}) = E\left[\frac{X_1+\cdots+X_n}{n}\right]^2$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}E(X_iX_j)$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\gamma_{i-j}$$

$$= \frac{1}{n^2}\left[n\gamma_0+(n-1)\gamma_{-1}+(n-1)\gamma_1+(n-2)\gamma_{-2}+(n-2)\gamma_2+\cdots+(n-(n-1))\gamma_{-(n-1)}+(n-(n-1))\gamma_{n-1}\right]$$

$$= \frac{1}{n^2}\sum_{k=-(n-1)}^{n-1}\left(n-|k|\right)\gamma_k$$

$$= \frac{\sigma^2}{n}\sum_{k=-(n-1)}^{n-1}\left(1-\frac{|k|}{n}\right)\rho_k.$$